Folha 4

1

$$P_n = F_0 + F_2 + F_4 + \dots + F_{2n-2}$$

$$= \sum_{k=0}^{n-1} F_{2k}$$

$$= F_0 + \sum_{k=0}^{n-1} F_{2k}$$

$$F_2 = \cancel{F_3} - \boxed{F_1}$$
$$+\ F_4 = \cancel{F_5} - \cancel{F_3}$$
$$+\ F_6 = \cancel{F_7} - \cancel{F_5}$$
$$\vdots$$
$$+\ F_{2n-4} = \cancel{F_{2n-3}} - \cancel{F_{2n-5}}$$
$$F_{2n-2} = \boxed{F_{2n-1}} - \cancel{F_{2n-3}}$$

$$= F_0 - F_1 + F_{2n-1}$$

$$= F_{2n-1}$$

II

a) $\frac{a_n}{n!} = \frac{n\, a_{n-1}}{n!} + \frac{n!}{n!}, \quad a_0 = 2$

$$\Leftrightarrow \frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)} + 1$$

$$\boxed{b_n = \frac{a_n}{n!}}$$

$$\hookrightarrow \begin{cases} b_n = b_{n-1} + 1, \quad n \geq 1 \\ b_0 = \frac{a_0}{0!} = 2 \end{cases}$$

Falta resolver

Saia: $b_n = 2 + n$

$$\hookrightarrow a_n = n!\,(2+n), \quad n \geq 0$$

b)

$$\begin{cases} 5n\, a_n + 2n\, a_{n-1} = 2a_{n-1}, \quad n \geq 3 \\ a_2 = -30 \end{cases}$$

$$5n\, a_n + 2n\, a_{n-1} - 2a_{n-1} = 0$$

$$\Leftrightarrow 5\underbrace{n\, a_n}_{b_n} + \underbrace{2a_{n-1}(n-1)}_{b_{n-1}} = 0$$

$$\boxed{b_n = n\, a_n}$$

$$\hookrightarrow 5b_n + 2b_{n-1} = 0, \quad b_2 = 2a_2 = -60$$

$$5q + 2 = 0$$
$$\Leftrightarrow q = -\frac{2}{5}$$

$$b_n = c\left(-\frac{2}{5}\right)^n$$

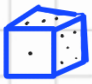
$$b_2 = c\,\frac{4}{25} \Leftrightarrow c = \frac{-60}{4/25} \Leftrightarrow c = -15 \times 25 \Leftrightarrow c = -375$$

Logo: $b_n = -375\left(-\frac{2}{5}\right)^n, \quad n \geq 2$

Regressando a a_n:

$$a_n = \frac{b_n}{n} = \frac{-375}{n}\left(-\frac{2}{5}\right)^n, \quad n \geq 2$$

4

 $\rightsquigarrow$ até que tenhamos 2 pares

Equação de recorrência: $\boxed{\text{para o n.º de exp que terminam no } n\text{-ésimo de experiências ou antes}}$ → n.º de experiências

Sucessão:

$$(a_n)_{n \in \mathbb{N}} : \begin{cases} a_n = \\ \text{cond. iniciais} \end{cases}$$

$$a_n = \underbrace{a_{n-1} \dots}?$$

$$\underbrace{\_\ \ \_\ \ \_\ \ (\dots)\ \ \_}_{(n-1)\text{ lançamentos}}\ \ \_$$

↑ Par

$$\left.\begin{array}{l} \text{Sair 1 par} \longmapsto 3(n-1) \\ \quad \text{e} \\ (n-2) \text{ ímpares} \longmapsto 3^{n-2} \\ \quad \text{e} \\ n\text{-ésimo} \longmapsto 3 \end{array}\right\} \quad a_n = \underbrace{a_{n-1}}_{} + 3(n-1) \times 3^{n-2} \times 3, \ n \geq 1$$

número de exp. que terminam antes do n-ésimo lançamento

$a_1 = 0$

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$p(x) = 2x^2 + x$

$$S_n = \sum_{i=0}^{n} p(i) = \underbrace{\underbrace{p(0)}_{0} + \underbrace{p(1)}_{3} + \underbrace{p(2)}_{9} + \dots + p(n-1)}_{S_{n-1}} + \underbrace{p(n)}_{2n^2+n}$$

$$+ \ 2n^2 + n$$

Logo: $\boxed{1}$ $\begin{cases} S_n = S_{n-1} + \underbrace{2n^2+n}_{\text{pol. de grau 2}}, \ n \geq 1 \\ S_0 = 0 \end{cases}$

termo não homogénio

$$S_n = S_n^{(h)} + S_n^{(p)}$$

Solução da equação geral da eq. de recorrência homogénia associada a $\boxed{1}$ $\Longrightarrow$ $S_n - S_{n-1} = 0$

Eq. característica: $q - 1 = 0 \Leftrightarrow q = 0$

Logo: $S_n^{(h)} = c_1 \times (1)^n = c_1, \ c_1 \in \mathbb{R}$

$$S_n^{(p)} = n^{\overset{m=1 \to \text{mult. de 1 enquanto raiz característica}}{m}} (A_0 + A_1 n + A_2 n^2)$$
$$= A_0 n + A_1 n^2 + A_2 n^3$$

Substituir em $\boxed{\text{I}}$

$$A_0 n + A_1 n^2 + A_2 n^3 = A_0 (n-1) + A_1 (n-1)^2 + A_2 (n-1)^3 + 2n^2 + n$$

(:)

$$\Leftrightarrow (-A_0 + A_1 - A_2) + (-2A_1 + 3A_2 + 1)n + (-3A_2 + 2) n^2 = 0$$

$$\Leftrightarrow \begin{cases} -A_0 + A_1 - A_2 = 0 \\ -2A_1 + 3A_2 + 1 = 0 \\ -3A_2 + 2 = 0 \end{cases} \quad \begin{cases} A_0 = 5/6 \\ A_1 = 3/2 \\ A_2 = 2/3 \end{cases}$$

Donde:

$$S_n = \underbrace{C_1}_{S_n^{(h)}} + \frac{5}{6} n + \frac{3}{2} n^2 + \frac{2}{3} n^3, \quad n \geq 0$$

Como: $S_0 = 0$, vem $C_1 = 0$

Assim:

$$S_n = \frac{5}{6} n + \frac{3}{2} n^2 + \frac{2}{3} n^3$$

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n cadeiras:

$$\underbrace{\underset{1}{\_\_}\ \underset{2}{\_\_}\ \underset{3}{\_\_}\ \underset{4}{\_\_}\ \cdots\ \underset{n-1}{\_\_}}_{n-1}\ \underbrace{\underset{n-2}{\_\_}}_{n}$$

k pacientes: k

k \ n	1	2	3	4	5	6	7	(...)
1	1	2	3	4	5	6	7	...
2	0	0	2!	...?				
3	0	0	0	0	3!	...?		
4	0	0	0	0	0	0	4!	...?
(...)								

$h(k, n) = 0, \quad n < 2k - 1, \quad k \geq 2$

$k(1, n) = n, \quad n \geq 1$

$$k(1, n) = \underbrace{h(k, n-1)}_{\text{Sentar k pac. em (n-1) cadeiras}} + \underbrace{h(k-1, n-2)}_{\text{Sentar k pacientes em (n-2) cadeiras}}$$

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a)

$$x_1 + x_2 + x_3 + x_4 = 27$$
$$x_i \in \{0, 1, 2, \ldots, 27\}$$

pol. gerador dos votos no candidato 1

$$x_1 \longmapsto 1 + x + x^2 + \ldots + x^{27} = p_1(x)$$

$$x_2 \longmapsto 1 + x + x^2 + \ldots + x^{27} = p_2(x)$$

$$x_3 \longmapsto 1 + x + x^2 + \ldots + x^{27} = p_3(x)$$

$$x_4 \longmapsto 1 + x + x^2 + \ldots + x^{27} = p_4(x)$$

$$\sum_{n=0}^{+\infty} a_n x^n = p_1 p_2 p_3 p_4 = \underbrace{(1 + x + x^2 + \ldots + x^{27})^4}_{\text{na prática será um pol. gerador}}$$

$\boxed{27 \times 4 = 108}$

$$\sum_{n=0}^{108} a_n x^n,$$ cujo coeficiente a_{27} dá a solução.

Extra exercício: (determinar a_{27})

$$(\underbrace{1 + x + x^2 + \ldots + x^{27}}_{S_n = a_1 \times \frac{(1-r^n)}{1-r}})^4 = \left(\frac{(1 - x^{28})}{(1-x)}\right)^4 = (1 - x^{28})^4 \times \left(\frac{1}{1-x}\right)^4$$

$$= (1 - x^{28})^4 \times \sum_{n=0}^{+\infty} \binom{n+4-1}{n} x^n = (*)$$

$$\hookrightarrow (\ldots) + \underbrace{a_{27}}_{\text{Só queremos o } a_{27}!} x^{27} + (\ldots)$$

Como:

$$(1 - x^{28})^4 = \sum_{k=0}^{4} \binom{4}{k} (-x^{28})^k = \boxed{\binom{4}{0}} - \binom{4}{1} x^{28} + \binom{4}{2} x^{28 \times 2} + \ldots$$

↳ O único que contribui para $a_{27} x^{27}$

Assim:

$$(*) = (\ldots) + \underbrace{\binom{4}{0}\binom{27+4-1}{27}}\, x^{27} + \ldots$$

$$a_{27} = \binom{30}{27} = \binom{30}{3} = 4060$$

b)

$$x_1 + x_2 + x_3 + x_4 = 27$$
$$x_i \in \{1, 2, 3, \ldots, 27\}$$

$$x_1 \longmapsto \cancel{1} + x + x^2 + \ldots + x^{27} = p_1(x) \qquad x_3 \longmapsto \cancel{1} + x + x^2 + \ldots + x^{27} = p_3(x)$$
$$x_2 \longmapsto \cancel{1} + x + x^2 + \ldots + x^{27} = p_2(x) \qquad x_4 \longmapsto \cancel{1} + x + x^2 + \ldots + x^{27} = p_4(x)$$

$$f(x) = (1 + x + \dots + x^{24})^4$$

$$= \dots = \sum_{n=0}^{+\infty} c_K x^K, \quad K = 27$$

$$= \sum_{n=0}^{+\infty} c_{27} x^{27}$$

$(\dots)$

c)

$$f(x) = (1 + x + \dots + x^{13})^4$$

$(\dots)$

Falta acabar!

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a)

$$b_n = n K^n, \quad n \in \mathbb{N}$$

$$(b_n)_{n\in\mathbb{N}} : 0, K, 2K^2, 3K^3, \dots, nK^n, \dots$$

$$B(x) = \sum_{n=0}^{+\infty} \underbrace{(nK^n)}_{b_n} x^n$$

$$= \sum_{n=0}^{+\infty} n (Kx)^n = \sum_{n=1}^{+\infty} n (Kx)^n$$

$$= x \sum_{n=1}^{+\infty} n K^n x^{n-1}$$

$$= x \sum_{n=1}^{+\infty} (nK) \times (Kx)^{n-1}$$

$$= x \sum_{n=0}^{+\infty} \underbrace{[K(n+1)] \times (Kx)^n}$$

$$= \left[(Kx)^{n+1}\right]' = (n+1) \times (Kx)' \times (Kx)^{n+1-1} = K(n+1)(Kx)^n$$

$$= x \left[\sum_{n=0}^{+\infty} (Kx)^{n+1}\right]'$$

$$= x \left[Kx \sum_{n=0}^{+\infty} (Kx)^n\right]'$$

$$= x \left[Kx \frac{1}{Kx}\right]'$$

Sabemos:

$$\sum_{n=0}^{+\infty} (Kx)^n = \frac{1}{1 - Kx}$$

$$= x \left(\frac{K(1-Kx) - Kx(-K)}{(1-Kx)^2}\right)$$

$$= x \left(\frac{K - \cancel{K^2 x} + \cancel{K^2 x}}{(1-Kx)^2}\right)$$

Logo: $B(x) = \dfrac{Kx}{(1-Kx)^2}$

b)

$$c_n = \underbrace{K}_{b_1} + \underbrace{2K^2}_{b_2} + \underbrace{3K^3}_{b_3} + \dots + \underbrace{nK^n}_{b_n \text{ de (a)}}, \quad n \in \mathbb{N}$$

$b_0 + b_1 + b_2 + b_3$, $b_0 = 0$

s.g. $\dfrac{Kx}{(1-Kx)^2}$

Resultado teórico:

Se $g(x)$ for a série geométrica associada à sucessão $(a_n)_{n\in\mathbb{N}}$, então

$\boxed{(1 + x + x^2 + \dots)\, g(x)}$ é a série geradora

da sucessão $(a_0 + a_1 + \dots + a_n)_{n\in\mathbb{N}}$

$$(1 + x + x^2 + \dots)(a_0 + a_1 x + a_2 x^2 + \dots)$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots + (a_0 + a_1 + a_n)x^n$$

Aplicando o resultado anterior tem-se a série geradora associada a $(c_n)_{n\in\mathbb{N}}$ é

$$C(x) = (1 + x + x^2 + \dots)\, B(x) = \frac{1}{1-x} \cdot \frac{Kx}{(1-Kx)^2} \to \frac{Kx}{(1-x)(1-Kx)^2}$$

pois,

$$C(x) = b_0 + (b_0 + b_1)x + \dots + \underbrace{(b_0 + b_1 + \dots + b_n)}_{c_n} x^n$$

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b)

$$F(x) = \frac{6x}{(1+2x)^2} + 2 - x^2$$

$$= 6x \frac{x}{(1+2x)(1+2x)} + 2 - x^2$$

$$\frac{x}{(1+2x)^2} = \frac{A}{(1+2x)} + \frac{B}{(1+2x)^2}$$

$$\Leftrightarrow x = A(1+2x) + B$$
$$\Leftrightarrow x = A + B + 2Ax$$

$$\Leftrightarrow \begin{cases} A + B = 0 \\ 2A = 1 \end{cases} \quad \begin{cases} B = -\frac{1}{2} \\ A = \frac{1}{2} \end{cases}$$

$$= \frac{3}{1+2x} - \frac{3}{(1-(-2x))^2} + 2 - x^2$$

$$= 3\sum_{n=0}^{+\infty} (-2x)^n - 3\sum_{n=0}^{+\infty} \binom{2+n-1}{n} (-2x)^n + 2 - x^2$$

$$= 3\sum_{n=0}^{+\infty} (-2x)^n - 3\sum_{n=0}^{+\infty} \underbrace{\binom{1+n}{n}}_{\frac{(1+n)!}{n!\,1!} = (1+n)} (-2x)^n + 2 - x^2$$

$$= 3\sum_{n=0}^{+\infty}(-2x)^n - 3\sum_{n=0}^{+\infty}(1+n)(-2x)^n + 2 - x^2$$

$$= 3\sum_{n=0}^{+\infty}\left[(-2)^n x^n - (1+n)(-2)^n x^n\right] + 2 - x^2$$

$$= 3\sum_{n=0}^{+\infty}\left[(-2)^n x^n (\cancel{1} - \cancel{1} - n)\right] + 2 - x^2$$

$$= -3\sum_{n=0}^{+\infty}\left[n(-2)^n x^n\right] + 2 - x^2$$

$$= \cancel{(-3)\times 0}^{\,0} + (-3)(-2)x + (-3)\times 2\times 4\times x^2 + 2 - x^2 - 3\sum_{n=3}^{+\infty}\left[n(-2)^n x^n\right]$$

$$= 2x^0 + 6x^1 - 25x^2 + \sum_{n=3}^{+\infty} -3n(-2)^n x^n$$

$\hookrightarrow a_n,\ n \geq 3$

A sucessão associada é:

$$\begin{cases} a_0 = 2 \\ a_1 = 6 \\ a_2 = -25 \\ a_n = -3n(-2)^n, & n \geq 3 \end{cases}$$

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a) Resolvido Aula 18

b) $\begin{cases} a_n = a_{n-1} + n, & n \geq 1 \\ a_0 = 1 \end{cases}$

$$A(x) = \sum_{n=0}^{+\infty} a_n x^n = 1 + \sum_{n=1}^{+\infty}(a_{n-1} + n)x^n$$

$$= 1 + \sum_{n=1}^{+\infty} a_{n-1}x^n + \sum_{n=1}^{+\infty} n x^n$$

$$= 1 + x\sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} n x^n$$

$$= 1 + xA(x) + \sum_{n=0}^{+\infty} n x^n$$

$$= 1 + xA(x) + x\sum_{n=0}^{+\infty} \underbrace{n x^{n-1}}_{(x^n)'}$$

$$= 1 + xA(x) + x\sum_{n=0}^{+\infty}(x^n)'$$

$$= 1 + xA(x) + x\left(\sum_{n=0}^{+\infty} x^n\right)'$$

$$= 1 + xA(x) + x\left(\frac{1}{1-x}\right)'$$

$$= 1 + xA(x) + \frac{x}{(1-x)^2}$$

$$\Rightarrow A(x)(1-x) = 1 + \frac{x}{(1-x)^2}$$

$$\Rightarrow A(x) = \frac{1}{(1-x)} + \frac{x}{(1-x)^3}$$

$$\overset{???}{(:)}\ \frac{x}{(1-x)^3} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} \Leftrightarrow x = A(1-x)^2 + B(1-x) + C$$

$$\Leftrightarrow x = (A+B+C) + (-B)x + (-A)x^2$$

$$\begin{cases} A+B+C=0 \\ -B=1 \\ -A=0 \end{cases} \quad \begin{cases} C=1 \\ B=-1 \\ A=0 \end{cases}$$

Logo:

$$A(x) = \frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1-x)^3} = \sum_{n=0}^{+\infty} x^n - \sum_{n=0}^{+\infty} \overbrace{\binom{2+n-1}{n}}^{=n+1} x^n + \sum_{n=0}^{+\infty} \overbrace{\binom{3+n-1}{n}}^{\binom{2+n}{n} = \frac{(2+n)!}{n!\,2!} = \frac{(2+n)(1+n)}{2}} x^n$$

$$= \sum_{n=0}^{+\infty} x^n - \sum_{n=0}^{+\infty} (n+1)x^n + \sum_{n=0}^{+\infty} \frac{(2+n)(1+n)}{2} x^n$$

$$= \sum_{n=0}^{+\infty} x^n \left[\cancel{1} - (n + \cancel{1}) + (2+n)(1+n) \right]$$

$$= \sum_{n=0}^{+\infty} x^n \left((-n) + (2+n)(1+n) \right)$$

$\pm$ → Tenho foto 8/5

$\boxed{26}$

a)

Mostrar que, para todos os $n, r \in \mathbb{N}$

$$\binom{-n}{r} = (-1)^r \underbrace{\binom{n+r-1}{r}}_{\left(\binom{n}{r}\right)}$$

$$x_k = \underbrace{x(x-1)(x-2)\cdots(x-(k-1))}_{k \text{ termos}}$$

Ora, nº bin generalizado

$$\binom{-n}{r} = \frac{(-n)_r}{r!} = \frac{(-n)(-n-1)(-n-2)\cdots\overbrace{(-n-(r-1))}^{-n-r+1}}{r!} = \frac{\overbrace{(-1)(n)(-1)(n+1)(-1)(n+2)\cdots(-1)(n+(r-1))}^{r \text{ termos}}}{r!}$$

$$= (-1)^r \frac{n(n+1)(n+2)(n+(r-1))}{r!} = (-1)^r \frac{(n+(r-1))\cdots(n+2)(n+1)(n) \times \boxed{(n-1)!}}{r!\,\boxed{(n-1)!}}$$

$$= (-1)^r \frac{\overbrace{(n+r-1)\cdots(n+1)\,n\,(n-1)\cdots 1}^{(n+r-1)!}}{r!\,(n-1)!} = (-1)^r \times \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$= (-1)^r \times \binom{n+r-1}{r}$$

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$$x_K = \underbrace{x\,(x-1)\,(x-2)\cdots(x-(K-1))}_{K \text{ termos}}$$

$$\binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)_3}{3!} = \frac{\overbrace{\frac{1}{2}\times\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}^{3 \text{ termos}}}{6} = \frac{-\frac{1}{4}\times\left(-\frac{3}{2}\right)}{6} = \frac{\frac{3}{8}}{6} = \frac{1}{16}$$

$$\binom{-2}{3} = \frac{(-2)_3}{3!} = \frac{\overbrace{(-2)(-2-1)(-2-2)}^{3 \text{ termos}}}{6} = \frac{\cancel{6}\,(-4)}{\cancel{6}} = -4$$

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Sabemos que:

$$F(x) = \sum_{n=1}^{+\infty} F_n x^n \overset{\text{slide 25}}{=} (\dots) = \frac{x}{1-x-x^2}$$

série geradora associada à sucessão $(F_n)_{n\in\mathbb{N}}$,

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, & n \geq 2 \\ F_0 = F_1 \end{cases}$$

Mostrar que:

$$F_n = \sum_{j=0}^{n} \binom{n-j}{j} \qquad \binom{n}{K} = \frac{n!}{K!\,(n-K)!}$$

$$F_0 = \binom{0-0}{0} = 1$$

$$F_1 = \underbrace{\binom{1-0}{0}}_{1} + \underbrace{\binom{1-1}{1}}_{0} = 1$$

$$F_2 = \underbrace{\binom{2-0}{0}}_{1} + \underbrace{\binom{2-1}{1}}_{1} + \underbrace{\binom{2-2}{2}}_{0} = 2$$

Pretende-se: $F(x) = \dfrac{x}{1-x-x^2} = (\dots) = \displaystyle\sum_{n=0}^{+\infty} \underbrace{(\quad)}_{F_n = \sum_{j=0}^{n}\binom{n-j}{j}} x^n$

Ora,

$$F(x) = F_0 x^0 + F_1 x^1 + F_2 x^2 + F_3 x^3 + \dots$$

$$= \frac{x}{1-x-x^2} = x\left(\frac{1}{1-(x+x^2)}\right)$$

$$= x \times \sum_{K=0}^{+\infty} (x+x^2)^K = x \sum_{K=0}^{+\infty} x^K (1+x)^K$$

$$= \sum_{K=0}^{+\infty} x^{K+1} \underbrace{(1+x)^K}_{\sum_{j=0}^{K}\binom{K}{j}x^j} = \sum_{K=0}^{+\infty} x^{K+1} \sum_{j=0}^{K} \binom{K}{j} x^j$$

$$= \underbrace{\sum_{K=0}^{+\infty} \sum_{j=0}^{K} \binom{K}{j} x^{K+1+j}}_{\longrightarrow\; x + x^2(1+x) + x^3(1+x)^2 + \dots}$$

(*) Tendo em conta que, para $K \geq 0$:

$$F(x) = \sum_{j=0}^{0} \binom{0}{j} x^{0+1+j} + \sum_{j=0}^{1} \binom{1}{j} x^{1+1+j} + \sum_{j=0}^{2} \binom{2}{j} x^{2+1+j} + \sum_{j=0}^{3} \binom{3}{j} x^{3+1+j} + \dots$$

$$= \binom{0}{0}x^1 + \binom{1}{0}x^2 + \binom{1}{1}x^3 + \binom{2}{0}x^3 + \binom{2}{1}x^4 + \binom{2}{2}x^5 + \binom{3}{0}x^4 + \binom{3}{1}x^5 + \binom{3}{2}x^6 + \binom{3}{3}x^7 + \dots$$

Pelo que,

$$F(x) = x_1 + x_2 + \left[\binom{1}{1} + \binom{2}{0}\right]x^3 + \left[\binom{2}{1} + \binom{3}{0}\right]x^4 + \left[\binom{2}{2} + \binom{3}{1} + \binom{4}{0}\right]x^5 + \dots$$

$$= \sum_{n=0}^{+\infty} \underbrace{\left(\sum_{j=0}^{n} \binom{n-j}{j} \right)}_{F_n} x^n$$

Duvidoso!