Practice 2 PAN

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Q1

From the satetment we know that $X_i \in \{0, 1\}$. Such as it is shown in the randomize mechanism, Y_i (RR) can be either ther real answer (X_i) or an altered answer ($1 - X_i$) with a certain probability ($1 - \gamma$, in the case of the strategy). If $X_i = 1$, then $1 - X_i = 0$ and viceversa, it seems that the strategy success defining the mechanism.

Now, let's focus at how it works the mechanism:

- After the first coin we have a 50% of chances of obtaing truth.
- After the second coin, the result are going to be true in half of the cases.

Therefore, we know that the final result are going to be tampered in 25% of the cases, i.e. $\gamma = \frac{3}{4}$.

```
p = 0.5 #Probabilidad de haber copiado
gamma = 0.5 + p*0.25 + (1-p)*0.25
print(gamma)
```

```
0.75
```

Q2

```
import numpy as np

def avg(v):
    return sum(v)/len(v)

def rr(X):
    Y = []
    for i in X:
        coin = np.random.binomial(n=1, p=0.5)
        # coin = 1 => Face
        if not coin:
            Y.append(i)
        else:
            coin = np.random.binomial(n=1, p=0.5)
            if coin:
                Y.append(0) # Not cheated
            else:
                Y.append(1) # Cheated
```

```
    return np.array(Y)

n = 100000
p = 0.2
X = np.random.binomial(n=1, p=p, size=n)

Y = rr(X)

gamma_hat = 1 - avg(Y ^ X)
gamma_hat
```

```
0.74979
```

Q3

The value q shows the probability that a random answer is *CHEATED* or not. We already know that:

- In the 50% of the cases the answer does not change and it has a probability p that be *CHEATED*.
- In the other 50%, the probability to be *CHEATED* is 50%.

Thus, we have to take into account two probabilities:

- Do not change the answer and cheat: γp
- Change the answer and do not cheat: $(1-\gamma)\cdot(1-p)$

Which gives rise to:

$$q = \gamma p + (1-\gamma)\cdot(1-p) = 2\gamma p + 1 - p - \gamma \iff$$
$$\iff \hat{p} = \frac{q+\gamma-1}{2\gamma-1}$$

If we apply the result of the last question ($\gamma = \frac{3}{4}$):

$$q = \frac{1}{2}p + \frac{1}{4}$$
$$\hat{p} = 2q - \frac{1}{2}$$

Q4

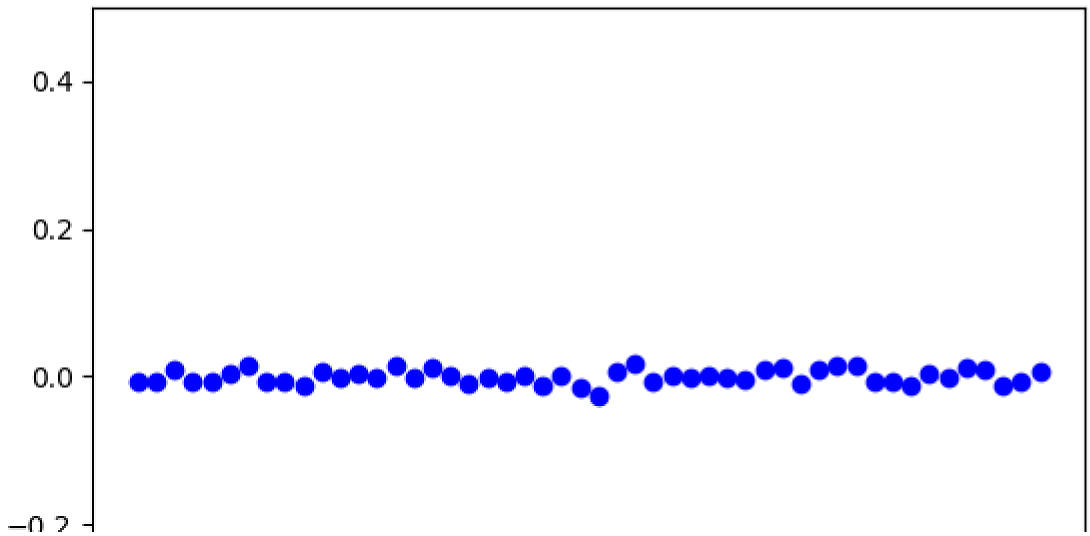
```
import matplotlib.pyplot as plt


differences = []
n = 10000
```

```
p = 0.2
for i in range(50):
    X = np.random.binomial(n=1, p=p, size=n)
    Y = rr(X)
    q = sum(Y)/len(Y)
    p_hat = 2*q - 1/2
    differences.append(p-p_hat)

print("Average of results: "+str(avg(differences)))
```

```
Average of results: -0.00011999999999999567
```

```
# Plot
plt.ylim(-0.5,0.5)
plt.scatter(range(len(differences)), differences, marker='o', color='b')
```

```
<matplotlib.collections.PathCollection at 0x792fd52f1990>
```



Q5

Our goal in this exercise is compute $Var[p - \hat{p}]$. Knowing that p is the average of X and the expectation for $\hat{p}$ ($E[\hat{p}] = p$), its variance in $0 \Rightarrow Var[p - \hat{p}] = Var[\hat{p}]$.

- From Q5: $\hat{p} = \frac{q+\gamma-1}{2\gamma-1}$

▾ -

$$Var[\hat{p}] = Var\left[\frac{q+\gamma-1}{2\gamma-1}\right]$$

$$= \left(\frac{1}{2\gamma-1}\right)^2 \cdot \big(Var[q] + Var[\gamma-1]\big)$$

$$=^{*_{1,2}}$$

$$= \left(\frac{1}{2\gamma-1}\right)^2 \frac{q\cdot(1-q)}{n} =^{*_3}$$

$$= \left(\frac{1}{2\gamma-1}\right)^2$$

$$\cdot \frac{(2\gamma p+1-p-\gamma)(-2\gamma p+p+\gamma)}{n}$$

- For $\gamma = \frac{3}{4}$: $Var[p-\hat{p}] = 4 \cdot \frac{-4p^2+4p+3}{4n}$
  $= \frac{-4p^2+4p+3}{n}$

($*_1$) As long as $\gamma - 1$ is a constant $\Rightarrow$ $Var[\gamma-1] = 0$.

($*_2$)

$$Var[q] = Var\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n^2}$$

$$\sum_{i=1}^{n} Var\left[Y_i\right] =^{*_4}$$

$$= \frac{1}{n} Var\left[Y_i\right] =^{*_5} \frac{q\cdot(1-q)}{n}$$

($*_3$) In Q3 we obtained that $q = \gamma p + (1-\gamma)\cdot(1-p) = 2\gamma p + 1 - p - \gamma$

($*_4$) The variables Y_i are independent.

($*_5$)

$$Var[Y_i] = E[Y_i^2] - E[Y_i]^2 = q - q^2$$
$$= q(1-q)$$

- $E[Y_i] = P[Y_i = 1]\cdot 1 + P[Y_i = 0] \cdot 0 = P[Y_i = 1] = q$

- $E[Y_i^2] = P[Y_i^2 = 1] \cdot 1$
  $+ P[Y_i^2 = 0] \cdot 0 =$
  $= P[Y_i^2 = 1] = P[Y_i = 1] = q$

-

When n grows the second part of $Var[p - \hat{p}]$ tends to decrease in absolute value (the result is closest to 0). Therefore:

$$\lim_{n \to \infty} (Var[p - \hat{p}]) = 0$$

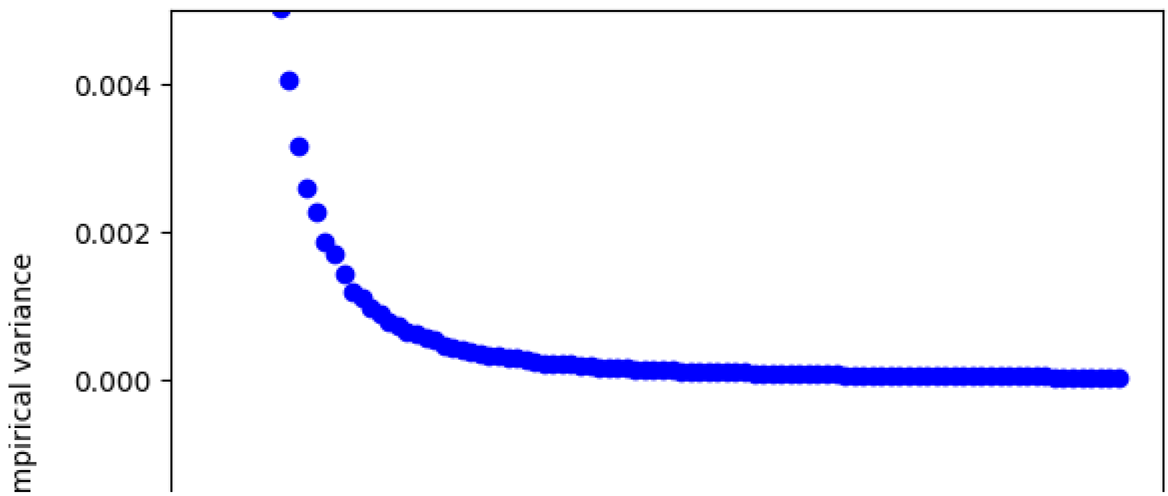
I.e. the variance of is null and as long as $E[\hat{p}] = p$, then it means that for $n \to \infty \Rightarrow \hat{p} = p$.

Q6

```
def compute_emp_and_th_vars(p: float, n: int, times: int):
  p_array = []

  for i in range(times):
    X = np.random.binomial(n=1, p=p, size=n)
    p_array.append(avg(X))

  avg_p = avg(p_array)
  emp_var = sum((p_array - avg_p) ** 2)/n
  th_var = (-4*p**2 + p + 3)/n

  return emp_var, th_var

p=0.2; times=5000; n = 1000

emp_var, th_var = compute_emp_and_th_vars(p, n, times)
print('Result for n=%i: \n'
      '\t - Empirical variance: %f\n'
      '\t - Theoretical variance: %f\n' % (n, emp_var, th_var))

n_array = np.linspace(50, 5000, 100)
p_plot = []
for i in n_array:
    emp_var, _ = compute_emp_and_th_vars(p=p, n=int(i), times=times)
    p_plot.append(emp_var)

plt.xlabel('n values')
plt.ylabel('Empirical variance')
plt.scatter(n_array, p_plot, marker='o', color='b', label='p-p_hat')
plt.ylim(-0.005,0.005)
```

```
Result for n=1000:
         - Empirical variance: 0.000809
         - Theoretical variance: 0.003040
```

```
(-0.005, 0.005)
```



Q7

On the one hand, when $\gamma = 1$ the result is deterministic and is the truth, and viceversa for $\gamma = 0$ (the result is always tamped):

$$q = \gamma p + (1-\gamma)\cdot(1-p)$$

$$Var[p-\hat{p}] = \left(\frac{1}{2\gamma - 1}\right)^2$$

$$\cdot \frac{(2\gamma p + 1 - p - \gamma)(-2\gamma p + p + \gamma)}{n}$$

$$\gamma = 1 \Rightarrow q = p, \quad Var[p-\hat{p}]$$

$$= \frac{p(1-p)}{n}$$

$$\gamma = 0 \Rightarrow q = 1 - p \quad Var[p-\hat{p}]$$

$$= \frac{p(1-p)}{n}$$

We can recovery the number of students who *CHEATED* (the best accuracy but no privacy).

On the other mand, when $\gamma = \frac{1}{2}$ we don't have any clue about the real answers:

$$\gamma = \frac{1}{2} \Rightarrow q = \frac{1}{2}, \quad Var[p - \hat{p}] = \infty$$

We have that q is independent of p and the variance is ∞ achiving the maximum privacy but the lower accuracy.

Q8

I know, as a professor, the values of Y_i when students trust me. When they don't trust me, Prof. Trustworthy give me q. The values of Y_i are anonymous so the order of the answers are not important an we can already compute the number of $Y_i = CHEATED$ as long as we know the number of students and q:

$$\sum_{i=1}^{n} Y_i = q/n$$

Therefore, students does not increase their security because we can obtain the same information in both scenarios.