17. ON THE NOTION OF MEAN*

All known types of mean, such as the arithmetic, geometric and harmonic means and the root-mean-square are of the form

$$M(x_1, x_2, \dots, x_n) = \psi\left(\frac{\phi(x_1) + \phi(x_2) + \dots + \phi(x_n)}{n}\right), \tag{1}$$

where ϕ is a continuous strictly monotone function and ψ is its inverse function.

It will be shown in this paper that each type of mean is necessarily of the form (1) whenever it satisfies some natural conditions (axioms of the mean).

We assume that the function $M_n(x_1, x_2, \dots, x_n)$ or simply $M(x_1, \dots, x_n)$, is defined for any $n \geq 1$ and all values of $x_1, \dots, x_n$ in the interval $a \leq x \leq b$.

The result established below can also be extended to means defined on the infinite intervals $-\infty < x < \infty$, $a \leq x < \infty$, $-\infty < x \leq b$ (for example, the arithmetic and geometric means). In such a case the proof needs only a minor modification.

A sequence of functions M_n determines a *regular* type of mean if the following conditions hold:

i) $M(x_1, \dots, x_n)$ is continuous and monotone in each variable. For the sake of being specific, we will assume that M increases in each variable.

ii) $M(x_1, \dots, x_n)$ is a symmetric function.

iii) The mean of identical numbers is equal to their common value: $M(x, \dots, x) = x$.

iv) A subset of values can be replaced by their mean with no effect on the total mean:

$$M(x_1, \dots, x_m, y_1, \dots, y_n) = M_{n+m}(x, \dots, x, y_1, \dots, y_n),$$

where $x = M(x_1, \dots, x_n)$.

Theorem. *If conditions* i) *to* iv) *hold, then the mean* $M(x_1, \dots, x_n)$ *has the form* (1) *where* ϕ *is a continuous increasing function and* ψ *is its inverse function.*