Solutions

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7 August 2024

Problem (4-3).

For notation, take $\langle \cdot, \cdot \rangle$ to be $T(\cdot, \cdot)$. Let $e_1, ..., e_n$ be an orthonormal basis for V. By definition, $|\omega(e_1, ..., e_n)| = 1$, and applying Theorem 4-6 gives

$$|\omega(e_1, ..., e_n)| = \left| \det \begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \right|.$$

Using the common identity (see any quantum mechanics textbook)

$$\begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \langle e_i, w_j \rangle & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \langle w_i, w_j \rangle & \\ & & \ddots \end{bmatrix},$$

we arrive at the desired result.

Problem (4-4).

$$\begin{aligned} f^*\omega(v_1, ..., v_n) &= \omega(f(v_1), ..., f(v_n)) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \omega(f(e_1), ..., f(e_n)) \quad \text{(Theorem 4-3)} \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \\ &= \det \begin{bmatrix} \ddots & & \\ & \langle v_i, e_j \rangle & \\ & & \ddots \end{bmatrix}, \end{aligned}$$

where the last two equalities come from the fact that ω is a volume element and $f^*T(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.

Problem (4-5).

Because det is continuous, the image of $\det \circ c$ on the path must be of the same sign.

Problem (4-6)**.**

(a) We have $v_1 \times v_2 = \det[v_1\ v_2]$.
(b) By definition, $\det[v_1, ..., v_{n-1}, v_1 \times \cdots \times v_{n-1}] = ||v_1 \times \cdots \times v_{n-1}||^2$.

Problem (4-7)**.**

Fix $\omega \in \wedge^n(V)$. Let S be any inner product on V. Construct inner product T by scaling S by $\frac{1}{\omega(e_1,...,e_n)^2}$, where $e_1, ..., e_n$ is an orthonormal basis for V. Now, it's easy to see that ω is a volume element with respect to T.

Problem (4-8)**.**

Take T to be the inner product on V. Let $\varphi \in V^*$ be given by $\varphi(v) = \omega(v_1, ..., v_{n-1}, v)$, where $v_1, ..., v_{n-1}$ are fixed. Then, by the Riesz representation theorem, $\varphi(v) = T(v, w)$ for some $w \in V$. We can now define $v_1 \times \cdots \times v_{n-1} = w$.