1 Introduction

Motivation Persistent homology is, as of this time of writing, a well-studied mathematical structure. From its unassuming history starting with postniov towers, the body of research created over the past two decades have established perisstence as not only an intrinsic quantity, but a useful tool. From **, ** , to **, applications abound with persistence; see for an survey. PH is more than just a homology inference tool.

Since its inception, a popular application of persistence in data analysis is its use as a featurization tool. In machine learning, featurization is a means of converting various data representations to a vector format amenable for learning and enhanced training. Classical examples include word2vec in natural language processing, Scale-Invariant Feature Transform (SIFT) in computer vision, extended-connectivity fingerprints (ECFs) in used in chemical informatics and molecular modeling, etc. More recent results include transformers... Through no small feat of engineering, many of these techniques have been incrementally improved and adapted throughout the past decades, and tend to do quite well in terms of their efficiency. As such, they have seen widespread-adoption from more scientific fields trying to harness their power. While certainly useful, one of the pitfalls with such featurizations if the difficulty that comes with interpretation. Difficulty in heavy featurization has lead to qualitative comparisons in scientific fields of the featurization outputs. Many are lead by the same equation: exactly what isa featurization tool capturing that is so useful for training?

Persistent homology is, in some sense, a natural tool for featurization. persistence is a stable invariant that comes equipped mathematical guarantees; thus featurization of diagrams can be interpreted as mapping persistence diagrams to Euclidean space in such a way that maximally preserves the topological information conveyed by the diagram. Moreover, we also know persistence diagrams retain some amount of geometry, such as those of curvature sets and the quasi-isometry theorems in distributed persistence. These results suggest an inverse theory related to persistence. Indeed, a recent injectivity result shows that collections of persistence diagrams are sufficient to uniquely characterize data sets in 2- and 3-dimensions[1], establishes persistence as truly an intrinsic description of shape.

Our Contribution: A few of the drawbacks with persistence-related featurization tools, such as persistence images [] or persistence landscapes [], is that they require as input the persistence diagram to produce the featurization. As a $O(m^3)$ operation, the traditional reduction computation can become infeasible, a property exacerbating greatly in the time-varying setting.

The Relaxation & Main Results We give below the the relaxation proposed in the paper. The rest of the paper is devoted to its properties and use cases.

Given an input data set X that varies along a 1-parameter family $\{X_{(t)}\}_{t \in T}$, for each $t \in T$, do the following:

1. Choose a geometric realization $\mathcal{K}$ of X. This could be a simple Rips filtration, or a Delaunay complex, a simplicial mesh, a neighborhood graph, etc.

2. Fix a set of coordinates $\mathcal{I} = \{(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)\}$ from the upper-half plane Δ_+ representing areas where significant topological changes are expected to occur.

3. Compute our relaxation of the PBNs at each $(a, b) \in \mathcal{I}$,

$$\beta_p^{i,j} = f(\hat{\partial}_p) - f(\hat{\partial}_p)$$

   where a) $A^{i,j}$ denotes the lower-left submatrix and b) $\hat{\partial}_p^*$ represents the p-th boundary matrix of K whose elementary chains $c(\sigma)$ take on values from a relaxation of the step function. and c) $f : K \to \mathbb{R}$ is a a function that approximates $\mathrm{rank}(A)$ using the spectrum $\sigma(A)$ of A. We give a few examples of suitable functions in section **??**.

The main observations our relaxation is based on is that the (1) PBNs $\beta_p^{i,j}$ can be written as sum of ranks of boundary matrices, (2) by working with real-valued coefficients, we can parameterize the non-zero entries of ∂_* with smoothly varying values whose signs match (or approximate) the signs of ∂, and (3) there exist a myriad of spectrum-based relaxations of the *rank* function that are easier to compute.

That $\beta_p^{i,j}$ is estimated using the spectrum(s) of $\hat{\partial}_*$ is the primary step to reducing the complexity of our relaxation; the special forms of the matrices involved in our relaxation have connections back to spectral graph theory.

[1]

Organization In what follows, we introduce a relaxation of the persistent Betti number (PBN) invariant that has certain advantages. Namely, we showing that a simple augmentation of traditional PBN computation leads to a continuous relaxation that $(1+\epsilon)$-approximates the PBN. Moreover, we show that this relaxation is permutation invariant, obeys and inclusion-exclusion principles, and admits a notion of stability in a certain sense—all properties useful in time-varying settings. Moreover, we show our relaxation is *interpetretable* as it satisfies certain basic "Betti-like" properties and we illicit its connections back to spectral graph theory.

2 Background & Notation

A *simplicial complex* $K \subseteq \mathcal{P}(V)$ over a vertex set $V = \{v_1, v_2, \dots, v_n\}$ is a collection of simplices $\{\sigma : \sigma \in \mathcal{P}(V)\}$ such that $\tau \subseteq \sigma \in K \implies \tau \in K$. A *filtration* $K_\bullet = \{K_i\}_{i \in I}$ of a simplicial complexes indexed by a totally ordered set I is a family of complexes such that $i < j \in I \implies K_i \subseteq K_j$. $K_\bullet$ is called *simplexwise* if $K_j \smallsetminus K_i = \{\sigma_j\}$ whenever j is the immediate successor of i in I and $K_\bullet$ is called *essential* if $i \neq j$ implies $K_i \neq K_j$:

$$\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet, \quad K_i = K_{i-1} \cup \{\sigma_i\} \tag{1}$$

Filtrations may be equivalently defined as functions $f : K \to I$ satisfying $f(\tau) \leq f(\sigma)$ whenever $\tau \subseteq \sigma$. Here, we consider two index sets for I: $\mathbb{R}$ and $[n] = \{1, \dots, n\}$. Any finite filtration may be trivially converted into an essential, simplexwise filtration via a set of *condensing*, *refining*, and *reindexing* maps [**?**]. Thus, without loss of generality, we exclusively consider essential simplexwise filtrations and for brevity refer to them as filtrations.

For K a simplicial complex and $\mathbb{F}$ a field, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K. The collection of p-chains under addition yields an $\mathbb{F}$-vector space denoted $C_p(K)$. The p-boundary $\partial_p(\sigma)$ of an oriented p-simplex $\sigma \in K$ is defined as the alternating sum of its oriented co-dimension 1 faces:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \dots, v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p] \tag{2}$$

where $\hat{v}_i$ indicates the removal of v_i from the ith summand. Similarly, the p-boundary of a p-chain is defined linearly in terms of its constitutive simplices. A p-chain with zero boundary is called a p-cycle, and together they form $Z_p(K) = \mathrm{Ker}\, \partial_p$. Similarly, the collection of p-boundaries forms $B_p(K) = \mathrm{Im}\, \partial_{p+1}$. Since $\partial_p \circ \partial_{p+1} = 0$ for all $p \geq 0$, the quotient space $H_p(K) = Z_p(K)/B_p(K)$ is well-defined, and $H_p(K)$ is called the p-th homology of K with coefficients in $\mathbb{F}$. The dimension of the p-th homology group $\beta_p(K) = \dim(H_p(K))$ of K is called the p-th *Betti* number of K.

Let $K_\bullet = \{K_i\}_{i \in [m]}$ denote a filtration of size $|K_\bullet| = m$. Let $\Delta_+^m = \{(i,j) : 0 \leq i < j \leq m\}$ denote the set of valid pairs of filtration indices. For every such pair $(i,j) \in \Delta_+^m$, the inclusions $K_i \subsetneq K_{i+1} \subsetneq \cdots \subsetneq K_j$ induce linear transformations $h_p^{i,j}$ at the level of homology:

$$0 = H_p(K_0) \to \cdots \to H_p(K_i) \underbrace{\to \cdots \to}_{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_m) = H_p(K_\bullet) \tag{3}$$

When $\mathbb{F}$ is a field, this sequence of homology groups admits a unique decomposition of $K_\bullet$ into a pairing of simplices (σ_i, σ_j) [] demarcating the evolution of homology classes: σ_i marks the creation of a homology class, σ_j marks its destruction, and the difference $|i - j|$ records the lifetime of the class, called its *persistence*. The p-th persistent homology groups are the images of these transformations and the p-th persistent Betti numbers are their dimensions:

$$H_p^{i,j} = \begin{cases} H(K_i) & i = j \\ \mathrm{Im}\, h_p^{i,j} & i < j \end{cases}, \qquad \beta_p^{i,j} = \begin{cases} \beta_p(K_i) & i = j \\ \dim(H_p^{i,j}) & i < j \end{cases} \tag{4}$$

For a fixed $p \geq 0$, the collection of persistent pairs (i,j) together with unpaired simplices (l, ∞) form a summary representation $\mathrm{dgm}_p(K_\bullet)$ called the *p-th persistence diagram of* $K_\bullet$. Note that the persistent Betti numbers can be read off directly given $\mathrm{dgm}_p(K_\bullet)$; conceptually, $\beta_p^{i,j}$ simply counts the number of persistent pairs lying inside the box $[0, i] \times (j, \infty)$ (see Figure **??**)—the number of persistent homology groups born at or before i that died sometime after j.

Remark 2: Persistence has been viewed from many different perspectives, and may be defined in a variety of ways. For example, Carlsson et al. [] observed persistence is simply a graded module under a particular polynomial

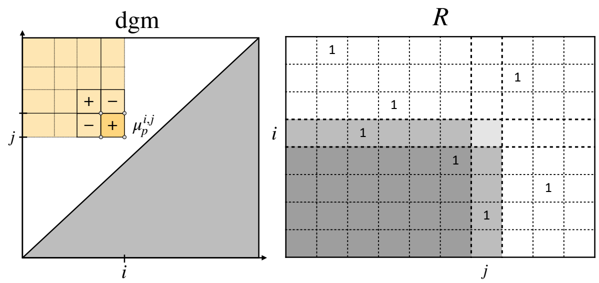

Figure 1: Depicting the consequence of the inclusion-exclusion property of PBNs. Left: the multiplicity $\mu_p^{i,j}$ is given by cancellations of the PBNs,

ring. More recently, Baur studied persistence in a form a matching. Use follow the presentation from Cohen-Steiner et al []: given a tame function $f : K \to \mathbb{R}$, its homological critical values $\{a_i\}_{i=1}^n$, and an interleaved sequence $\{b_i\}_{i=0}^n$ satisfying $b_{i-1} < a_i < b_i$ for all i, the p-th persistence diagram $\mathrm{dgm}_p(f) \subset \bar{\mathbb{R}}^2$ of a filtration induced by f is defined as:

$$\mathrm{dgm}_p(K_\bullet) = \{\, (a_i, a_j) : \mu_p^{i,j} \neq 0 \,\} \,\cup\, \mathcal{L} \tag{5}$$

where $\mathcal{L}$ denotes the points on the diagonal, counted with infinite multiplicity, and $\mu_p^{i,j}$ is defined as:

$$\mu_p^{i,j} = \left(\beta_p^{i,j\text{-}1} - \beta_p^{i,j}\right) - \left(\beta_p^{i\text{-}1,j\text{-}1} - \beta_p^{i\text{-}1,j}\right) \qquad \text{for } 0 < i < j \leq n+1 \tag{6}$$

where $\beta_p^{i,j}$ is the persistent Betti number defined at the values of the interleaved sequence, i.e. $\beta_p^{i,j} = \dim(\mathrm{Im}(h_p^{b_i,b_j}))$. More generally, by interpreting μ_p^* as function defined over $\bar{\mathbb{R}}$, Chazal [] view the multiplicity μ as a counting measure. This interpretation (as well as Cohen-Steiners) is perhaps the most relvent to the work we present here.

3 Main Result

In what follows, we briefly outline the computation of the

Derivation:

As in section 2, let $B_p(K_*) \subseteq Z_p(K_*) \subseteq C_p(K_*)$ denote the p-th boundary, cycle, and chain groups of K_*, respectively. Given a simplicial filtration $K_\bullet$, let $\partial_p : C_p(K_\bullet) \to C_p(K_\bullet)$ denote the boundary operator sending p-chains to their respective boundaries. With a slight abuse of notation, we also use ∂_p to also denote the filtration boundary matrix with respect to an ordered basis $(\sigma_i)_{1 \leq i \leq m_p}$. The p-th persistent Betti number between scales (i, j), $i < j$, is defined as:

$$\begin{aligned} \beta_p^{i,j} &= \dim(H_p^{i,j}) \\ &= \dim\left(Z_p(K_i)/(Z_p(K_i) \cap B_p(K_j)\right) \\ &= \dim\left(Z_p(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \end{aligned} \tag{7}$$

While the dimension of $Z_p(K_i)$ is given by the dimension of the kernel of $\partial_p(K_i)$ and is thus easily computed, there are a variety of ways to address the computation of the intersection term (the persistence part). Zomorodian et al [] give a procedure to compute a basis for $Z_p(K_i) \cap B_p(K_j)$ via a sequence of Gaussian-elimination reductions. Conceptually intuitive alternatives for obtaining such basis is to use projectors, as studied by Ben Israel [].

We require more notation. If A is a $m \times n$ matrix, let $A^{i,j}$ denote the lower-left submatrix defined by last $m-i+1$ rows (rows i through m, inclusive) and the first j columns, and let $A^{*,j}$ denote the submatrix given by including all m rows and the first j columns. For any $1 \leq i < j \leq m$, define $r_A(i, j)$ as follows:

$$r_A(i,j) = \mathrm{rank}(A^{i,j}) - \mathrm{rank}(A^{i+1,j}) + \mathrm{rank}(A^{i+1,j\text{-}1}) - \mathrm{rank}(A^{i,j\text{-}1}) \tag{8}$$

A consequence of the Pairing Uniqueness Lemma [] is that the quantity $r_R(i, j)$ has the effect of detecting whether $R[i, j] \neq 0$, see Figure 3. In fact, the this lemma may be generalized to provide *μ-queries*, which count the number of . Moreover, the structure theorem from [] shows that 1-parameter persistence modules can be decomposed in an *essentially unique* way []. As a result, we have our first lemma:

Lemma 1. *Let $R = \partial V$ denote the matrix decomposition of a given filtered boundary matrix ∂ derived from the associated filtration $K_\bullet$. For any pair (i, j) satisfying $1 \leq i < j \leq m$, we have:*

$$\mathrm{rank}(R^{i,j}) = \mathrm{rank}(\partial^{i,j}) \tag{9}$$

Equivalently, all lower-left submatrices of ∂ have the same rank as their corresponding submatrices in R.

Lemma 1 was the essential motivating step used by Chen et al [] in their rank-based persistence algorithm, which was the first output-sensitive algorithm given for computing persistent homology of a filtered complex. We show this result allows us to write the persistent Betti number as a sum of rank functions defined over *unfactored* matrices.

Proposition 1. *For some fixed $p \geq 0$, let $R_p = \partial_p V_p$ denote the reduced decomposition of the p-dimension boundary matrix ∂_p of some filtration $K_\bullet$.*

$$\beta_p^{i,j} = \mathrm{rank}(I_p^{*,i}) - \mathrm{rank}(\partial_p^{*,i}) - \mathrm{rank}(\partial_{p+1}^{*,j}) + \mathrm{rank}(\partial_{p+1}^{i+1,j}) \tag{10}$$

where $I_p^{*,i}$ denotes the identity matrix with i columns.

In conclusion, we may write the persistent Betti number as a combination of rank computations performed directly on the (ordered) dimension p and $(p+1)$ boundary matrices. As we will show in the next section, this formulation has a few advantages over the approaches given by Zomorodian Carlsson, or the projector approaches [].

A Time-varying Boundary Matrix Relaxation

Thus, we require alternative expressions for each of the terms in equation (10) to extend its applicability to the time-varying setting. Towards deriving these expression, we first require a replacement of the standard boundary matrix formulation.

Recall that the boundary operator ∂_p for a finite simplicial filtration $K_\bullet$ with $m = |C_p(K_\bullet)|$ and $n = |C_{p-1}(K_\bullet)|$ can be represented by an $(n \times m)$ boundary matrix ∂_p whose columns and rows correspond to p-simplices and $(p-1)$-simplices, respectively. The entries of ∂_p depend on the choice of $\mathbb{F}$; in general, after orientating the simplices of K arbitrarily, they have the form:

$$\partial_p[i, j] = \begin{cases} c(\sigma_j) & \text{if } \sigma_i \in \partial_p(\sigma_j) \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $c(\sigma_*) \in \mathbb{F}$ is an arbitrary constant satisfying $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex, typically set to ± 1. Towards relaxing the persistent Betti computation in dynamic setting, we seek an alternative choice for $c(\sigma)$ which endows continuity in the entries of ∂_p in T.

Observe that one of the consequence of the formulation **??** with is that we may take advantage of the the various properties of the rank function. One such property is permutation invariance, i.e. $\mathrm{rank}(A) = \mathrm{rank}(P^T A P)$. Thus, we need not represent the boundary matrices ∂_p in the filtration order. Moreover, unlike the vineyards algorithm [], this also implies we do not need to *maintain* the filtration order in the time-varying setting—as long as the matrices have the correct non-zero pattern as their filtered equivalents, the PBNs may be deduced via rank functions.

We revisit the choice of index sets. We formalize this with the following definition.

Definition 1 (Time-varying boundary matrix)**.** *Let $\mathbb{F} = \mathbb{R}$ denote the field, and let $(\mathcal{P}(X), \preceq^*)$ be a linear extension of the face poset of $K_\bullet$. For some constant $\omega > 0$, a time-varying p-th boundary matrix ∂_p^t is an $\binom{n}{p} \times \binom{n}{p+1}$ matrix whose entries $c(\sigma)$ satisfy:*

$$\partial_p^t[i, j] = \begin{cases} \pm\, S_\omega(\epsilon) & \textit{if } \sigma_i \in \partial_p(\sigma_j), \textit{ where } \epsilon = f(\sigma_j) \\ 0 & \textit{otherwise} \end{cases}$$

where $S_{\omega,\epsilon}(\sigma_j)$ is a smooth-step *function whose range decreases smoothly from $1 \to 0$ in the interval $f(\sigma_j) \pm \frac{\omega}{2}$, and where rows and columns of ∂_p^t follow fixed order given by $\preceq^*$ for all $t \in T$.*

We now show a few properties that ∂_p^t exhibits which is advantageous for time-varying systems. Clearly the entries of ∂_p^t must vary continuously in $t \in T$. Moreover, for fixed $p \geq 0$, we have:

1. $\mathrm{rank}(\partial_p^t) \to \dim(\mathrm{B}_{p-1}(K_t))$ as $\omega \to 0$, for all $t \in T$
2. $\|\partial_p^t - \partial_p^{t'}\|_F \sim O(m_p)$ when $\delta_\mathcal{X}$ is C-Lipshitz over T and $|t - t'|$ is small,
3. $\|\partial_p^t\|_2 \leq \epsilon\sqrt{\kappa}\,(p+1)$ where $\kappa = \max \sum\limits_{t \in T} \sum\limits_{\sigma \in K_t} \mathbb{1}(\mathrm{diam}(\sigma) \leq \epsilon)$

Rank Relaxation (TODO)

By fixed our field of coefficients $\mathbb{F} = \mathbb{R}$ and changing the boundary chain formulation from (**??**), we're able to express the PBN as a sum of ranks of unfactored, continuously varying matrices. As integer-valued invariants, Betti numbers pose several difficulties to vectorization. This is perhaps best illustrated.... The excessive freedom associated with pure topological equivalence makes discrimination difficult. Contrary to a topologists intuition, we seek a relaxation whose sensitivity to geometry is adjustable.

We opt for the generic rank approximation method proposed by []; for any $n \times n$ matrix A and some fixed $\alpha > 0$:

$$\Phi_\epsilon(A) = \mathrm{tr}\left(A(A^TA + \epsilon I)^{-1}A^T\right) = \sum_{i=1}^{r} \frac{\sigma_i^2(A)}{\sigma_i^2(A) + \epsilon}$$

where $\sigma_i(A)$ are the singular values of A. This ϵ-approximation scheme has a few advantages; namely, the smoothness of $\Phi_\epsilon(A)$ now depends on the spectrum of A, $0 \leq \Phi_\epsilon(A) \leq \mathrm{rank}(A)$ for all $\epsilon > 0$, and $\mathrm{rank}(A) - \Phi_\epsilon(A) \leq \epsilon \sum_{i=1}^{r} \sigma_i(A)^{-2}$. Plugging this into equation, we have following relaxation:

$$\hat{\beta}_p^{i,j} = \Phi_\epsilon(\hat{I}_p^{*,i}) - \Phi_\epsilon(\hat{\partial}_p^{*,i}) - \Phi_\epsilon(\hat{\partial}_{p+1}^{*,j}) + \Phi_\epsilon(\hat{\partial}_{p+1}^{i+1,j}) \tag{12}$$

Moreover, we immediately have the following Corollary:

Corollary 1. *For any filtration $K_\bullet$, there exists an $\epsilon^* > 0$ such that $\beta_p^{i,j} = \lceil \hat{\beta}_p^{i,j} \rceil$ for all $\epsilon \in (0, \epsilon^*]$.*

Basic properties

Corollary 2 (Approximately monotone)**.** *Let $\beta_p^{i,j}$ denote the persistent Betti number, and let $\hat{\beta}_p^{i,j}$ denote our relaxation. It is known that the PBN satisfies []:*

$$\beta_p^{i,j} \leq \beta_p^{i',j} \text{ for all } i \leq i'$$

Given fixed non-negative constants (ϵ, ω), the relaxation in equation **??** *satisfies:*

$$\hat{\beta}_p^{i,j} \leq f_{\epsilon,\omega}(i' - i)^k + \hat{\beta}_p^{i',j} \text{ for all } i \leq i'$$

It is known that PBNs satisfy a certain inclusion/exclusion property related to the fact they are subadditive. Namely, for any $0 < a < b \leq c < d$, it is known that:

$$\mu_p$$

As discussed in section **??**, this leads naturally to the interpretation of PBNs as *counting measures* defined over the upper half-plane. We show that the induced multiplicity function:

Corollary 3 (Subadditivity)**.** *For any $0 < a < b \leq c < d$, the relaxed multiplicity $\hat{\mu}_p$ function satisfies:*

$$\hat{\mu}_p$$

Interpretation

4 Applications

A Appendix

Proofs

A.1.1 Proof of lemma1

Proof. The Pairing Uniqueness Lemma [] asserts that if $R = \partial V$ is a decomposition of the total $m \times m$ boundary matrix ∂, then for any $1 \leq i < j \leq m$ we have $\mathrm{low}_R[j] = i$ if and only if $r_\partial(i,j) = 1$. As a result, for $1 \leq i < j \leq m$, we have:

$$\mathrm{low}_R[j] = i \iff r_R(i,j) \neq 0 \iff r_\partial(i,j) \neq 0 \tag{13}$$

Extending this result to equation (9) can be seen by observing that in the decomposition, $R = \partial V$, the matrix V is full-rank and obtained from the identity matrix I via a sequence of rank-preserving (elementary) left-to-right column additions. □

Proof of Proposition 1

Proof. We first need to show that $\beta_p^{i,j}$ can be expressed as a sum of rank functions. Note that by the rank-nullity theorem, so we may rewrite (7) as:

$$\beta_p^{i,j} = \dim\left(C_p(K_i)\right) - \dim\left(B_{p-1}(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)$$

The dimensions of groups $C_p(K_i)$ and $B_p(K_i)$ are given directly by the ranks of diagonal and boundary matrices, yielding:

$$\beta_p^{i,j} = \mathrm{rank}(I_p^{*,i}) - \mathrm{rank}(\partial_p(K_i)) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)$$

where I_p^i is a $m_p \times m_p$ copy of the identity matrix with diagonal entries $I_p[k,k] = 1$ for all $k < i$. To express the intersection term, note that we need to find a way to express the number of p-cycles born at or before index i that became boundaries before index j. Observe that the non-zero columns of R_{p+1} with index at most j span $B_p(K_j)$. Now, since the low entries of the non-zero columns of R_{p+1} are unique, we have:

$$\dim(Z_p(K_i) \cap B_p(K_i)) = |\Gamma_p^{i,j}| \tag{14}$$

where $\Gamma_p^{i,j} = \{\, \mathrm{col}_{R_{p+1}[k]} \neq 0 \mid\ k \in [j],\, 1 \leq \mathrm{low}_{R_{p+1}}[k] \leq i \,\}$. Consider the complementary matrix $\bar{\Gamma}_p^{i,j}$, given by the non-zero columns of R_{p+1} with index at most j that are not in $\Gamma_p^{i,j}$, i.e. have $\mathrm{low}_{R_{p+1}}[k] > i$. Combining with the observation above, we have:

$$\dim(B_p(K_j)) - |\Gamma_p^{i,j}| = |\bar{\Gamma}_p^{i,j}| = \mathrm{rank}(R_{p+1}^{i+1,j}) \tag{15}$$

Combining equations (14) and (15) yields:

$$\dim(Z_p(K_i) \cap B_p(K_j)) = |\Gamma_p^{i,j}| = \dim(B_p(K_j)) - |\bar{\Gamma}_p^{i,j}| = \mathrm{rank}(R_{p+1}^{*,j}) - \mathrm{rank}(R_{p+1}^{i+1,j}) \tag{16}$$

Observing the final matrices in (16) are *lower-left* submatrices of R_{p+1}, the final expression (10) follows by applying Lemma 1 repeatedly. □

A.1.2 Proof of boundary matrix properties

Proof. First, consider property (1). For any $t \in T$, applying the boundary operator ∂_p to $K_t = \mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t))$ with non-zero entries satisfying (11) by definition yields a matrix ∂_p satisfying $\mathrm{rank}(\partial_p) = \dim(\mathrm{B}_{p-1}(K_t))$. In contrast, definition (1) always produces p-boundary matrices of Δ_n; however, notice that the only entries which are non-zero are precisely those whose simplices σ that satisfy $\mathrm{diam}(\sigma) < \epsilon$. Thus, $\mathrm{rank}(\partial_p^t) = \dim(\mathrm{B}_{p-1}(K_t))$ for all $t \in T$. < (show proof of (2))> Property (3) follows from the construction of ∂_p and from the inequality $\|A\|_2 \leq \sqrt{m}\|A\|_1$ for an $n \times m$ matrix A, as $\|\partial_p^t\|_1 \leq (p+1)\,\epsilon$ for all $t \in T$.

□

Dynamic Metric Spaces Consider an $\mathbb{R}$-parameterized metric space $\delta_X = (X, d_X(\cdot))$ where X is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_+$, satisfying:

1. For every $t \in \mathbb{R}, \delta_X(t) = (X, d_X(t))$ is a pseudo-metric space[2]
2. For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous.

When the parameter $t \in \mathbb{R}$ is interpreted as *time*, the above yields a natural characterization of a "time-varying" metric space. More generally, we refer to an $\mathbb{R}^h$-parameterized metric space as *dynamic metric space*(DMS). Such space have been studied more in-depth [] and have been shown...

Rank relaxation A common approach in the literature to optimize quantities involving $\mathrm{rank}(A)$ for some $m \times n$ matrix A is to consider optimizing its *nuclear norm* $\|A\|_* = \mathrm{tr}(\sqrt{A^T A}) = \sum_{i=1}^r |\sigma_i|$, where σ_i denotes the ith singular value of A and $r = \mathrm{rank}(A)$. One of the primary motivations for this substitution is that the nuclear norm is a convex envelope of the rank function over the set:

$$S := \{A \in \mathbb{R}^{n \times m} \mid \|A\|_2 \leq m\}$$

[2]This is required so that if one can distinguish the two distinct points $x, x' \in X$ incase $d_X(t)(x, x') = 0$ at some $t \in \mathbb{R}$.

That is, for an appropriate $m > 0$, the function $A \mapsto \frac{1}{m}\|A\|_*$ is a lower convex envelope of the rank function over S. The nuclear norm also admits a subdifferential... thus, we may consider replacing (**??**) with:

$$\beta_p^{i,j}(t) = |\partial_{p,t}^{1,i}| - m_1^{-1}\|\partial_{p,t}^{1,i}\|_* - m_2^{-1}\|\partial_{\bar{p},t}^{1,j}\|_* - m_3^{-1}\|\partial_{\bar{p},t}^{\bar{i},j}\|_* \tag{17}$$

where $\bar{c} = c + 1$. Now, if $t \mapsto \partial_p^*(t)$ is a non-decreasing, convex function in t, then the composition ... is convex, as each of the individual terms are convex. Moreover, we have...

< Insert proof about this relaxation always lower-bounding β >

Application: Time-varying Let $\delta_\mathcal{X}$ denote an T-parameterized metric space $\delta_\mathcal{X}(\cdot) = (X, d_X(\cdot))$, where $d_X : \mathrm{T} \times X \times X \to \mathbb{R}_+$ is called a *time-varying metric* and X is a finite set with fixed cardinality $|X| = n$. δ_X as called a *dynamic metric space* (DMS) iff $d_X(\cdot)(x, x')$ is continuous for every pair $x, x' \in X$ and $\delta_\mathcal{X}(t) = (X, d_X(t))$ is a pseudo-metric space for every $t \in \mathrm{T}$. For a fixed $t \in \mathrm{T}$, the Rips complex at scale $\epsilon \in \mathbb{R}$ is the abstract simplicial complex given by

$$\mathrm{Rips}_\epsilon(\delta_\mathcal{X}(t)) := \{\sigma \subset X : d_X(t)(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{18}$$

As before, the family of Rips complexes for varying $\epsilon > 0$ yields a filtration whose inclusion maps induce linear maps at the level of homology. The time-varying counterpart is analogous. In this context, we write the p-th persistent Betti number with respect to fixed values $i, j \in I$ as a function of $t \in \mathrm{T}$:

$$\beta_p^{i,j}(t) = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ \delta_\mathcal{X}\right)(t) \tag{19}$$

A Boundary matrix factroization

Definition 2 (Boundary matrix decomposition)**.** *Given a filtration $K_\bullet$ with m simplices, let ∂ denote its $m \times m$ filtered boundary matrix. We call the factorization $R = \partial V$ the* boundary matrix decomposition *of ∂ if:*

I1. V is full-rank upper-triangular

I2. R satisfies $\mathrm{low}_R[i] \neq \mathrm{low}_R[j]$ iff its i-th and j-th columns are nonzero

where $\mathrm{low}_R(i)$ denotes the row index of lowest non-zero entry of column i in R or null *if it doesn't exist. Any matrix R satisfying property (I2) is said to be* reduced*; that is, no two columns share the same low-row indices.*