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1 Power Series

Definition 1. *Power series are of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

The goal of this section is to explain the circumstances where the continuity and differentiability properties of polynomials are kept with a power series (think polynomial of infinite degree).

1. Consider sequence of functions $g_n(x) = x^n$ on $[0, 1]$. Now

$$\lim_{n\to\infty} g_n(x) = g(x)$$

   where $g(x) = 0$ if $0 \leq x < 1$ and $g(x) = 1$ if $x = 1$. Here, we see example of sequence of continuous functions converging to $g(x)$ which is discontinuous at $g(1)$.

2. Another example of the limit function not inheriting its sequence's properties can be seen with $h_n(x) = x^{\frac{2n}{2n-1}}$. Since

$$h(x) = \lim_{n\to\infty} h_n(x) = |x|,$$

   we see that h is not differentiable at $h(0)$.

Definition 2. *A sequence of functions $f_n : X \to R$ converges pointwise to f on X if for all $x \in X$, the sequence $f_n(x)$ converges to $f(x)$.*

Definition 3. *Our sequence f_n converges uniformly to f on X if for any $\epsilon > 0$, there exists N such that $|f_{n\geq N}(x) - f(x)| < \epsilon$ for all $x \in X$.*

Since with pointwise convergence, neither continuity nor differentiability are guaranteed to be preserved, we consider uniform convergence.

Theorem 1. *(Cauchy Criterion for Uniform Convergence) If for any $\epsilon > 0$, there exist N such that $|f_{n\geq N}(x) - f_{m\geq N}(x)| < \epsilon$, then $(f_n) \to f$ uniformly.*

Proof. Suppose the above were true. Fix an arbitrary $x \in X$. For any $\epsilon > 0$, there is N where $|f_{n\geq N}(x) - f_{m\geq N}(x)| < \epsilon$. By Cauchy Criterion for Convergent Sequences, $(f_n(x))$ converges to some value $f(x)$. Since N is not dependant on x, we can say this is true for all $x \in X$ and so $(f_n(x)) \to f(x)$ uniformly.

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