Lecture 3- Dynamic Linear Regression

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today

Special Topics in Time Series Econometrics - Forecasting

FGV EESP
CENTRO DE ESTUDOS QUANTITATIVOS EM ECONOMIA E FINANÇAS

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Introduction

- Analyse forecasting with dynamic linear regresion model
- In the Autoregressive distribued lags model (ADL(p,q)) the dependent variable, y, is allowed to depend on p lags of itself $(y_{t-1}, \cdots, y_{t-p})$, the autoregressive component, and q lags of the regresssors, x_t $(x_{t-1}, \cdots, x_{t-q})$ the distributed lag component.(see [Valls Pereira, 2024][1])

[1] This notes can be downloaded here ADL Model

Dynamic Models typology I

- Starting with an ADL(1,1), i.e.:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \tag{1}$$

- where $\varepsilon_t \sim NI(0, \sigma_\varepsilon^2)$ and for simplicity we are assuming that $\alpha_0 = 0$.
- There are 10 sub-models nested in (1) which can be obtained by testing restrictions in the coefficients.
- Three important observations must be made before commenting on each of these models.
  - First is related to the status of variable x_t
  - Second the dependence of the model properties and the data properties.
  - Third if the model is correctly specified or is an approximation of a more general D.G.P.
- Since we did not start from a general model and because the D.G.P. is in general unknown this observations are needed.

Dynamic Models typology II

- The inference in these models will be valid if the conditional model is a valid one. The implicit assumption is that x_t is weakly exogenous (see [Engle et al., 1983] or [Valls Pereira, 2024]) for the parameter of interest which are $\theta = (\alpha_1, \beta_0, \beta_1, \sigma_\varepsilon^2)$.
- Assuming weakly exogeneity for x_t with respect to the parameters of interest, if any sub-model in the typology is valid then all other models lest restrict are also valid.
- We can have x_t stationary or $I(1)$ and in this case we have $\Delta x_t \sim I(0)$.
- This solve the second observation because some economic time series are $I(0)$, like the unemployment rate, so the case of stationary x_t is included.
- When x_t and y_t are $I(1)$ the correct representation in the Equilibrium Correction Model (EqCM) if the two variable cointegrate that is if $u_t = y_t - \gamma x_t$ is $I(0)$.

Dynamic Models typology III

- If $u_t \sim I(1)$ the error correction term in the EqCM model will not be present, and y_t and x_t are $I(1)$ and do not cointegrate.
- The typology treats each model by its one so we are assuming correct specification of the model. Then the third observation does not apply.
- Since we are assuming that the models are correctly specified the error are white noise, by construction. Therefore we can estimate the model by OLS, except the static regression with correlated errors. In this case we can use GLS or iterative procedure.
- For all model we do not present specification test or mis-specification tests. But for the static regression with correlated error we present a test to detect autocorrelation in the errors which can also be used to test dynamic mis-specification

Static Regression I

- The model is given by:
$$y_t = \beta_0 x_t + \varepsilon_t \tag{2}$$
- (in general both β_0 and x_t are vectors) It is important in macroeconometric systems since it can be interpreted as a structural equation.
- In practice the hypothesis that ε_t is white noise is unlikely since lack of dynamic induces correlation in the errors and spurious correlation between variables can induce high correlation in the errors making the usual inference invalid.
- In the end of the 80's static regression reappeared in the literature as part of the two-step procedure of Engle and Granger (see [Engle and Granger, 1987]) to test cointegration.
- In this way β_0 is an estimator of γ the long-run elasticity.

Static Regression II

- This procedure to obtain the long-run elasticity has been questioned in the literature since the static relationship instead of the dynamic regression imposes common factor restrictions that may not be valid (see [Ericsson and MacKinnon, 2002]).
- Another reason for the static model is related to the approach of autocorrelation, that is estimate the static model and correct the possible autocorrelation of the residuals through an autoregressive model. But this type of procedure can impose common factors restrictions that may be invalid.

Example using Simulated Data

Consider the model

$$y_t = \alpha_0 + \beta_0 x_t + \varepsilon_t$$

- the following R commands are used to estimate this model by OLS.
  - `# Sample @first+1 to keep estimation sample the same for all models`
  - DLM

Univariate Autoregressive Model I

- The D. G. P. is given by:

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t \tag{3}$$

- which is a possible parametrization of univariate time series models.
- Note that in the case $\alpha_1 = 1$ this model is a random walk if ε_t is a white noise.
- Autoregressive models are used in ex-ante forecast in univariate and multivariate cases and are an alternative to econometric models in terms of predictive power.
- The economic decisions are interdependent inducing autoregressive models. But these models are not autonomous, because changes in the data generated process of x_t can affect the data generating process of y_t. Also the autoregressive process for y_t can be obtained through elimination of marginalization of the data generating process for x_t.

Univariate Autoregressive Model II

- For example, suppose that $x_t = x_{t-1} + v_t$ with v_t ~ $NI(0, \sigma_t^2)$ and if the restrictions $\alpha_1 = 1$ and $\beta_0 = -\beta_1$ are valid the data generating process for y_t is given by $y_t = y_{t-1} + \varepsilon_t + \beta_0 v_t$ which has non constant variance given by $t(\sigma_\varepsilon^2 + \beta_0^2 \sigma_v^2)$.
- So the econometric model should have better fit than the autoregressive model , or it is less dynamically mis-specified than the autoregressive model and, the forecasts for the econometric model are better that the ones obtained using the autoregressive model or the non constant variance of the econometric model need to be tested.

Example using Simulated Data - Continuation

- To estimate an $AR(1)$ for y using OLS. The following commands in R have to be used.
  - `# estimate an AR(1)`
  - DLM

Leading Indicators

- The D. G. P. is given by:

$$y_t = \beta_1 x_{t-1} + \varepsilon_t \tag{4}$$

- and it can be used to obtain a one-step ahead forecast for y_t because x_t leads y_t by one period.
- In the absence of a behavioral theory it is less probable that β_1 be constant.
- So the forecast with this model will not be good in moments of structural breaks.
- Note that the variable y_{t-1} was excluded a priori and this exclusion made implies in a dynamically mis-specified model

Example using Simulated Data - Continuation

- To estimate a model for CONS using INC as leading indicator. The following commands in R have to be used.
  - `# estimate a leading indicator model`
  - DLM

Rate of Change Model I

- Economic Series have an underline deterministic or stochastic trends. So before any statistical analysis on the date we need to differentiate the series in order to induce stationarity.
- There are some transformations that can take out the trend from the series, for example, ratio between series, which do not exclude the long run relationship between the variables. It will be seen that these transformations are related to the concept of cointegration.
- The rate of change model is given by:

$$\Delta y_t = \beta_0 \, \Delta x_t + \varepsilon_t \tag{5}$$

- and this model does not have the problem of spurious regression which can be present in models with $I(1)$ variables.

Rate of Change Model II

- On the other hand, if the variance of Δx_t is large in relationship to the variance of Δy_t, β_0 will be small even if y_t and x_t cointegrate with $\gamma = 1$.
- Furthermore, even if $y_t = \gamma x_t$ implies that $_t = \gamma \, \Delta x_t$ the reciprocal is not true.
- A priori there is no reason to exclude the level of the series, given that initial disequilibrium cannot be assumed as irrelevant

Example using Simulated Data - Continuation

- To estimate a rate of change model for y and x. The commands in R are given by:
  - `# estimate a rate of change model`
  - DLM

Distributed Lags

- The D. G. P. is given by:

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \tag{6}$$

- Note that the criticism to the static regression also apply here.
- The a priori exclusion of y_{t-1} may, one more time, implies dynamically mis-specification.

Example using Simulated Data - Continuation

- The distributed lags model for y and x can be estimated using the following commands in R
  - `# estimate the distributed lag model`
  - DLM

- The D. G. P. is given by:

$$y_t = \beta_0 x_t + \alpha_1 y_{t-1} + \varepsilon_t \tag{7}$$

- This model can be obtained through an adjustment process where long-run desired relationship is given by $y_t^* = \gamma x_t$ and the cost of adjustment are quadratic.
- Another way to get the partial adjustment model is to assume that the variable y_t is the realized variable and y_t^* is the desired long-run and there is the factor x_t.
- The long-run equilibrium relationship is given by:

$$y_t^* = \gamma x_t \tag{8}$$

Partial Adjustment II

- The adjustment between the observable and desired is given by:

$$y_t - y_{t-1} = \phi(y_t^* - y_{t-1}) + \varepsilon_t \tag{9}$$

- Using (9) is possible to explain the variable y_t^* as a function of y_t and y_{t-1} in the following way:

$$y_t^* = \frac{1}{\phi} y_t + (\frac{\phi - 1}{\phi}) y_{t-1} - \frac{1}{\phi} \varepsilon_t \tag{10}$$

- and substituting (10) into (8) we have:

$$\frac{1}{\phi} y_t + (\frac{\phi - 1}{\phi}) y_{t-1} - \frac{1}{\phi} \varepsilon_t = \gamma x_t \Rightarrow y_t = (1 - \phi) y_{t-1} + \phi \gamma x_t + \varepsilon_t$$

Example using Simulated Data - Continuation

- To estimate the partial adjustment for y and x, we have the following commands in R
  - `# estimate the partial adjustment model`
  - DLM

Static Regression with AR(1) Errors or Common Factor Model I

- Written (1) in lag operator we have:

$$(1 - \alpha_1 L)y_t = \beta_0(1 + \frac{\beta_1}{\beta_0}L)x_t + \varepsilon_t \tag{11}$$

- Now if and only if $\alpha_1 = -\frac{\beta_1}{\beta_0}$ or $\alpha_1\beta_0 + \beta_1 = 0$, the two polynomials $(1 - \alpha_1 L)$ and $(1 + \frac{\beta_1}{\beta_0}L)$ have the same root, i. e., there is a common factor in the two polynomials and, therefore we can divide both sides of (11) by $(1 - \alpha_1 L)$ to have:

$$y_t = \beta_0 x_t + \frac{\varepsilon_t}{(1 - \alpha_1 L)} \tag{12}$$

Static Regression with AR(1) Errors or Common Factor Model II

- and letting $\rho = \alpha_1$, we can rewrite (12) in the following way:

$$\begin{aligned} y_t &= \beta_0 x_t + u_t \\ u_t &= \rho u_{t-1} + \varepsilon_t \end{aligned} \tag{13}$$

- which is a static regression with autocorrelated errors.
- Note that $(1 - \alpha_1 L)$ is one (and the only) factor of the polynomial $A(L)$ and similarly $(1 + \frac{\beta_1}{\beta_0} L)$ is one (and the only) factor of the polynomial $B(L)$, and when there are equal the two polynomials $A(L)$ and $B(L)$ have a common factor.
- Because (13) impose a testable restriction in the model $ADL(1,1)$ and if this restriction is rejected the sub-model is an invalid reduction of the general model.

Static Regression with AR(1) Errors or Common Factor Model III

- If the restriction is valid the static regression with autocorrelated error is a valid submodel of the general *ADL*.
- Note that the methodology **Specific to General (STGE)** which starts with a static regression and test autocorrelation in the error and this hypothesis is not rejected, then the model is corrected by autocorrelation in the errors. This will impose common factor restrictions that could be invalid.
- If the estimate the static regression by OLS without taking into account the autocorrelation of the errors, the estimates will still be consistent but they are inefficient.

Static Regression with AR(1) Errors or Common Factor Model IV

- Consider the estimation of (13) and note that in this case the variance-covariance matrix for the errors is not scalar and is given by:

$$Var(\mathbf{u}) = \sigma^2\Omega \Rightarrow \Omega = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & . & . & . & \rho^{T-1} \\ \rho & 1 & \rho & . & . & . & \rho^{T-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \rho^{T-2} & \rho^{T-3} & . & . & \rho & 1 & \rho \\ \rho^{T-1} & \rho^{T-2} & . & . & . & \rho & 1 \end{bmatrix} \tag{14}$$

- and the estimation of the parameters are obtained by Generalized least Squared (G.L.S.), i. e.

$$\widehat{\beta}_0 = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y \tag{15}$$

Static Regression with AR(1) Errors or Common Factor Model V

- Note that in (15), we are implicitly assuming that ρ is known which is not true in practice.
- If $\widehat{\rho}$ is an estimator of ρ, the estimator of $\widehat{\beta}_0$ is given by $(X'\widehat{\Omega}^{-1}X)^{-1}X'\widehat{\Omega}^{-1}y$ where $\widehat{\Omega}$ is given by:

$$\widehat{\Omega} = \frac{1}{1-\widehat{\rho}^2}\begin{bmatrix} 1 & \widehat{\rho} & \widehat{\rho}^2 & . & . & . & \widehat{\rho}^{T-1} \\ \widehat{\rho} & 1 & \widehat{\rho} & . & . & . & \widehat{\rho}^{T-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \widehat{\rho}^{T-2} & \widehat{\rho}^{T-3} & . & . & \widehat{\rho} & 1 & \widehat{\rho} \\ \widehat{\rho}^{T-1} & \widehat{\rho}^{T-2} & . & . & . & \widehat{\rho} & 1 \end{bmatrix} \tag{16}$$

Static Regression with AR(1) Errors or Common Factor Model VI

- One possible estimator of ρ is

$$\widehat{\rho} = \frac{\sum_{t=1}^{T} \widehat{u}_t \widehat{u}_{t-1}}{\sum_{t=1}^{T} \widehat{u}_{t-1}^2} \tag{17}$$

- where $\widehat{u}_t = y_t - \widehat{\beta}_0 x_t$.
- Note that the estimator of β_0 depends upon the estimator of ρ and vice-versa.

Static Regression with AR(1) Errors or Common Factor Model VII

- Another form to obtain these estimators is using Weighted Least Squared since the inverse of the matrix Ω does have a close form and is given by:

$$\Omega^{-1} = \begin{bmatrix} 1 & -\rho & 0 & \dots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \dots & 0 & -\rho & 1 \end{bmatrix} \quad (18)$$

Static Regression with AR(1) Errors or Common Factor Model VIII

- We known that $\Omega^{-1} = P'P$ where P is given by:

$$P = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\rho & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\rho & 1 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & -\rho & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -\rho & 1 \end{bmatrix} \tag{19}$$

Static Regression with AR(1) Errors or Common Factor Model IX

- which implies in the following transformation of the data:

$$y^* = \begin{bmatrix} \sqrt{1-\rho^2} \cdot y_1 \\ y_2 - \rho y_1 \\ . \\ . \\ . \\ y_T - \rho y_{T-1} \end{bmatrix} \tag{20}$$

- and

$$x^* = \begin{bmatrix} \sqrt{1-\rho^2} \cdot x_1 \\ x_2 - \rho x_1 \\ . \\ . \\ . \\ x_T - \rho x_{T-1} \end{bmatrix} \tag{21}$$

Static Regression with AR(1) Errors or Common Factor Model X

- Using OLS in the transformed model the estimator is equivalent to GLS.
- Note that if we do not use the first row of P the estimator of β_0 is easily obtained, since it will be the OLS estimator for $t = 2, ..., T$.
- Note also that this estimator is an approximation since we are not using the first observation.
- As note above ρ is unknown and can be estimated by NLLSE in the following transformed model of (13):

$$y_t = \rho y_{t-1} + \beta_0 x_t - \rho \beta_0 x_t + \varepsilon_t \tag{22}$$

- assuming $\varepsilon_t \sim NI(0, \sigma^2)$.
- Nowadays it is easy to estimate (22) using NLLSE but in the past a iterative optimization procedure was needed.

Static Regression with AR(1) Errors or Common Factor Model XI

- This procedure is given by:

1. assuming $\rho = 0$ the consistent estimator of β_0 is the OLS estimator in the static regression of y_t in x_t;
2. save the residuals form the regression in (1), and denoted it by $\widehat{u}_t$;
3. regress $\widehat{u}_t$ into $\widehat{u}_{t-1}$, to obtain an estimator of ρ;
4. transform the variables using (20) and (21) to obtain a new estimator of β_0;
5. repeat items (2)-(4), and stop if there is a small change in the parameters.

Example using Simulated Data - Continuation

- To estimate the static regression with $AR(1)$ errors we have the following commands in R
  - `# estimate the static regression with` $AR(1)$ `errors`
  - DLM

Reduced Form

- The D.G.P. is given by:

$$y_t = \alpha_1 y_{t-1} + \beta_1 x_{t-1} + \varepsilon_t \tag{23}$$

- and in this model the current information about x is excluded. Note that (23) can be interpreted as one of equations of a Vector Autoregressive and we are not specifying the other equation. Thus if the D.G.P. of x_t does not have constant parameters, the parameter (23) will also not be constant.

Example using Simulated Data - Continuation

- To estimate the reduced form for y and x we have the following commands in R
  - `# estimate the reduced form model`
  - DLM

Unrestricted ADL(1,1) Model

- The Unrestricted ADL(1,1) is given by:

$$y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \alpha_1 y_{t-1} + \varepsilon_t \tag{24}$$

- And if the variables are $I(0)$ this is right specification
- If they are $I(1)$ we need to reparametrized the model into a Equilibrium Correction Model given by

$$\Delta y_t = \beta_0 \Delta x_t + (\alpha_1 - 1)(y_{t-1} - \frac{\alpha_0}{1-\alpha_1} - \frac{\beta_0 + \beta_1}{1-\alpha_1} x_{t-1}) + \varepsilon_t \tag{25}$$

Example using Simulated Data - Continuation

- Starting with an ADL(1,1) for y and x, we have the following results using the following commands in R
  - `# estimate an ADL(1,1) for y and x`
  - DLM

Unrestricted ADL(1,1) Model - Unit Root Test for y and x

- Testing unit roots for y and x to check Equilibrium Correction Model
  - `# testing unit roots for y`
  - DLM

Unrestricted ADL(1,1) Model - Equilibrium Correction Model Specification

- In order to rewrite the Unrestricted ADL(1,1) Model as an Equilibrium Correction Model we need to test the following hypotheses

$$H_0 : \alpha_1 = 1 \text{ and } \beta_0 + \beta_1 = 0 \tag{26}$$

- Testing these hypotheses
  - `# Testing if the Error Correction Specification is valid`
  - DLM

Unrestricted ADL(1,1) Model - Omitted Variable - Dummy and (Dummy*x)

- Testing omitted variables
  - `# Testing omitted variables dummy and (dummy*x)`
  - DLM

Unrestricted ADL(1,1) Model with dummies

- The Unrestricted ADL(1,1) with dummies is given by:

$$y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \alpha_1 y_{t-1} + \delta_1 D_t + \delta_2 D_t x_t + \varepsilon_t \qquad (27)$$

Example using Simulated Data - Continuation

- The ADL(1,1) for y and x with dummies , we have the following results using the following commands in R
  - `# estimate an ADL(1,1) for y and x and dummies`
  - DLM

Model Selection for all models

Model	AIC	SCHWARZ
Static Regression	7.315891	7.348990
AR(1)	7.218772	7.251871
Leading Indicator	7.494658	7.527757
Rate of Change - Right	6.797012	6.830111
Distributed Lags	6.378449	6.428097
Partial Adjustment	5.499322	5.548970
Static Regression with AR(1) Errors	6.696042	6.762239
Reduced Form Model	7.183311	7.232959
Unrestricted ADL	5.129634	5.195831
Unrestricted ADL with dummies	**2.943383**	**3.042678**

- Let us illustrate forecasting with dynamic models using the ADL(1,1) specification, i.e.:

$$y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \alpha_1 y_{t-1} + \varepsilon_t \quad \text{with} \quad \varepsilon_t \overset{iid}{\sim} (0, \sigma_\varepsilon^2) \quad (28)$$

- we make the additional assumptions that $\alpha_0 = 0$ (the mean of all variables are zero) and $\alpha_1 = 0$ (only exogenous variables are present) and the model simplies to an AD(0,1) that is:

$$y_t = \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \quad (29)$$

- Let the variables be grouped as

$$\mathbf{z}_t = (x_t, x_{t-1}) \quad (30)$$

Forecasting with Dynamic Models II

- and the parameters into

$$\beta = (\beta_0, \beta_1)' \tag{31}$$

- Model (29) is rewrite as

$$y_t = \mathbf{z}_t \beta + \varepsilon_t \tag{32}$$

- Can use the same reasoning as in the linear regression model to obtain the optimal (in the MSFE sense) forecast for y_{T+h} that is

$$\widehat{y}_{T+h} = \mathbf{z}_{T+h} \widehat{\beta} \tag{33}$$

- where $\widehat{\beta}$ is the *OLS* estimator of β

Forecasting with Dynamic Models III

- If future values of $\mathbf{z}_{T+h}$ are unknown, they should be replaced by forecasts using the marginal model for x, and in this case we need that x be strong exogenous (see [Engle et al., 1983] or [Valls Pereira, 2024]) for the parameters of interest
- Let add y_{t-1} back and write the ADL(1,1) as

$$y_t = \mathbf{z}_t \beta + \alpha_1 y_{t-1} + \varepsilon_t$$

- and the optimal one-step ahead forecast is

$$\widehat{y}_{T+1} = \mathbf{z}_{T+1} \widehat{\beta} + \widehat{\alpha}_1 y_T$$

- for the two-step ahead forecast we have

$$\widehat{y}_{T+2} = \mathbf{z}_{T+2} \widehat{\beta} + \widehat{\alpha}_1 y_{T+1}$$

- but y_{T+1} is unknown and we replace by its conditional expectation given all the information up to T which coincides with $\widehat{y}_{T+1}$, hence:

$$\widehat{y}_{T+2} = \mathbf{z}_{T+2}\widehat{\beta} + \widehat{\alpha}_1 \widehat{y}_{T+1}$$

- and the optimal $h-$step ahead forecast is

$$\widehat{y}_{T+h} = \mathbf{z}_{T+h}\widehat{\beta} + \widehat{\alpha}_1 \widehat{y}_{T+h-1} \tag{34}$$

- The forecast error is given by:

$$e_{T+h} = y_{T+h} - \widehat{y}_{T+h} \tag{35}$$

- and assuming $\mathbf{z}_{T+h}$ and the parameters are known, there is not estimation uncertainty we can write:

$$e_{T+h} = \alpha_1 (y_{T+h-1} - \widehat{y}_{T+h-1}) + \varepsilon_{T+h} = \alpha_1 e_{T+h-1} + \varepsilon_{T+h} \tag{36}$$

Forecasting with Dynamic Models V

- the presence of an autoregressive component in the model creates correlation in the $h-$step ahead forecast error.
- If in (36) we replace e_{T+h-1} with its expression we obtain:

$$e_{T+h} = \alpha_1^2 e_{T+h-2} + \alpha_1 \varepsilon_{T+h-1} + \varepsilon_{T+h}$$

- and by repeat substitution yields:

$$e_{T+h} = \alpha_1^{h-1} \varepsilon_{T+1} + \cdots + \alpha_1 \varepsilon_{T+h-1} + \varepsilon_{T+h} \tag{37}$$

- since $e_{T+h} = 0$ for $h \leq 0$ and (37) is a $MA(h-1)$ representation for forecast error e_{T+h}.

Forecasting with Dynamic Models VI

- If the parameters are unknown as well as the future values of x then the forecast error becomes:

$$\begin{aligned} e_{T+h} &= (\mathbf{z}_{T+h}\beta - \widehat{\mathbf{z}}_{T+h}\widehat{\beta}) + (\alpha_1 y_{T+h-1} - \widehat{\alpha}_1 \widehat{y}_{T+h-1}) + \varepsilon_{T+h} \\ &= \mathbf{z}_{T+h}(\beta - \widehat{\beta}) + (\mathbf{z}_{T+h} - \widehat{\mathbf{z}}_{T+h})\widehat{\beta} + (\alpha_1 - \widehat{\alpha}_1) y_{T+h-1} \\ &\quad + \widehat{\alpha}_1 e_{T+h-1} + \varepsilon_{T+h} \end{aligned} \tag{38}$$

- Expressions for the variance of the forecast error when the parameters are unknown are complicated, but can use state space representation to obtain.
- Assuming normality and if the parameters and futures values for x are known then we have:

$$\left(\frac{y_{T+h} - \widehat{y}_{T+h}}{\sqrt{Var(e_{T+h})}} \right) \sim N(0,1) \tag{39}$$

Forecasting with Dynamic Models VII

- which implies:

$$y_{T+h} \sim N(\widehat{y}_{T+h}, Var(e_{T+h})) \tag{40}$$

Example using Simulated Data - Forecasting

- We used the following commands in R
  - `# Forecasting for the best model`
  - `# Data Generating Process:`
  - Aula3_2022

Example using Simulated Data - Dynamic Forecasting

- We used the following commands in R
  - `# dynamic forecast`
  - Aula3_2022

Example using Simulated Data - Comparing Static and Dynamic Forecasting

- We used the following commands in R
  - `# estimate model forecasting performance`

	Forecasting	Method
	Static	Recursive
RMSE	0.2993306	**0.2888852**
MAFE	0.2393963	**0.2366004**

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