# *Redes de Computadores*

## Delay Models in Computer Networks

*Manuel P. Ricardo*

*Faculdade de Engenharia da Universidade do Porto*

» *What are the common multiplexing strategies?*
» *What is a Poisson process?*
» *What is the Little theorem?*
» *What is a queue?*
» *What is the meaning of service time $1/\mu$ in a queue of packets?*
» *What is the meaning of traffic intensity $\rho$ in a queue model?*
» *What is the probability of a M/M/1 queue being in a given state n ?*
» *What is the mean number of clients in a M/M/1 queue? What is the mean waiting time in a M/M/1 queue? What is the relationship between N and $\rho$ in a M/M/1 queue?*
» *What are the differences between M/M/1 and M/G/1 queues? How to estimate mean number of packets and mean delay in a M/G/1 queue?*
» *How to model a network of transmission lines? How to calculate the mean number of packets and mean delay in this case?*
» *What is a Jackson Network? Why is it important?*

# *Multiplexing Traffic on a Link*

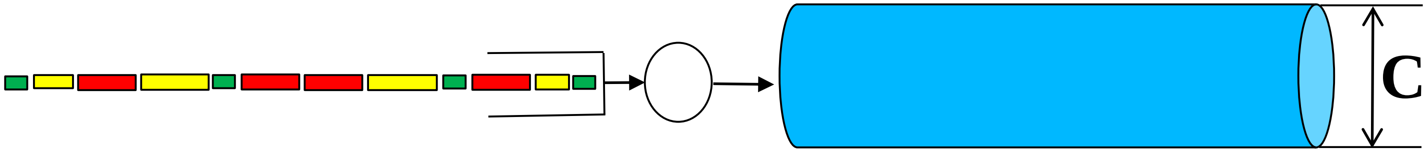

- Communication link
  - Bit pipe with a given capacity C (bit/s)
  - Link capacity → rate at which bits are transmitted to the link
  - Link may transport multiplexed traffic streams

- Multiplexing strategies
  - Statistical Multiplexing
  - Frequency Division Multiplexing
  - Time Division Multiplexing

- Multiplexing strategy affects traffic delay

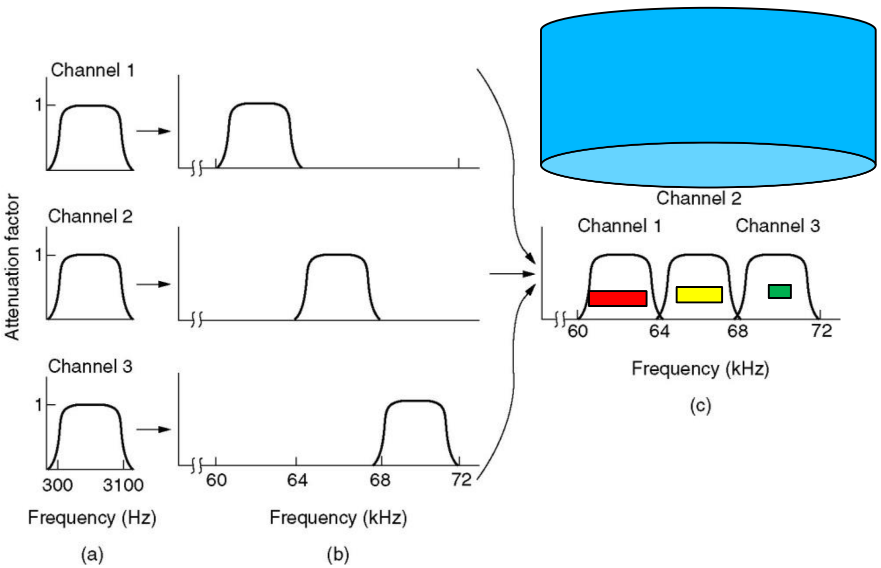
# *Statistical Multiplexing*

- Packets of all traffic streams merged in a single queue
- Packets transmitted on a first-come first-served basis
- Time required to transmit a packet of length L ➔ $T_{frame}$=L/C



# *FDM – Frequency Division Multiplexing*

- Link capacity C subdivided into ***m*** portions
- Channel bandwidth W subdivided into m channels of W/m Hz
- Capacity of each channel ➔ C/m
- Time required to transmit a packet of length L ➔ $T_{frame}$= Lm/C



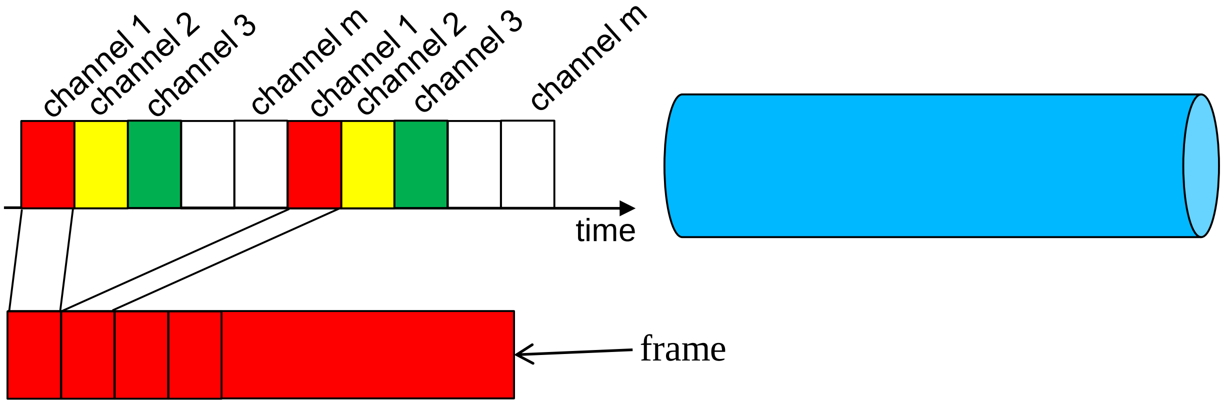
# *TDM – Time Division Multiplexing*

- Time axis divided into ***m*** slots of fixed length

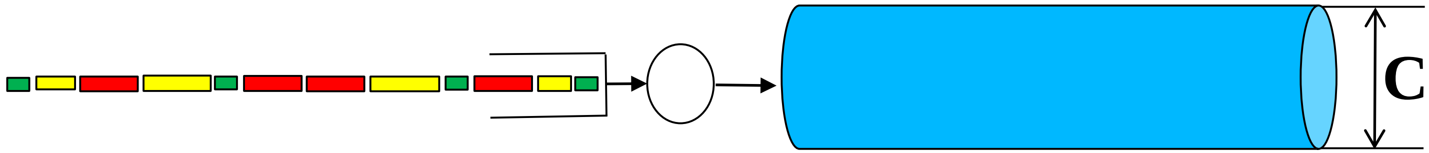
  (usually one octet long)

- Communication ➔ m channels with capacity C/m
- Time required to transmit a packet of length L ➔ $T_{frame}$=Lm/C

one octet

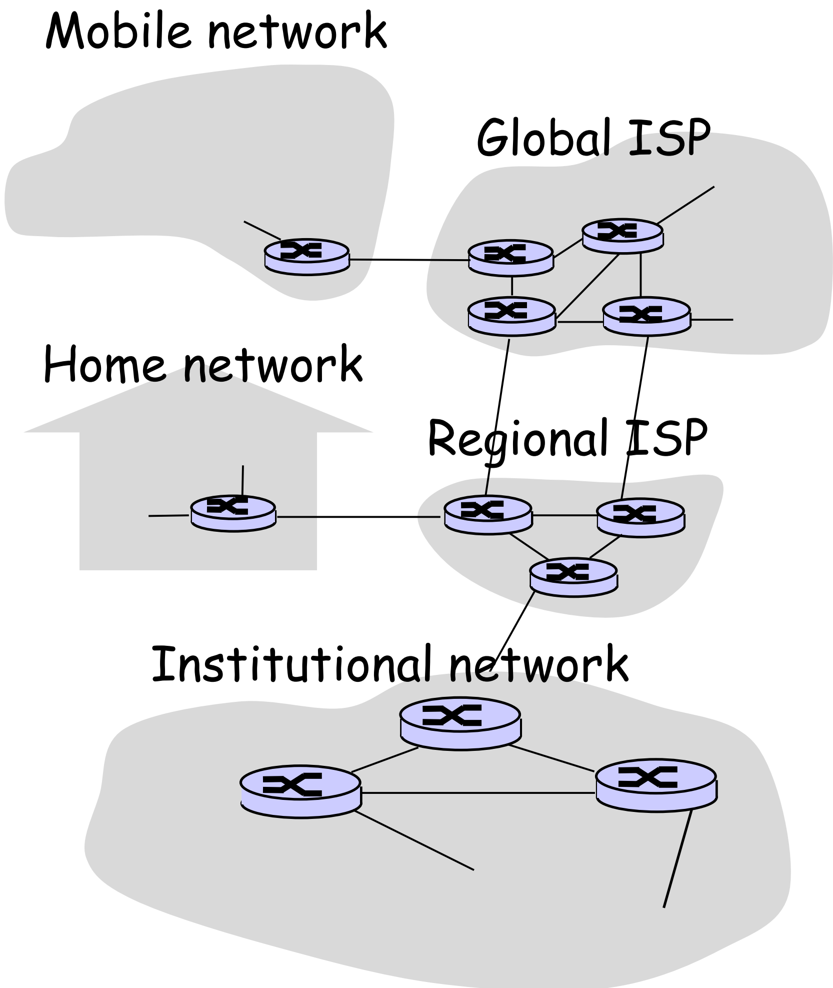


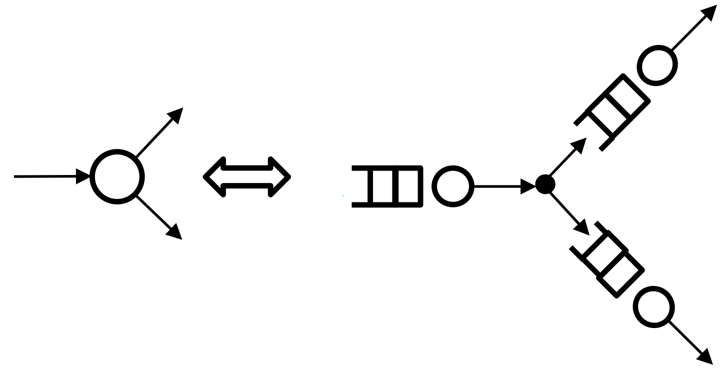
# *Delay on Computer Networks*

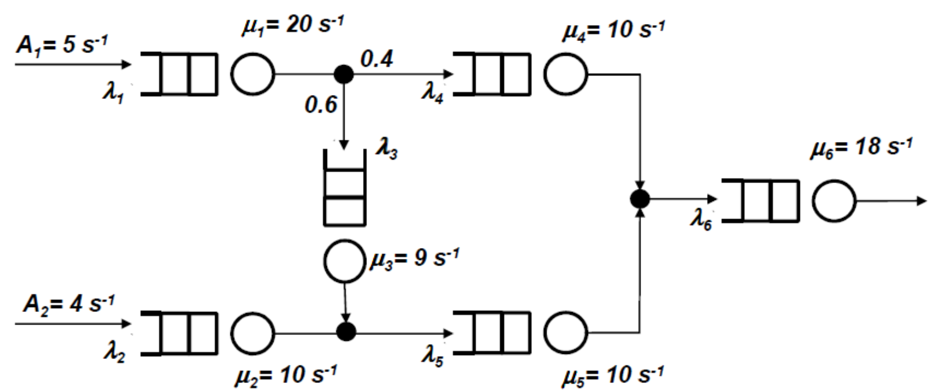



- Delay
  - » Important performance parameter in computer networks
  - » Characterized using queue models
- Queue model
  - » Customers arrive at random times to obtain service
  - » Customer ➔ packet to be transmitted through a link
  - » Serve a packet = transmit a packet
  - » Service time ➔ **packet transmission time =$T_{pac(frame)}$= L/C**
- Queue models enable the quantification of
  - » Average number of customers/packets in the network
  - » Average delay per packet ➔ waiting plus service times

# *Computer Networks Modeled as Queue Networks*



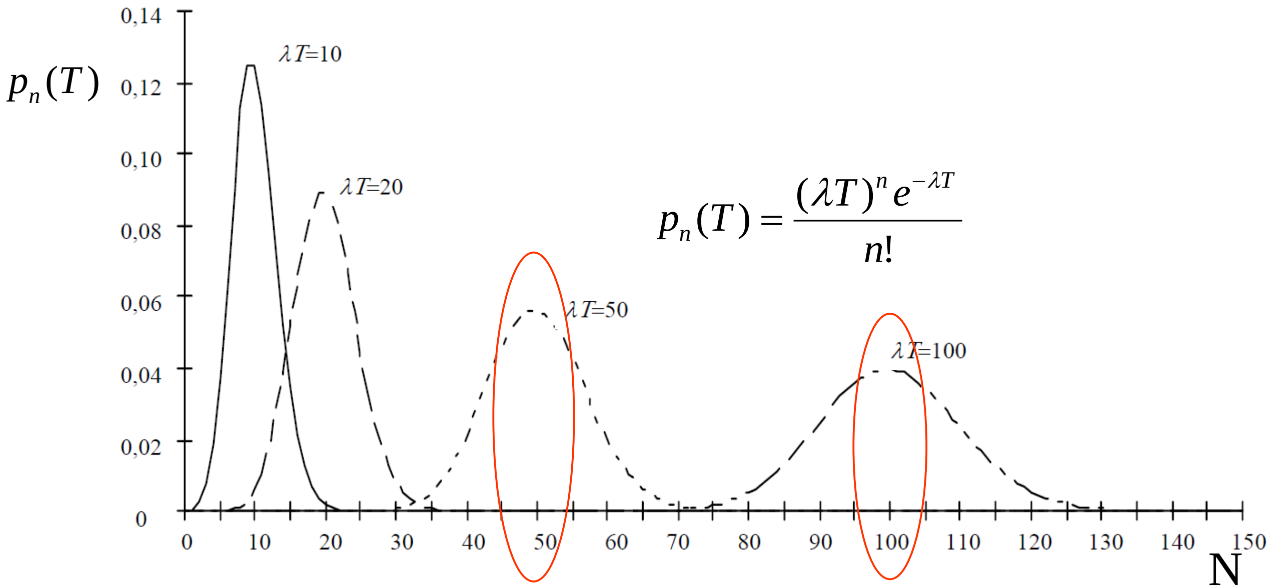


# *Poisson Distribution and Poisson Process*

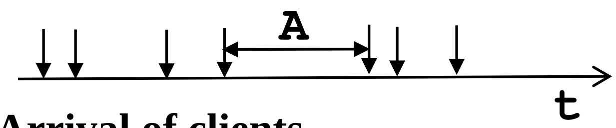
- Poisson distribution with parameter m

$$P[N=n]=p_n=\frac{m^n e^{-m}}{n!},\quad n=0,1,\ldots \qquad E[N]=Var[N]=m$$

- Poisson process
  - » $\lambda T=m$ , (e.g. $\lambda \rightarrow$ arrivals/s )
  - » P[ n arrivals in interval T ] = $p_n(T)=p_n=\frac{(\lambda T)^n e^{-\lambda T}}{n!}$ $\qquad E[N]=Var[N]=\lambda T$



# *Inter-Arrival Interval A – Statistical Characterization*




**Arrival of clients**

**A – time interval between the arrival of consecutive clients**

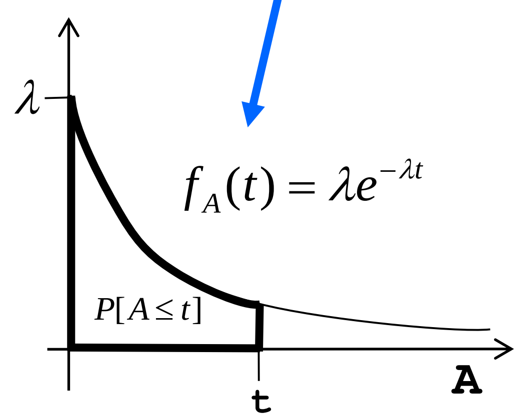
$$F_A(t) = P[A \le t] = 1 - P[A > t] = 1 - p_0(t) = 1 - e^{-\lambda t}$$

$$f_A(t) = pdf = \frac{\partial F_A(t)}{\partial t} = \lambda e^{-\lambda t}$$

***Exponential distribution***

$$E[A] = {}^{1}\!/_{\lambda}$$

$$Var[A] = {}^{1}\!/_{\lambda^2}$$



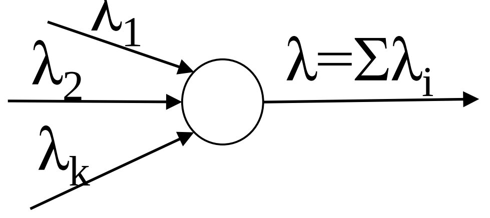
- What is the difference between

  Deterministic arrivals and

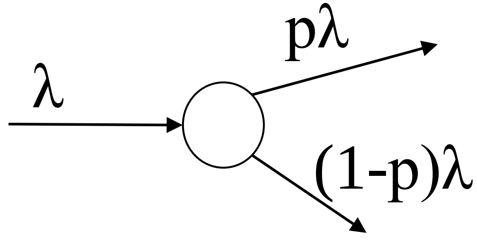
  Poisson arrivals?

# *Markov Process - Properties*

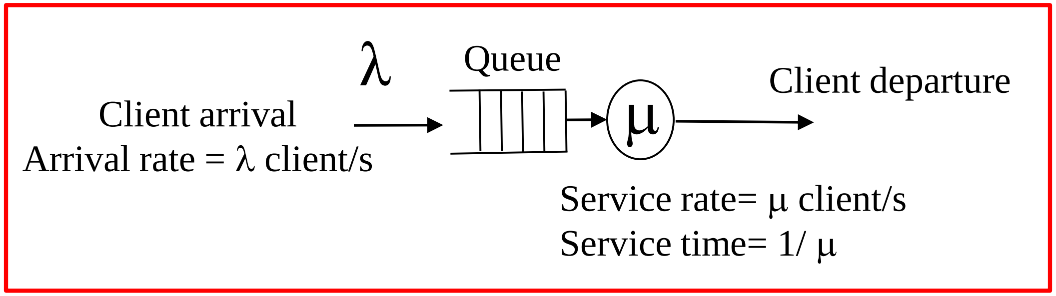
- Merging Property
  - » $A_1, A_2, \ldots A_k$ are independent Poisson Processes with rates $\lambda_1, \lambda_2, \ldots \lambda_k$
  - » $A=\Sigma A_i$ still is a Poisson process, with rate $\lambda=\Sigma\lambda_i$




- Splitting property
  - » Packets arrive to a router according to a Poisson Process $(A,\lambda)$
  - » They are routed randomly to two output lines with probabilities **p** and **1-p**
  - » Packets leaving the router still are Poisson Processes, characterized by $(A,p\lambda)$ and $(A,(1-p)\lambda)$



# *Queue Model*




- Queue – model used for
  - Customers waiting in line
  - Packets in a network
- Used to determine
  - Average number of clients in the system → N
  - Average delay experienced by a client → T
- Queue characterized in terms of
  - $\lambda$ - arrival rate of client (average number of clients per time unit)
  - $\mu$ - service rate (average number of clients the server processes per time unit)
  - $\rho=\lambda/\mu$ – traffic intensity (occupation of the server)
- Kendall notation → **A/S/s/K**
  - A – arrival statistical process
  - S – service statistical process
  - s – number of servers
  - K – capacity of the system in buffers

# *Little's Theorem*

- **$N=\lambda T$**
  - » N- average number of clients in a system
  - » T – average amount of time a client spends in the system
  - » $\lambda$ – arrival rate of clients to the system
- $T=T_w+T_s$
  - » $T_w$ – time a client waits in the queue for being served
  - » $T_s$ – service time
- $N=N_w+N_s$
  - » $N_w$ – number of clients waiting in the queue for being served
  - » $N_s$ – number of clients being served
- **$N_w=\lambda T_w$**

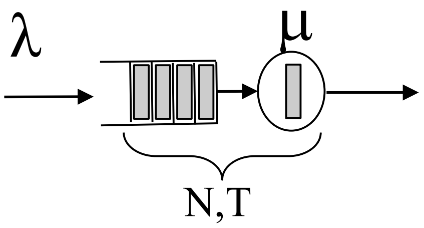
$$N_w = \lambda T_w \rightarrow T_w = N_w / \lambda$$

- The (mean) time a client has to wait before being served (**$T_w$**) depends on the number of clients waiting (**$N_w$**) and on the arrival rate of clients ($\lambda$ )

- No dependence on the service rate?!
- Can you explain it?

# *Little's Theorem*

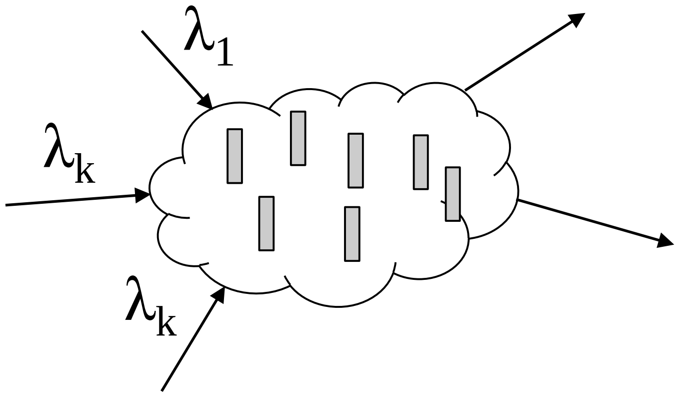
- Can be applied to a single Queue




- Can be applied to a complex system
  - For each stream i → $N_i=\lambda_i T_i$
  - For the system:

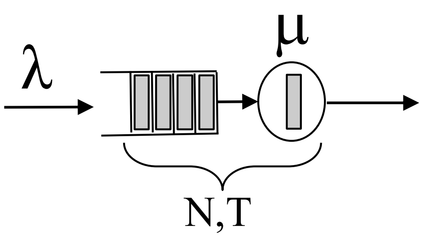
    $\lambda=\Sigma\ \lambda_i \qquad N=\Sigma\ N_i$

    $T=(\Sigma\ N_i)/\ (\Sigma\ \lambda_i) \quad \rightarrow T=N/\lambda$



# *M/M/1 Queue*

- M/M/1
  - » Poisson arrival, exponential service time

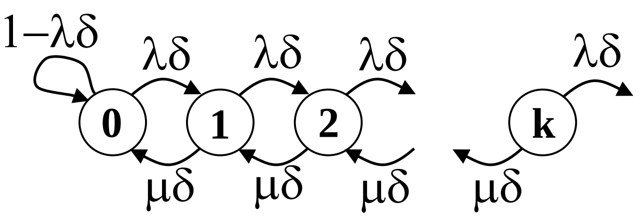



- Modeled by a Markov Chain
  - » State **k** - k clients in the queue
  - » p(i,j) – probability of transition from state i to state j
  - » When δ → 0

    p(i, i+1)= λδ            p(i, i-1)= μδ

    p(i, i)= 1- λδ–μδ            p(0, 0)= 1- λδ

    p(i, j)=0 for other values i, j




  - » Birth-death chain
    - – Transitions between adjacent states
    - – λδ and μδ become flow rates between states

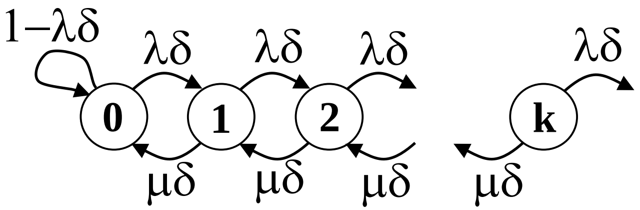
$$p(i, i+1) = p_1(\delta) = (\lambda\delta)e^{-\lambda\delta} \approx \lambda\delta$$

$$p(0,0) = p_0(\delta) = e^{-\lambda\delta} \approx 1 - \lambda\delta$$

# *M/M/1 Queue – Equilibrium Analysis*

- P(j) – probability of the Markov chain be in state j
- Markov Chain - global balance equations

$$P(j)\sum_{\substack{i=0\\ i\neq j}}^{\infty} p(j,i) = \sum_{\substack{i=0\\ i\neq j}}^{\infty} P(i)\,p(i,j)$$




- In the case of M/M/1

$$P(0)\lambda\delta = P(1)\mu\delta \quad \Rightarrow P(1) = \rho P(0)$$

$$P(2) = \rho P(1) = \rho^2 P(0)$$

$$P(n) = \rho^n P(0)$$

$$\sum_{i=0}^{\infty} P(i) = 1$$

$$\sum_{i=0}^{\infty} \rho^i P(0) = \frac{P(0)}{1-\rho} = 1$$

$$P(0) = 1-\rho$$

$$\boxed{P(n) = \rho^n (1-\rho)}$$

# *M/M/1 Queue*

- Average Queue size N

$$N = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n\rho^n (1-\rho) = \frac{\rho}{1-\rho} \qquad N = \sum_{n=0}^{\infty} nP(n) = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

- Average amount of time the client spends in the system, T

  » Little's formula, T=N/λ ➔ $T = \dfrac{1}{\mu-\lambda}$

- Average waiting time $T_w$ ➔ $T_w = T - T_s = \dfrac{1}{\mu-\lambda} - \dfrac{1}{\mu} = \dfrac{\rho}{\mu(1-\rho)}$

- Average number of clients waiting in the queue, $N_w$

$$N_w = T_w \lambda = \frac{\lambda}{\mu-\lambda} - \frac{\lambda}{\mu} = N - \rho$$

# M/M/1 Queue – N=f(ρ)

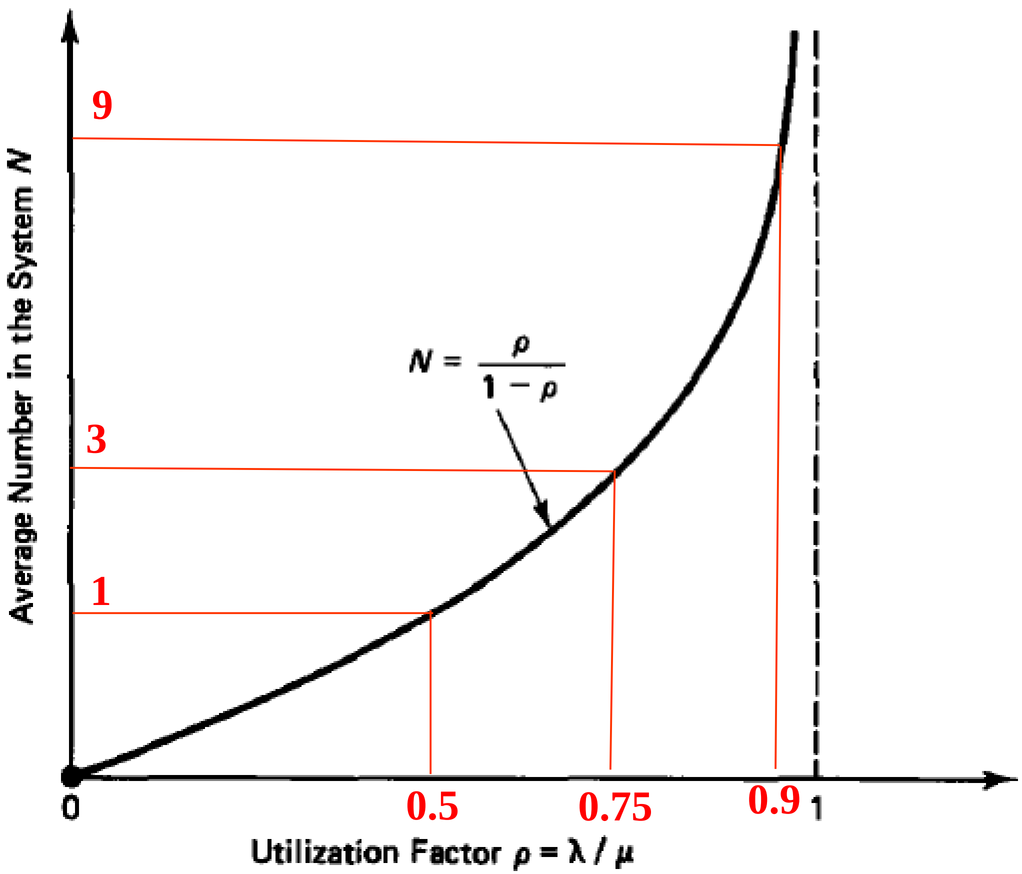



**Figure 3.6** The average number in the system versus the utilization factor in the $M/M/1$ system. As $\rho \rightarrow 1$, $N \rightarrow \infty$.

- M/M/1: $\rho=0.9 \Rightarrow N=9$

- Why have clients to wait if the server is busy only 90% of his time?

- What would happen for D/D/1, $\rho=0.9$?

# *Packet Length, Service Time, Speed*

- 100 packet/s are required to be transmitted through a link
- Packets arrive according to a Poisson process
- Packet lengths are exponentially distributed ➔ $E[L]=10^4$ bit/packet
- Link has capacity C=10 Mbit/s

- Then
  - Arrival rate: $\lambda=100$ packet/s
  - Service rate: $\mu=C/E[L]= 10^7/10^4= 10^3$ packet/s
  - $\rho=\lambda/\mu=0.1$, $N=\rho/(1-\rho)=1/9$, $T=N/\lambda=1/900$ s

- Assume now: **$\lambda'=10\lambda$** and **$C'=10C$** ➔ **$\mu'=10C/E[L]=10\mu$**
  - Then $\rho'=\rho$ and $N'=N$ but $T'=N'/\lambda'=T/10$

    The speed of the system increases!

# *M/M/1/B Queue*

- M/M/1 queue has limited capacity (B buffers)
  - » Packets can be lost
  - » Probability of packet being lost = P(B) ➔ Queue is full
- Analysis similar to M/M/1

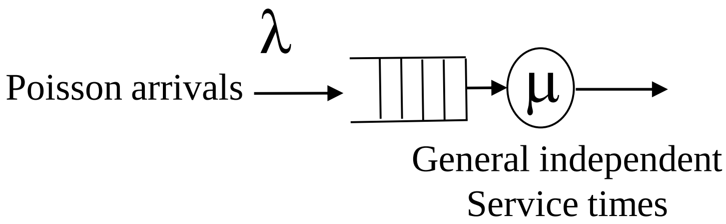
$$\sum_{i=0}^{B} P(i) = 1 \qquad P(n) = \rho^n P(0)$$

$$P(0) = \frac{1-\rho}{1-\rho^{B+1}} \qquad P(B) = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}}$$

- Particular cases

$$\rho = 1, \quad P(B) = \frac{1}{B+1} \qquad \rho >> 1, \quad P(B) \approx \frac{\rho - 1}{\rho} = \frac{\lambda - \mu}{\lambda}$$

# *M/G/1 Queue*




- Poisson arrivals at rate λ
- Service time X has arbitrary distribution with given E[X] and E[$X^2$]
  - » Service times Independent and Identically Distributed (IID)
  - » Independent of arrival times
  - » E[service time] =E[X]= 1/μ
  - » Single Server queue

# *M/G/1 Queue – Pollaczek-Khinchin (P-K) Formula*

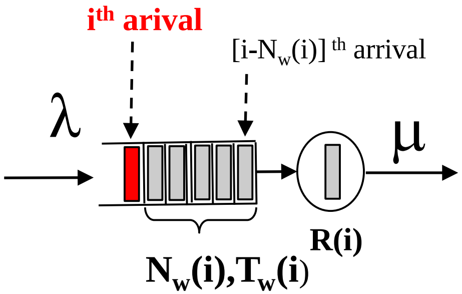
$$\boxed{T_w = \frac{\lambda E[X^2]}{2(1-\rho)}}$$

- where $\rho = \lambda/\mu = \lambda E[X]$ = line utilization
- From Little's Theorem
  - » $N_w = \lambda T_w$
  - » $T = T_w + E[X] = T_w + 1/\mu$
  - » $\mathbf{N = \lambda T} = \lambda(T_w + 1/\mu) \mathbf{= N_w + \rho}$

# *M/G/1 Queue – Proof of (P-K) Formula*

$$T_w = \frac{\lambda E[X^2]}{2(1-\rho)}$$

- Let
  - $T_w(i)$ - waiting time in queue of $i^{th}$ arrival
  - $R(i)$ – residual service time seen by the $i^{th}$ arrival
  - $N_w(i)$ – number of clients found in queue by the $i^{th}$ arrival
  - $X(i)$ – service time of the $i^{th}$ arrival




$$T_w(i) = \sum_{j=i-N_w(i)}^{i-1} X(j) + R(i)$$

$$E[T_w(i)] = T_w = E[N_w(i)] \times E[X(i)] + E[R(i)] = \frac{N_w}{\mu} + E[R(i)]$$

» Using Little's formula

$$T_w = \frac{\lambda T_w}{\mu} + E[R(i)] \qquad T_w = \frac{E[R(i)]}{1-\rho}$$

# *M/G/1 Queue – Proof of (P-K) Formula*

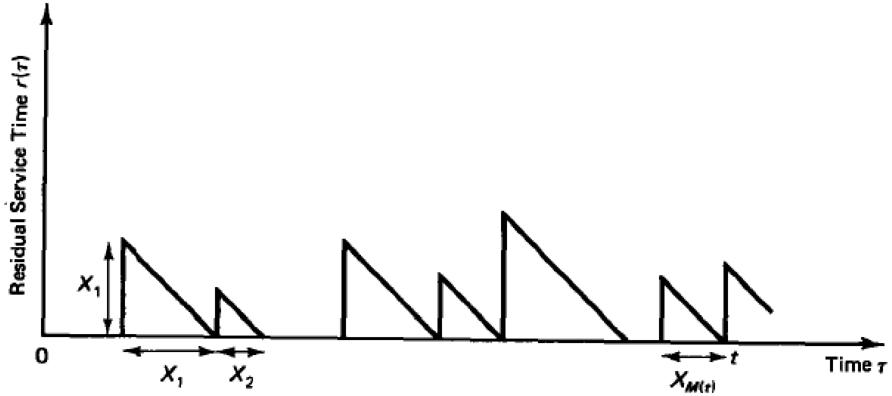



**Figure 3.10** Derivation of the mean residual service time. During period $[0, t]$, the time average of the residual service time $r(\tau)$ is

M(t) – number of clients served by time t

$$E[R(i)] = R_t = \frac{1}{t}\int_0^t r(\tau)\partial\tau = \frac{1}{t}\sum_{i=1}^{M(t)}\frac{X_i^2}{2} = \frac{M(t)}{2t}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)}$$

$t \to \infty, \quad \frac{M(t)}{t} = \lambda$ = arrival rate= departure rate

$$E[R(i)] = \frac{\lambda}{2}\sum_{i=1}^{M(t)}\frac{X_i^2}{M(t)} = \frac{\lambda}{2}\times E\left[X^2\right]$$

$$T_w = \frac{E[R(i)]}{1-\rho}$$

$$\boxed{T_w = \frac{\lambda E[X^2]}{2(1-\rho)}}$$

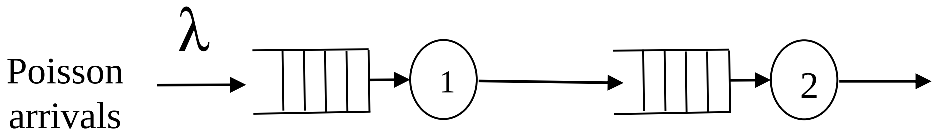
# *M/G/1 Examples*

- Case M/M/1
  - » E[X]= 1/μ ; E[X$^2$]= 2/μ$^2$

$$T_w = \frac{\lambda}{\mu^2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}$$

- Case M/D/1
  - » Deterministic, constant service time 1/μ
  - » E[X]= 1/μ ; E[X$^2$]= 1/μ$^2$

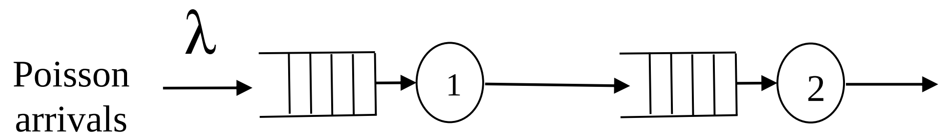
$$T_w = \frac{\lambda}{2\mu^2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}$$




- Assume Queue 1 is M/D/1.

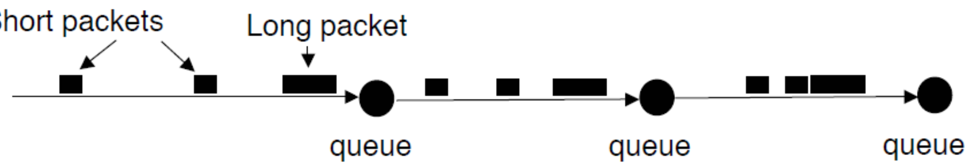
- Can the arrival of packets to Queue 2 be described as a Poisson process?

# *Networks of Transmission Lines - Problems*




- Case 1
  - » Arrival to $Q_1$ → Poisson, $\lambda$
  - » Assume contant packet length ➔ $Q_1$= M/D/1
  - » Arrival to $Q_2$ is not Poisson; $\lambda_2 < \mu_2$ ➔ $1/\lambda_2 > 1/\mu_2$

    ➔ no waiting at $Q_2$

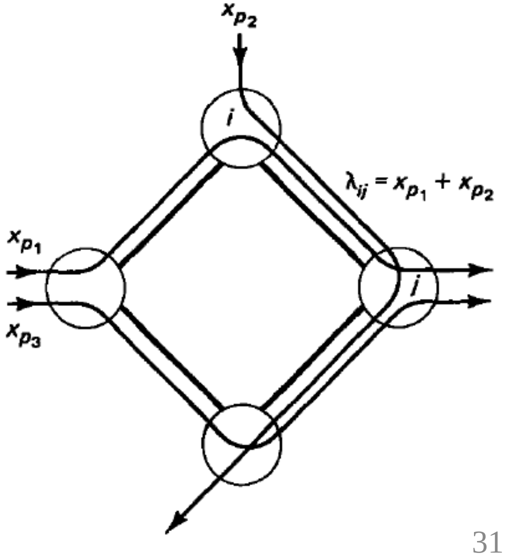



- Case 2
  - » $Q_1$=M/M/1
  - » arrival to $Q_2$ strongly related to packet length
  - » long packets require long service at each node
  - » shorter packets will catch up long packets ➔ interarrival times change

  ➔ $Q_2$ cannot be modeled as M/M/1

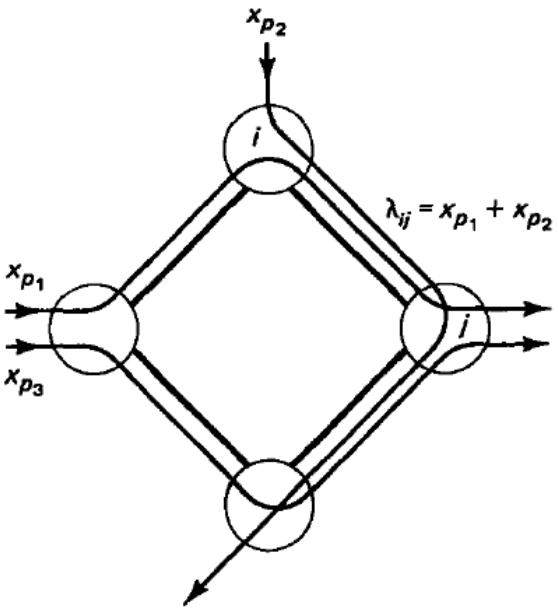
# *Kleinrock Independence Approximation*

- Merging several packet streams on a transmission line
  restores independence of interarrival times and packet lengths
- M/M/1 can be used to model each communication link
- Approximation good for
  - systems involving Poisson stream arrivals at the entry points
  - packet lengths nearly exponentially distributed
  - densely connected networks
  - Moderate to heavy traffic loads



# *Kleinrock Independence Approximation*

- Let
  - » $x_p$ = arrival rate of packets along path p
  - » $\lambda_{ij}$ = arrival rate of packets to link (i,j)
  - » $\mu_{ij}$ = service rate on link (i,j)




- Link queues ➔ <u>independent M/M/1 queues</u>

$$\lambda_{ij} = \sum_{\substack{\text{all p traversing} \\ \text{link(i,j)}}} x_p \qquad \rho_{ij} = \frac{\lambda_{ij}}{\mu_{ij}} \qquad N_{ij} = \frac{\rho_{ij}}{1-\rho_{ij}}$$

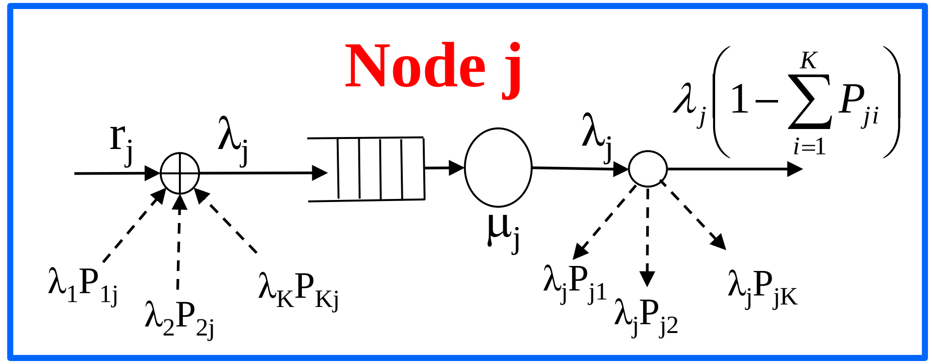
- And
  - » N= Average number of packets in network
  - » T – Average packet delay in network

$$N = \sum_{i,j} N_{ij} \qquad \lambda = \sum_{\text{all paths p}} x_p = \text{total external arrival rate} \qquad T = \frac{N}{\lambda}$$

# *Jackson Networks*

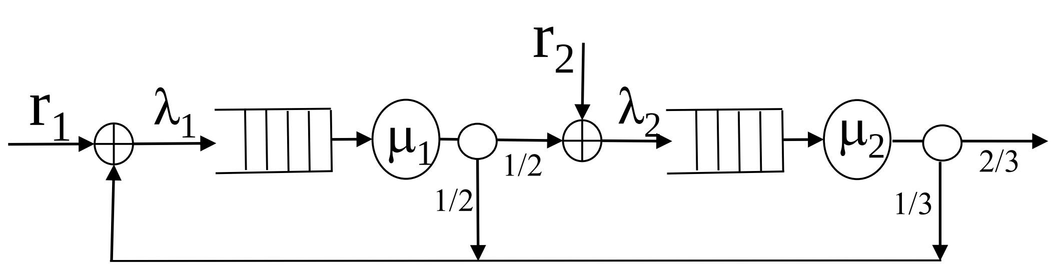
- Arrival rate at node j

$$\lambda_j = r_j + \sum_{i=1}^{K} \lambda_i P_{ij} \quad , j = 1,2,\ldots,K$$




- Independent routing of packets
  - When a packet leaves node ***i*** it comes to node ***j*** with probability $\boldsymbol{P_{ij}}$
  - Packets can loop inside network
  - Packet leaves the system at node j with probability

$$P = 1 - \sum_{i=1}^{K} P_{ji}$$



# *Jackson Networks*

- Let the state of the system be defined by $\vec{n} = (n_1, n_2, \ldots, n_K)$

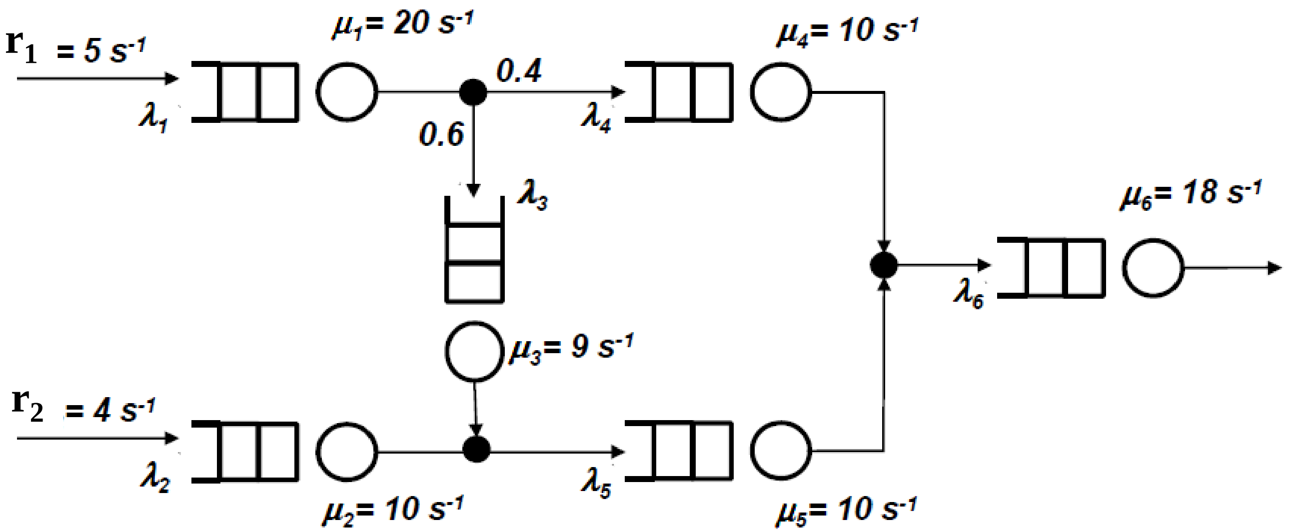
  $n_j$ – number of clients in $Q_j$

- Jackson's theorem: $P(\vec{n}) = \prod_{j=1}^{K} P_j(n_j) = \prod_{j=1}^{K} \rho_j^{n_j}(1-\rho_j)$, where $\rho_j = \frac{\lambda_j}{\mu_j}$
  - » State of $Q_j$ ($n_j$) is independent $\left(\prod_{j=1}^{K}\right)$ of state of other queues
  - » Similar to independent M/M/1 queues!
  - » Similar to Kleinrock's independence

- Again, by Little's theorem

$$N_j = \frac{\rho_j}{1-\rho_j} \qquad N = \sum_{j=1}^{K} N_j \qquad \lambda = \sum_{j=1}^{K} r_j \qquad T = \frac{N}{\lambda}$$

# *Jackson Network - Example*




$$\lambda = \sum_{i=1}^{6} r_i = 9\ \mathrm{s}^{-1}$$

$$N = \sum_{i=1}^{6} N_i = 5.08$$

$$T = \frac{N}{\lambda} = \frac{5.08}{9} = 0.56\ \mathrm{s}$$

| Queue $i$ | $\mathbf{r_i}$ $(s^{-1})$ | $\lambda_i$ $(s^{-1})$ | $\mu_i$ $(s^{-1})$ | $\rho_i = \lambda_i/\mu_i$ | $\mathbf{N_i} = \rho_i/(1-\rho_i)$ |
|---|---|---|---|---|---|
| 1 | 5 | 5 | 20 | 0.25 | 0.33 |
| 2 | 4 | 4 | 10 | 0.40 | 0.67 |
| 3 | - | 3 | 9 | 0.33 | 0.50 |
| 4 | - | 2 | 10 | 0.20 | 0.25 |
| 5 | - | 7 | 10 | 0.70 | 2.33 |
| 6 | - | 9 | 18 | 0.50 | 1 |

# *Homework*

1. Review slides

2. Read *Bertsekas&Gallager*
   » Sections 3.1, 3.2, 3.3, 3.5, 3.6, 3.8

3. Answer questions at moodle