

# Introduction to Digital Systems

## Part II (4 lectures)

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### Combinational Logic Blocks

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# Lecture 7 contents

- Block oriented combinational logic design
- Arithmetic Circuits

# Addition

- Addition is a very common arithmetic operation in digital systems
- Let's recall some concepts ...

# Addition of Binary Numbers

- Addition and subtraction of non-decimal numbers by hand uses the same technique that you know from school for decimal numbers.
- The only catch is that the addition and subtraction tables are different.
- To add two **unsigned binary numbers**  $X$  and  $Y$ , we add together the least significant bits with an initial carry ( $c_{in}$ ) of 0, producing carry ( $c_{out}$ ) and sum ( $s$ ) bits according to the table. We continue processing bits from right to left, adding the carry out of each column into the next column's sum.

Example:

$$\begin{array}{r} 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \\ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \\ + \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \\ \hline 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \end{array}$$

$c_{in}$	x	y	$c_{out}$	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

# Subtraction of Binary Numbers

- Binary subtraction is performed similarly, using borrows ( $b_{in}$  and  $b_{out}$ ) instead of carries between steps, and producing a difference bit  $d$ .

Examples:

$$\begin{array}{r} 0111100 \\ 11100001 \\ - 10101101 \\ \hline 00110100 \end{array}$$

$$\begin{array}{r} 111 \\ 1000 \\ - 0011 \\ \hline 0101 \end{array}$$

$b_{in}$	$x$	$y$	$b_{out}$	$d$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

# Overflow

- With  $n$  bits it is possible to represent **unsigned integer numbers** ranging from 0 to  $2^n-1$ .
- If an arithmetic operation produces a result that exceeds the range of the number system, **overflow** is said to occur.
- Overflows can easily be detected by analyzing **a carry or borrow from the most significant bit**.
  - the carry bit  $c_{out}$  or the borrow bit  $b_{out}$  out of the MSB = 1

## Examples:

$n=8: [0..255]$

$$173_{10} + 97_{10} = 270_{10}$$

$$\begin{array}{r} 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \\ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \\ + 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \\ \hline 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \end{array}$$

overflow

$n=4: [0..15]$

$$4_{10} - 11_{10} = -7_{10}$$

$$\begin{array}{r} 1 \ 0 \ 1 \ 1 \\ 0 \ 1 \ 0 \ 0 \\ - 1 \ 0 \ 1 \ 1 \\ \hline 1 \ 0 \ 0 \ 1 \end{array}$$

overflow

# Representation of Negative Numbers

- There are many ways to represent negative numbers.
- In everyday business we use the **signed-magnitude system** (i.e. reserve a special symbol to indicate whether a number is negative).
- However, most computers use **two's-complement representation**:
  - The **most significant bit (MSB)** of a number in this system serves as the sign bit; a number is negative if and only if its MSB is 1.
  - The weight of the MSB is negative: for an  $n$ -bit number the weight is  $-2^{n-1}$ .
  - The decimal equivalent for a two's-complement binary number is computed the same way as for an unsigned number, except that the weight of the MSB is negative:
    - $D = d_{n-1}d_{n-2} \dots d_1d_0 = -2^{n-1} + \sum_{i=0}^{n-2} d_i \times 2^i$

## Examples:

$$1010_2 = ???_{10}$$

$$1010_2 = -2^3 + 2^1 = -8 + 2 = -6_{10}$$

$$1111_2 = ???_{10}$$

$$1111_2 = -2^3 + 2^2 + 2^1 + 2^0 = -8 + 4 + 2 + 1 = -1_{10}$$

$$0111_2 = ???_{10}$$

$$0111_2 = 2^2 + 2^1 + 2^0 = 4 + 2 + 1 = 7_{10}$$

# Two's Complement Representation

- For  $n$  bits, the range of representable numbers is  $[-2^{n-1}, 2^{n-1}-1]$ .
- For  $n=4$ , the range is  $[-8, 7]$ :

0	0	0	0	0
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
-8	1	0	0	0
-7	1	0	0	1
-6	1	0	1	0
-5	1	0	1	1
-4	1	1	0	0
-3	1	1	0	1
-2	1	1	1	0
-1	1	1	1	1

# Towards implementation

- The immediate approach
  - Digit-wise addition and carry propagation
  - Iterative hardware
  - Building blocks
    - Half-Adder
    - Full-Adder

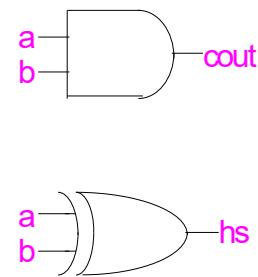
# The Half-Adder (HA)

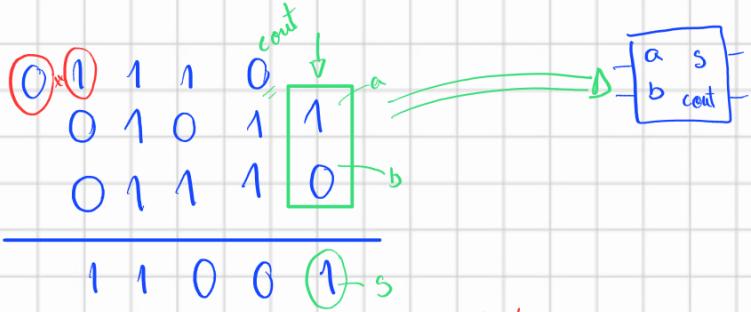
- Inputs: 2 single bit operands (a, b)
- Outputs
  - A 2 bit result:
    - The Half-Sum (hs)
    - The Carry-out (Cout)
    - Note that:  $0 \leq (\text{Cout}, \text{hs})_{10} \leq 2$

a	b	$c_{\text{out}}$	hs
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

$$c_{\text{out}} = a \cdot b$$

$$\text{hs} = a \oplus b$$





Son dos complementos  $\Rightarrow$  está correcto!

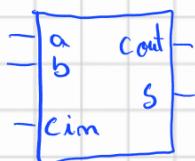
Son dos complement  $\Rightarrow$  Overflow

a	b	cout	s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

$$\text{cout} = ab$$

$$s = a \oplus b$$

Mos tenemos o  
problema do  
cout!



a	b	cim	cout	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

		cout					
		ab	00	01	11	10	
cim	ab	00	0	0	0	1	0
		01	1	1	1	1	1

$$\text{cout} = ab + a \text{cim} + b \text{cim}$$

$$s = \bar{a}\bar{b}\text{cim} + \bar{a}b\bar{\text{cim}} + a\bar{b}\bar{\text{cim}} + ab\text{cim}$$

$$= \bar{a}(b \oplus \text{cim}) + a(\bar{b} \oplus \text{cim})$$

$$= a \oplus b \oplus \text{cim}$$

		s				
		ab	00	01	11	10
cim	ab	00	0	1	0	1
		01	1	0	1	0

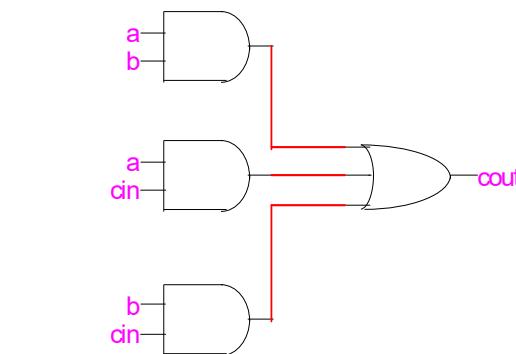
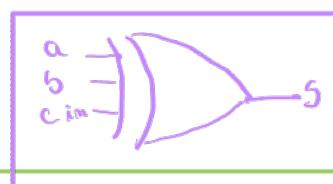
# The Full-Adder (FA)

- Inputs: 2 single bit operands ( $a$ ,  $b$ ) and a Carry-in bit ( $C_{in}$ )
- Outputs
  - A 2 bit result:
    - The Sum ( $S$ )
    - The Carry-out ( $C_{out}$ )
    - Note that:  $0 \leq (C_{out}, S)_{10} \leq 3$

$C_{in}$	$a$	$b$	$C_{out}$	$s$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$c_{out} = a \cdot b + a \cdot c_{in} + b \cdot c_{in}$$

$$s = a \oplus b \oplus c_{in}$$

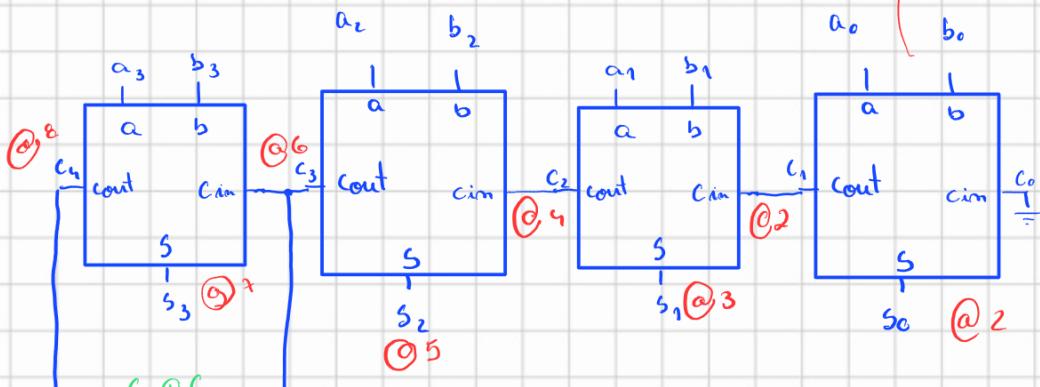


$$C_{out} = ab + aC_{in} + bC_{in}$$

$$S = a \oplus b \oplus C_{in}$$

$$\begin{array}{c} C_4 \ C_3 \ C_2 \ C_1 \ C_0 \\ \textcircled{1} \ \textcircled{2} \ \textcircled{3} \ \textcircled{4} \ \textcircled{5} \\ A: a_3 \ a_2 \ a_1 \ a_0 \\ B: b_3 \ b_2 \ b_1 \ b_0 \\ S_3 \ S_2 \ S_1 \ S_0 \end{array}$$

Usando estos!



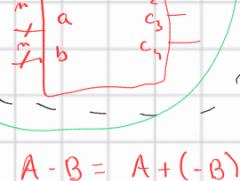
Problema: É muito demorado

⑤ = muitos de atraso

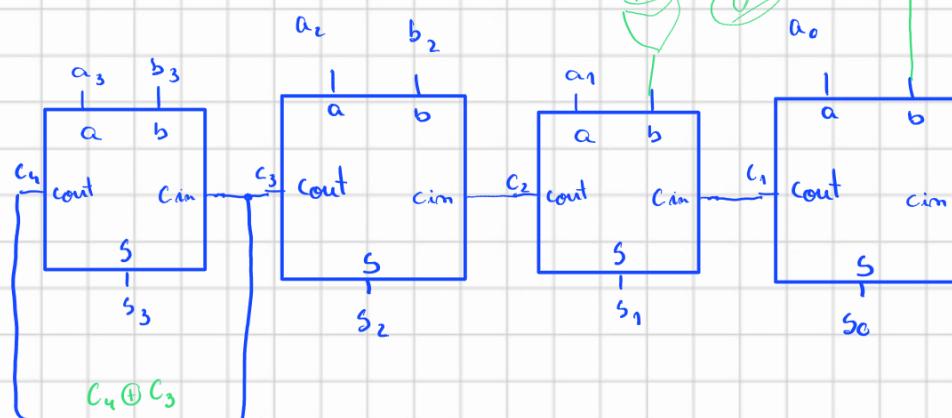
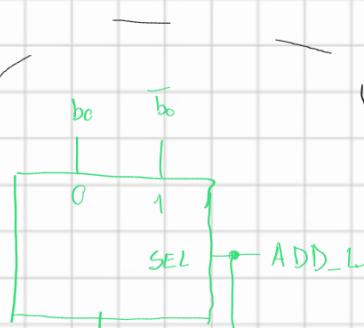
Overflow  
1 de fórmula iguala

ADD\_L, utiliza-se para distinguir se queremos somar ou subtrair.

Queremos:



$$A - B = A + (-B)$$



Overflow  
0 de fórmula iguala

$$C_{out} = ab + a \cdot c_{in} + b \cdot c_{in} = g + c_{in}(\underbrace{A+B}_P)$$

$$g_i = a_i \cdot b_i$$

$$p_i = a_i + b_i$$

$$c_1 = a_0 b_0 + a_0 c_0 + b_0 c_0 = (g_0 + c_0 p_0) = (g) 3$$

$$c_2 = g_1 + c_1 p_1 = (g_1) + (g p_1) + (c_0 p_0 p_1) = (g) 3$$

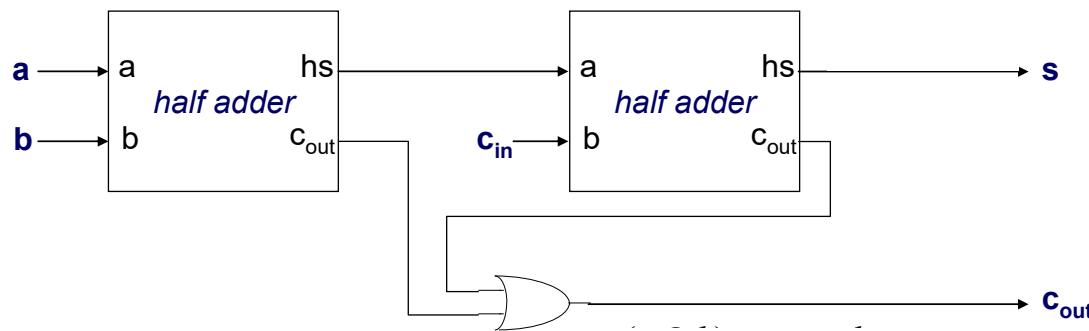
$$c_3 = g_2 + c_2 p_2 = \dots$$

$$c_4 = g_3 + c_3 p_3 = \dots$$

Como as fórmulas não gigantes, só se faz até aos 4 bits

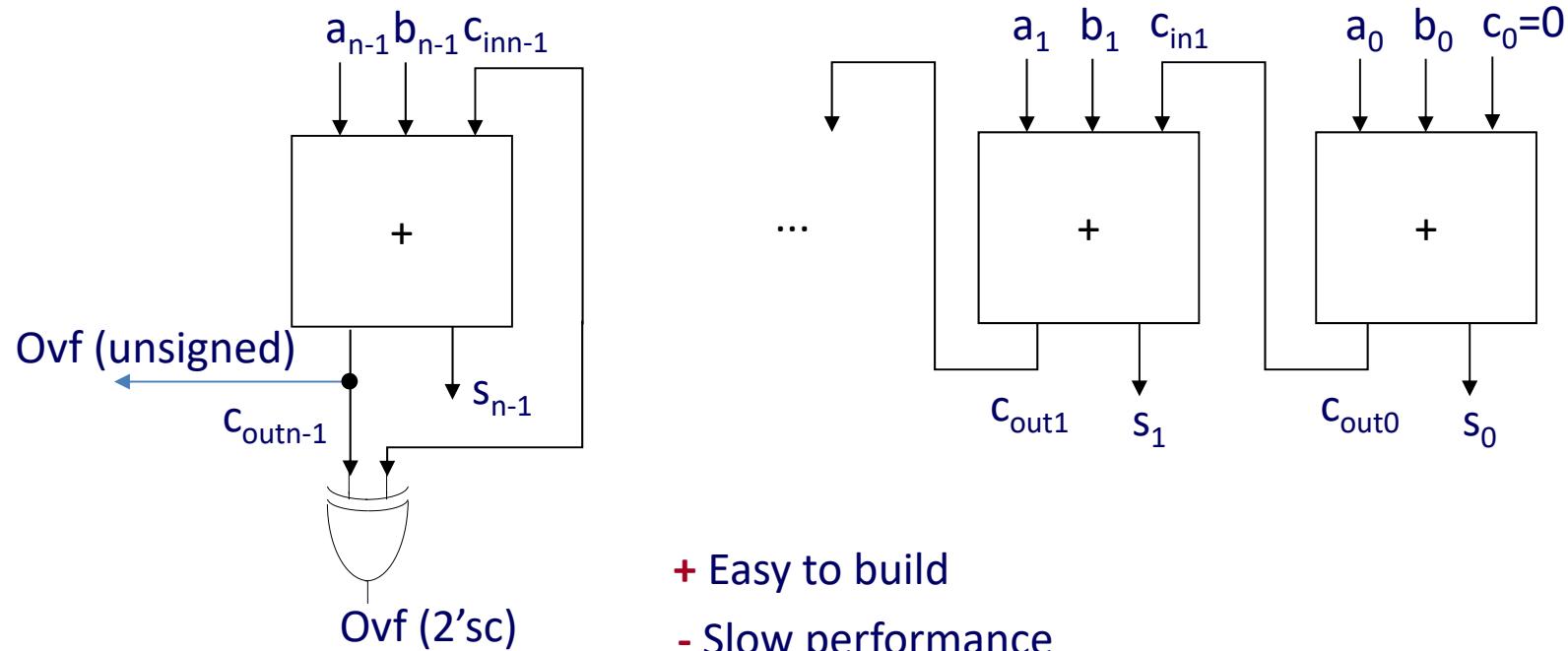
# Exercise

- Show that the following circuit is Full-Adder



# Ripple Adders

- The bit-wise addition and carry-propagation is implemented by a cascade of Full-Adders
- An iterative circuit paradigm



# Full Subtractor

- Inputs: 2 single bit operands ( $a$ ,  $b$ ) and a Borrow-in bit ( $b_{in}$ )

- Outputs

- A 2 bit result:

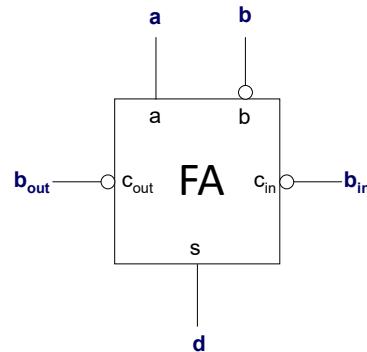
- The difference ( $d$ )
    - The Borrow-out ( $b_{out}$ )

- Exercise

$b_{in}$	$a$	$b$	$b_{out}$	$d$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	1
0	1	1	0	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	1	1

$$b_{out} = \bar{a} \cdot b + \bar{a} \cdot b_{in} + b \cdot b_{in}$$

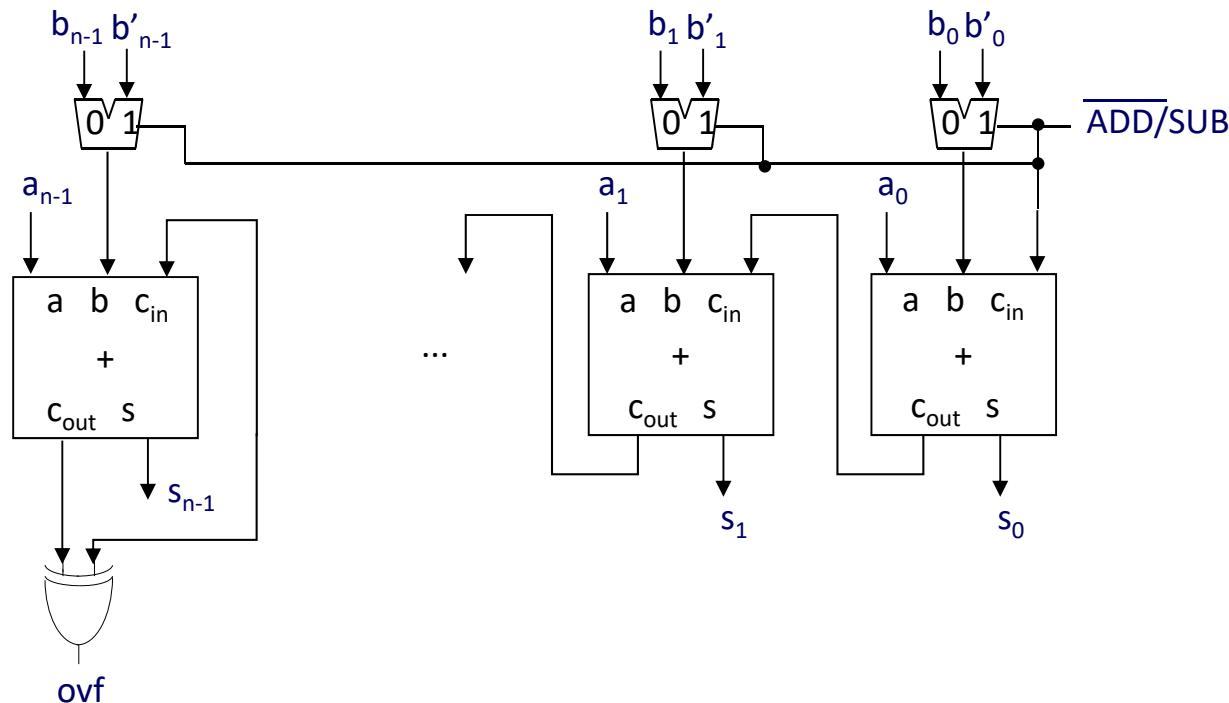
$$d = a \oplus b \oplus b_{in}$$



Write the output equations and show that the Full-Adder with modified inputs is a subtractor

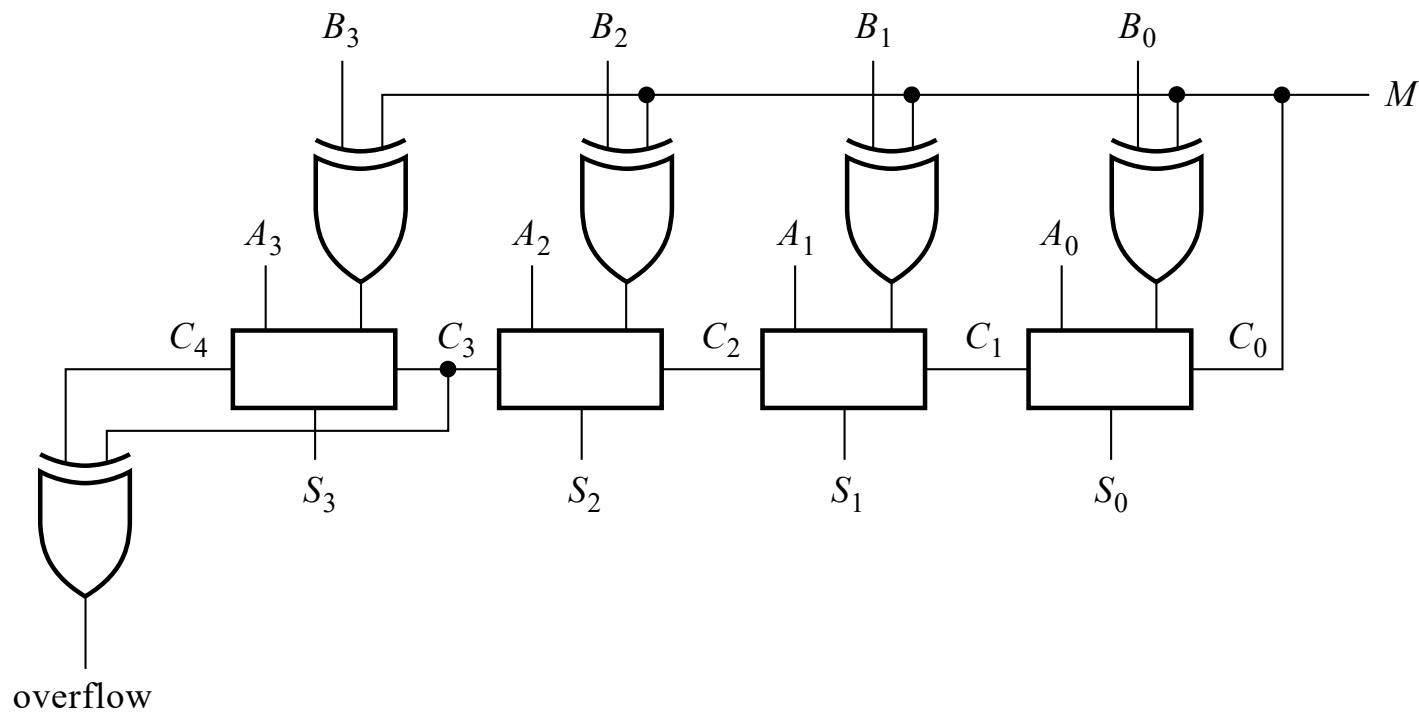
# Ripple Adder/Subtractor

- Using 2's complement representation means the same hardware for addition and subtraction
  - “Muxed”  $b$  and  $b'$  inputs.
  - Initial carry = 1 if subtraction

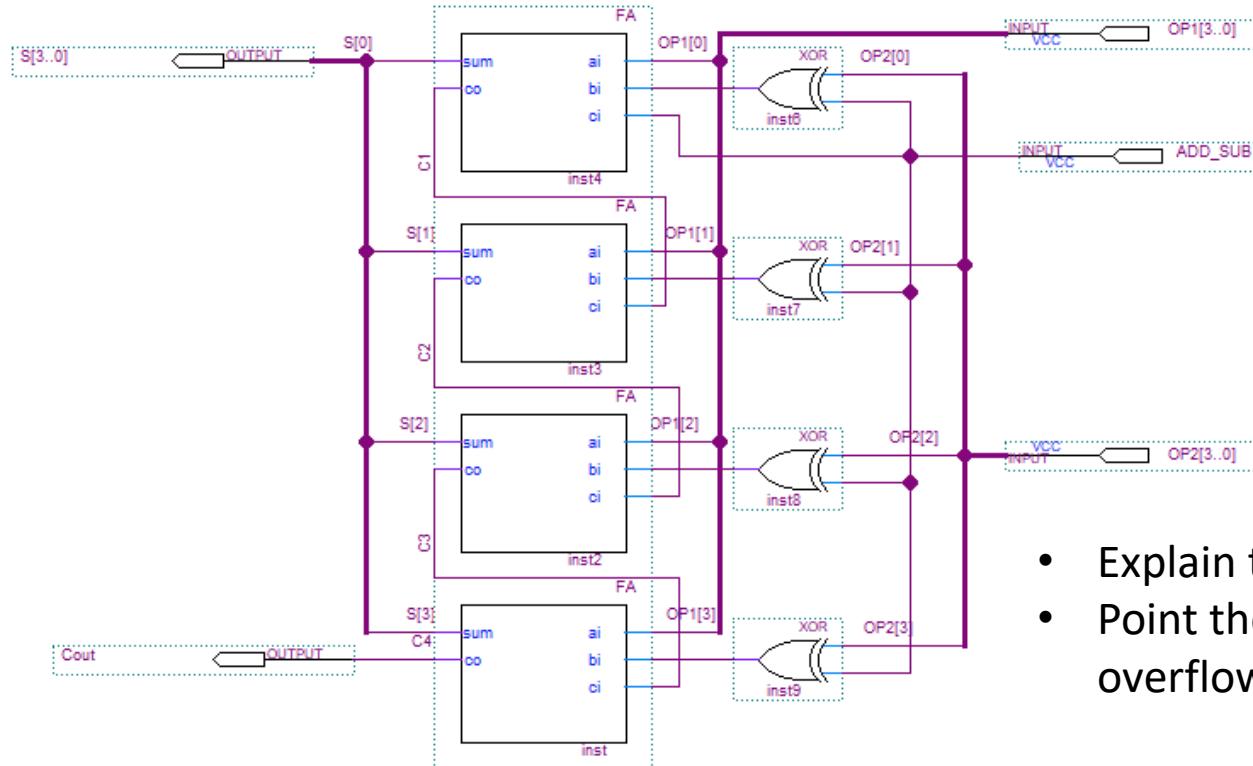


# Exercise

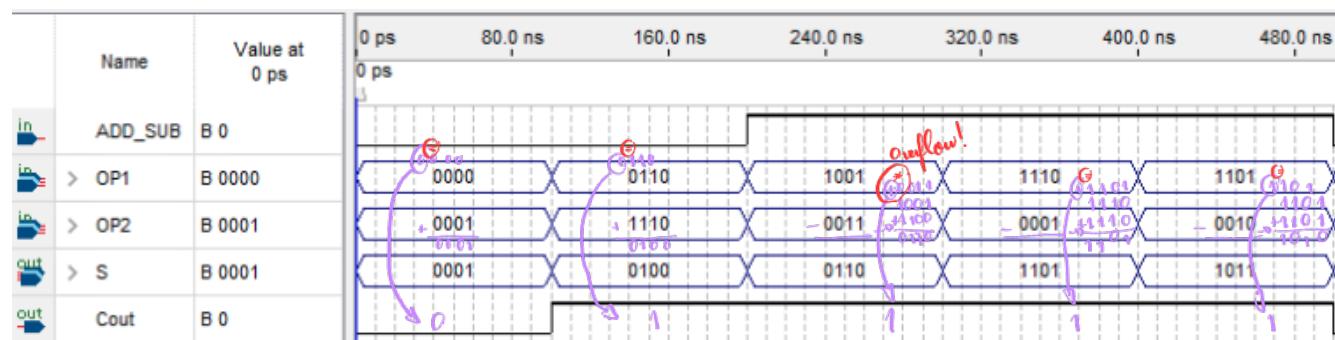
- Verify that the following implementation is equivalent to the adder/subtractor circuit of the previous slide



# Exercise



- Explain the timing diagram.
- Point the time intervals where overflow occurs



# Carry-Lookahead Adders (CLA)

- The idea: Compute the  $i$ th carry non-iteratively
- The starting points:
  - The usual equations

$$s_i = a_i \oplus b_i \oplus c_i$$

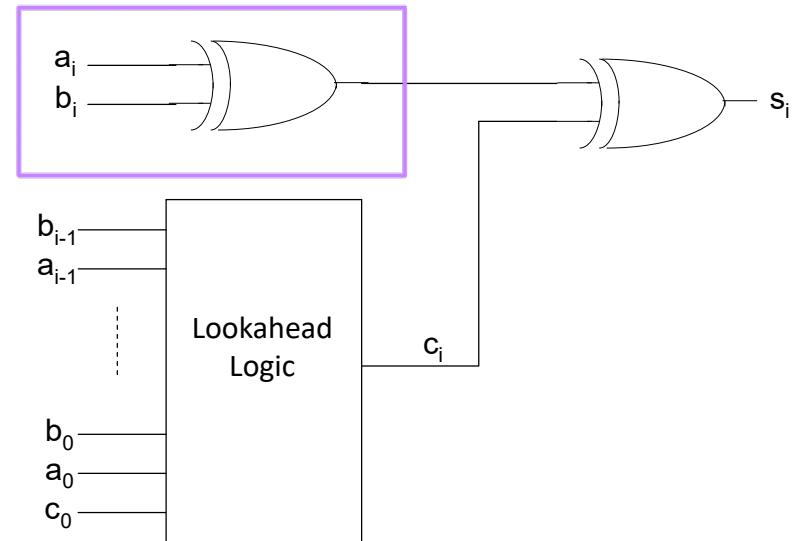
$$c_{i+1} = a_i \cdot b_i + a_i \cdot c_i + b_i \cdot c_i$$

- Intermediate functions

- Carry Generation

- Carry Propagation     $g_i = a_i \cdot b_i$

$$p_i = a_i + b_i$$



# Remarks about the Carry

- At any stage, we necessarily have a carry generation  $C_{i+1} = 1$  whenever  $a_i = b_i = 1$ .  
So  $g_i = a_i \cdot b_i$
- If  $a_i \neq b_i$  but  $a_i + b_i = 1$  the carry propagates since  $C_{i+1} = C_i$  so  $p_i = a_i + b_i$
- Finally the Carry equation becomes

$$C_{i+1} = g_i + p_i \cdot C_i$$

# The 4 bit CLA

- Carry equations

$$c_1 = \underbrace{g_0 + p_0 \cdot c_0}_{\text{Carry 1}} = \text{Carry 2}$$

$$c_2 = g_1 + p_1 \cdot c_1 = g_1 + p_1 \cdot (g_0 + p_0 \cdot c_0) = g_1 + p_1 \cdot g_0 + p_1 \cdot p_0 \cdot c_0$$

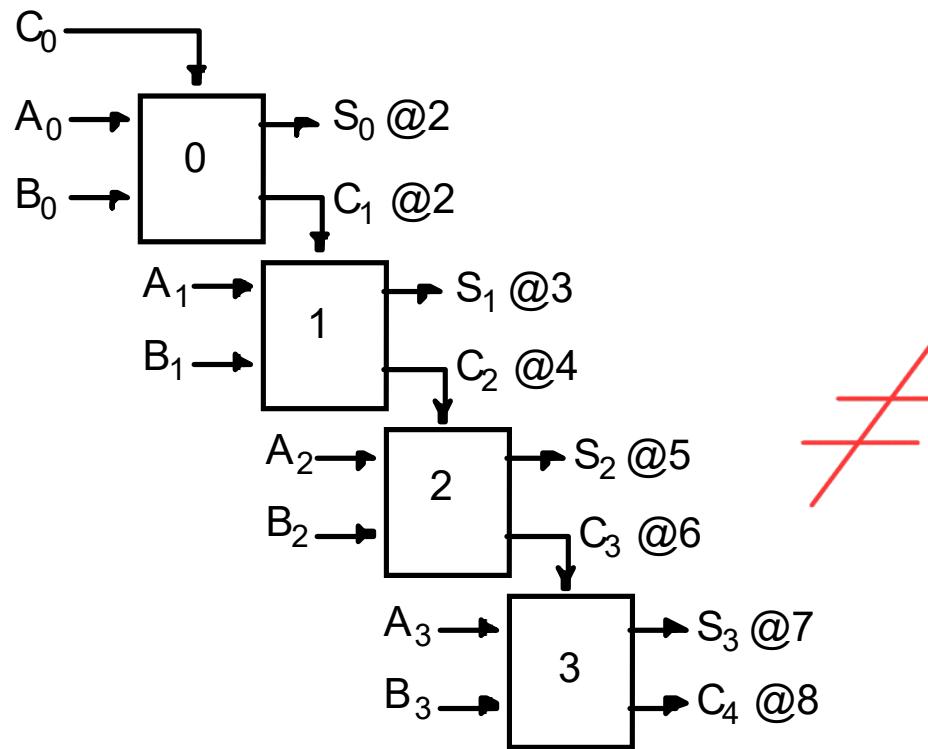
$$\begin{aligned} c_3 &= g_2 + p_2 \cdot c_2 = g_2 + p_2 \cdot (g_1 + p_1 \cdot g_0 + p_1 \cdot p_0 \cdot c_0) = \\ &= g_2 + p_2 \cdot g_1 + p_2 \cdot p_1 \cdot g_0 + p_2 \cdot p_1 \cdot p_0 \cdot c_0 \end{aligned}$$

$$\begin{aligned} c_4 &= g_3 + p_3 \cdot c_3 = g_3 + p_3 \cdot (g_2 + p_2 \cdot g_1 + p_2 \cdot p_1 \cdot g_0 + p_2 \cdot p_1 \cdot p_0 \cdot c_0) = \\ &= g_3 + p_3 \cdot g_2 + p_3 \cdot p_2 \cdot g_1 + p_3 \cdot p_2 \cdot p_1 \cdot g_0 + p_3 \cdot p_2 \cdot p_1 \cdot p_0 \cdot c_0 \end{aligned}$$

- Notice that any Carry is determined after 3 delay levels

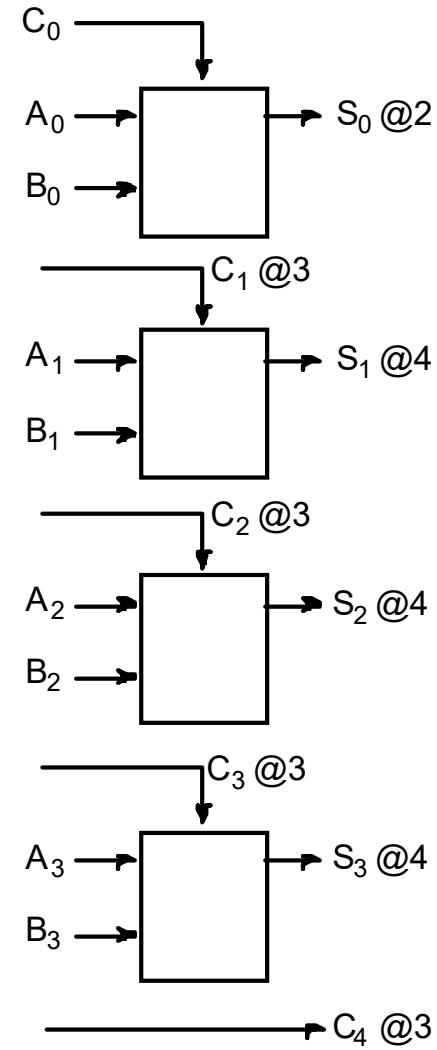
# Ripple Adder versus CLA

- 4 bit ripple adder



Final result always after  $2 \times 4 = 8$  delays

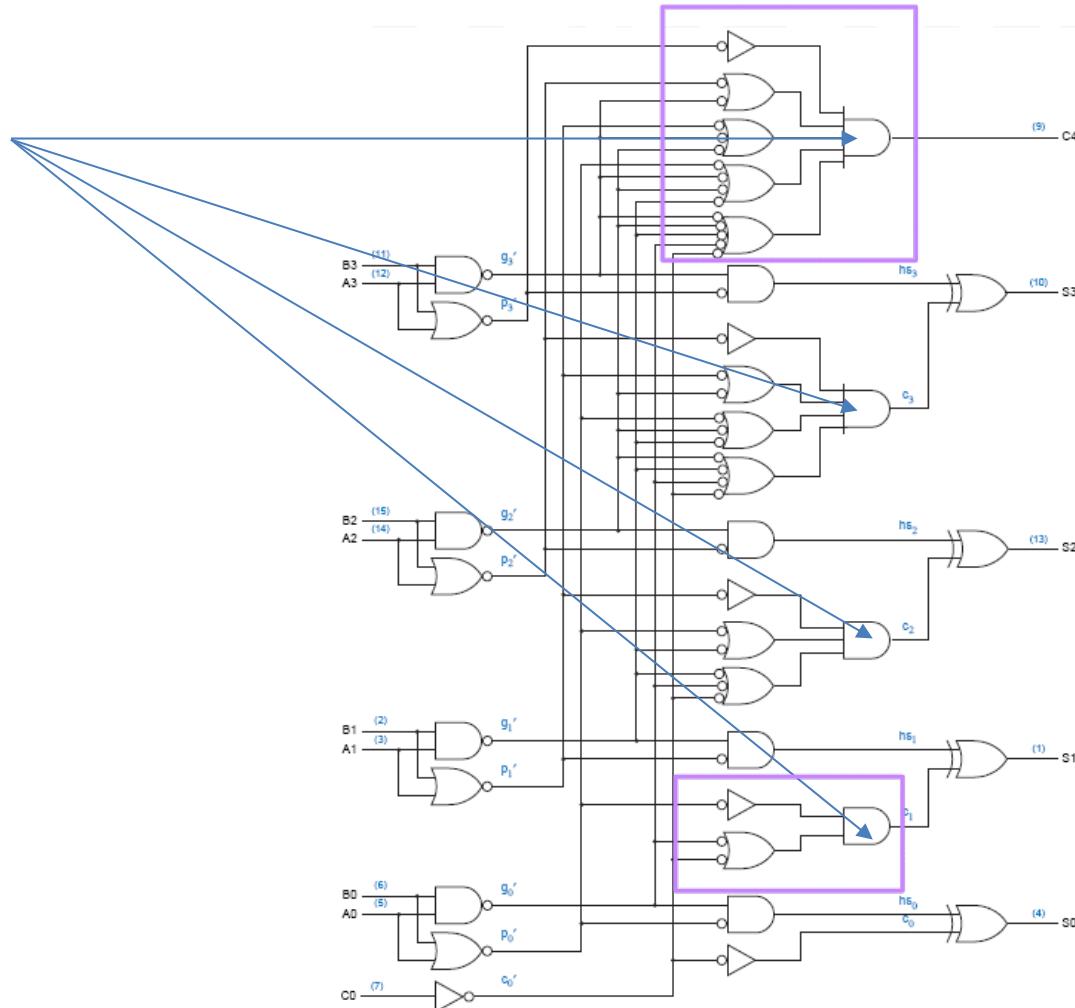
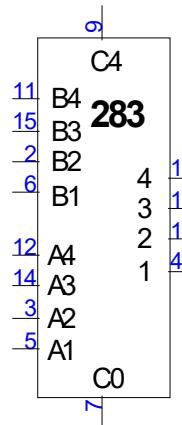
- 4 bit CLA



Final result  
always after 4  
delays

# The 74283 model

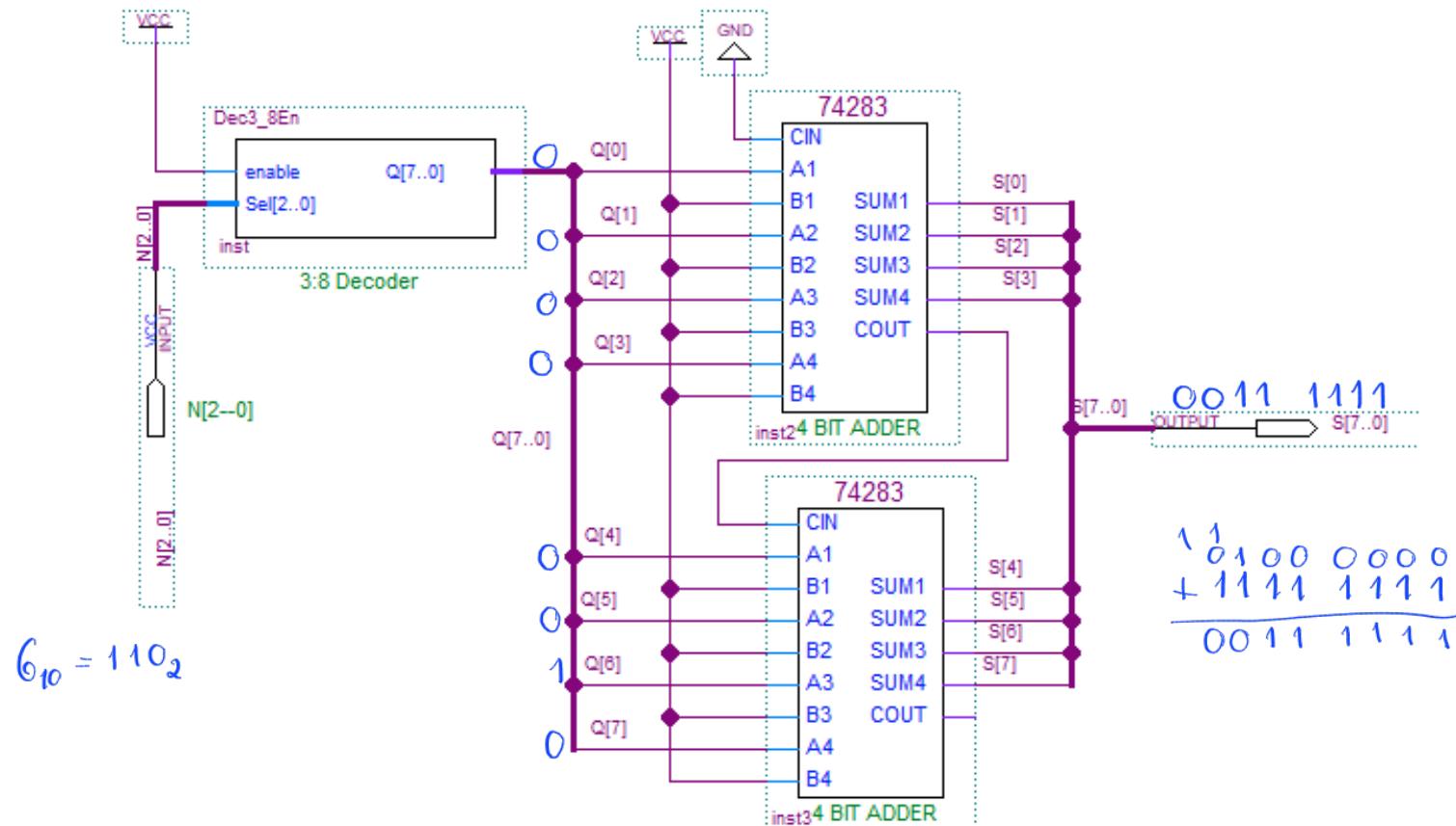
- CLA logic



# Exercise !

Sai SEMPRE  
no teste!

- In a 2's complement representation, what's the decimal value of the result S with  $N = 6_{10}$



# BCD Addition

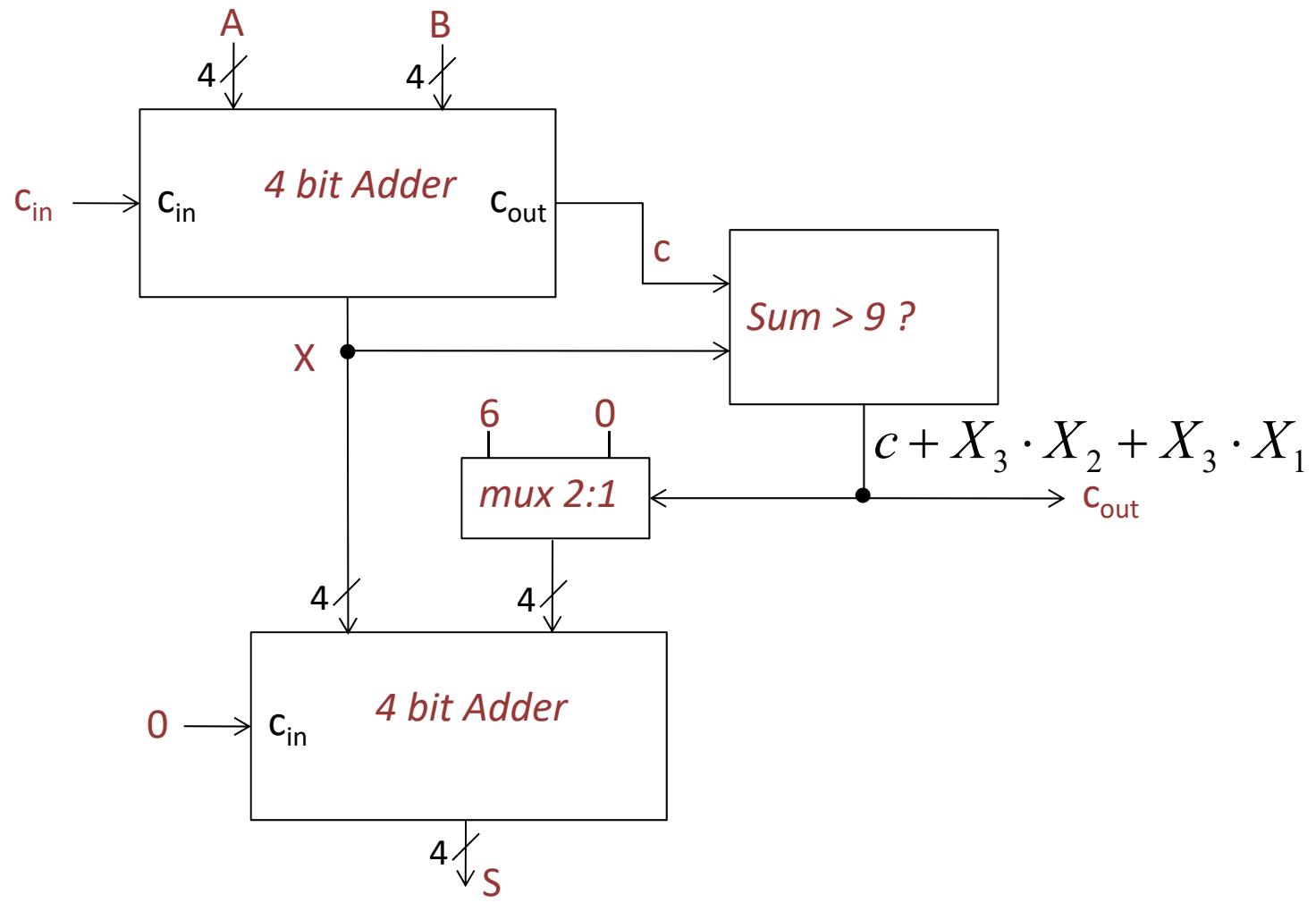
- Possible results for 2 digit BCD addition with carry

soma	binário		BCD	
	<i>carry out</i>	soma	<i>carry out</i>	soma
0	0	0000	0	0000
1	0	0001	0	0001
2	0	0010	0	0010
3	0	0011	0	0011
4	0	0100	0	0100
5	0	0101	0	0101
6	0	0110	0	0110
7	0	0111	0	0111
8	0	1000	0	1000
9	0	1001	0	1001
10	0	1010	1	0000
11	0	1011	1	0001
12	0	1100	1	0010
13	0	1101	1	0011
14	0	1110	1	0100
15	0	1111	1	0101
16	1	0000	1	0110
17	1	0001	1	0111
18	1	0010	1	1000
19	1	0011	1	1001

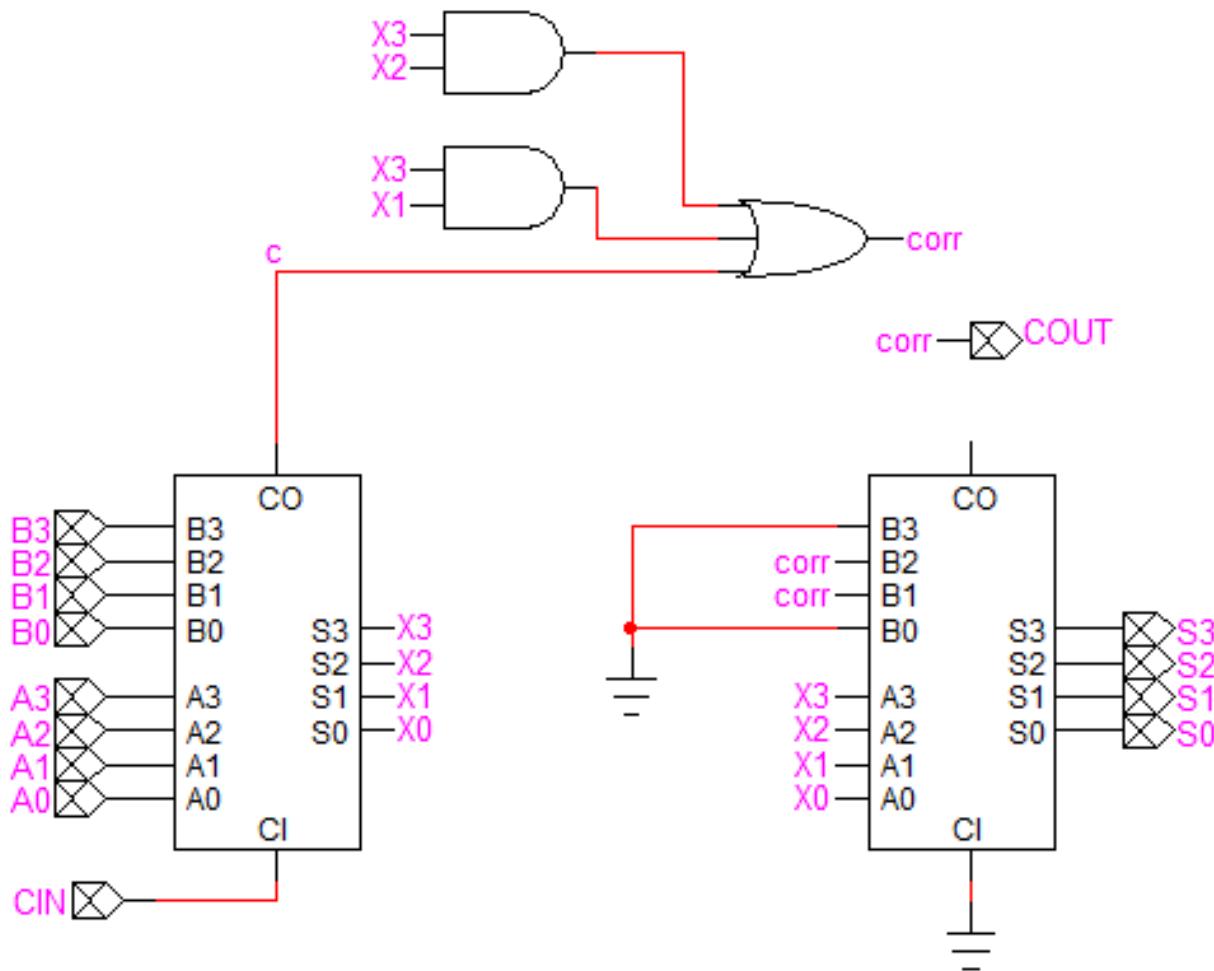


Offset correction required:  
add 6 to the binary result

# BCD addition algorithm



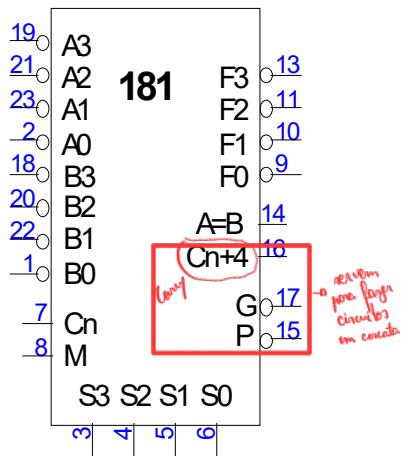
# Possible implementation



# ALU

- An arithmetic logic unit is a combinational block that executes any logical or arithmetic operation over a pair of  $b$  bits operands.
- There is a mode input that chooses between the logical or arithmetic behavior
- There is a op-code set of inputs that choose a particular operation from a limited “operation” set.

# The 74181 ALU



S3	S2	S1	S0	M=0 (op. aritm.)	M=1 (op. lógica)
0	0	0	0	$F = A - 1 + \text{CIN}$	$F = \bar{A}$
0	0	0	1	$F = A \text{ AND } B - 1 + \text{CIN}$	$F = \bar{A} \text{ OR } \bar{B}$
0	0	1	0	$F = A \text{ AND } \bar{B} - 1 + \text{CIN}$	$F = \bar{A} \text{ OR } B$
0	0	1	1	$F = 1111 + \text{CIN}$	$F = 1111$
0	1	0	0	$F = A + (A \text{ OR } \bar{B}) + \text{CIN}$	$F = \bar{A} \text{ AND } \bar{B}$
0	1	0	1	$F = A \text{ AND } B + (A \text{ OR } \bar{B}) + \text{CIN}$	$F = \bar{B}$
0	1	1	0	$F = A - B - 1 + \text{CIN}$	$F = A \text{ XOR } \bar{B}$
0	1	1	1	$F = A \text{ OR } \bar{B} + \text{CIN}$	$F = A \text{ OR } \bar{B}$
1	0	0	0	$F = A + (A \text{ OR } B) + \text{CIN}$	$F = \bar{A} \text{ AND } B$
1	0	0	1	$F = A + B + \text{CIN}$	$F = A \text{ XOR } B$
1	0	1	0	$F = A \text{ AND } \bar{B} + (A \text{ OR } B) + \text{CIN}$	$F = B$
1	0	1	1	$F = A \text{ OR } B + \text{CIN}$	$F = A \text{ OR } B$
1	1	0	0	$F = A + A + \text{CIN}$	$F = 0000$
1	1	0	1	$F = A \text{ AND } B + A + \text{CIN}$	$F = A \text{ AND } \bar{B}$
1	1	1	0	$F = A \text{ AND } \bar{B} + A + \text{CIN}$	$F = A \text{ AND } B$
1	1	1	1	$F = A + \text{CIN}$	$F = A$

# Unsigned Multiplication

- We follow the same rules of the decimal system

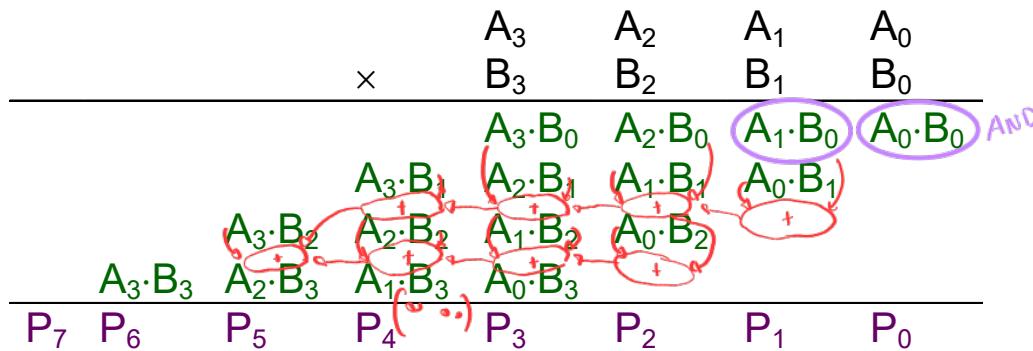
$$12 \times 13 = 156$$

$$\begin{array}{r} 1 & 1 & 0 & 0 \\ \times & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}$$

Diagram illustrating the multiplication process:

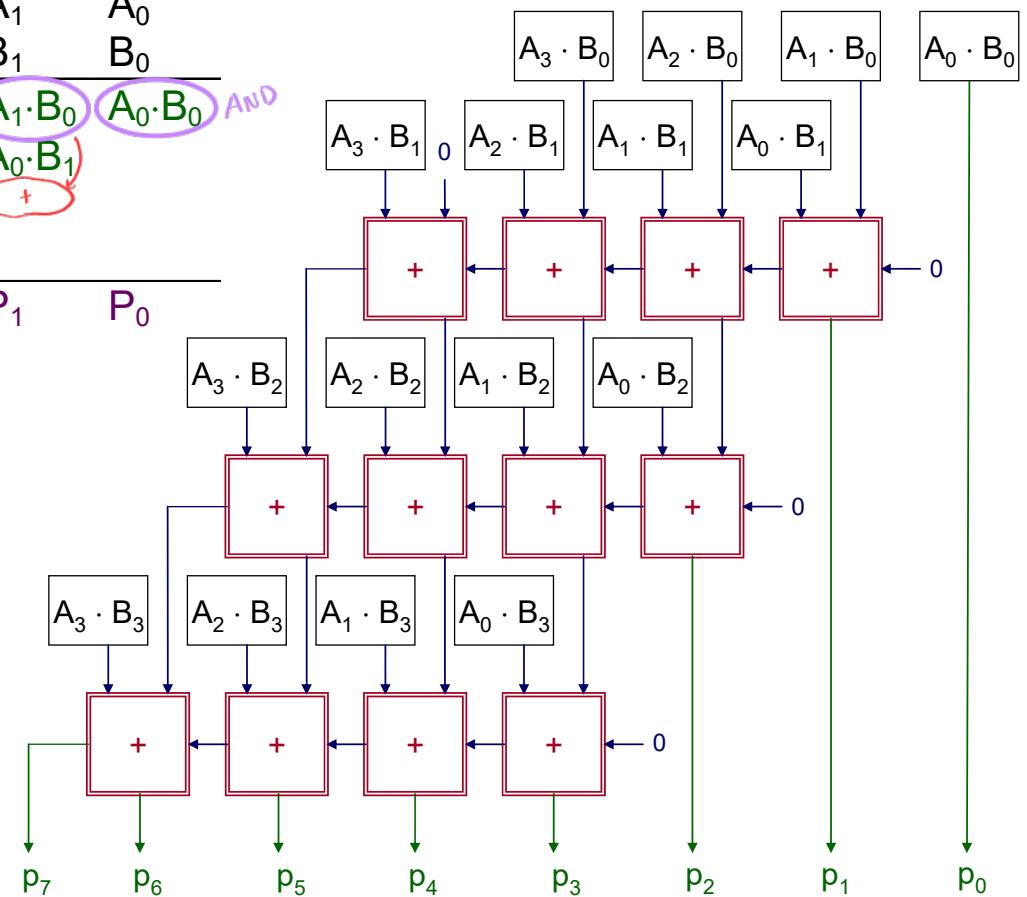
- The multiplicand is 1100.
- The multiplier is 1101.
- The partial products are 1100, 0000, and 1100, which are grouped by a brace under the label "Partial products".
- The final product is 10011100.
- A blue vertical line separates the multiplication steps from the final result.
- A blue bracket on the right side groups the final product and the text "Igual a decimal".

# Combinational Multipliers



$$\begin{array}{r}
 & \times \\
 & \quad 1 \quad 1 \quad 0 \quad 0 \\
 & \quad 1 \quad 1 \quad 0 \quad 1 \\
 \hline
 & \quad 1 \quad 1 \quad 0 \quad 0 \\
 & \quad 0 \quad 0 \quad 0 \quad 0 \\
 & \quad 1 \quad 1 \quad 0 \quad 0 \\
 \hline
 & \quad 1 \quad 1 \quad 0 \quad 0 \\
 \hline
 & \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0
 \end{array}$$

$16 \times \text{AND-2}$   
12 adders



# Final Remarks

- Always recall
  - The block symbol
  - The types of inputs (operands) and outputs
  - Distinguish between iterative and non-iterative solutions
- Design with encapsulated logic requires mastering all the functional details of each block