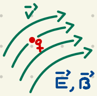
Tensor de Maxwell



Carga puntual $\longrightarrow$ $\vec{F} = q\vec{E} + q\frac{\vec{v}}{c} \times \vec{B}$

Distribución en volumen $\longrightarrow$ $\vec{f} = \rho\vec{E} + \frac{\vec{j}}{c} \times \vec{B}$

$$\frac{d\vec{P}_{mec}}{dt} = \underset{\substack{\downarrow \\ \text{origen} \\ \text{no EM}}}{\vec{F}_0} + \vec{F}_{EM} = \vec{F}_0 + \int dv \left(\rho\vec{E} + \frac{\vec{j}}{c} \times \vec{B}\right)$$

$\longrightarrow$ de las ecuaciones de Maxwell (vacío)
$$\begin{cases} \rho = \frac{1}{4\pi}\vec{\nabla}\cdot\vec{E} \\ \frac{\vec{j}}{c} = \frac{1}{4\pi}\left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) \end{cases}$$

$$\Longrightarrow \frac{d\vec{P}_{mec}}{dt} + \frac{d\vec{P}_{EM}}{dt} = \vec{F}_0 + \frac{1}{4\pi}\int_V dv\, \vec{\nabla}\cdot\overline{\overline{T}} = \vec{F}_0 + \frac{1}{4\pi}\int_{S(V)} \overline{\overline{T}}\cdot d\vec{S}$$

$$\frac{dP_{mec\,i}}{dt} + \frac{dP_{EM\,i}}{dt} = F_{0\,i} + \frac{1}{4\pi}\int_{S(V)} T_{ij}\, n_j\, ds$$

el campo EM puede transportar Energía e Impulso

Impulso lineal EM: $\vec{P}_{EM} \equiv \frac{1}{4\pi c}\int_V dv\, \vec{E}\times\vec{B}$

Tensor de Maxwell: $T_{ij} \equiv E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}(E^2 + B^2) \longrightarrow$ rango 2

$\longrightarrow$ Si $\frac{d\vec{P}}{dt} = 0 \Longrightarrow \vec{F}_0 = -\frac{1}{4\pi}\int_{S(V)} \overline{\overline{T}}\cdot d\vec{S}$

Fuerza electro-magnética estática: $\boxed{\vec{F} = \frac{1}{4\pi}\int_{S(V)} \overline{\overline{T}}\cdot d\vec{S}}$

S: cualquier sup. que encierre la distribución

En electrostática ($B_i = 0$)

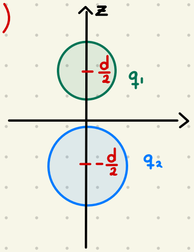
$\longrightarrow$ Campo ortogonal a la superficie $\vec{E} = E\,\hat{n}$

$$\overline{\overline{T}}\cdot\hat{n} = (\vec{E}\cdot\hat{n})\vec{E} - \frac{1}{2}E^2\hat{n} = \frac{1}{2}E^2\hat{n}$$

$\longrightarrow$ Campo paralelo a la superficie $\vec{E}\cdot\vec{n} = 0$

$$\overline{\overline{T}}\cdot\vec{n} = (\underbrace{\cancel{\vec{E}\cdot\hat{n}}}_{0})\vec{E} - \frac{1}{2}E^2\hat{n} = -\frac{1}{2}E^2\hat{n}$$

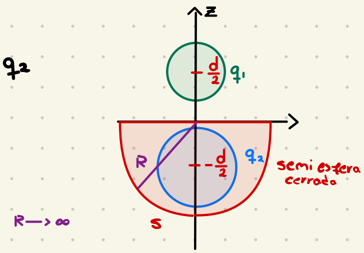
1)



1. Usar el tensor de Maxwell para encontrar:

   (a) La fuerza entre dos esferas de radios a y b, con cargas q_1 y q_2 y separadas una distancia $d > a+b$.

Fuerza sobre q_2



$$\int_S = \int_{SE} + \int_{DR}$$

$\hat{n} = \hat{r}$ $\qquad$ $\hat{n} = \hat{z}$

$$\vec{F}_2 = \frac{1}{4\pi}\int_{DR} ds\, \overline{\overline{T}}\cdot\hat{n} + \frac{1}{4\pi}\int_{SE} ds\, \overline{\overline{T}}\cdot\hat{n}$$

$$\left|\int_{SE} ds\, \overline{\overline{T}}\cdot\hat{n}\right| \le \int_{SE} ds\, \left|\overline{\overline{T}}\cdot\hat{n}\right| \le \int_{SE} ds\, \frac{3}{2}E^2$$

Campo total !!

$$\left|\overline{\overline{T}}\cdot\hat{n}\right| = \left|(\vec{E}\cdot\hat{n})\vec{E} - \frac{1}{2}E^2\hat{n}\right| = \underbrace{\left|\vec{E}\cdot\hat{n}\right|}_{\le E}\left|\vec{E}\right| + \frac{1}{2}E^2\overbrace{\left|\hat{n}\right|}^{1} \le \frac{3}{2}E^2$$

Desarrollo multipolar:

$$\left|\vec{E}\right| \le \frac{\overset{q_1+q_2}{\overset{\shortparallel}{Q}}}{R^2} + \left|3\vec{P}\cdot\hat{R} - \vec{P}\right|\frac{1}{R^3} + \cdots$$

$$\left|\int_{SE} ds\, \overline{\overline{T}}\cdot\hat{n}\right| \le \int_{SE} ds\, \frac{3}{2}E^2 \le \frac{3}{2}\left(\frac{Q}{R^2} + \left|3\vec{P}\cdot\hat{R} - \vec{P}\right|\frac{1}{R^3} + \cdots\right)^2 \cdot \overbrace{\frac{1}{2}4\pi R^2}^{\text{área SE}} \xrightarrow[R\to\infty]{} 0$$

$$\therefore \boxed{\vec{F}_2 = \frac{1}{4\pi}\int_{DR} ds\, \overline{\overline{T}}\cdot\hat{n}}$$

$$\vec{F}_2 = \frac{1}{4\pi}\int_{D_R} ds\, \overline{\overline{T}}\cdot\hat{n} = \frac{1}{4\pi}\int_{D_R} ds\left((\vec{E}_D\cdot\hat{n})\vec{E}_D - \frac{1}{2}E_D^2\hat{n}\right)$$

$\vec{E}_D = \vec{E}(z=0)$

campo sobre el plano

En cilíndricas:

$$\vec{E}_D = \frac{q_1}{(d^2/4+\rho^2)^{3/2}}\left(\rho\hat{\rho}+\frac{d}{2}\hat{z}\right) + \frac{q_2}{(d^2/4+\rho^2)^{3/2}}\left(\rho\hat{\rho}-\frac{d}{2}\hat{z}\right) = \frac{Q\rho\hat{\rho}+q\frac{d}{2}\hat{z}}{(d^2/4+\rho^2)^{3/2}}$$

$Q \equiv q_1+q_2$, $q \equiv q_1 - q_2$

$\hat{\rho}\cdot\hat{z}=0$

$$\Rightarrow E_D^2 = \frac{Q^2\rho^2 + q^2 d^2/4}{(d^2/4+\rho^2)^3}$$

$$\longrightarrow \frac{1}{8\pi}\int_{D_R} ds\, E_D^2\hat{n} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^{\infty}\rho\, d\rho\, d\phi\, E_D^2\,\hat{z} = \frac{1}{8\pi}\,2\pi\hat{z}\left(Q^2\underbrace{\int_0^{\infty}\frac{\rho^3 d\rho}{(d^2/4+\rho^2)^3}}_{1/d^2} + q^2 d^2/4\underbrace{\int_0^{\infty}\frac{\rho\, d\rho}{(d^2/4+\rho^2)^3}}_{4/d^4}\right) = \frac{1}{4d^2}(Q^2+q^2)\hat{z}$$

$$\longrightarrow \vec{E}_D\cdot\hat{n} = \vec{E}_D\cdot\hat{z} = \frac{q\frac{d}{2}}{(d^2/4+\rho^2)^{3/2}}$$

$$\Rightarrow \frac{1}{4\pi}\int_{D_R} ds\,(\vec{E}_D\cdot\hat{n})\vec{E}_D = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\rho\, d\rho\, d\phi\,\frac{q\frac{d}{2}}{(d^2/4+\rho^2)^{3/2}}\,\frac{Q\rho\hat{\rho}+q\frac{d}{2}\hat{z}}{(d^2/4+\rho^2)^{3/2}} = \frac{q\,d}{8\pi}\int_0^{2\pi}\int_0^{\infty}\rho\, d\rho\, d\phi\,\frac{Q\rho\hat{\rho}+q\frac{d}{2}\hat{z}}{(d^2/4+\rho^2)^3}$$

$\int_0^{2\pi}\hat{\rho}\, d\phi = 0$

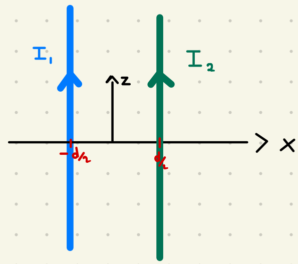
$$= \frac{q\,d}{8\pi}\,2\pi\, q\,\frac{d}{2}\,\hat{z}\underbrace{\int_0^{\infty}\frac{\rho\, d\rho}{(d^2/4+\rho^2)^3}}_{4/d^4} = \frac{q^2}{2d^2}\hat{z}$$

$$\Rightarrow \vec{F}_2 = \frac{1}{4\pi}\int_{D_R} ds\left((\vec{E}_D\cdot\hat{n})\vec{E}_D - \frac{1}{2}E_D^2\hat{n}\right) = \frac{q^2}{2d^2}\hat{z} - \frac{1}{4d^2}(Q^2+q^2)\hat{z} = \boxed{\frac{-1}{2d^2}q_1 q_2\hat{z}}$$

$\longrightarrow$ notar: indep. $\rightarrow$ radios

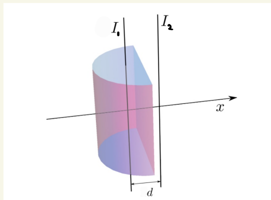
solo usamos los campos sobre el plano!

b)



$$E = 0$$

$$\overline{\overline{T}}\cdot\hat{n} = (\vec{B}\cdot\hat{n})\vec{B} - \tfrac{1}{2}B^2\hat{n} =$$



$$\vec{F}_2 = \frac{1}{4\pi}\int_{\text{plano}} ds\,\overline{\overline{T}}\cdot\hat{n} + \underbrace{\cancel{\frac{1}{4\pi}\int_{SC} ds\,\overline{\overline{T}}\cdot\hat{n}}}_{0}$$

$$\int_0^L dz\int_0^{2\pi} R\,d\phi\;\overline{\overline{T}}\cdot\hat{n} \sim LR\frac{I^2}{R^2}\xrightarrow[R\to\infty]{} 0$$

$$\vec{F}_2 = \frac{1}{4\pi}\int_{\text{plano}} ds\,\overline{\overline{T}}\cdot\hat{n}$$

El campo de un hilo conductor

$$\vec{B} = \frac{I}{c\rho}\hat{\theta} = \frac{I}{c\rho}\hat{z}\times\hat{\rho}$$

en el plano x=0:

$$I_\pm = I_1 \pm I_2$$

$$\vec{B}_p = \vec{B}(x=0) = \frac{2I_1}{c(d^2/4+y^2)}\hat{z}\times\left(-\frac{d}{2}\hat{x}+y\hat{y}\right) + \frac{2I_2}{c(d^2/4+y^2)}\hat{z}\times\left(\frac{d}{2}\hat{x}+y\hat{y}\right) = \frac{2}{c(d^2/4+y^2)}\hat{z}\times\left(-\frac{d}{2}I_-\hat{x}+I_+y\hat{y}\right)$$

$$= \frac{-2}{c(d^2/4+y^2)}\left(\frac{d}{2}I_-\hat{y}+I_+y\hat{x}\right)$$

$$\longrightarrow\quad B_p^2 = \frac{4}{c^2(d^2/4+y^2)^2}\left(\frac{d^2}{4}I_-^2+I_+^2y^2\right)$$

$$\Longrightarrow\quad \frac{1}{4\pi}\int_{\text{plano}} ds\left(-\frac{1}{2}B_p^2\right)\hat{x} = \frac{-1}{8\pi}\int_0^L dz\int_{-\infty}^{\infty} dy\, B_p^2\,\hat{x} = \frac{-L}{8\pi}\frac{4}{c^2}\left(I_-^2\frac{d^2}{4}\underbrace{\int_{-\infty}^{\infty} dy\,\frac{1}{(d^2/4+y^2)^2}}_{4\pi/d^3} + I_+^2\underbrace{\int_{-\infty}^{\infty} dy\,\frac{y^2}{(d^2/4+y^2)^2}}_{\pi/d}\right) = \frac{-L}{2c^2d}\left(I_-^2+I_+^2\right)\hat{x}$$

$$\longrightarrow \vec{B}_p \cdot \hat{n} = \vec{B}_p \cdot \hat{x} = \frac{-2I_+ y}{c(d^2/4 + y^2)}$$

$$\Longrightarrow \frac{1}{4\pi}\int_{\text{plano}} ds \left(\vec{B}_p \cdot \hat{n}\right)\vec{B}_p = \frac{L}{4\pi}\int_{-\infty}^{\infty} dy \left(\frac{-2I_+ y}{c(d^2/4+y^2)}\right)\frac{(-2)}{c(d^2/4+y^2)}\left(\frac{d}{2}I_- \hat{y} + I_+ y\,\hat{x}\right)$$

$$= \frac{L}{c^2\pi} I_+ \left(\frac{d}{2} I_- \hat{y} \underbrace{\cancel{\int_{-\infty}^{\infty} dy \frac{y}{(d^2/4+y^2)^2}}}_{0 \text{ impar}} + I_+ \hat{x} \underbrace{\int_{-\infty}^{\infty} dy \frac{y^2}{(d^2/4+y^2)}}_{\pi/d} \right) = \frac{L}{c^2 d} I_+^2 \hat{x}$$

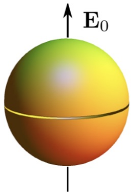
$$\Longrightarrow \vec{F}_2 = \frac{1}{4\pi}\int_{\text{plano}} ds\, \overline{\overline{T}} \cdot \hat{n} = \frac{-L}{2c^2 d}\left(I_-^2 + I_+^2\right)\hat{x} + \frac{L}{c^2 d} I_+^2 \hat{x}$$

$$\boxed{\frac{\vec{F}_2}{L} = \frac{2}{c^2 d} I_1 I_2 \hat{x}}$$

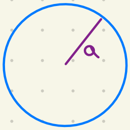
3. Una esfera conductora descargada tiene radio a y está en un campo eléctrico externo uniforme $\mathbf{E}_0$.

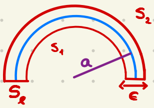
(a) Calcular la fuerza que tiende a separar o a unir los hemisferios según la dirección de $\mathbf{E}_0$.

(b) Calcular la fuerza si ahora la esfera tiene carga neta Q ¿Puede obtenerse este resultado sumando a la fuerza obtenida en el ítem anterior la fuerza calculada en la segunda parte del problema 2?



Fuerza sobre el hemisferio superior





$$\underbrace{\vec{F} = \frac{1}{4\pi}\int_S ds\, \overline{\overline{T}}\cdot\hat{n} + \frac{1}{4\pi}\int_{S_1} ds\, \overline{\overline{T}}\cdot\hat{n}} + \frac{1}{4\pi}\int_{S_2} ds\, \overline{\overline{T}}\cdot\hat{n} + \underbrace{\cancel{\frac{1}{4\pi}\int_{S_\ell} ds\, \overline{\overline{T}}\cdot\hat{n}}}_{\epsilon \to 0}$$

en el límite la contribución es la misma con $\hat{n}_1 = -\hat{n}_2$

$$\vec{F} = \frac{1}{4\pi}\int_S ds\, (\overline{\overline{T}}_2 - \overline{\overline{T}}_1)\cdot\hat{n}_2$$

pero dentro del conductor $\vec{E}_1 = 0$ $\Rightarrow$ $\overline{\overline{T}}_1\cdot\hat{n} = (\vec{E}_1\cdot\hat{n})\vec{E}_1 - \frac{1}{2}E_1^2\hat{n} = 0$

$$\boxed{\vec{F} = \frac{1}{4\pi}\int_S ds\, \overline{\overline{T}}_2\cdot\hat{n}_2}$$

a) Esfera conductora $\Longrightarrow$ potencial sobre la superficie es constante $:= V$

Campo externo $\vec{E}_0 = E_0 \hat{z}$

$\Longrightarrow$ potencial total : suma de 1 que da origen a $\vec{E}_0$ y otro dado por la distribución de cargas sobre la esfera

sim. azimutal

$$\varphi = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} A_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

$$\varphi(r=a) = V \Longrightarrow A_1 = E_0 a^2 \; ; \; A_0 = V a$$

$$r > a: \begin{cases} \varphi = -E_0 r \cos\theta + \frac{Va}{r} + \frac{E_0 a^3}{r^2}\cos\theta \\ \vec{E} = \vec{E}_0 + \frac{Va}{r^2}\hat{r} + \left(\frac{a}{r}\right)^3 \left(3 E_0 \cos\theta \, \hat{r} - \vec{E}_0\right) \end{cases} \qquad r < a: \begin{cases} \varphi = V \\ E = 0 \end{cases}$$

término monopolar: $\frac{Va}{r} \Longrightarrow$ esfera descargada $\boxed{V=0}$

$$\Longrightarrow \vec{E} = \vec{E}_0 + \left(\frac{a}{r}\right)^3 \left(3 E_0 \cos\theta \, \hat{r} - \vec{E}_0\right) \qquad r > a$$

y en el límite $\epsilon \to 0$: $\vec{E} = 3 E_0 \cos\theta \, \hat{r}$

$$\overline{\overline{T}} \cdot \hat{n} = \overline{\overline{T}} \cdot \hat{r} = (\vec{E} \cdot \hat{r})\vec{E} - \tfrac{1}{2} E^2 \hat{r} = \tfrac{1}{2} E^2 \hat{r} = \tfrac{9}{2} E_0^2 \cos^2\theta \, \hat{r}$$

$$\int_0^{2\pi} d\phi \, \hat{r} = 2\pi \cos \hat{z}$$

$$\Longrightarrow \vec{F} = \frac{1}{4\pi}\int_{S_2} ds \; \overline{\overline{T}}_2 \cdot \hat{n}_2 = \frac{9}{8\pi}\int_0^{2\pi}\int_0^{\pi/2} a^2 \sin\theta \, d\theta \, d\phi \; E_0^2 \cos^2\theta \, \hat{r} = \frac{9}{\cancel{8\pi}_{\,4}} \cancel{2\pi} \int_0^{\pi/2} a^2 \sin\theta \, d\theta \; E_0^2 \cos^3\theta \, \hat{z}$$

$$= \frac{9}{4} E_0^2 a^2 \int_0^1 d(\cos\theta) \cos^3\theta \, \hat{z} = \boxed{\frac{9}{16} E_0^2 a^2 \hat{z} = \vec{F}}$$

b) $Q \neq 0$

$$\vec{E} = \vec{E}_0 + \frac{\overset{Q}{\overset{\shortparallel}{Va}}}{r^2}\,\hat{r} + \left(\frac{a}{r}\right)^3 \left(3 E_0 \cos\theta\, \hat{r} - \vec{E}_0\right) \qquad r > a$$

en el límite $\epsilon \longrightarrow 0$: $\vec{E} = \left(\frac{Q}{a^2} + 3 E_0 \cos\theta \right) \hat{r}$

$$\overline{\overline{T}} \cdot \hat{n} = \overline{\overline{T}} \cdot \hat{r} = \frac{1}{2} E^2 \hat{r}$$

$$\Rightarrow \quad \vec{F} = \frac{1}{4\pi} a^2 \int_0^{2\pi}\!\!\int_0^1 d(\cos\theta)\, d\phi\, \frac{1}{2}\left(\frac{Q}{a^2} + 3 E_0 \cos\theta \right)^2 \hat{r} = \frac{a^2}{4} \int_0^1 d(\cos\theta) \left(\frac{Q^2}{a^4} + 6 \frac{Q}{a^2} E_0 \cos\theta + 9 E_0^2 \cos^2\theta \right) \cos\theta\, \hat{z}$$

$$\boxed{\vec{F} = \left(\frac{Q^2}{a^2} + \frac{Q E_0}{2} + \frac{9}{16} E_0^2 a^2 \right) \hat{z}}$$