

Solutions

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Problem (4-3).

For notation, take $\langle \cdot, \cdot \rangle$ to be $T(\cdot, \cdot)$. Let e_1, \dots, e_n be an orthonormal basis for V . By definition, $|\omega(e_1, \dots, e_n)| = 1$, and applying Theorem 4-6 gives

$$|\omega(e_1, \dots, e_n)| = \left| \det \begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \right|.$$

Using the common identity (see any quantum mechanics textbook)

$$\begin{bmatrix} \ddots & & \\ & \langle w_i, e_j \rangle & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & \\ & \langle e_i, w_j \rangle & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & \langle w_i, w_j \rangle & \\ & & \ddots \end{bmatrix},$$

we arrive at the desired result.

Problem (4-4).

$$\begin{aligned} f^*\omega(v_1, \dots, v_n) &= \omega(f(v_1), \dots, f(v_n)) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \omega(f(e_1), \dots, f(e_n)) \quad (\text{Theorem 4-3}) \\ &= \det \begin{bmatrix} \ddots & & \\ & T(f(v_i), f(e_j)) & \\ & & \ddots \end{bmatrix} \\ &= \det \begin{bmatrix} \ddots & & \\ & \langle v_i, e_j \rangle & \\ & & \ddots \end{bmatrix}, \end{aligned}$$

where the last two equalities come from the fact that ω is a volume element and $f^*T(\cdot, \cdot) = \langle \cdot, \cdot \rangle$.

Problem (4-5).

Because \det is continuous, the image of $\det \circ c$ on the path must be of the same sign.

Problem (4-6).

- (a) We have $v_1 \times v_2 = \det[v_1 \ v_2]$.
 (b) By definition, $\det[v_1, \dots, v_{n-1}, v_1 \times \dots \times v_{n-1}] = \|v_1 \times \dots \times v_{n-1}\|^2$.

Problem (4-7).

Fix $\omega \in \wedge^n(V)$. Let S be any inner product on V . Construct inner product T by scaling S by $\frac{1}{\omega(e_1, \dots, e_n)^2}$, where e_1, \dots, e_n is an orthonormal basis for V . Now, it's easy to see that ω is a volume element with respect to T .

Problem (4-8).

Take T to be the inner product on V . Let $\varphi \in V^*$ be given by $\varphi(v) = \omega(v_1, \dots, v_{n-1}, v)$, where v_1, \dots, v_{n-1} are fixed. Then, by the Riesz representation theorem, $\varphi(v) = T(v, w)$ for some $w \in V$. We can now define $v_1 \times \dots \times v_{n-1} = w$.

Problem (4-9).

- (a) and (b) By definition,

$$\langle v \times w, e_i \rangle = \det[v \ w \ e_i],$$

and the formula in (b) follows. (a) is obtained by applying (b).

- (c) We have

$$\begin{aligned} \frac{1}{\|v\|^2 \|w\|^2} \|v \times w\|^2 &= \frac{1}{\|v\|^2 \|w\|^2} \left\| \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \right\|^2 \\ &= \frac{1}{\|v\|^2 \|w\|^2} (v_2^2 w_3^2 + v_3^2 w_2^2 - 2v_2 v_3 w_2 w_3 \\ &\quad + v_3^2 w_1^2 + v_1^2 w_3^2 - 2v_1 v_3 w_1 w_3 \\ &\quad + v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 v_2 w_1 w_2) \\ &= 1 - \frac{\langle v, w \rangle^2}{\|v\|^2 \|w\|^2} \\ &= 1 - \cos^2 \theta \\ &= \sin^2 \theta \end{aligned}$$

so $\|v \times w\| = \|v\| \|w\| \sin \theta$. By definition,

$$\langle v \times w, v \rangle = \det[v \ w \ v] = 0,$$

and $\langle v \times w, w \rangle = 0$ follows similarly.

- (d) We have

$$\langle v, w \times z \rangle = \det[w \ z \ v] = \langle w, z \times v \rangle = \det[z \ v \ w] = \langle z, v \times w \rangle = \det[v \ w \ z].$$

To prove the vector triple product identity, we have to bash out some algebra. I'm too lazy to type it out, so I will leave it as an exercise to the reader.

- (e) We have

$$\begin{aligned} \|v \times w\|^2 &= \|v\|^2 \|w\|^2 \sin^2 \theta \\ &= \|v\|^2 \|w\|^2 (1 - \cos^2 \theta) \\ &= \|v\|^2 \|w\|^2 - \langle v, w \rangle^2 \\ &= \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2. \end{aligned}$$

Problem (4-10).

By definition,

$$\|w_n\|^2 = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix},$$

where $w_n = w_1 \times \cdots \times w_{n-1}$. Observe that $w_1, \dots, w_{n-1} \in \text{span}\{w_n\}^\perp$, since

$$\langle w_i, w_n \rangle = \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_i \end{bmatrix} = 0$$

if $1 \leq i \leq n-1$.

By problem 4-3, we have

$$\begin{aligned} \det \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} &= \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i, j \leq n}} \\ &= \sqrt{\langle w_n, w_n \rangle \det[\langle w_i, w_j \rangle]_{1 \leq i, j \leq n-1}} \\ &= \|w_n\| \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i, j \leq n-1}}. \end{aligned}$$

Thus, $\|w_n\| = \sqrt{\det[\langle w_i, w_j \rangle]_{1 \leq i, j \leq n-1}}$.

Problem (4-11).

Since v_1, \dots, v_n is an orthonormal basis, this result follows by the definition, since $a_{i,j} = \langle f(v_i), v_j \rangle = \langle f(v_j), v_i \rangle = a_{j,i}$.

Problem (4-13).

(a) We have

$$\begin{aligned} (g \circ f)_*(v_p) &= ((g \circ f)'(p)v)_{g(f(p))} \\ &= (g'(f(p))f'(p)v)_{g(f(p))} \\ &= g_*(f'(p)v_{f(p)}) \\ &= g_*(f_*(v_p)). \end{aligned}$$

Let ω be a k -form on \mathbb{R}^p . Then,

$$\begin{aligned} ((g \circ f)^*\omega)(p)(v_1, \dots, v_k) &= \omega(g(f(p)))(g_*(f_*(v_1)), \dots, g_*(f_*(v_k))) \\ &= (g^*\omega)(f(p))(f_*(v_1), \dots, f_*(v_k)) \\ &= (f^*g^*\omega)(p)(v_1, \dots, v_k). \end{aligned}$$

(b) Follows from Theorem 4-7.