Interpreting the spectrum of the Graph Laplacian

Matt Piekenbrock

Let $G = (V, E)$ denote a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E \subseteq V \times V$. A weighted graph is a pair (G, μ) where $\mu : V \times V \to \mathbb{R}_+$ is an weight function satisfying $\mu_{v,v'} = \mu_{v',v}$, $\mu_{v,v'} > 0$ iff $(v, v') \in E$. Note that the last condition implies μ completely characterizes the connectivity of G, i.e. positive values of μ indicate the presence of edges.

We recall the Spectral Theorem, which characterizes the eiegnvlaues of a linear operator in terms of *Rayleigh quotients*. If U is a finite dimensional vector space and A a linear operator on U, then for any non-zero $u \in U$, the Rayleigh quotient of u is defined as:

$$\mathcal{R}(u) = \frac{\langle Au, u\rangle}{\langle u, u\rangle}$$

Any Laplacian operator $\mathcal{L} = A - D$ has a spectrum of $\lambda \in [0, 2]$ for any $\lambda \in \Lambda(L)$.

Let (λ, f) denote a eigenvalue/eigenfunction pair of L, respectively.

$$\langle Lf, f\rangle = \frac{1}{2}\sum_{v \in V}\sum_{v' \in V} \mu(v, v') \cdot (f(v) - f(v'))^2$$

One can interpret the eigenvalues of the Laplacian physically as the frequencies of vibration of a membrane, as energy levels of a Hamiltonian in an infinite potential well, as rates of decay for the heat (or mass diffusion) equation, and as cut-off frequencies for waveguides.

Two Laplacians L_1 and L_2 are cospectral if and only if $\alpha L_1 + \beta I + \gamma J$ and $\alpha L_2 + \beta I + \gamma J$ are (where $\alpha \neq 0$).

The following can be deduced from the spectrum of the adjacency matrix or Laplacian matrix of G:

1. The number of vertices
2. Number of edges
3. Whether G is a regular
4. Whether G is regular with any fixed girth

Additional, the spectrum of the Laplacian contains information about 1. the number of connected components and 2. the number of unique spanning trees. The former is given by the algebraic multiplicity of the 0 eigenvalue of L (the corank of L), and the latter is given by—if G is connected—$1/n$ times the product of L non-zero eigenvalues.

There are many graphs which are determined by their spectrum (DS).