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1 Power Series

Definition 1. *Power series are of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

The goal of this section is to explain the circumstances where the continuity and differentiability properties of polynomials are kept with a power series (think polynomial of infinite degree).

1. Consider sequence of functions $g_n(x) = x^n$ on $[0, 1]$. Now

$$\lim_{n\to\infty} g_n(x) = g(x)$$

   where $g(x) = 0$ if $0 \leq x < 1$ and $g(x) = 1$ if $x = 1$. Here, we see example of sequence of continuous functions converging to $g(x)$ which is discontinuous at $g(1)$.

2. Another example of the limit function not inheriting its sequence's properties can be seen with $h_n(x) = x^{\frac{2n}{2n-1}}$. Since

$$h(x) = \lim_{n\to\infty} h_n(x) = |x|,$$

   we see that h is not differentiable at $h(0)$.

Definition 2. *A sequence of functions $f_n : X \to R$ converges pointwise to f on X if for all $x \in X$, the sequence $f_n(x)$ converges to $f(x)$.*

Definition 3. *Our sequence f_n converges uniformly to f on X if for any $\epsilon > 0$, there exists N such that $|f_{n\geq N}(x) - f(x)| < \epsilon$ for all $x \in X$.*

Since with pointwise convergence, neither continuity nor differentiability are guaranteed to be preserved, we consider uniform convergence.

Theorem 1. *(Cauchy Criterion for Uniform Convergence) If for any $\epsilon > 0$, there exist N such that $|f_{n\geq N}(x) - f_{m\geq N}(x)| < \epsilon$, then $(f_n) \to f$ uniformly.*

Proof. Suppose the above were true. Fix an arbitrary $x \in X$. For any $\epsilon > 0$, there is N where $|f_{n\geq N}(x) - f_{m\geq N}(x)| < \epsilon$. By Cauchy Criterion for Convergent Sequences, $(f_n(x))$ converges to some value $f(x)$. Since N is not dependant on x, we can say this is true for all $x \in X$ and so $(f_n(x)) \to f(x)$ uniformly.

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Indeed, continuity is preserved with uniform convergence.

Theorem 2. *Suppose each function in the sequence of functions $f_n : X \to R$ is continuous at $c \in X$. If $(f_n(x)) \to f(x)$ uniformly, then f is continuous at c.*

Proof. Fix $\frac{\epsilon}{3} > 0$. There must exist N such that $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ and $|f_N(c) - f(c)| < \frac{\epsilon}{3}$. Furthermore, since f_N is continuous, it follows that there exists δ where $|x - c| < \delta \implies |f_N(x) - f_N(c)| < \frac{\epsilon}{3}$. It follows that

$$|f(x) - f(c) + f_N(x) - f_N(x) + f_N(c) - f_N(c)| \leq |f_N(x) - f(x)| + |f_N(c) - f(c)| + |f_N(x) - f_N(c)| < \epsilon.$$

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Theorem 3. *Let $X = \{x_1, x_2, x_3, ...\}$, a countable set of real numbers. If there is a bounded sequence of functions $f_n : X \to R$, it follows that there exist a pointwise convergent subsequence.*

Proof. Consider the sequence $f_n(x_1)$. Since this sequence is bouded, by Bolzano-Weierstrass theorem, there is a convergent subsequence $F_n(x_1)$. Now consider $F_n(x_2)$ (ie. the functions part of the convergent subsequence for x_1 applied to x_2). Like with x_1, we can generate a subsequence of F_n (so we're nesting subsequences). Let us keep generating subsequences in this manner. Now, let us create a new sequence of functions in the following manner. Enumerate through $x_1, x_2, x_3, ...$: for x_1, pick f_{k_1} where $f_{k_1}(x_1)$ is part of x_1's subsequence. For x_n, pick f_{k_n} in the same manner but make sure that $k_n > k_{n-1}$. Notice how for any point x_i, the sequence $f_{k_i}(x_i), f_{k_{i+1}}(x_i), f_{k_{i+2}}(x_i)...$ is convergent. Therefore, (f_{k_n}) converges pointwise to some function.

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This is one condition where the idea of Bolzano-Weierstrass can be applied to sequences of functions. Do pay special attention to the fact that f_n was bounded. We present another condition, but this time it implies uniform continuity.

Definition 4. *A function $f_n : X \to R$ is called equicontinuous if for all $\epsilon > 0$, there exist $\delta > 0$ such that*

$$|x - c| < \delta \implies |f_{n \geq N}(x) - f_{n \geq N}(c)| < \epsilon$$

for any x and c in domain.

Note that this is stronger than uniform continuity since δ is decided for all the functions as well.

Theorem 4. *(Arzela-Ascoli Theorem) Consider bounded sequence of functions $f_n : [A, B] \to R$ where each function is equicontinuous. There exists a uniformly convergent subsequence of f_n.*

Proof. Fix $E > 0$. Let $\epsilon = \frac{E}{2}$. Consider the countable set of rational numbers within the domain $\{r_1, r_2, r_3, ...\}$. From theorem 3, it follows that there exist subsequence of functions f_{n_k} that is pointwise convergent on these rational numbers. Due to equicontinuity, there exist δ such that

$$|x - c| < \delta \implies |f_n(x) - f_n(c)| < \epsilon$$

for all x and c in $[A, B]$. Now let $\{V_\delta(r_{n_1}), V_\delta(r_{n_2}), V_\delta(r_{n_3}), ..., V_\delta(r_{n_L})\}$ be a finite open cover of $[A, B]$ consisting of δ neighborhoods of r_{n_i}, who are rational numbers within $[A, B]$. For each r_{n_i}, let there be corresponding N_i such that $|f_{n \geq N_i}(r_{n_i}) - f_{m \geq N_i}(r_{n_i})| < \frac{\epsilon}{3}$ (we know such value exist from our proof about pointwise convergence earlier). Now, define

$$N = \max\{N_1, N_2, N_3, ..., N_L\}.$$

Consider two real numbers a, b in $[A, B]$. Because rational numbers are dense in the reals, we can pick two distinct rational numbers R_1, R_2 in the δ neighborhoods of a and b respectively. We know, from triangle inequality, that for $E > 0$ there exist N such that

$$\begin{aligned}
|f_{n\geq N}(a) - f_{m\geq N}(b)| &= |f_{n\geq N}(a) - f_{m\geq N}(b) - f_{n\geq N}(R_1) + f_{n\geq N}(R_1) - f_{m\geq N}(R_2) + f_{m\geq N}(R_2)| \\
&\leq |f_{n\geq N}(a) - f_{n\geq N}(R_1)| + |f_{m\geq N}(b) - f_{n\geq N}(R_2)| + |f_{n\geq N}(R_1) - f_{m\geq N}(R_2)| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq \epsilon \\
&< E.
\end{aligned}$$

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Now we examine the conditions where differentiability is preserved.

Theorem 5. *If $f_n : [A, B]$ is a sequence of differentiable functions with the sequence $(f_n') \to g$ uniformly and for some $x \in [A, B]$ the sequence $f_n(x)$ is convergent, then $(f_n) \to f$ uniformly and $f'(x) = g(x)$.*

Proof.

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