Aula 18

- Usar funções (séries) geradoras para resolver eq. de recorrência (método da função (série) geradora)

Eq. de recorrência $\longrightarrow$ $A = \sum_{n=0}^{+\infty} \underbrace{a_n}_{(a_n)_{n\in\mathbb{N}}} x^n = (\cdots) = f(x)$ $\boxed{1}$

$(\cdots) \sum_{n=0}^{+\infty} (\,?\,)\, x^n$ $\boxed{2}$

↳ fórmula fechada solução de a_n

ex 1: O problema das Torres de Hanoi

$$\begin{cases} a_n = 2a_{n-1} + 1, & n \geq 2 \\ a_1 = 1 \end{cases}$$

Ora, a função geradora associada a $(a_n)_{n\in\mathbb{N}}$ é

$$A(x) = \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} a_n x^n, \quad (a_0 = 0)$$

$$= a_1 x + \sum_{n=2}^{+\infty} \underbrace{a_n}_{\text{sabemos } a_n \text{ para } n \geq 2} x^n$$

$$= a_1 x + \sum_{n=2}^{+\infty} (2a_{n-1} + 1) x^n$$

$$= a_1 x + 2\sum_{n=2}^{+\infty} a_{n-1} x^n + \sum_{n=2}^{+\infty} x^n$$

$$= a_1 x + 2x \sum_{n=2}^{+\infty} a_{n-1} x^{n-1} + \sum_{n=2}^{+\infty} x^n$$

$$= a_1 x + 2x \sum_{n=1}^{+\infty} a_n x^n + \sum_{n=2}^{+\infty} x^n$$

$$A(x) = a_1 x + 2x \sum_{n=1}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} x^{n+2}$$

$$\Leftrightarrow A(x) = \cancel{a_1}^{1} x + 2x\, A(x) + \frac{x^2}{1-x}$$

$$\Leftrightarrow A(x)(1-2x) = x + \frac{x^2}{1-x}$$

Donde vem:

$$A(x)(1-2x) = x + \frac{x^2}{1-x} = \frac{(1-x)x + x^2}{1-x} = \frac{x}{1-x}$$

$$\Leftrightarrow A(x) = \frac{x}{(1-x)(1-2x)} = f(x)$$

Agora:

$$A(x) = \frac{A}{1-x} + \frac{B}{1-2x}$$

$\longrightarrow$ $x = A(1-2x) + B(1-x)$

$\Leftrightarrow x = A + B + (-2A - B)x$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ -2A - B = 1 \end{cases} \quad \begin{cases} A = -1 \\ B = 1 \end{cases}$$

Assim, tem-se:

$$A(x) = \frac{1}{1-2x} - \frac{1}{1-x}$$

$$\Leftrightarrow A(x) = \sum_{n=0}^{+\infty} \underbrace{(2x)^n}_{2^n x^n} - \sum_{n=0}^{+\infty} x^n = (1-1) + \sum_{n=1}^{+\infty} 2^n x^n - \sum_{n=1}^{+\infty} x^n$$

$$\Leftrightarrow A(x) = \sum_{n=1}^{+\infty} \underbrace{(2^n - 1)}_{a_n} x^n$$

$$\boxed{a_n = 2^n - 1, \; n \geq 1}$$

ex: $\boxed{20}$ a) da Folha 4

coef. não constante ⟶ Usamos a: função geradora exponêncial

$$\begin{cases} a_n = n\,a_{n-1}, & n \geq 1 \\ a_0 = 1 \end{cases}$$

Neste caso:

$$A(x) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n = 1 + \sum_{n=1}^{+\infty} \frac{(n\,a_{n-1})}{n!} x^n$$

$$= 1 + \sum_{n=1}^{+\infty} \frac{a_{n-1}}{(n-1)!} x^n$$

$$= 1 + x \sum_{n=1}^{+\infty} \frac{a_{n-1}}{(n-1)!} x^{n-1}$$

$$= 1 + x \sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n$$

$$\Rightarrow A(x) = 1 + x\,A(x)$$

$$\Leftrightarrow A(x)(1-x) = 1$$

$$\Leftrightarrow A(x) = \frac{1}{1-x}$$

$$\Leftrightarrow A(x) = \sum_{n=0}^{+\infty} x^n$$

$$\Leftrightarrow A(x) = \sum_{n=0}^{+\infty} \frac{\overbrace{n!}^{a_n}}{n!} x^n$$

Logo: $a_n = n!, \; n \geq 0$

ex 3: Ramos de n folhas que se podem obter com 2 Rosas, 1 Margarida e um nº infinito de Dálias

$$F(x) = \underbrace{(1 + x + x^2)(1+x)}(1 + x + x^2 + \ldots + x^n) = (\ldots) = \sum_{n=0}^{+\infty} (?)\, x^n$$

$$= \frac{(1+x+x^2)\times(1-x)}{(1-x)} \times (1+x) \times \frac{1}{1-x} = \frac{(1 - \cancel{x} + \cancel{x} - \cancel{x^2} + \cancel{x^2} - x^3)}{1-x} \times (1+x) \times \frac{1}{1-x}$$

$$= \frac{(1-x^3)}{(1-x)^2} \times (1+x)$$

Regra de Ruffini:

	1	0	0	-1
1		1	1	1
	1	1	1	0

$\Rightarrow (1-x^3) = (1-x) \times (1+x+x^2)$

Assim tem-se:

$$= \frac{\cancel{(1-x)}(1+x+x^2)}{(1-x)^{\cancel{2}}} \times (1+x) = \underbrace{(1+x+x^2+x+x^2+x^3)}_{1+2x+2x^2+x^3} \times \sum_{n=0}^{+\infty} x^n$$

$$\Rightarrow F(x) = \sum_{n=0}^{+\infty} x^n + \sum_{n=0}^{+\infty} 2x^{n+1} + \sum_{n=0}^{+\infty} 2x^{n+2} + \sum_{n=0}^{+\infty} x^{n+3}$$

$$= \underbrace{1}_{a_0} + \underbrace{(1+2)}_{a_1}x + \underbrace{(1+2+2)}_{a_2}x^2 + \underbrace{(1+2+2+1)}_{a_3}x^3 + (1+2+2+1)x^4 + \dots + \underbrace{(1+2+2+1)}_{=6}\underbrace{x^n}_{n\geq 3}$$

Resposta: 6