Solutions

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Problem (4-3).

For notation, take $\langle \cdot, \cdot \rangle$ to be $T(\cdot, \cdot)$. Let $e_1, ..., e_n$ be an orthonormal basis for V. By definition, $|\omega(e_1, ..., e_n)| = 1$, and applying Theorem 4-6 gives

$$|\omega(e_1, ..., e_n)| = \left| \det \begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \right|.$$

Using the common identity (see any quantum mechanics textbook)

$$\begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \langle e_i, w_j \rangle & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \langle w_i, w_j \rangle & \\ & & \ddots \end{bmatrix},$$

we arrive at the desired result.

Problem (4-4).

$$\begin{aligned} f^*\omega(v_1, ..., v_n) &= \omega(f(v_1), ..., f(v_n)) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \omega(f(e_1), ..., f(e_n)) \quad \text{(Theorem 4-3)} \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \\ &= \det \begin{bmatrix} \ddots & & \\ & \langle v_i, e_j \rangle & \\ & & \ddots \end{bmatrix}, \end{aligned}$$

where the last two equalities come from the fact that ω is a volume element and $f^*T(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.

Problem (4-5).

Because det is continuous, the image of $\det \circ c$ on the path must be of the same sign.

Problem (4-6)**.**

(a) We have $v_1 \times v_2 = \det[v_1\ v_2]$.
(b) By definition, $\det[v_1, ..., v_{n-1}, v_1 \times \cdots \times v_{n-1}] = ||v_1 \times \cdots \times v_{n-1}||^2$.

Problem (4-7)**.**

Fix $\omega \in \wedge^n(V)$. Let S be any inner product on V. Construct inner product T by scaling S by $\frac{1}{\omega(e_1,...,e_n)^2}$, where $e_1, ..., e_n$ is an orthonormal basis for V. Now, it's easy to see that ω is a volume element with respect to T.

Problem (4-8)**.**

Take T to be the inner product on V. Let $\varphi \in V^*$ be given by $\varphi(v) = \omega(v_1, ..., v_{n-1}, v)$, where $v_1, ..., v_{n-1}$ are fixed. Then, by the Riesz representation theorem, $\varphi(v) = T(v, w)$ for some $w \in V$. We can now define $v_1 \times \cdots \times v_{n-1} = w$.

Problem (4-9)**.**

(a) and (b) By definition,

$$\langle v \times w, e_i \rangle = \det[v\ w\ e_i],$$

and the formula in (b) follows. (a) is obtained by applying (b).
(c) We have

$$\begin{aligned} \frac{1}{\|v\|^2 \|w\|^2} \|v \times w\|^2 &= \frac{1}{\|v\|^2 \|w\|^2} \left\| \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \right\|^2 \\ &= \frac{1}{\|v\|^2 \|w\|^2} (v_2^2w_3^2 + v_3^2w_2^2 - 2v_2v_3w_2w_3 \\ &+ v_3^2w_1^2 + v_1^2w_3^2 - 2v_1v_3w_1w_3 \\ &+ v_1^2w_2^2 + v_2^2w_1^2 - 2v_1v_2w_1w_2) \\ &= 1 - \frac{\langle v, w \rangle^2}{\|v\|^2 \|w\|^2} \\ &= 1 - \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

so $\|v \times w\| = \|v\| \|w\| \sin\theta$. By definition,

$$\langle v \times w, v \rangle = \det[v\ w\ v] = 0,$$

and $\langle v \times w, w \rangle = 0$ follows similarly.
(d) We have

$$\langle v, w \times z \rangle = \det[w\ z\ v] = \langle w, z \times v \rangle = \det[z\ v\ w] = \langle z, v \times w \rangle = \det[v\ w\ z].$$

To prove the vector triple product identity, we have to bash out some algebra. I'm too lazy to type it out, so I will leave it as an exercise to the reader.
(e) We have

$$\begin{aligned} \|v \times w\|^2 &= \|v\|^2 \|w\|^2 \sin^2\theta \\ &= \|v\|^2 \|w\|^2 (1 - \cos^2\theta) \\ &= \|v\|^2 \|w\|^2 - \langle v, w \rangle^2 \\ &= \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2. \end{aligned}$$

Problem (4-10)**.**

By definition,

$$\|w_n\|^2 = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix},$$

where $w_n = w_1 \times \cdots \times w_{n-1}$. Observe that $w_1, ..., w_{n-1} \in \operatorname{span}\{w_n\}^\perp$, since

$$\langle w_i, w_n \rangle = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_i \end{bmatrix} = 0$$

if $1 \leq i \leq n-1$.

By problem 4-3, we have

$$\begin{aligned} \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} &= \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n}} \\ &= \sqrt{\langle w_n, w_n \rangle \det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}} \\ &= \|w_n\| \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}}. \end{aligned}$$

Thus, $\|w_n\| = \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i,j \leq n-1}}$.

Problem (4-11)**.**

Since $v_1, ..., v_n$ is an orthonormal basis, this result follows by the definition, since $a_{i,j} = \langle f(v_i), v_j \rangle = \langle f(v_j), v_i \rangle = a_{j,i}$.

Problem (4-13)**.**

(a) We have

$$\begin{aligned} (g \circ f)_*(v_p) &= ((g \circ f)'(p)v)_{g(f(p))} \\ &= (g'(f(p))f'(p)v)_{g(f(p))} \\ &= g_*(f'(p)v_{f(p)}) \\ &= g_*(f_*(v_p)). \end{aligned}$$

Let ω be a k-form on $\mathbb{R}^p$. Then,

$$\begin{aligned} ((g \circ f)^*\omega)(p)(v_1, ..., v_k) &= \omega(g(f(p)))(g_*(f_*(v_1)), ..., g_*(f_*(v_k))) \\ &= (g^*\omega)(f(p))(f_*(v_1), ..., f_*(v_k)) \\ &= (f^*g^*\omega)(p)(v_1, ..., v_k). \end{aligned}$$

(b) Follows from Theorem 4-7.

Problem (4-14)**.**

Follows immediately from problem 4-13.

Problem (4-15)**.**

Tangent vector is $c_*((e_1)_t) = (1, f'(t))_{c(t)}$. Endpoint is $(1+t, f'(t)+f(t))$. Tangent line $y = f'(t)(x-t)+f(t)$. One may verify that the endpoint lies on the tangent line.

Problem (4-16)**.**

We have $\frac{d}{dt}\|c(t)\|^2 = 0 = 2\langle c(t), c'(t)\rangle$.

Problem (4-17)**.**

(a) Trivial. (b) $\nabla \cdot \mathbf{f} = \sum_i \frac{\partial}{\partial x^i} f^i = \operatorname{Tr} f'$ by definition.

Problem (4-18)**.**

$D_v f(p) = \frac{d}{dt} f(p+tv)|_{t=0} = f'(p)v$ via chain rule. The rest follows from Cauchy-Schwarz.

Problem (4-19)**.**

(a) $df = \sum_i \frac{\partial}{\partial x^i} f^i dx^i = \omega^1_{\nabla f}$.

$$\begin{aligned}
d\omega^1_F &= dF^1 dx + dF^2 dy + dF^3 dz \\
&= \frac{\partial}{\partial y}F^1 dy \wedge dx + \frac{\partial}{\partial z}F^1 dz \wedge dx + \frac{\partial}{\partial x}F^2 dx \wedge dy + \frac{\partial}{\partial z}F^2 dz \wedge dy + \frac{\partial}{\partial x}F^3 dx \wedge dz + \frac{\partial}{\partial y}F^3 dy \wedge dz \\
&= \left(\frac{\partial}{\partial x}F^2 - \frac{\partial}{\partial y}F^1\right) dx \wedge dy + \left(\frac{\partial}{\partial y}F^3 - \frac{\partial}{\partial z}F^2\right) dy \wedge dz + \left(\frac{\partial}{\partial z}F^1 - \frac{\partial}{\partial x}F^3\right) dz \wedge dx \\
&= \omega^2_{\nabla \times F}.
\end{aligned}$$

$$\begin{aligned}
d\omega^2_F &= dF^1 dy \wedge dz + dF^2 dz \wedge dx + dF^3 dx \wedge dy \\
&= \frac{\partial}{\partial x}F^1 dx \wedge dy \wedge dz + \frac{\partial}{\partial y}F^2 dy \wedge dz \wedge dx + \frac{\partial}{\partial z}F^3 dz \wedge dx \wedge dy \\
&= \left(\frac{\partial}{\partial x}F^1 + \frac{\partial}{\partial y}F^2 + \frac{\partial}{\partial z}F^3\right) dx \wedge dy \wedge dz \\
&= \nabla \cdot F dx \wedge dy \wedge dz.
\end{aligned}$$

(b) We have $d^2 f = 0 = \omega^2_{\nabla \times \nabla f}$. We have $d^2 \omega^1_{\nabla f} = 0 = \nabla \cdot f dx \wedge dy \wedge dz$.

(c) Since $\nabla \times F = 0, d\omega^1_F = 0$. By Poincare's lemma, there exist a function f such that $df = \omega^1_F$. We have $df = \omega^2_{\nabla f}$, so $\nabla f = F$.

Problem (4-20)**.**

Suppose ω is a closed k-form on $f(U)$. Then, $f^*(d\omega) = d(f^*\omega) = 0$, so $f^*\omega = d\eta$ for some k-form η on the domain. Using problem 4-13, we have $(f^{-1})^*(d\eta) = \omega = d(f^{-1})^*\eta$. Thus, ω is exact.

Problem (4-23)**.**

By drawing a picture, we can guess that (you may need to switch R_1 with R_2 if unlucky)

$$c(x,y) = ([(R_2 - R_1)y + R_1]\cos(2\pi n x), [(R_2 - R_1)y + R_1]\sin(2\pi n x)).$$

Indeed

$$\begin{aligned}
(\partial c)(t) &= -c(0,t) + c(1,t) + c(t,0) - c(t,1) \\
&= c(t,0) - c(t,1) \\
&= (R_1 \cos(2\pi n t), R_1 \sin(2\pi n t)) - (R_2 \cos(2\pi n t), R_2 \sin(2\pi n t))
\end{aligned}$$