Persistent Betti Numbers *over time*

Theory, computation, and applications

Preliminaries

A *simplicial complex* $K = \{\sigma : \sigma \in \mathcal{P}(V)\}$ over set $V = \{v_1, \ldots, v_n\}$ satisfies:

1. $v \in V \implies \{v\} \in K$
2. $\tau \subseteq \sigma \in K \implies \tau \in K$

A *filtration* $K_\bullet$ is a family $\{\, K_i \,\}_{i \in I}$ indexed over totally ordered index set I:

$$\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet$$

Additional requirements here:

1. *Essential*: $i \neq j$ implies $K_i \neq K_j$
2. *Simplexwise*: $K_j \setminus K_i = \{\sigma_j\}$ whenever $j = \mathrm{successor}(i)$
3. I may be $\mathbb{R}_+$ or $[m] = \{\, 1, 2, \ldots, m \,\}$

Three ~~Mountains of Moria~~ Groups

Given a pair $(K, \mathbb{F})$, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K

Given an *oriented* p-simplex $\sigma \in K$, define its p-boundary as the alternating sum:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \ldots, v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots v_p]$$

Three groups of interest:

1. $C_p(K) = (K, +, \times, \mathbb{F})$ (vector space of p-chains)
2. $B_p(K) = (\mathrm{Im} \circ \partial_{p+1})(K)$ (boundary group)
3. $Z_p(K) = (\mathrm{Ker} \circ \partial_p)(K)$ (cycle group)

Homology

The p-th homology of K is the quotient group $H_p(K) = Z_p(K)/B_p(K)$

Note: $H_p(K)$ depends on the choice of $\mathbb{F}$!

Persistent Homology

Inclusions $K_i \subsetneq K_j$ induce linear transformations $h_p^{i,j}$ between homology groups

$$H_p(K_0) \to \cdots \to H_p(K_i) \underbrace{\to \cdots \to}_{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_m) = H_p(K_\bullet)$$

1. $H_p^{i,j}(K) = \mathrm{Im}(h_p^{i,j}) \Leftrightarrow$ *persistent* homology groups
2. $H_p(K_\bullet)$ admits a *decomposition* $\mathrm{dgm}(K) = \{(\sigma_i, \sigma_j) :\}$
3. (*unique* iff $\mathbb{F}$ is a field!)
4. (σ_i, σ_j) = birth/death pair, $|i - j|$ = persistence of the pair
5. $\beta_p^{i,j} = \dim(H_p^{i,j})$ the *persistent* Betti number

So far, we have two defined two topological invariants of K + a summary rep. $\mathrm{dgm}_p(K)$:

$$H_p^{i,j} = \operatorname{Im} h_p^{i,j}, \quad \beta_p^{i,j} = \dim(H_p^{i,j})$$

Note we only need $\beta_p^{i,j}$ to define $\mathrm{dgm}_p(K)$.

To see this, define the *multiplicity* $\mu_p^{i,j}$, for all $0 < i < j \leq m+1$:

$$\mu_p^{i,j} = (\beta_p^{i,j-1} - \beta_p^{i,j}) - (\beta_p^{i-1,j-1} - \beta_p^{i-1,j})$$

The alternative definition:

Definition The p-th persistence diagram $\mathrm{dgm}_p(K) \subset \bar{\mathbb{R}}^2$ of K is the set of points (i, j)

The question: Suppose you wanted a continuous version of $\beta_p^{i,j}$