Redes de Computadores

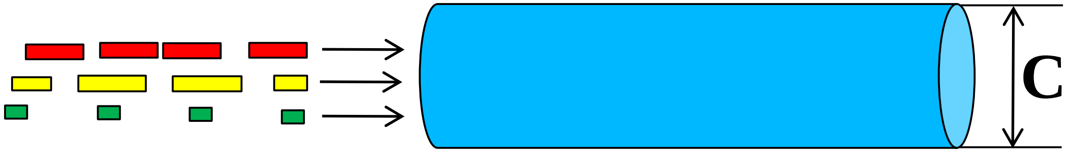
Delay Models in Computer Networks

Manuel P. Ricardo

Faculdade de Engenharia da Universidade do Porto

- *What are the common multiplexing strategies?*
- *What is a Poisson process?*
- *What is the Little theorem?*
- *What is a queue?*
- *What is the meaning of service time $1/\mu$ in a queue of packets?*
- *What is the meaning of traffic intensity ρ in a queue model?*
- *What is the probability of a M/M/1 queue being in a given state n ?*
- *What is the mean number of clients in a M/M/1 queue? What is the mean waiting time in a M/M/1 queue? What is the relationship between N and ρ in a M/M/1 queue?*
- *What are the differences between M/M/1 and M/G/1 queues? How to estimate mean number of packets and mean delay in a M/G/1 queue?*
- *How to model a network of transmission lines? How to calculate the mean number of packets and mean delay in this case?*
- *What is a Jackson Network? Why is it important?*

Multiplexing Traffic on a Link



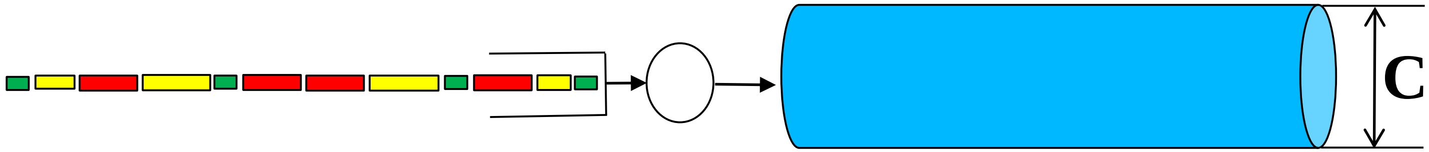
- Communication link
  - Bit pipe with a given capacity C (bit/s)
  - Link capacity → rate at which bits are transmitted to the link
  - Link may transport multiplexed traffic streams

- Multiplexing strategies
  - Statistical Multiplexing
  - Frequency Division Multiplexing
  - Time Division Multiplexing

- Multiplexing strategy affects traffic delay

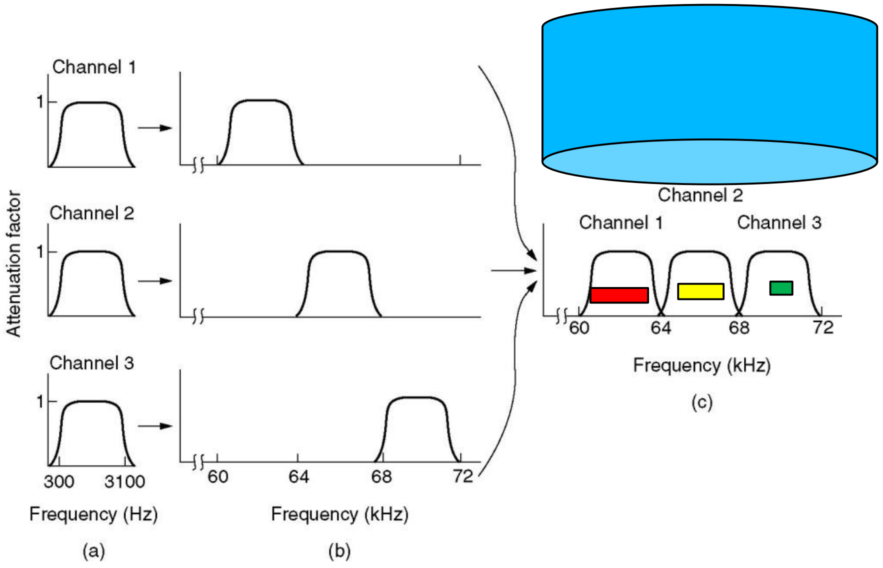
Statistical Multiplexing

- Packets of all traffic streams merged in a single queue
- Packets transmitted on a first-come first-served basis
- Time required to transmit a packet of length L ➔ $T_{frame}=L/C$



FDM – Frequency Division Multiplexing

- Link capacity C subdivided into ***m*** portions
- Channel bandwidth W subdivided into m channels of W/m Hz
- Capacity of each channel ➔ C/m
- Time required to transmit a packet of length L ➔ $T_{frame} = Lm/C$

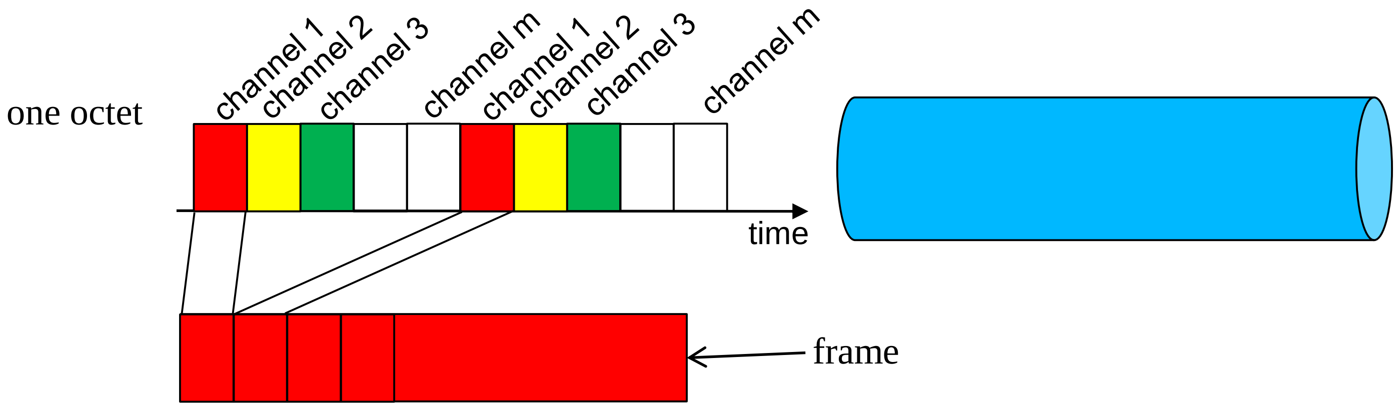


TDM – Time Division Multiplexing

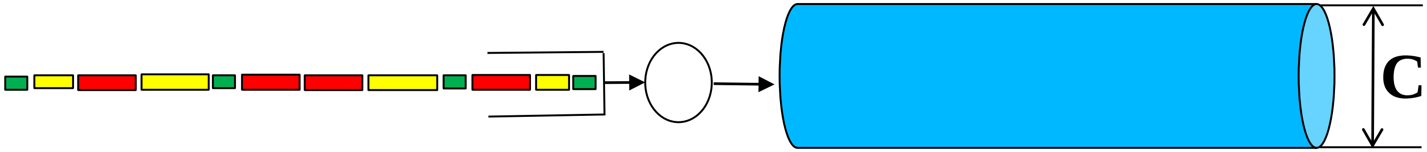
- Time axis divided into ***m*** slots of fixed length

  (usually one octet long)

- Communication ➔ m channels with capacity C/m
- Time required to transmit a packet of length L ➔ T_{frame}=Lm/C

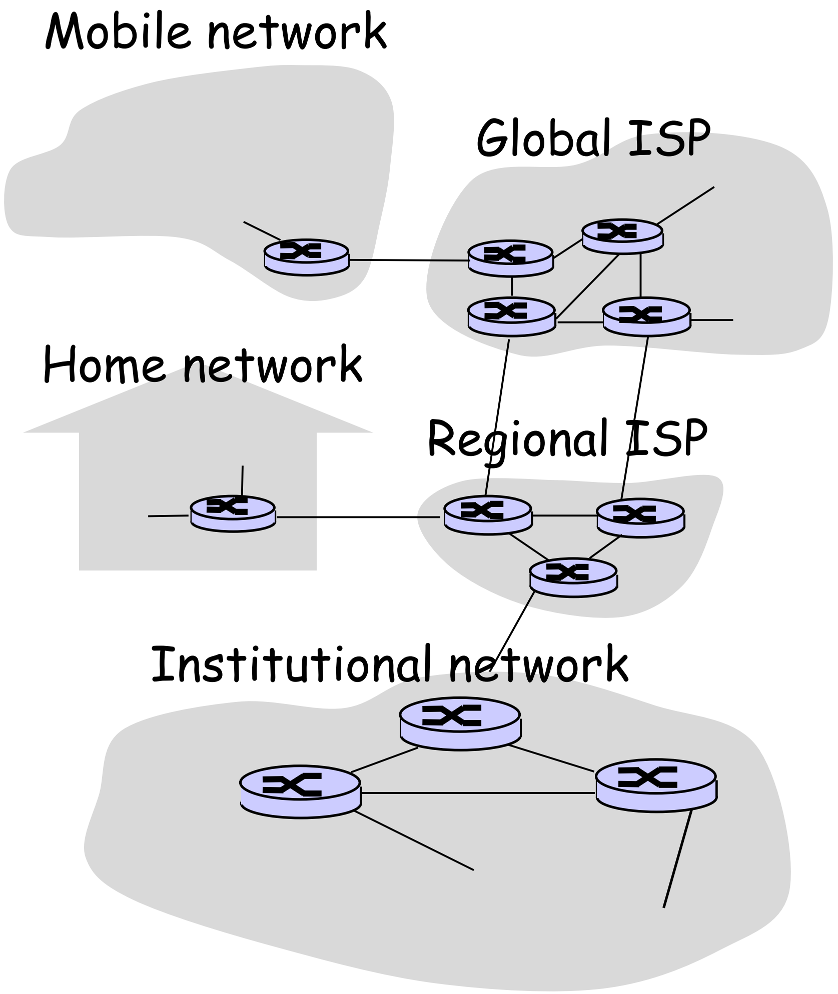


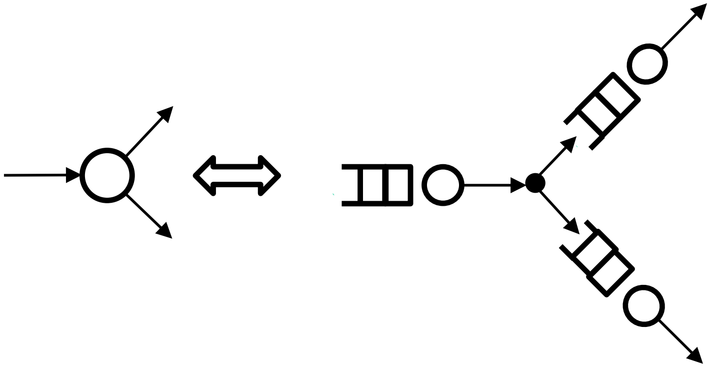
Delay on Computer Networks

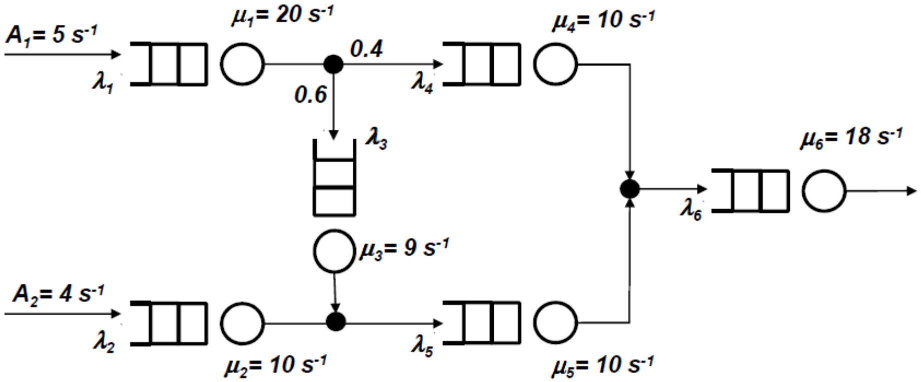


- Delay
  - Important performance parameter in computer networks
  - Characterized using queue models
- Queue model
  - Customers arrive at random times to obtain service
  - Customer ➔ packet to be transmitted through a link
  - Serve a packet = transmit a packet
  - Service time ➔ **packet transmission time =$T_{pac(frame)}$= L/C**
- Queue models enable the quantification of
  - Average number of customers/packets in the network
  - Average delay per packet ➔ waiting plus service times

Computer Networks Modeled as Queue Networks





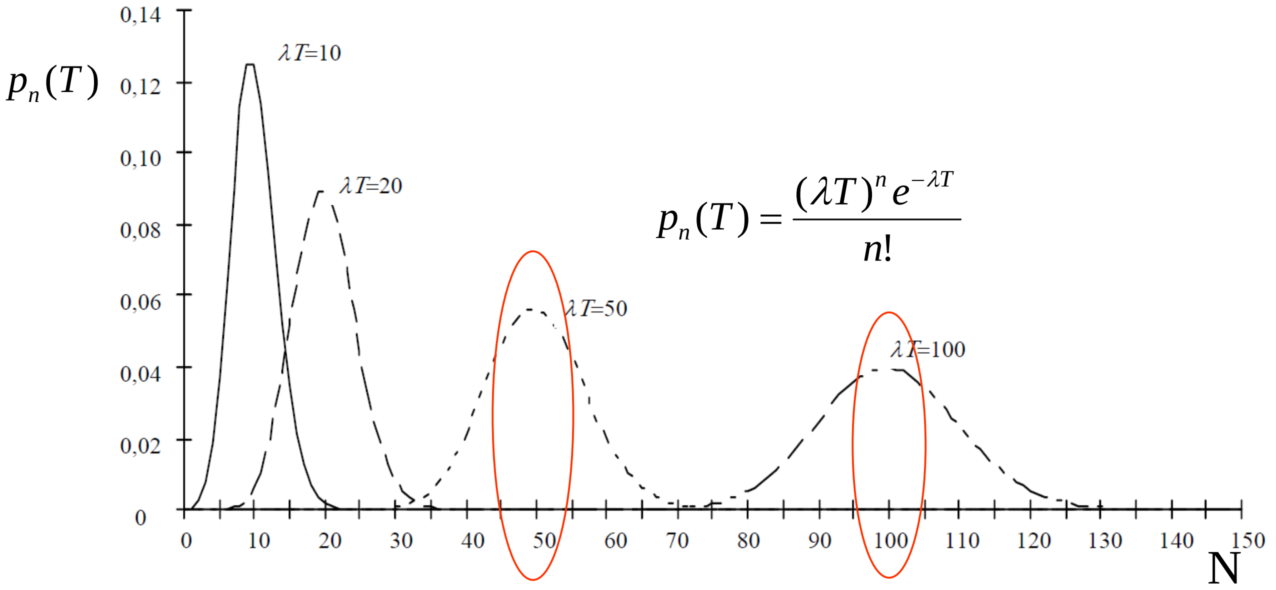


Poisson Distribution and Poisson Process

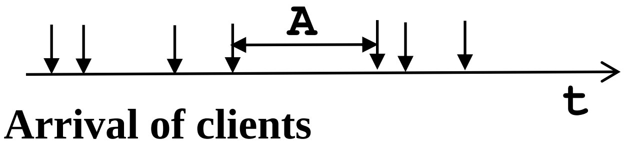
- Poisson distribution with parameter m

$$P[N=n]=p_n=\frac{m^n e^{-m}}{n!}, \quad n=0,1,\ldots \qquad E[N]=Var[N]=m$$

- Poisson process
  - » λT=m , (e.g. λ → arrivals/s)
  - » P[n arrivals in interval T] = $p_n(T)=p_n=\dfrac{(\lambda T)^n e^{-\lambda T}}{n!}$ $\qquad E[N]=Var[N]=\lambda T$



Inter-Arrival Interval A – Statistical Characterization



Arrival of clients

A - time interval between the arrival of consecutive clients

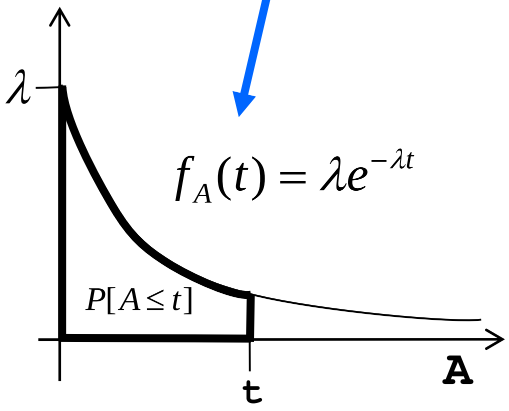
$$F_A(t) = P[A \le t] = 1 - P[A > t] = 1 - p_0(t) = 1 - e^{-\lambda t}$$

$$f_A(t) = pdf = \frac{\partial F_A(t)}{\partial t} = \lambda e^{-\lambda t}$$

Exponential distribution

$$E[A] = 1/\lambda$$

$$Var[A] = 1/\lambda^2$$



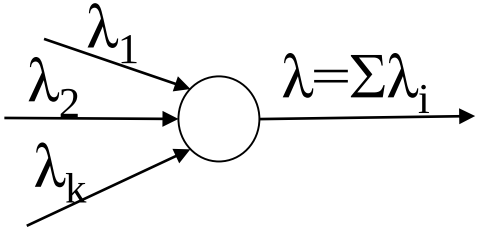
- What is the difference between

  Deterministic arrivals and

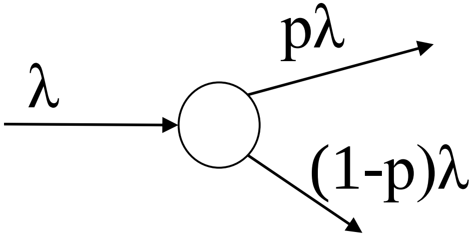
  Poisson arrivals?

Markov Process - Properties

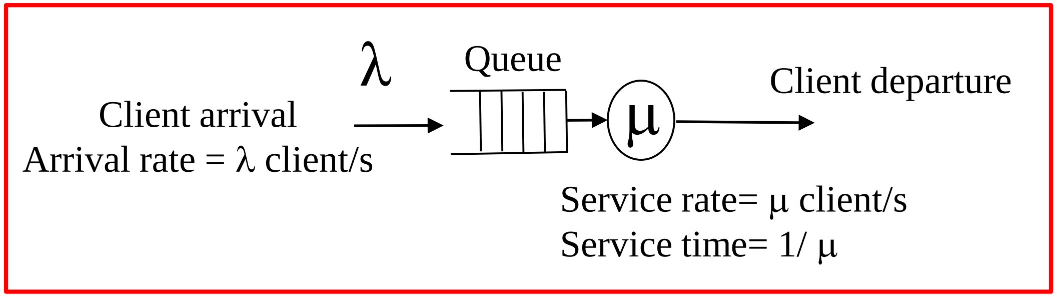
- Merging Property
  - » A_1, A_2, … A_k are independent Poisson Processes with rates λ_1, λ_2, … λ_k
  - » A=Σ A_i still is a Poisson process, with rate $\lambda=\Sigma\lambda_i$



- Splitting property
  - » Packets arrive to a router according to a Poisson Process (A,λ)
  - » They are routed randomly to two output lines with probabilities **p** and **1-p**
  - » Packets leaving the router still are Poisson Processes, characterized by (A,pλ) and (A,(1-p)λ)



Queue Model



λ

Queue

Client departure

Client arrival
Arrival rate = λ client/s

μ

Service rate= μ client/s
Service time= 1/ μ

- Queue – model used for
  - Customers waiting in line
  - Packets in a network

- Used to determine
  - Average number of clients in the system → N
  - Average delay experienced by a client → T

- Queue characterized in terms of
  - λ - arrival rate of client (average number of clients per time unit)
  - μ - service rate (average number of clients the server processes per time unit)
  - ρ=λ/μ – traffic intensity (occupation of the server)

- Kendall notation ➔ **A/S/s/K**
  - A – arrival statistical process
  - S – service statistical process
  - s – number of servers
  - K – capacity of the system in buffers

Little's Theorem

- **$N=\lambda T$**
  - » N- average number of clients in a system
  - » T – average amount of time a client spends in the system
  - » λ – arrival rate of clients to the system

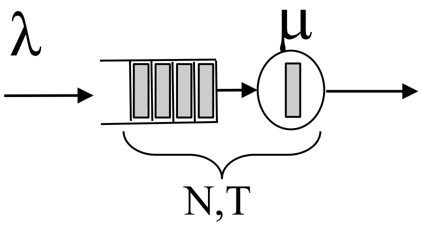
- $T=T_w+T_s$
  - » T_w – time a client waits in the queue for being served
  - » T_s – service time
- $N=N_w+N_s$
  - » N_w – number of clients waiting in the queue for being served
  - » N_s – number of clients being served
- **$N_w=\lambda T_w$**

$$N_w = \lambda T_w \rightarrow T_w = N_w / \lambda$$

- The (mean) time a client has to wait before being served (T_w) depends on the number of clients waiting (N_w) and on the arrival rate of clients (λ)
- No dependence on the service rate?!
- Can you explain it?

Little's Theorem

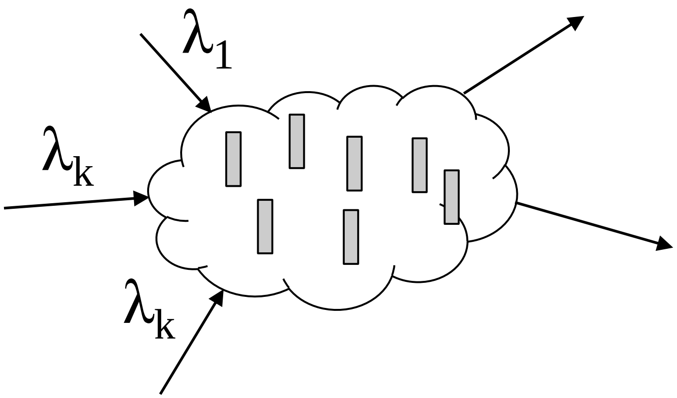
- Can be applied to a single Queue



- Can be applied to a complex system
  - » For each stream i → $N_i=\lambda_i T_i$
  - » For the system:

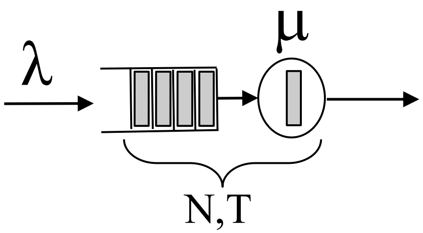
    $\lambda=\Sigma\,\lambda_i$ $N=\Sigma\,N_i$

    $T=(\Sigma\,N_i)/(\Sigma\,\lambda_i)$ → $T=N/\lambda$



M/M/1 Queue

- M/M/1
  - » Poisson arrival, exponential service time

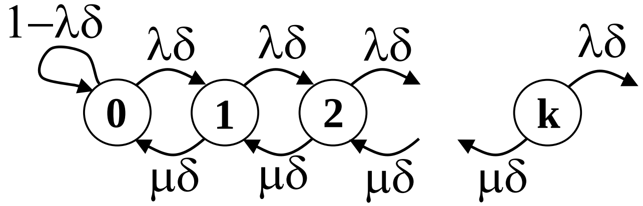


- Modeled by a Markov Chain
  - » State **k** - k clients in the queue
  - » p(i,j) – probability of transition from state i to state j
  - » When δ → 0

p(i, i+1)= λδ p(i, i-1)= μδ

p(i, i)= 1- λδ–μδ p(0, 0)= 1- λδ

p(i, j)=0 for other values i, j



  - » Birth-death chain
    - – Transitions between adjacent states
    - – λδ and μδ become flow rates between states

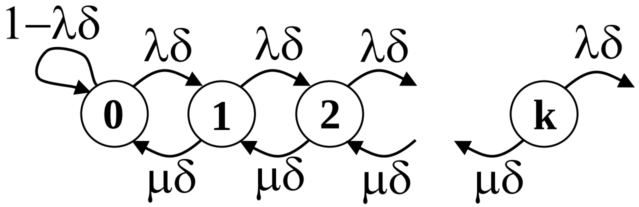
$$p(i,i+1) = p_1(\delta) = (\lambda\delta)e^{-\lambda\delta} \approx \lambda\delta$$

$$p(0,0) = p_0(\delta) = e^{-\lambda\delta} \approx 1-\lambda\delta$$

M/M/1 Queue – Equilibrium Analysis

- P(j) – probability of the Markov chain be in state j
- Markov Chain - global balance equations

$$P(j)\sum_{\substack{i=0\\ i\neq j}}^{\infty} p(j,i) = \sum_{\substack{i=0\\ i\neq j}}^{\infty} P(i)\,p(i,j)$$



- In the case of M/M/1

$$P(0)\lambda\delta = P(1)\mu\delta \quad \Rightarrow P(1) = \rho P(0)$$

$$P(2) = \rho P(1) = \rho^2 P(0)$$

$$P(n) = \rho^n P(0)$$

$$\sum_{i=0}^{\infty} P(i) = 1$$

$$\sum_{i=0}^{\infty} \rho^i P(0) = \frac{P(0)}{1-\rho} = 1$$

$$P(0) = 1-\rho$$

$$\boxed{P(n) = \rho^n(1-\rho)}$$

M/M/1 Queue

- Average Queue size N

$$N=\sum_{n=0}^{\infty}nP(n)=\sum_{n=0}^{\infty}n\rho^{n}(1-\rho)=\frac{\rho}{1-\rho} \qquad N=\sum_{n=0}^{\infty}nP(n)=\frac{\rho}{1-\rho}=\frac{\lambda/\mu}{1-\lambda/\mu}=\frac{\lambda}{\mu-\lambda}$$

- Average amount of time the client spends in the system, T
  - » Little’s formula, T=N/λ ➔ $T=\dfrac{1}{\mu-\lambda}$

- Average waiting time $\mathrm{T_w}$ ➔ $T_w=T-T_s=\dfrac{1}{\mu-\lambda}-\dfrac{1}{\mu}=\dfrac{\rho}{\mu(1-\rho)}$

- Average number of clients waiting in the queue, $\mathrm{N_w}$

$$N_w=T_w\lambda=\frac{\lambda}{\mu-\lambda}-\frac{\lambda}{\mu}=N-\rho$$

M/M/1 Queue – N=f(ρ)

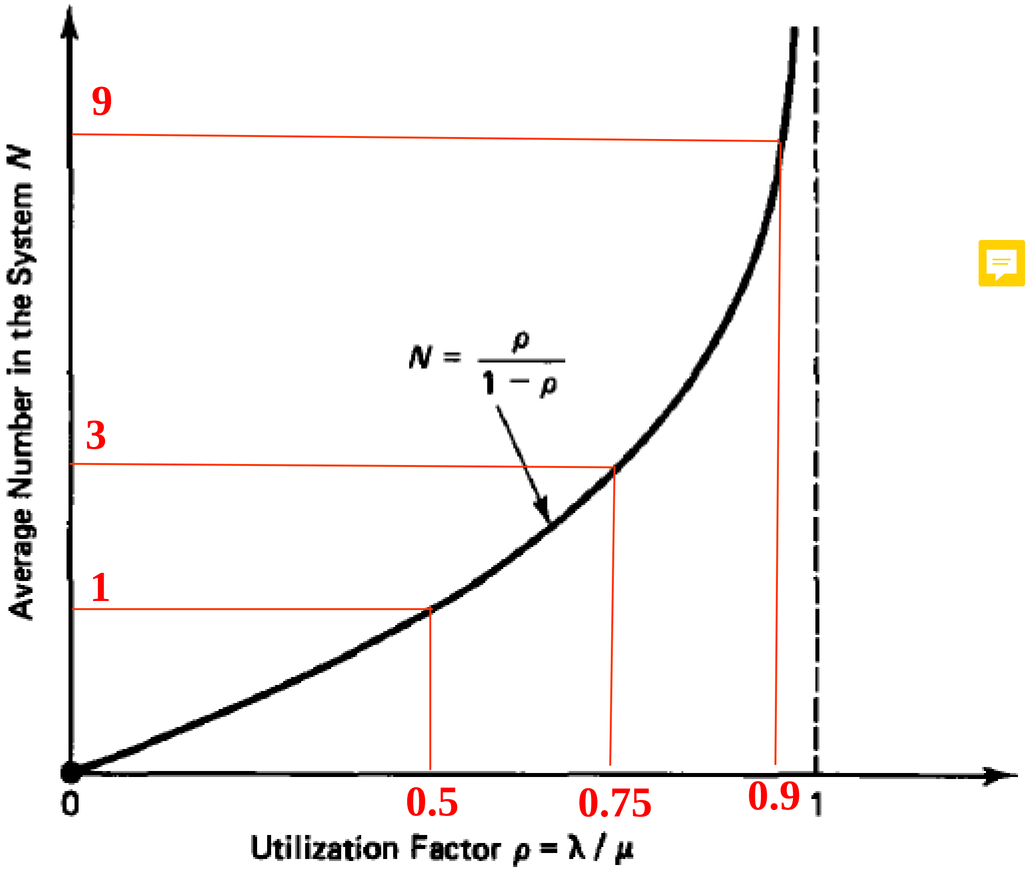


Figure 3.6 **The average number in the system versus the utilization factor in the *M*/*M*/1 system. As $\rho \rightarrow 1$, $N \rightarrow \infty$.**

- M/M/1: $\rho=0.9$ → $N=9$

- Why have clients to wait if the server is busy only 90% of his time?

- What would happen for D/D/1, $\rho=0.9$?

Packet Length, Service Time, Speed

- » 100 packet/s are required to be transmitted through a link
- » Packets arrive according to a Poisson process
- » Packet lengths are exponentially distributed ➔ $E[L]=10^4$ bit/packet
- » Link has capacity C=10 Mbit/s

- ♦ Then
  - » Arrival rate: $\lambda=100$ packet/s
  - » Service rate: $\mu=C/E[L]= 10^7/10^4= 10^3$ packet/s
  - » $\rho=\lambda/\mu=0.1$, $N=\rho/(1-\rho)=1/9$, $T=N/\lambda=1/900$ s

- ♦ Assume now: **$\lambda'=10\lambda$** and **$C'=10C$** ➔ **$\mu'=10C/E[L]=10\mu$**
  - » Then $\rho'=\rho$ and $N'=N$ but $T'=N'/\lambda'=T/10$

    The speed of the system increases!

M/M/1/B Queue

- M/M/1 queue has limited capacity (B buffers)
  - » Packets can be lost
  - » Probability of packet being lost = P(B) ➔ Queue is full
- Analysis similar to M/M/1

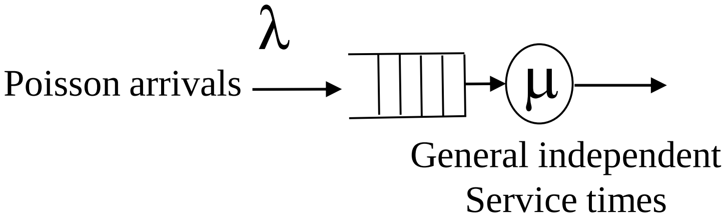
$$\sum_{i=0}^{B} P(i) = 1 \qquad P(n) = \rho^n P(0)$$

$$P(0) = \frac{1-\rho}{1-\rho^{B+1}} \qquad P(B) = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}}$$

- Particular cases

$$\rho = 1, \quad P(B) = \frac{1}{B+1} \qquad \rho >> 1, \quad P(B) \approx \frac{\rho - 1}{\rho} = \frac{\lambda - \mu}{\lambda}$$

M/G/1 Queue



- Poisson arrivals at rate λ
- Service time X has arbitrary distribution with given E[X] and E[X^2]
  - » Service times Independent and Identically Distributed (IID)
  - » Independent of arrival times
  - » E[service time] =E[X]= 1/μ
  - » Single Server queue

M/G/1 Queue – Pollaczek-Khinchin (P-K) Formula

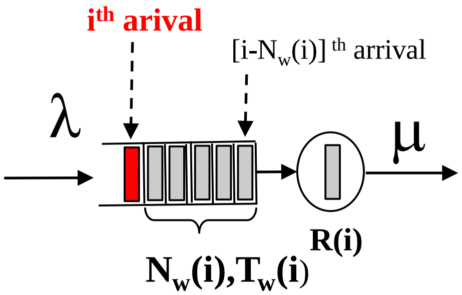
$$\boxed{T_w = \frac{\lambda E[X^2]}{2(1-\rho)}}$$

- where ρ = λ/μ = λE[X] = line utilization
- From Little’s Theorem
  - » $N_w = \lambda T_w$
  - » $T = T_w + E[X] = T_w + 1/\mu$
  - » **$N = \lambda T$**$= \lambda(T_w + 1/\mu\,)$**$= N_w + \rho$**

M/G/1 Queue – Proof of (P-K) Formula

$$\boxed{T_w = \frac{\lambda E[X^2]}{2(1-\rho)}}$$

- Let
  - $T_w(i)$ - waiting time in queue of ith arrival
  - R(i) – residual service time seen by the ith arrival
  - $N_w(i)$ – number of clients found in queue by the ith arrival
  - X(i) – service time of the ith arrival



$$T_w(i) = \sum_{j=i-N_w(i)}^{i-1} X(j) + R(i)$$

$$E[T_w(i)] = T_w = E[N_w(i)] \times E[X(i)] + E[R(i)] = \frac{N_w}{\mu} + E[R(i)]$$

» Using Little’s formula

$$T_w = \frac{\lambda T_w}{\mu} + E[R(i)] \qquad\qquad T_w = \frac{E[R(i)]}{1-\rho}$$

M/G/1 Queue – Proof of (P-K) Formula

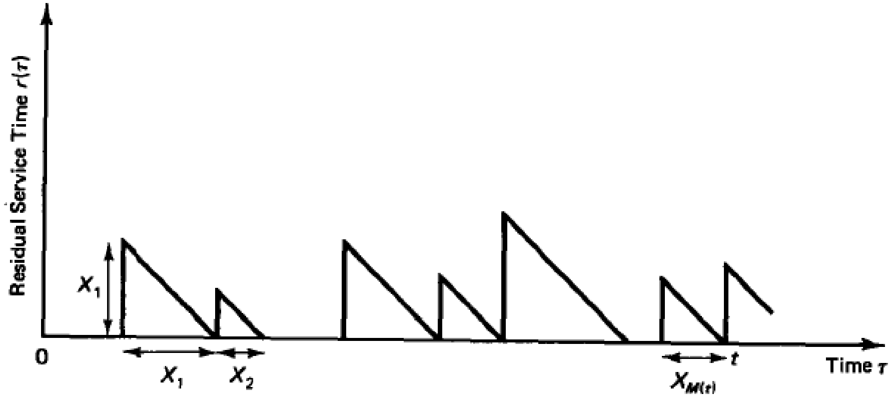


Figure 3.10 Derivation of the mean residual service time. During period $[0, t]$, the time average of the residual service time $r(\tau)$ is

M(t) – number of clients served by time t

$$E\left[R(i)\right] = R_t = \frac{1}{t}\int_0^t r(\tau)\partial\tau = \frac{1}{t}\sum_{i=1}^{M(t)}\frac{X_i^2}{2} = \frac{M(t)}{2t}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)}$$

$t \rightarrow \infty,\quad \dfrac{M(t)}{t} = \lambda$ = arrival rate= departure rate

$$\left.\begin{aligned} E\left[R(i)\right] &= \frac{\lambda}{2}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)} = \frac{\lambda}{2}\times E\left[X^2\right] \\ T_w &= \frac{E\left[R(i)\right]}{1-\rho} \end{aligned}\right\} \quad \boxed{T_w = \frac{\lambda E[X^2]}{2(1-\rho)}}$$

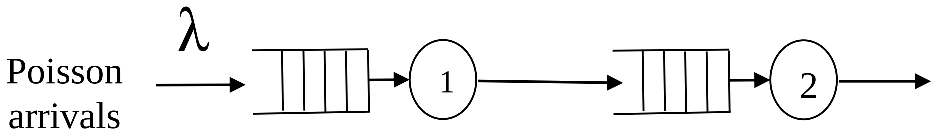
M/G/1 Examples

- Case M/M/1
  - » E[X]= 1/μ ; E[X^2]= 2/μ^2

$$T_w = \frac{\lambda}{\mu^2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}$$

- Case M/D/1
  - » Deterministic, constant service time 1/μ
  - » E[X]= 1/μ ; E[X^2]= 1/μ^2

$$T_w = \frac{\lambda}{2\mu^2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}$$

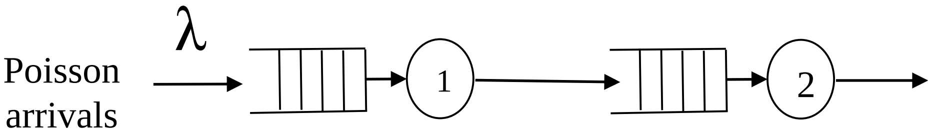


- Assume Queue 1 is M/D/1.

- Can the arrival of packets to Queue 2 be described as a Poisson process?

Networks of Transmission Lines - Problems

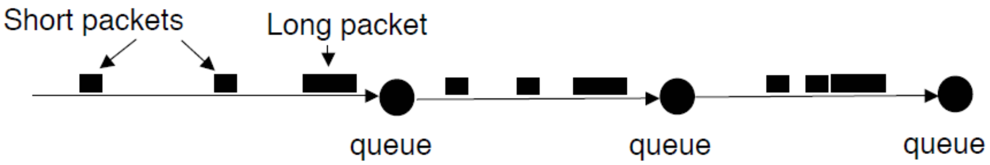
- Case 1



  - » Arrival to Q_1 → Poisson, λ
  - » Assume contant packet length → Q_1= M/D/1
  - » Arrival to Q_2 is not Poisson; $\lambda_2 < \mu_2$ → $1/\lambda_2 > 1/\mu_2$

    → no waiting at Q_2

- Case 2



  - » Q_1=M/M/1
  - » arrival to Q_2 strongly related to packet length
  - » long packets require long service at each node
  - » shorter packets will catch up long packets → interarrival times change

  → Q_2 cannot be modeled as M/M/1

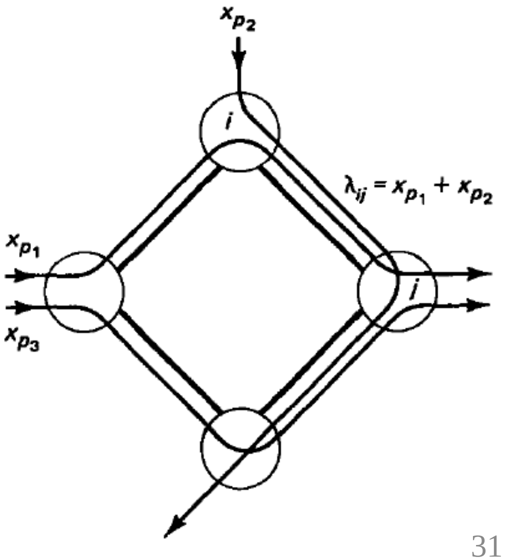
Kleinrock Independence Approximation

- Merging several packet streams on a transmission line

  restores independence of interarrival times and packet lengths

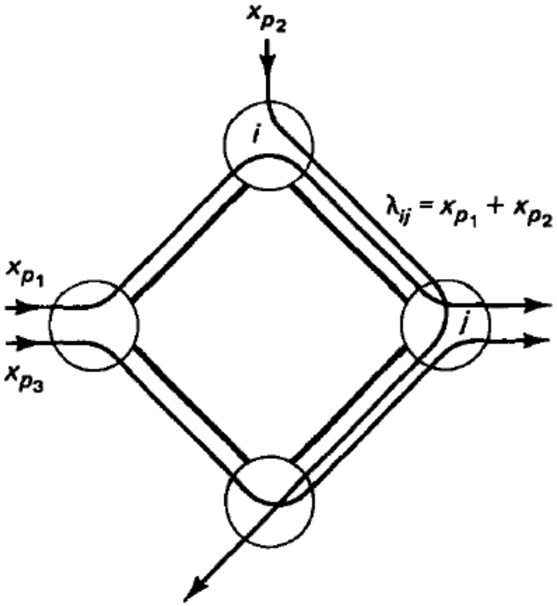
- M/M/1 can be used to model each communication link

- Approximation good for
  - systems involving Poisson stream arrivals at the entry points
  - packet lengths nearly exponentially distributed
  - densely connected networks
  - Moderate to heavy traffic loads



Kleinrock Independence Approximation

- Let
  - » x_p = arrival rate of packets along path p
  - » λ_{ij} = arrival rate of packets to link (i,j)
  - » μ_{ij} = service rate on link (i,j)



- Link queues ➔ <u>independent M/M/1 queues</u>

$$\lambda_{ij} = \sum_{\substack{\text{all p traversing} \\ \text{link (ij)}}} x_p \qquad \rho_{ij} = \frac{\lambda_{ij}}{\mu_{ij}} \qquad N_{ij} = \frac{\rho_{ij}}{1-\rho_{ij}}$$

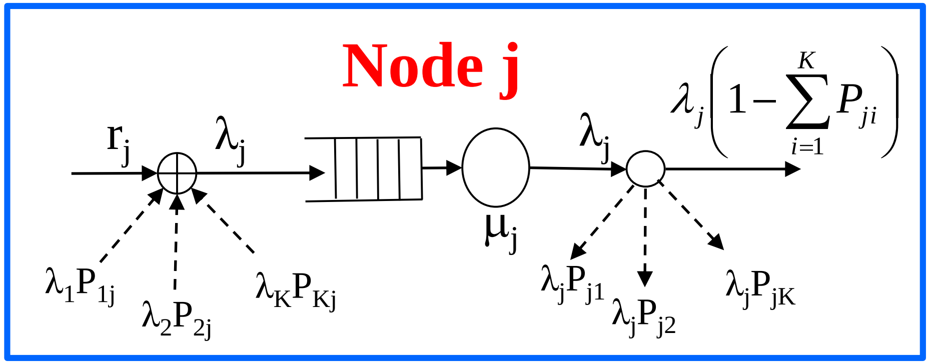
- And
  - » N= Average number of packets in network
  - » T – Average packet delay in network

$$N = \sum_{i,j} N_{ij} \qquad \lambda = \sum_{\text{all paths p}} x_p = \text{total external arrival rate} \qquad T = \frac{N}{\lambda}$$

Jackson Networks

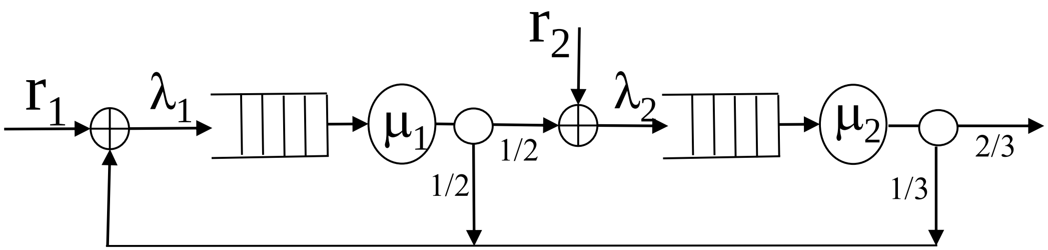
- Arrival rate at node j

$$\lambda_j = r_j + \sum_{i=1}^{K} \lambda_i P_{ij} \quad , j = 1,2,\ldots,K$$



- Independent routing of packets
  - » When a packet leaves node ***i*** it comes to node ***j*** with probability $\boldsymbol{P_{ij}}$
  - » Packets can loop inside network
  - » Packet leaves the system at node j with probability

$$P = 1 - \sum_{i=1}^{K} P_{ji}$$



Jackson Networks

- Let the state of the system be defined by $\vec{n} = (n_1, n_2, \dots, n_K)$

  n_j – number of clients in Q_j

- Jackson's theorem: $P(\vec{n}) = \prod_{j=1}^{K} P_j(n_j) = \prod_{j=1}^{K} \rho_j^{n_j}(1-\rho_j),$ where $\rho_j = \frac{\lambda_j}{\mu_j}$

  » State of Q_j (n_j) is independent $\left(\prod_{j=1}^{K}\right)$ of state of other queues

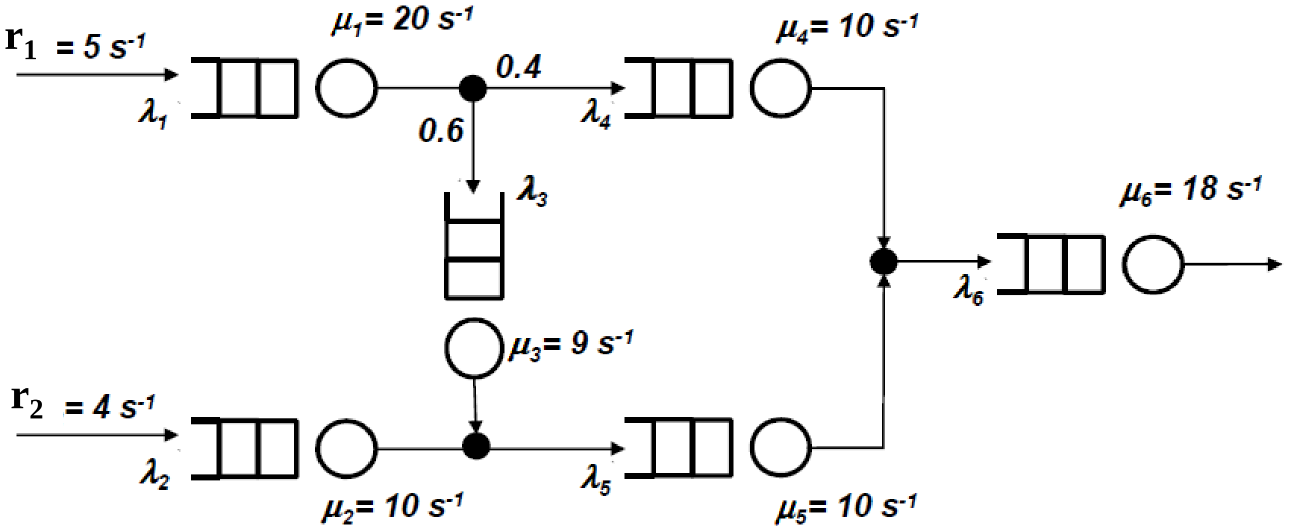
  » Similar to independent M/M/1 queues!

  » Similar to Kleinrock's independence

- Again, by Little's theorem

$$N_j = \frac{\rho_j}{1-\rho_j} \qquad N = \sum_{j=1}^{K} N_j \qquad \lambda = \sum_{j=1}^{K} r_j \qquad T = \frac{N}{\lambda}$$

Jackson Network - Example



$$\lambda = \sum_{i=1}^{6} r_i = 9\ \mathrm{s}^{-1}$$

$$N = \sum_{i=1}^{6} N_i = 5.08$$

$$T = \frac{N}{\lambda} = \frac{5.08}{9} = 0.56\ \mathrm{s}$$

Queue i	$\mathbf{r_i}$ (s^{-1})	λ_i (s^{-1})	μ_i (s^{-1})	$\rho_i = \lambda_i/\mu_i$	$\mathbf{N_i} = \rho_i/(1-\rho_i)$
1	5	5	20	0.25	0.33
2	4	4	10	0.40	0.67
3	-	3	9	0.33	0.50
4	-	2	10	0.20	0.25
5	-	7	10	0.70	2.33
6	-	9	18	0.50	1

Homework

1. Review slides

2. Read *Bertsekas&Gallager*
   - » Sections 3.1, 3.2, 3.3, 3.5, 3.6, 3.8

3. Answer questions at moodle