A Utility Proportional Fairness Approach for Resource Allocation in 4G-LTE

Ahmed Abdel-Hadi and Charles Clancy
Bradley Department of Electrical and Computer Engineering
Virginia Tech, Arlington, VA, 22203, USA
{aabdelhadi, tcc}@vt.edu

***Abstract*—In this paper, we introduce an approach for resource allocation of elastic and inelastic adaptive real-time traffic in fourth generation long term evolution (4G-LTE) system. In our model, we use logarithmic and sigmoidal-like utility functions to represent the users applications running on different user equipments (UE)s. We present a resource allocation optimization problem with utility proportional fairness policy, where the fairness among users is in utility percentage (i.e user satisfaction with the service) of the corresponding applications. Our objective is to allocate the resources to the users with priority given to the adaptive real-time application users. In addition, a minimum resource allocation for users with elastic and inelastic traffic should be guaranteed. Our goal is that every user subscribing for the mobile service should have a minimum quality-of-service (QoS) with a priority criterion. We prove that our resource allocation optimization problem is convex and therefore the optimal solution is tractable. We present a distributed algorithm to allocate evolved NodeB (eNodeB) resources optimally with a priority criterion. Finally, we present simulation results for the performance of our rate allocation algorithm.**

***Index Terms*—Resource Allocation, Inelastic Traffic, Convex Optimization**

I. INTRODUCTION

The area of resource allocation optimization has received significant interest since the seminal network utility maximization problem presented in [1]. The network utility maximization problem allocates the resources among users based on bandwidth proportional fairness and using Lagrange multiplier methods of optimization theory. An iterative algorithm based on the dual problem has been proposed to solve the resource allocation optimization problem in [2]. The utility functions used in early research work, as in [1] and [2], are logarithmic utility functions that are good approximations of the elastic Internet traffic for wired communication networks. Therefore, all the utility functions are strictly concave functions and the algorithms proposed converge to the optimal solution.

In recent years, there has been an increasing demand for wireless adaptive real-time applications. The utility functions that approximate real-time applications are non-concave functions. Applications with utility functions that are not strictly concave are presented in [3]. For example, voice-over-IP (VoIP) can be approximated as a step function where the utility percentage is zero below a certain rate threshold and is 100% above that threshold. While rate-adaptive applications, e.g. video streaming, have utility functions that can be approximated as a sigmoidal-like function according to [3]. The sigmoidal-like function is a convex function for rates below the curve inflection point and is a concave function for rates above that inflection point.

In this paper, we focus on finding the optimal solution for the resource allocation problem that includes users with non-concave utility functions (i.e. sigmoidal-like functions) and users with strictly concave utility functions (i.e. logarithmic utility functions). The optimization problem is formulated to ensure fair utility percentage with the available eNodeB resources allocated for all users. Therefore, our rate allocation algorithm gives priority to real-time application users who have non-concave utility functions approximated by sigmoidal-like functions with different parameters for different real-time applications. In addition, the optimization problem formulation guarantees that all users are assigned a fraction of the bandwidth, as the eNodeB should provide a minimum QoS for all the users subscribing for the mobile service.

A. Related Work

In [4], the authors presented a distributed power allocation algorithm for a mobile cellular system. They used non-concave sigmoidal-like utility functions. The proposed algorithm approximates the global optimal solution but could drop users to maximize the overall system utilities, therefore, it does not guarantee minimum QoS for all users.

In [5], the author presented a weighted aggregation of elastic and inelastic utility functions in each UE. These aggregated utility functions are then approximated to the nearest concave utility function from a set of functions using minimum mean-square error. That approximate utility function is used to solve the rate allocation problem using a modified version of the distributed rate allocation algorithm presented in [1].

In [6] and [7], the authors presented a non-convex optimization formulation for maximization of utility functions in wireless networks. They used both elastic and sigmoidal-like utility functions and proposed a distributed algorithm to solve it when the duality gap is zero. But the algorithm doesn't converge to the optimal solution for a positive duality gap. A fair allocation heuristic is included to ensure network stability which resulted in a high aggregated utility.

In [8], the authors proposed a utility max-min fairness resource allocation for users with elastic and real-time traffic sharing a single path in the network. In [9], the authors proposed a utility proportional fair optimization formulation for high-SINR wireless networks using a utility max-min architecture. They compared their algorithm to the traditional bandwidth proportional fair algorithms [10] and presented a closed form solution that prevents oscillations in the network.

B. Our Contributions

Our contributions in this paper are summarized as:

- We introduce a utility proportional fairness optimization problem that solves for utility functions that are both strictly concave and non-concave (i.e. sigmoidal-like). In addition, the optimization problem gives priority to real-time application users (i.e. with sigmoidal-like utility functions) while allocating resources.
- We prove that the proposed optimization problem is convex and therefore the global optimal solution is tractable. We present a distributed rate allocation algorithm to solve it.

The remainder of this paper is organized as follows. Section II presents the problem formulation. Section III proves the global optimal solution exists and is tractable. In Section V, we present our distributed rate allocation algorithm for the utility proportional fairness optimization problem. Section VI discusses simulation setup and provides quantitative results along with discussion. Section VII concludes the paper.

II. Problem Formulation

Without loss of generality, we consider a single cell 4G-LTE mobile system consisting of a single eNodeB and M UEs. The bandwidth allocated by the eNodeB to i^{th} UE is given by r_i. Each UE has its own utility function $U_i(r_i)$ that corresponds to the type of traffic being handled by the UE. Our objective is to determine the bandwidth the eNodeB should allocate to the UEs. We assume the utility functions $U_i(r_i)$ to be strictly concave or a sigmoidal-like functions. The utility functions have the following properties:

- $U_i(0) = 0$ and $U_i(r_i)$ is an increasing function of r_i.
- $U_i(r_i)$ is twice continuously differentiable in r_i.

In our model, we use the normalized sigmoidal-like utility function, as in [4], that can be expressed as

$$U_i(r_i) = c_i\Big(\frac{1}{1+e^{-a_i(r_i-b_i)}} - d_i\Big) \tag{1}$$

where $c_i = \frac{1+e^{a_i b_i}}{e^{a_i b_i}}$ and $d_i = \frac{1}{1+e^{a_i b_i}}$. So, it satisfies $U(0) = 0$ and $U(\infty) = 1$. In Figure 1, the normalized sigmoidal-like utility function with $a = 5$ and $b = 10$ is a good approximation for a step function (e.g. VoIP), and $a = 0.5$ and $b = 20$ is a good approximation to an adaptive real-time application (e.g. video streaming). In addition, we use the normalized logarithmic utility function, as in [9], that can be expressed as

$$U_i(r_i) = \frac{\log(1+k_i r_i)}{\log(1+k_i r_{max})} \tag{2}$$

where r_{max} is the required rate for the user to achieve 100% utility percentage and k_i is the rate of increase of utility percentage with the allocated rate r_i. So, it satisfies $U(0) = 0$ and $U(r_{max}) = 1$. The logarithmic utility functions with

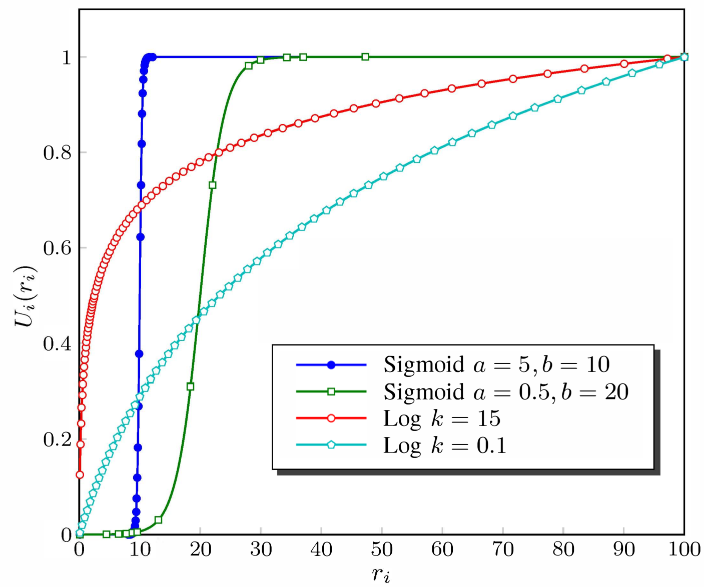

Fig. 1. The sigmoidal-like utility functions (representing real-time traffic) and logarithmic utility functions (representing delay-tolerant traffic) $U_i(r_i)$.

$k = 15$ and $k = 0.1$ are shown in Figure 1. We consider the utility proportional fairness objective function given by

$$\max_{\mathbf{r}} \prod_{i=1}^{M} U_i(r_i) \tag{3}$$

where $\mathbf{r} = \{r_1, r_2, ..., r_M\}$ and M is the number of UEs in the coverage area of the eNodeB. The goal of this resource allocation objective function is to allocate the resources for each UE that maximize the total mobile system objective (i.e. the product of the utilities of all the UEs) while ensuring proportional fairness between individual utilities. This resource allocation objective function ensures non-zero resource allocation for all users. Therefore, the corresponding resource allocation optimization problem guarantees minimum QoS for all users. In addition, this approach allocates more resources to users with real-time applications providing improvement to the QoS of 4G-LTE system.

The basic formulation of the utility proportional fairness resource allocation problem is given by the following optimization problem:

$$\begin{aligned} \max_{\mathbf{r}} \quad & \prod_{i=1}^{M} U_i(r_i) \\ \text{subject to} \quad & \sum_{i=1}^{M} r_i \leq R \\ & r_i \geq 0, \quad i = 1, 2, ..., M. \end{aligned} \tag{4}$$

where R is the total rate of the eNodeB covering the M UEs, and $\mathbf{r} = \{r_1, r_2, ..., r_M\}$.

We prove in Section III that there exists a tractable global optimal solution to the optimization problem (4).

III. The Global Optimal Solution

In the optimization problem (4), since the objective function $\arg\max_{\mathbf{r}} \prod_{i=1}^{M} U_i(r_i)$ is equivalent to

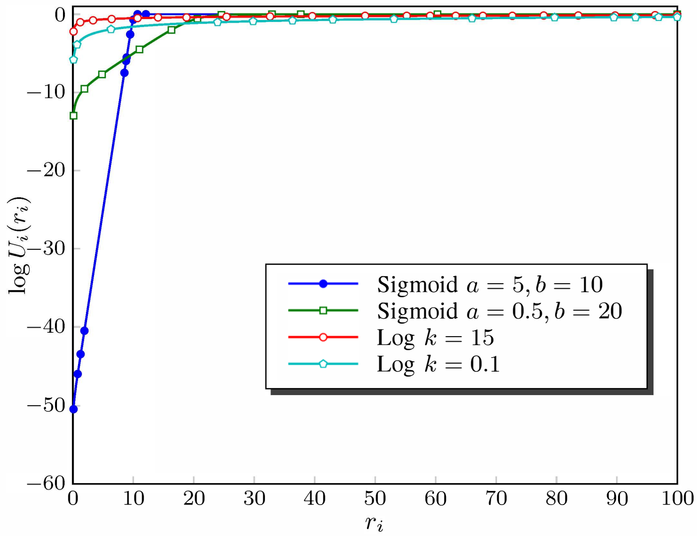

Fig. 2. The natural logarithm of sigmoidal-like and logarithmic utility functions $\log U_i(r_i)$.

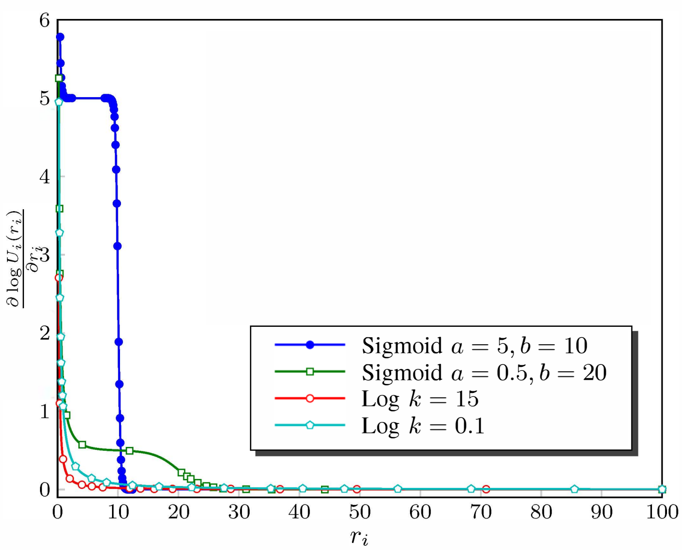

Fig. 3. The first derivative of the natural logarithm of sigmoidal-like and logarithmic utility functions $\frac{\partial \log U_i(r_i)}{\partial r_i}$.

$\arg\max_{\mathbf{r}} \sum_{i=1}^{M} \log(U_i(r_i))$, then optimization problem (4) can be written as:

$$
\begin{aligned}
\max_{\mathbf{r}} \quad & \sum_{i=1}^{M} \log(U_i(r_i)) \\
\text{subject to} \quad & \sum_{i=1}^{M} r_i \leq R \\
& r_i \geq 0, \quad i = 1, 2, ..., M.
\end{aligned} \tag{5}
$$

Lemma III.1. *The utility functions* $\log(U_i(r_i))$ *in the optimization problem (5) are strictly concave functions.*

Proof: In Section II, we assume that all the utility functions of the UEs are strictly concave or sigmoidal-like functions.

In the strictly concave utility function case, recall the utility function properties in Section II, the utility function is positive $U_i(r_i) > 0$, increasing and twice differentiable with respect to r_i. Then, it follows that $U_i'(r_i) = \frac{dU_i(r_i)}{dr_i} > 0$ and $U_i''(r_i) = \frac{d^2U_i(r_i)}{dr_i^2} < 0$. It follows that, the utility function $\log(U_i(r_i))$ in the optimization problem (5) has $\frac{d\log(U_i(r_i))}{dr_i} = \frac{U_i'(r_i)}{U_i(r_i)} > 0$ and $\frac{d^2\log(U_i(r_i))}{dr_i^2} = \frac{U_i''(r_i)U_i(r_i) - U_i'^2(r_i)}{U_i^2(r_i)} < 0$. Therefore, the strictly concave utility function $U_i(r_i)$ natural logarithm $\log(U_i(r_i))$ is also strictly concave. It follows that the natural logarithm of the logarithmic utility function in equation (2) is strictly concave.

In the sigmoidal-like utility function case, the utility function of the normalized sigmoidal-like function is given by equation (1) as $U_i(r_i) = c\Big(\frac{1}{1+e^{-a_i(r_i-b_i)}} - d\Big)$. For $0 < r_i < R$, we have

$$
\begin{aligned}
& 0 < c_i\Big(\frac{1}{1+e^{-a_i(r_i-b_i)}} - d_i\Big) < 1 \\
& d_i < \frac{1}{1+e^{-a_i(r_i-b_i)}} < \frac{1+c_id_i}{c_i} \\
& \frac{1}{d_i} > 1+e^{-a_i(r_i-b_i)} > \frac{c_i}{1+c_id_i} \\
& 0 < 1-d_i(1+e^{-a_i(r_i-b_i)}) < \frac{1}{1+c_id_i}
\end{aligned}
$$

It follows that for $0 < r_i < R$, we have the first and second derivative as

$$
\begin{aligned}
\frac{d}{dr_i}\log U_i(r_i) =& \frac{a_id_ie^{-a_i(r_i-b_i)}}{1-d_i(1+e^{-a_i(r_i-b_i)})} \\
&+ \frac{a_ie^{-a_i(r_i-b_i)}}{(1+e^{-a_i(r_i-b_i)})} > 0 \\
\frac{d^2}{dr_i^2}\log U_i(r_i) =& \frac{-a_i^2d_ie^{-a_i(r_i-b_i)}}{c_i\Big(1-d_i(1+e^{-a(r_i-b_i)})\Big)^2} \\
&+ \frac{-a_i^2e^{-a_i(r_i-b_i)}}{(1+e^{-a_i(r_i-b_i)})^2} < 0
\end{aligned}
$$

Therefore, the sigmoidal-like utility function $U_i(r_i)$ natural logarithm $\log(U_i(r_i))$ is strictly concave function. Therefore, all the utility functions in our system model have strictly concave natural logarithms. ■

The natural logarithms of the utility functions of Figure 1 are shown in Figure 2 and the derivatives of natural logarithms of the utility functions are shown in Figure 3.

Theorem III.2. *The optimization problem (4) is a convex optimization problem and there exists a unique tractable global optimal solution.*

Proof: It follows from Lemma III.1 that for all UEs utility functions are strictly concave. Therefore, the optimization problem (5) is a convex optimization problem [11]. The optimization problem (5) is equivalent to optimization problem (4), therefore it is also a convex optimization problem. For a convex optimization problem there exists a unique tractable global optimal solution [11]. ■

IV. The Dual Problem

The key to a distributed and decentralized optimal solution of the primal problem in (5) is to convert it to the dual problem, similar to [1] and [2]. The optimization problem (5) can be divided into two simpler problems by using the dual problem. We define the Lagrangian

$$\begin{aligned} L(\mathbf{r}, p) &= \sum_{i=1}^{M} \log(U_i(r_i)) - p(\sum_{i=1}^{M} r_i + z - R) \\ &= \sum_{i=1}^{M} \Big(\log(U_i(r_i)) - pr_i\Big) + p(R - z) \\ &= \sum_{i=1}^{M} L_i(r_i, p) + p(R - z) \end{aligned} \tag{6}$$

where $z \geq 0$ is the slack variable and p is Lagrange multiplier or the shadow price (i.e. the total price per unit bandwidth for all the M channels). Therefore, the i^{th} UE bid for bandwidth can be given by $w_i = pr_i$ and we have $\sum_{i=1}^{M} w_i = p\sum_{i=1}^{M} r_i$. The first term in equation (6) is separable in r_i. So we have $\max_{\mathbf{r}} \sum_{i=1}^{M}(\log(U_i(r_i)) - pr_i) = \sum_{i=1}^{M} \max_{r_i}(\log(U_i(r_i)) - pr_i)$. The dual problem objective function can be written as

$$\begin{aligned} D(p) &= \max_{\mathbf{r}} L(\mathbf{r}, p) \\ &= \sum_{i=1}^{M} \max_{r_i}\Big(\log(U_i(r_i)) - pr_i\Big) + p(R - z) \\ &= \sum_{i=1}^{M} \max_{r_i}(L_i(r_i, p)) + p(R - z) \end{aligned} \tag{7}$$

The dual problem is given by

$$\begin{aligned} &\min_{p} && D(p) \\ &\text{subject to} && p \geq 0. \end{aligned} \tag{8}$$

So we have

$$\frac{\partial D(p)}{\partial p} = R - \sum_{i=1}^{M} r_i - z = 0 \tag{9}$$

substituting by $\sum_{i=1}^{M} w_i = p\sum_{i=1}^{M} r_i$ we have

$$p = \frac{\sum_{i=1}^{M} w_i}{R - z} \tag{10}$$

Now, we divide the primal problem (5) into two simpler optimization problems in the UEs and the eNodeB. The i^{th} UE optimization problem is given by:

$$\begin{aligned} &\max_{r_i} && \log U_i(r_i) - pr_i \\ &\text{subject to} && p \geq 0 \\ & && r_i \geq 0, \quad i = 1, 2, ..., M. \end{aligned} \tag{11}$$

The eNodeB optimization problem is given by:

$$\begin{aligned} &\min_{p} && D(p) \\ &\text{subject to} && p \geq 0. \end{aligned} \tag{12}$$

The minimization of shadow price p is achieved by the minimization of the slack variable $z \geq 0$ from equation (10). Therefore, the maximum utility percentage for the available eNodeB bandwidth is achieved by setting the slack variable $z = 0$. In this case, we replace the inequality in primal problem (5) constraint by equality constraint and so we have $\sum_{i=1}^{M} w_i = pR$. Therefore, we have $p = \frac{\sum_{i=1}^{M} w_i}{R}$ where $w_i = pr_i$ is transmitted by the i^{th} UE to the eNodeB. The utility proportional fairness in the objective function of the optimization problem (4) is guaranteed in the solution of the optimization problems (11) and (12).

V. Distributed Optimization Algorithm

The distributed resource allocation algorithm for optimization problems (11) and (12) is a modified version of the distributed algorithms in [1] and [2], which is an iterative solution for allocating the network resources with bandwidth proportional fairness. Our algorithm allocates resources with utility proportional fairness, which is the objective of our problem formulation. The algorithm is divided into an UE algorithm shown in Algorithm (1) and an eNodeB algorithm shown in Algorithm (2). For the Algorithm in (1) and (2), each UE starts with an initial bid $w_i(1)$ which is transmitted to the eNodeB. The eNodeB calculates the difference between the received bid $w_i(n)$ and the previously received bid $w_i(n-1)$ and exits if it is less than a pre-specified threshold δ. We set $w_i(0) = 0$. If the value is greater than the threshold δ, eNodeB calculates the shadow price $p(n) = \frac{\sum_{i=1}^{M} w_i(n)}{R}$ and sends that value to all the UEs. Each UE receives the shadow price to solve for the rate r_i that maximizes $\log U_i(r_i) - p(n)r_i$. That rate is used to calculate the new bid $w_i(n) = p(n)r_i(n)$. Each UE sends the value of its new bid $w_i(n)$ to the eNodeB. This process is repeated until $|w_i(n) - w_i(n-1)|$ is less than the pre-specified threshold δ.

Algorithm 1 UE Algorithm

```
Send initial bid w_i(1) to eNodeB
loop
  Receive shadow price p(n) from eNodeB
  if STOP from eNodeB then
    Calculate allocated rate r_i^opt = w_i(n)/p(n)
    STOP
  else
    Solve r_i(n) = arg max_{r_i} ( log U_i(r_i) − p(n) r_i )
    Send new bid w_i(n) = p(n) r_i(n) to eNodeB
  end if
end loop
```

The solution r_i of the optimization problem $r_i(n) = \arg\max_{r_i}\Big(\log U_i(r_i) - p(n)r_i\Big)$ in Algorithm (1), is the value of r_i that solves equation $\frac{\partial \log U_i(r_i)}{\partial r_i} = p(n)$. It is the intersection of the horizontal line $y = p(n)$ with the curve $y = \frac{\partial \log U_i(r_i)}{\partial r_i}$, in Figure 3, which is calculated in the i^{th} UE.

Algorithm 2 eNodeB Algorithm

loop
  Receive bids $w_i(n)$ from UEs {Let $w_i(0) = 0\ \forall i$}
  if $|w_i(n) - w_i(n-1)| < \delta\ \forall i$ **then**
    Allocate rates, $r_i^{\text{opt}} = \frac{w_i(n)}{p(n)}$ to user i
    STOP
  else
    Calculate $p(n) = \frac{\sum_{i=1}^{M} w_i(n)}{R}$
    Send new shadow price $p(n)$ to all UEs
  end if
end loop

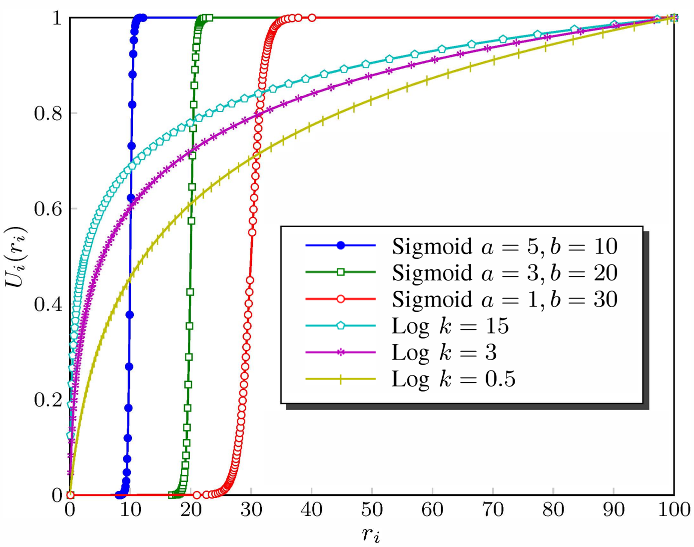

Fig. 4. The users utility functions $U_i(r_i)$ used in the simulation (three sigmoidal-like functions and three logarithmic functions).

VI. Simulation Results

The algorithm in (1) and (2) was applied to various logarithmic and sigmoidal-like utility functions with different parameters in MATLAB. The simulation results showed convergence to the global optimal solution. In this section, we present the simulation results of six utility functions corresponding to six UEs shown in Figure 4. We use three normalized sigmoidal-like function that are expressed by equation (1) with different parameters, $a = 5$, $b = 10$ which is an approximation to a step function at rate $r = 10$ (e.g. VoIP), $a = 3$, $b = 20$ which is an approximation of an adaptive real-time application with inflection point at rate $r = 20$ (e.g. standard definition video streaming), and $a = 1$, $b = 30$ which is also an approximation of an adaptive real-time application with inflection point at rate $r = 30$ (e.g. high definition video streaming). We use three logarithmic functions that are expressed by equation (2) with r_{max} =100 and different k_i parameters which are approximations for delay tolerant applications (e.g. FTP). We use $k = \{15, 3, 0.5\}$.

A. Algorithm convergence for R =100

In the following simulations, we set $R = 100$ and the number of iterations $n = 20$. In Figure 5, we show the rates of different

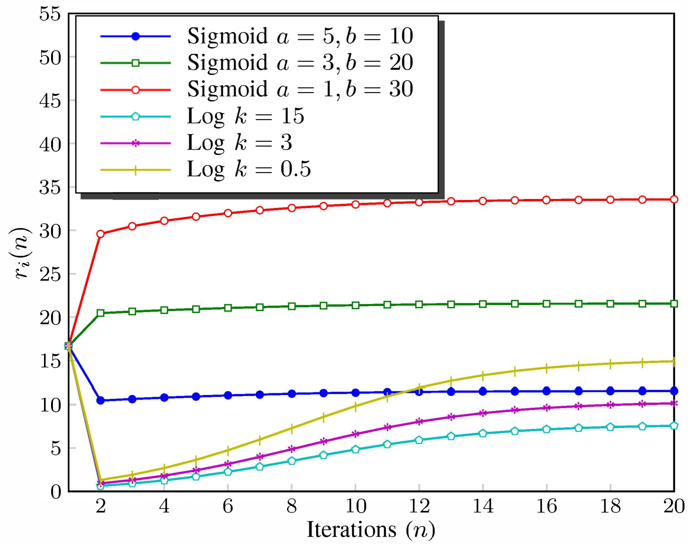

Fig. 5. The users allocated rates convergence $r_i(n)$ with number of iterations n for eNodeB rate $R = 100$.

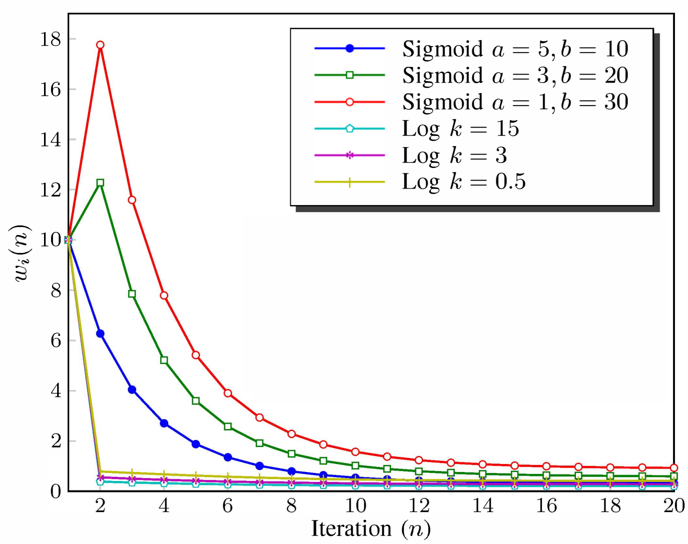

Fig. 6. The users bids convergence $w_i(n)$ with number of iterations n for eNodeB rate $R = 100$.

users with the number of iterations. The sigmoidal-like utility functions have priority over the logarithmic utility functions in the rate allocation process. The steady state rates of all the sigmoidal-like functions exceed the corresponding inflection point b_i. In Figure 6, we show the bids of different users with the number of iterations. The higher the user bids the higher the rate allocated. The users with adaptive real-time applications bid higher until they reach their inflection points then the elastic traffic can divide the remaining resources according to their utility parameters. In Figure 7, we show the shadow price with the number of iterations.

B. For $60 \le R \le 100$

In the following simulations, we set $\delta = 10^{-3}$ and the eNodeB rate R takes values between 60 and 100 with step of 5. In Figure 8, we show the steady state rates of different users with different eNodeB rate R. Our distributed algorithm

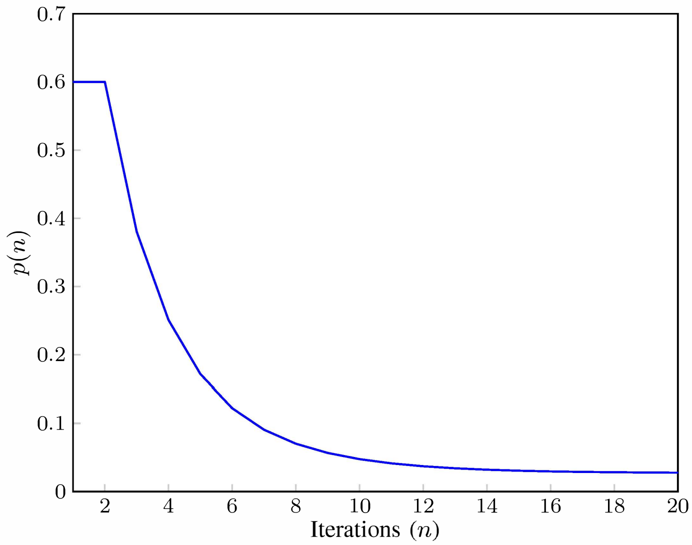

Fig. 7. The shadow price $p(n)$ convergence with the number of iterations n.

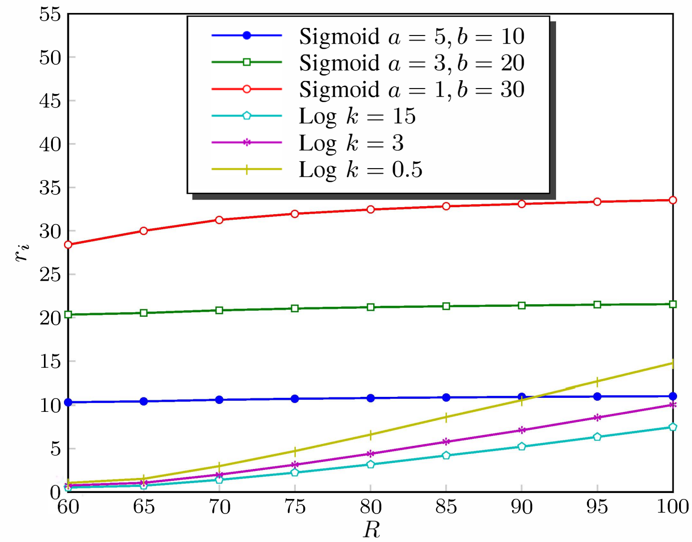

Fig. 8. The allocated rates r_i with eNodeB rate $60 \leq R \leq 100$ and a pre-specified threshold $\delta = 10^{-3}$.

is set to avoid the situation of allocating zero rate to any user (i.e. no user is dropped). However, the eNodeB allocates the majority of the resources to the UEs running adaptive real-time applications until they reach their corresponding inflection rates $r_i = b_i$. When the eNodeB rate R exceed the sum of the inflection rates $\sum b_i$ of all the adaptive real-time applications, the eNodeB allocates more resources to the UEs with elastic application, as shown in Figure 8, for $R \geq 65$. In Figure 9, we show the steady state bids of different users with different eNodeB rate R. The higher the user bid the higher the allocated rate. The real-time application users bid higher when the eNodeB resources are scare and their bids decrease as R increases.

C. For $\delta = \{10^{-2}, 10^{-3}\}$

In Figure 10, we show the number of iterations n for different thresholds $\delta = \{10^{-2}, 10^{-3}\}$ with the eNodeB rate R. The number of iterations increase with the decrease in the threshold value δ and therefore the algorithm provides a more accurate solution to the optimization problem. For $R = 100$, in Figure 10, we have number of iterations $n = 16$ for $\delta = 10^{-3}$ and therefore the allocated rates, from Figure 5, are $\mathbf{r}(16) = \{11.5, 21.5, 33.5, 7.2, 9.6, 14.2\}$ and $n = 8$ for $\delta = 10^{-2}$ and therefore the allocated rates are $\mathbf{r}(8) = \{11.2, 21.2, 32.6, 3.5, 4.8, 7.2\}$. The increase in the number of iterations n leads to a decrease in the error in the allocated rates r_i but on the other hand leads to an increase in the allocation process time. So, we have a trade-off between the rate allocation delay and accuracy.

VII. Conclusion

In this paper, we introduced a utility proportional fairness optimization problem for UEs with delay-tolerant and real-time applications in 4G-LTE. We proved that the global optimal solution exists and is tractable for the resource allocation optimization problem for UEs with logarithmic (delay-tolerant) and sigmoidal-like (real-time) utility functions. We presented a distributed algorithm for allocating the eNodeB resources optimally to the UEs. Our algorithm ensures fairness in the utility percentage achieved by the allocated resources for all the users. Therefore, the algorithm gives priority to the users with adaptive real-time applications. In addition, a minimum resource allocation for users with elastic or inelastic traffic is guaranteed to satisfy a minimum QoS for all service subscribers. We showed through simulations that our algorithm converges to the optimal rates and allocates the eNodeB resources with priority to users running real-time applications.

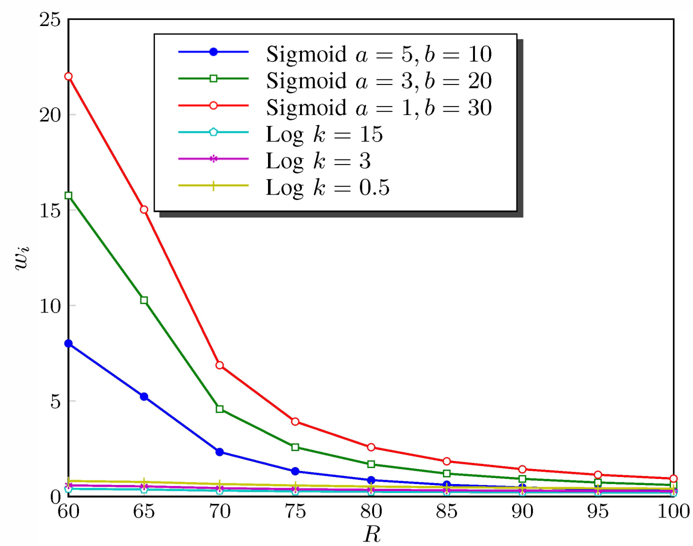

Fig. 9. The final users bids w_i with eNodeB rate $60 \leq R \leq 100$ and a pre-specified threshold $\delta = 10^{-3}$.

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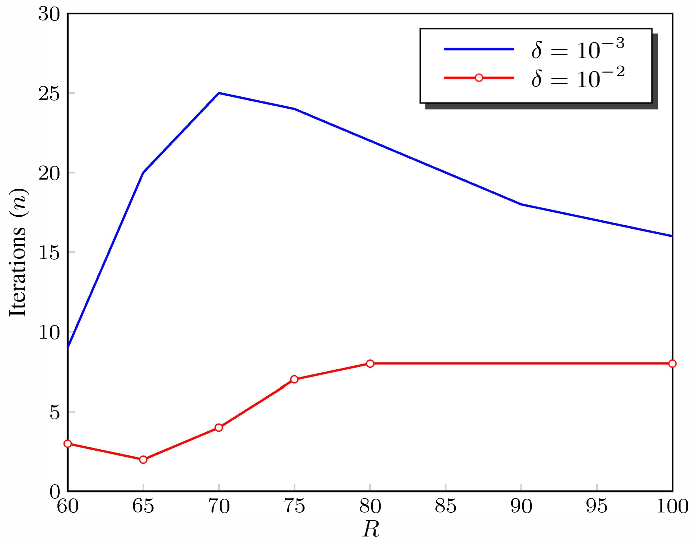

Fig. 10. Number of iterations n with eNodeB rate $60 \leq R \leq 100$ for pre-specified thresholds $\delta = 10^{-2}$ and $\delta = 10^{-3}$

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