1 Introduction

Motivation Persistent homology is, as of this time of writing, a well-studied mathematical structure. From its unassuming history starting with postniov towers, the body of research created over the past two decades have established perisstence as not only an intrinsic quantity, but a useful tool. From **, ** , to **, applications abound with persistence; see for an survey. PH is more than just a homology inference tool.

Since its inception, a popular application of persistence in data analysis is its use as a featurization tool. In machine learning, featurization is a means of converting various data representations to a vector format amenable for learning and enhanced training. Classical examples include word2vec in natural language processing, Scale-Invariant Feature Transform (SIFT) in computer vision, extended-connectivity fingerprints (ECFs) in used in chemical informatics and molecular modeling, etc. More recent results include transformers... Through no small feat of engineering, many of these techniques have been incrementally improved and adapted throughout the past decades, and tend to do quite well in terms of their efficiency. As such, they have seen widespread-adoption from more scientific fields trying to harness their power. While certainly useful, one of the pitfalls with such featurizations if the difficulty that comes with interpretation. Difficulty in heavy featurization has lead to qualitative comparisons in scientific fields of the featurization outputs. Many are lead by the same equation: exactly what isa featurization tool capturing that is so useful for training?

Persistent homology is, in some sense, a natural tool for featurization. persistence is a stable invariant that comes equipped mathematical guarantees; thus featurization of diagrams can be interpreted as mapping persistence diagrams to Euclidean space in such a way that maximally preserves the topological information conveyed by the diagram. Moreover, we also know persistence diagrams retain some amount of geometry, such as those of curvature sets and the quasi-isometry theorems in distributed persistence. These results suggest an inverse theory related to persistence. Indeed, a recent injectivity result shows that collections of persistence diagrams are sufficient to uniquely characterize data sets in 2- and 3-dimensions[1], establishes persistence as truly an intrinsic description of shape.

Our Contribution: A few of the drawbacks with persistence-related featurization tools, such as persistence images [] or persistence landscapes [], is that they require as input the persistence diagram to produce the featurization. As a $O(m^3)$ operation, the traditional reduction computation can become infeasible, a property exacerbating greatly in the time-varying setting.

The Relaxation & Main Results We give below the the relaxation proposed in the paper. The rest of the paper is devoted to its properties and use cases.

Given an input data set X that varies along a 1-parameter family $\{X_{(t)}\}_{t \in T}$, for each $t \in T$, do the following:

1. Choose a geometric realization $\mathcal{K}$ of X. This could be a simple Rips filtration, or a Delaunay complex, a simplicial mesh, a neighborhood graph, etc.

2. Fix a set of coordinates $\mathcal{I} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ from the upper-half plane Δ_+ representing areas where significant topological changes are expected to occur.

3. Compute our relaxation of the PBNs at each $(a, b) \in \mathcal{I}$,

$$\beta_p^{i,j} = f(\hat{\partial}_p) - f(\hat{\partial}_p)$$

   where a) $A^{i,j}$ denotes the lower-left submatrix and b) $\hat{\partial}_p^*$ represents the p-th boundary matrix of K whose elementary chains $c(\sigma)$ take on values from a relaxation of the step function. and c) $f : K \to \mathbb{R}$ is a a function that approximates $\mathrm{rank}(A)$ using the spectrum $\sigma(A)$ of A. We give a few examples of suitable functions in section **??**.

The main observations our relaxation is based on is that the (1) PBNs $\beta_p^{i,j}$ can be written as sum of ranks of boundary matrices, (2) by working with real-valued coefficients, we can parameterize the non-zero entries of ∂_* with smoothly varying values whose signs match (or approximate) the signs of ∂, and (3) there exist a myriad of spectrum-based relaxations of the *rank* function that are easier to compute.

That $\beta_p^{i,j}$ is estimated using the spectrum(s) of $\hat{\partial}_*$ is the primary step to reducing the complexity of our relaxation; the special forms of the matrices involved in our relaxation have connections back to spectral graph theory.

[1]

Organization In what follows, we introduce a relaxation of the persistent Betti number (PBN) invariant that has certain advantages. Namely, we showing that a simple augmentation of traditional PBN computation leads to a spectral relaxation that $(1-\epsilon)$-approximates the PBN. Moreover, we show that this relaxation is permutation invariant, obeys certain inclusion-exclusion principles, and admits a notion of stability in a certain sense—all properties useful in parameterized settings.

2 Background & Notation

A *simplicial complex* $K \subseteq \mathcal{P}(V)$ over a vertex set $V = \{v_1, v_2, \dots, v_n\}$ is a collection of simplices $\{\sigma : \sigma \in \mathcal{P}(V)\}$ such that $\tau \subseteq \sigma \in K \implies \tau \in K$. A *filtration* $K_\bullet = \{K_i\}_{i \in I}$ of a simplicial complexes indexed by a totally ordered set I is a family of complexes such that $i < j \in I \implies K_i \subseteq K_j$. $K_\bullet$ is called *simplexwise* if $K_j \setminus K_i = \{\sigma_j\}$ whenever j is the immediate successor of i in I and $K_\bullet$ is called *essential* if $i \neq j$ implies $K_i \neq K_j$:

$$\emptyset = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K_\bullet, \quad K_i = K_{i-1} \cup \{\sigma_i\} \tag{1}$$

Filtrations may be equivalently defined as functions $f : K \to I$ satisfying $f(\tau) \leq f(\sigma)$ whenever $\tau \subseteq \sigma$. Here, we consider two index sets for I: $\mathbb{R}$ and $[n] = \{1, \dots, n\}$. Any finite filtration may be trivially converted into an essential, simplexwise filtration via a set of *condensing*, *refining*, and *reindexing* maps [1]. Thus, without loss of generality, we exclusively consider essential simplexwise filtrations and for brevity refer to them as filtrations.

For K a simplicial complex and $\mathbb{F}$ a field, a p-chain is a formal $\mathbb{F}$-linear combination of p-simplices of K. The collection of p-chains under addition yields an $\mathbb{F}$-vector space denoted $C_p(K)$. The p-boundary $\partial_p(\sigma)$ of an oriented p-simplex $\sigma \in K$ is defined as the alternating sum of its oriented co-dimension 1 faces:

$$\partial_p(\sigma) = \partial_p([v_0, v_1, \dots, v_p]) := \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v}_i, \dots v_p] \tag{2}$$

where $\hat{v}_i$ indicates the removal of v_i from the ith summand. Similarly, the p-boundary of a p-chain is defined linearly in terms of its constitutive simplices. A p-chain with zero boundary is called a p-cycle, and together they form $Z_p(K) = \operatorname{Ker} \partial_p$. Similarly, the collection of p-boundaries forms $B_p(K) = \operatorname{Im} \partial_{p+1}$. Since $\partial_p \circ \partial_{p+1} = 0$ for all $p \geq 0$, the quotient space $H_p(K) = Z_p(K)/B_p(K)$ is well-defined, and $H_p(K)$ is called the p-th homology of K with coefficients in $\mathbb{F}$. The dimension of the p-th homology group $\beta_p(K) = \dim(H_p(K))$ of K is called the p-th *Betti* number of K.

Let $K_\bullet = \{K_i\}_{i \in [m]}$ denote a filtration of size $|K_\bullet| = m$, and let $\Delta_+^m = \{(i,j) : 0 \leq i < j \leq m\}$ denote the set of valid pairs of filtration indices. For every such pair $(i,j) \in \Delta_+^m$, the inclusions $K_i \subsetneq K_{i+1} \subsetneq \cdots \subsetneq K_j$ induce linear transformations $h_p^{i,j}$ at the level of homology:

$$0 = H_p(K_0) \to \cdots \to H_p(K_i) \underbrace{\to \cdots \to}_{h_p^{i,j}} H_p(K_j) \to \cdots \to H_p(K_m) = H_p(K_\bullet) \tag{3}$$

When $\mathbb{F}$ is a field, this sequence of homology groups admits a unique decomposition of $K_\bullet$ into a pairing of simplices (σ_i, σ_j) [6] demarcating the evolution of homology classes: σ_i marks the creation of a homology class, σ_j marks its destruction, and the difference $|i-j|$ records the lifetime of the class, called its *persistence*. The p-th persistent homology groups are the images of these transformations and the p-th persistent Betti numbers are their dimensions:

$$H_p^{i,j} = \begin{cases} H(K_i) & i = j \\ \operatorname{Im} h_p^{i,j} & i < j \end{cases}, \qquad \beta_p^{i,j} = \begin{cases} \beta_p(K_i) & i = j \\ \dim(H_p^{i,j}) & i < j \end{cases} \tag{4}$$

For a fixed $p \geq 0$, the collection of persistent pairs (i,j) together with unpaired simplices (l, ∞) form a summary representation $\mathrm{dgm}_p(K_\bullet)$ called the *p-th persistence diagram of* $K_\bullet$. Note that the persistent Betti numbers can be read off directly given $\mathrm{dgm}_p(K_\bullet)$; conceptually, $\beta_p^{i,j}$ counts the number of persistent pairs lying inside the box $(-\infty, i\,] \times (\,j, \infty)$ (see Figure 1)—the number of persistent homology groups born at or before i that died sometime after j.

3 Main Result

Let $B_p(K_\bullet) \subseteq Z_p(K_\bullet) \subseteq C_p(K_\bullet)$ denote the p-th boundary, cycle, and chain groups of a given filtration $K_\bullet$, respectively. Additionally, let $\partial_p : C_p(K_\bullet) \to C_p(K_\bullet)$ denote the boundary operator sending p-chains to their

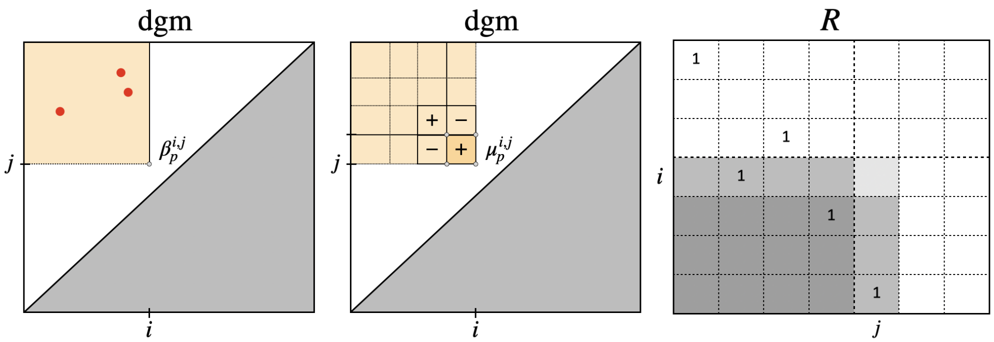

Figure 1: (Left) the persistent Betti number $\beta_p^{i,j}$ counts the number of points (3) in upper left-corner of $\mathrm{dgm}_p(K_\bullet)$. (Middle) The additivity of PBNs can be used to express multiplicity $\mu_p^{i,j}$ of any given box. (Right) The computational interpretation of the Pairing Uniqueness Lemma; in this case $r_R(i,j) = 3 - 2 + 1 - 2 = 0$ yields whether the entry $R[i,j]$ is non-zero.

respective boundaries. With a slight abuse of notation, we also use ∂_p to also denote the filtration boundary matrix with respect to an ordered basis $(\sigma_i)_{1 \leq i \leq m_p}$. The p-th persistent Betti number $\beta_p^{i,j}$ at some index $(i,j) \in \Delta_+^m$ is defined as:

$$\begin{aligned} \beta_p^{i,j} &= \dim(H_p^{i,j}) \\ &= \dim\left(Z_p(K_i)/(Z_p(K_i) \cap B_p(K_j)\right) \\ &= \dim\left(Z_p(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right) \end{aligned} \tag{5}$$

While $Z_p(K_i) = \mathrm{nullity}(\partial_p(K_i))$ and is thus easily obtained, efficient computation of the intersection term (the persistence part) is a bit more subtle. Zomorodian et al [6] give a procedure to compute a basis for $Z_p(K_i) \cap B_p(K_j)$ via a sequence of boundary matrix reductions; the subsequent Theorem (5.1) reduces the complexity of computing PH groups with coefficients in any PID to that of computing homology groups. However, the standard homology computations exhibit $O(m^2)$ space and $O(m^3)$ time complexities, making the PBN computation no more efficient than the full persistence computation.

Alternatively, both iterative and explicit projector-based methods [2] may also for the intersection term in (5), though these projectors may still be expensive to compute.

In what follows, we outline a different approach to computing (5) that is both simpler and computationally more attractive. To illustrate our approach, we require more notation. If A is a $m \times n$ matrix, let $A^{i,j}$ denote the lower-left submatrix defined by last $m-i+1$ rows (rows i through m, inclusive) and the first j columns. For any $1 \leq i < j \leq m$, define the quantity $r_A(i,j)$ as follows:

$$r_A(i,j) = \mathrm{rank}(A^{i,j}) - \mathrm{rank}(A^{i+1,j}) + \mathrm{rank}(A^{i+1,j-1}) - \mathrm{rank}(A^{i,j-1}) \tag{6}$$

The structure theorem from [6] shows that 1-parameter persistence modules can be decomposed in an *essentially unique* way into indecomposables. Computationally, a consequence of this phenomenon is the Pairing Uniqueness Lemma [4], which asserts that if $R = \partial V$ is the decomposition of the boundary matrix, then:

$$r_R(i,j) \neq 0 \Leftrightarrow R[i,j] \neq 0$$

Since the persistence diagram is derived completely from R, the immediate result of this result is that information about a diagram can be obtained through rank computations alone. For a more geometric description of this idea, see the third picture in Figure 1. We record a non-trivial fact that follows from this observation:

Lemma 1 (Dey & Wang [5])**.** *Let $R = \partial V$ denote the matrix decomposition of a given filtered boundary matrix ∂ derived from the associated filtration $K_\bullet$. For any pair (i,j) satisfying $1 \leq i < j \leq m$, we have:*

$$\mathrm{rank}(R^{i,j}) = \mathrm{rank}(\partial^{i,j}) \tag{7}$$

Equivalently, all lower-left submatrices of ∂ have the same rank as their corresponding submatrices in R.

Lemma 1 was the essential motivating step used by Chen et al [3] in their rank-based persistence algorithm—the first output-sensitive algorithm given for computing persistent homology of a filtered complex. In fact, Lemma 1 may be further generalized to arbitrary rectangles in Δ_+ via *μ-queries* [3]: box-parameterized rank-based queries that count the number of persistence pairs that intersect a fixed "box" placed in the upper half-plane. We show this Lemma allows us to write the persistent Betti number as a sum of rank functions.

Proposition 1. *For any fixed $p \geq 0$, let ∂_p denote the p-dimensional boundary matrices of filtration $K_\bullet$ of size $m_p = |K_{(p)}|$. For any pair $(i,j) \in ([m_p],[m_{p+1}])$, the persistent Betti number $\beta_p^{i,j}$ at (i,j) is given by:*

$$\beta_p^{i,j} = \mathrm{rank}(I_p^{1,i}) - \mathrm{rank}(\partial_p^{1,i}) - \mathrm{rank}(\partial_{p+1}^{1,j}) + \mathrm{rank}(\partial_{p+1}^{i+1,j}) \tag{8}$$

where $I_p^{1,i}$ denotes the first i columns of the $m_p \times m_p$ identity matrix.

A detailed proof using Lemma 1 is given in the appendix. The main utility this proposition provides is that it enables the persistent Betti number as a combination of rank computations performed directly on the *unfactored* dimension p and $(p+1)$ boundary matrices.

To get some intuition on what these matrices look like, we include a picture of each of the terms in Equation (8).

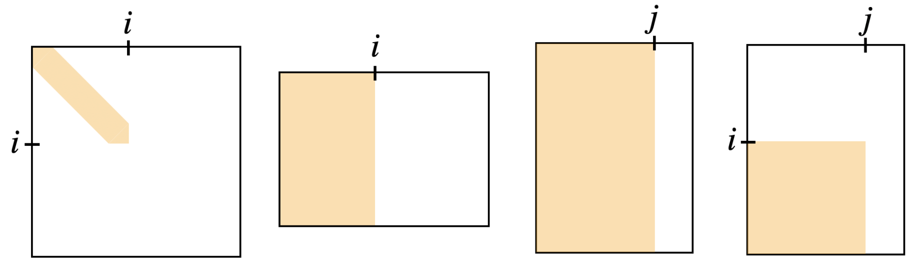

Figure 2: The four matrices whose ranks yield $\beta_p^{i,j}$ in the same order as given in (8). Each solid portion represents (sparse) blocks of non-zero entries, while each white portion is zero. Observe $\partial_{p+1}^{i+1,j}$ can be obtained by intersecting the non-zero entries of $\partial_{p+1}^{1,j}$ with the non-zero entries in the complement of $\partial_p^{1,i}$.

A Parameterized Boundary Matrix Relaxation

One advantage of expressing the PBN via equation (8) is that certain properties of rank function may be exploited in *parameterized* settings, i.e. settings where the scalar-valued filter function and corresponding filtration belong to a parametric family. One simple such property is permutation invariance: given any square matrix $A \in \mathbb{R}^{n \times n}$, $\mathrm{rank}(A) = \mathrm{rank}(P^T A P)$ for any permutation matrix P. Observe the boundary matrices ∂_p in (8) need not be in filtration order induced by $(K_\bullet, f)$ to be evaluated—the rank function is permutation invariant so long as the involved matrices have the same essential non-zero pattern. Thus, unlike the vineyards algorithm [4]—which requires a $\approx O(m^2)$ maintenance procedure to simulate persistence across a homotopy[2]—PBNs need no such procedure even in parameterized settings. We elaborate in the following section.

Recall that the boundary operator ∂_p for a finite simplicial filtration $K_\bullet$ with $m = |C_p(K_\bullet)|$ and $n = |C_{p-1}(K_\bullet)|$ can be represented by an $(n \times m)$ boundary matrix ∂_p whose columns and rows correspond to p-simplices and $(p-1)$-simplices, respectively. The entries of ∂_p depend on the choice of $\mathbb{F}$; in general, after orientating the simplices of K arbitrarily, they have the form:

$$\partial_p[k,l] = \begin{cases} c(\sigma_j) & \text{if } \sigma_l \in \partial_p(\sigma_k) \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $c(\sigma_*) \in \mathbb{F}$ is an arbitrary constant satisfying $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same simplex, typically set to ± 1. In what follows, we assume a fixed orientation is given on K, and write $\pm c(\sigma)$ to indicate the sign of $c(\sigma)$ depends on the orientation of σ.

The typical input to the persistence computation is a fixed complex/filter pair $(K_\bullet, f)$, from which the corresponding boundary matrices are constructed. These filter functions typically have a particular geometrical interpretation. For example, a common scenario in practice is let $K_\bullet$ via a Rips filtration a metric space (X, d_X); in this case $f : \mathcal{P}(X) \to \mathbb{R}_+$ is given as the diameter $f(\sigma) = \max_{x,x' \in \sigma} d_X(x,x')$. In many applications it is of interest to study

[2]Strictly speaking, the bound $O(m^2)$ assumes the homotopy changes each filtration value in a monotone way throughout the homotopy.

such filter function in parameterized settings, e.g. given some set of parameters $\mathcal{H}$, the goal is to understand a given topological invariant at many parameters $h \in \mathcal{H}$, i.e. treat $f : \mathcal{P}(X) \times \mathcal{H} \to \mathbb{R}_+$. We include several example in section **??**.

Suppose that instead of being given a fixed filtration $(K_\bullet, f)$, the filter was parameterized $f : \mathcal{H} \times K \to \mathbb{R}$ and you wanted to compute $\beta_p^{i,j}$ over $\mathcal{H}$. We give several instantiations of this in section **??**.

Definition 1 (Parameterized boundary matrix)**.** *Let X denote a data set of interest of size $|X| = n$, equipped with parameterized filtering function $f : \mathcal{P}(X) \times \mathcal{H} \to \mathbb{R}$. Define $(\mathcal{P}(X), \preceq^*)$ be a fixed linear extension of the face poset of the standard $(n$-$1)$-simplex. For fixed $i, j \in \mathbb{F}$, define the $\mathcal{H}$-*parameterized p-th boundary matrix $\hat{\partial}_p^{i,j}(t)$ at scale *(i, j) to be the $\binom{n}{p} \times \binom{n}{p+1}$ matrix ordered by $\preceq^*$ for all $h \in \mathcal{H}$, and whose entries (k, l) satisfy:*

$$\hat{\partial}_p^{i,j}(t)[k,l] = \begin{cases} \pm\, S_{i,j}(\sigma_k, \sigma_l) & \text{if } \sigma_k \in \partial_p(\sigma_l) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $S_{i,j} : \mathcal{P}(X) \times \mathcal{P}(X) \to \{0, 1\}$ is a step *function that accepts a face/coface pair (σ_k, σ_l) and returns a* 1 *if $f(\sigma_k) \geq i$ and $f(\sigma_l) \leq j$, and* 0 *otherwise.*

We now show a few properties that $\hat{\partial}_p^*(t)$ exhibits which is advantageous for parameterized families. To simplify the notation, we write $A^x = A^{1,x}$ for the setting where only columns up to x of A are being selected, and let $q = p + 1$. The first such property is a simple parameterized relaxation of PBN:

$$\begin{aligned} \hat{\beta}_p^{i,j} &= |K_i^{(p)}| - \mathrm{rank}(\partial_p^i) - \mathrm{rank}(\partial_q^j) + \mathrm{rank}(\partial_q^{i+1,j}) && (11)\\ &= |K_i^{(p)}| - \mathrm{rank}((\partial_p^i)(\partial_p^i)^T) - \mathrm{rank}((\partial_q^j)(\partial_q^j)^T) + \mathrm{rank}((\partial_q^{i+1,j})(\partial_q^{i+1,j})^T) && (12)\\ &= |K_i^{(p)}| - \mathrm{rank}(L_p^i) - \mathrm{rank}(L_q^j) + \mathrm{rank}(L_q^{i+1,j}) && (13) \end{aligned}$$

Rank Relaxation As integer-valued invariants, Betti numbers pose several difficulties to vectorization. For example, the rank function typically defined on matrices M is given as:

Much is known about the spectrum of Laplacian matrices. From discerning the beat of drums to understanding the chemical makeup of Benzene.... etc, the normalize graph Laplacian has found many interpretations of its spectrum.

Typically, one often needs substantial and precise domain-specific knowledge about the phenonum in order to write it down as, e.g. a partial differential equation (PDE) and to interpret its corresponding Laplace operator(s). For example, one may need to know the underyling system producing the observed data follow some prescribed laws of motion, or conversation laws of energy, etc. However, information about such laws is either absent, obfuscated, or non-existent in many modern data science applications. This is invariably the case for many human-generated data sets: movie ratings, newsfeeds, social media updates, etc.

implicitly regarded a graph as a discrete analogue of a Riemannian manifold and cohomology as a discrete analogue of PDEs: standard partial differential operators on Riemannian manifolds — gradient, divergence, curl, Jacobian, Hessian, scalar and vector Laplace operators, Hodge Laplacians — all have natural counterparts on graphs

For example, it is known that one of them completely characterizes the other 2 sets... (**??**)

Interpretation: Multiplicity

Complexity

4 Applications

A Appendix

Proofs

Proof of Lemma 1

Proof. The Pairing Uniqueness Lemma [5] asserts that if $R = \partial V$ is a decomposition of the total $m \times m$ boundary matrix ∂, then for any $1 \leq i < j \leq m$ we have $\text{low}_R[j] = i$ if and only if $r_\partial(i,j) = 1$. As a result, for $1 \leq i < j \leq m$, we have:

$$\text{low}_R[j] = i \iff r_R(i,j) \neq 0 \iff r_\partial(i,j) \neq 0 \tag{14}$$

Extending this result to equation (7) can be seen by observing that in the decomposition, $R = \partial V$, the matrix V is full-rank and obtained from the identity matrix I via a sequence of rank-preserving (elementary) left-to-right column additions. ☐

Proof of Proposition 1

Proof. We first need to show that $\beta_p^{i,j}$ can be expressed as a sum of rank functions. Note that by the rank-nullity theorem, so we may rewrite (5) as:

$$\beta_p^{i,j} = \dim\left(C_p(K_i)\right) - \dim\left(B_{p-1}(K_i)\right) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)$$

The dimensions of groups $C_p(K_i)$ and $B_p(K_i)$ are given directly by the ranks of diagonal and boundary matrices, yielding:

$$\beta_p^{i,j} = \text{rank}(I_p^{1,i}) - \text{rank}(\partial_p^{1,i}) - \dim\left(Z_p(K_i) \cap B_p(K_j)\right)$$

To express the intersection term, note that we need to find a way to express the number of p-cycles born at or before index i that became boundaries before index j. Observe that the non-zero columns of R_{p+1} with index at most j span $B_p(K_j)$, i.e $\{ \text{col}_{R_{p+1}[k]} \neq 0 \mid k \in [j] \} \in \text{Im}(\partial_{p+1}^{1,j})$. Now, since the low entries of the non-zero columns of R_{p+1} are unique, we have:

$$\dim(Z_p(K_i) \cap B_p(K_i)) = |\Gamma_p^{i,j}| \tag{15}$$

where $\Gamma_p^{i,j} = \{ \text{col}_{R_{p+1}[k]} \neq 0 \mid k \in [j], 1 \leq \text{low}_{R_{p+1}}[k] \leq i \}$. Consider the complementary matrix $\bar{\Gamma}_p^{i,j}$, given by the non-zero columns of R_{p+1} with index at most j that are not in $\Gamma_p^{i,j}$, i.e. the columns satisfying $\text{low}_{R_{p+1}}[k] > i$. Combining rank-nullity with the observation above, we have:

$$|\bar{\Gamma}_p^{i,j}| = \dim(B_p(K_j)) - |\Gamma_p^{i,j}| = \text{rank}(R_{p+1}^{i+1,j}) \tag{16}$$

Combining equations (15) and (16) yields:

$$\dim(Z_p(K_i) \cap B_p(K_j)) = |\Gamma_p^{i,j}| = \dim(B_p(K_j)) - |\bar{\Gamma}_p^{i,j}| = \text{rank}(R_{p+1}^{1,j}) - \text{rank}(R_{p+1}^{i+1,j}) \tag{17}$$

Observing the final matrices in (17) are *lower-left* submatrices of R_{p+1}, the final expression (8) follows by applying Lemma 1 repeatedly. ☐

Proof of boundary matrix properties

Proof. First, consider property (1). For any $t \in T$, applying the boundary operator ∂_p to $K_t = \text{Rips}_\epsilon(\delta_\mathcal{X}(t))$ with non-zero entries satisfying (9) by definition yields a matrix ∂_p satisfying $\text{rank}(\partial_p) = \dim(\text{B}_{p-1}(K_t))$. In contrast, definition (1) always produces p-boundary matrices of Δ_n; however, notice that the only entries which are non-zero are precisely those whose simplices σ that satisfy $\text{diam}(\sigma) < \epsilon$. Thus, $\text{rank}(\partial_p^t) = \dim(\text{B}_{p-1}(K_t))$ for all $t \in T$. < (show proof of (2))> Property (3) follows from the construction of ∂_p and from the inequality $\|A\|_2 \leq \sqrt{m}\|A\|_1$ for an $n \times m$ matrix A, as $\|\partial_p^t\|_1 \leq (p+1)\,\epsilon$ for all $t \in T$.

☐

Dynamic Metric Spaces Consider an $\mathbb{R}$-parameterized metric space $\delta_X = (X, d_X(\cdot))$ where X is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_+$, satisfying:

1. For every $t \in \mathbb{R}, \delta_X(t) = (X, d_X(t))$ is a pseudo-metric space[3]

[3] This is required so that if one can distinguish the two distinct points $x, x' \in X$ incase $d_X(t)(x, x') = 0$ at some $t \in \mathbb{R}$.

2. For fixed $x, x' \in X$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_+$ is continuous.

When the parameter $t \in \mathbb{R}$ is interpreted as *time*, the above yields a natural characterization of a "time-varying" metric space. More generally, we refer to an $\mathbb{R}^h$-parameterized metric space as *dynamic metric space*(DMS). Such space have been studied more in-depth [] and have been shown...

Rank relaxation A common approach in the literature to optimize quantities involving $\mathrm{rank}(A)$ for some $m \times n$ matrix A is to consider optimizing its *nuclear norm* $\|A\|_* = \mathrm{tr}(\sqrt{A^T A}) = \sum_{i=1}^{r} |\sigma_i|$, where σ_i denotes the ith singular value of A and $r = \mathrm{rank}(A)$. One of the primary motivations for this substitution is that the nuclear norm is a convex envelope of the rank function over the set:

$$S := \{A \in \mathbb{R}^{n \times m} \mid \|A\|_2 \leq m\}$$

That is, for an appropriate $m > 0$, the function $A \mapsto \frac{1}{m}\|A\|_*$ is a lower convex envelope of the rank function over S. The nuclear norm also admits a subdifferential... thus, we may consider replacing (**??**) with:

$$\beta_p^{i,j}(t) = |\partial_{p,t}^{1,i}| - m_1^{-1}\|\partial_{p,t}^{1,i}\|_* - m_2^{-1}\|\partial_{\bar{p},t}^{1,j}\|_* - m_3^{-1}\|\partial_{\bar{p},t}^{\bar{i},j}\|_* \tag{18}$$

where $\bar{c} = c + 1$. Now, if $t \mapsto \partial_p^*(t)$ is a non-decreasing, convex function in t, then the composition ... is convex, as each of the individual terms are convex. Moreover, we have...

< Insert proof about this relaxation always lower-bounding β >

Application: Time-varying Let $\delta_{\mathcal{X}}$ denote an T-parameterized metric space $\delta_{\mathcal{X}}(\cdot) = (X, d_X(\cdot))$, where $d_X : \mathrm{T} \times X \times X \to \mathbb{R}_+$ is called a *time-varying metric* and X is a finite set with fixed cardinality $|X| = n$. δ_X as called a *dynamic metric space* (DMS) iff $d_X(\cdot)(x, x')$ is continuous for every pair $x, x' \in X$ and $\delta_{\mathcal{X}}(t) = (X, d_X(t))$ is a pseudo-metric space for every $t \in \mathrm{T}$. For a fixed $t \in \mathrm{T}$, the Rips complex at scale $\epsilon \in \mathbb{R}$ is the abstract simplicial complex given by

$$\mathrm{Rips}_\epsilon(\delta_{\mathcal{X}}(t)) := \{\sigma \subset X : d_X(t)(x, x') \leq \epsilon \text{ for all } x, x' \in \sigma\} \tag{19}$$

As before, the family of Rips complexes for varying $\epsilon > 0$ yields a filtration whose inclusion maps induce linear maps at the level of homology. The time-varying counterpart is analogous. In this context, we write the p-th persistent Betti number with respect to fixed values $i, j \in I$ as a function of $t \in \mathrm{T}$:

$$\beta_p^{i,j}(t) = \left(\dim \circ \mathrm{H}_p^{i,j} \circ \mathrm{Rips} \circ \delta_{\mathcal{X}}\right)(t) \tag{20}$$

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A Boundary matrix factorization

Definition 2 (Boundary matrix decomposition). *Given a filtration $K_\bullet$ with m simplices, let ∂ denote its $m \times m$ filtered boundary matrix. We call the factorization $R = \partial V$ the* boundary matrix decomposition *of ∂ if:*

I1. V is full-rank upper-triangular

I2. R satisfies $\mathrm{low}_R[i] \neq \mathrm{low}_R[j]$ iff its i-th and j-th columns are nonzero

where $\mathrm{low}_R(i)$ denotes the row index of lowest non-zero entry of column i in R or null *if it doesn't exist. Any matrix R satisfying property (I2) is said to be* reduced*; that is, no two columns share the same low-row indices.*