1. Motivation

Persistent homology is, as of this time of writing, a well-studied mathematical structure. From its unassuming history starting with postniov towers, the body of research created over the past two decades have established perisstence as not only an intrinsic quantity, but a useful tool. From **, ** , to **, applications abound with persistence; see for an survey. PH is more than just a homology inference tool.

Since its inception, a popular application of persistence in data analysis is its use as a featurization tool. In machine learning, featurization is a means of converting various data representations to a vector format amenable for learning and enhanced training. Classical examples include word2vec in natural language processing, Scale-Invariant Feature Transform (SIFT) in computer vision, Extended-connectivity fingerprints in used in chemical informatics and molecular modeling, etc. More recent results include transformers... Through no small feat of engineering, many of these techniques have been incrementally improved and adapted throughout the past decades, and tend to do quite well in terms of their efficiency. As such, they have seen widespread-adoption from more scientific fields trying to harness their power. While certainly useful, one of the pitfalls with such featurizations if the difficulty that comes with interpretation. Difficulty in heavy featurization has lead to qualitative comparisons in scientific fields of the featurization outputs. Many are lead by the same equation: exactly what isa featurization tool capturing that is so useful for training?

Persistent homology is, in some sense, a natural tool for featurization. persistence is a stable invariant that comes equipped mathematical guarentees; thus featurization of diagrams can be interpreted as mapping persistence diagrams to Euclidean space in such a way that maximally preserves the topological information conveyed by the diagram. Moreover, we also know persistence diagrams retain some amount of geometry, such as those of curvature sets and the quasi-isometry theorems in distributed persistence. These results suggest an inverse theory related to persistence. Indeed, a recent injectivity result shows that collections of persistence diagrams are sufficient to uniquely characterize data sets in 2- and 3-dimensions[1], establishes persistence as truly an intrinsic description of shape.

In what follows, we introduce a relaxation of the persistent Betti number (PBN) invariant that has certain advantages. Namely, we showing that a simple augmentation of traditional PBN computation leads to a continuous relaxation that $(1+\epsilon)$-approximates the PBN that is permutation invariant, a property useful to have in time-varying settings. Moreover, we show our relaxation satisfies certain basic properties, and we illicit its connections back to spectral graph theory.

2. Background & Notation

A *simplicial complex* $K \subseteq \mathcal{P}(V)$ over a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is a collection of simplices $\{\sigma : \sigma \in \mathcal{P}(V)\}$ such that $\tau \subseteq \sigma \in K \implies \tau \in K$. A *filtration* $K_\bullet = \{K_i\}_{i \in I}$ of a simplicial complexes indexed by a totally ordered set I is a family of complexes such that $i < j \in I \implies K_i \subseteq K_j$. $K_\bullet$ is called *simplexwise* if $K_j \smallsetminus K_i = \{\sigma_j\}$ whenever j is the immediate successor of i in I and $K_\bullet$ is called *essential* if $i \neq j$ implies $K_i \neq K_j$:

$$\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet, \quad K_i = K_{i-1} \cup \{\sigma_i\} \tag{1}$$

Filtrations may be equivalently defined as functions $f : K \to I$ satisfying $f(\tau) \leq f(\sigma)$ whenever $\tau \subseteq \sigma$. Here, we consider two index sets for I: $\mathbb{R}$ and $[n] = \{1, \ldots, n\}$. Any finite filtration may be trivially converted into an essential, simplexwise filtration via a set of *condensing*, *refining*, and *reindexing* maps [**?**]. Thus, without loss of generality, we exclusively consider essential simplexwise filtrations and for brevity refer to them as filtrations.

Remark 1: In practice, filtrations often arise from triangulations parameterized by geometric scaling parameters. For example, given a finite metric space $\mathcal{X} = (X, d_X)$, the *Rips complex* at scale $\epsilon \in \mathbb{R}_+$ is the complex given by:

$$\mathrm{Rips}_\epsilon(\mathcal{X}) := \{\sigma \subseteq X : d_X(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{2}$$

Connecting successive complexes via inclusions $\mathrm{Rips}_\epsilon(\mathcal{X}) \hookrightarrow \mathrm{Rips}_{\epsilon'}(\mathcal{X})$ for $\epsilon < \epsilon'$ yields a family of complexes $\mathrm{Rips}_\alpha := \{\, \mathrm{Rips}_\epsilon(\mathcal{X}) \,\}_{\epsilon \leq \alpha}$ called the *Rips filtration*. We keep the notation general by letting $K_\bullet$ denote any filtration.

For K a simplicial complex and $\mathbb{F}$ a field, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K. The collection of p-chains under addition yields an $\mathbb{F}$-vector space denoted $C_p(K)$. The p-boundary $\partial_p(\sigma)$ of an oriented

[1]

p-simplex $\sigma \in K$ is defined as the alternating sum of its oriented co-dimension 1 faces:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \dots, v_p]) = \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p] \tag{3}$$

where $\hat{v}_i$ indicates the removal of v_i from the ith summand. Similarly, the p-boundary of a p-chain is defined linearly in terms of its constitutive simplices. A p-chain with zero boundary is called a p-cycle, and together they form $Z_p(K) = \mathrm{Ker}\, \partial_p$. Similarly, the collection of p-boundaries forms $B_p(K) = \mathrm{Im}\, \partial_{p+1}$. Since $\partial_p \circ \partial_{p+1} = 0$ for all $p \geq 0$, the quotient space $H_p(K) = Z_p(K)/B_p(K)$ is well-defined, and $H_p(K)$ is called the p-th homology of K with coefficients in $\mathbb{F}$. The dimension of the p-th homology group $\beta_p(K) = \dim(H_p(K))$ of K is called the p-th *Betti* number of K.

Let $K_\bullet = \{K_i\}_{i \in [m]}$ denote a filtration of size $|K_\bullet| = m$. For every pair $i, j \in [m]$ with $i < j$, the inclusions $K_i \subsetneq K_{i+1} \subsetneq \dots \subsetneq K_j$ induce linear transformations $h_p^{i,j}$ at the level of homology:

$$0 = H_p(K_0) \to \dots \to H_p(K_i) \underbrace{\to \dots \to}_{h_p^{i,j}} H_p(K_j) \to \dots \to H_p(K_m) = H_p(K_\bullet) \tag{4}$$

When $\mathbb{F}$ is a field, this sequence of homology groups admits a unique decomposition of $K_\bullet$ into a pairing of simplices (σ_i, σ_j) [] demarcating the evolution of homology classes: σ_i marks the creation of a homology class, σ_j marks its destruction, and the difference $|i - j|$ records the lifetime of the class, called its *persistence*. The p-th persistent homology groups are the images of these transformations and the p-th persistent Betti numbers are their dimensions:

$$H_p^{i,j} = \begin{cases} H(K_i) & i = j \\ \mathrm{Im}\, h_p^{i,j} & i < j \end{cases}, \qquad \beta_p^{i,j} = \begin{cases} \beta_p(K_i) & i = j \\ \dim(H_p^{i,j}) & i < j \end{cases} \tag{5}$$

For a fixed $p \geq 0$, the collection of persistent pairs (i, j) together with unpaired simplices (l, ∞) form a summary representation $\mathrm{dgm}_p(K_\bullet)$ called the *p-th persistence diagram of* $K_\bullet$. Note that the persistent Betti numbers can be read off directly given $\mathrm{dgm}_p(K_\bullet)$; conceptually, $\beta_p^{i,j}$ simply counts the number of persistent pairs lying inside the box $[0, i] \times (j, \infty)$ (see Figure **??**)—the number of persistent homology groups born at or before i that died sometime after j.

Remark 2: Persistence has been viewed from many different perspectives, and may be defined in a variety of ways. For example, Carlsson et al. [] observed persistence is simply a graded module under a particular polynomial ring. More recently, Baur studied persistence in a form a matching. Use follow the presentation from Cohen-Steiner et al []: given a tame function $f : K \to \mathbb{R}$, its homological critical values $\{a_i\}_{i=1}^n$, and an interleaved sequence $\{b_i\}_{i=0}^n$ satisfying $b_{i-1} < a_i < b_i$ for all i, the p-th persistence diagram $\mathrm{dgm}_p(f) \subset \bar{\mathbb{R}}^2$ of a filtration induced by f is defined as:

$$\mathrm{dgm}_p(K_\bullet) = \{\, (a_i, a_j) : \mu_p^{i,j} \neq 0 \,\} \,\cup\, \mathcal{L} \tag{6}$$

where $\mathcal{L}$ denotes the points on the diagonal, counted with infinite multiplicity, and $\mu_p^{i,j}$ is defined as:

$$\mu_p^{i,j} = \left(\beta_p^{i,j\text{-}1} - \beta_p^{i,j}\right) - \left(\beta_p^{i\text{-}1,j\text{-}1} - \beta_p^{i\text{-}1,j}\right) \qquad \text{for } 0 < i < j \leq n+1$$

where $\beta_p^{i,j}$ is the persistent Betti number defined at the values of the interleaved sequence, i.e. $\beta_p^{i,j} = \dim(\mathrm{Im}(h_p^{b_i, b_j}))$. More generally, by interpreting μ_p^* as function defined over $\bar{\mathbb{R}}$, Chazal [] view the multiplicity μ as a counting measure. This interpretation (as well as Cohen-Steiners) is perhaps the most relvent to the work we present here.

3. Methodology

Persistent Betti Numbers:

As in section 2, let $B_p(K_*) \subseteq Z_p(K_*) \subseteq C_p(K_*)$ denote the p-th boundary, cycle, and chain groups of K_*, respectively. Given a simplicial filtration $K_\bullet$, let $\partial_p : C_p(K_\bullet) \to C_p(K_\bullet)$ denote the boundary operator sending p-chains to their respective boundaries. With a slight abuse of notation, we also use ∂_p to also denote the filtration boundary matrix with respect to an ordered basis $(\sigma_i)_{1 \leq i \leq m_p}$. The p-th persistent Betti number between scales (i, j) is defined as:

$$\begin{aligned} \beta_p^{i,j} &= \dim(H_p^{i,j}) \\ &= \dim\left(Z_p(K_i)/(Z_p(K_i) \cap B_p(K_j)\right) \\ &= \dim\left(Z_p(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \end{aligned} \tag{7}$$

Note that $\dim(C_p(K_*)) = \dim(B_{p-1}(K_*)) + \dim(Z_p(K_*))$ by the rank-nullity theorem, so we may rewrite (7) as:

$$\beta_p^{i,j} = \dim\left(C_p(K_i)\right) - \dim\left(B_{p-1}(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \tag{8}$$

The dimension of the boundary group $B_{p-1}(K_i)$ may be directly inferred from the rank of ∂_p^i, and the dimension of $C_p(K_i)$ is simply the number of p-simplices with filtration values $f(\sigma) \leq i$. To express the intersection term, we require more notation. If A is a $m \times m$ matrix, then let $A^{i,j}$ denote the lower-left submatrix of A given by the first j columns and last $m - i + 1$ rows (rows i through m, inclusive). For any $1 \leq i < j \leq m$, define the quantity $r_A(i,j)$:

$$r_A(i,j) = \mathrm{rank}(A^{i,j}) - \mathrm{rank}(A^{i+1,j}) + \mathrm{rank}(A^{i+1,j-1}) - \mathrm{rank}(A^{i,j-1}) \tag{9}$$

The Pairing Uniqueness Lemma [] asserts that if $R = \partial V$ is a decomposition of the total $m \times m$ boundary matrix ∂, then $\mathrm{low}_R[j] = i$ if and only if $r_\partial(i,j) = 1$. Moreover, from the algebraic perspective, the structure theorem from [] shows that 1-parameter persistence modules can be decomposed in an *essentially unique* way []. As a result, for $1 \leq i < j \leq m$, we have:

$$\mathrm{low}_R[j] = i \iff r_\partial(i,j) = 1 \iff \mathrm{rank}(R^{i,j}) = \mathrm{rank}(\partial^{i,j})$$

Thus, all lower-left submatrices of the filtered boundary matrix ∂ have the same rank as their corresponding submatrices in R. Thus, we may write $\beta_p^{i,j}$ as:

$$\beta_p^{i,j} = \mathrm{rank}(I_p^i) - \mathrm{rank}(\partial_p^{*,i}) - \mathrm{rank}(\partial_p^{i,j}) + \mathrm{rank}(\partial_p^{i+1,j}) \tag{10}$$

where $I_p^i = \mathrm{diag}(\mathbb{1}(\mathrm{diam}(\sigma) \leq i))$ denotes the order of the matrix. In conclusion, we may write the persistent Betti number as a combination of rank computations performed directly on the dimension p and $(p+1)$ boundary matrices.

A Time-varying Boundary Matrix Relaxation

Let $\delta_\mathcal{X}$ denote an T-parameterized metric space $\delta_\mathcal{X}(\cdot) = (X, d_X(\cdot))$, where $d_X : \mathrm{T} \times X \times X \to \mathbb{R}_+$ is called a *time-varying metric* and X is a finite set with fixed cardinality $|X| = n$. δ_X as called a *dynamic metric space* (DMS) iff $d_X(\cdot)(x,x')$ is continuous for every pair $x, x' \in X$ and $\delta_X(t) = (X, d_X(t))$ is a pseudo-metric space for every $t \in \mathrm{T}$. For a fixed $t \in \mathrm{T}$, the Rips complex at scale $\epsilon \in \mathbb{R}$ is the abstract simplicial complex given by

$$\mathrm{Rips}_\epsilon(\delta_\mathcal{X}(t)) := \{\sigma \subset X : d_X(t)(x,x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{11}$$

As before, the family of Rips complexes for varying $\epsilon > 0$ yields a filtration whose inclusion maps induce linear maps at the level of homology. The time-varying counterpart is analogous. In this context, we write the p-th persistent Betti number with respect to fixed values $i, j \in I$ as a function of $t \in \mathrm{T}$:

$$\beta_p^{i,j}(t) = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ \delta_\mathcal{X}\right)(t) \tag{12}$$

As integer-valued invariants, Betti numbers pose several difficulties to vectorization. Thus, we require alternative expressions for each of the terms in equation (10) to extend its applicability to the time-varying setting. Towards deriving these expression, we first require a replacement of the standard boundary matrix formulation.

Recall that the boundary operator ∂_p for a finite simplicial filtration $K_\bullet$ with $m = |C_p(K_\bullet)|$ and $n = |C_{p-1}(K_\bullet)|$ can be represented by an $(n \times m)$ boundary matrix ∂_p whose columns and rows correspond to p-simplices and $(p-1)$-simplices, respectively. The entries of ∂_p depend on the choice of $\mathbb{F}$; in general, after orientating the simplices of K arbitrarily, they have the form:

$$\partial_p[i,j] = \begin{cases} c(\sigma_j) & \text{if } \sigma_i \in \partial_p(\sigma_j) \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where $c(\sigma_*) \in \mathbb{F}$ is an arbitrary constant satisfying $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex, typically set to ± 1. Towards relaxing the persistent Betti computation in dynamic setting, we propose an alternative choice for $c(\sigma)$ which endows continuity in the entries of ∂_p in T.

Definition 1 (Time-varying boundary matrix)**.** *Let $\mathbb{F} = \mathbb{R}$ denote the field, $\delta_\mathcal{X}(\cdot) = (X, d_X(\cdot))$ a DMS over a finite set X of fixed size $|X| = n$, and let $(\mathcal{P}(X), \preceq^*)$ be a linear extension of the face poset of the $(n-1)$-simplex Δ_n. For some constant $\epsilon > 0$, a time-varying p-th boundary matrix ∂_p^t is an $\binom{n}{p} \times \binom{n}{p+1}$ matrix whose entries $c(\sigma)$ satisfy:*

$$\partial_p^t[i,j] = \begin{cases} \pm\, |\epsilon - \mathrm{diam}_t(\sigma_j)|_+ & \text{if } \sigma_i \in \partial_p(\sigma_j) \\ 0 & \text{otherwise} \end{cases}$$

and whose rows and columns are ordered by $\preceq^*$ for all $t \in T$.

We now show a few properties that ∂_p^t exhibits which is advantageous for optimization. Clearly the entries of ∂_p^t must vary continuously in $t \in T$. Moreover, for fixed $p \geq 0$, we have:

1. $\mathrm{rank}(\partial_p^t) = \dim(\mathrm{B}_{p-1}(K_t))$ for all $t \in T$, where $K_t = \mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t))$,
2. $\|\partial_p^t - \partial_p^{t'}\|_F \sim O(m_p)$ when $\delta_{\mathcal{X}}$ is C-Lipshitz over T and $|t - t'|$ is small,
3. $\|\partial_p^t\|_2 \leq \epsilon\sqrt{\kappa}\,(p+1)$ where $\kappa = \max \sum\limits_{t \in T} \sum\limits_{\sigma \in K_t} \mathbb{1}(\mathrm{diam}(\sigma) \leq \epsilon)$

Proof. First, consider property (1). For any $t \in T$, applying the boundary operator ∂_p to $K_t = \mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t))$ with non-zero entries satisfying (13) by definition yields a matrix ∂_p satisfying $\mathrm{rank}(\partial_p) = \dim(\mathrm{B}_{p-1}(K_t))$. In contrast, definition (1) always produces p-boundary matrices of Δ_n; however, notice that the only entries which are non-zero are precisely those whose simplices σ that satisfy $\mathrm{diam}(\sigma) < \epsilon$. Thus, $\mathrm{rank}(\partial_p^t) = \dim(\mathrm{B}_{p-1}(K_t))$ for all $t \in T$. < (show proof of (2))> Property (3) follows from the construction of ∂_p and from the inequality $\|A\|_2 \leq \sqrt{m}\|A\|_1$ for an $n \times m$ matrix A, as $\|\partial_p^t\|_1 \leq (p+1)\,\epsilon$ for all $t \in T$.

□

We now re-write equation using this relaxation. Fix persistence parameters $a, b \in \mathbb{R}^+$. Since our boundary matrices now follow a constant order, we write ∂_p.

Rank Relaxation (TODO)

In light of expression (10), we may interpret many of the terms of $\beta_p^{i,j}$ from a function composition perspective:

$$t \overset{f}{\mapsto} \partial_*^t \overset{g}{\mapsto} \mathrm{rank}(\partial_*^t)$$

In this sense, by modifying the entries of ∂_p^* via 1, we ensure that f is both continuous and inherits the the smoothness of $\partial_{\mathcal{X}}(\cdot)$. We now address g.

A common relaxation of the rank function found in the literature is the nuclear norm, $\|A\|_* = \mathrm{tr}(S)$, where $A = USV^T$. This is due to the fact that the nuclear norm is the tightest convex envelope of the rank function over the set of matrices whose spectral norm is less than $m > 0$:

$$\mathcal{A} = \{A \in \mathbb{R}^{n_1 \times n_2} \, : \, \|A\|_2 \leq m\}$$

Equivalently, $\mathcal{A}$ may be thought of as the set of matrices $A \in \mathbb{R}^{n_1 \times n_2}$ such that $\|\frac{1}{m}A\|_2 \leq 1$, for some appropriate choice of $m > 0$.

Smoothness

Given a function $f : \mathbb{R} \to (-\infty, \infty]$ and a fixed $\mu > 0$, the *proximal operator* or *prox* of f is given by:

$$\mathrm{prox}_f^\mu(x) = \underset{u \in \mathbb{R}}{\arg\min} \left\{ f(u) + \frac{1}{2\mu}\|u - x\|^2 \right\} \tag{14}$$

When f is proper closed convex, $\mathrm{prox}_f^\mu(x)$ is single-valued and yields the solution to the *Moreau envelope* of f:

$$M_f^\mu(x) = \min_{u \in \mathbb{R}} \left\{ f(u) + \frac{1}{2\mu}\|x - u\|^2 \right\} = f(\mathrm{prox}_f^\mu(x)) + \frac{1}{2\mu}\|x - \mathrm{prox}_f^\mu(x)\|^2$$

$M_f^\mu(x)$ exhibits a number of properties suitable for optimization: it is $\frac{1}{\mu}$-Lipshitz over $\mathbb{R}$, it retains the same minima as f, and for any $x \in \mathbb{R}$ it admits a gradient $\nabla M_f^\mu(x)$ which again is expressible via the proximal operator:

$$\nabla M_f^\mu(x) = \frac{1}{\mu}(x - \mathrm{prox}_f^\mu(x)) \tag{15}$$

Moreover, if ∇f is L_f-Lipshitz, then $\nabla M_f^\mu(x)$ is L_f/μ-Lipshitz, and if f is L_f-Lipshitz, then $|f(x) - M_f^\mu(x)| \leq L_f^2\mu$. Thus, the Moreau envelope acts as a smooth approximation of f, which makes it an excellent candidate for smoothing (**??**) if prox_f^μ can be efficiently computed. Fortunately, the prox of $\|\cdot\|_*$ admits a simple characterization:

$$\begin{aligned} \mathrm{prox}_{\|\cdot\|_*}^\mu(X) &= \underset{Y \in \mathbb{R}^{n \times m}}{\arg\min} \left\{ \|Y\|_* + \frac{1}{2\mu}\|Y - X\|_F^2 \right\} & (16) \\ &= U\mathcal{D}_\mu(S)V^T & (17) \end{aligned}$$

where $X = USV^T$ is the SVD of an $(n \times m)$ matrix X and $\mathcal{D}_\mu(S) = \mathrm{diag}(\{|\sigma_i - \mu|_+\}_{1 \leq i \leq r})$ is the application of the *soft-thresholding operator* to the singular values $S = \mathrm{diag}(\{\sigma_i\}_{1 \leq i \leq r})$. Equation (16) yields the prox operator for nuclear norm $\|A\|_*$, which is not necessarily a convex function. However, it is known that $\|\cdot\|_*$ is convex over the set $\{X \in \mathbb{R}^{n\times m} : \|X\|_2 \leq m\}$, thus we may inherit the smoothing properties of M_f^μ by considering the prox operator for the function $f : X \mapsto \frac{1}{m}\|X\|_*$. Letting $\alpha = m^{-1}$, the proximal operator of this function is given by:

$$\mathrm{prox}_{\alpha f}^{\mu}(X) = \underset{Y \in \mathbb{R}^{n\times m}}{\arg\min} \left\{ \alpha \|Y\|_* + \frac{1}{2\mu}\|Y - X\|_F^2 \right\} = \alpha \cdot \mathrm{prox}_f^{\mu\alpha}(X) \tag{18}$$

Remark 1. *Show example of Moreau envelope*

4. Persistent 1-Betti Number Approximation

In this section we demonstrate that there exists a convenient output-sensitive approximation the $p = 1$ Persistent Betti number using the formulation we given in Equation 20. In what follows, we assume was have as input a $d = 2$ Rips complex $\mathcal{R}_b$ constructed up to some threshold $b > 0$ over a fixed metric space (X, d_X) with $n_v = |X|$ vertices. Recall we can write the $p = 1$ persistent Betti number as a sum of rank functions:

$$\beta_p^{a,b} = \mathrm{rank}(I_1^a) - \mathrm{rank}(\partial_1^a) - \mathrm{rank}(\partial_2^b) + \mathrm{rank}(\partial_2^{\bar{a},b}) \tag{19}$$

Thus, any algorithm which can approximate the rank function provides an approximation algorithm for $\beta_p^{a,b}$.

A. Appendix

Dynamic Metric Spaces

Consider an $\mathbb{R}$-parameterized metric space $\delta_X = (X, d_X(\cdot))$ where X is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_+$, satisfying:

1. For every $t \in \mathbb{R}, \delta_X(t) = (X, d_X(t))$ is a pseudo-metric space[2]
2. For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous.

When the parameter $t \in \mathbb{R}$ is interpreted as *time*, the above yields a natural characterization of a "time-varying" metric space. More generally, we refer to an $\mathbb{R}^h$-parameterized metric space as *dynamic metric space*(DMS). Such space have been studied more in-depth [] and have been shown...

Rank relaxation

A common approach in the literature to optimize quantities involving $\mathrm{rank}(A)$ for some $m \times n$ matrix A is to consider optimizing its *nuclear norm* $\|A\|_* = \mathrm{tr}(\sqrt{A^T A}) = \sum_{i=1}^r |\sigma_i|$, where σ_i denotes the ith singular value of A and $r = \mathrm{rank}(A)$. One of the primary motivations for this substitution is that the nuclear norm is a convex envelope of the rank function over the set:

$$S := \{A \in \mathbb{R}^{n\times m} \mid \|A\|_2 \leq m\}$$

That is, for an appropriate $m > 0$, the function $A \mapsto \frac{1}{m}\|A\|_*$ is a lower convex envelope of the rank function over S. The nuclear norm also admits a subdifferential... thus, we may consider replacing (**??**) with:

$$\beta_p^{i,j}(t) = |\partial_{p,t}^{1,i}| - m_1^{-1}\|\partial_{p,t}^{1,i}\|_* - m_2^{-1}\|\partial_{\bar{p},t}^{1,j}\|_* - m_3^{-1}\|\partial_{\bar{p},t}^{\bar{i},j}\|_* \tag{20}$$

where $\bar{c} = c + 1$. Now, if $t \mapsto \partial_p^*(t)$ is a non-decreasing, convex function in t, then the composition ... is convex, as each of the individual terms are convex. Moreover, we have...

< Insert proof about this relaxation always lower-bounding β >

[2]This is required so that if one can distinguish the two distinct points $x, x' \in X$ incase $d_X(t)(x, x') = 0$ at some $t \in \mathbb{R}$.

Computation

In this section, we discuss the computation of suitable bases for the subspaces $Z_p(X_*)$, $B_p(K_*)$, and $Z_p(X_*) \cap B_p(X_*)$. In particular, we address two cases: the *dense* case, wherein the aforementioned bases are represented densely in memory, and the *sparse* case, which uses the structure of a particular decomposition of the boundary matrices to derive bases whose size in memory inherits the sparsity pattern of the decomposition.

Sparse case: We require an appropriate choice of bases for the groups $B_{p-1}(K_*)$ and $Z_p(X_*) \cap B_p(X_*)$. For some fixed $t \in T$, let $R_p = \partial_p V_p$ denote the decomposition discussed above, and let $b, d \in \mathbb{R}_+$ be fixed constants satisfying $b \leq d$. Since the boundary group $B_{p-1}(K_b)$ lies in the image of the ∂_p, it can be shown that a basis for the boundary group $B_{p-1}(K_*)$ is given by:

$$M_p^b = \{ \, \mathrm{col}_{R_{p+1}}(j) \neq 0 \mid j \leq b \, \} \qquad \mathit{span}() \quad (21)$$

Moreover, since $B_{p-1}(K_b) = \mathrm{Im}(\partial_p^b)$, we have $\mathrm{span}(M_p^b) = B_{p-1}(K_b)$ and thus $\mathrm{rank}(M_p^b) = \mathrm{rank}(\partial_p^b)$. Indeed, it can be shown that every lower-left submatrix of ∂_p^* satisfies $\mathrm{rank}(\partial_p^*) = \mathrm{rank}(R_p^*)$. Thus, although M_p^b does provide a minimal basis for the boundary group $B_{p-1}(K_b)$, it is unneeded here.

A suitable basis for the cycle group can also be read off from the reduced decomposition directly as well. Indeed, let $R_p = \partial_p V_p$ as before. Then the cycle group is spanned by linear combinations of columns of V_p:

$$Z_p^b = \{ \, \mathrm{col}_{V_p}(j) \mid \mathrm{col}_{R_p}(j) = 0, j \leq b \, \} \tag{22}$$

The formulation of a basis spanning $Z_p(K_i) \cap B_p(K_j)$ is more subtle, as we can no longer use the fact that every lower-left submatrix of R_p has the same rank as the same lower-left submatrix of ∂_p. Nonetheless, a basis for this group can be obtained by reading off specific columns from R_p:

$$M_p^{b,d} := \{ \, \mathrm{col}_{R_{p+1}}(k) \neq 0 \mid 1 \leq k \leq d \text{ and } 1 \leq \mathrm{low_{R_{p+1}}}(k) \leq b \, \} \tag{23}$$

One can show that M_b^d does indeed span $Z_p(X_*) \cap B_p(X_*)$ by using the fact that the non-zero columns of R_p with indices at most at most d form a basis for $B_p(K_d)$, and that each low-row index for every non-zero is unique.

Dense case: In general, persistent homology groups and its various factor groups are well-defined and computable with the reduction algorithm with coefficients chosen over any ring. By applying operations with respect to a field $\mathbb{F}$, both the various group structures $Z_p(K_\bullet) \subseteq B_p(K_\bullet) \subseteq C_p(K_\bullet)$ and their induced quotient groups $H_p(K_\bullet)$ are vector spaces; thus, the computation of suitable bases can be approached from a purely linear algebraic perspective. In particular, by fixing $\mathbb{F} = \mathbb{R}$, we inherit not only many useful tools for obtaining suitable bases for these groups, but also access to their corresponding optimized implementations as well.

Consider the p-th boundary operator $\partial_p^* : C_p(K_*) \to C_{p-1}(K_*)$ whose matrix realization with respect to some choice of simplex ordering $\{\sigma_i\}_{1 \leq i \leq m}$ we also denote with ∂_p. By definition, the boundary group $B_p(K_*)$ is given by the image $\mathrm{Im}(\partial_{p+1}^*) = B_p(K_*)$, thus one may basis for $B_p(K_*)$ by computing the considering the first $r > 0$ columns of the reduced SVD:

$$M_p^* = [\, u_1 \mid u_2 \mid \cdots \mid u_r \,] = \{ \; \} \tag{24}$$

A.1. Old

We consider the problem of maximizing the p-th *persistent* Betti number $\beta_p^{i,j}$ over some set $T \subseteq \mathrm{T}$:

$$t_* = \operatorname*{arg\,max}_{t \in T} \beta_p^{i,j}(t) \tag{25}$$

As an illustrative example, see Figure. < insert SW1Pers vineyards plot >