Solutions

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Problem (4-3)**.**

For notation, take $\langle \cdot, \cdot \rangle$ to be $T(\cdot, \cdot)$. Let $e_1, ..., e_n$ be an orthonormal basis for V. By definition, $|\omega(e_1, ..., e_n)| = 1$, and applying Theorem 4-6 gives

$$|\omega(e_1, ..., e_n)| = \left| \det \begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \right|.$$

Using the common identity (see any quantum mechanics textbook)

$$\begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \langle e_i, w_j \rangle & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \langle w_i, w_j \rangle & \\ & & \ddots \end{bmatrix},$$

we arrive at the desired result.

Problem (4-4)**.**

$$\begin{aligned} f^*\omega(v_1, ..., v_n) &= \omega(f(v_1), ..., f(v_n)) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \omega(f(e_1), ..., f(e_n)) \quad \text{(Theorem 4-3)} \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \\ &= \det \begin{bmatrix} \ddots & & \\ & \langle v_i, e_j \rangle & \\ & & \ddots \end{bmatrix}, \end{aligned}$$

where the last two equalities come from the fact that ω is a volume element and $f^*T(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.