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Erro Truncatura Euler - Cromer

a)

Comparando com a série de Taylor:

$$x(t+\delta t)_{EXATO} = x(t) + x^{(1)}(t)\,\delta t + \frac{x^{(2)}(t)}{2}\delta t^2 + o(\delta t^3)$$

$$\boxed{1} \rightarrow v_x(t+\delta t)_{EXATO} = v_x(t) + v_x^{(1)}(t)\,\delta t + \frac{v_x^{(2)}(t)}{2}\delta t^2 + o(\delta t^3)$$

Fazendo aproximação com o método de Euler-Cromer (EC):

$$x(t+\delta t)_{EC} = x(t) + v_x(t+\delta t)\,\delta t$$

Usando $\boxed{1} \longrightarrow$ $= x(t) + \left[v_x(t) + v_x^{(1)}(t)\delta t + \frac{v_x^{(2)}(t)}{2}\delta t^2 + o(\delta t^3)\right]\delta t$

$$= x(t) + v_x(t)\delta t + v_x^{(1)}(t)\delta t^2 + \frac{v_x^{(2)}(t)}{2}\delta t^3 + o(\delta t^4)$$

$$v_x(t+\delta t)_{EC} = v_x(t) + a_x(t)\,\delta t$$

b)

Cálculo dos erros locais na posição e velocidade:

erro local da posição $= \left| x(t+\delta t)_{EXATO} - x(t+\delta t)_{EC} \right|$

$$= \Big| \cancel{x(t)} + x^{(1)}\cancel{(t)\delta t} + \frac{x^{(2)}(t)}{2}\delta t^2 + \underbrace{o(\delta t^3)}_{= \frac{x^{(3)}(t)}{3!}\delta t^3 + o(\delta t^4)}$$

$$- \cancel{x(t)} - \underbrace{\cancel{v_x(t)\delta t}}_{x^{(1)}(t)} - \underbrace{v_x^{(1)}(t)}_{x^{(2)}(t)}\delta t^2 - \frac{v_x^{(2)}(t)}{2}\delta t^3 - o(\delta t^4) \Big|$$

$$= \left| -\frac{x^{(2)}(t)}{2}\delta t^2 + \frac{x^{(3)}(t)}{3!}\delta t^3 - \frac{v_x^{(2)}(t)}{2}\delta t^3 + o(\cancel{\delta t^4}) - o(\cancel{\delta t^4}) \right|$$

$$= \left| -\frac{x^{(2)}(t)}{2}\delta t^2 - \frac{x^{(3)}(t)}{3}\delta t^3 \right|$$

$$= \frac{x^{(2)}(t)}{2}\delta t^2 + \frac{x^{(3)}(t)}{3}\delta t^3$$

$$= o(\delta t^2)$$

erro local da velocida $= \left| v(t+\delta t)_{EXATO} - v(t+\delta t)_{EC} \right|$

$$= \left| \cancel{v_x(t)} + \cancel{v_x^{(1)}(t)\delta t} + \frac{v_x^{(2)}(t)}{2}\delta t^2 + o(\delta t^3) \right.$$

$$\left. - \cancel{v_x(t)} - \underbrace{\cancel{a_x(t)}}_{v_x^{(1)}(t}\delta t \right|$$

$$= \frac{v_x^{(2)}(t)}{2}\delta t^2 + o(\delta t^3)$$

$$= o(\delta t^2)$$

c) Cálculo do erro global da posição e velocidade:

Quer na posição quer na velocidade o erro de truncatura local é de $o(\delta t^2)$.

No instante (qualquer) $t_f = n \times \delta t$ o erro acumulado depois de n passos é:

$$n \times o(\delta t^2) = \frac{t_f}{\cancel{\delta t}} o(\delta t^{\cancel{2}}) = o(\delta t)$$

Nota: quer para a posição quer para a velocidade!

$\boxed{2}$

Erro Truncatura Feymmon - Newton

a) Comparando com a série de Taylor:

$$x(t+\delta t)_{EXATO} = x(t) + x^{(1)}(t)\,\delta t + \frac{x^{(2)}(t)\,\delta t^2}{2} + \cancel{\frac{x^{(3)}(t)}{3!}}\, o(\delta t^3)$$

$$v\left(t+\frac{\delta t}{2}\right)_{EXATO} = v(t) + v^{(1)}(t)\,\frac{\delta t}{2} + \frac{v^{(2)}(t)}{2}\left(\frac{\delta t}{2}\right)^2 + o\left(\frac{\delta t}{2}\right)^3$$

$$v\left(t-\frac{\delta t}{2}\right)_{EXATO} = v(t) - v^{(1)}(t)\,\frac{\delta t}{2} + \frac{v^{(2)}(t)}{2}\left(\frac{\delta t}{2}\right)^2 - o\left(\frac{\delta t}{2}\right)^3$$

Fazendo aproximação com o método de Feymmon - Newton (FN):

$$x(t+\delta t)_{FN} = x(t) + v_x\left(t+\frac{\delta t}{2}\right)\delta t$$

$$= x(t) + v(t)\,\delta t + v^{(1)}(t)\frac{\delta t^2}{2} + \frac{v^{(2)}(t)}{2}\frac{(\delta t)^3}{2^2} + o\,\frac{(\delta t)^4}{2^3}$$

$$v\left(t+\frac{\delta t}{2}\right)_{FN} = v_x\left(t-\frac{\delta t}{2}\right) + a_x(t)\,\delta t$$

$$= v(t) - v^{(1)}(t)\,\frac{\delta t}{2} + \frac{v^{(2)}(t)}{2}\left(\frac{\delta t}{2}\right)^2 - o\left(\frac{\delta t}{2}\right)^3 + a_x(t)\delta t$$

Cálculo dos erros locais da velocidade e posição:

erro local da posição $= \left| x(t+\delta t)_{EXATO} - x(t+\delta t)_{FN} \right|$

$$= \cancel{x(t)} + \cancel{x^{(1)}(t)\,\delta t} + \frac{x^{(2)}(t)\,\delta t^2}{2} + o(\delta t^3)$$
$$- \cancel{x(t)} - \cancel{v(t)\,\delta t} - \cancel{\frac{v^{(1)}(t)\,\delta t^2}{2}} - \frac{v^{(2)}(t)}{2}\frac{(\delta t)^3}{2^2} - o\,\frac{(\delta t)^4}{2^3}$$

$$= \frac{x^{(3)}(t)}{\underbrace{3!}_{=6}}\,\delta t^3 + o(\delta t^4) - \frac{v^{(2)}(t)}{\underbrace{2^3}_{=8}}\,\delta t^3 - \frac{1}{8}\,o(\delta t^4)$$

$$= \frac{x^{(3)}(t)\,\delta t^3}{24} + \frac{7}{8}(\delta t^4) = o(\delta t^3)$$

erro local da velocidade $= \left| v(t+\frac{\delta t}{2})_{EXATO} - v(t+\frac{\delta t}{2})_{FN} \right|$

$$= \cancel{v(t)} + v^{(1)}(t)\frac{\delta t}{2} + \cancel{\frac{v^{(2)}(t)}{2}}\left(\frac{\delta t}{2}\right)^2 + o\left(\frac{\delta t}{2}\right)^3$$
$$- \cancel{v(t)} + v^{(1)}(t)\frac{\delta t}{2} - \cancel{\frac{v^{(2)}(t)}{2}}\left(\frac{\delta t}{2}\right)^2 + o\left(\frac{\delta t}{2}\right)^3 - a_x(t)\,\delta t$$

$$= \cancel{2v^{(1)}(t)\frac{\delta t}{2}} + 2\,o\left(\frac{\delta t}{2}\right)^3 - \cancel{v^{(1)}(t)\,\delta t}$$

$$= o(\delta t^3)$$

b)

<u>Cálculo do erro global da velocidade e posição:</u>

Quer na posição quer na velocidade o erro de truncatura local é de ~~...~~ $o(\delta t^3)$.

No instante (qualquer) $t_f = n\,\delta t$ o erro acumulado depois de n passos é:

$$n\,o(\delta t^3) = \frac{t_f}{\delta t}\,o(\delta t^3) = o(\delta t^2)$$

Nota: quer para a posição quer para a velocidade!

c)

<u>Comparando com a série de Taylor:</u>

$$\cancel{v_x\left(\frac{\delta t}{2}\right) = v_x(t) + v_x^{(1)}(t)\left(\frac{\delta t}{2}-t\right) + \frac{v_x^{(2)}(t)}{2!}\left(\frac{\delta t}{2}-t\right)^2 + o\left(\frac{\delta t}{2}-t\right)^3}$$

~~como $t_0 = 0$~~

$v_x\left(t+\frac{\delta t}{2}\right)$, quand $t = t_0 = 0$

$$\Rightarrow v_x\left(\frac{\delta t}{2}\right) = v_{0x}(0) + v^{(1)}(0)\frac{\delta t}{2} + \frac{v^{(2)}(t)}{2}\left(\frac{\delta t}{2}\right)^2 + o\left(\frac{\delta t}{2}\right)^3$$

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Erro Aproximação Retangular

Comparando com a série de Taylor centrada em $x = x_i$:

$$f(x)_{EXATO} = f(x_i) + f^{(1)}(x_i)(x - x_i) + \frac{f^{(2)}(x_i)}{2}(x - x_i)^2 + o(x - x_i)^3$$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} = \int_{x_i}^{x_{i+1}} \left[f(x_i) + f^{(1)}(x_i)(x - x_i) + \frac{f^{(2)}(x_i)}{2}(x - x_i)^2 + o(x - x_i)^3 \right] dx$$

$$= \int_{x_i}^{x_{i+1}} f(x_i)\,dx + \int_{x_i}^{x_{i+1}} f^{(1)}(x)(x - x_i)\,dx + \int_{x_i}^{x_{i+1}} \frac{f^{(2)}(x_i)}{2}(x - x_i)^2\,dx + \int_{x_i}^{x_{i+1}} o(x - x_i)^3\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + f^{(1)}(x)\left(\frac{(x_{i+1})^2}{2} - \frac{(x_i)^2}{2} - x_i(x_{i+1} - x_i) \right) + \int_{x_i}^{x_{i+1}} o(x - x_i)^2\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + \frac{f^{(1)}(x)}{2}\left((x_{i+1})^2 - 2x_{i+1}x_i + (x_i)^2 \right) + \int_{x_i}^{x_{i+1}} o(x - x_i)^2\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + \frac{f^{(1)}(x)}{2}(x_{i+1} - x_i)^2 + o(x_{i+1} - x_i)^3$$

Como $\delta x = x_{i+1} - x_i$, se $a = x_i$, $b = x_{i+1}$, $n = 1$:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} = f(x_i)\,\delta x + f^{(1)}(x)\frac{(\delta x)^2}{2} + o(\delta x^3)$$

Fazendo a aproximação com a aproximação retangular:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{AR} = f(x_i)\,\delta x$$

Cálculo do erro local:

$$\text{erro local} = \left| \int_{x_i}^{x_{i+1}} f(x)\,dx_{\text{EXATO}} - \int_{x_i}^{x_{i+1}} f(x)\,dx_{\text{AR}} \right|$$

$$= \cancel{f(x_i)\delta x} + \frac{f^{(1)}(x)(\delta x)^2}{2} + o(\delta x^3) - \cancel{f(x_i)\delta x}$$

$$= \frac{f^{(1)}(x)(\delta x)^2}{2} + o(\delta x)^3$$

$$= o(\delta x^2)$$

Cálculo do erro global:

O erro global é um somatório de n fatios, é o acumular dos erros locais, logo após n fatios o erro é:

$$\delta x = \frac{b-a}{n}$$

$$\Leftrightarrow n = \frac{b-a}{\delta x}$$

$$n\,o(\delta x^2) = \frac{b-a}{\delta x}\, o(\delta x^2)$$

$$= (b-a)\, o(\delta x)$$

$$= o(\delta x)$$

↳ ordem de grandeza de δx

$\boxed{\text{Erro Aproximação Trapezoidal}}$

$$I = \int_a^b f(x)\,dx$$

• Comparando com a série de Taylor centrada em $x = x_i$:

$$f(x)_{EXATO} = f(x_i) + f^{(1)}(x_i)(x - x_i) + o(x - x_i)^2$$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} = f(x_i)(x_{i+1} - x_i) + f^{(1)}(x_i) \times \left(\left[\frac{x^2}{2}\right]_{x_i}^{x_{i+1}} - x_i(x_{i+1} - x_i) \right) + \int_{x_i}^{x_{i+1}} o(x - x_i)^2\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + f^{(1)}(x_i) \times \left(\frac{(x_{i+1})^2}{2} - \frac{(x_i)^2}{2} - x_i x_{i+1} + x_i^2 \right) + \int_{x_i}^{x_{i+1}} o(x - x_i)^2\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + \frac{f^{(1)}(x_i)}{2}\left((x_{i+1})^2 - 2x_i x_{i+1} + (x_i)^2 \right) + \int_{x_i}^{x_{i+1}} o(x - x_i)^2\,dx$$

$$= f(x_i)(x_{i+1} - x_i) + \frac{f^{(1)}(x_i)}{2}(x_{i+1} - x_i)^2 + o(x - x_i)^3$$

• Fazendo a aproximação trapezoidal:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} \simeq \frac{f(x_{i+1}) + f(x_i)}{2}\,\delta x \rightarrow \int_{x_i}^{x_{i+1}} f(x)\,dx_{APROXIMADO}$$

• Como para $a = x_i$, $b = x_{i+1}$, $n = 1$: $\delta x = x_{i+1} - x_i$:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} = f(x_i)\,\delta x + \frac{f^{(1)}(x_i)}{2}\,\delta x^2 + o(\delta x)^3$$

$$\boxed{f(x_{i+1}) = f(x_i) + f^{(1)}(x_i)(x_{i+1} - x_i) + o(x_{i+1} - x_i)^2}$$

$$\Downarrow$$

$$\frac{f(x_{i+1}) + f(x_i)}{2}\,\delta x = f(x_i)\,\delta x + \frac{f^{(1)}(x_i)}{2}\,\delta x^2 + o(\delta x)^3$$

• Erro local de truncatura com o passo δx:

$$\text{erro local} = \left| \int_{x_i}^{x_{i+1}} f(x)\,dx_{EXATO} - \int_{x_i}^{x_{i+1}} f(x)\,dx_{APROX} \right| =$$

$$= \left| \cancel{f(x_i)\,\delta x} + \cancel{\frac{f^{(1)}(x_i)\,\delta x^2}{2}} + o(\delta x)^3 - \cancel{f(x_i)\,\delta x} - \cancel{\frac{f^{(1)}(x_i)\,\delta x^2}{2}} + o(\delta x)^3 \right|$$

$$= o(\delta x)^3$$

Fazer ordem 4 para depois cortar

Global $\times$ n fatias

$$\delta x = \frac{b-a}{n}$$

$$n \times o(\delta x)^3 = \frac{b-a}{\delta x}\,o(\delta x)^3 = (\delta x^2) \Rightarrow n = \frac{b-a}{\delta x}$$