Notes about convergence rate

Matt Piekenbrock

If the value x^* that a sequence $\{x_n\} = x_1, x_2, \ldots, x_n$ approaches exists, then $\{x_n\}$ is called *convergent*. Moreover, if there exist real numbers $\mu \leq 1$ and $\alpha \geq 1$ such that:

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = \mu$$

then sequence is convergent, and the value of α is called the *rate of convergence*. When $\mu \in (0, 1)$, the When $\alpha = 1$, 2, and 3, we say the sequence exhibits *linear*, *quadratic*, and *cubic* convergence, respectively. A sequence with rate $1 < \alpha < 2$ is said to converge *superlinearly*.

1 Spectral Sparsifiers & Effective Resistance

Given a weighted undirected graph $G = (V, E, w)$ with $w : E \to \mathbb{R}_+$, denote with L_G its weighted *graph Laplacian* $L_G = D - W$, where $D = \mathrm{diag}(\deg^w(v_1), \deg(v_2), \ldots, \deg(v_n))$ is a diagonal (weighted) degree matrix and W is the weighted adjacency matrix satisfying $W[i, j] = -w_{ij}$ for $i \neq j$ and $v_i \sim v_j$. An intuitive goal is to choose a sparse subgraph $H = (V, E')$ with $E' \subseteq E$ satisfying:

$$(1 - \epsilon)L_G \preceq L_H \preceq (1 + \epsilon)L_G \quad \Leftrightarrow \quad (1 - \epsilon)x^T L_G x \leq x^T L_H x \leq (1 + \epsilon)x^T L_G x$$

If we can find a subgraph H satisfying this, then L_H is said to be an ϵ*-spectral sparsifier* of G. Such sparsifiers are naturally very appealing in that they satisfying the following properties:

1. (Cut approximation) The weight of every cut $w_G(E(S, S'))$ is within $1 \pm \epsilon$ of $L_H(E(S, S'))$
2. (κ-approximation) $(1 - \epsilon)^{-1}H$ is a $(1 + \epsilon)(1 - \epsilon)^{-1}$-approximation of G; H is called a κ*-approximation* of G if:

$$L_G \preceq L_H \preceq \kappa L_G \quad \Leftrightarrow \quad \kappa^{-1} L_G^+ \preceq L_H^+ \preceq L_G^+$$

3. (Spectral approximation) If $\Lambda(L_G) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\Lambda(L_H) = \{\lambda'_1, \lambda'_2, \ldots, \lambda'_n\}$, then $(1 - \epsilon)\lambda'_k \leq \lambda_k \leq (1 + \epsilon)\lambda'_k$

Since the definition does not explicitly bound the size of E', clearly a subgraph $H \subseteq G$ always exists. It's not immediately clear whether one can obtain sparser graphs H with e.g. edge sparsities $|E'| \sim O(n \log(n) \cdot \epsilon^{-1})$. Surprisingly, Spielman showed a positive existential result towards this direction: every connected, *unweighted* graph $G = (V, E)$ admits a *weighted* subgraph $H = (V, E', w')$ that is an ϵ-spectral sparsifier, and the number of edges of these sparsifiers $|E'|$ is the order of $O(\epsilon^{-2} n \log n)$. Moreover, they gave a simple randomized sampling algorithm to obtain such sparsifiers.

The idea of their approach is as follows: Let L_e denote its restriction to edge $e \in E$, i.e. the $n \times n$ matrix with $L_e[i, i] = L_e[i, j] = 1$ and $L_e[i, i] = L_e[j, j] = -1$. This matrix is given by the outer product $L_e = (\xi_i - \xi_j)(\xi_i - \xi_j)^T$, where ξ_i is the characteristic vector with a 1 in the i-th component and 0 otherwise. Observe that:

$$L_G = \sum_{e \in E} w_e L_e$$

Now, suppose we have edge probabilites $\{p_e\}_{e \in E}$ satisyfing $\sum_{e \in E} p_e = 1$ and we construct H by sampling k edges iid from E with respect to these probabilities. By definition of epxectation, we have:

$$\mathbb{E}[L_H] = \sum_{e \in E} p_e \cdot \frac{1}{kp_e} L_e = \sum_{e \in E} \frac{1}{k} L_G$$

Thus L_H approximates L_G via a sum of iid random matrices. By utilizing random matrix theory, one may readily apply concentration inequalities, such as Chernoff bounds, to bound the degree of the approximation. In particular, if $L_H^{(i)} \preceq \delta \mathbb{E}[L_H^{(i)}]$ with probability 1 for any $\delta \geq 1$, then with $k \sim O(\delta \cdot \epsilon^{-2} \cdot n \log n)$ edges sampled, one has:

$$P(L_H \subset_\epsilon L_G) \geq 1 - 2ne^{-\epsilon^2 k}/4\delta$$

Computing the effective resistance of a given edge requires—almost by definition—the solution of a linear system on the graph Laplacian.