Lorenz Equations and Strange Attractors

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The Lorentz Equations and Weather Modelling

In the 1950s, pioneering meteorologist Edward Lorenz endeavored to use computational methods to model the atmosphere. Employing a system of 12 differential equations, Lorenz sought trajectories that were neither periodic nor asymptotically periodic, in an effort to comprehend the unpredictable nature of weather [1]. His extensive research led to a groundbreaking discovery of such trajectories. This further motivated him to simplify his model, ultimately resulting in a seminal publication [3], in which he investigated the Rayleigh-Bénard convection.

The Rayleigh-Bénard convection is a physical phenomenon that occurs in a fluid contained between two plates. The lower plate, heated to a higher temperature T_l, warms the fluid, creating an instability as the upper plate remains at a cooler temperature T_u, as depicted in Figure 1. Initially, with minimal temperature difference, the fluid remains static. However, as the temperature difference escalates, the fluid commences to move in an increasingly chaotic manner.

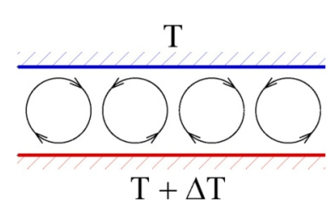

Figure 1. Illustration of the Rayleigh-Bénard convection phenomenon.

By utilizing Fourier series and truncating infinite terms, Lorenz ingeniously derived a more simplified system of three differential equations encapsulating the chaotic behaviour he was investigating. This system, called the Lorenz system, is represented as:

$$x' = \sigma(y - x) \tag{1}$$
$$y' = rx - y - xz \tag{2}$$
$$z' = xy - bz, \tag{3}$$

where $\sigma, b, r \in \mathbb{R}$ are the parameters of the model, usually assumed to be positive.

Properties of Lorentz Equations

- **Non-Linearities:** The system of equations contains two non-linearities: The term xz in equation 2 and the term xy in equation 3.
- **Symmetry:** There is a symmetry consisting in a reflection through the z-axis. It is defined by the transformation $f(x, y, z) = (-x, -y, z)$ [4].
- **Invariance of the z-axis:** The z-axis defined by $x = y = 0$ is invariant and all the trajectories starting in this axis tend to the origin $(0, 0, 0)$.
- **Fixed Points:** The origin $(0, 0, 0)$ is an fixed point for every value of the parameters. If we have that $r > 1$, by setting the equations 1, 2 and 3 to 0, a simple calculation shows that there are two other stationary points given by $A_+ = (c, c, r-1)$ and $A_- = (-c, -c, r-1)$, where $c = \sqrt{b(r-1)}$.
- **Linear Stability of the Origin:** To investigate stability, we linearize the system at the origin. This simplifies the nonlinear Lorenz system to a more tractable linear system in the neighborhood of $(0, 0, 0)$. Nonlinear terms are disregarded since their influence is negligible near this point. We thus obtain

$$\begin{aligned} x' &= \sigma(y-x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned} \quad \Rightarrow \quad \begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ z &\to 0 \end{aligned}$$

  This is a decoupled system, and we can see that the z decays to 0 exponentially. The other equations are determined by the linear system above. The trace is $\alpha = -1 - \sigma$ and the determinant $\beta = (1 - r)\sigma$. If $r < 1$ we have that the origin is an stable node or sink, as $\alpha^2 - 4\beta > 0$, and thus all the directions are incoming. If $r > 1$ the origin is a saddle point, with two incoming and one outgoing directions.

Satbility of the Fixed Points A_+ and A_-

We now want to study the stability of the fixed points A_+ and A_-. We assume that $r > 1$ and $\sigma > b + 1$ [2]. We have to study the eigenvalues of the Jacobian matrix at these points.

$$J(x,y,z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix} \Rightarrow J(A_\pm) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{bmatrix}$$

We can now write the characteristic polynomial of the Jacobian:

$$\lambda^3 + (1 + b + \sigma)\lambda^2 + (r + \sigma)b\lambda + 2(r-1)b\sigma = 0.$$

We note that when $r = 1$, the roots of the polynomial are $0, -b, -1-\sigma$ and thus all the eigenvalues are negative. This means in particular that these fixed points are sinks when $r = 1$. We want to study when they stop being sinks, for which they need to have eigenvalues with a non-negative real part. By continuity we know that as r increases, the eigenvalues will continue to have negative real part until any of them becomes purely imaginary. Thus, we want look for purely imaginary eigenvalues, so we assume $\lambda = i\xi$ and we must have:

$$-\xi^3 i - (1 + b + \sigma)\lambda^2 + (r + \sigma)bi\xi + 2(r-1)b\sigma = 0.$$

Setting the real and imaginary part of the equation we obtain that $\xi = \sqrt{b(\sigma + r)}$ and that $r = \frac{\sigma(\sigma+3\sigma+1)}{\sigma-b-1}$. We have obtained that for $1 < r < R = \frac{\sigma(\sigma+3\sigma+1)}{\sigma-b-1}$ the fixed points $A_\pm$ are sinks. At $r = R$ it occurs what is called a Hopf bifurcation and the fixed points $A_\pm$ stop being sinks.

Strange Attractors and Chaos

As it has been shown, for $r > R = \frac{\sigma(\sigma+3\sigma+1)}{\sigma-b-1}$ the stable points $A_\pm$ stop being sinks and we move into unknown terrain. Lorenz originally studied the case where $\sigma = 10, b = 8/3$ and $r = 28$, and thus had $r > 24.74 \approx R$. Figure 2 shows the projection of two three-dimensional trajectories following the Lorenz equations with the above mentioned parameters.

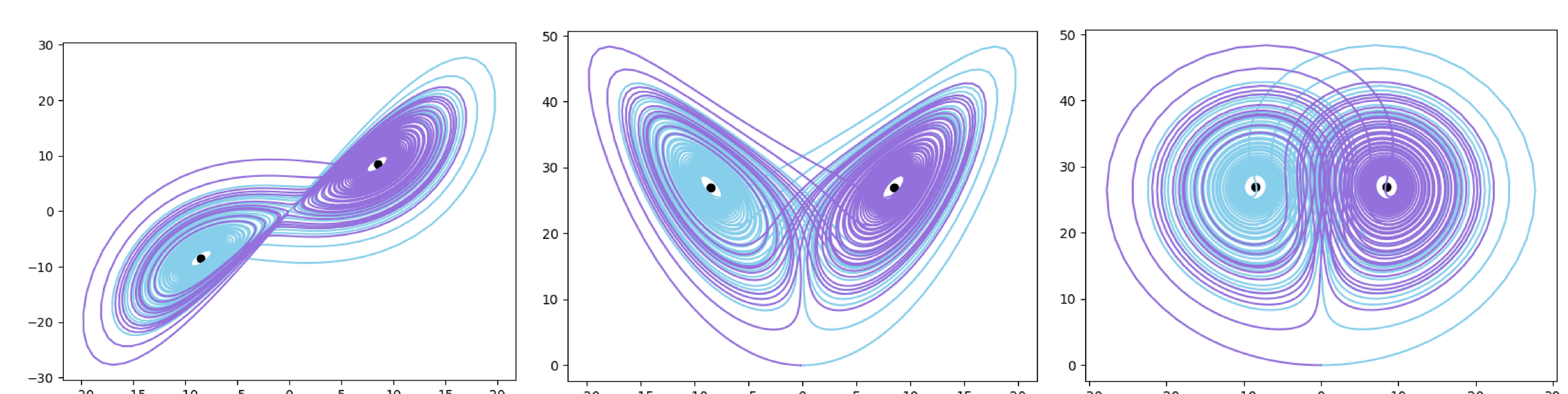

Figure 2. Projections on the XY, XZ and YZ planes (from left to right) of the phase diagrams of the Lorenz equations with two trajectories starting at $(0.1, 0, 0)$ and $(-0.1, 0, 0)$.

We define a strange attractors following [5]. An attractor is a closed set A with the following properties:

- A is an invariant set: Any trajectory $y(t)$ which starts in A stays in A for all time.
- A attracts an open set of initial conditions: there is an open set U containing A such that if $y(0) \in U$ then $\lim_{t\to\infty} d(A, y(t)) = 0$.
- A is minimal in the sense that there is no proper subset of A which satisfies the two first conditions.
- A exhibits sensitive dependence to the initial conditions.

The illustrations presented in Figure 2 offer instances of strange attractors. It also shows chaotic behaviour as defined in [5]: *aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions*. However, a formal verification of this facts is beyond the scope of this project.

Global Stability of Origin

We now want to show that whenever $r < 1$ the fixed point $(0, 0, 0)$ is globally stable. We follow the proof in [5].

We consider ellipsoid defined by $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$. These ellipsoids are centred in the origin. The idea is that it suffices to prove that along the trajectories defined by the Lorenz equations $V'(x, y, z) < 0$ for every (x, y, z) different from the origin. This would mean that these Lorentz trajectories enter smaller and smaller ellipsoids as $t \to \infty$, making $(0, 0, 0)$ a gobally stable point. Let's compute $V'(x, y, z)$ along the Lorenz trajectories:

$$\frac{1}{2}V'(x,y,z) = \frac{1}{\sigma}xx' + yy' + zz' = -x^2 + xy - y^2 - xyz + rxy + xyz - bz^2$$
$$= -(x^2 + y^2 + bz^2) + (r+1)xy = -\left(x - \frac{r+1}{2}y\right)^2 - \left(1 - \left(\frac{r+1}{2}\right)^2\right)y^2 - bz^2.$$

It is clear from the last expression that $\frac{1}{2}V'(x, y, z) \le 0$ for all (x, y, z). In order to show that $\frac{1}{2}V'(x, y, z) \ne 0$ for $(x, y, z) \ne (0, 0, 0)$ we argue by contradiction. Suppose $\frac{1}{2}V'(x, y, z) = 0$. Thanks to $r < 1$, we must have $y = 0 = z$ due to the last two terms of the previous expression. This in turn implies that $x = 0$ due to the first term of the expression, implying $(x, y, z) = (0, 0, 0)$, and thus completing the proof.

Additional Properties

- **Transient Chaos:** In the Lorenz system with parameters $\sigma = 10$ and $b = 8/3$, we have shown that the origin remains stable for $r < 1$ while for $1 < r < 24.74$, two new fixed points $A_\pm$ emerge and the origin becomes unstable. Additionally, the system experiences a phenomenon named transient chaos for $13.93 < r < 24.06$, by which some trajectories exhibit initial chaotic behaviour which eventually decays into a non-chaotic behaviour. Beyond this range, when $r > 24.06$, the fixed points $A_\pm$ evolve into strange attractors.
- **Periodic Behaviour for $r > 24.74$:** When $\sigma = 10$ and $b = 8/3$, there are small ranges of $r > 24.74$ for which there appear periodic orbits. This include ranges such as $[99.53, 100.79]$ or $[145, 166]$.
- **Volume Contraction:** Every volume in the phase space of the Lorenz system 'dissipates', that is, the volume tends to 0. The divergence of the Lorenz system is given by $-(1 + b + \sigma)$ which in turn implies that $V' = -(1 + b + \sigma)V$, were V is a volume in the phase space. This means that volumes in the Lorenz system contract exponentially. It also turns out that there exists a bounded ellipsoid where all trajectories enter at some point.

References

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