< insert motivating examples >

Suppose one observes points in a geometric space whose position is driven by some unknown continuous-time system. Towards understanding its dynamic, one may ask whether one can infer properties of the underlying evolving system

Introduction

To formalize this, suppose one has an T-parameterized metric space $\delta_X = (X, d_X(\cdot))$, where $d_X : \mathrm{T} \times X \times X \to \mathbb{R}_+$ such that $(X, d_X(t))$ is a pseudo-metric space for every $t \in \mathrm{T}$ and $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous for every pair $x, x' \in X$. For a fixed $t \in \mathrm{T}$, the *Vietoris-Rips* complex at scale $\epsilon \in \mathbb{R}$ is the abstract simplicial complex given by

$$\mathrm{Rips}_{\epsilon,\mathrm{t}}(X, d_X(t)) := \{\sigma \subset X : d_X(t)(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\}$$

By connecting successive complexes via inclusion maps $\mathrm{Rips}_{\epsilon,\mathrm{t}} \hookrightarrow \mathrm{Rips}_{\epsilon',\mathrm{t}}$ for $\epsilon < \epsilon'$, one obtains a family of complexes is called the *Vietoris-Rips filtration* $\mathrm{Rips}_{\alpha,t} := \{\mathrm{Rips}_\epsilon\}_{\epsilon \leq \alpha}$ as some fixed $t \in \mathrm{T}$. These inclusions induce maps at level of homology, i.e.

$$\mathrm{H}_p(\mathrm{Rips}_{\epsilon,t}) \hookrightarrow \mathrm{H}_p(\mathrm{Rips}_{\epsilon',t}) \hookrightarrow \cdots \hookrightarrow \mathrm{H}_p(\mathrm{Rips}_{\alpha,t})$$

where $0 \leq \epsilon \leq \epsilon' \leq \alpha$. The pth *Betti number* is defined as the dimension of any of these homology groups $\beta_p = \dim(H_p(\mathrm{Rips}_{\alpha,t})))$. By restricting our attention to the persistent homology groups which were born before $b \in \mathbb{R}$ and died after $d \in \mathbb{R}$, we obtain the p-th *persistent Betti number* with respect to (b, d) at time $t \in \mathrm{T}$:

$$\beta_p^{b,d} = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ d_X\right)(t)$$

This quantity can be readily visualized as the number of persistent pairs lying inside the box $[0, b] \times (d, \infty)$ on the collection of all persistence diagrams for varying $t \in \mathrm{T}$. We consider the problem of maximizing the p-th *persistent* Betti number $\beta_p^{b,d}$ over T:

$$t_* = \operatorname*{arg\,max}_{t \in \mathrm{T}} \beta_p^{b,d}(t) \tag{1}$$

Since Betti numbers are integer-valued invariants, direct optimization is difficult. Moreover, the space of persistence diagrams is [banach space statement].... Nonetheless, the differentiability of persistence has been studied extensively in [show chain rule paper on persistence diagrams]...

A motivating derivation

For the moment, we omit the subscript $t \in \mathrm{T}$ and focus our attention on a particular instance in time. Let $B_p(X_*) \subseteq Z_p(X_*) \subseteq C_p(X_*)$ denote the p-th boundary, cycle, and chain groups of X_*, respectively. Given a simplicial filtration $X_\bullet$, let boundary operator $\partial_p : C_p(X_\bullet) \to C_p(X_\bullet)$ denote the boundary operator sending p-chains to their respective boundaries. With a slight abuse of notation, we use ∂_p to also denote the filtration boundary matrix with respect to the ordered basis $(\sigma_i)_{1 \leq i \leq m_p}$. Recall the p-th persistent Betti number between scales (b, d) is defined as:

$$\begin{aligned}
\beta_p^{b,d} &= \dim(H_p^{b,d}) \\
&= \dim\left(Z_p(X_b)/(Z_p(X_b) \cap B_p(X_d)\right) \\
&= \dim\left(Z_p(X_b)\right) - \dim\left(Z_p(X_b) \cap B_p(X_d)\right)
\end{aligned} \tag{2}$$

Note we may rewrite 2 with a straightforward application of the rank-nullity theorem:

$$\beta_p^{b,d} = \dim\left(C_p(X_b)\right) - \dim\left(B_p(X_b)\right) - \dim\left(Z_p(X_b) \cap B_p(X_d)\right) \tag{3}$$

Let ∂_p^b and $\partial_p^{b,d}$ denote matrices whose columns span the subspaces $B_p(X_b)$ and $Z_p(X_b) \cap B_p(X_d)$, respectively. We address their computation in section **??**. Observe that equation 3 can be written as:

$$\beta_p^{b,d} = |\,\partial_p^b\,| - \mathrm{rank}(\partial_p^b) - \mathrm{rank}(\partial_p^{b,d}) \tag{4}$$

$$= |\,\partial_p^b\,| - \left(\mathrm{rank}(\partial_p^b) + \mathrm{rank}(\partial_p^{b,d})\right) \tag{5}$$

$$= |\,\partial_p^b\,| - \mathrm{rank}\left(\left[\begin{array}{c|c} \partial_p^b & 0 \\ \hline 0 & \partial_p^{b,d} \end{array}\right]\right) \tag{6}$$

Thus, the persistence Betti number can be expressed as a difference between a simple quantity to compute and the rank of a particular matrix. A remarkable result established by [] show that the rank$(\cdot)$ function is lower-bounded by the convex envelope... [describe this more in detail]

Relaxation

We would like a continuous relaxation of equation 6 amenable to optimization.

Bases computation

In this section, we discuss the computation of suitable bases for the subspaces $Z_p(X_*)$, $B_p(X_*)$, and $Z_p(X_*) \cap B_p(X_*)$. In particular, we address two cases: the *dense* case, wherein the aforementioned bases are represented densely in memory, and the *sparse* case, which uses the structure of a particular decomposition of the boundary matrices to derive bases whose size in memory inherits the sparsity pattern of the decomposition.

Dense case:

Dynamic Metric Spaces

Consider an $\mathbb{R}$-parameterized metric space $\delta_X = (X, d_X(\cdot))$ where X is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_+$, satisfying:

1. For every $t \in \mathbb{R}, \delta_X(t) = (X, d_X(t))$ is a pseudo-metric space[1]
2. For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous.

When the parameter $t \in \mathbb{R}$ is interpreted as *time*, the above yields a natural characterization of a "time-varying" metric space. More generally, we refer to an $\mathbb{R}^h$-parameterized metric space as *dynamic metric space*(DMS). Such space have been studied more in-depth [] and have been shown...

Homology

Let K be an abstract simplicial complex and $\mathbb{F}$ a field. A p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K. The collection of p-chains under addition yields an $\mathbb{F}$-vector space denoted $C_p(K)$. The p-boundary $\partial_p(\sigma)$ of a p-simplex $\sigma \in K$ is the alternating sum of its oriented co-dimension 1 faces, and the p-boundary of a p-chain is defined linearly in terms of its constitutive simplices. A p-chain with zero boundary is called a p-cycle, and together they form $Z_p(K) = \operatorname{Ker} \partial_p$. Similarly, the collection of p-boundaries forms $B_p(K) = \operatorname{Im} \partial_{p+1}$. Since $\partial_p \circ \partial_{p+1} = 0$ for all $p \geq 0$, then the quotient space $H_p(K) = Z_p(K)/B_p(K)$ is well-defined, and called the p-th homology of K with coefficients in $\mathbb{F}$. If $\{K_i\}_{i \in [m]}$ is a filtration, then the inclusion maps $K_i \subset K_{i+1}$ induce linear transformations at the level of homology:

$$H_p(K_1) \to H_p(K_2) \to \cdots \to H_p(K_m) \tag{7}$$

Simplices whose inclusion in the filtration creates a new homology class are called *creators*, and simplices that destroy homology classes are called *destroyers*. The filtration indices of these creators/destroyers are referred to as *birth* and *death* times, respectively. The collection of birth/death pairs (i, j) is denoted $\mathrm{dgm}_p(K)$, and referred to as the p-th *persistence diagram* of K. If a homology class is born at time i and dies entering time j, the difference $|i - j|$ is called the *persistence* of that class. In practice, filtrations often arise from triangulations parameterized by geometric scaling parameters, and the "persistence" of a homology class actually refers to its lifetime with respect to the scaling parameter.

Rips complex

$$\mathrm{Rips}_\epsilon(X) = \{S \subseteq X : S \neq \emptyset \ \text{ and } \ \mathrm{diam}(S) \leq \epsilon\} \tag{8}$$

Letting the scale parameter $\epsilon \in \mathbb{R}$ vary, one obtains a filtration of simplicial complexes connected by inclusion maps:

$$\mathrm{Rips}_\epsilon(X) \to \mathrm{Rips}_{\epsilon'}(X) \to \cdots \to \mathrm{Rips}_{\epsilon''}(X)$$

[1]This is required so that if one can distinguish the two distinct points $x, x' \in X$ incase $d_X(t)(x, x') = 0$ at some $t \in \mathbb{R}$.