

Introduction to Digital Systems

Part I (4 lectures)

2022/2023

Introduction
Number Systems and Codes
Combinational Logic Design Principles

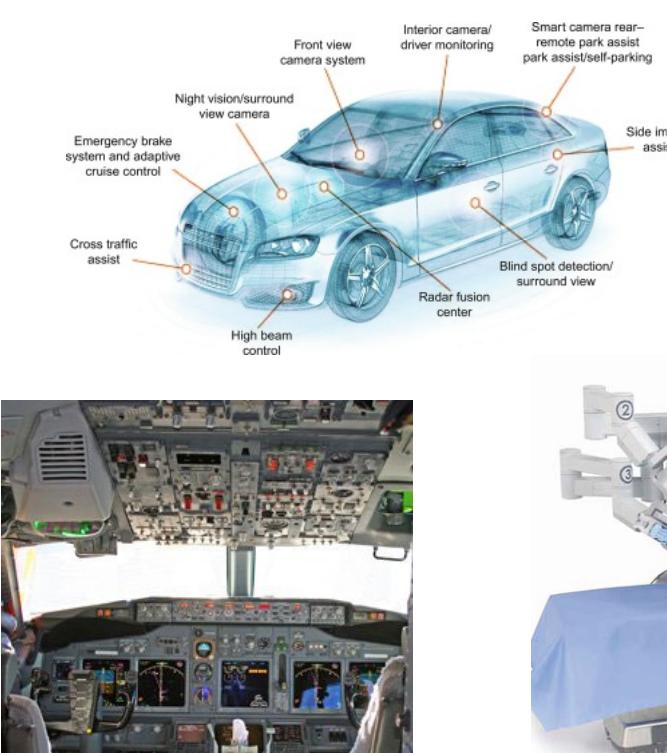
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Lecture 1 contents

- About digital design
- Number systems
 - Positional number systems
 - Binary, octal, and hexadecimal numbers
 - General positional-number-system conversions

Motivation

Electronics all around us: automotive, consumer products, communications, military and aerospace, medicine, etc.



- <https://www.sciencedirect.com/topics/engineering/automotive-electronics>
- <http://www.beadelectronics.com/military-and-aerospace-electronics-applications>
- <https://electronicsforu.com/technology-trends/medical-electronics-key-concerns-opportunities>
- <https://www.itotechnologynews24.com/tag/online-consumer-electronics-market-review/>
- <http://www.indiaeducation.net/engineering/engineering-branch/electronics-engineering-versus-electronics-communication-engineering.html>

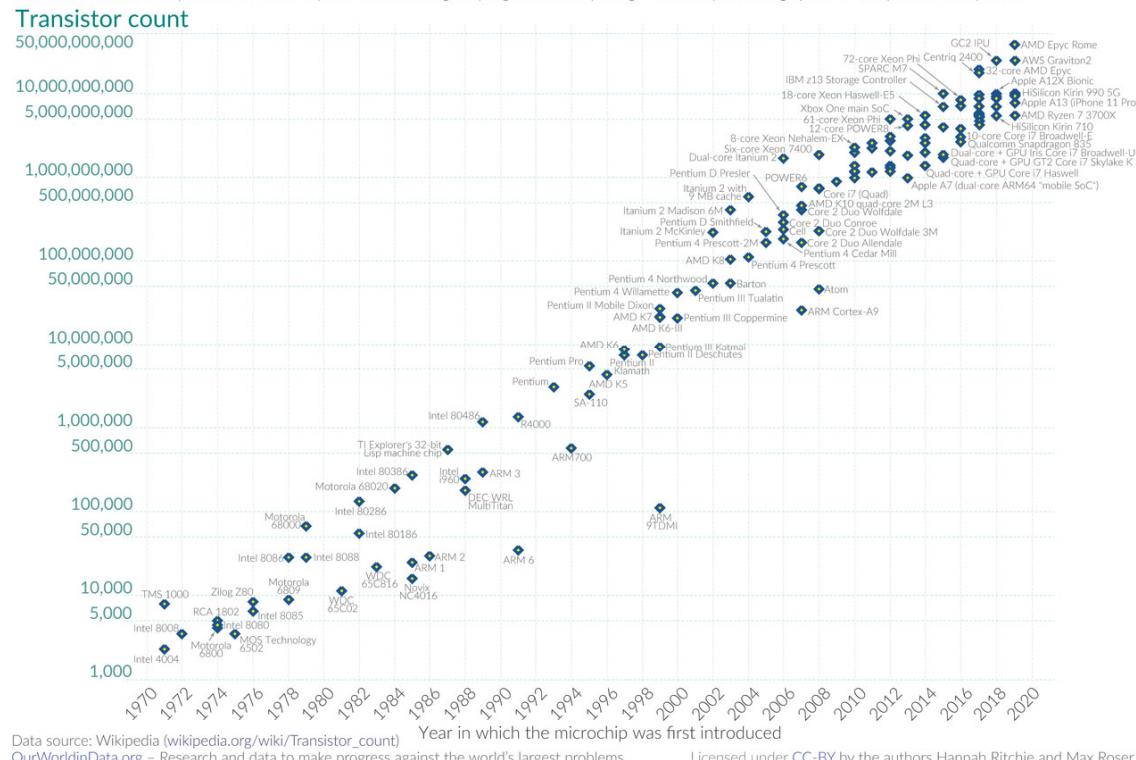
Exponential Growth

Moore's law (observed in 1965): the number of transistors on a computer chip doubles every 1.5-2 years

Moore's Law: The number of transistors on microchips doubles every two years

Moore's law describes the empirical regularity that the number of transistors on integrated circuits doubles approximately every two years. This advancement is important for other aspects of technological progress in computing – such as processing speed or the price of computers.

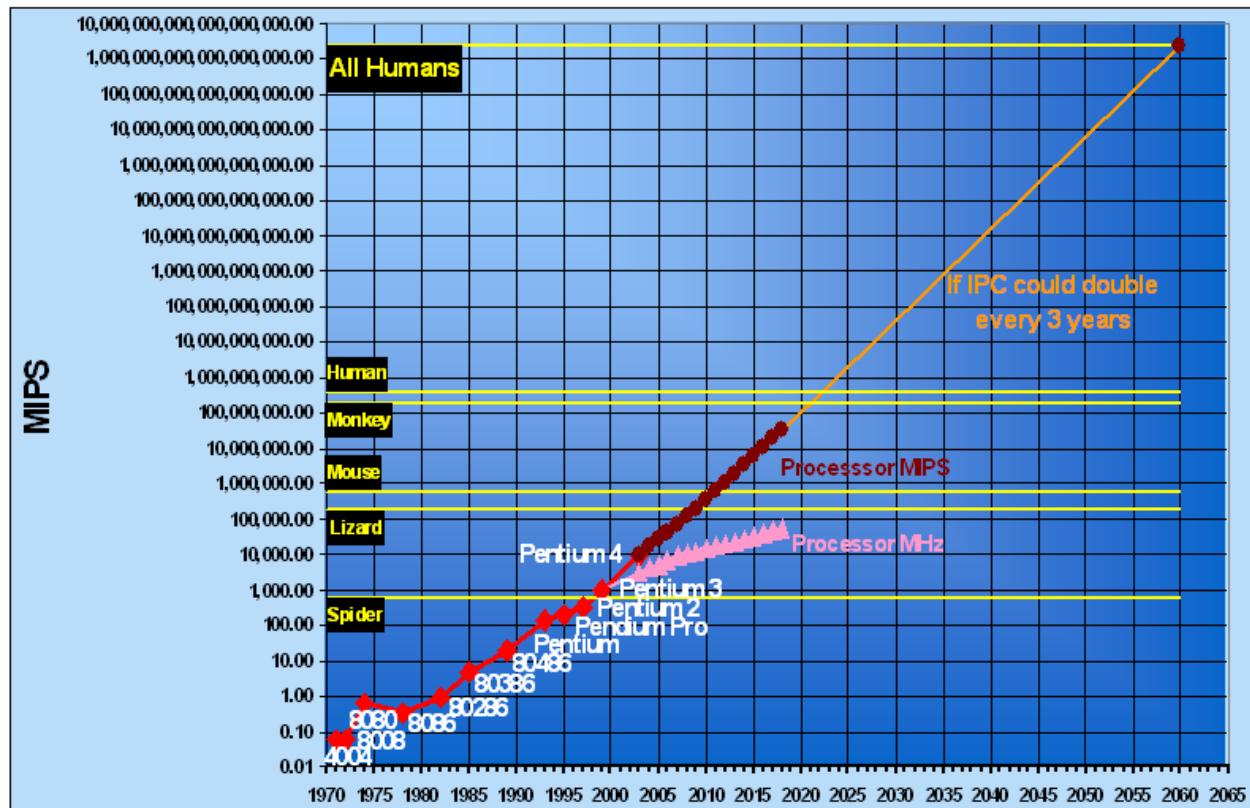
Our World
in Data



https://en.wikipedia.org/wiki/Moore%27s_law#/media/File:Moore's_Law_Transistor_Count_1970-2020.png

Exponential Growth

Processing power increases at about the same rate (~40%)



<http://www.captec-group.com/>

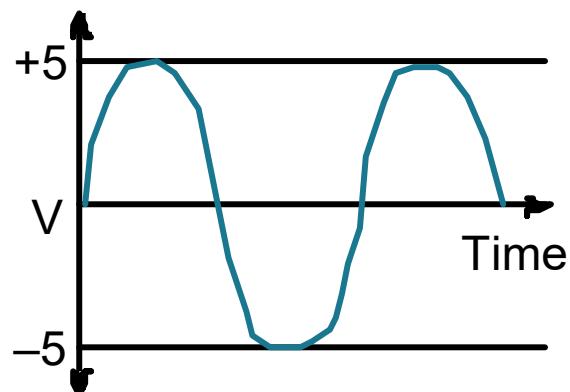
Course content

Application software (programs)
Operating systems (device drivers)
Instruction Set Arch (instructions, registers)
Machine organization (datapaths, controllers)
Logic (adders, encoders)
Digital circuits (gates)
Analog circuits (amplifiers)
Devices (transistors)
Physics (electrons)

- Fundamentals of Boolean logic
- Encoding
- Combinational circuits
- Arithmetic units
- Synchronous circuits
- Finite state machines
- Timing and clocking
- Simulation

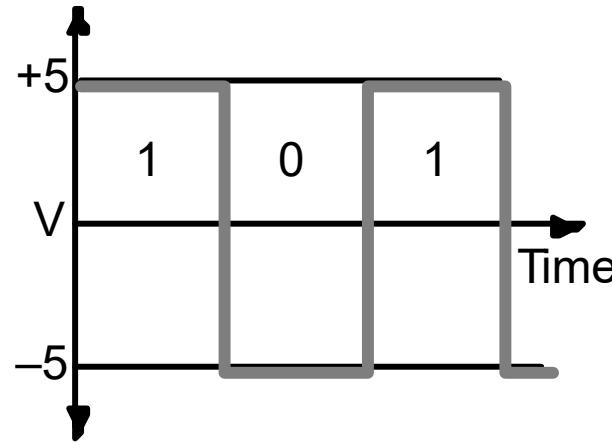
Digital abstraction

Analog: values vary over a broad range continuously



Digital: only assumes discrete values

- reproducibility of results
- ease of design
- programmability
- speed

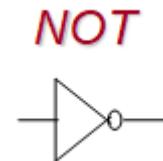
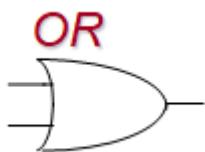
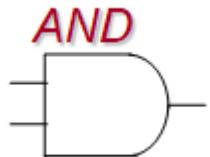


Boolean Algebra

- Inputs and outputs can only have two discrete values:
 - physical domain (usually, voltages) (0V/5V)
 - mathematical domain : Boolean variables (true/ false, 0/1)
- **Boolean algebra** is used to analyze and describe the behavior of digital circuits
 - a symbolic variable, such as x , represents the condition of a logic signal
 - algebraic operators, such as AND, OR and NOT, represent **logic gates**
 - a **gate** is the most basic digital device and has one or more input and produces an output that is a function of the current input value(s)

Logic Gates

- AND – logical product
- OR – logical sum
- NOT - inversion



x	y	x and y
		xy
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x or y
		x+y
0	0	0
0	1	1
1	0	1
1	1	1

x	not x
	\bar{x}
0	1
1	0

Number Systems

- Digital systems are built from circuits that process binary digits
- Very few real-life problems are based on binary numbers or any numbers at all
- Some correspondence must be established between the binary digits processed by digital circuits and real-life numbers, events, and conditions
 - How to represent familiar numeric quantities?
 - number systems: binary, octal, and hexadecimal
 - How to represent nonnumeric data?

Positional Number Systems

- In a positional number system, a number is represented by a string of digits, where each digit position has an associated weight.
- The value of a number is a weighted sum of the digits. For a decimal system with **radix $r = 10$** :
 - $1734_{10} = 1 \cdot 1000 + 7 \cdot 100 + 3 \cdot 10 + 4 \cdot 1$
 - $5185.68_{10} = 5 \cdot 10^3 + 1 \cdot 10^2 + 8 \cdot 10^1 + 5 \cdot 10^0 + 6 \cdot 10^{-1} + 8 \cdot 10^{-2}$
- In a general positional number system, the radix may be any integer $r \geq 2$, and a digit in position i has weight r^i .
- The general form of a number D in such a system is

$$D = d_{p-1}d_{p-2}\dots d_1d_0.d_{-1}d_{-2}\dots d_{-n} = \sum_{i=-n}^{p-1} d_i * r^i$$

, where there are p digits to the left of the point and n digits to the right of the point, called the **radix point**.

Radices and Sets of Symbols

number system	radix	symbols
binary	2	0,1
octal	8	0, 1,..., 7
decimal	10	0, 1,..., 9
hexadecimal	16	0, 1,..., 9, A, B, C, D, E, F

Examples:

Decimal

$$2007_{10} = 2*1000 + 0*100 + 0*10 + 7*1$$

$$19.85_{10} = 1*10 + 9*1 + 8*0.1 + 5*0.01$$

Binary

$$1100110_2 = 1*2^6 + 1*2^5 + 1*2^2 + 1*2^1 = 64 + 32 + 4 + 2 = 102_{10}$$

$$101.0011_2 = 1*2^2 + 1*2^0 + 1*2^{-3} + 1*2^{-4} = 4 + 1 + 0.125 + 0.0625 = ..._{10}$$

Most significant bit

$$D = \sum_{i=-n}^{p-1} d_i * 10^i$$

$$D = \sum_{i=-n}^{p-1} d_i * 2^i$$

Least significant bit

$$\begin{aligned}1100110_2 &= 111_{10} \\&= 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^2 + 1 \times 2 \\&= 64 + 32 + 4 = 102_{10}\end{aligned}$$

$$\begin{aligned}101.0011 &= 1 \times 2^2 + 1 + 2^{-3} + 2^{-4} \\&= 4 + 1 + \frac{1}{8} + \frac{1}{16} \\&= 5 + 0,125 + 0,0625 \\&= 5,1875\end{aligned}$$

Number Systems Examples

Examples:

Octal

$$D = \sum_{i=-n}^{p-1} d_i * 8^i$$

$$3577_8 = 3*8^3 + 5*8^2 + 7*8^1 + 7*8^0 = 1919_{10}$$

$$35.77_8 = 3*8^1 + 5*8^0 + 7*8^{-1} + 7*8^{-2}$$

Hexadecimal

$$D = \sum_{i=-n}^{p-1} d_i * 16^i$$

$$2007_{16} = 2*16^3 + 7*16^0 = 8199_{10}$$

$$7D7_{16} = 7*16^2 + 13*16^1 + 7*16^0 = 2007_{10}$$

$$A.2C_{16} = 10*16^0 + 2*16^{-1} + 12*16^{-2} = 10 + 0.125 + 0.046875 = \dots_{10}$$

Conversion from any Positional Number System to Decimal

- Radix 10 is important because we use it in everyday life.
- Radix 2 is important because binary numbers can be processed directly by digital circuits.
- Numbers in other radices are not often processed directly but may be important for documentation or other purposes.
- In particular, the radices 8 and 16 provide convenient shorthand representations for multibit numbers in a digital system.
- The value of the number expressed in radix r can be found by converting each digit of the number to its radix-10 equivalent and expanding the formula ($D = \sum_{i=-n}^{p-1} d_i \times r^i$) using radix-10 arithmetic.

Binary, Decimal, Octal , and Hexadecimal Numbers

binary	decimal	octal	hexadecimal
0	0	0	0
(1)	1	1	1
10	2	2	2
11	3	3	3
100	4	4	4
101	5	5	5
110	6	6	6
111	7	7	7
1000	8	10	8
1001	9	11	9
1010	10	12	A
1011	11	13	B
1100	12	14	C
1101	13	15	D
1110	14	16	E
1111	15	17	F

Conversion from Decimal to any Positional Number System: Integral Part

- The **integral part** of a number D expressed in decimal can be converted to any radix $r \geq 2$ by **successful division of D by r** (using radix-10 arithmetic, until the result is 0) with **reverse recording** (in radix r) of all the obtained **remainders**.

Examples:

Conversion to binary

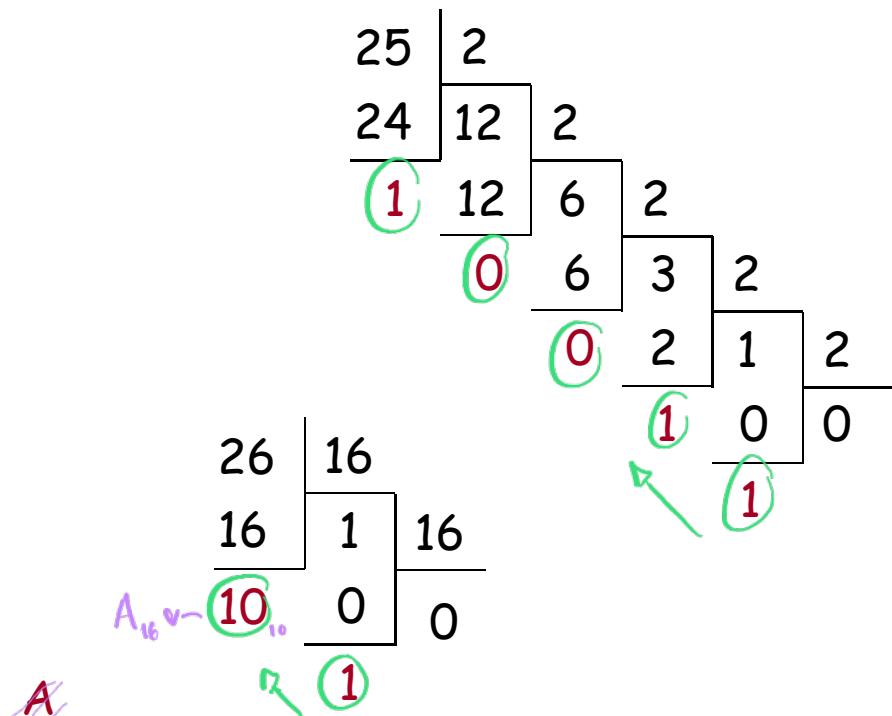
$$25_{10} = ???_2$$

$$25_{10} = 11001_2$$

Conversion to hexadecimal

$$26_{10} = ???_{16}$$

$$26_{10} = 1A_{16}$$



$$x_n = b_{n-1} 2^{n-1} + b_{n-2} 2^{n-2} \dots + b_0$$

$$(b_{n-1}, b_{n-2}, b_0)_2$$

$$= 2 \left(b_{n-1} 2^{n-2} + b_{n-2} 2^{n-3} \dots \right) + b_0$$

$\underbrace{\hspace{10em}}$
Q

$$= 2 Q + b_0$$

$$25_{10} = ???_2$$

$$25_{10} = 110001_2$$

$$\begin{array}{r} 25 \\ 24 \quad | \quad 2 \\ \hline 1 \quad | \quad 12 \quad | \quad 2 \\ \hline 0 \quad | \quad 6 \quad | \quad 2 \\ \hline 0 \quad | \quad 2 \quad | \quad 1 \quad | \quad 2 \\ \hline 1 \quad 0 \quad | \quad 0 \\ \hline 1 \end{array}$$

$$26_{10} = ???_6$$

$$26_{10} = 1A$$

$$\begin{array}{r} 26 \\ 16 \quad | \quad 16 \\ \hline 10 \quad | \quad 0 \quad | \quad 0 \\ \hline 1 \end{array}$$

Conversion from Decimal to any Positional Number System: Fractional Part

- The **fractional part** of a number D expressed in decimal can be converted to any radix $r \geq 2$ by **successful multiplication of D by r** (using radix-10 arithmetic, until the desired precision is reached) with **direct recording** (in radix r) of all the obtained **integral parts**.

Examples:

Conversion to binary

$$0.6875_{10} = ???_2$$

$$0.6875_{10} = 0.\quad_2$$

Conversion to hexadecimal

$$0.25_{10} = ???_{16}$$

$$0.25_{10} = 0.\quad_{16}$$

$$\begin{array}{r} 0.6875 \\ \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} 0.3750 \\ \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} 0.750 \\ \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} 0.50 \\ \times 2 \\ \hline \end{array}$$

$$\begin{array}{r} 0.00 \\ \times 2 \\ \hline \end{array}$$

Outro processo



$$\begin{array}{r} 0.25 \\ \times 16 \\ \hline \end{array}$$

$$\begin{array}{r} 0.00 \\ \times 16 \\ \hline \end{array}$$

Outro processo !

$4,25$

$$\begin{array}{r} 4 \\ \hline 2 \\ | \\ 2 \\ \hline 0 \\ | \\ 2 \\ | \\ 1 \\ | \\ 2 \\ \hline 0 \\ | \\ 0 \\ | \\ 0 \\ \hline 1 \end{array}$$

$$0,25 \times 2 = 0,50$$

$$0,50 \times 2 = 1,00$$

mas tem resto acaba !

$$4_{10} = 100_2$$

$$0,25_{10} = .01_2$$

$$4,25_{10} = 100,01_2$$

$10,75$

$$\begin{array}{r} 10 \\ \hline 7 \\ | \\ 5 \\ | \\ 2 \\ \hline 1 \\ | \\ 2 \\ | \\ 1 \\ | \\ 2 \\ \hline 0 \\ | \\ 0 \\ | \\ 0 \\ \hline 1 \end{array}$$

$$0,75 \times 2 = 1,50$$

$$0,50 \times 2 = 1,00$$

$$10_{10} = 1010_2$$

$$0,75_{10} = .11_2$$

$$10,75_{10} = (1010,11)_2$$

$$0,6875 \times 2 = 1,375$$

$$0,375 \times 2 = 0,75$$

$$0,75 \times 2 = 1,50$$

$$0,5 \times 2 = 1,00$$

$$(0,6875)_{10} = (101011)_2$$

Rounding Error

- If the successive multiplication processes does not seem to be heading towards a final zero, the fractional number will have an infinite length. Try representing 0.33_{10} in binary.
- So the number of multiplication steps should be limited depending on the degree of accuracy required.
- In order to keep the accuracy of the original presentation, the following formula is applied:

r_1 – initial radix

r_2 – final radix

n_1 – number of fractional digits in the original number, expressed in radix r_1

n_2 – number of fractional digits in the converted number, expressed in radix r_2

$$n_2 = \lfloor n_1 * \log_{r_2} r_1 \rfloor$$

$$\begin{aligned} 5.9 &\approx 5 \\ 5.3 &\approx 5 \end{aligned}$$

Examples:

$$0.6875_{10} = ???_2$$

$$0.6875_{10} = 0.1011_2$$

$$0.6875_{10} = 0.1011000000000_2$$

$$m_2 = 13$$

$$A.2C_{16} = 10 * 16^0 + 2 * 16^{-1} + 12 * 16^{-2} = 10 + 0.125 + 0.046875 = 10.171875 = 10.17_{10}$$

$$101.0011_2 = 1 * 2^2 + 1 * 2^0 + 1 * 2^{-3} + 1 * 2^{-4} = 4 + 1 + 0.125 + 0.0625 = 5.1875 = 5.2_{10}$$

$$m_2 = \lfloor 2 \log_{10} 16 \rfloor = 2$$

4

$$m_2 = \lfloor 4 * \log_{10} 2 \rfloor = 1$$



$$m_2 = \left\lceil 4 \times \log_2 10 \right\rceil = \left\lceil 4 \times 3.3 \right\rceil = 13$$

APP - Wolframe alpha

Special Conversion Cases

- When a number is converted from radix r_1 to radix r_2 and $r_1=r_2^x$, then each digit in radix r_1 can be **substituted directly** with x radix r_2 digits.
- The **octal** and **hexadecimal** number systems are useful for representing multibit numbers because their radices are powers of 2.
- Each **octal digit is represented with 3 binary digits** ($8=2^3$). Each **hexadecimal digit is represented with 4 binary digits** ($16=2^4$).

Examples:

Conversion from octal to binary

$$r_1 = 8, r_2 = 2, 8 = 2^3 \rightarrow 3 \text{ bits por digito}$$

$$753.6_8 = 111\ 101\ 011 . 110_2$$

Conversion from hexadecimal to binary

$$r_1 = 16, r_2 = 2, 16 = 2^4 \rightarrow 4 \text{ bits por digito}$$

$$A5.E_{16} = 1010\ 0101 . 1110_2$$

Special Conversion Cases (cont.)

- When a number is converted from radix r_1 to radix r_2 and $r_1^x = r_2$, then a sequence of x digits in radix r_1 can be **substituted directly** with one radix r_2 digit.
- To convert a **binary number to octal**, start at the binary point and work left, separating the bits into groups of three and replacing each group with the corresponding octal digit; then work right. Freely add zeroes on the left or right to make the total number of bits a multiple of 3.
- To convert a **binary number to hexadecimal**, start at the binary point and work left, separating the bits into groups of four and replacing each group with the corresponding hexadecimal digit; then work right. Freely add zeroes on the left or right to make the total number of bits a multiple of 4.

Examples:

Conversion from binary to octal

$$r_1 = 2, r_2 = 8, 2^3 = 8$$

$$\begin{array}{r} \overbrace{1}^1 \quad \overbrace{101}^5 \quad . \quad \overbrace{010}^2 \\ 00 \end{array} = 15.2_8$$
$$n_2 = \lfloor 3 \times \log_8 2 \rfloor = 1$$

Conversion from binary to hexadecimal

$$r_1 = 2, r_2 = 16, 2^4 = 16$$

$$\begin{array}{r} \overbrace{110}^6 \quad \overbrace{0101}^5 \quad \overbrace{1100}^C \\ 0 \end{array} = 65C_{16}$$

Summary of Conversion Methods

From	To	Method
Binary	Octal	Substitution (replace each group of 3 bits with the corresponding octal digit)
	Decimal	Summation ($D = \sum_{i=-n}^{p-1} d_i \times 2^i$)
	Hexadecimal	Substitution (replace each group of 4 bits with the corresponding hexadecimal digit)
Octal	Binary	Substitution (each octal digit is represented with 3 bits)
	Decimal	Summation ($D = \sum_{i=-n}^{p-1} d_i \times 8^i$)
	Hexadecimal	Convert to binary, then to hexadecimal
Decimal	Binary	Division by 2 for integral part, multiplication by 2 for fractional part
	Octal	Division by 8 for integral part, multiplication by 8 for fractional part
	Hexadecimal	Division by 16 for integral part, multiplication by 16 for fractional part
Hexadecimal	Binary	Substitution (each hexadecimal digit is represented with 4 bits)
	Octal	Convert to binary, then to octal
	Decimal	Summation ($D = \sum_{i=-n}^{p-1} d_i \times 16^i$)



Exercises

- Explain Moore's Law.
- What are the advantages of digital systems compared to analog systems?
- When an OR gate output is 0?
- When an AND gate output is 1?

Exercises (cont.)

- How to convert an integer octal number into hexadecimal?
- Consider that expression $1234_r < 1234_8$ is true. Is it possible to determine the value of r ? *No.*
- Is it possible to convert the number 39_8 into decimal?
- What is the relationship ($>$, $=$, or $<$) between 34_8 and 34_{16} ?
- Is the following affirmation true: $20=14_{16}$? *Yes.*

Se fosse $20_{10} = 14_{16}$ era verdadeiro

$$\sqrt{41} \approx 5$$

$$(\sqrt{4b+1})^2 = 25$$

$$\Leftrightarrow 4b+1 = 25$$

$$\Rightarrow b = 6$$

Exercises (cont.)

- Convert the following numbers into binary, octal, decimal, and hexadecimal:

$$10111011001_2 = 2731_8 = 5D9_{16} = 1497_{10}$$

$$1234_8 = 001010011100_2 = 29C_{16} = 668_{10}$$

$$\text{CODE}_{16} = 1100000011011110_2 = 140336_8 = 49374_{10}$$

$$108_{10} = 1101100_2 = 154_8 = 6C_{16}$$

$$15.46_{10} = 1111.011101_2 = 17.35_8 = F.7_{16}$$

$$1234_8 = \underbrace{0010}_2 \underbrace{1001}_9 \underbrace{1100}_c = 29C_{16}$$

$$= 2 \times 16^2 + 9 \times 16 + 12$$

$$= 668_{10}$$

$$1111 \quad 1111 = 2^8 - 1 = 255$$

$$1000 \quad 0000 = 2^7 = 128 = 2^{m-1}$$

$$0000 \quad 1000 = 2^3 = 8$$

$$1000 \quad 0011 = 2^7 + 2^2 - 1 = 128 + 3 = 131$$

2 Forms \rightarrow

$$1000 \quad 1111 = 2^7 + 2^4 - 1 = 128 + 15 = 143$$

$$\underbrace{1000}_{8} \quad \underbrace{1111}_{F_{16}} = 8 F_{16} = 8 \times 16 + 15 = 143$$