1. Motivation

< insert motivating examples, etc >

2. Background & Notation

A *simplicial complex* $K \subseteq \mathcal{P}(V)$ over a vertex set $V = \{v_1, v_2, \dots, v_n\}$ is a collection of simplices $\{\sigma : \sigma \in \mathcal{P}(V)\}$ such that $\tau \subseteq \sigma \in K \implies \tau \in K$. A *filtration* $K_\bullet = \{K_i\}_{i \in I}$ of a simplicial complexes indexed by a totally ordered set I is a family of complexes such that $i < j \in I \implies K_i \subseteq K_j$. $K_\bullet$ is called *simplexwise* if $K_j \smallsetminus K_i = \{\sigma_j\}$, and $K_\bullet$ is *essential* if $i \neq j$ implies $K_i \neq K_j$ whenever j is the immediate successor of i in I:

$$\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet, \quad K_i = K_{i-1} \cup \{\sigma_i\} \tag{1}$$

Filtrations may be equivalently defined as functions $f : K \to I$ satisfying $f(\tau) \leq f(\sigma)$ whenever $\tau \subseteq \sigma$. Here, we consider two index sets for I: $\mathbb{R}$ and $[n] = \{1, \dots, n\}$. Any finite filtration may be trivially converted into an essential, simplexwise filtration via a set of *condensing*, *refining*, and *reindexing* maps [?]. Thus, without loss of generality, we exclusively consider essential simplexwise filtrations and for brevity refer to them as filtrations.

For K a simplicial complex and $\mathbb{F}$ a field, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K. The collection of p-chains under addition yields an $\mathbb{F}$-vector space denoted $C_p(K)$. The p-boundary $\partial_p(\sigma)$ of an oriented p-simplex $\sigma \in K$ is defined as the alternating sum of its oriented co-dimension 1 faces:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \dots, v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p] \tag{2}$$

where $\hat{v}_i$ indicates the removal of v_i from the ith summand. Similarly, the p-boundary of a p-chain is defined linearly in terms of its constitutive simplices. A p-chain with zero boundary is called a p-cycle, and together they form $Z_p(K) = \operatorname{Ker} \partial_p$. Similarly, the collection of p-boundaries forms $B_p(K) = \operatorname{Im} \partial_{p+1}$. Since $\partial_p \circ \partial_{p+1} = 0$ for all $p \geq 0$, the quotient space $H_p(K) = Z_p(K)/B_p(K)$ is well-defined, and $H_p(K)$ is called the p-th homology of K with coefficients in $\mathbb{F}$. The dimension of the p-th homology group $\beta_p(K) = \dim(H_p(K))$ of K is called the p-th *Betti* number of K.

Let $K_\bullet = \{K_i\}_{i \in [m]}$ denote a filtration of size $|K_\bullet| = m$. For every pair $i, j \in [m]$ with $i < j$, the inclusions $K_i \subsetneq K_{i+1} \subsetneq \cdots \subsetneq K_j$ induce linear transformations $h_p^{i,j}$ at the level of homology:

$$0 = H_p(K_0) \to \cdots \to H_p(K_i) \underbrace{\to \cdots \to}_{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_m) = H_p(K_\bullet) \tag{3}$$

When $\mathbb{F}$ is a field, this sequence of homology groups admits a unique decomposition of $K_\bullet$ into a pairing of simplices (σ_i, σ_j) [] demarcating the evolution of homology classes: σ_i marks the creation of a homology class, σ_j marks its destruction, and the difference $|i - j|$ records the lifetime of the class, called its *persistence*. The p-th persistent homology groups are the images of these transformations and the p-th persistent Betti numbers are their dimensions:

$$H_p^{i,j} = \begin{cases} H(K_i) & i = j \\ \operatorname{Im} h_p^{i,j} & i < j \end{cases}, \qquad \beta_p^{i,j} = \begin{cases} \beta_p(K_i) & i = j \\ \dim(H_p^{i,j}) & i < j \end{cases} \tag{4}$$

For a fixed $p \geq 0$, the collection of persistent pairs (i, j) together with unpaired simplices (l, ∞) form a summary representation $\mathrm{dgm}_p(K_\bullet)$ called the *p-th persistence diagram of $K_\bullet$*.

Remark 1. In practice, filtrations often arise from triangulations parameterized by geometric scaling parameters, and references to the "persistence" of a given homology class are with respect to these parameterizations. For example, given a finite metric space $\mathcal{X} = (X, d_X)$, the *Rips complex* at scale $\epsilon \in \mathbb{R}_+$ is the complex given by:

$$\mathrm{Rips}_\epsilon(\mathcal{X}) := \{\sigma \subseteq X : d_X(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{5}$$

Connecting successive complexes via inclusions $\mathrm{Rips}_\epsilon(\mathcal{X}) \hookrightarrow \mathrm{Rips}_{\epsilon'}(\mathcal{X})$ for $\epsilon < \epsilon'$ yields a family of complexes $\mathrm{Rips}_\alpha := \{\, \mathrm{Rips}_\epsilon(\mathcal{X}) \,\}_{\epsilon \leq \alpha}$ called the *Rips filtration*. As in equation (**??**), these inclusions induce linear maps at level of homology. Though we consider primarily Rips filtrations in this effort, we will at times keep the notation simple and general by letting $K_\bullet$ denote any simplicial filtration.

3. Methodology

In this section, we derive the relaxed objective function we seek to maximize.

Let $\delta_\mathcal{X}$ denote an T-parameterized metric space $\delta_\mathcal{X}(\cdot) = (X, d_X(\cdot))$, where $d_X : \mathrm{T} \times X \times X \to \mathbb{R}_+$ is called a *time-varying metric* and X is a finite set with fixed cardinality $|X| = n$. δ_X as called a *dynamic metric space* (DMS) iff $d_X(\cdot)(x, x')$ is continuous for every pair $x, x' \in X$ and $\delta_\mathcal{X}(t) = (X, d_X(t))$ is a pseudo-metric space for every $t \in \mathrm{T}$. For a fixed $t \in \mathrm{T}$, the Rips complex at scale $\epsilon \in \mathbb{R}$ is the abstract simplicial complex given by

$$\mathrm{Rips}_\epsilon(\delta_\mathcal{X}(t)) := \{\sigma \subset X : d_X(t)(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{6}$$

As before, the family of Rips complexes for varying $\epsilon > 0$ yields a filtration whose inclusion maps induce linear maps at the level of homology. The time-varying counterpart is analogous. In this context, we write the p-th persistent Betti number with respect to fixed values $i, j \in I$ as a function of $t \in \mathrm{T}$:

$$\beta_p^{i,j}(t) = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ \delta_\mathcal{X}\right)(t) \tag{7}$$

This quantity can be readily visualized as the number of persistent pairs lying inside the box $[0, i] \times (j, \infty)$, representing the persistent homology groups which were born at or before i and died sometime after j. We consider the problem of maximizing the p-th *persistent* Betti number $\beta_p^{i,j}$ over some set $T \subseteq \mathrm{T}$:

$$t_* = \arg\max_{t \in T} \beta_p^{i,j}(t) \tag{8}$$

As an illustrative example, see Figure. < insert SW1Pers vineyards plot >

Persistent Betti Numbers:

As in section **??**, let $B_p(K_*) \subseteq Z_p(K_*) \subseteq C_p(K_*)$ denote the p-th boundary, cycle, and chain groups of K_*, respectively. Given a simplicial filtration $K_\bullet$, let $\partial_p : C_p(K_\bullet) \to C_p(K_\bullet)$ denote the boundary operator sending p-chains to their respective boundaries. With a slight abuse of notation, we also use ∂_p to also denote the filtration boundary matrix with respect to an ordered basis $(\sigma_i)_{1 \leq i \leq m_p}$. The p-th persistent Betti number between scales (i, j) is defined as:

$$\begin{aligned}
\beta_p^{i,j} &= \dim(H_p^{i,j}) \\
&= \dim\left(Z_p(K_i)/(Z_p(K_i) \cap B_p(K_j)\right) \\
&= \dim\left(Z_p(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)
\end{aligned} \tag{9}$$

Note that $\dim(C_p(K_*)) = \dim(B_{p-1}(K_*)) + \dim(Z_p(K_*))$ by the rank-nullity theorem, so we may rewrite (**??**) as:

$$\beta_p^{i,j} = \dim\left(C_p(K_i)\right) - \dim\left(B_{p-1}(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \tag{10}$$

The dimension of the boundary group $B_{p-1}(K_i)$ may be directly inferred from the rank of ∂_p^i, and the dimension of $C_p(K_i)$ is simply the number of p-simplices with filtration values $f(\sigma) \leq i$. To express the intersection term, we require more notation. If A is a $m \times m$ matrix, then let A_i^j denote the lower-left submatrix of A given by the first j columns and last $m - i + 1$ rows (rows i through m, inclusive). For any $1 \leq i < j \leq m$, define the quantity $r_A(i, j)$:

$$r_A(i, j) = \mathrm{rank}(A_i^j) - \mathrm{rank}(A_{i+1}^j) + \mathrm{rank}(A_{i+1}^{j-1}) - \mathrm{rank}(A_i^{j-1}) \tag{11}$$

One of the seminal results from the Pairing Uniqueness lemma [] asserts that if $R = \partial V$ is a decomposition of the total $m \times m$ boundary matrix ∂, then for any $1 \leq i < j \leq m$, we have (todo: elaborate more):

$$\mathrm{low}_R[j] = i \iff r_\partial(i, j) = 1 \iff \mathrm{rank}(R_i^j) = \mathrm{rank}(\partial_i^j)$$

Thus, the lower-left submatrices of the filtered boundary matrix ∂ has the same rank as R. Let $\hat{\partial} = \partial_p$ and $\bar{\partial} = \partial_{p+1}$. Utilizing the result above, observe $\beta_p^{i,j}$ may be written as:

$$\beta_p^{i,j} = |\hat{\partial}_1^i| - \mathrm{rank}(\hat{\partial}_1^i) - \mathrm{rank}(\bar{\partial}_1^j) + \mathrm{rank}(\bar{\partial}_{i+1}^j) \tag{12}$$

where $|\,A\,| = \dim(\mathrm{dom}(A))$ denotes the order of the matrix. Thus, we may write the persistent Betti number as a combination of rank computations performed directly on the dimension p and $(p + 1)$ boundary matrices.

Boundary Matrix Relaxation

As integer-valued invariants, Betti numbers pose several difficulties to direct optimization. Thus, we require alternative expressions for each of the terms in equation (**??**) to extend its applicability to the time-varying setting. Towards deriving these expression, we first require a replacement of the standard boundary matrix formulation.

Recall that the boundary operator ∂_p for a finite simplicial filtration $K_\bullet$ with $m = |C_p(K_\bullet)|$ and $n = |C_{p-1}(K_\bullet)|$ can be represented by an $(n \times m)$ boundary matrix ∂_p whose columns and rows correspond to p-simplices and $(p-1)$-simplices, respectively. The entries of ∂_p depend on the choice of $\mathbb{F}$; in general, after orientating the simplices of K arbitrarily, they have the form:

$$\partial_p[i,j] = \begin{cases} c(\sigma) & \text{if } \sigma_i \in \partial_p(\sigma_j) \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $c(\sigma) \in \mathbb{F}$ is an arbitrary constant satisfying $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex. In practice, $c(\sigma)$ is typically set to ± 1. Towards relaxing the persistent Betti computation in dynamic setting, we propose an alternative choice for $c(\sigma)$ which endows continuity in the entries of ∂_p in T.

Definition 1 (Time-varying boundary matrix)**.** *Let $\mathbb{F} = \mathbb{R}$ denote the field, $\delta_{\mathcal{X}}(\cdot) = (X, d_X(\cdot))$ a DMS over a finite set X of fixed size $|X| = n$, and let $(\mathcal{P}(X), \preceq^*)$ be a linear extension of the face poset of the $(n-1)$-simplex Δ_n. For a fixed $\epsilon > 0$, a time-varying p-th boundary matrix ∂_p^t is an $\binom{n}{p} \times \binom{n}{p+1}$ matrix whose entries $c(\sigma)$ satisfy:*

$$\partial_p^t[i,j] = \begin{cases} |\epsilon - \mathrm{diam}_t(\sigma)|_+ & \textit{if } \sigma_i \in \partial_p(\sigma_j) \\ 0 & \textit{otherwise} \end{cases}$$

and whose rows and columns are ordered by $\preceq^*$ for all $t \in T$.

We now show a few properties that ∂_p^t exhibits which is advantageous for optimization. Clearly the entries of ∂_p^t must vary continuously in $t \in T$. Moreover, for all $t, t' \in T$, we have:

1. $\mathrm{rank}(\partial_p^t) = \dim(\mathrm{B}_{p-1}(K_t))$ where $K_t = \mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t))$
2. $\|\partial_p^t - \partial_p^{t'}\|_F \leq C|t - t'|(p+1)$ when $\delta_{\mathcal{X}}$ is C-Lipshitz over T
3. $\|\partial_p^t\|_2 \leq \sqrt{\kappa}\,(p+1)\,\epsilon$ where $\kappa = \max \sum\limits_{t \in T} \sum\limits_{\sigma \in K_t} \mathbb{1}(\mathrm{diam}(\sigma) \leq \epsilon)$

Proof. First, consider property (1). For any $t \in T$, applying the boundary operator ∂_p to $K_t = \mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t))$ with non-zero entries satisfying (**??**) by definition yields a matrix ∂_p satisfying $\mathrm{rank}(\partial_p) = \dim(\mathrm{B}_{p-1}(K_t))$. In contrast, definition (**??**) always produces p-boundary matrices of Δ_n; however, notice that the only entries which are non-zero are precisely those whose simplices σ that satisfy $\mathrm{diam}(\sigma) < \epsilon$. Thus, $\mathrm{rank}(\partial_p^t) = \dim(\mathrm{B}_{p-1}(K_t))$ for all $t \in T$. < (show proof of (2))> Property (3) follows from the construction of ∂_p and from the inequality $\|A\|_2 \leq \sqrt{m}\|A\|_1$ for an $n \times m$ matrix A, as $\|\partial_p^t\|_1 \leq (p+1)\,\epsilon$ for all $t \in T$. □

Rank Relaxation (TODO)

In light of the formulation of (**??**), we may interpret $\beta_p^{i,j}$ from a function composition perspective:

$$t \overset{f}{\mapsto} \partial_*^t \overset{g}{\mapsto} \mathrm{rank}(\partial_*^t)$$

In this sense, by modifying the entries of ∂_p^* via **??**, we ensure that f is both continuous and inherits the the smoothness of $\partial_{\mathcal{X}}(\cdot)$. We now address g.

Smoothness

Given a function $f : \mathbb{R} \to (-\infty, \infty]$ and a fixed $\mu > 0$, the *proximal operator* or *prox* of f is given by:

$$\mathrm{prox}_f^\mu(x) = \operatorname*{arg\,min}_{u \in \mathbb{R}} \left\{ f(u) + \frac{1}{2\mu}\|u - x\|^2 \right\} \tag{14}$$

When f is proper closed convex, $\mathrm{prox}_f^\mu(x)$ is single-valued and yields the solution to the *Moreau envelope* of f:

$$M_f^\mu(x) = \min_{u \in \mathbb{R}} \left\{ f(u) + \frac{1}{2\mu}\|x - u\|^2 \right\} = f(\mathrm{prox}_f^\mu(x)) + \frac{1}{2\mu}\|x - \mathrm{prox}_f^\mu(x)\|^2$$

$M_f^\mu(x)$ exhibits a number of properties suitable for optimization: it is $\frac{1}{\mu}$-Lipshitz over $\mathbb{R}$, it retains the same minima as f, and for any $x \in \mathbb{R}$ it admits a gradient $\nabla M_f^\mu(x)$ which again is expressible via the proximal operator:

$$\nabla M_f^\mu(x) = \frac{1}{\mu}(x - \mathrm{prox}_f^\mu(x)) \tag{15}$$

Moreover, if ∇f is L_f-Lipshitz, then $\nabla M_f^\mu(x)$ is L_f/μ-Lipshitz, and if f is L_f-Lipshitz, then $|f(x) - M_f^\mu(x)| \leq L_f^2\mu$. Thus, the Moreau envelope acts as a smooth approximation of f, which makes it an excellent candidate for smoothing (**??**) if prox_f^μ can be efficiently computed. Fortunately, the prox of $\|\cdot\|_*$ admits a simple characterization:

$$\mathrm{prox}_{\|\cdot\|_*}^\mu(X) = \underset{Y \in \mathbb{R}^{n\times m}}{\arg\min} \left\{ \|Y\|_* + \frac{1}{2\mu}\|Y - X\|_F^2 \right\} \tag{16}$$

$$= U\mathcal{D}_\mu(S)V^T \tag{17}$$

where $X = USV^T$ is the SVD of an $(n \times m)$ matrix X and $\mathcal{D}_\mu(S) = \mathrm{diag}(\{|\sigma_i - \mu|_+\}_{1\leq i \leq r})$ is the application of the *soft-thresholding operator* to the singular values $S = \mathrm{diag}(\{\sigma_i\}_{1\leq i\leq r})$. Equation (**??**) yields the prox operator for nuclear norm $\|A\|_*$, which is not necessarily a convex function. However, it is known that $\|\cdot\|_*$ is convex over the set $\{X \in \mathbb{R}^{n\times m} : \|X\|_2 \leq m\}$, thus we may inherit the smoothing properties of M_f^μ by considering the prox operator for the function $f : X \mapsto \frac{1}{m}\|X\|_*$. Letting $\alpha = m^{-1}$, the proximal operator of this function is given by:

$$\mathrm{prox}_{\alpha f}^\mu(X) = \underset{Y \in \mathbb{R}^{n\times m}}{\arg\min} \left\{ \alpha\|Y\|_* + \frac{1}{2\mu}\|Y - X\|_F^2 \right\} = \alpha \cdot \mathrm{prox}_f^{\mu\alpha}(X) \tag{18}$$

Remark 2. *Show example of Moreau envelope*

A. Appendix

Dynamic Metric Spaces

Consider an $\mathbb{R}$-parameterized metric space $\delta_X = (X, d_X(\cdot))$ where X is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_+$, satisfying:

1. For every $t \in \mathbb{R}, \delta_X(t) = (X, d_X(t))$ is a pseudo-metric space[1]
2. For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous.

When the parameter $t \in \mathbb{R}$ is interpreted as *time*, the above yields a natural characterization of a "time-varying" metric space. More generally, we refer to an $\mathbb{R}^h$-parameterized metric space as *dynamic metric space*(DMS). Such space have been studied more in-depth [] and have been shown...

Rank relaxation

A common approach in the literature to optimize quantities involving $\mathrm{rank}(A)$ for some $m \times n$ matrix A is to consider optimizing its *nuclear norm* $\|A\|_* = \mathrm{tr}(\sqrt{A^TA}) = \sum_{i=1}^r |\sigma_i|$, where σ_i denotes the ith singular value of A and $r = \mathrm{rank}(A)$. One of the primary motivations for this substitution is that the nuclear norm is a convex envelope of the rank function over the set:

$$S := \{A \in \mathbb{R}^{n\times m} \mid \|A\|_2 \leq m\}$$

That is, for an appropriate $m > 0$, the function $A \mapsto \frac{1}{m}\|A\|_*$ is a lower convex envelope of the rank function over S. The nuclear norm also admits a subdifferential... thus, we may consider replacing (**??**) with:

$$\beta_p^{i,j}(t) = |\partial_{p,t}^{1,i}| - m_1^{-1}\|\partial_{p,t}^{1,i}\|_* - m_2^{-1}\|\partial_{\bar{p},t}^{1,j}\|_* - m_3^{-1}\|\partial_{\bar{p},t}^{\bar{i},j}\|_* \tag{19}$$

where $\bar{c} = c + 1$. Now, if $t \mapsto \partial_p^*(t)$ is a non-decreasing, convex function in t, then the composition ... is convex, as each of the individual terms are convex. Moreover, we have...

$<$ Insert proof about this relaxation always lower-bounding β $>$

[1] This is required so that if one can distinguish the two distinct points $x, x' \in X$ incase $d_X(t)(x, x') = 0$ at some $t \in \mathbb{R}$.

Computation

In this section, we discuss the computation of suitable bases for the subspaces $Z_p(X_*)$, $B_p(K_*)$, and $Z_p(X_*) \cap B_p(X_*)$. In particular, we address two cases: the *dense* case, wherein the aforementioned bases are represented densely in memory, and the *sparse* case, which uses the structure of a particular decomposition of the boundary matrices to derive bases whose size in memory inherits the sparsity pattern of the decomposition.

Sparse case: We require an appropriate choice of bases for the groups $B_{p-1}(K_*)$ and $Z_p(X_*) \cap B_p(X_*)$. For some fixed $t \in T$, let $R_p = \partial_p V_p$ denote the decomposition discussed above, and let $b, d \in \mathbb{R}_+$ be fixed constants satisfying $b \leq d$. Since the boundary group $B_{p-1}(K_b)$ lies in the image of the ∂_p, it can be shown that a basis for the boundary group $B_{p-1}(K_*)$ is given by:

$$M_p^b = \{ \operatorname{col}_{R_{p+1}}(j) \neq 0 \mid j \leq b \} \qquad \textit{span}() \tag{20}$$

Moreover, since $B_{p-1}(K_b) = \operatorname{Im}(\partial_p^b)$, we have $\operatorname{span}(M_p^b) = B_{p-1}(K_b)$ and thus $\operatorname{rank}(M_p^b) = \operatorname{rank}(\partial_p^b)$. Indeed, it can be shown that every lower-left submatrix of ∂_p^* satisfies $\operatorname{rank}(\partial_p^*) = \operatorname{rank}(R_p^*)$. Thus, although M_p^b does provide a minimal basis for the boundary group $B_{p-1}(K_b)$, it is unneeded here.

A suitable basis for the cycle group can also be read off from the reduced decomposition directly as well. Indeed, let $R_p = \partial_p V_p$ as before. Then the cycle group is spanned by linear combinations of columns of V_p:

$$Z_p^b = \{ \operatorname{col}_{V_p}(j) \mid \operatorname{col}_{R_p}(j) = 0, j \leq b \} \tag{21}$$

The formulation of a basis spanning $Z_p(K_i) \cap B_p(K_j)$ is more subtle, as we can no longer use the fact that every lower-left submatrix of R_p has the same rank as the same lower-left submatrix of ∂_p. Nonetheless, a basis for this group can be obtained by reading off specific columns from R_p:

$$M_p^{b,d} := \{ \operatorname{col}_{R_{p+1}}(k) \neq 0 \mid 1 \leq k \leq d \text{ and } 1 \leq \operatorname{low}_{\mathrm{R_{p+1}}}(k) \leq b \} \tag{22}$$

One can show that M_b^d does indeed span $Z_p(X_*) \cap B_p(X_*)$ by using the fact that the non-zero columns of R_p with indices at most at most d form a basis for $B_p(K_d)$, and that each low-row index for every non-zero is unique.

Dense case: In general, persistent homology groups and its various factor groups are well-defined and computable with the reduction algorithm with coefficients chosen over any ring. By applying operations with respect to a field $\mathbb{F}$, both the various group structures $Z_p(K_\bullet) \subseteq B_p(K_\bullet) \subseteq C_p(K_\bullet)$ and their induced quotient groups $H_p(K_\bullet)$ are vector spaces; thus, the computation of suitable bases can be approached from a purely linear algebraic perspective. In particular, by fixing $\mathbb{F} = \mathbb{R}$, we inherit not only many useful tools for obtaining suitable bases for these groups, but also access to their corresponding optimized implementations as well.

Consider the p-th boundary operator $\partial_p^* : C_p(K_*) \to C_{p-1}(K_*)$ whose matrix realization with respect to some choice of simplex ordering $\{\sigma_i\}_{1 \leq i \leq m}$ we also denote with ∂_p. By definition, the boundary group $B_p(K_*)$ is given by the image $\operatorname{Im}(\partial_{p+1}^*) = B_p(K_*)$, thus one may basis for $B_p(K_*)$ by computing the considering the first $r > 0$ columns of the reduced SVD:

$$M_p^* = [\, u_1 \mid u_2 \mid \cdots \mid u_r \,] = \{ \ \} \tag{23}$$