Online Appendix

The Misbehavior of Simple Bid-Ask Spread Estimators

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A Additional Performance Evaluation in US Stocks

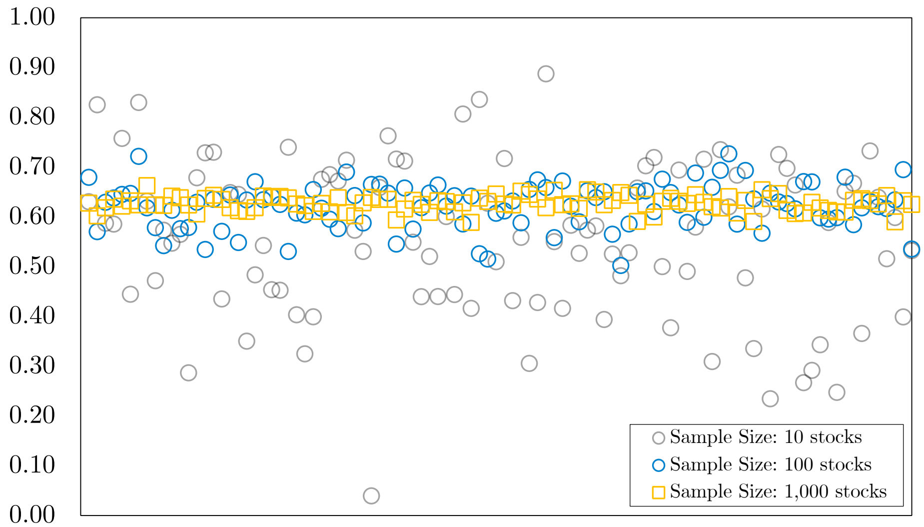
This section builds on Section 3 in the main paper. We conduct additional tests designed to assess the robustness and stability of the performance of the high-low spread estimator in US stocks. This sample is the standard setting used in horse-race type studies that evaluate the performance of multiple bid-ask spread estimators and recommend out-of-sample use — i.e., in other settings — based on in-sample performance.

Cross-section size and performance stability. We begin with a second factor that accounts for the performance of high-low spread estimates benchmarked in stock data. Our goal is to consider how the estimator's performance varies were a practitioner to observe random samples of an underlying asset class - i.e., the cross-section - with different sizes. In particular, we are interested in draws that are "small" compared to the full sample used in performance evaluation studies, which includes thousands of units (stocks). In most applications, a cross-section available to researchers includes no more than few dozen assets, as with US commodity or treasury futures. Thus, the stability of correlation estimates in samples of different sizes is a desired feature, since inference over the performance of spread estimators in the evaluation sample implies out-of-sample extrapolation.

To do so, Figure (A.1) repeats the following experiment under three different settings. We draw 100 random samples of 10 stocks from the full DTAQ dataset, calculating the cross-section correlation between monthly HL estimates and effective spreads in each sample. We repeat the process for another 100 random draws with 100 and 1,000 stocks each time. The variability in performance in samples of 10 stocks is remarkable: the high-low estimator generates both exceptional cross-sectional correlations upwards of 80% and no correlation in other cases. Larger sample draws lead to more stable average cross-sectional correlation estimates, quickly converging to the full sample correlation of about 62%.

Liquidity heterogeneity in the sample. A robust finding for *all* moment-based bid-ask spread estimators that rely on a combination of closing, high, and low prices is a much poorer performance in the time-series compared to the cross-section. The larger the number of assets relative to the number of periods (usually months), the wider this gap. Although we are not interested in the reasons for this difference in performance, we exploit the fact that spread size heterogeneity is much higher between units than within to demonstrate the importance of heterogeneity in spread levels in the application context. This also purges the analysis from potential time-invariant unobserved factors within each stock.

First, from Figure (2) in the main paper, we learn that the high-low estimator performs better when effective spreads are larger. From Figure (A.1), we also know that larger cross-sections have higher and more stable correlations, suggesting that the proxy requires large variations in the spread *level* to be more accurately estimated. Mechanically, larger cross-sections tend to have more variation in spread levels



A researcher with a random sample of 10 US stocks would obtain very distinct average cross-sectional correlations between monthly high-low estimates and monthly effective spreads every time a new sample would be drawn. Correlations become stable across randomly chosen samples when the stock data includes more than 100 units.

Notes: This figure shows cross-sectional correlations between the high-low spread estimator (using the zeros adjustment) and the effective spreads from 100 random samples of 10, 100, and 1,000 stocks from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

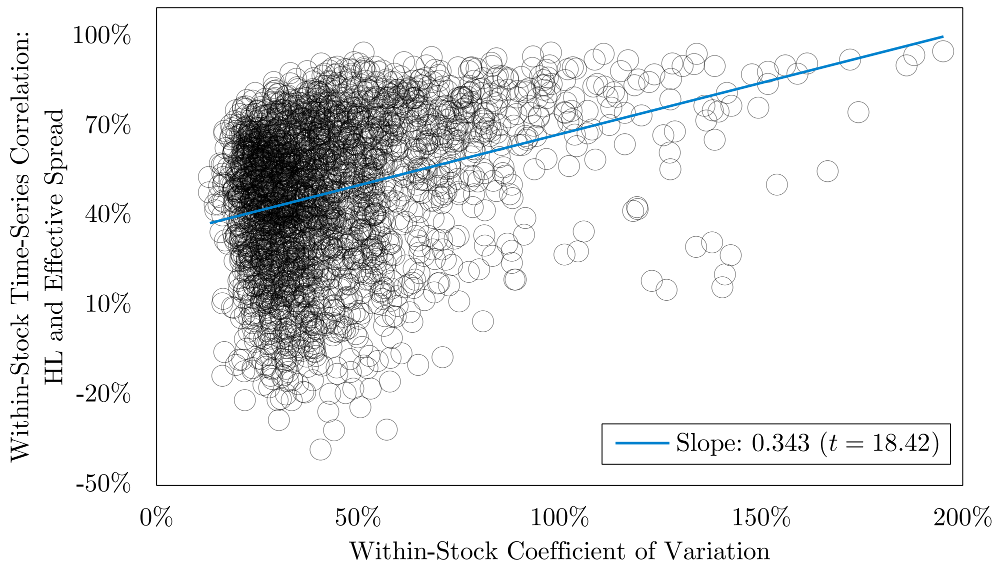
Figure A.1: Cross-Sectional Correlation: Stability Across Different Sample Sizes

than spread levels do when varying over time for the same stock, which might offer one explanation as to why performance is worse in the time series.

We test that below. Figure (A.2) plots within-stock average monthly time-series correlations between the high-low estimator and effective spreads against within-stock coefficients of variation of monthly effective spreads (standard deviation of spreads divided by the sample mean across months). The strong positive estimated slope confirms that the spread measure performs better in stocks with more dispersion in underlying effective spreads. Applying this rationale to the full cross-section generates an important result. Samples of assets with relatively homogeneous trading costs are likely to result in poor accuracy for the high low estimator.

Suggestions for performance validation. The exercises we perform in this section are simple and would greatly increase confidence in application of estimators across different markets and when benchmarking performance of new spread measures. Some suggestions for implementation are the following. When the cross-section is sufficiently large, researchers should conduct a systematic analysis of the performance of the estimator in subsamples with different levels of liquidity.

Most preferably, newly-developed estimators should address how the measure may suffer from systemic bias with respect to the latent spread, and empirically analyze the extent to which such de-



Stocks with greater dispersion in effective spreads are associated with higher time series correlation between monthly high-low estimates and the spread.

Notes: This figure compares within-stock dispersion in the spread (sample standard deviation divided by mean) with time series correlations between the high-low spread estimator (using the zeros adjustment) and effective spreads.There are 3,398 stocks in total from DTAQ US stock data and sample construction details are described in Appendix (I).

Figure A.2: HL Performance in the Time Series and Dispersion of Effective Spread

pendence may exist. They should also carefully report the stability of correlation coefficients in random subsamples from the full cross-section. This exercise confronts the fact that often times researchers in empirical finance may only have partial or sample data available. In such "small sample" applications, proxies with large variability of estimated spreads may yield somewhat spurious performance benchmarks, incorrectly concluding that the measure performs adequately in that context or market.

B Negative Spread Estimates: Determinants & Empirical Use

Why do high-low estimates turn out negative? Negative high-low spread estimates are inconsistent with theory. Like many of other simple spread estimators, negative (or indeterminate) daily computed spreads with the high-low estimator are pervasive. The standard approach is to consider these point estimates as “meaningless”, either replacing them with zeros or discarding them. This implies that in many instances nearly half of the estimating sample is excluded from the analysis and with that, potential relevant information.

Negative spread estimates are not “random” or purely the result of model assumption failures. This is clear in Figure (B.1), which replicates the ideal simulated data we use in the main paper and computes the average frequency of negative high-low estimates by each effective spread bin. For small spreads the frequency of spread estimates that turn out negative is as high as 40%, even without any idiosyncratic source of bias. Based on that, we build on the following simple insight. If the intensity of negative estimates is a result of any aspect in the underlying data, we may use how often negative estimates arise to learn about the latent price process.

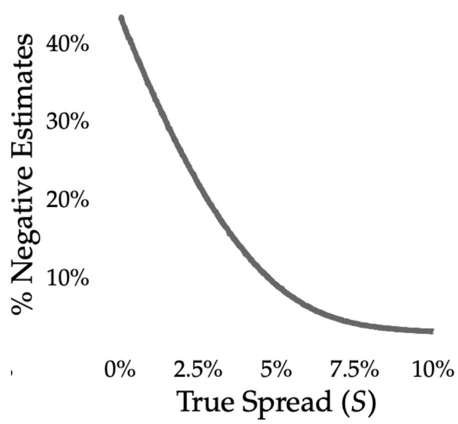


Figure B.1: High-Low Negative Estimates Under Ideal Conditions

To fix ideas, we begin with mechanical relationships that account for generated negative spread estimates, but that offer no insight on mechanisms. We do that next. The high-low estimator depends upon both the first and second-day ranges, r_t and r_{t+1}, and the two-day range r_t^*. Small perturbations at price boundaries have small effects on daily range values, but may alter the domain of the parameter γ and lead to considerable swings in the HL point-estimate. Let $t \in \{1,2\}$. If changes in the second-day high, for example, are sufficiently large so that h_2 becomes higher than the previous day high, the mapping of γ shifts from h_1 to h_2, and the relationship between β and γ is updated. From the expression for α in the main paper or in Appendix (II), it is clear that the high-low estimator is positively defined only when $\beta > \gamma$.

Negative spread estimates do not arise only if the volatility-proportionality assumption in the high-low estimator is violated. Consider the following example. Suppose the first-day range is $r_1 = 0.01$, which is obtained when $(h_1, l_1) = (350, 345)$, and the second-day range is $r_2 = 0.27$, for $(h_2, l_2) = (401, 305)$. Clearly, the second day is much more volatile than the previous day, but the HL estimate is $0.09\% > 0$ over the two days. The simple explanation is that the square of the two-day range cannot be larger than itself added to the square of the first day range: $\gamma = r_2^2 < \beta = r_1^2 + r_2^2$. This happens because r_1 is enclosed in the second-day range. Note that it follows from this reasoning that the HL estimator can only be negative when the parameter γ combines boundaries from both consecutive days (i.e., either $h_1 - l_2$ or $h_2 - l_1$).

However, this is not sufficient for negativity. Consider the example on the left-hand side of Figure (B.2). With a partial overlap between r_t and r_{t+1}, the high-low estimator is not necessarily negative. Particularly, it is straightforward to show that some combination of high and low prices will produce a positive HL estimate when $l_{t+1} \in (l_t, h_t)$ and $h_{t+1} > h_t$. When we decrease the spread S by half (right-hand side panel), the new relative position of the second-day low, $l_{t+1} > h_t$, implies that the HL estimate is negative.

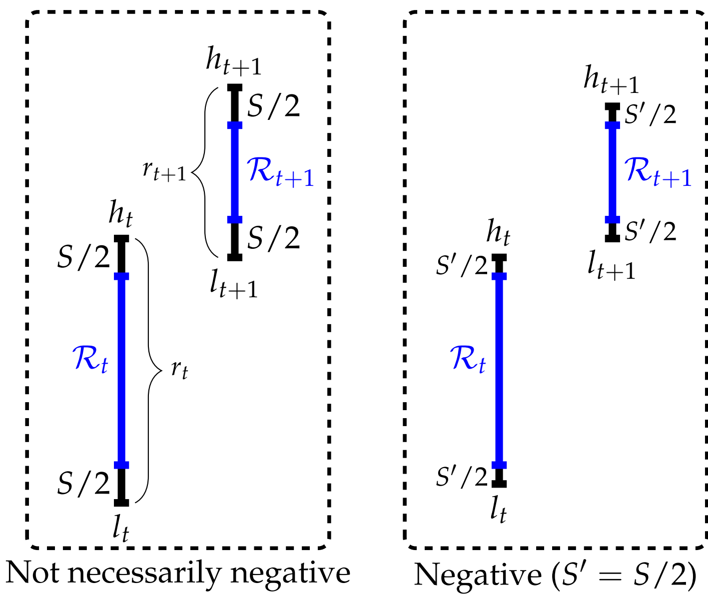


Figure B.2: Negativity Induced by the Spread Size in the HL Estimator

To organize these ideas, we derive two equivalent conditions that determine when the high-low spread is negative. While the first condition in Proposition B.1 is very intuitive and arises directly from the functional form of the implementable high-low measure, the second condition will be useful for us to prove the driving factors of negative estimates, in similar fashion to our treatment of the bias.

PROPOSITION B.1. The high-low spread estimator is negative if and only if the following equivalent conditions hold:

$$\phi r_t^{min} < r_t^* \text{ or } |\eta_{t+1} - \eta_t| > \sqrt{r_t^2 + r_{t+1}^2} - \left(\frac{r_t + r_{t+1}}{2}\right)$$

where η_t is the (log) mid-range, $\eta_t \equiv \dfrac{h_t + l_t}{2}$, and r_t is the log range on day t.

Proof. The range of the high-low spread estimator $S(\alpha)$ only contains negative values when S is evaluated at negative values of α. Hence, negativity of α defines non-positive spread estimates. The parameter α is expressed as $\alpha = (1+\sqrt{2})(\sqrt{\beta} - \sqrt{\gamma})$, which implies that $\gamma > \beta$ determines when $\alpha < 0$. The parameter β always maps each day's range onto $\mathbb{R}_+$. Since γ includes extreme-valued functions, it may take the following values: $\gamma = r_t^2$, $\gamma = r_{t+1}^2$, $\gamma = (h_t - l_{t+1})^2$, or $\gamma = (h_{t+1} - l_t)^2$. In the first two cases, β is always greater than γ. For $\gamma = (h_{t+1} - l_t)^2$, we can write $\gamma > \beta$ as:

$$\underbrace{h_{t+1} - \frac{r_{t+1}}{2}}_{\eta_{t+1}} - \underbrace{\left(\frac{r_t}{2} + l_t\right)}_{\eta_t} + \frac{r_t}{2} + \frac{r_{t+1}}{2} > \sqrt{\beta} \tag{1}$$

and similarly, for $\gamma = (h_t - l_{t+1})^2$:

$$\underbrace{h_t - \frac{r_t}{2}}_{\eta_t} - \underbrace{\left(\frac{r_{t+1}}{2} + l_{t+1}\right)}_{\eta_{t+1}} + \frac{r_t}{2} + \frac{r_{t+1}}{2} > \sqrt{\beta} \tag{2}$$

so that (1) and (2) combined can be stated as:

$$|\eta_{t+1} - \eta_t| > \sqrt{\beta} - \left(\frac{r_t + r_{t+1}}{2}\right) \tag{3}$$

Note that the right-hand side of the negativity condition is always positive, since

$$\left(r_t^2 - r_{t+1}^2\right)^2 + 2\left(r_t^2 + r_{t+1}^2\right) > 0$$

which completes the proof of the proposition.

Generality. Throughout we assumed no consecutive-day price ties and well-defined daily ranges. Here, we relax this assumption and show that any price tie yields strictly positive spread values.

Identical consecutive-day low prices Let $h_t > h_{t+1}$ (w.l.o.g.) and $l_t = l_{t+1} \equiv a$. Given the definitions for β and γ, and the negativity condition slightly restated as $\gamma \geq \beta$, we have:

$$(h_t - a)^2 \geq (h_t - a)^2 + (h_{t+1} - a)^2 \tag{4}$$

which cannot hold since $h_{t+1} > a$.

Identical consecutive-day high prices Consider when $h_t = h_{t+1} = b$ and $l_t < l_{t+1}$ (w.l.o.g.). The negativity conditions becomes:

$$(b - l_t)^2 \geq (b - l_t)^2 + (b - l_{t+1})^2 \tag{5}$$

which is unfeasible since $b > l_{t+1}$.

Identical consecutive-day low and high prices Now, let $l_t = l_{t+1} = a$ and $h_t = h_{t+1} = b$. It is straightforward to see that $\gamma = 2\beta$, $\beta > 0$. Therefore, the spread is always strictly positive. ■

Using Proposition B.1, we now show that the negativity condition is more easily attained when the underlying spread is smaller or volatility is higher.

PROPOSITION B.2. Let $f \equiv f(\eta_t, \eta_{t+1})$ and $g \equiv g(r_t, r_{t+1})$ be functions that define the left-hand and right-hand sides, respectively, of the negativity condition for the high-low spread estimator $\widehat{S}^{HL}$:

$$f(\eta_t, \eta_{t+1}) > g(r_t, r_{t+1})$$

For *ex-ante* changes in the bid-ask spread level S and variance σ^2, the following relationships hold:

$$\frac{\partial f}{\partial S} = 0, \quad \frac{\partial g}{\partial S} > 0, \quad \frac{\partial E[f]}{\partial \sigma^2} = (2 - 2\ln 2), \quad \frac{\partial E[g]}{\partial \sigma^2} > 0, \quad \text{and} \quad \frac{\partial^2 E[g]}{\partial \sigma^2 \partial S} > 0$$

Proof. Because the efficient mid-range is identical to the observed mid-range, the left-hand side of the inequality does not depend on the spread level. That is, volatility estimated with squared returns of mid-prices is independent of the spread level. Thus $\partial f / \partial S = 0$ follows. On the other hand, the right-hand side of the inequality is a function of the observed range, $g(r_t, r_{t+1})$, which depends on the spread. If we rewrite observed ranges in terms of efficient ranges, the modified function g depends on S in the following way:

$$g \equiv \sqrt{(\mathcal{R}_t + S)^2 + (\mathcal{R}_{t+1} + S)^2} - \left(\frac{\mathcal{R}_t + \mathcal{R}_{t+1} + 2S}{2} \right)$$

and therefore

$$\frac{\partial g}{\partial S} = \frac{(\mathcal{R}_t + S) + (\mathcal{R}_{t+1} + S)}{\sqrt{(\mathcal{R}_t + S)^2 + (\mathcal{R}_{t+1} + S)^2}} - 1 > 0$$

for well-defined efficient ranges. That implies that the right-hand side in the negativity condition increases in the spread; hence as the spread widens, $\sqrt{\beta}$ relatively exceeds $(r_t + r_{t+1})/2$, so that the inequality becomes more difficult to be attained. For relatively greater spreads, the HL estimator is more likely to be positive.

The second part of Proposition B.2 is also simple. Denote $r_{t+1} = \kappa r_t$, $\kappa > 0$, but not necessarily bounded by 1. After squaring both sides of the negativity condition with the proper substitutions for r_{t+1}, we have

$$(\eta_{t+1} - \eta_t)^2 > \frac{1}{4}\left(\kappa - 2\sqrt{\kappa^2 + 1} + 1\right)^2 r_t^2$$

for which we can replace the observed range with the efficient range. To simplify the calculations, assume $\overline{\kappa} = 1$. Under the maintained hypothesis of constant volatility, daily observed range values will be very close, and κ will differ from the unity when the spread is very small. The average value of κ in 210,000 days of simulated data when $\sigma = 3\%$ and with an infinitesimal spread is 1.09. The simplification is therefore empirically consistent with historical stock data. With $\overline{\kappa} = 1$, the term multiplying r_t^2 collapses to $(3 - 2\sqrt{2})$ and after taking expectations of both sides, we arrive at

$$E\left[(\eta_{t+1} - \eta_t)^2\right] > (3 - 2\sqrt{2})E\left[\mathcal{R}_t^2\right] + 2S(3 - 2\sqrt{2})E\left[\mathcal{R}_t\right] + (3 - 2\sqrt{2})S^2.$$

Direct substitutions for the moments above using (4) and (11) yield

$$\left(2 - \frac{k_2}{2}\right)\sigma^2 > (3 - 2\sqrt{2})k_2\sigma^2 + 2(3 - 2\sqrt{2})k_1 S\sqrt{\sigma^2} + (3 - 2\sqrt{2})S^2. \tag{6}$$

Since the left-hand side of (6) is now $E\left[f\right]$, we have $\partial E\left[f\right]/\partial\sigma^2 > 0$. Similarly, the right-hand side is given by $E[g]$; hence $\partial E[g]/\partial\sigma^2 > 0$ follows. The growth rate of $E[g]$ in σ^2 may exceed $2 - k_2/2$, which largely depends on the magnitude of S. Since $\partial^2 E[g]/\partial\sigma^2\partial S > 0$, smaller spreads will reduce $E[g]$ even while volatility increases, so that the net effect of volatility depends on the relative size between S and σ^2. This completes the proof of the proposition.

Varying κ. Without setting $\kappa = 1$, we can replace $r_t = \mathcal{R}_t + S$ in $(\eta_{t+1} - \eta_t)^2 > \frac{1}{4}r_t^2\left[(\kappa + 1) - 2\sqrt{1 + \kappa^2}\right]^2$ and manipulate it as:

$$(\eta_{t+1} - \eta_t)^2 > \mathcal{R}_t^2\Psi(\kappa) + 2\mathcal{R}_t S\Psi(\kappa) + S^2\Psi(\kappa) \tag{7}$$

where $\Psi(\kappa) \equiv \frac{1}{4}\left(\kappa - 2\sqrt{\kappa^2 + 1} + 1\right)^2$. We can then take expectations of both sides and substitute $E[(\eta_{t+1} - \eta_t)^2] \approx 0.61\sigma^2$, $E[\mathcal{R}_t^2] = 4\ln 2\sigma^2$ and $E[\mathcal{R}_t] = (\sqrt{8/\pi})\sigma$:

$$0.61\sigma^2 > \sigma^2\Psi(\kappa)4\ln 2 + 2S\Psi(\kappa)\left(\sqrt{\frac{8}{\pi}}\right)\sqrt{\sigma^2} + \Psi(\kappa)S^2. \tag{8}$$

We assume $\overline{\kappa} = 1$ to make (8) more tractable. For $\kappa \in [0.8, 1.2]$, the function $\Psi(\kappa)$ varies from 0.15 to 0.21, with $\Psi(1) = 0.17$. After plugging 0.17 into the expression above we have

$$0.61\sigma^2 > 0.17 \left(2.77\sigma^2 + 3.19S\sqrt{\sigma^2} + S^2\right) \tag{9}$$

which clearly yields the first-order derivatives shown in the proposition. ■

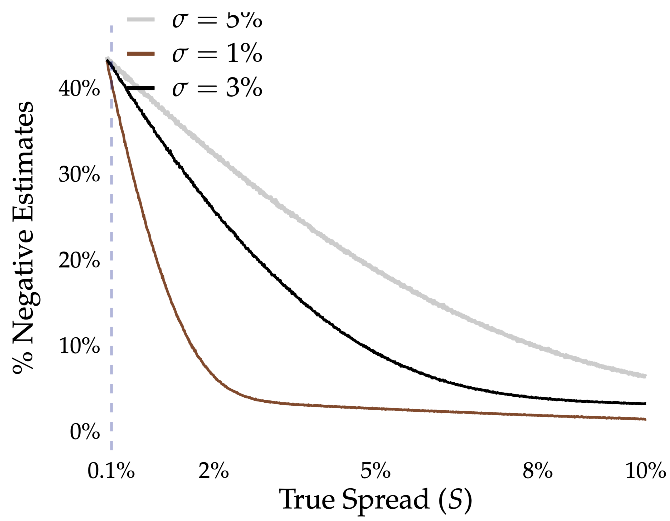
We let two arbitrary functions f and g represent the inequality condition for the HL estimator to make the proposition more intuitive. The left-hand side f contains only mid-range prices, η, while the right-hand side g contains only ranges r. Mid-range prices are independent of the spread, since $\eta_t \equiv (h_t + l_t)/2 = (H_t + L_t)/2$. The greater f is compared to g, the easier the negativity condition is attained. In turn, decreasing the spread makes it easier for the negativity condition to hold and therefore generate a negative estimate.

While greater volatility also increases the frequency of negative estimates, even if volatility approached zero, a small spread would still generate a large number of non-positive spreads. That is because increasing volatility has an ambiguous effect on the proportion of negative estimates, as $E[g]$ may or may not exceed $E[f]$. Thus, S regulates the relative contribution of σ^2 in $E[g]$. This implies that ultimately the spread is more important than volatility to determine when the high-low estimator is negative.

To see this point formally, because S acts as a limiting factor to the growth of $E[g]$ with respect to σ^2, we should expect a high proportion of negative spreads for a narrow spread, regardless of the volatility level. In the limit, as $S \to 0$, the right-hand side of (6) converges to $(3 - 2\sqrt{2})k_2\sigma^2$, which is the minimum of $E[g]$ with respect to S and smaller than $2 - k_2/2$. In the theoretical case, the expected frequency of negative spreads is 100%. In practice, either with simulated or actual data that can only be observed at a discrete set of times, the proportion of negative estimates is maximized when the underlying asset is very liquid. In Figure (B.3), when the true spread is 0.1%, increasing volatility from 1% to 5% produces almost no impact in the proportion of negative HL estimates.

Observable differences in the frequency of negative estimates resulting from varying σ^2 start to appear as the spread grows. Locally widening the spread enables volatility to contribute toward the growth rate of $E[g]$. The difference $E[f] - E[g]$ increases in σ^2 up to a spread level where $E[g]$ is greater than $E[f]$. At this point, the frequency of negative estimates is minimized, and different levels of volatility are again redundant to (6).

Figure (B.3) illustrates the proposition's prediction, where the frequency of negative estimates in the ideal simulated setting for each spread level is plotted using three different price volatility levels. The monotonicity given in Proposition C2 clearly holds up. While the negative share with a daily volatility of 1% decays faster than shares at higher volatility levels, when spreads are around the median value of US stocks, the frequency of negatives is about the same, around 40%.



For very small spreads (below 0.1%), the frequency of negative spread estimates is always greater than 40%, regardless of price volatility. For reference, the median effective spread in US stocks during 2003-2015 is 0.3%.

Notes: This figure computes the average frequency of negative high-low estimates over 10,000 trading months for each 0.1-percentage-point increment in the spread between 0.1% and 10% from simulated data. The data generating process is described in detail in the text and is designed to maintain all model assumptions, including constant true spreads in each sample. We report the relationship with three different volatility values.

Figure B.3: Relationship Between Spread Size, % of Negative Estimates Under Ideal Conditions

C Supplementary Analysis: Close-High-Low Estimator

This section replicates most of our main results for the high-low and roll estimators using the close-high-low (CHL) measure from Abdi and Ranaldo (2017). We also consider this effective spread proxy because in the original paper, Abdi and Ranaldo (2017) show results, especially in the cross-section, that suggest a slight edge of the measure over the high-low estimator. Unfortunately, Figures (D.1) and (D.4) show that the estimator suffers the same significant decreases in performance in the cross-section of monthly, annual, and changes in effective spreads as the high-low proxy. Performance in the entire cross-section is significantly driven by few large-spread stocks, with correlation between the measure and the benchmark spread decreasing as the effective spread narrows. Indeed, the patters are remarkably similar to the Roll and high-low estimators.

Although we do not offer formal proofs for the systematic biases we study in this paper - the moment and small sample bias - we show that the empirical and simulated results implied by our previous derivations apply almost identically to the close-high-low estimator. Furthermore, we present a complete analysis of the estimator's negativity determinants, which reveal the same driving forces behind degenerate estimates in this case as with the high-low proxy. Those are smaller spreads and higher volatility, with spreads contributing more significantly to generating negative spreads. Simulation results show that the bias behaves in the same fashion as with the high-low estimator. All in all, our results confirm the incidence of the same misbehavior patterns and driving forces for the CHL measure as in the HL and Roll estimators.

The estimator. The close-high-low spread estimator from Abdi and Ranaldo (2017) combines the use of high and low prices from the HL measure with daily closing prices from Roll (1984). In a sense, the measure uses the Parkinson-Garman-Klauss framework more broadly by integrating the range and daily returns to estimate bid-ask spreads along the lines of HL. The intuition for augmenting the information set is that closing prices are more "contaminated" by the bid-ask bounce than the range (Alizadeh et al. (2002)). Let the mid-range, or average price on day t, be defined as

$$\eta_t \equiv \frac{h_t + l_t}{2} = \frac{r_t}{2} + l_t \tag{10}$$

which clearly coincides with the mid-range of daily efficient extreme prices, $\eta_t = (\mathcal{H}_t + \mathcal{L}_t)/2$. A crucial result in Abdi and Ranaldo (2017) establishes how the variance of the mid-range returns relates to the efficient price volatility σ^2:

$$E\left[(\eta_{t+1} - \eta_t)^2\right] = \left(2 - \frac{k_2}{2}\right)\sigma^2 \tag{11}$$

where the variance may be replaced with $\widehat{\sigma}_{\eta}^{2}$ for estimated squared returns of consecutive-day mid-range prices. Under the validity of (11), and because the average of consecutive mid-ranges,

$$S^{CHL} = 2\sqrt{\left(c_t - \eta_t\right)\left(c_t - \eta_{t+1}\right)}. \tag{12}$$

Negativity of the close-high-low estimator. We follow the same steps for the close-high-low estimator to prove the drivers of negative estimates as we did with the high-low proxy. Even though the CHL measure does not directly depend on the range, it does depend on the mid-range. Moreover, our simulation results show a behavior dynamic of CHL similar to HL with respect to negative spreads, true spread size, and bias Figure (C.2). This suggests a common channel through which the spread-to-volatility ratio affects both proxies. From the expression for CHL_t, the straightforward negativity condition of the close-high-low estimator is:

$$c_t \in \left(\eta_t, \eta_{t+1}\right) \tag{13}$$

when the second-day mid-range is above the first-day mid-range, and $c_t \in \left(\eta_{t+1}, \eta_t\right)$ in the symmetric case. When the price variation across two days is large enough (measured by the difference in mid-prices), a wide range of values for c_t yields negative CHL spread estimates. We can rewrite (13) as (by analogy the symmetric case follows)

$$\eta_{t+1} - \eta_t > c_t - \eta_t \tag{14}$$

and similarly to HL, define an inequality condition to investigate the effects of spread and volatility levels in generating negative spread estimates.

Proposition C.1 Let $f \equiv f\left(\eta_t, \eta_{t+1}\right)$ and $v \equiv v\left(c_t, \eta_t\right)$ be functions that define the left-hand and right-hand sides, respectively, of the negativity condition for the close-high-low spread estimator:

$$f\left(\eta_t, \eta_{t+1}\right) > v\left(c_t, \eta_t\right) \tag{15}$$

For *ex-ante* changes in the bid-ask spread level S and variance σ^2, the following relationships hold:

$$\frac{\partial f}{\partial S} = 0, \quad \frac{\partial E[v]}{\partial S} > 0, \quad \frac{\partial E[f]}{\partial \sigma^2} = k_3, \quad \frac{\partial E[v]}{\partial \sigma^2} = \frac{k_3}{2} \quad \text{and} \quad \frac{\partial^2 E[v]}{\partial \sigma^2 \partial S} = 0. \tag{16}$$

Proof. The close-high-low spread estimator is negatively defined if the following holds:

$$\eta_{t+1} > c_t > \eta_t \tag{17}$$

or, equivalently,

$$\eta_{t+1} - \eta_t > c_t - \eta_t > 0. \tag{18}$$

We can replace observed mid-ranges and close prices with true values:

$$\eta_{t+1} - \eta_t > \mathcal{C}_t + q_t \frac{S}{2} - \left(\frac{\mathcal{H}_t + \mathcal{L}_t}{2} \right) \tag{19}$$

and square both sides such that

$$(\eta_{t+1} - \eta_t)^2 > \left[\left(q_t \frac{S}{2} + \frac{\mathcal{C}_t}{2} - \frac{\mathcal{H}_t}{2} \right) + \left(\frac{\mathcal{C}_t}{2} - \frac{\mathcal{L}_t}{2} \right) \right]^2 \tag{20}$$

and further simplify it as

$$\begin{aligned} (\eta_{t+1} - \eta_t)^2 >& \frac{1}{4} \left[q_t^2 S^2 + 2 q_t S \left(\mathcal{C}_t - \mathcal{H}_t \right) + \left(\mathcal{C}_t - \mathcal{H}_t \right)^2 \right] + \\ &+ \frac{1}{2} \left[q_t S \mathcal{C}_t + \mathcal{C}_t^2 - q_t S \mathcal{L}_t + \mathcal{C}_t \mathcal{L}_t - \mathcal{C}_t \mathcal{H}_t + \mathcal{H}_t \mathcal{L}_t \right] + \frac{1}{4} \left(\mathcal{C}_t - \mathcal{L}_t \right)^2 . \end{aligned}$$

After taking expectations of the above, we arrive at the intermediate expression

$$E\left[(\eta_{t+1} - \eta_t)^2 \right] > \frac{S^2}{4} + \frac{1}{4} E\left[\left(\mathcal{C}_t - \mathcal{H}_t \right)^2 \right] + \frac{1}{2} \left(E\left[\mathcal{C}_t \mathcal{L}_t \right] - E\left[\mathcal{C}_t \mathcal{H}_t \right] + E\left[\mathcal{H}_t \mathcal{L}_t \right] \right) + \frac{1}{4} E\left[\left(\mathcal{C}_t - \mathcal{L}_t \right)^2 \right] \tag{21}$$

which can be easily solved for with the moments given in Garman and Klass (1980). The final inequality is

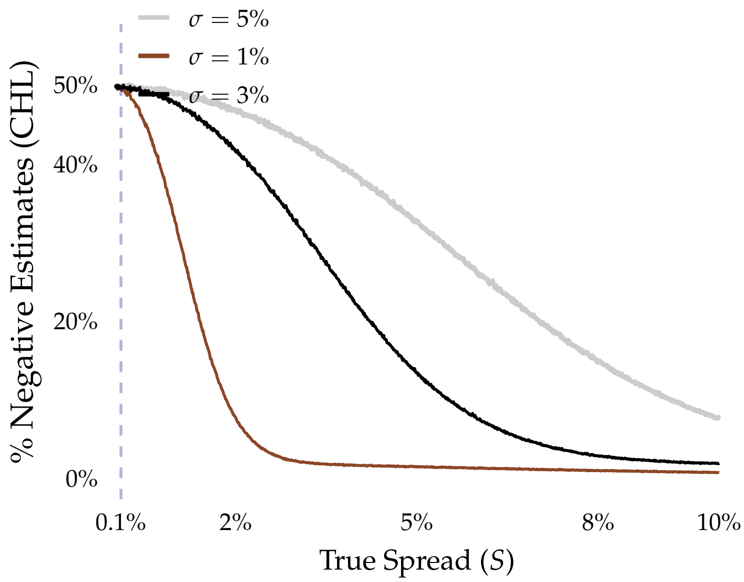
$$(2 - 2\ln 2)\,\sigma^2 > \frac{S^2}{4} + \frac{\sigma^2}{4} + \frac{1}{2}(1 - 2\ln 2)\,\sigma^2 + \frac{\sigma^2}{4} \tag{22}$$

and therefore:

$$(2 - 2\ln 2)\,\sigma^2 > \frac{S^2}{4} + (1 - \ln 2)\,\sigma^2. \tag{23}$$

■

Proposition C.1 is intended to mimic the structure used in Proposition B.2. The generic function f is intentionally identical to f in the HL estimator. The proof, however, is slightly different from the proposition in the high-low case. Since we do not assume the value of the trade indicator q_t in $c_t = \mathcal{C}_t + q_t S/2$, we can only work with the expected value of v, without assuming a deterministic q_t. We also use moments derived for $E\left[\mathcal{C}^a \mathcal{H}^b \mathcal{L}^c\right]$ in Garman and Klass (1980).



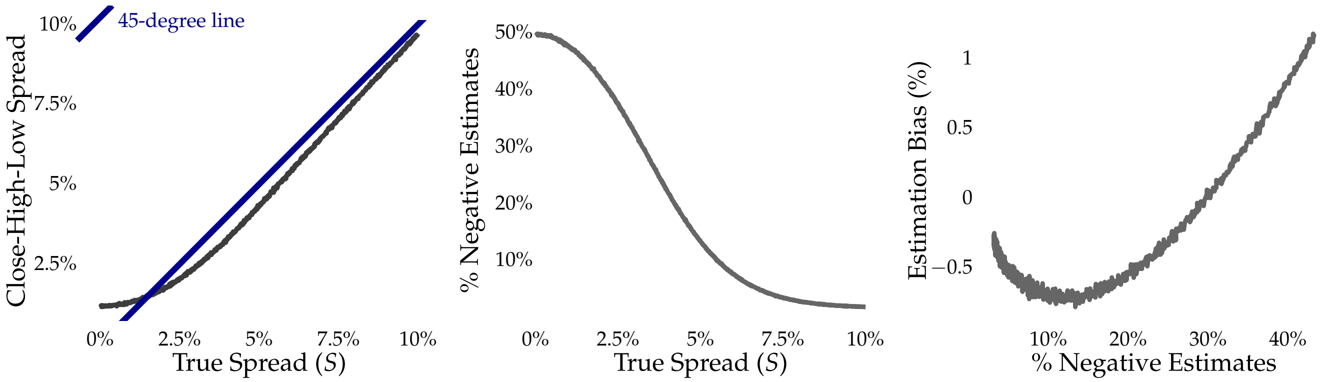
For very small spreads (below 0.1%), the frequency of negative close-high-low spread estimates is always greater than 50%, regardless of price volatility. For reference, the median effective spread in US stocks during 2003-2015 is 0.3%.

Notes: This figure computes the average frequency of negative close-high-low estimates over 10,000 trading months for each 0.1-percentage-point increment in the spread between 0.1% and 10% from simulated data. The data generating process is described in detail in the text, being identical to the DGP used in the exercises for the high-low estimator. Moreover, it is also designed to maintain all model assumptions, including constant true spreads in each sample. We report the relationship with three different volatility values.

Figure C.1: Relationship Between Spread Size, % of Negative Estimates Under Ideal Conditions

The interpretation of Proposition C.1 is also very similar to Proposition B.2, although simpler. Decreasing the spread size lowers the expected value of the right-hand side of the negativity condition, contributing to more negative estimates. When the spread is small, increases in σ^2 cannot induce $E[f] < E[v]$, which explains why in Figure (C.1) the proportion of negative CHL estimates is around 50% when $S = 0.1\%$, regardless of volatility size. Note that the figure shows the same monotonic patterns as with the high-low measure, revealing an even higher proportion of negative estimates in the close-high-low case for most spread sizes.

Moreover, for small spreads, any value of σ^2 increases $E[f]$ faster than $E[v]$ – a pattern that is eventually reversed when the spread is large enough so that the right-hand side becomes greater than $E[f]$. For completeness, we also provide the following alternative result that shows why smaller spreads induce negative CHL estimates more often.



For small spreads (below 1%), the close-high-low estimator suffers from an upward bias and a higher fraction of negative estimates. As the spread becomes larger, the bias decreases and fewer estimates turn out negative. For reference, the median effective spread in US stocks during 2003-2015 is 0.3%.

Notes: This figure computes 10,000 monthly averages of the close-high-low spread estimator (with zeros adjustment) and share of negative daily estimates for each 0.1-percentage-point increment in the spread between 0.1% and 10% from simulated data. The data generating process is described in detail in the text and is designed to maintain all model assumptions, including constant true spreads in each sample. Bias is defined by the difference between the estimated spread and the true spread.

Figure C.2: Bias, Negative Estimates, and True Spread Under Ideal Conditions

Proposition C.2. The probability that c_t falls in the interval (η_t, η_{t+1}), and therefore the negativity condition for CHL is attained, increases as the spread size decreases.

Proof. The probability that the negativity condition of CHL is attained, $\Pr\left[\eta_t < c_t < \eta_{t+1}\right]$, is given by

$$\int_{\eta_t}^{\eta_{t+1}} f\left(c_t\right) dc_t = \int_{l_t}^{h_t} f\left(c_t\right) dc_t - \underbrace{\left(\int_{\eta_{t+1}}^{h_t} f\left(c_t\right) dc_t + \int_{l_t}^{\eta_t} f\left(c_t\right) dc_t\right)}_{\mathcal{K}}.$$

Let $h_t^* = h_t - \delta$ and $l_t^* = l_t + \delta$ represent modified daily high and low prices from a decrease of 2δ in the spread S. Let $\Pr\left[\eta_t^* < c_t < \eta_{t+1}^*\right]$ denote the probability that c_t falls within the negativity interval *ex post* the spread decrease. Then, it follows that

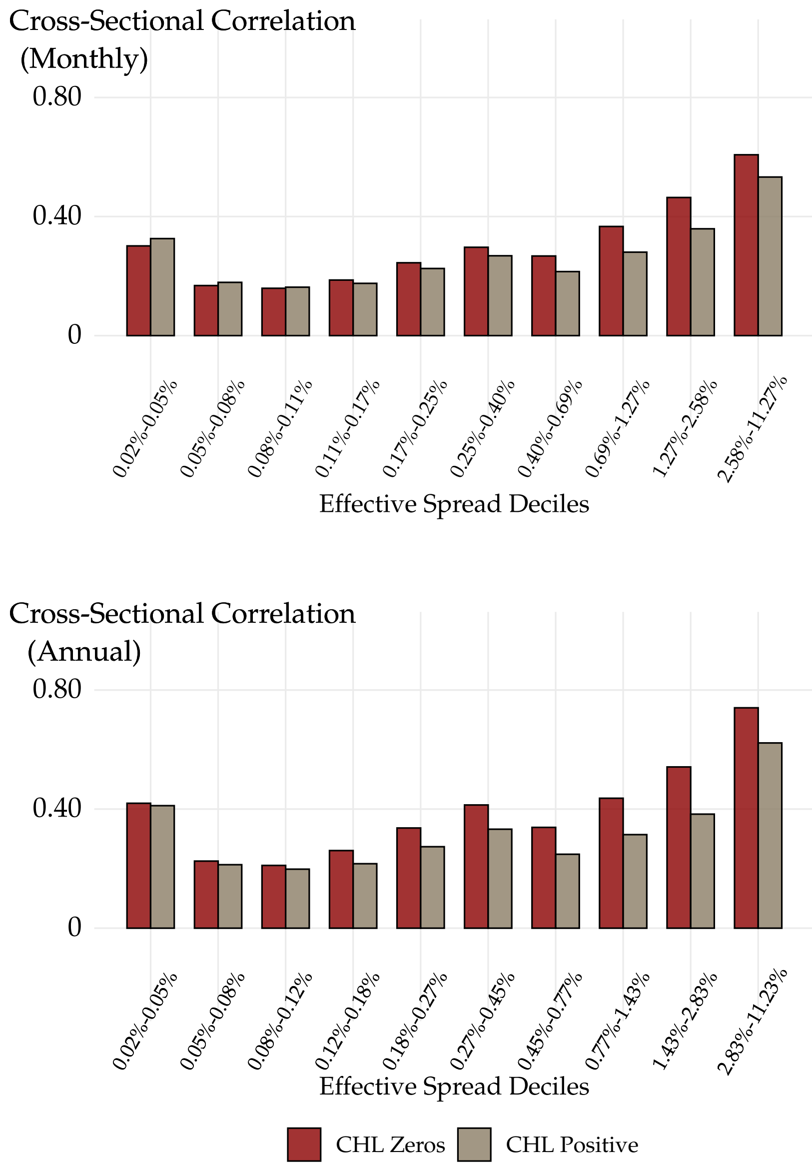
$$\left(\int_{\eta_{t+1}}^{h_t} f\left(c_t\right) dc_t + \int_{l_t}^{\eta_t} f\left(c_t\right) dc_t\right) \geq \left(\int_{\eta_{t+1}}^{h_t^*} f\left(c_t\right) dc_t + \int_{l_t^*}^{\eta_t} f\left(c_t\right) dc_t\right)$$

since

$$\left(\int_{\eta_{t+1}}^{h_t^*} f\left(c_t\right) dc_t + \int_{l_t^*}^{\eta_t} f\left(c_t\right) dc_t\right) \leq \left(\int_{h_t-\delta}^{h_t} f\left(c_t\right) dc_t + \int_{\eta_{t+1}}^{h_t-\delta} f\left(c_t\right) dc_t + \right.$$
$$\left. \int_{l_t+\delta}^{\eta_t} f\left(c_t\right) dc_t + \int_{l_t}^{l_t+\delta} f\left(c_t\right) dc_t\right)$$

and therefore $\int_{\eta_t}^{\eta_{t+1}} f\left(c_t\right) dc_t \leq \int_{\eta_t^*}^{\eta_{t+1}^*} f\left(c_t\right) dc_t$. The equality holds if and only if $\mathcal{K} = 0$. This can be the case only when $\eta_t = \eta_{t+1}$. Hence, $\Pr\left[\eta_t^* < c_t < \eta_{t+1}^*\right] > \Pr\left[\eta_t < c_t < \eta_{t+1}\right]$ and we conclude the proof. ■

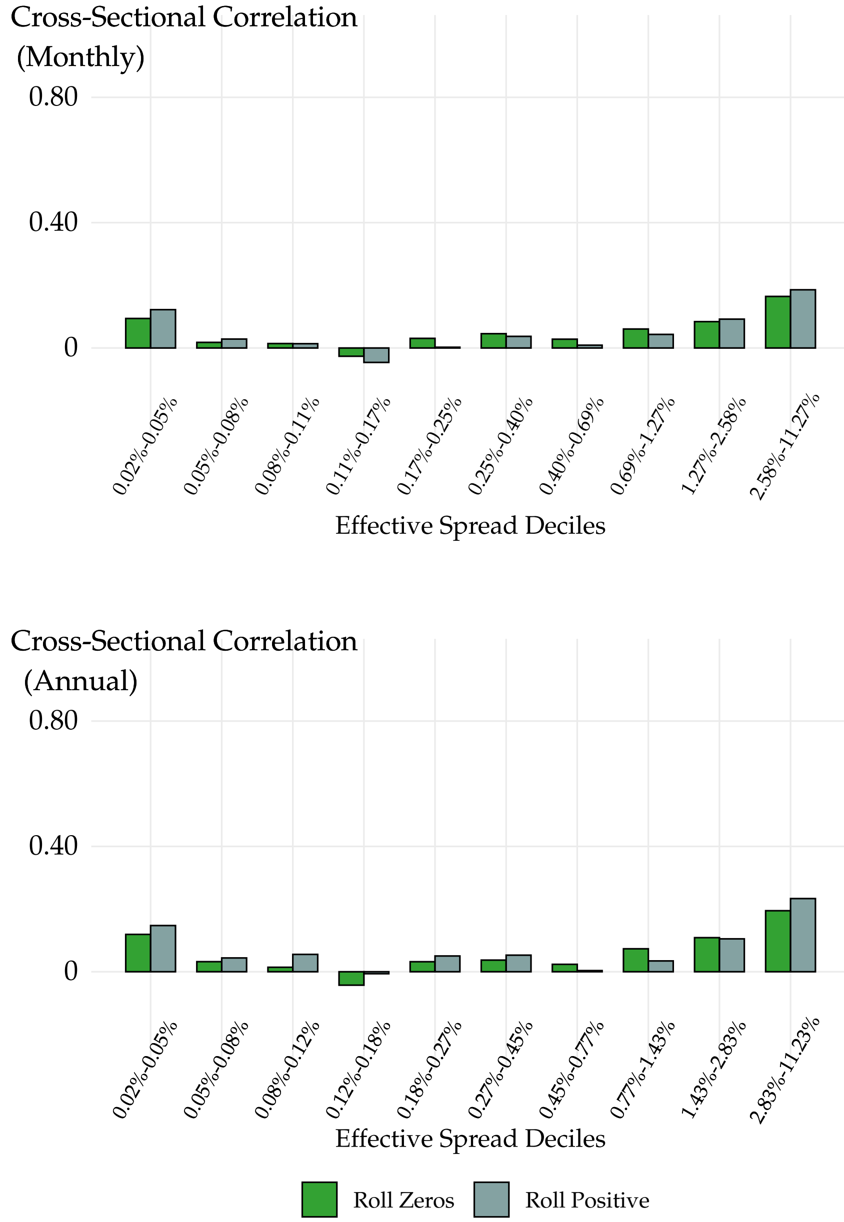
D Appendix Figures



In all effective spread size deciles, monthly averages of close-high-low estimates without ad hoc adjustments correlate poorly with monthly average effective spreads. Correlations from the estimates with the positives and zeros adjustments behave similarly, with the best performance in the least liquid stocks (above 50% correlation with effective spreads). The frequency of daily estimated spreads that turn out negative is above 50% in the most liquid stocks, decreasing as effective spread bands increase.

Notes: This figure shows cross-sectional correlations between three versions of the close-high-low spread estimator and effective spreads from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

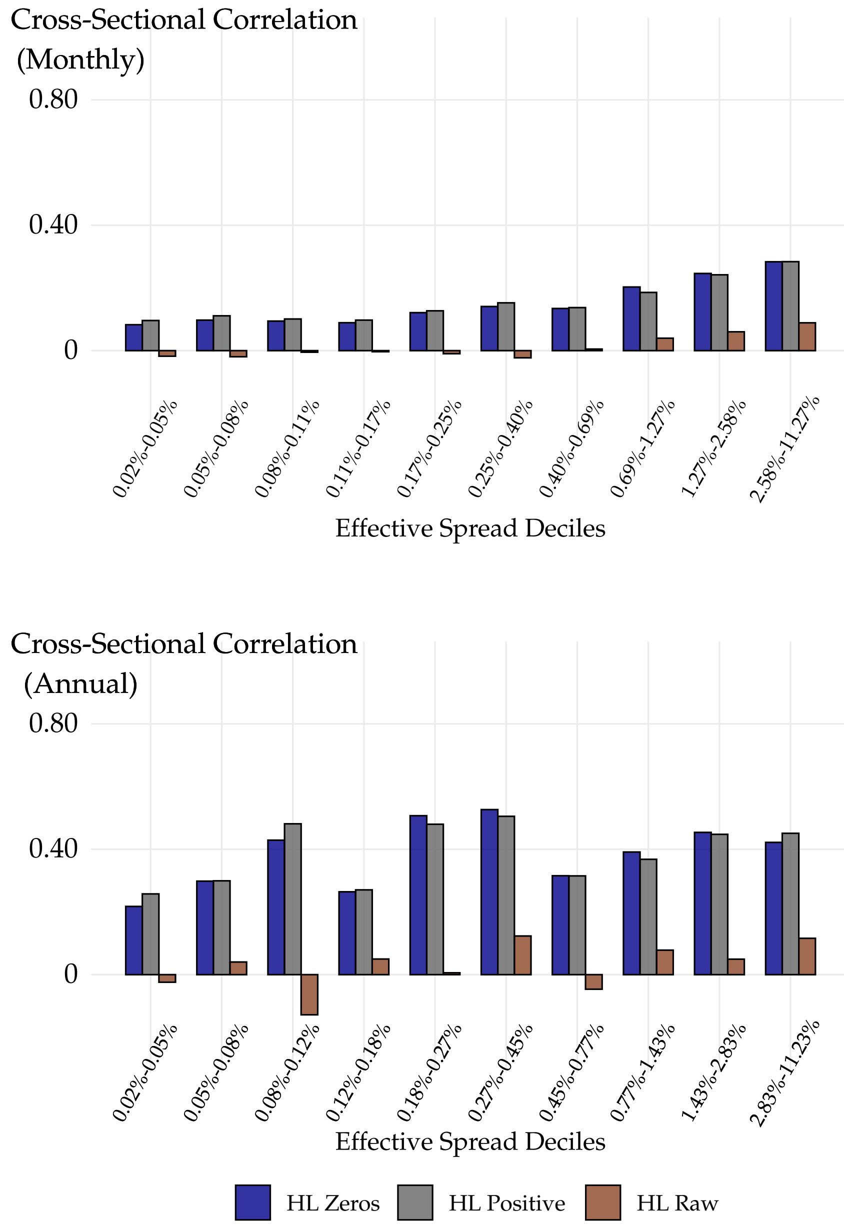
Figure D.1: Cross-Sectional Correlation: CHL Estimates and Effective Spread By Effective Spread Decile Size



In all effective spread size deciles, monthly averages of Roll estimates without ad hoc adjustments correlate poorly with monthly average effective spreads. Correlations from the estimates with the positives and zeros adjustments behave similarly, with the best performance in the least liquid stocks (above 15% correlation with effective spreads). The frequency of daily estimated spreads that turn out negative is above 40% in the most liquid stocks, decreasing as effective spread bands increase.

Notes: This figure shows cross-sectional correlations between three versions of the Roll spread estimator and effective spreads from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

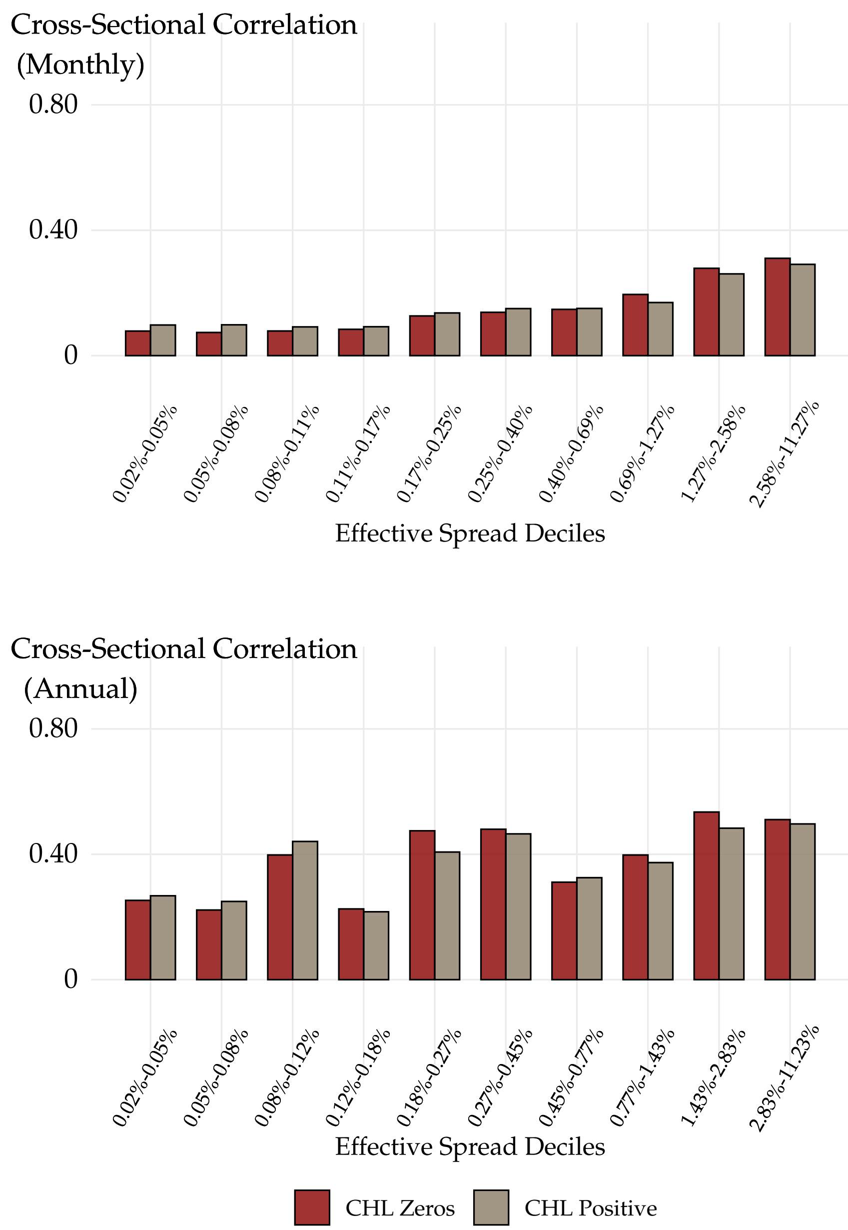
Figure D.2: Cross-Sectional Correlation: Roll Estimates and Effective Spread By Effective Spread Decile Size



In all effective spread size deciles, monthly averages of changes in high-low estimates without ad hoc adjustments correlate poorly with monthly average change in effective spreads. Correlations from the estimates with the positives and zeros adjustments behave similarly, with the best performance in the least liquid stocks (above 30% correlation with effective spreads). The frequency of daily estimated spreads that turn out negative is above 40% in the most liquid stocks, decreasing as effective spread bands increase.

Notes: This figure shows cross-sectional correlations between three versions of the close-high-low spread estimator and effective spreads from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

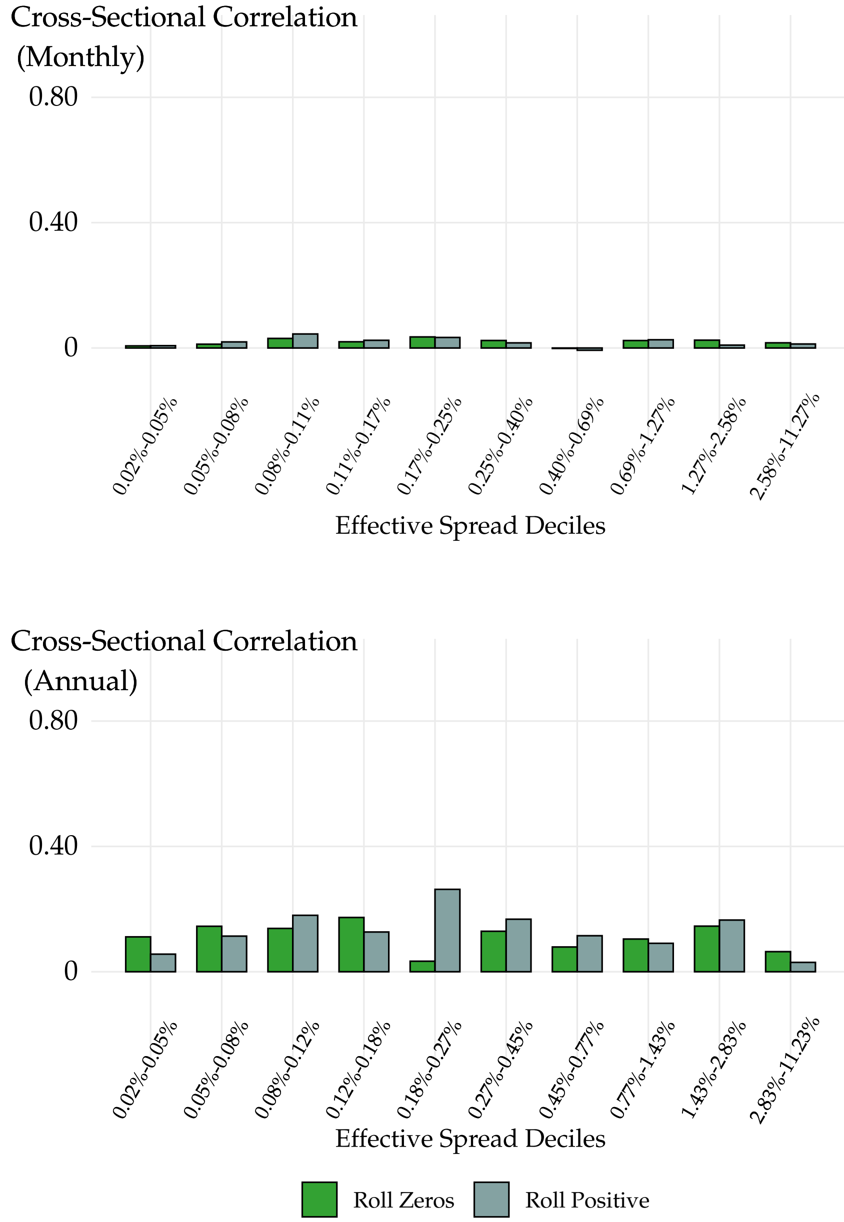
Figure D.3: Cross-Sectional Correlation: Changes in HL Estimates and Changes in Effective Spreads By Effective Spread Decile Size



In all effective spread size deciles, monthly averages of changes in close-high-low estimates without ad hoc adjustments correlate poorly with monthly changes of average effective spreads. Correlations from the estimates with the positives and zeros adjustments behave similarly, with the best performance in the least liquid stocks (above 30% correlation with effective spreads). The frequency of daily estimated spreads that turn out negative is above 50% in the most liquid stocks, decreasing as effective spread bands increase.

Notes: This figure shows cross-sectional correlations between three versions of the close-high-low spread estimator and effective spreads from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

Figure D.4: Cross-Sectional Correlation: Changes in CHL Estimates and Effective Spreads By Effective Spread Decile Size



In all effective spread size deciles, monthly averages of changes in Roll estimates without ad hoc adjustments correlate poorly with monthly changes of average effective spreads. Correlations from the estimates with the positives and zeros adjustments behave similarly, with the best performance in the least liquid stocks (above 1% correlation with effective spreads). The frequency of daily estimated spreads that turn out negative is above 40% in the most liquid stocks, decreasing as effective spread bands increase.

Notes: This figure shows cross-sectional correlations between three versions of the Roll spread estimator and effective spreads from DTAQ US stock data. There are 3,398 stocks in total and sample construction details are described in Appendix (I).

Figure D.5: Cross-Sectional Correlation: Changes in Roll Estimates and Changes in Effective Spreads By Effective Spread Decile Size