Algorithm 1 Vineyards: Tranposition Framework

Require: Matrices R, V satisfying $R = DV$, $1 \leq i \leq m-1$
Ensure: Output (R, V) maintains the decomposition invariants

function TRANSPOSE(R, V, i)
  pos $\leftarrow$ columns satisfying $\text{col}_R = 0$
  if pos$[i]$ **and** pos$[i+1]$ **then**
    if $V[i, i+1] \neq 0$ **then**
      $\text{col}_V(i+1)$ += $\text{col}_V(i)$
    if $\exists\ k, l$ s.t. $\text{low}_R(k) = i, \text{low}_R(l) = i+1$ and $R[i,l] \neq 0$ **then** ▷ O(m)
      if $k < l$ **then**
        return $(R, V) \leftarrow (PRPS^l_k, PVPS^l_k)$
      else
        return $(R, V) \leftarrow (PRPS^k_l, PVPS^k_l)$
  else if !pos$[i]$ **and** !pos$[i+1]$ **then**
    if $V[i, i+1] \neq 0$ **then** ▷ O(m)
      if $\text{low}_R(i) < \text{low}_R(i+1)$ **then**
        return $(R, V) \leftarrow (PRS^{i+1}_i P, PVS^{i+1}_i P)$
      else
        return $(R, V) \leftarrow (PRS^{i+1}_i PS^{i+1}_i, PVS^{i+1}_i PS^{i+1}_i)$
  else if !pos$[i]$ **and** pos$[i+1]$ **then**
    if $V[i, i+1] \neq 0$ **then** ▷ O(m)
      return $(R, V) \leftarrow (PRS^{i+1}_i PS^{i+1}_i, PVS^{i+1}_i PS^{i+1}_i)$
  else if pos$[i]$ **and** !pos$[i+1]$ **then**
    if $V[i, i+1] \neq 0$ **then** ▷ O(m)
      $\text{col}_V(i+1)$ += $\text{col}_V(i)$
  return $(R, V) \leftarrow (PRP, PVP)$