Práctica 3

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Q1

In this case, sensitivity is the maximum difference that can exist between the average of two neighboring databases (databases that only differ in one row). In the different row, for a maximum difference in the average, the difference between the values in both databases may be the highest possible (one may be 0, and the other 125). $\Rightarrow \Delta^{(average)} = \frac{125}{n}$

Q2

Again we have to take into account the extreme case. Let's imagine a dataset D with an odd number of rows such us the rows with $age = 0$ are exactly one more than the number of rows with $age = 125$ and the only ages that exist in the dataset are those ones, i.e, the median of D is 0. However if we change one of the rows that have $age = 0$ for $age = 125$, then the median will be $125 \Rightarrow \Delta^{(median)} = 125$.

Q3

The $l_1 - sensitivity$, in this case, will be achived when SB1 and SB2 of D are different of SB1' and SB2' of D' in the row that differs. So, if $g(D) = [\sum SB1, \sum SB2]$ and $g(D) = [\sum SB1', \sum SB2'] \Rightarrow \Delta^{(bitcout)} = 2$.

E.g, in one database $SB1 = SB2 = 0$ and in the other one $SB1' = SB2' = 1$.

In the second part of the exercise that tell us what happend if SB1 is the complement of the bit in SB2, we will have the same answer. If $SB1 = 0$ & $SB2 = 1$ and $SB1' = 1$ & $SB2' = 0$, then we have the same distance as in the general case.

Q4

If one age changes, then one bin of the histogram decreases by one and other bin increases by one. Therefore, the maximum diference between $g(D)$ and $g(D')$ is 2, because all the values are equal except those two. $\Rightarrow \Delta^{(histogram)} = 2$.

Q5

$$\frac{f(R|D)}{f(R|D')} =^{(*_1)} \prod_{i=1}^{126} \frac{\frac{\epsilon}{2\Delta^{(g)}} exp\big(\frac{-\epsilon|r_i - g_i(D)|}{\Delta^{(g)}}\big)}{\frac{\epsilon}{2\Delta^{(g)}} exp\big(\frac{-\epsilon|r_i - g_i(D')|}{\Delta^{(g)}}\big)} = \prod_{i=1}^{126} \frac{exp\big(\frac{-\epsilon|r_i - g_i(D)|}{\Delta^{(g)}}\big)}{exp\big(\frac{-\epsilon|r_i - g_i(D')|}{\Delta^{(g)}}\big)} =^{(*_2)}$$

$$= \prod_{i=1}^{126} exp\big(\frac{\epsilon}{\Delta^{(g)}}(|r_i - g_i(D')| - |r_i - g_i(D)|)\big) \leq^{(*_3)} \prod_{i=1}^{126} exp\big(\frac{\epsilon}{\Delta^{(g)}}(g_i(D') - g_i(D))\big) =$$

$$= exp\big(\frac{\epsilon}{\Delta^{(g)}} \cdot \sum_{i=1}^{126} |g_i(D') - g_i(D)|\big) = exp\big(\frac{\epsilon}{\Delta^{(g)}} \cdot ||g_i(D') - g_i(D)||_1\big) \leq$$

$$\leq e^{\frac{\epsilon}{\Delta^{(g)}} \cdot \Delta^{(g)}} \leq e^{\epsilon} \quad \checkmark$$

$(*_1)$ As the elements of the random vectors are independent and identically distributed, the pdf is the product of the individual pdf's.

$(*_2) \frac{x^{-i}}{x^{-j}} = x^{j-i}$

$(*_3)$ The greatest result that we can achive is with the combination $r_i < g_i(D) < g_i(D')$

Q6

$$\frac{f(R|D)}{f(R|D')} =^{(*_1)} \prod_{i=1}^{126} \frac{\frac{1}{\sigma\sqrt{2\pi}} exp\big(\frac{-\epsilon^2|r_i - g_i(D)|^2}{2(\Delta^{(g)})^2}\big)}{\frac{1}{\sigma\sqrt{2\pi}} exp\big(\frac{-\epsilon^2|r_i - g_i(D')|^2}{2(\Delta^{(g)})^2}\big)} = \prod_{i=1}^{126} \frac{exp\big(\frac{-\epsilon^2|r_i - g_i(D)|^2}{2(\Delta^{(g)})^2}\big)}{exp\big(\frac{-\epsilon^2|r_i - g_i(D')|^2}{2(\Delta^{(g)})^2}\big)} =^{(*_2)}$$

$$= \prod_{i=1}^{126} exp\big(\frac{\epsilon^2}{2(\Delta^{(g)})^2}(|r_i - g_i(D')|^2 - |r_i - g_i(D)|^2)\big) \leq^{(*_2)} \prod_{i=1}^{126} exp\big(\frac{\epsilon^2}{2(\Delta^{(g)})^2}(g_i(D')^2 - g_i(D)^2)\big) =$$

$$= exp\big(\frac{\epsilon^2}{2(\Delta^{(g)})^2} \cdot \sum_{i=1}^{126}(g_i(D') - g_i(D)) \cdot (g_i(D') + g_i(D))\big) \leq^{(*_3)}$$

$$\leq exp\big(\frac{\epsilon^2}{2(\Delta^{(g)})^2} \cdot \Delta^{(g)} \cdot \sum_{i=1}^{126}(g_i(D') + g_i(D))\big) = exp\big(\frac{\epsilon^2}{2\Delta^{(g)}} \cdot \sum_{i=1}^{126}(g_i(D') + g_i(D))\big)$$

The value of ϵ used to be greater than 1. Therefore, $exp\big(\frac{\epsilon^2}{2\Delta^{(g)}} \cdot \sum_{i=1}^{126}(g_i(D') + g_i(D))\big) > e^{\epsilon} \Rightarrow$ Gaussian mechanism is not secure.

$(*_1)$ As the elements of the random vectors are independent and identically distributed, the pdf is the product of the individual pdf's.

$(*_2) \frac{x^{-i}}{x^{-j}} = x^{j-i}$

$(*_3)$ The greatest result that we can achive is with the combination $r_i < g_i(D) < g_i(D')$

Q7

```
!pip install faker
import warnings
warnings.filterwarnings('ignore')


import pandas as pd
from faker import Faker
import random

def createDatabase(n:int = 1000):
  data = {'Name': [], 'Age': [], 'SB1': [], 'SB2': []}
  fake = Faker()
  for _ in range(n):
    name = fake.name()
    age = random.randint(0, 125)
    sb1 = random.choice([0,1])
    sb2 = random.choice([0,1])

    data['Name'].append(name)
    data['Age'].append(age)
    data['SB1'].append(sb1)
    data['SB2'].append(sb2)

  return pd.DataFrame(data)

createDatabase(n=10)
```

	Name	Age	SB1	SB2
0	Chelsea Lamb PhD	120	1	0
1	Kimberly Sampson	105	0	1
2	Deborah Park	123	1	1
3	Gail Schultz	0	0	0
4	Bradley Manning	57	0	1
5	Lori Chambers	61	1	1
6	Tara Sharp	26	0	0
7	Jose Jones	23	0	0
8	Joshua Gay	89	1	1
9	Phillip Wallace	111	1	0

Q8

```
import numpy as np

def Q1_curated(database, epsilon):

    #Compute average
    total_sum = sum(database['Age'])
    n = len(database)
    average = total_sum / n
```

```
    #Add Laplace noise
    noise = np.random.laplace(scale=(125/n) / epsilon)

    return average, int(np.round(average + noise))

def Q2_curated(database, epsilon):

    #Compute median
    sorted_db = sorted(database['Age'])
    n = len(sorted_db)
    median = sorted_db[n // 2]

    noise = np.random.laplace(scale=(125) / epsilon)

    return median, int(np.round(median + noise))

# Function to compute the ε-Differentially Private Bitcount
def Q3_curated(database, epsilon):

    bitcount = sum(int(SB1) for SB1 in database['SB1']) #Compute bitcount

    noise = np.random.laplace(scale=2/epsilon)

    return bitcount, int(np.round(bitcount + noise))

# Function to compute the ε-Differentially Private Histogram
def Q4_curated(database, epsilon):
    age_histogram = [0] * 126
    for i in range(0, 126):
      age_histogram[i] = len(database[database['Age']==i])

    #Add Laplace noise to each factor in the histogram
    noisy_histogram = []
    for n_age in age_histogram:
        noise = np.random.laplace(scale=2/epsilon)
        noisy_histogram.append(int(np.round(n_age + noise)))

    return age_histogram, noisy_histogram

# Example usage:
database = createDatabase(n=100)
epsilon = 10

real_average, noisy_average = Q1_curated(database, epsilon)
real_median, noisy_median = Q2_curated(database, epsilon)
real_bitcount, noisy_bitcount = Q3_curated(database, epsilon)
real_histogram, noisy_histogram = Q4_curated(database, epsilon)

print('Average:', real_average)
print('Noisy Average (Q1):', noisy_average)
print()
print('Median:', real_median)
print('Noisy Median (Q2):', noisy_median)
print()
print('Bitcount:', real_bitcount)
print('Noisy Bitcount (Q3):', noisy_bitcount)
print()
print('Histogram:', real_histogram)
print('Noisy Histogram (Q4):', noisy_histogram)
```

```
Average: 59.77
Noisy Average (Q1): 60

Median: 63
Noisy Median (Q2): 64

Bitcount: 64
Noisy Bitcount (Q3): 64

Histogram: [1, 2, 0, 1, 1, 1, 1, 0, 1, 2, 2, 0, 0, 0, 1, 1, 1, 1, 1, 0, 2, 0, 0, 2, 2, 0, 1, 0, 1, 0, 1, 0, 0, 0, 3, 1,
Noisy Histogram (Q4): [1, 1, 0, 1, 1, 1, 1, 0, 1, 2, 2, 0, 0, 0, 1, 1, 1, 1, 1, 0, 2, 0, 0, 2, 2, 0, 1, 0, 1, 0, 1, 0, 0
```

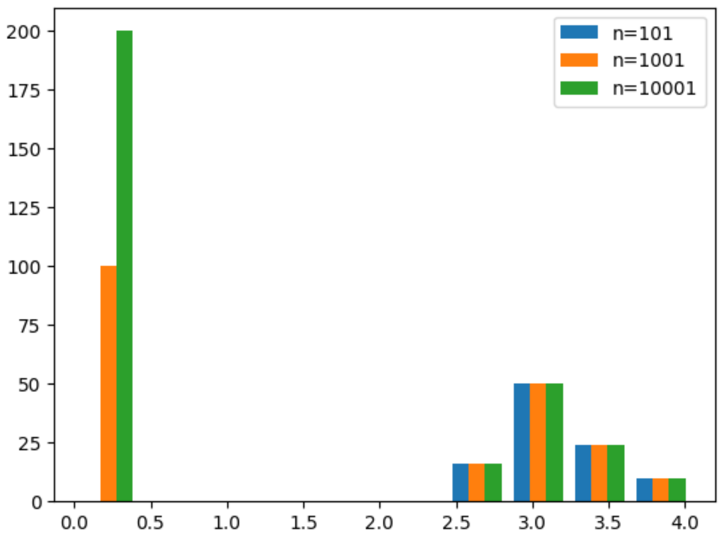
Q9

```
import math
import numpy as np
import matplotlib.pyplot as plt
```

```
n = [101, 1001, 10001]
epsilon = 1

error_Q1 = []; error_Q2 = []; error_Q3 = []; error_Q4 = []
real_Q1 = []; real_Q2 = []; real_Q3 = []; real_Q4 = []
nrmsd_Q1 = []; nrmsd_Q2 = []; nrmsd_Q3 = []; nrmsd_Q4 = []

for n_i in n:
  for m in range(1,101):
    database = createDatabase(n_i)
    real_q1, curated_q1 = Q1_curated(database, epsilon)
    real_q2, curated_q2 = Q2_curated(database, epsilon)
    real_q3, curated_q3 = Q3_curated(database, epsilon)
    real_q4, curated_q4 = Q4_curated(database, epsilon)

    real_Q1.append(real_q1)
    real_Q2.append(real_q2)
    real_Q3.append(real_q3)
    real_Q4.append(real_q4)
    error_Q1.append(curated_q1 - real_q1)
    error_Q2.append(curated_q2 - real_q2)
    error_Q3.append(curated_q3 - real_q3)
    error4 = []
    for i, j in zip(curated_q4, real_q4):
      error4.append(i - j)
    error_Q4.append(error4)

  nrmsd_Q1.append(math.sqrt(sum(np.array(error_Q1)**2)/100)/(sum(np.array(real_Q1))/100))
  nrmsd_Q2.append(math.sqrt(sum(np.array(error_Q2)**2)/100)/(sum(np.array(real_Q2))/100))
  nrmsd_Q3.append(math.sqrt(sum(np.array(error_Q3)**2)/100)/(sum(np.array(real_Q3))/100))
  nrmsd_q4 = []
  for num, i in enumerate(error_Q4):
    nrmsd_q4.append(math.sqrt(sum(np.array(error_Q4[num])**2)/100)/(sum(np.array(real_Q4[num]))/100))
  nrmsd_Q4.append(nrmsd_q4)


print('n\t|'+'\t  101\t\t|\t  1001\t\t|\t  10001\t\t|')
print('NRMSD_Q1| '+str(nrmsd_Q1[0])+'\t| '+str(nrmsd_Q1[1])+'\t| '+str(nrmsd_Q1[2])+'\t|')
print('NRMSD_Q2| '+str(nrmsd_Q2[0])+'\t| '+str(nrmsd_Q2[1])+'\t| '+str(nrmsd_Q2[2])+'\t|')
print('NRMSD_Q3| '+str(nrmsd_Q3[0])+'\t| '+str(nrmsd_Q3[1])+'\t| '+str(nrmsd_Q3[2])+'\t|')
print()
print("NRMSD_Q4")
```

```
n       |          101           |          1001          |          10001         |
NRMSD_Q1| 0.028609320085830545   | 0.014545230258120616   | 0.009826579145881 57   |
NRMSD_Q2| 2.6426679362937313     | 2.3203606268568144     | 1.7291056390561192     |
NRMSD_Q3| 0.04617110383283045    | 0.006933944189355779   | 0.0008412251164748324  |

NRMSD_Q4
```

```
plt.figure()
plt.hist(np.array(nrmsd_Q4))
plt.legend(labels=['n=101', 'n=1001', 'n=10001'])
plt.show()
```



The lacplacian mechanism increase the variance in the result of the queries, although the mean of the result for each querie (Q1, Q2, Q3 and Q4) for a long datasets since the NRMSD also decrease.

Q10

```
import math
import numpy as np
import matplotlib.pyplot as plt


n = [101, 1001, 10001]
epsilon = 1

original_error_Q1 = []
original_real_Q1 = []
original_nrmsd_Q1 = []
error_Q1 = []; error_Q2 = []; error_Q3 = []; error_Q4 = []
real_Q1 = []; real_Q2 = []; real_Q3 = []; real_Q4 = []
nrmsd_Q1 = []; nrmsd_Q2 = []; nrmsd_Q3 = []; nrmsd_Q4 = []

for n_i in n:
  for m in range(1,101):
    database = createDatabase(n_i)
    original_real_q1, original_curated_q1 = Q1_curated(database, epsilon)
    real_q4, curated_q4 = Q4_curated(database, epsilon)

    real_q1 = 0
    curated_q1 = 0
    count = 0
    for i, j in zip(real_q4, curated_q4):
      real_q1 += count*i
      curated_q1 += count*j
      count += 1

    real_q1 /= len(real_q4)
    curated_q1 /= len(curated_q4)

    real_Q1.append(real_q1)
    original_real_Q1.append(original_real_q1)
    error_Q1.append(curated_q1 - real_q1)
    original_error_Q1.append(original_curated_q1 - original_real_q1)

  nrmsd_Q1.append(math.sqrt(sum(np.array(error_Q1)**2)/100)/(sum(np.array(real_Q1))/100))
  original_nrmsd_Q1.append(math.sqrt(sum(np.array(original_error_Q1)**2)/100)/(sum(np.array(original_real_Q1))/100))


print('n\t\t |'+'\t  101\t\t|\t  1001\t\t|\t  10001\t\t|')
print('NRMSD_Q1\t | '+str(nrmsd_Q1[0])+'\t| '+str(nrmsd_Q1[1])+'\t| '+str(nrmsd_Q1[2])+'\t|')
print('ORIGINAL_NRMSD_Q1| '+str(original_nrmsd_Q1[0])+'\t| '+str(original_nrmsd_Q1[1])+'\t| '+str(original_nrmsd_Q1[2])+'\t|'
```

```
n                |        101           |         1001          |         10001         |
NRMSD_Q1         | 0.36372706215074446  | 0.045651526881951024  | 0.00543967573176553   |
ORIGINAL_NRMSD_Q1| 0.026037853840080484 | 0.013193828511223351  | 0.008924074551896621  |
```

The error with the original mechanism looks to be smaller for smaller values of n. However, when n increase the original and estimated values are more similar, indeed for $n = 10001$ the error in the stimated case is lower.

For smaller values of n its better to use the original mechanishm but for larger databases it can be used both because there are not significant differences.