Persistent Betti Numbers *over time*

Theory, computation, and applications

"Murmurations"

Persistent Homology is an *intrinsic invariant*

Persistent Homology (PH) is a well-established tool in the sciences[1,2]

PH has many attractive properties beyond homology:

1. *General* : Persistence can be generalized *via rank functions* [5,6]
2. *Descriptive* : $d_B(\mathrm{dgm}_p(X), \mathrm{dgm}_p(Y))$ lower-bounds $d_{GH}(X, Y)$[7]
3. *Geometric* : distributed dgm's interpolate local geometry $\leftrightarrow$ global topology[4]
4. *Stable* : $d_B(\mathrm{dgm}(f), \mathrm{dgm}(g)) \leq \|f - g\|_\infty$ between function f, g[3]

Collections of dgm's uniquely characterize simplicial complexes in $\mathbb{R}^d$[7]

$$\boxed{\text{PH is more then just a homology inference tool!}}$$

1. Wigner, Eugene P. "The unreasonable effectiveness of mathematics in the natural sciences." Mathematics and Science. 1990. 291-306.
2. Turkeš, Renata, Guido Montúfar, and Nina Otter. "On the effectiveness of persistent homology." arXiv preprint arXiv:2206.10551 (2022).
3. Cohen-Steiner, David, Herbert Edelsbrunner, and John Harer. "Stability of persistence diagrams." Discrete & computational geometry 37.1 (2007): 103-120.
4. Solomon, Elchanan, Alexander Wagner, and Paul Bendich. "From geometry to topology: Inverse theorems for distributed persistence." arXiv preprint arXiv:2101.12288 (2021).
5. Zomorodian, Afra, and Gunnar Carlsson. "Computing persistent homology." Discrete & Computational Geometry 33.2 (2005): 249-274.
6. Bergomi, Mattia G., and Pietro Vertechi. "Rank-based persistence." arXiv preprint arXiv:1905.09151 (2019).
7. Turner, Katharine, Sayan Mukherjee, and Doug M. Boyer. "Persistent homology transform for modeling shapes and surfaces." Information and Inference: A Journal of the IMA 3.4 (2014): 310-344.

Persistent Homology is a ~~intrinsic~~ *difficult invariant*

The persistence computation scales $\sim O(m^3)$ over K with $m = |K|$ simplices

Morozov gave an counter-example[1] showing this bound to be tight (i.e. $\Omega(m^3)$)

$$\implies \text{computing } \mathrm{dgm}_p(K) \sim \Theta(m^3) = \Theta(n^{3(p+2)}) \text{ over an } n\text{-point set, for } p \geq 1$$

Simple algorithm $\neq$ *simple implementation*

1. Computing $R = \partial V$ is *memory intensive*: $|V| \sim O(m^2)$
2. K's structure affects complexity (e.g. 2-manifolds $\sim O(n\alpha(n))$[2])
3. Theory is extensive: *clearing*[3], *apparent pairs*[4], *cohomology*[5], ...
4. $\mathbb{F}$ matters: $\mathbb{Z}_2$ columns $\leftrightarrow$ 64-arity bit-trees + DeBruijin "magic" tables[6]

1. Morozov, Dmitriy. "Persistence algorithm takes cubic time in worst case." BioGeometry News, Dept. Comput. Sci., Duke Univ 2 (2005).
2. Dey, Tamal Krishna, and Yusu Wang. Computational topology for data analysis. Cambridge University Press, 2022.
3. Chen, Chao, and Michael Kerber. "Persistent homology computation with a twist." Proceedings 27th European workshop on computational geometry. Vol. 11. 2011.
4. Bauer, Ulrich. "Ripser: efficient computation of Vietoris–Rips persistence barcodes." Journal of Applied and Computational Topology 5.3 (2021): 391-423.
5. De Silva, Vin, Dmitriy Morozov, and Mikael Vejdemo-Johansson. "Dualities in persistent (co) homology." Inverse Problems 27.12 (2011): 124003.
6. See PHAT's source: https://github.com/blazs/phat/blob/master/include/phat/representations/bit_tree_pivot_column.h

This Talk: Persistent Betti numbers *over time*

The persistence dgm of function f is *defined* by persistent Betti numbers (PBNs)

$$\mathrm{dgm}_p(f) \subset \overline{\mathbb{R}}^2 \Leftrightarrow (i, j) \text{ such that } \mu_p^{i,j} \neq 0$$

where $\mu_p^{i,j}$ is called the *multiplicity function*, defined as:

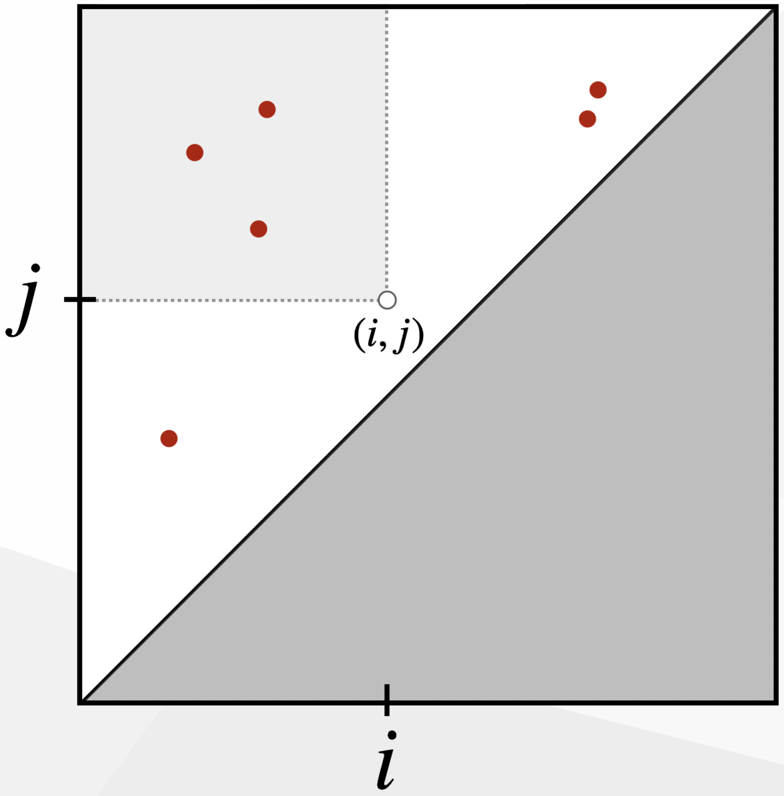
$$\mu_p^{i,j} = (\beta_p^{i,j\text{-}1} - \beta_p^{i,j}) - (\beta_p^{i\text{-}1,j\text{-}1} - \beta_p^{i\text{-}1,j})$$

$\beta_p : K_\bullet \times \mathbb{R} \to \mathbb{Z}_+ \implies$ *Betti curve* over filtration

$\beta_p : \mathcal{P}(X) \times \mathbb{R} \times \mathbb{R} \to \mathbb{Z}_+ \implies$ *Betti curv<u>es</u>* over 1-parameter family

$\beta_p^{i,j} : \mathcal{P}(X) \times \mathbb{R} \to \mathbb{Z}_+ \implies$ *Persistent Betti curves* for *fixed* $i, j \in I$

This talk will focus on relaxing $\beta_p^{i,j}$ for *time-varying settings* dgm



$$\beta_p^{i,j} = \dim(H_p(K_i) \to H_p(K_j))$$

Outline

$\longrightarrow$ Background $\longleftarrow$

- Simplicial Complexes
- Cyle, Boundary, and Chain Groups
- Filtrations and Persistent Homology

The Main Result

- $\beta_p^{i,j}$'s definition + computation
- A clever observation + trick
- Re-thinking chains and ranks w/ coefficients in $\mathbb{R}$
- Relaxation: definition and properties

Applications

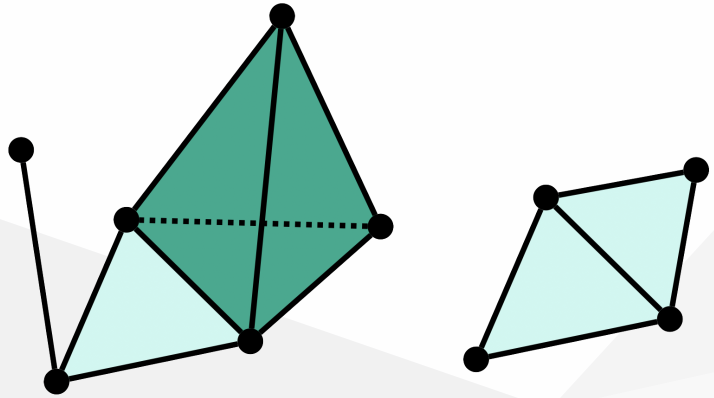
- A $(1-\epsilon)$-approximation of $\beta_p^{i,j}$
- Signatures of time-varying systems
- Leveraging PHT theory

Background: Simplicial Complexes

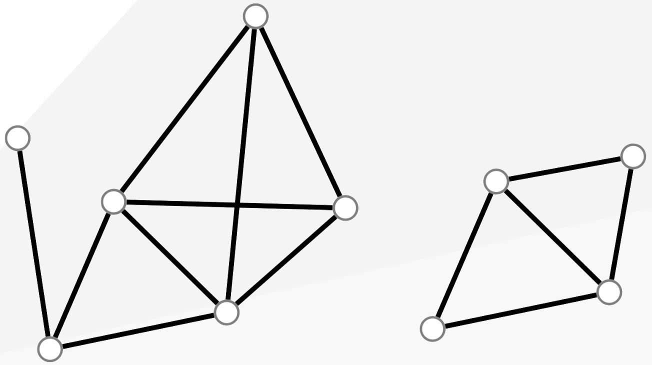
A *simplicial complex* $K = \{\sigma : \sigma \in \mathcal{P}(V)\}$ over set $V = \{v_1, \ldots, v_n\}$ satisfies:

$$\text{(vertex)} \quad v \in V \implies \{v\} \in K$$

$$\text{(face)} \quad \tau \subseteq \sigma \in K \implies \tau \in K$$



simplicial 3-complex



$K^{(1)}$

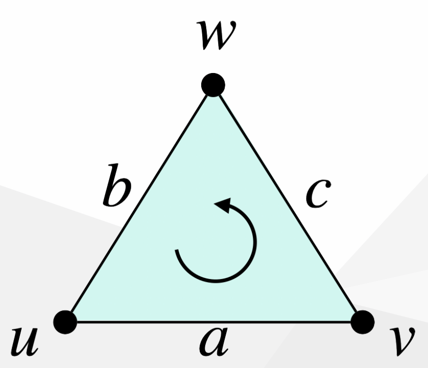
All computations here will be with *finite simplicial complexes*

Background: Boundaries

Given an *oriented* p-simplex $\sigma \in K$, define its p-boundary as the alternating sum:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \ldots, v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots v_p]$$

We will make heavy use of *oriented boundary matrices*



K

	a	b	c
u	1		1
v	-1	1	
w		-1	-1

∂_p

By default, we will work generically with the simplex-wise lexicographical order

Background: The Groups

Given a pair $(K , \mathbb{F})$, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K

The operator ∂_p extends linearly to p-chains via their constitutive simplices

$$c = \sum_{i=1}^{m_p} \alpha_i \sigma_i , \qquad c + c' = \sum_{i=1}^{m_p} (\alpha_i + \alpha_i') \sigma_i$$

Given $\mathbb{F}$ a field and K a simplicial complex, the following groups are defined[1]

$$C_p(K) = (K , + , \times , \mathbb{F}) \iff \textit{vector space of } p\textit{-chains}$$

$$B_p(K) = (\mathrm{Im} \circ \partial_{p+1})(K) \iff \textit{boundary group}$$

$$Z_p(K) = (\mathrm{Ker} \circ \partial_p)(K) \iff \textit{cycle group}$$

$$H_p(K) = Z_p(K) / B_p(K) \iff \textit{homology group}$$

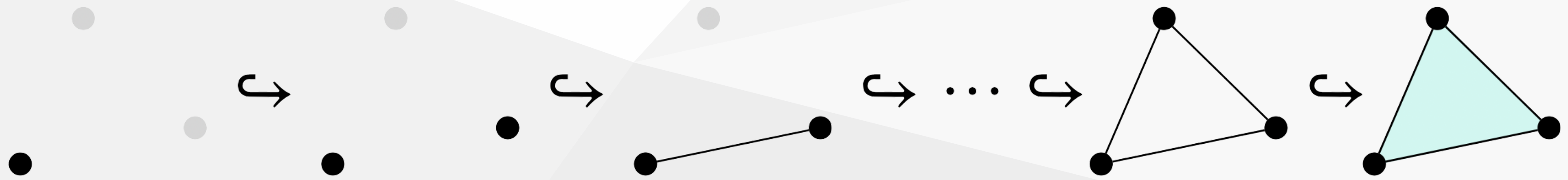
Munkres, James R. Elements of algebraic topology. CRC press, 2018.

Background: Filtrations

A *filtration* $K_\bullet$ is a family $\{ K_i \}_{i \in I}$ indexed over a totally ordered index set I:

$$\begin{aligned} \textit{Filtered} &\iff K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet \\ \textit{Essential} &\iff i \neq j \text{ implies } K_i \neq K_j \\ \textit{Simplexwise} &\iff K_j \setminus K_i = \{\sigma_j\} \text{ when } j = \operatorname{succ}(i) \end{aligned}$$

Any $K_\bullet \mapsto$ essential & simplexwise via *condensing* + *refining* + *reindexing maps* [1]



Note here that I may be $\mathbb{R}_+$ or $[m] = \{ 1, 2, \dots, m \}$, depending on the context!

Bauer, Ulrich. "Ripser: efficient computation of Vietoris–Rips persistence barcodes." Journal of Applied and Computational Topology 5.3 (2021): 391-423.

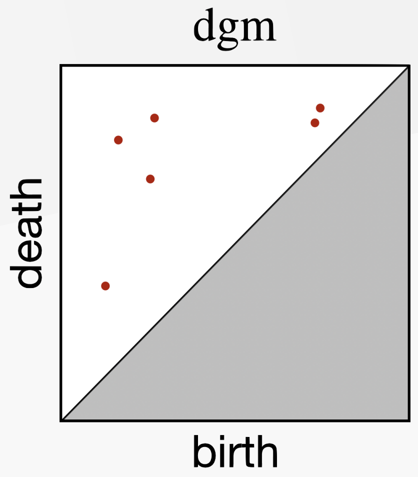
Background: Persistent Homology

Inclusions $K_i \hookrightarrow K_j$ induce linear transformations $h_p^{i,j}$ between homology groups

$$H_p(K_0) \to \cdots \to H_p(K_i) \underbrace{\to \cdots \to}_{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_m) = H_p(K_\bullet)$$

Properties of persistent homology groups:

1. $H_p(K_\bullet)$ admits a *pair decomposition* $\mathrm{dgm}(K) \subseteq \bar{\mathbb{R}}^2$
2. $\mathrm{dgm}(K)$ is *unique* iff $\mathbb{F}$ is a field
3. $\beta_p^{i,j}$ can be read-off directly for any i, j from $\mathrm{dgm}_p(K)$
4. Computed via matrix decomposition $R = \partial V$



For simplicity, we will use $\partial_p^i = \partial_p(K_i)$, $Z_p^i = Z_p(K^i)$, $B_p^i = B_p(K^i)$, etc.

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$\beta_p^{i,j}$: starting with the definition

Before extending $\beta_p^{i,j}$ to the time-varying setting, first consider its definition:

$$\begin{aligned}
\beta_p^{i,j} &= \dim(H_p(K_i) \to H_p(K_j)) \\
&= \dim\left(Z_p(K_i) \,/\, B_p(K_j)\right) \\
&= \dim\left(Z_p(K_i) \,/\, (Z_p(K_i) \cap B_p(K_j))\right) \\
&= \dim\left(Z_p(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \\
&= \dim\left(C_p(K_i)\right) - \dim\left(B_{p-1}(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)
\end{aligned}$$

Replacing the groups above with appropriate matrices / constants, we have:

$$\beta_p^{i,j} = |K_i^{(p)}| - \mathrm{rank}(\partial_p^i) - \mathrm{rank}(\partial_p^{i,j})$$

where $\partial_p^{i,j}$ is *some matrix* whose columns span $Z_p(K_i) \cap B_p(K_j)$...

Computing the *persistent* Betti number $\beta_p^{i,j}$

$$\boxed{\beta_p^{i,j} = \underbrace{\dim\left(C_p(K_i)\right)}_{(1)} - \underbrace{\dim\left(B_{p-1}(K_i)\right)}_{(2)} - \underbrace{\dim\left(Z_p(K_i) \cap B_p(K_j)\right)}_{(3)}}$$

Both (1) are (2) easy to obtain. Computing (3) is more subtle:

PH / reduction algorithm $\Longrightarrow$ $\sum_{k=1}^{j} \mathbf{1}(\mathrm{low}_{R_{p+1}}[k] \leq i)$

Gaussian elimination[1] $\Longrightarrow$ (see Zomorodian & Carlsson [1])

Anderson-Duffin formula[2] $\Longrightarrow$ $P_{\mathbf{Z}\cap\mathbf{B}} = 2P_{\mathbf{Z}}(P_{\mathbf{Z}} + P_{\mathbf{B}})^{\dagger} P_{\mathbf{B}}$

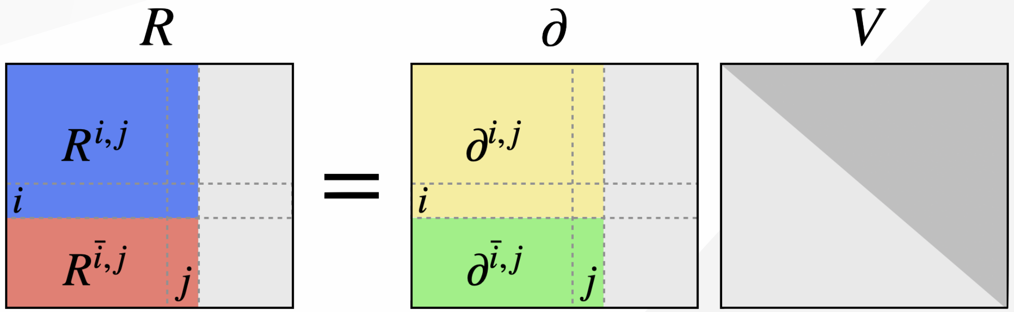
Alternative: $\beta_p^{i,j} = \mathrm{null}(\Delta_p^{i,j})$ where $\Delta_p^{i,j}$ is the *persistent Laplacian* [4]

All of these rely on <u>explicit reductions</u> or <u>expensive projectors</u>. Not great!

1. Zomorodian, Afra, and Gunnar Carlsson. "Computing persistent homology." Discrete & Computational Geometry 33.2 (2005): 249-274.
2. Ben-Israel, A., and A. Charnes. "On the intersections of cones and subspaces." Bulletin of the American Mathematical Society 74.3 (1968): 541-544.
3. Neumann, J. Von. "Functional Operators, Vol. II. The Geometry of Orthogonal Spaces. Annals of Math." Studies Nr. 22 Princeton Univ. Press (1950).
4. Mémoli, Facundo, Zhengchao Wan, and Yusu Wang. "Persistent Laplacians: Properties, algorithms and implications." SIAM Journal on Mathematics of Data Science 4.2 (2022): 858-884.

A clever observation

Let $R = \partial V$. Define the submatrices $R^{i,j}$, $R^{\bar{i},j}$, $\partial^{i,j}$, $\partial^{\bar{i},j}$ as follows:



The Pairing Uniqueness Lemma[2] can be used to show:

$$\mathrm{low}_R[j] = i \iff r_R(i,j) \neq 0 \iff r_\partial(i,j) \neq 0 \iff \mathrm{rank}(R^{\bar{i},j}) = \mathrm{rank}(\partial^{\bar{i},j})$$

where $r_A(i,j) := \mathrm{rank}(A^{\overline{i-1},j}) - \mathrm{rank}(A^{\bar{i},j}) + \mathrm{rank}(A^{\bar{i},j-1}) - \mathrm{rank}(A^{\overline{i-1},j-1})$

$$\boxed{\text{Take-a-way: } \mathrm{rank}(R^{\bar{i},j}) \text{ can be deduced from } \mathrm{rank}(\partial^{\bar{i},j}), \text{ for any } 1 \leq i < j < m}$$

This was the motivating exploit in first output-sensitive persistence algorithm[1,2]

1. Chen, Chao, and Michael Kerber. "An output-sensitive algorithm for persistent homology." Computational Geometry 46.4 (2013): 435-447.
2. Dey, Tamal Krishna, and Yusu Wang. Computational topology for data analysis. Cambridge University Press, 2022.

A clever trick

Pairing uniqueness[1] $\implies \operatorname{rank}(R^{\bar{i},j}) = \operatorname{rank}(\partial^{\bar{i},j})$, for any $1 \leq i < j < m$

Dey & Wang show[2] have shown the following:

$$\begin{aligned}\dim(Z_p^i \cap B_p^j) &= \dim(B_p^j) - \#(\mathrm{col}_{R_{p+1}}[k] \neq 0 \mid k \in [j],\ \mathrm{low}_{R_{p+1}}[k] > i) \\ &= \operatorname{rank}(R_{p+1}^{j,j}) - \operatorname{rank}(R_{p+1}^{\bar{i},j}) \\ &= \operatorname{rank}(\partial_{p+1}^{j,j}) - \operatorname{rank}(\partial_{p+1}^{\bar{i},j})\end{aligned}$$

Let I_p^i be the diagonal matrix with whose first i entries are 1. We can now write:

$$\beta_p^{i,j} = \operatorname{rank}(\mathrm{I}_p^i) - \operatorname{rank}(\partial_p^{i,i}) - \operatorname{rank}(\partial_{p+1}^{j,j}) + \operatorname{rank}(\partial_{p+1}^{\bar{i},j})$$

Thus, we may write $\beta_p^{i,j}$ completely in terms of *unfactored matrices*

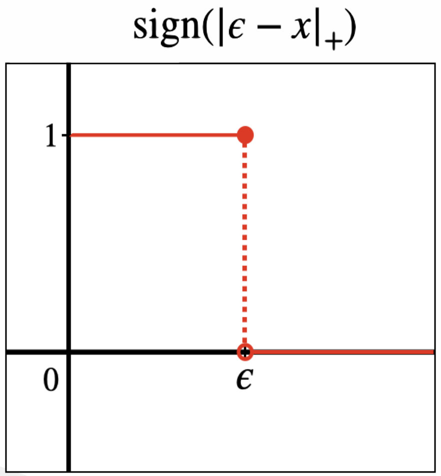
1. Edelsbrunner, Herbert, and John L. Harer. Computational topology: an introduction. American Mathematical Society, 2022.
2. Dey, Tamal Krishna, and Yusu Wang. Computational topology for data analysis. Cambridge University Press, 2022.

Parameterizing elementary p-chains

Suppose we fix $\mathbb{F} = \mathbb{R}$ and replace chain values with *smoothstep* functions $S_\epsilon(x)$

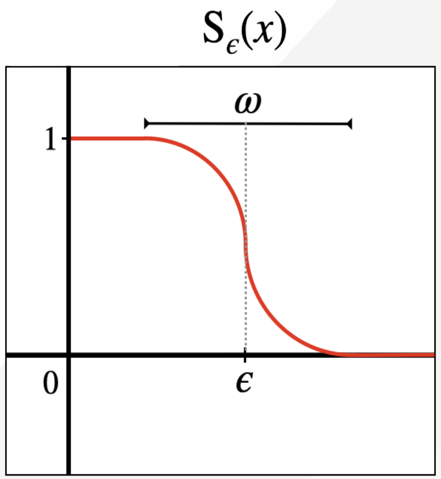
$$\begin{array}{c|ccc} & a & b & c \\ \hline u & 1 & & 1 \\ v & -1 & 1 & \\ w & & -1 & -1 \end{array} \quad \leftrightarrow$$

∂_p



Step

$\mapsto$



Smoothstep

$$\leftrightarrow \quad \begin{array}{c|ccc} & a & b & c \\ \hline u & S_a(x) & & S_c(x) \\ v & -S_a(x) & S_b(x) & \\ w & & -S_b(x) & -S_c(x) \end{array}$$

$\hat{\partial}_p$

$$\hat{\partial}_p[i,j] = \pm(S_{\epsilon_j} \circ f)(\sigma_j) \quad \text{if } \sigma_i \in \partial(\sigma_j), \quad \text{where } \epsilon_j = f(\sigma_j)$$

Advantage: If f varies continuous one-parameter family, $\hat{\partial}_p$ also varies continuously

A generic approximation of rank

Moreover, replace $\mathrm{rank}(A)$ with $\Phi_\epsilon(A)$, defined for some fixed $\epsilon > 0$ as:

$$\Phi_\epsilon(A) = \mathrm{tr}\left[A^T(AA^T + \epsilon I)^{-1}A\right] = \sum_i^n \frac{\sigma_i^2}{\sigma_i^2 + \epsilon}, \quad \text{where } \sigma_i^2 := \lambda_i(AA^T)$$

Observe $\Phi_\epsilon(A) \leq \mathrm{rank}(A)$, with equality when $\epsilon = 0$, yielding the final relaxation:

$$\hat{\beta}_p^{i,j} = \Phi_\epsilon(\mathrm{I}_p^i) - \Phi_\epsilon(\hat{\partial}_p^{i,i}) - \Phi_\epsilon(\hat{\partial}_{p+1}^{j,j}) + \Phi_\epsilon(\hat{\partial}_{p+1}^{\bar{i},j})$$

We use the spectrum of $\hat{\partial}_p^*$ is used to encode geometric information from f

Ex: Let $\delta_X = (X, d_X(\cdot))$, $d_X : X \times X \times \mathbb{R}$ be a *dynamic metric space*, and let:

$$\hat{\beta}_p^{i,j}(t) = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ \delta_{\mathcal{X}}\right)(t)$$

Observe that $\hat{\beta}_p^{i,j}(t) \in \mathbb{R}$ varies *continuously* with t—a time-varying relaxation!

Basic properties of $\hat{\beta}_p^{i,j}$

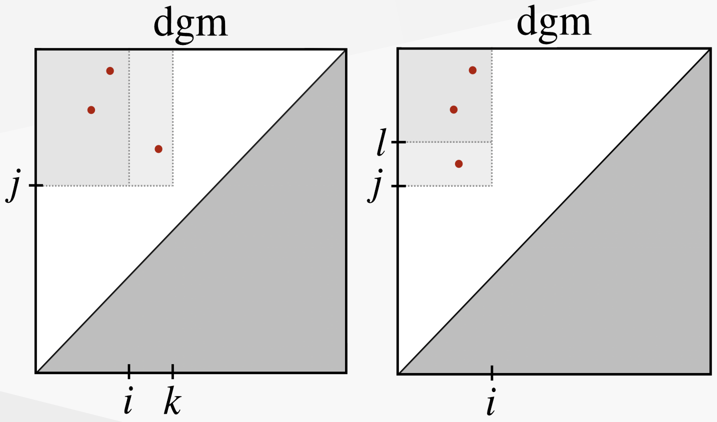
Observe $\Phi_\epsilon(A) = \sum_{i=1}^n \sigma_i^2/(\sigma_i^2 + \epsilon) \leq \mathrm{rank}(A)$, with equality obtained when $\epsilon = 0$

1. $\hat{\beta}_p^{i,j} \to \beta_p^{i,j} \quad \text{as} \quad \epsilon \to 0,\ \omega \to 0$
2. There $\exists$ an $\epsilon^* > 0$ such that $\lceil \hat{\beta}_p^{i,j} \rceil = \beta_p^{i,j}$ for all $\epsilon \in (0, \epsilon^*]$

$\hat{\beta}_p^{i,j}$ respects several of $\beta_p^{i,j}$ monotonicity properties *approximately*

$$\forall\ i < k,\ \beta_p^{i,j} \leq \beta_p^{k,j} \implies \hat{\beta}_p^{i,j} + \delta_\epsilon(k-i) \leq \hat{\beta}_p^{k,j}$$

$$\forall\ j < l,\ \beta_p^{i,j} \geq \beta_p^{i,l} \implies \hat{\beta}_p^{i,j} + \delta_\epsilon(l-j) \leq \hat{\beta}_p^{i,l}$$



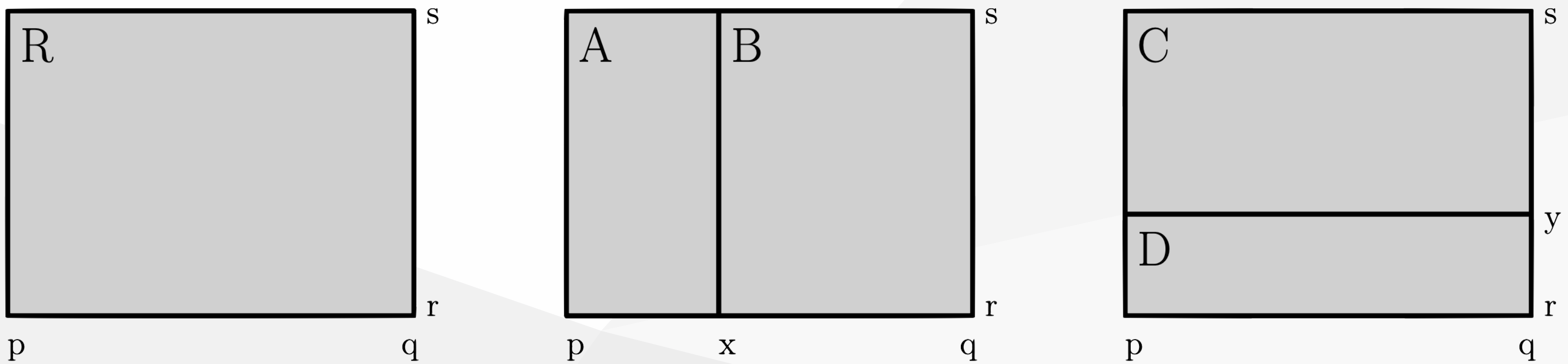
$\hat{\beta}_p^{i,j}$ also satisfies and ϵ-approximate version of *jump monotonicity*[1]

Persistence measures

Pairs (i, j) in dgm's can also be defined as limiting points w/ non-zero multiplicity

$$\mu_p^{i,j} = \min_{\epsilon > 0} \left\{ \beta_p^{i+\epsilon, j-\epsilon} - \beta_p^{i-\epsilon, j-\epsilon} - \beta_p^{i+\epsilon, j+\epsilon} + \beta_p^{i-\epsilon, j+\epsilon} \right\}$$

PBN's also yield "counting measures" in $\overline{\mathbb{R}}^2$, due to their additivity under splitting:



$$\mu(R) = \mu(A) + \mu(B) = \mu(C) + \mu(D)$$

$\hat{\mu}_\epsilon$ also obeys inclusion/exclusion—can be interpreted as a *persistence measure*

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$(1-\delta)$-approximation scheme for $\hat{\beta}_p^{i,j}$

The fixed parameters (ω, ϵ) completely determine the closeness of $|\hat{\beta}_p^{i,j} - \beta_p^{i,j}|$

The *Lanczos* method[1,2] computes q-largest $\sigma^2(A)$ of a sparse $m \times m$ matrix A in:

$$O(m \cdot T_m(A) + q^2 \cdot m)$$

where $T_m(A)$ is complexity of $v \mapsto Av$. Note ∂_* is highly structured, namely:

$$\mathrm{nnz}(\partial_p) \leq (p+1)m_p \sim O(m_p \log(m_p))$$

$v \mapsto \langle \partial_p, v \rangle$ takes $\sim O(\kappa_p)$ time where $\kappa_p = \sum \deg_p(\sigma_p)$

$\Delta_p = \mathrm{tr}(\partial_p \partial_p^T) = \sum \sigma_i^2(\partial_p)$ can be determined in $O(m_p)$ time

We deduce a $(1-\delta)$-approximation by computing the q-largest σ_i^2's such that:

$$\lceil \Delta_p^q / \Delta_p^{m_p} \rceil \geq (1-\delta)$$

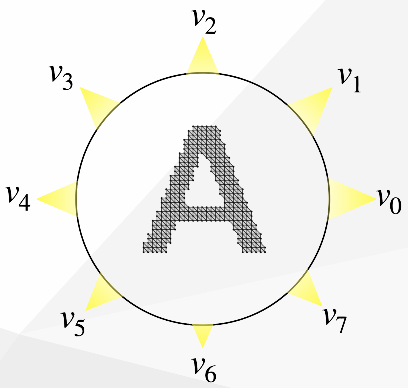
1. Golub, Gene H., and Charles F. Van Loan. Matrix computations. JHU press, 2013.
2. Trefethen, Lloyd N., and David Bau III. Numerical linear algebra. Vol. 50. Siam, 1997.

The Persistent Homology Transform

The *PHT* characterizes the set of embeddeable s.c.'s in $\mathbb{R}^d$ via a collection of dgm's

$$\mathrm{PHT}(M) : S^{d-1} \to \mathcal{D}^d$$
$$v \mapsto (X_0(M, v), X_1(M, v), \ldots, X_{d-1}(M, v))$$

where $\mathcal{D}^d$ is the space of dgm_p's up to dimension $p = d - 1$



The PHT is *injective* $\implies$ dgm-distances (e.g. integrated d_B) are *metrics*

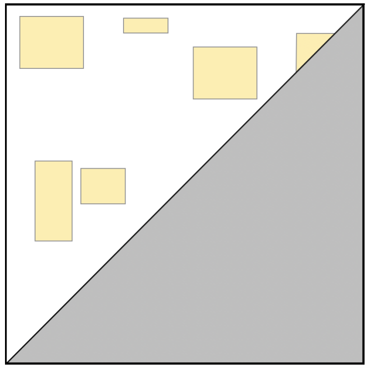
The injectivity PHT theory allows for comparison of *non-diffeomorphic* shapes

1. Turner, Katharine, Sayan Mukherjee, and Doug M. Boyer. "Persistent homology transform for modeling shapes and surfaces." Information and Inference: A Journal of the IMA 3.4 (2014): 310-344.

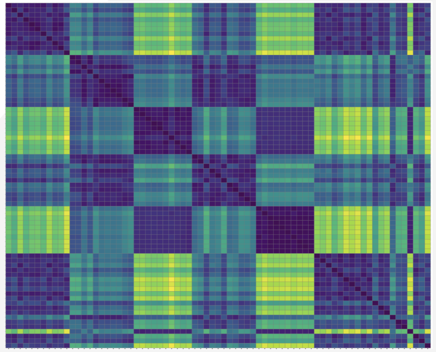
Applications: Leveraging PHT

Pro: PHT + it's associated distance metrics tend to do well at shape discrimination[1]

Con: Many dgm's + $\int d_B(\ldots)$ are highly non-trivial to compute



$\longmapsto$



(1) Choose a set rectangles $\mathcal{R} = \{r_1, r_2, \ldots, r_k\}$ in $\mathbb{R}^2$ representing "features"

(2) Compute multiplicities $\mathbf{u}_p(X) = \{\hat{\mu}_p^\epsilon(r_1), \hat{\mu}_p^\epsilon(r_2), \ldots, \hat{\mu}_p^\epsilon(r_k)\}$ for shapes X, Y

(3) Define $\hat{d}_{\mathcal{R}}(X, Y) = \|\mathbf{u}_p(X) - \mathbf{u}_p(Y)\|$, up to an optimal rotation[1]

We hope to do have more comparisons in the future

Thank you

Properties of the rank function

The *rank* of a linear map Φ is given as the dimension of its image:

$$\mathrm{r}(\Phi) = \mathrm{rank}(\Phi) = \dim(\mathrm{Im}(\Phi))$$

When $A, B \in \mathcal{M}_{(n\times n)}(\mathbb{R})$, the *rank function* has many convenient properties:

rank-nullity	$\iff$	$\mathrm{r}(A) = \lvert A\rvert - \mathrm{null}(A)$
subadditive	$\iff$	$\mathrm{r}(A+B) \leq \mathrm{r}(A) + \mathrm{r}(B)$
transposition invariance	$\iff$	$\mathrm{r}(A) = \mathrm{r}(A^T) = \mathrm{r}(A^TA) = \mathrm{r}(AA^T)$
orthogonal invariance	$\iff$	$\mathrm{r}(A) = \mathrm{r}(QA) = \mathrm{r}(AQ^T)$ $\quad$ (Q := orthogonal)
permutation invariance	$\iff$	$\mathrm{r}(A) = \mathrm{r}(P^{-1}AP)$ $\quad$ (P := permutation matrix)

Let's see if we can apply some of these.