Solutions

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Problem (4-3).

For notation, take $\langle \cdot, \cdot \rangle$ to be $T(\cdot, \cdot)$. Let $e_1, ..., e_n$ be an orthonormal basis for V. By definition, $|\omega(e_1, ..., e_n)| = 1$, and applying Theorem 4-6 gives

$$|\omega(e_1, ..., e_n)| = \left| \det \begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \right|.$$

Using the common identity (see any quantum mechanics textbook)

$$\begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \langle e_i, w_j \rangle & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \langle w_i, w_j \rangle & \\ & & \ddots \end{bmatrix},$$

we arrive at the desired result.

Problem (4-4).

$$\begin{aligned} f^*\omega(v_1, ..., v_n) &= \omega(f(v_1), ..., f(v_n)) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \omega(f(e_1), ..., f(e_n)) \quad \text{(Theorem 4-3)} \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \\ &= \det \begin{bmatrix} \ddots & & \\ & \langle v_i, e_j \rangle & \\ & & \ddots \end{bmatrix}, \end{aligned}$$

where the last two equalities come from the fact that ω is a volume element and $f^*T(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.

Problem (4-5).

Because det is continuous, the image of $\det \circ c$ on the path must be of the same sign.

Problem (4-6)**.**

(a) We have $v_1 \times v_2 = \det[v_1 \ v_2]$.
(b) By definition, $\det[v_1, ..., v_{n-1}, v_1 \times \cdots \times v_{n-1}] = ||v_1 \times \cdots \times v_{n-1}||^2$.

Problem (4-7)**.**

Fix $\omega \in \wedge^n(V)$. Let S be any inner product on V. Construct inner product T by scaling S by $\frac{1}{\omega(e_1,...,e_n)^2}$, where $e_1, ..., e_n$ is an orthonormal basis for V. Now, it's easy to see that ω is a volume element with respect to T.

Problem (4-8)**.**

Take T to be the inner product on V. Let $\varphi \in V^*$ be given by $\varphi(v) = \omega(v_1, ..., v_{n-1}, v)$, where $v_1, ..., v_{n-1}$ are fixed. Then, by the Riesz representation theorem, $\varphi(v) = T(v, w)$ for some $w \in V$. We can now define $v_1 \times \cdots \times v_{n-1} = w$.

Problem (4-9)**.**

(a) and (b) By definition,

$$\langle v \times w, e_i \rangle = \det[v \ w \ e_i],$$

and the formula in (b) follows. (a) is obtained by applying (b).
(c) We have

$$\begin{aligned}
\frac{1}{\|v\|^2 \|w\|^2} \|v \times w\|^2 &= \frac{1}{\|v\|^2 \|w\|^2} \left\| \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \right\|^2 \\
&= \frac{1}{\|v\|^2 \|w\|^2} (v_2^2w_3^2 + v_3^2w_2^2 - 2v_2v_3w_2w_3 \\
&+ v_3^2w_1^2 + v_1^2w_3^2 - 2v_1v_3w_1w_3 \\
&+ v_1^2w_2^2 + v_2^2w_1^2 - 2v_1v_2w_1w_2) \\
&= 1 - \frac{\langle v, w \rangle^2}{\|v\|^2 \|w\|^2} \\
&= 1 - \cos^2\theta \\
&= \sin^2\theta
\end{aligned}$$

so $\|v \times w\| = \|v\| \|w\| \sin\theta$. By definition,

$$\langle v \times w, v \rangle = \det[v \ w \ v] = 0,$$

and $\langle v \times w, w \rangle = 0$ follows similarly.
(d) We have

$$\langle v, w \times z \rangle = \det[w \ z \ v] = \langle w, z \times v \rangle = \det[z \ v \ w] = \langle z, v \times w \rangle = \det[v \ w \ z].$$

To prove the vector triple product identity, we have to bash out some algebra. I'm too lazy to type it out, so I will leave it as an exercise to the reader.
(e) We have

$$\begin{aligned}
\|v \times w\|^2 &= \|v\|^2 \|w\|^2 \sin^2\theta \\
&= \|v\|^2 \|w\|^2 (1 - \cos^2\theta) \\
&= \|v\|^2 \|w\|^2 - \langle v, w \rangle^2 \\
&= \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2.
\end{aligned}$$

Problem (4-10)**.**

By definition,

$$\|w_n\|^2 = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix},$$

where $w_n = w_1 \times \cdots \times w_{n-1}$. Observe that $w_1, ..., w_{n-1} \in \operatorname{span}\{w_n\}^\perp$, since

$$\langle w_i, w_n \rangle = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_i \end{bmatrix} = 0$$

if $1 \leq i \leq n-1$.

By problem 4-3, we have

$$\begin{aligned} \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} &= \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n}} \\ &= \sqrt{\langle w_n, w_n \rangle \det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}} \\ &= \|w_n\| \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}}. \end{aligned}$$

Thus, $\|w_n\| = \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}}$.

Problem (4-11)**.**

Since $v_1, ..., v_n$ is an orthonormal basis, this result follows by the definition, since $a_{i,j} = \langle f(v_i), v_j \rangle = \langle f(v_j), v_i \rangle = a_{j,i}$.

Problem (4-12)**.**