Image Filtering in Frequency Domain

FRA 626 Machine Vision in Smart Factory

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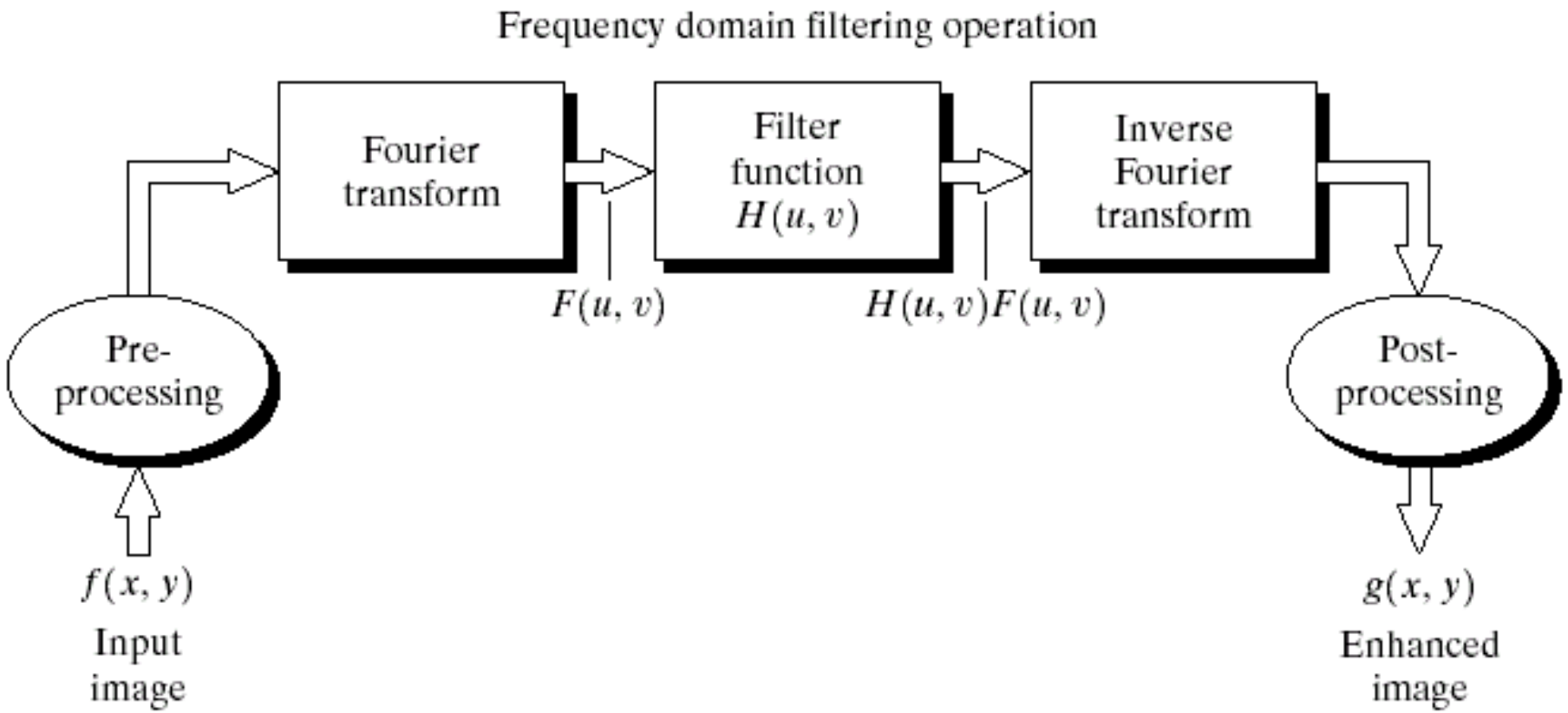
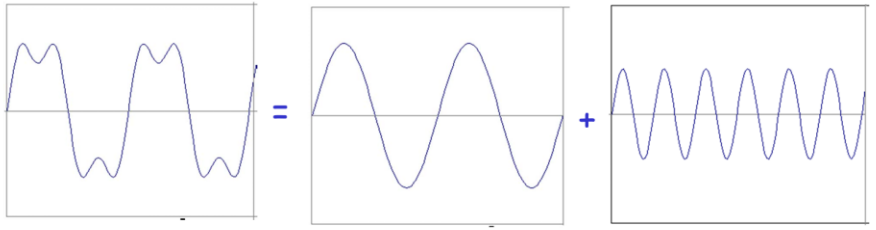
Image Filtering in Frequency Domain



Discrete Fourier Transform

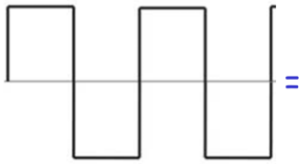


- Discrete Fourier Transform

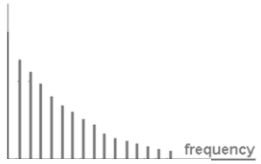
$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

for $u = 0,1,\ldots,M-1$ and $v = 0,1,\ldots,N-1$

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \ldots$$



$$f(x) = \sum_{n=1,3,5,\ldots} \frac{1}{n}\sin nx$$

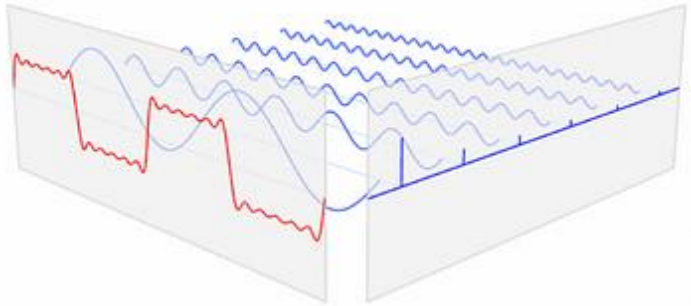


- The inverse DFT can be written as

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

for $x = 0,1,\ldots,M-1$ and $y = 0,1,\ldots,N-1$

Note: $e^{j\theta} = \cos\theta + j\sin\theta$



Discrete Fourier Transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \left[\cos\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right) - j\sin\left(2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)\right]$$

for $u = 0,1,\ldots,M-1$ and $v = 0,1,\ldots,N-1$

DFT in polar form:

$$F(u,v) = |F(u,v)|e^{-j\phi(u,v)}$$

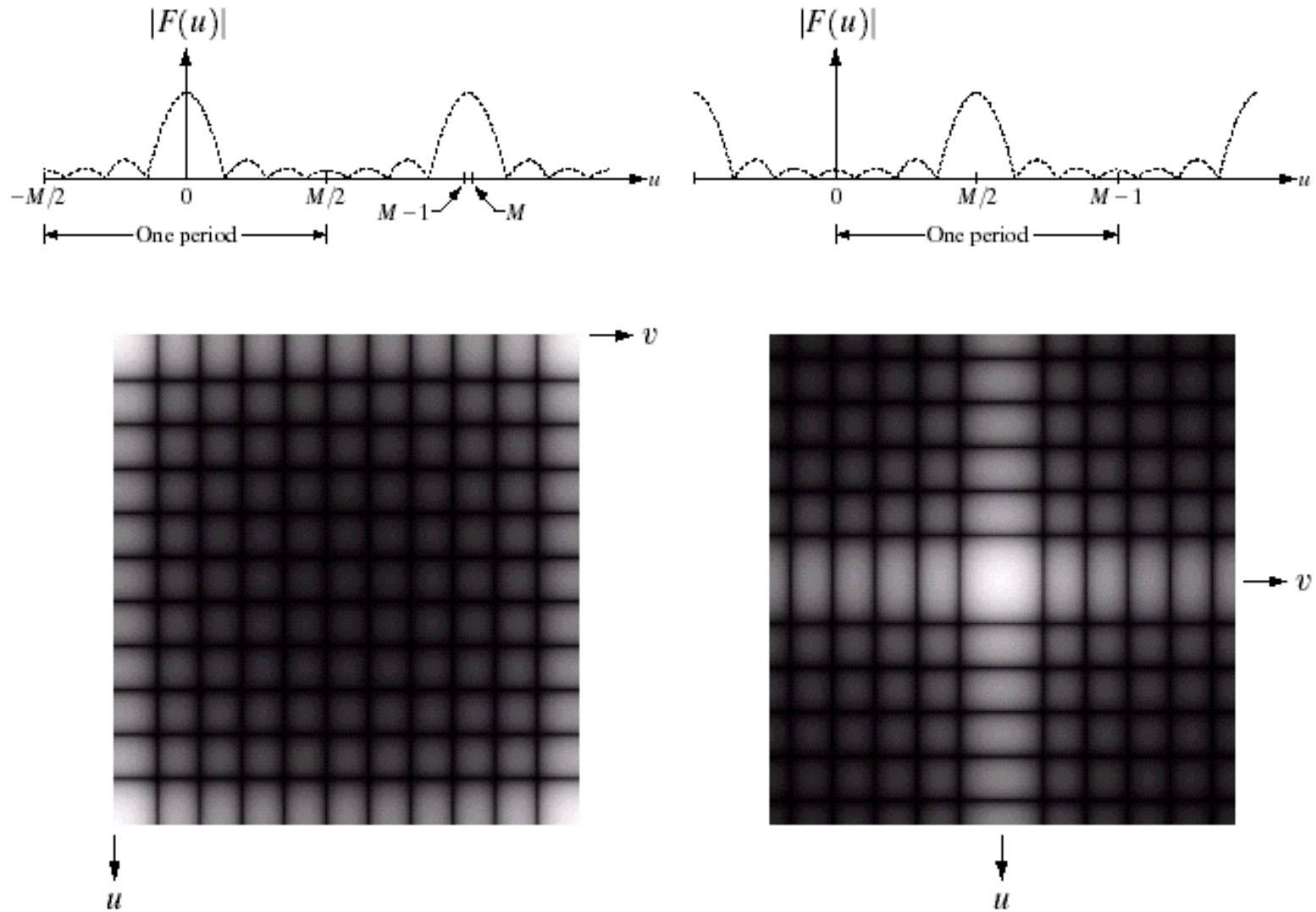
where the magnitude (Fourier spectrum)

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

and

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

Periodicity



Frequency Domain Filters & Enhancements

- **Lowpass Filter**
  - Ideal lowpass filters
  - Butterworth lowpass filters
  - Gaussian lowpass filters
- **Highpass Filter**
  - Ideal highpass filters
  - Butterworth highpass filters
  - Gaussian highpass filter
  - Laplacian
  - Unsharp masking, highboost filtering, and high-frequency-emphasis filtering
  - Homomorphic filtering
- **Selective Filtering**
  - Bandreject and bandpass filters
  - Notch filters

Example of Filtering in the Frequency Domain

Basic Steps:

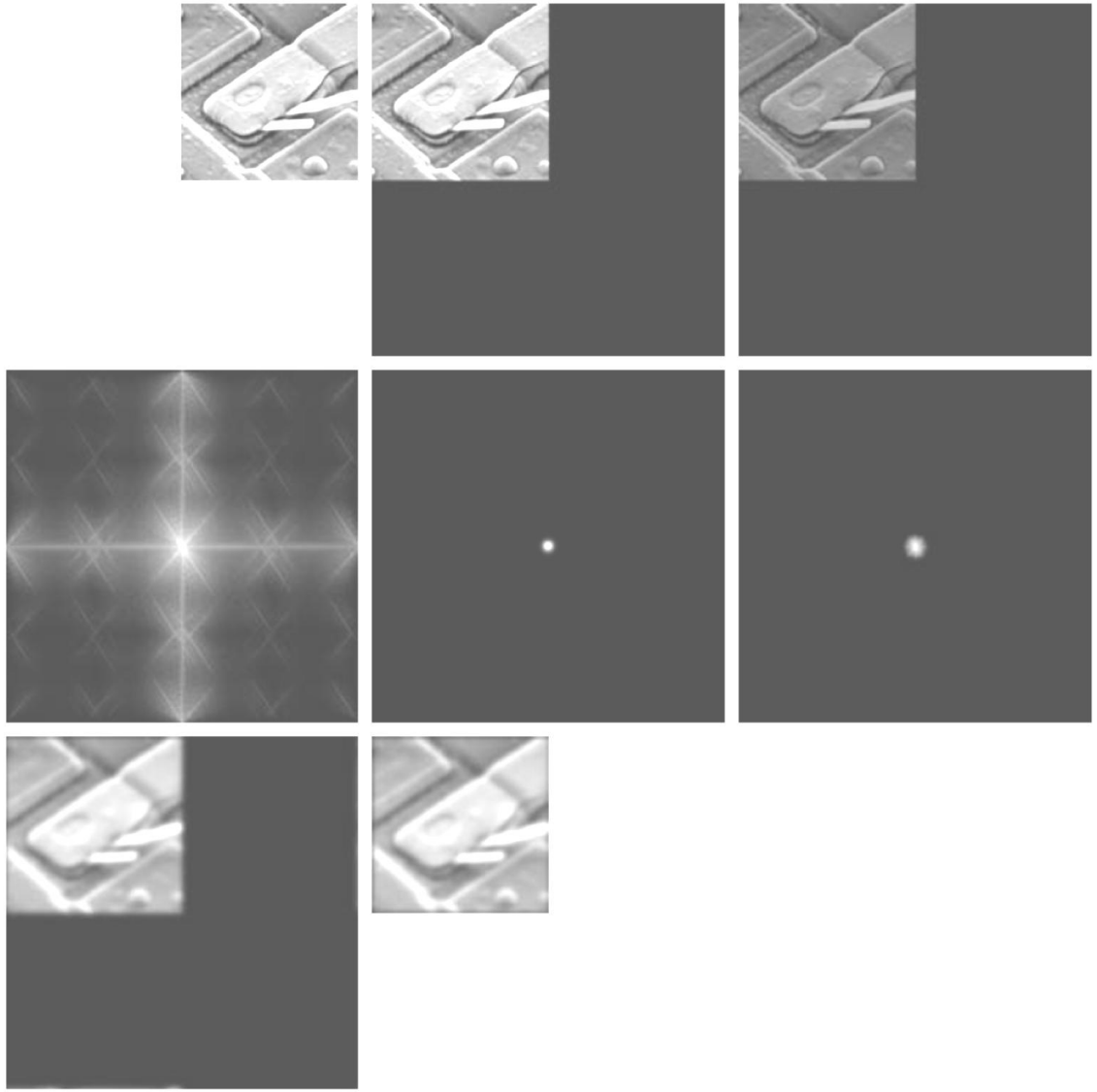
1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q $(P = 2M, Q = 2N)$.
2. Form a padded image $f_p(x, y)$ of size $P \times Q$.
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$
5. Generate a real, symmetric filter function, $H(u, v)$ of size $P \times Q$ with center at $(P/2, Q/2)$.
6. Form the product

$$G(u, v) = F(u, v)H(u, v)$$

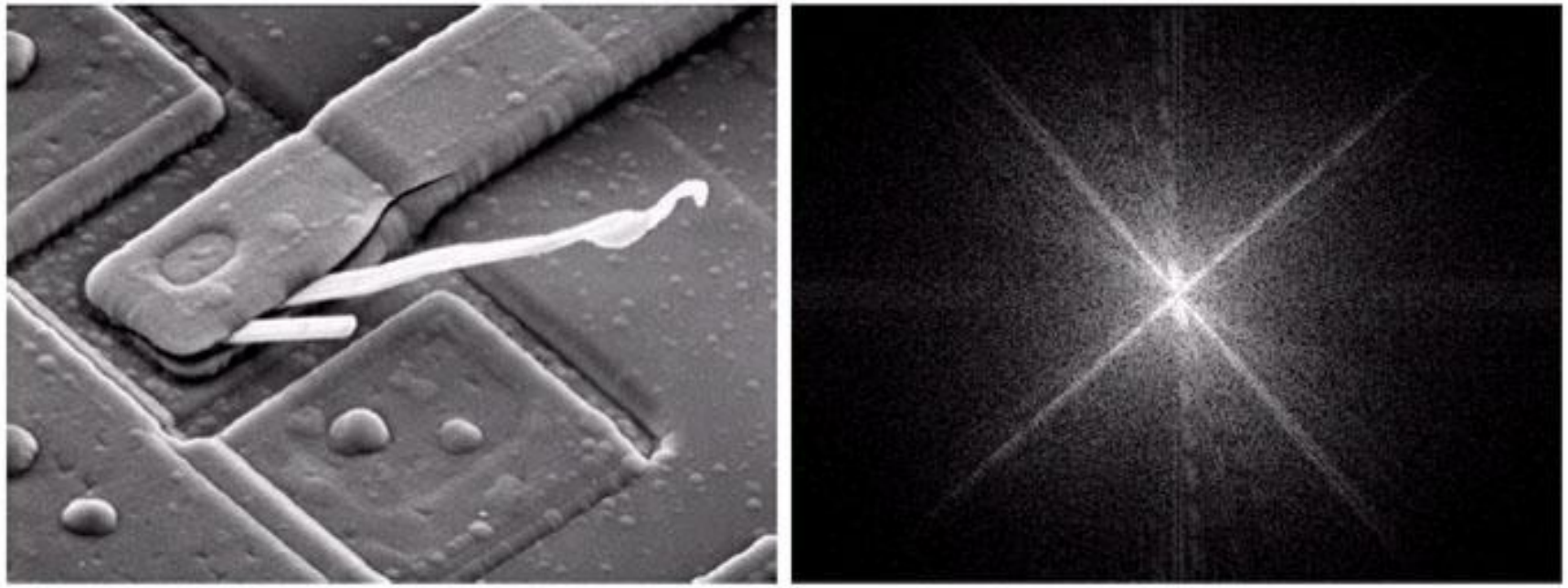
7. Obtain the precessed image

$$g_p(x, y) = \{\text{real}[F^{-1}\{G(u, v)\}]\}\,(-1)^{x+y}$$

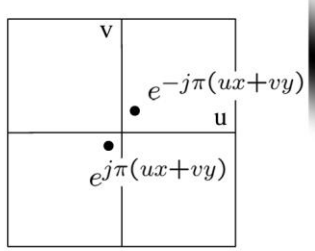
8. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

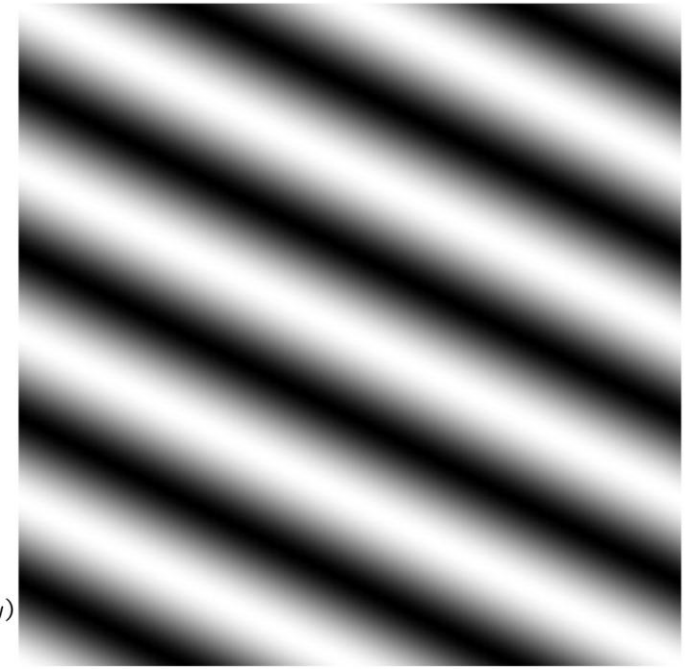


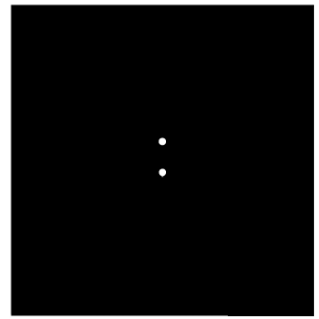
Original Image and its FT

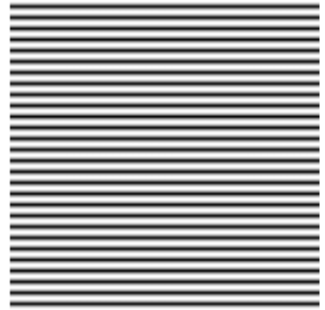


To get some sense of what basis elements look like, we plot a basis element --- or rather, its real part --- as a function of x,y for some fixed u, v. We get a function that is constant when (ux+vy) is constant. The magnitude of the vector (u, v) gives a frequency, and its direction gives an orientation. The function is a sinusoid with this frequency along the direction, and constant perpendicular to the direction.

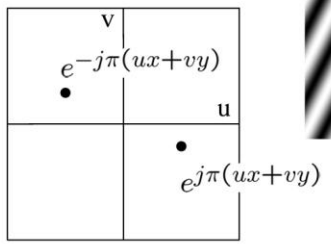


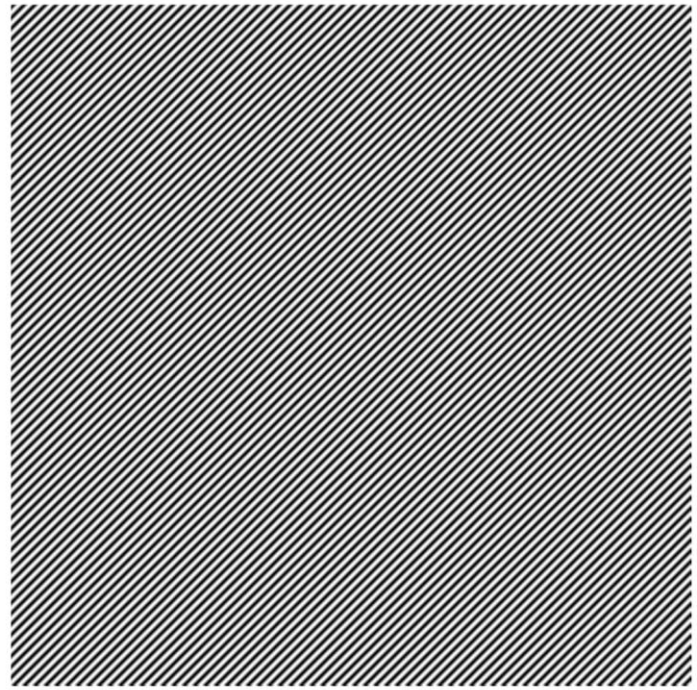


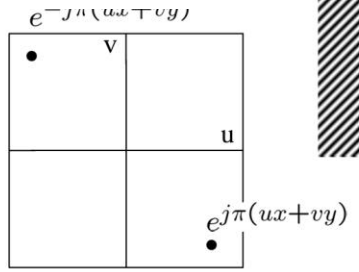




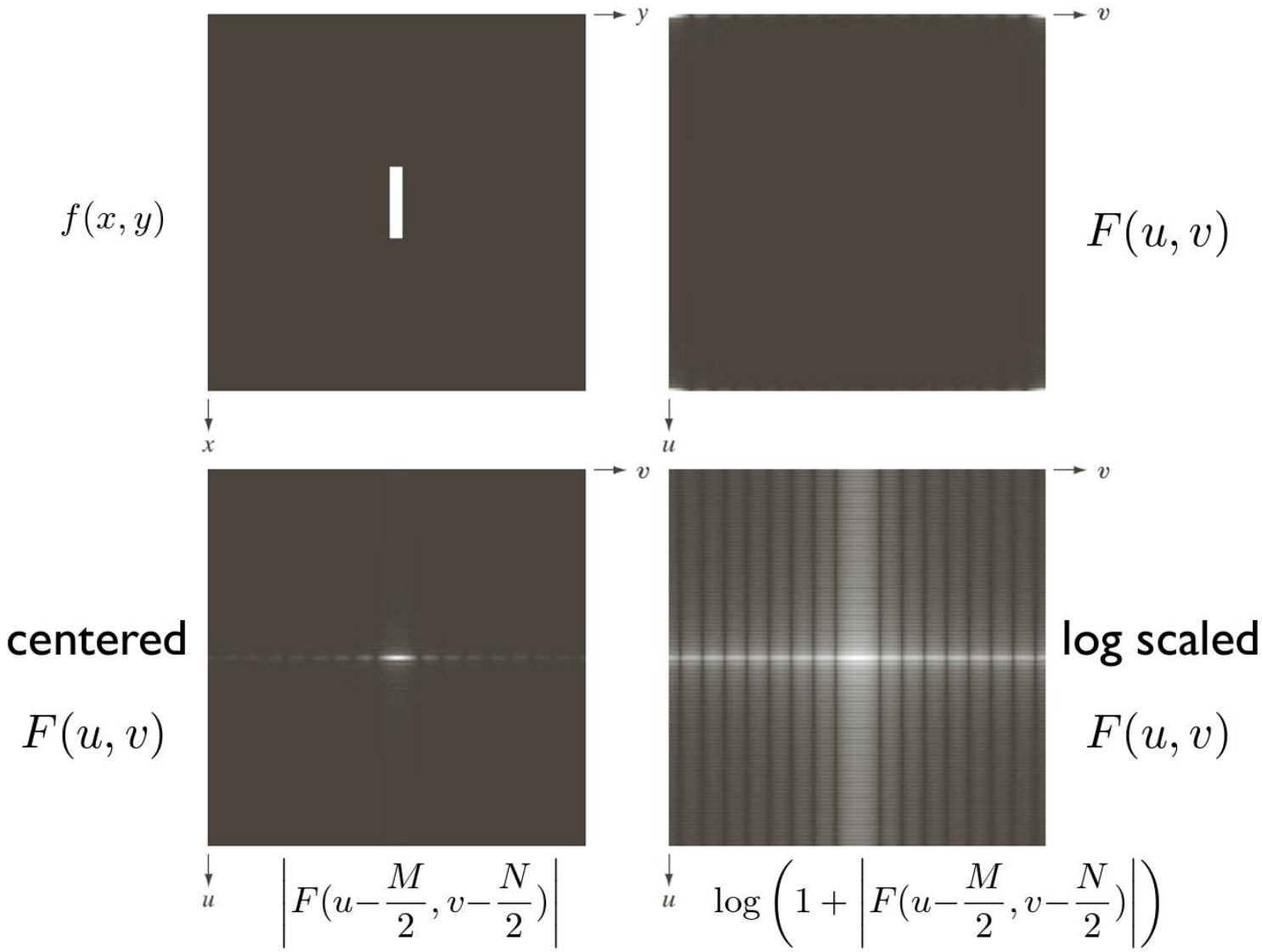
slide: B. Freeman

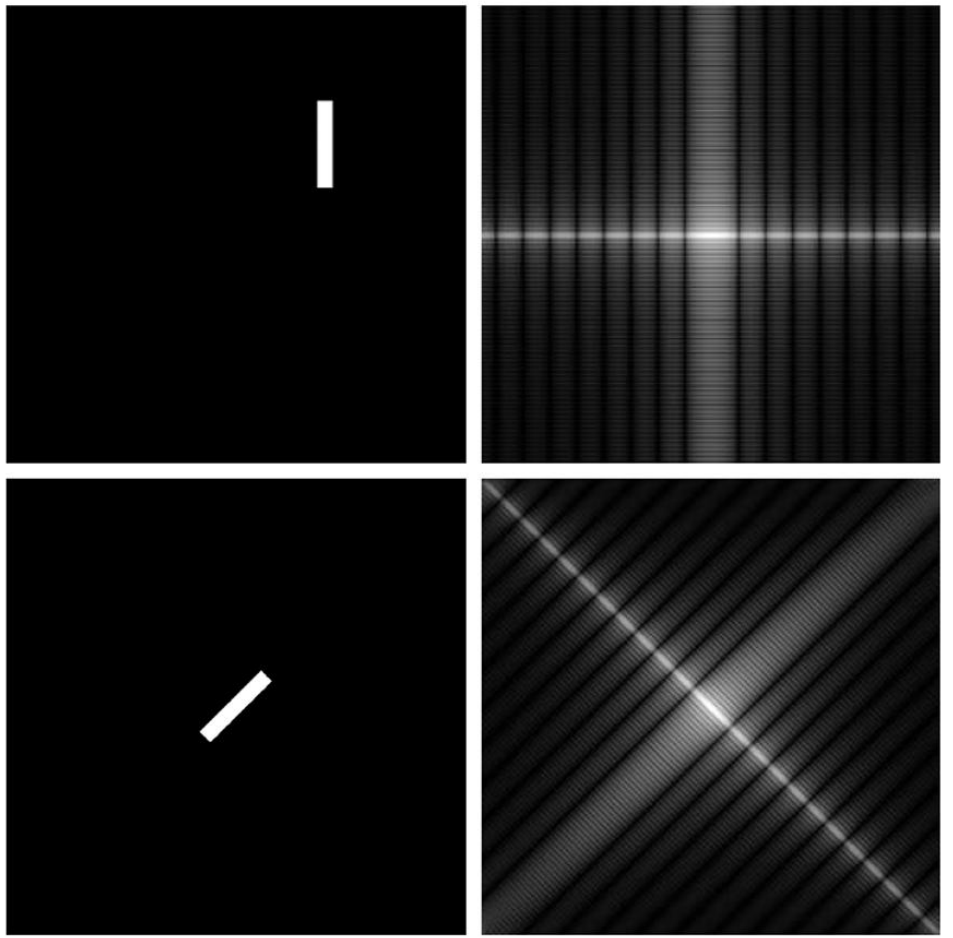






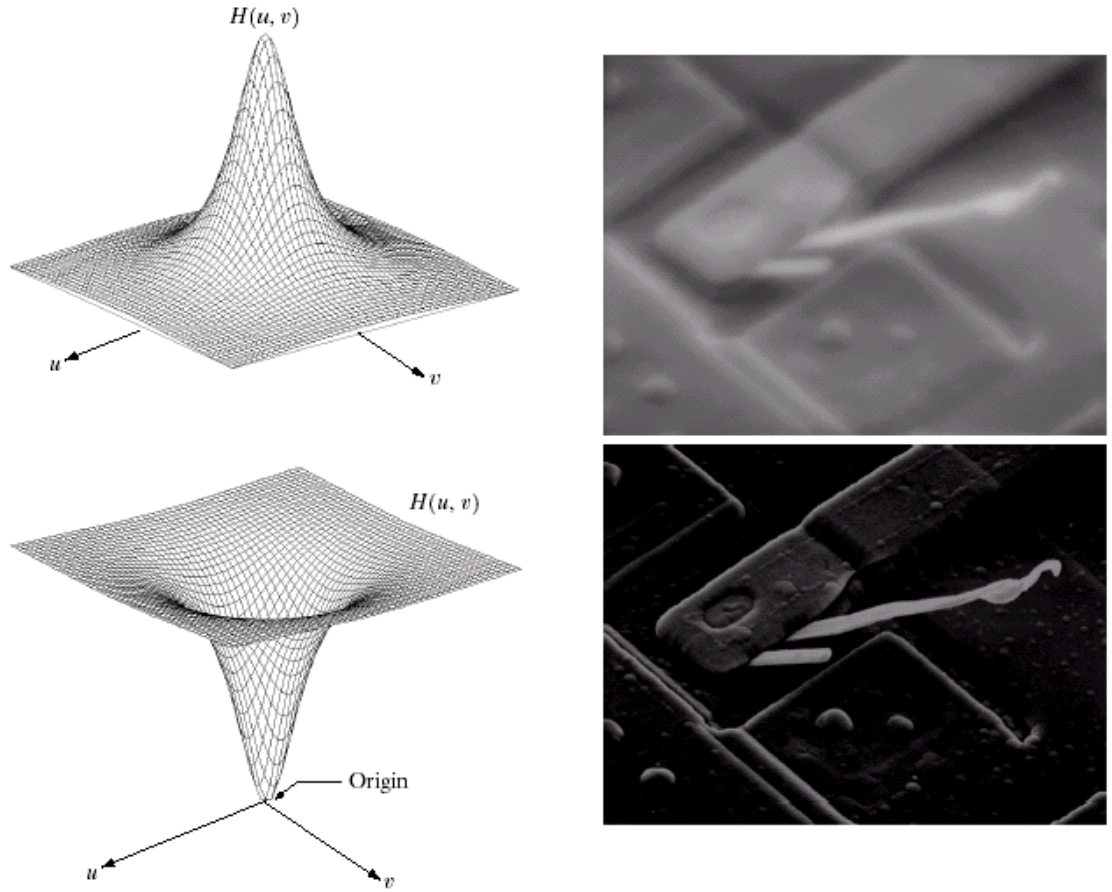
Ref: https://www.robots.ox.ac.uk/~az/lectures/ia/lect2.pdf





Ref: https://web.engr.oregonstate.edu/~sinisa/courses/OSU/ECE468/lectures/ECE468_11.pdf

Lowpass & Highpass Filtering

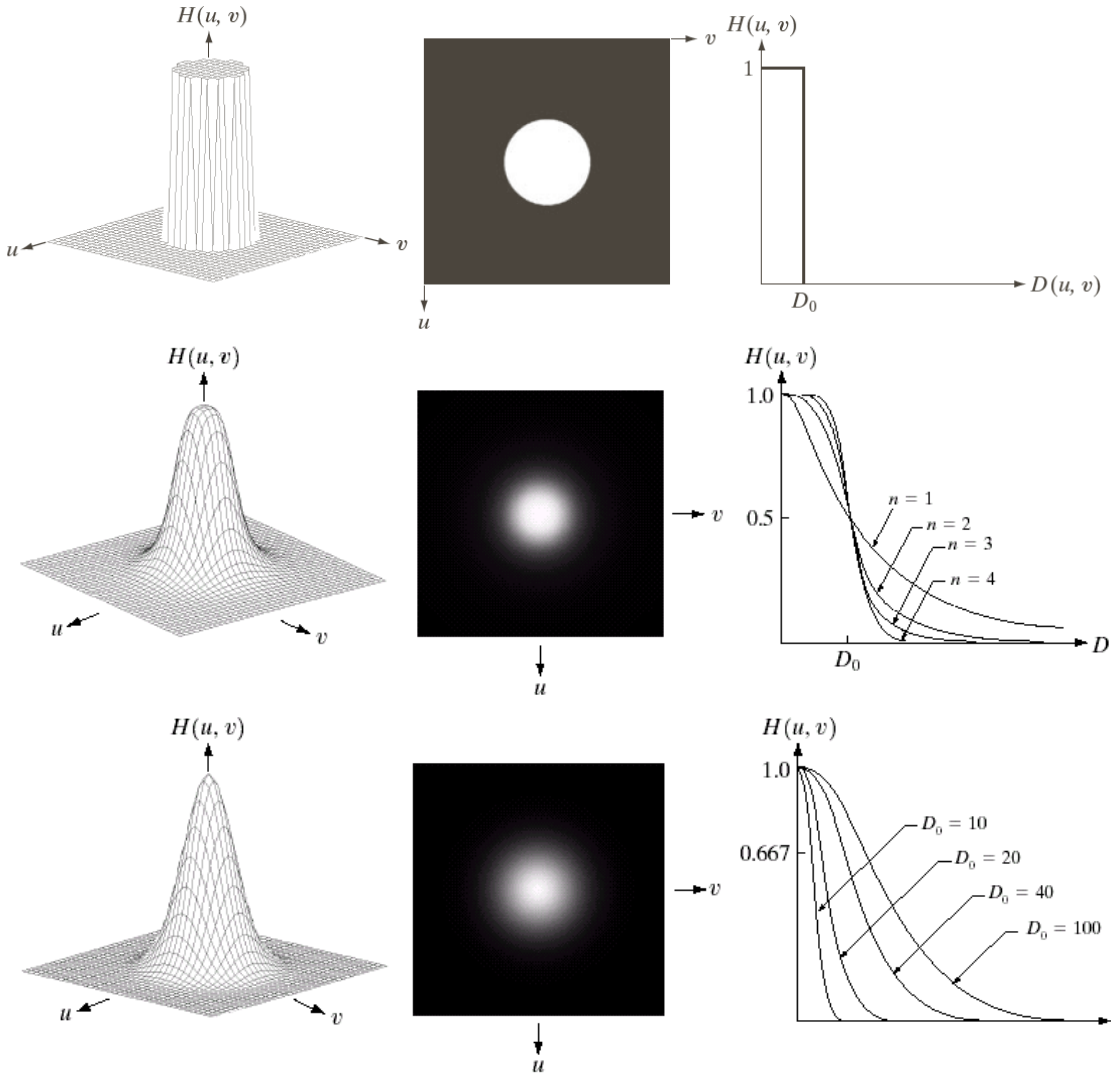


a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Lowpass Filters

$$D(u,v) = \left[\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2\right]^{\frac{1}{2}}$$



Ideal lowpass filter

$$H(u,v) = \begin{cases} 1 & \text{, if } D(u,v) \le D_0 \\ 0 & \text{, if } D(u,v) > D_0 \end{cases}$$

Butterworth lowpass filter

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

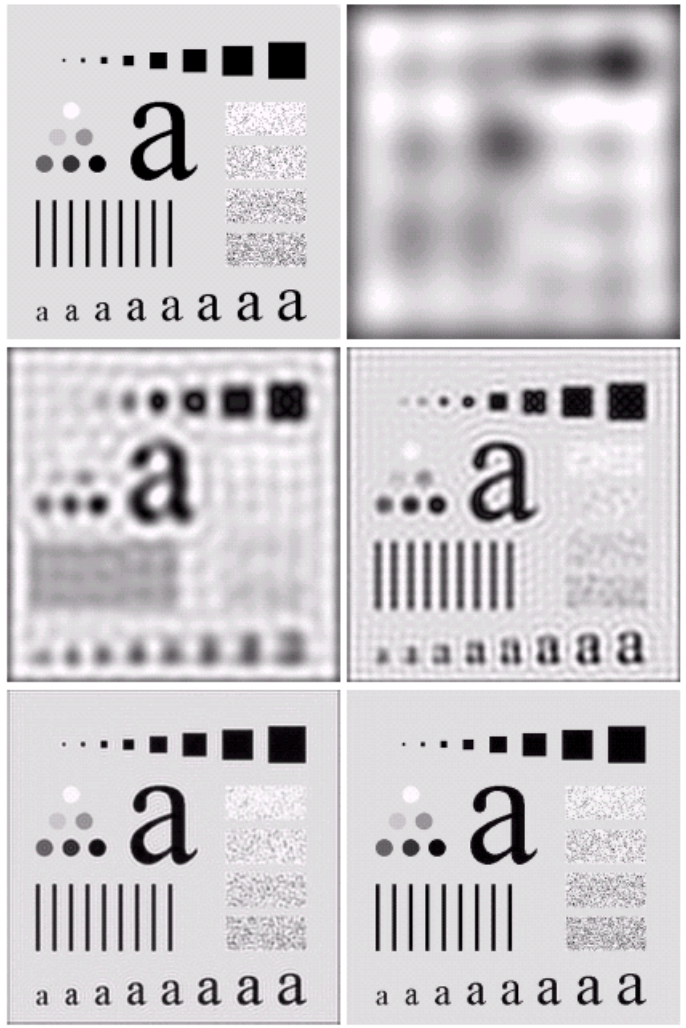
Gaussian lowpass filter

$$H(u,v) = e^{-\frac{D^2(u,v)}{2D_0^2}}$$

Ideal

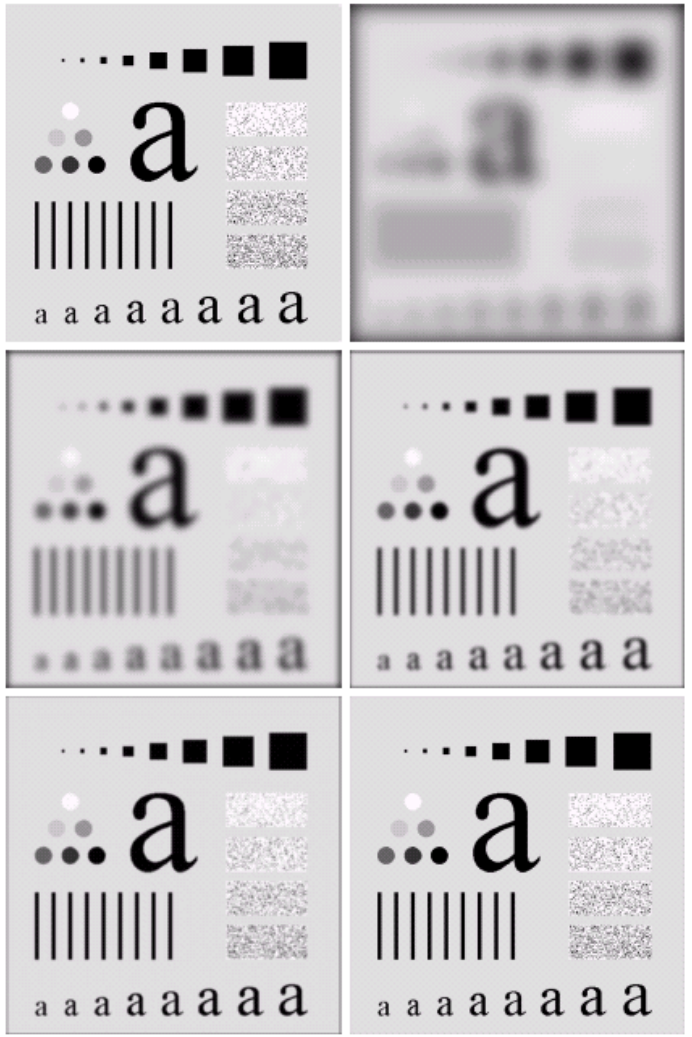
Butterworth

Gaussian



a b
c d
e f

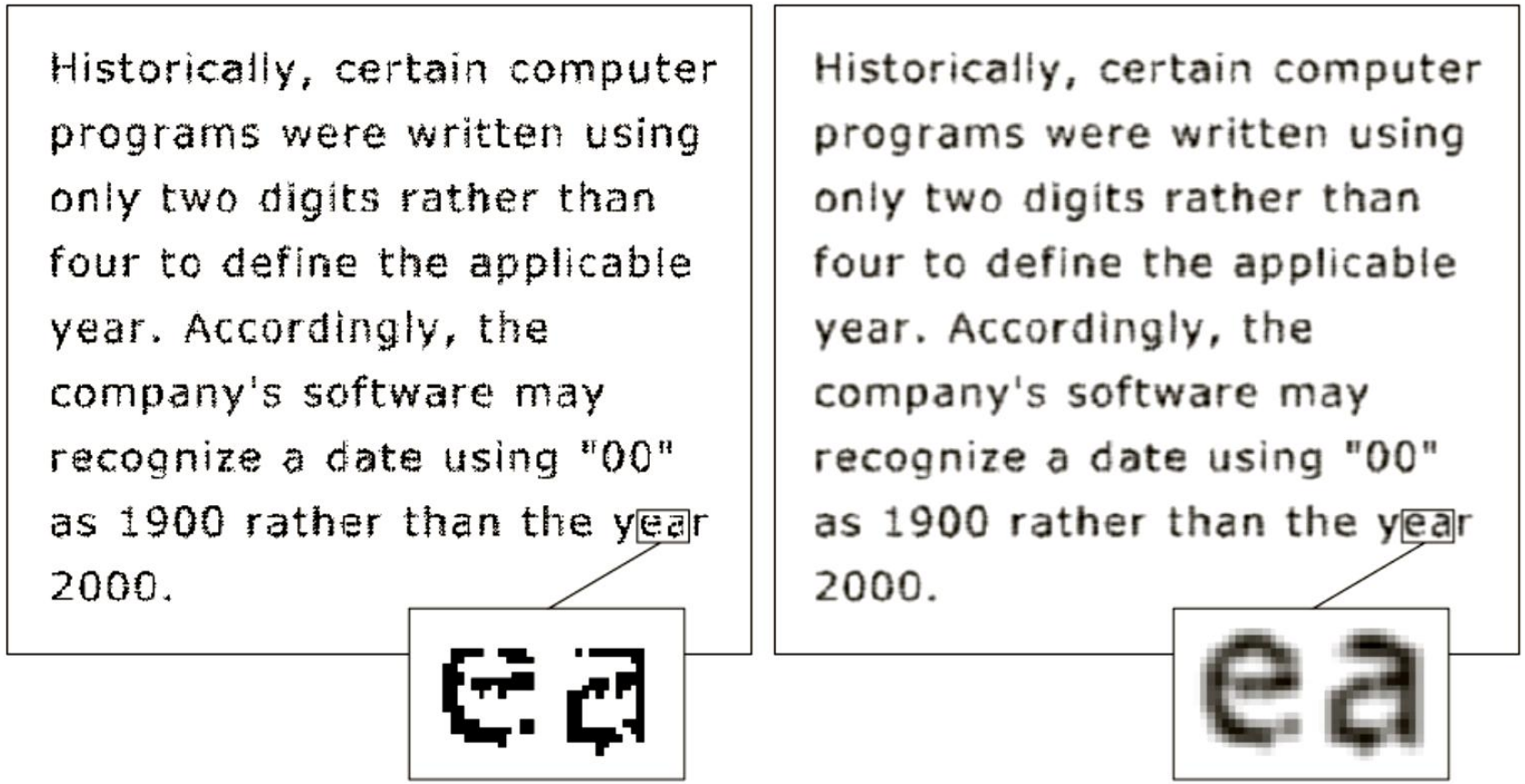
FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.



a b
c d
e f

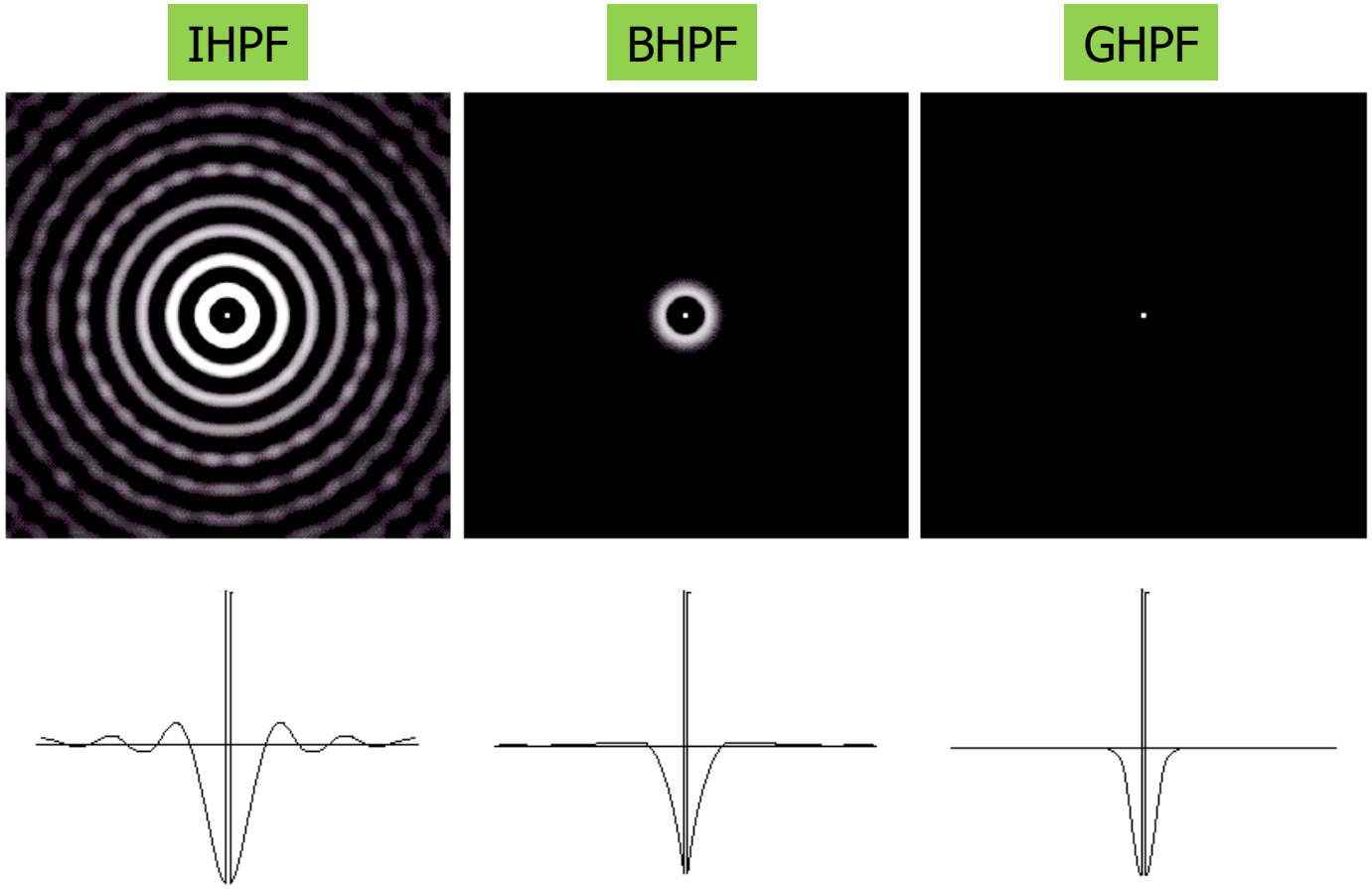
FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

Gaussian Lowpass Filters

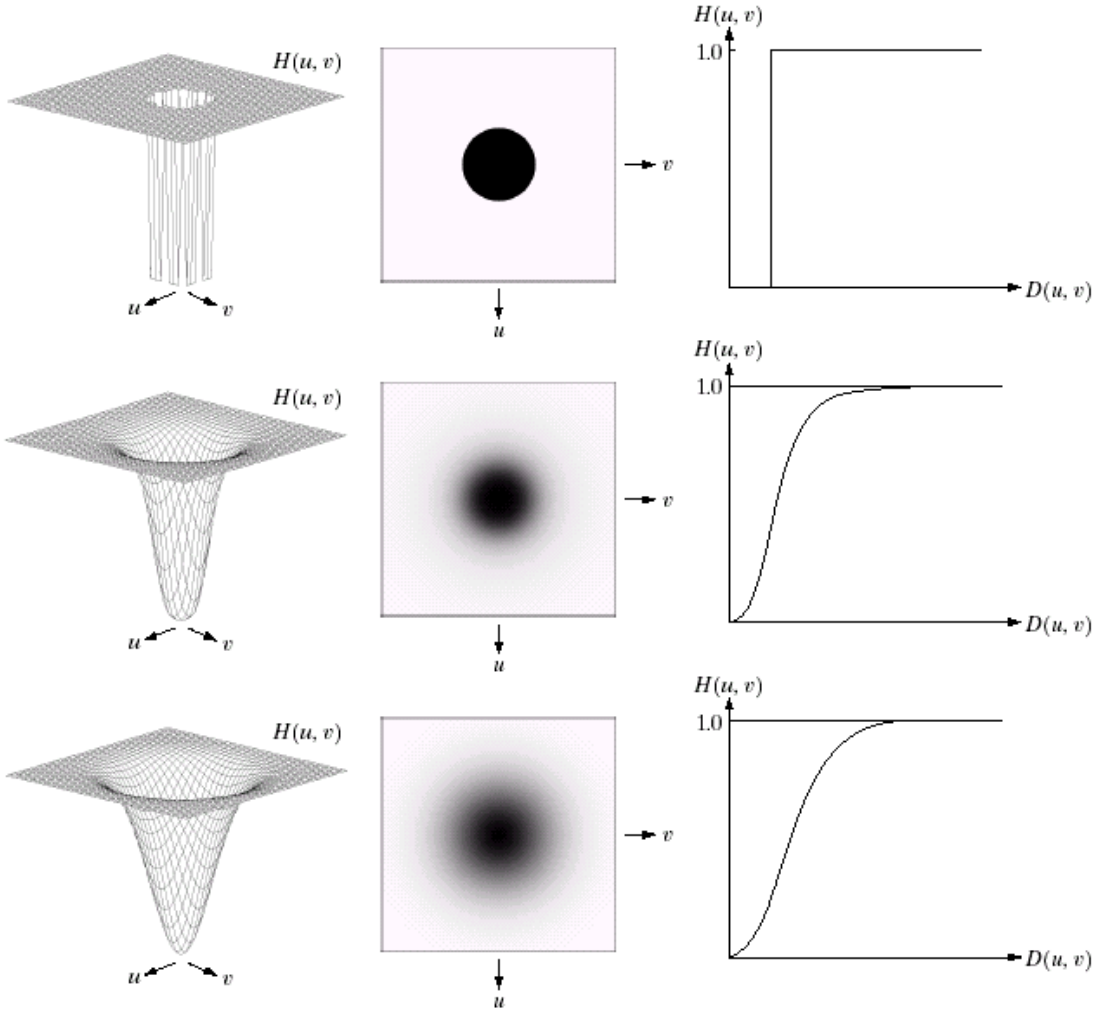


Highpass Filters

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$



Highpass Filters



Ideal highpass filter

$$H(u,v) = \begin{cases} 0 & \text{, if } D(u,v) \le D_0 \\ 1 & \text{, if } D(u,v) > D_0 \end{cases}$$

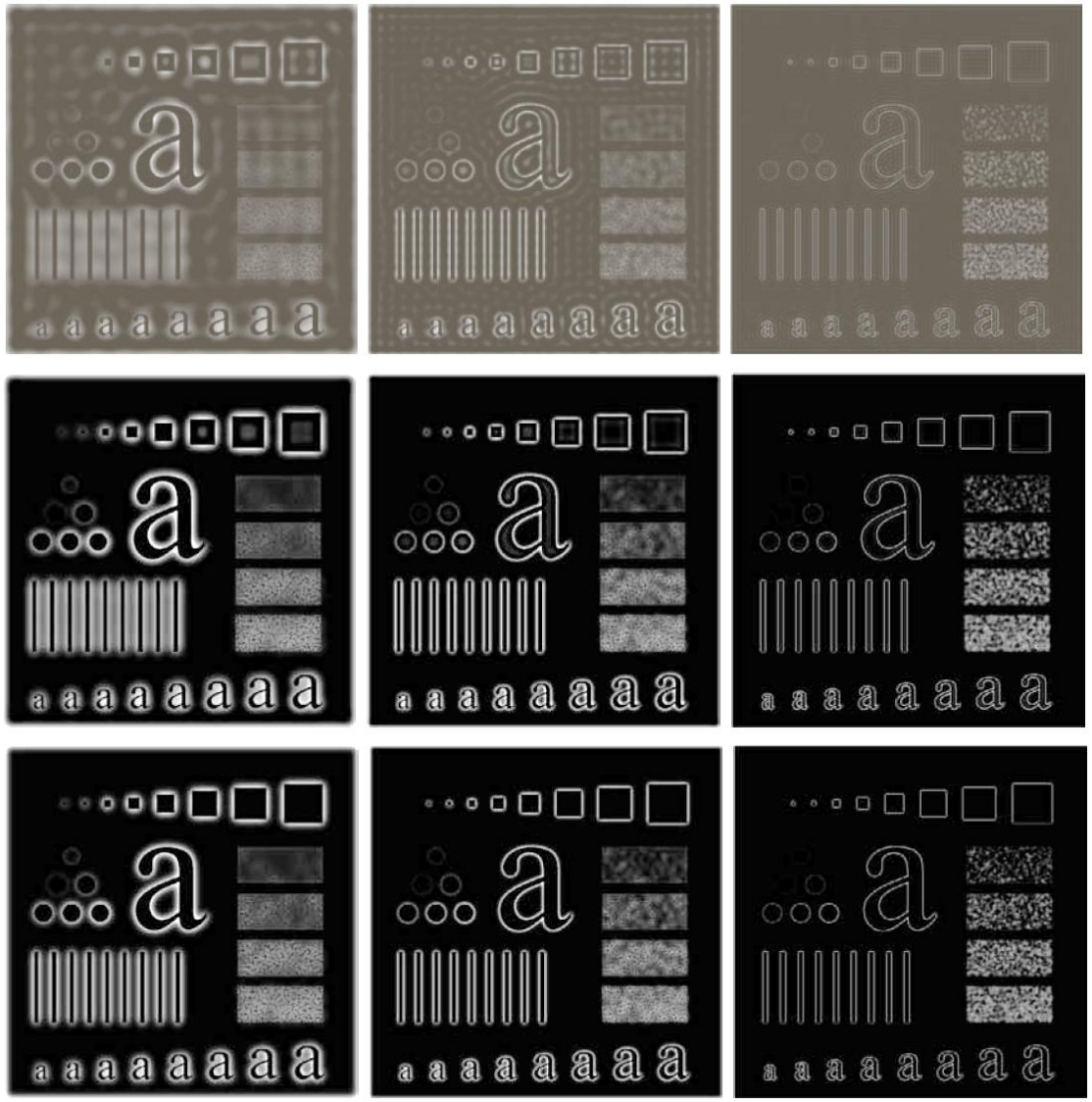
Butterworth highpass filter

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)\,]^{2n}}$$

Gaussian highpass filter

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$

Applications of HPFs



- Ideal HPF
  - D_o = 30, 60, 160
- Butterworth HPF
  - n = 2,
  - D_o = 30, 60, 160
- Gaussian HPF
  - D_o = 30, 60, 160

Applications of HPFs

Thumb print

Result of highpass filtering

Result of thresholding

Laplacian HPF

- The Laplacian of a function $f(x,y)$:

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

- The Fourier transform of the Laplacian:

$$\begin{aligned}\mathcal{F}\{\nabla^2 f(x,y)\} &= \mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\} \\ &= (j2\pi u)^2 F(u,v) + (j2\pi v)^2 F(u,v) \\ &= -4\pi^2(u^2 + v^2)F(u,v) \\ &= H(u,v)F(u,v)\end{aligned}$$

$$H(u,v) = -4\pi^2\left[\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2\right] = -4\pi^2 D^2(u,v)$$

Laplacian HPF

- Enhancement is achieved by:

$$g(x,y) = f(x,y) + \mathrm{c}\,\nabla^2 f(x,y)$$

- Because $H(u,v)$ is negative, $c = -1$:

$$g(x,y) = \mathcal{F}^{-1}\{F(u,v) - H(u,v)F(u,v)\}$$
$$= \mathcal{F}^{-1}\{[1 + 4\pi^2 D^2(u,v)]F(u,v)\}$$

Unsharp Masking and Highboost Filtering

$$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

where a smoothed image is

$$f_{LP}(x,y) = \mathcal{F}^{-1}\{H_{LP}(u,v)F(u,v)\}$$

The filtered image is

$$g(x,y) = f(x,y) + k * g_{mask}(x,y)$$

In the frequency domain:

$$\begin{aligned} g(x,y) &= \mathcal{F}^{-1}\{[1 + k * [1 - H_{LP}(u,v)]]F(u,v)\} \\ &= \mathcal{F}^{-1}\{[1 + k * H_{HP}(u,v)]F(u,v)\} \end{aligned}$$

when $k = 1$ is unsharp masking and $k > 1$ is highboost filtering.

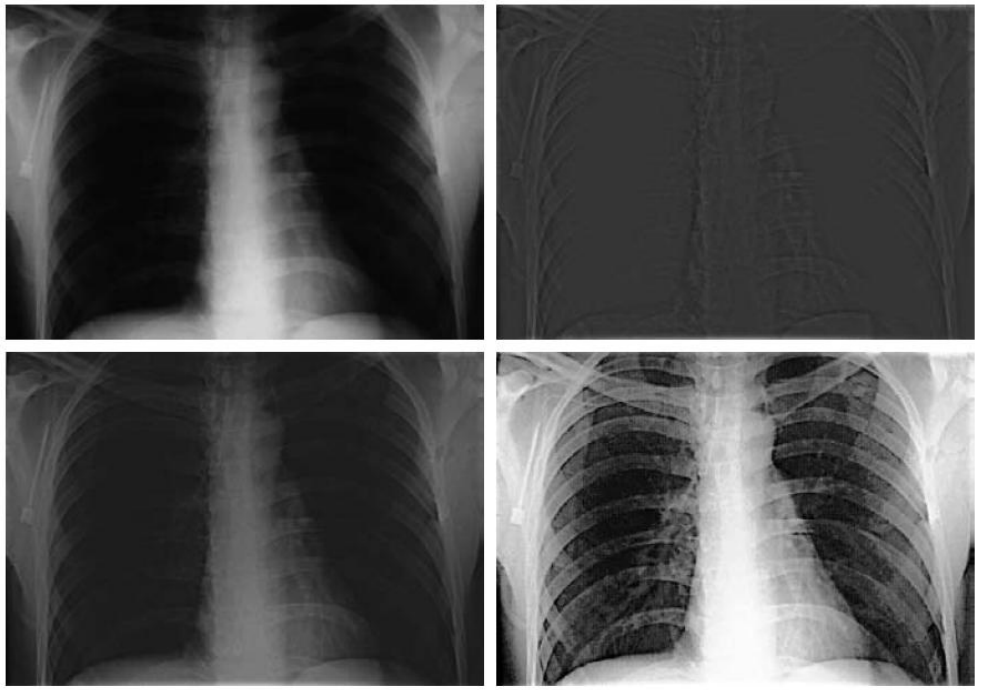
High-Frequency Emphasis Filtering

$$g(x,y) = \mathcal{F}^{-1}\{[k_1 + k_2 * HH_P(u,v)]F(u,v)\}$$

$$H_{HFE}(u,v) = k_1 + k_2 H_{HP}(u,v)$$

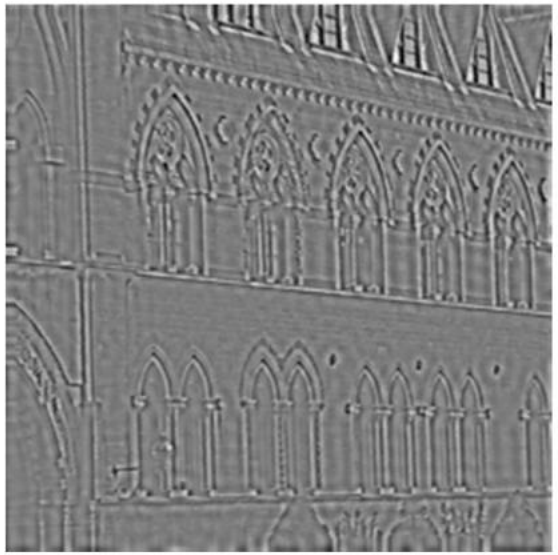
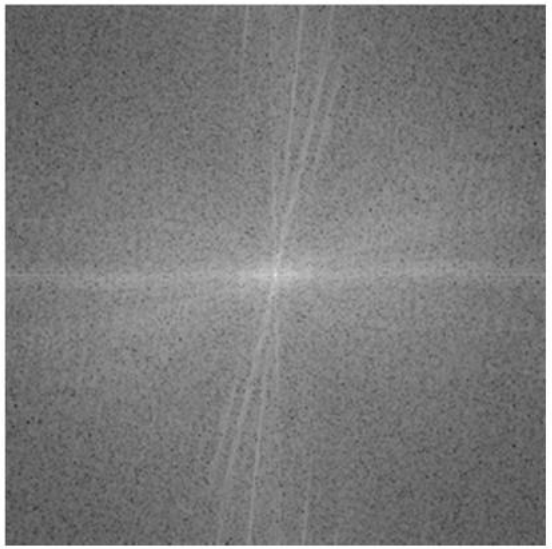
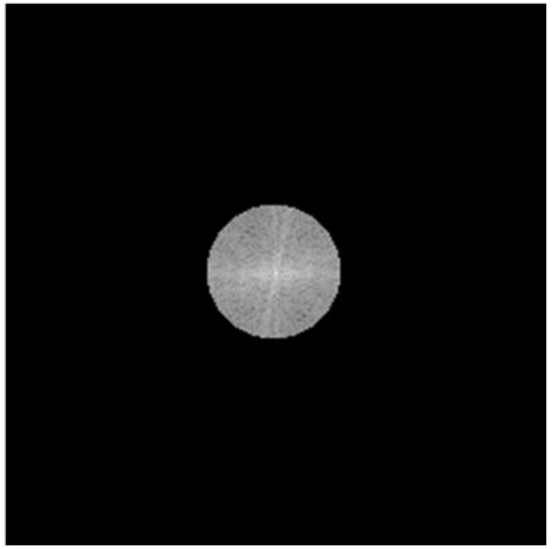
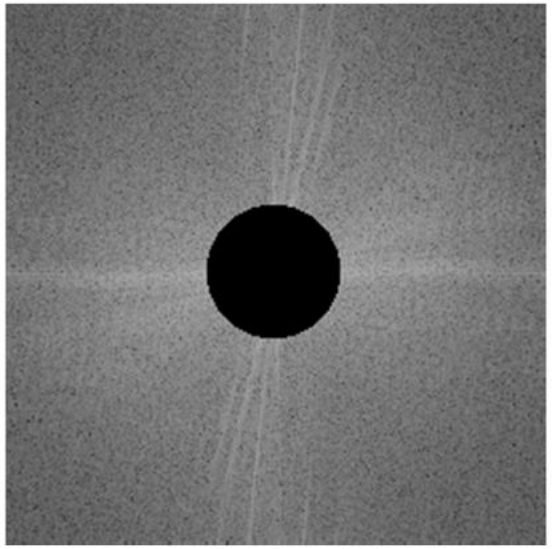
where $k_1 \geq 0$ controls of offset from the origin and $k_2 \geq 0$ controls the contribution of high frequencies.

Because highpass filters set dc term to zero which reducing the average intensity in the filtered image



a b
c d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

	original	low pass	high pass
$f(x,y)$			
$\|F(u,v)\|$			

Ref: https://www.robots.ox.ac.uk/~az/lectures/ia/lect2.pdf

Homomorphic Filtering

- An image $f(x, y)$ is created by "reflecting" the light from an object that has been “illuminated” by some light source:

$$f(x, y) = i(x, y)r(x, y)$$

- The illuminating intensity $i(x, y)$ is a slowly varying component (controlling overall dynamic range), $r(x, y)$ is a fast varying component (affecting the local contrast).

$$\mathcal{F}\{f(x, y)\} \neq \mathcal{F}\{i(x, y)\}\,\mathcal{F}\{r(x, y)\}$$

Homomorphic Filtering

$$f(x,y) = i(x,y)r(x,y)$$

$$\ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

$$Z(u,v) = F_i(u,v) + F_r(u,v)$$

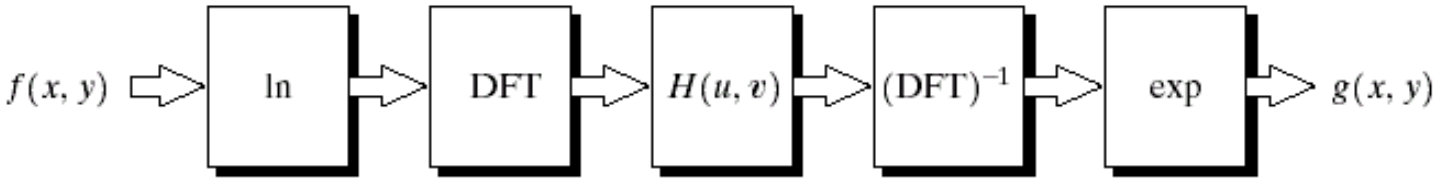
$$S(u,v) = H(u,v)Z(u,v)$$

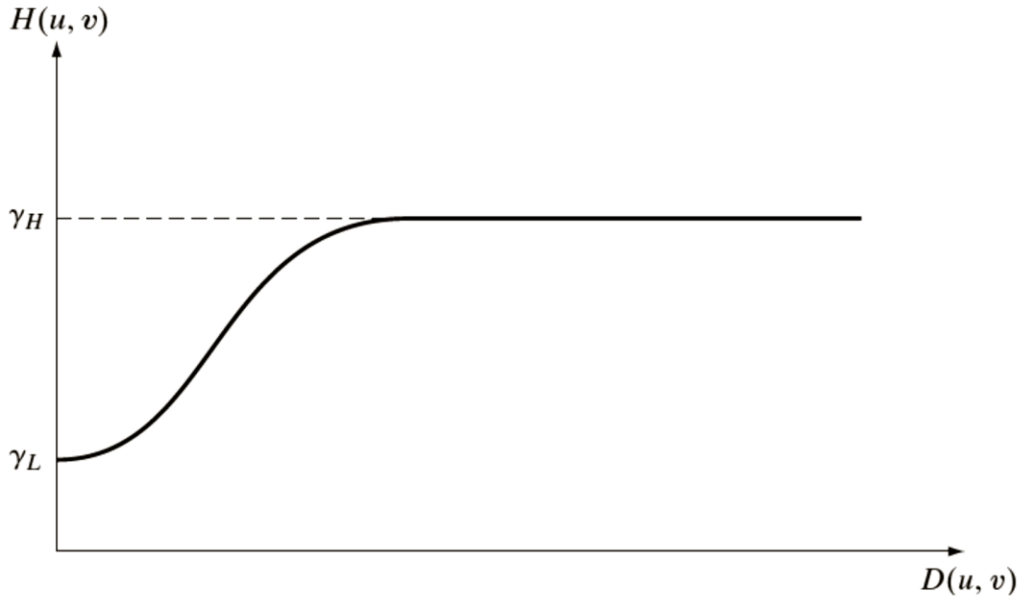
$$S(u,v) = H(u,v)F_i(u,v) + H(u,v)F_r(u,v)$$

$$s(x,y) = \mathcal{F}^{-1}\{S(u,v)\}$$

$$g(x,y) = e^{s(x,y)}$$

Homomorphic Filtering





$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left[D^2(u,v)/D_0^2\right]}\right] + \gamma_L$$

Homomorphic Filtering

Bandreject and Bandpass Filters

$$H_{BP}(u,v) = 1 - H_{BR}(u,v)$$

Ideal bandreject filter

$$H_{BR}(u,v) = \begin{cases} 0 & \text{, if } D_0 - \frac{W}{2} \le D \le D_0 + \frac{W}{2} \\ 1 & \text{, otherwise} \end{cases}$$

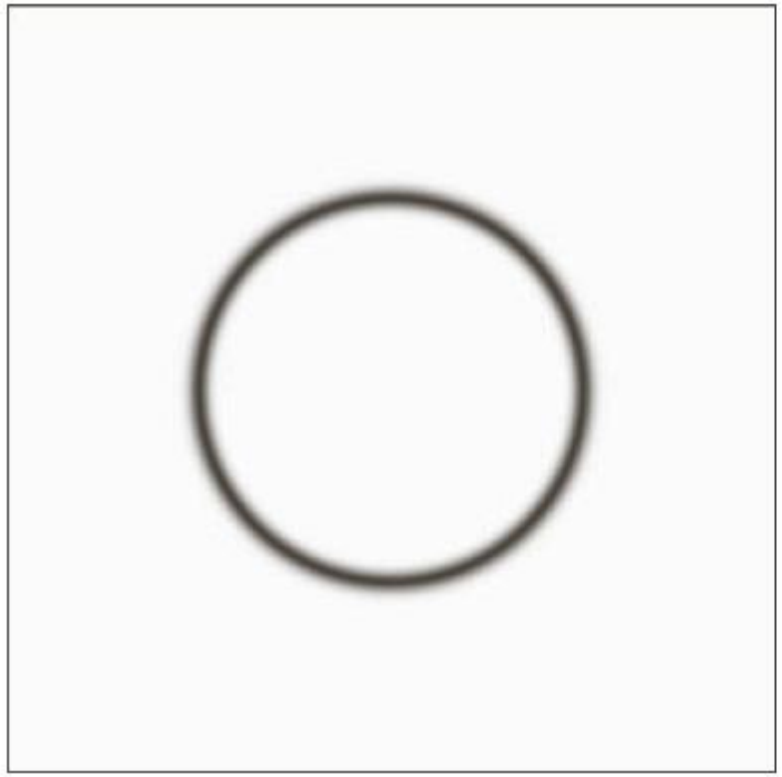
Butterworth bandreject filter

$$H(u,v) = \frac{1}{1 + \left[\frac{DW}{D^2 - D_0^2}\right]^{2n}}$$

Gaussian bandreject filter

$$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$$

Bandreject and Bandpass Filters



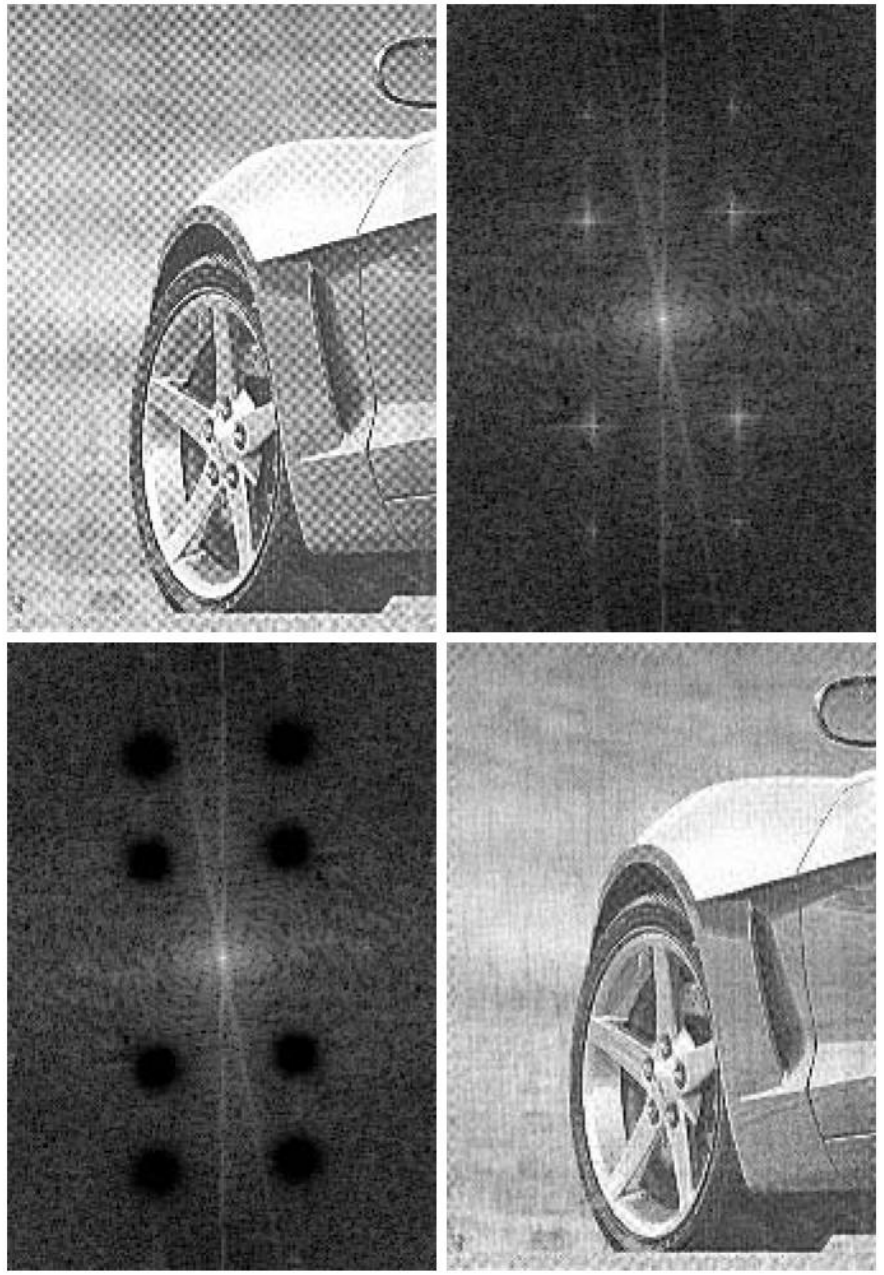
Bandreject Gaussian filter

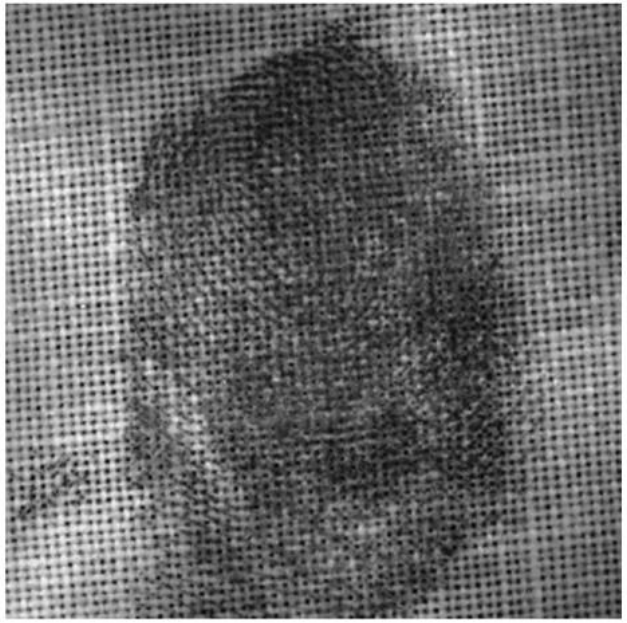
Corresponding Bandpass filter

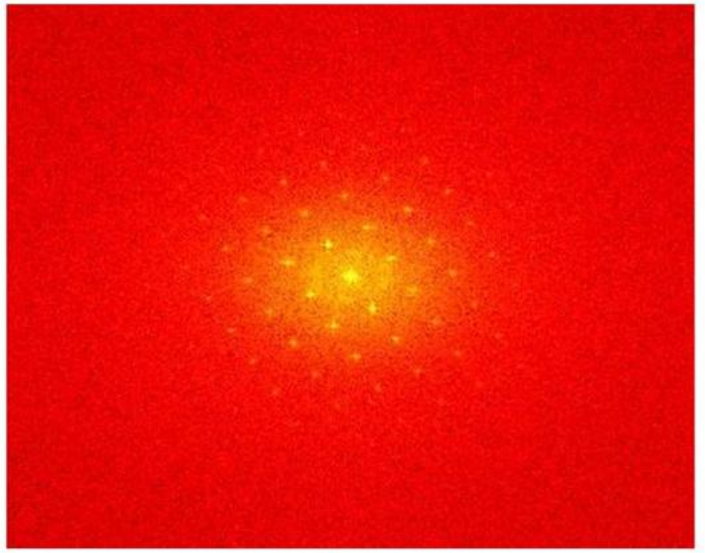
Notch Filters

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v)$$

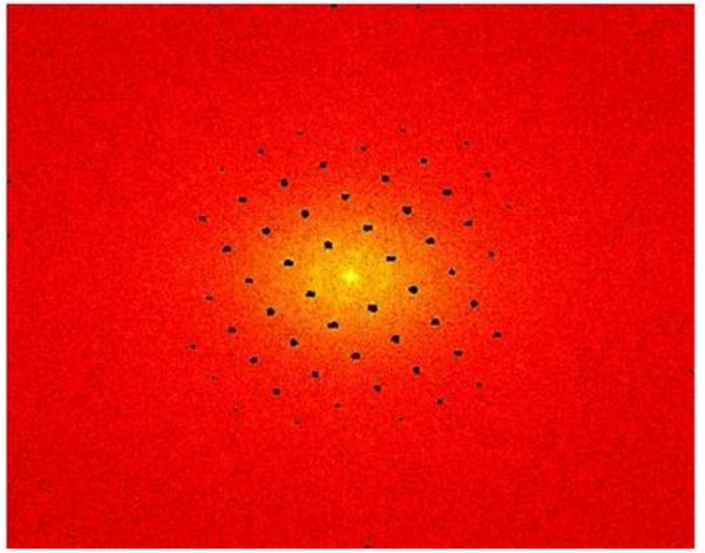
where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filters whose centers are at (u_k, v_k) and $(-u_k, -v_k)$, respectively.







$|F(u,v)|$

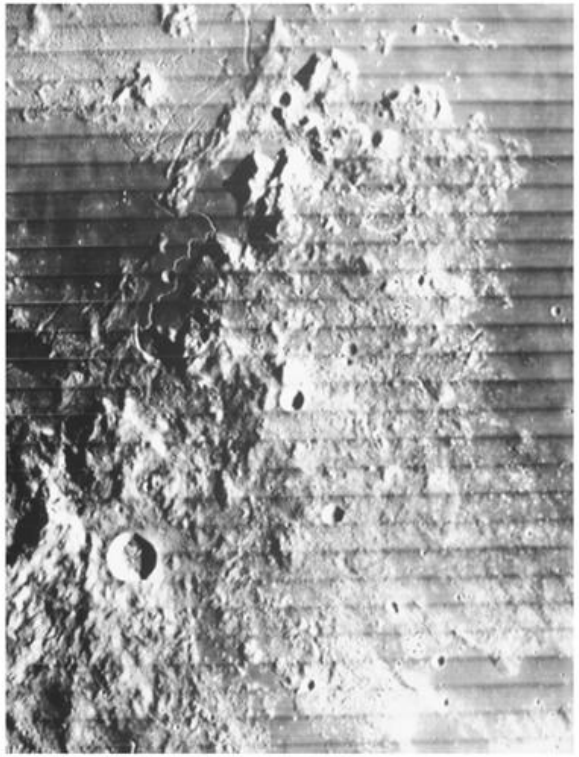


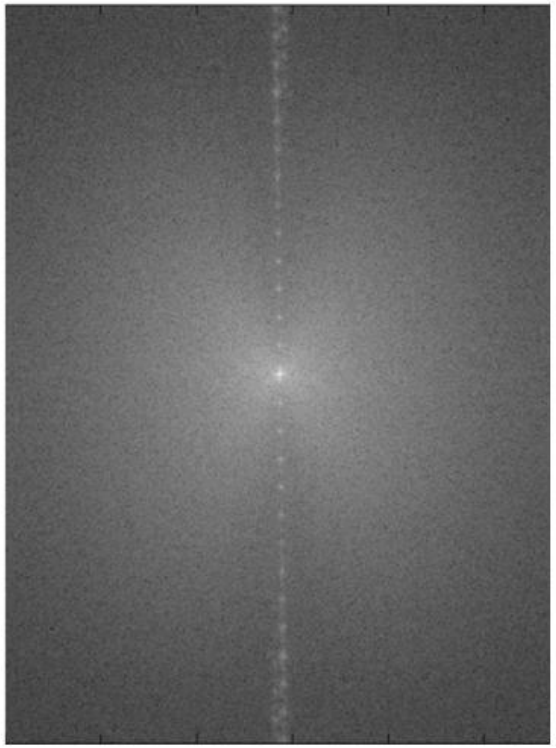
remove peaks

Periodic background removed

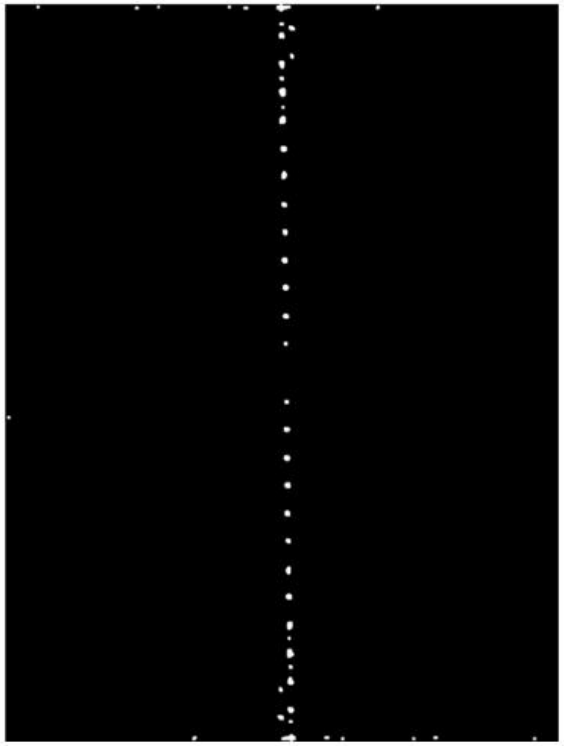
Ref: https://www.robots.ox.ac.uk/~az/lectures/ia/lect2.pdf

Lunar orbital image (1966)





$|F(u,v)|$

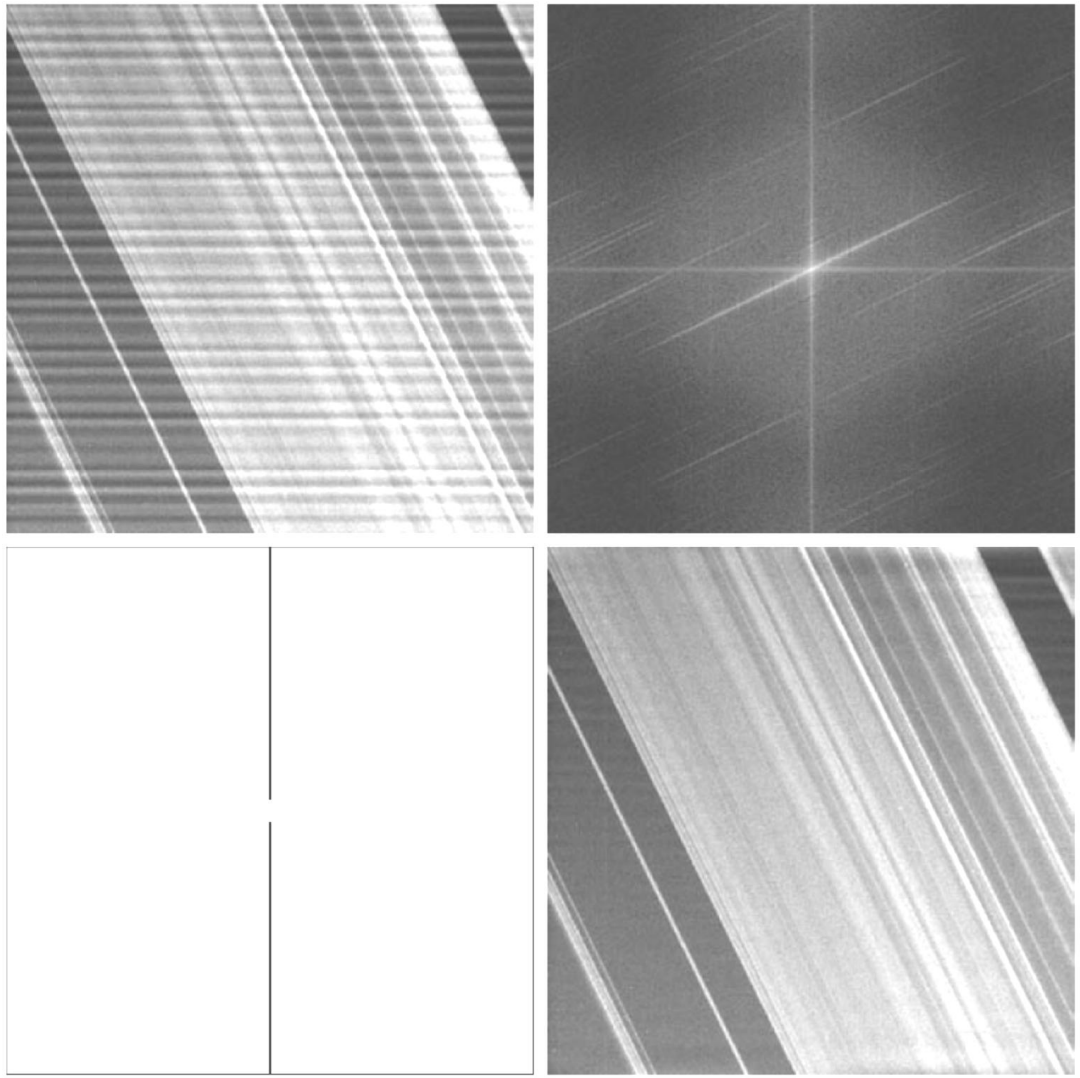


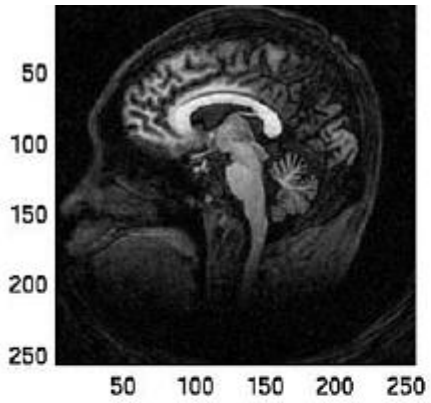
remove peaks

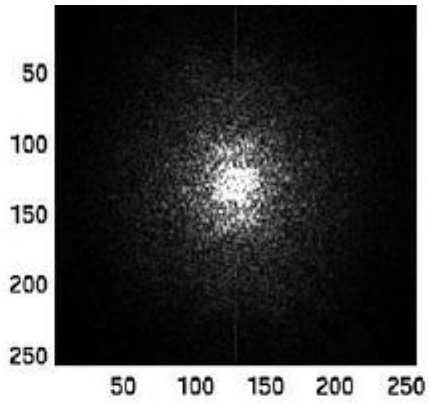
join lines removed

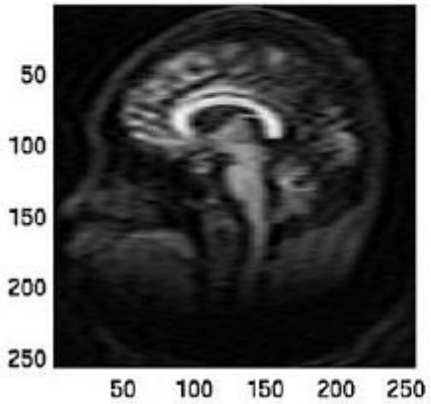
Ref: https://www.robots.ox.ac.uk/~az/lectures/ia/lect2.pdf

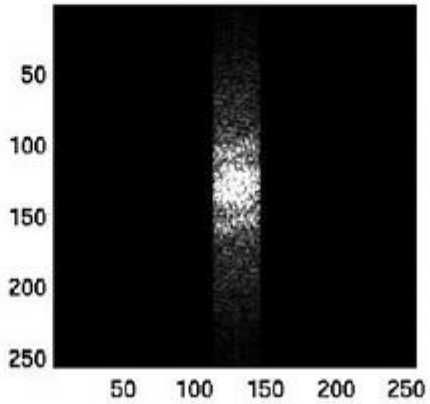
Notch Filters

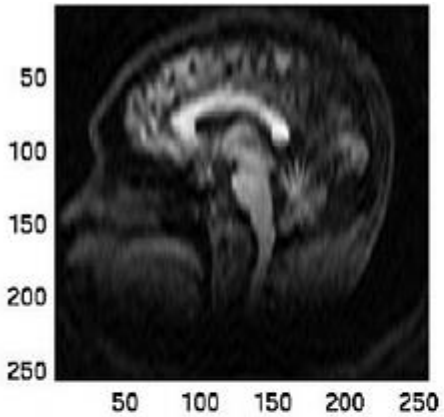


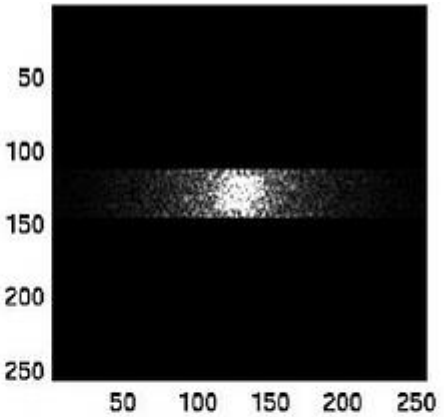


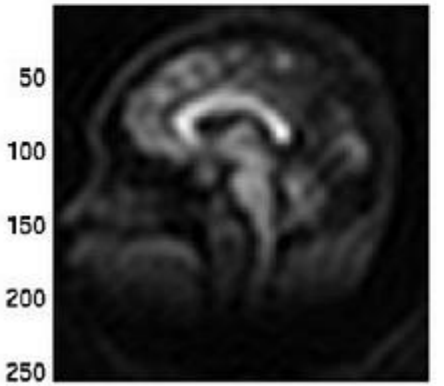


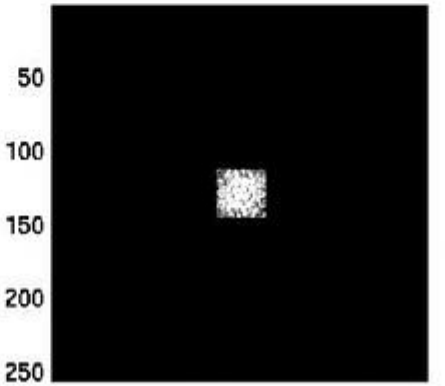




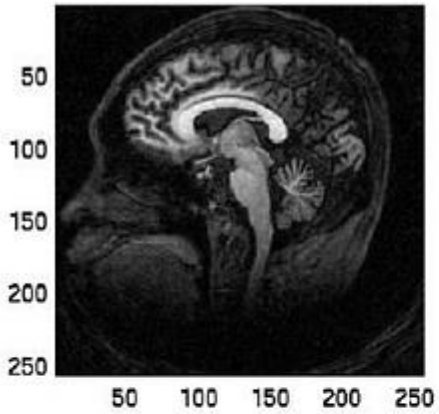


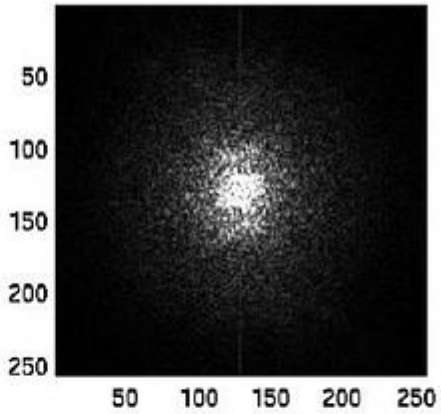


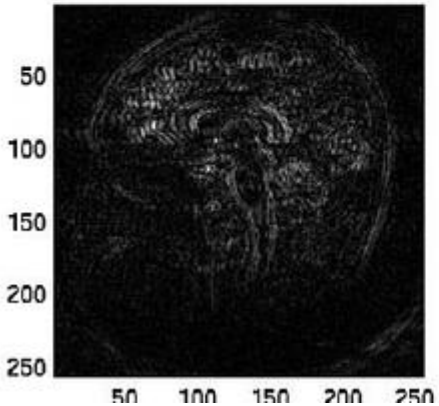


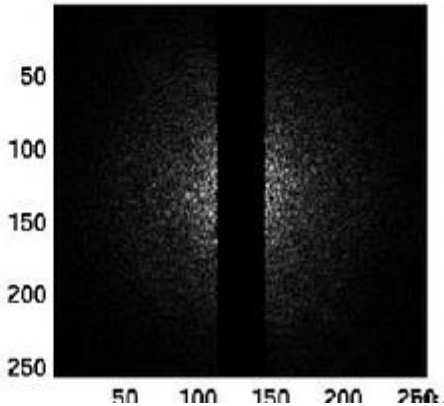


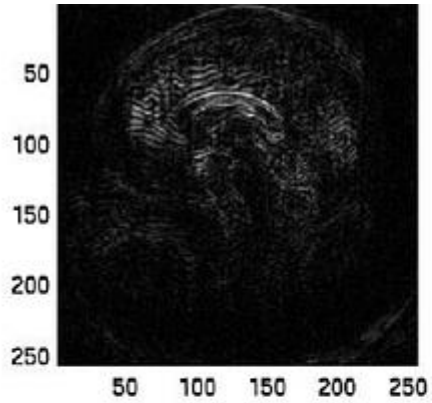
https://medium.com/@gokcenazakyol/what-are-filtering-in-frequency-domain-and-fourier-transform-image-processing-5-6f4cace43c91

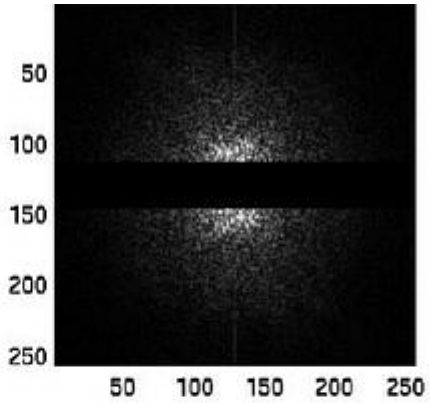


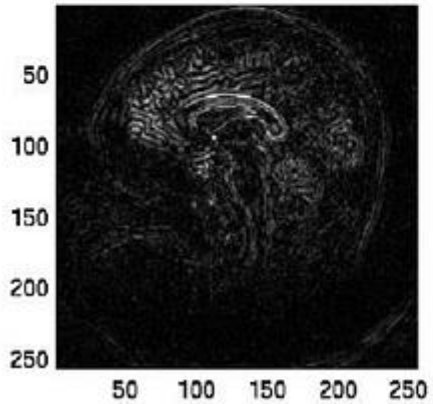


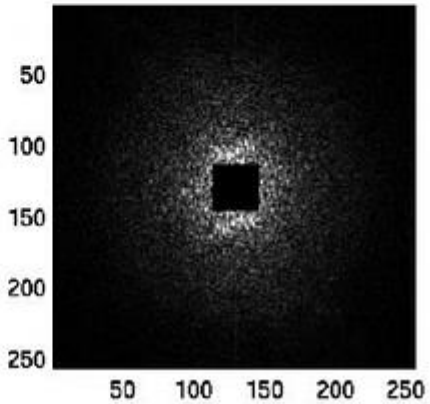




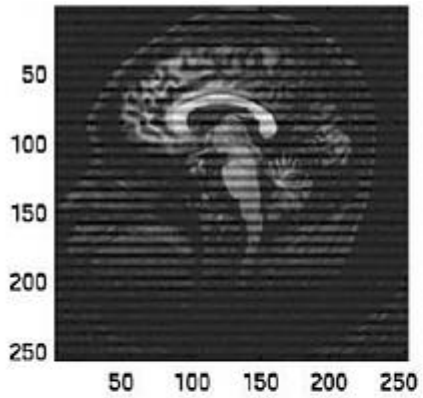


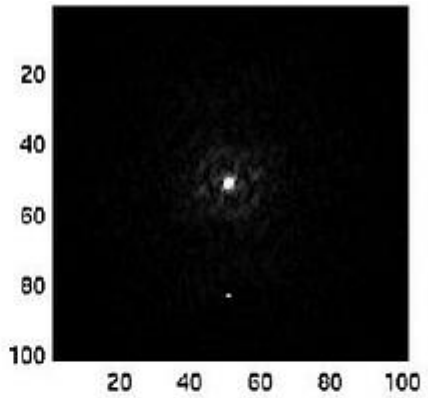




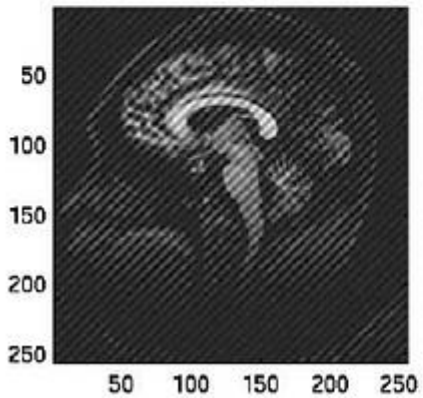


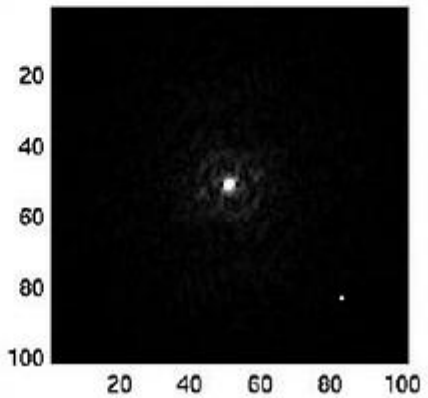
https://medium.com/@gokcenazakyol/what-are-filtering-in-frequency-domain-and-fourier-transform-image-processing-5-6f4cace43c91

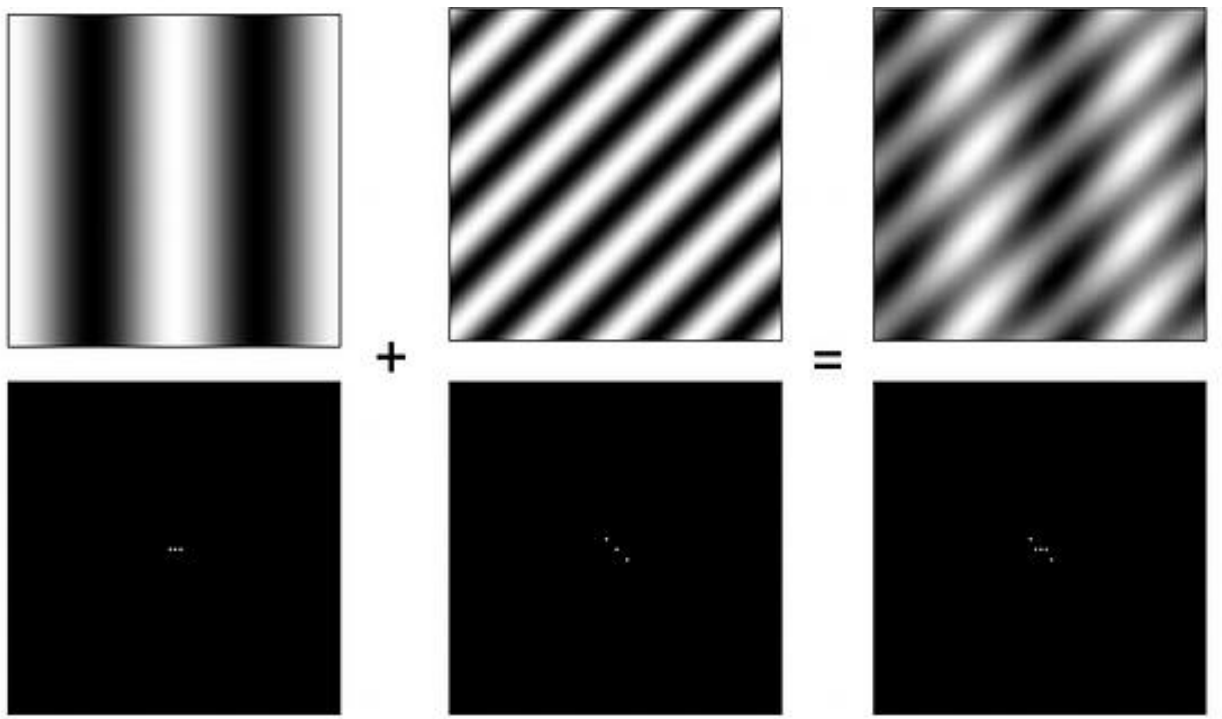




(d) spike off-center in kx







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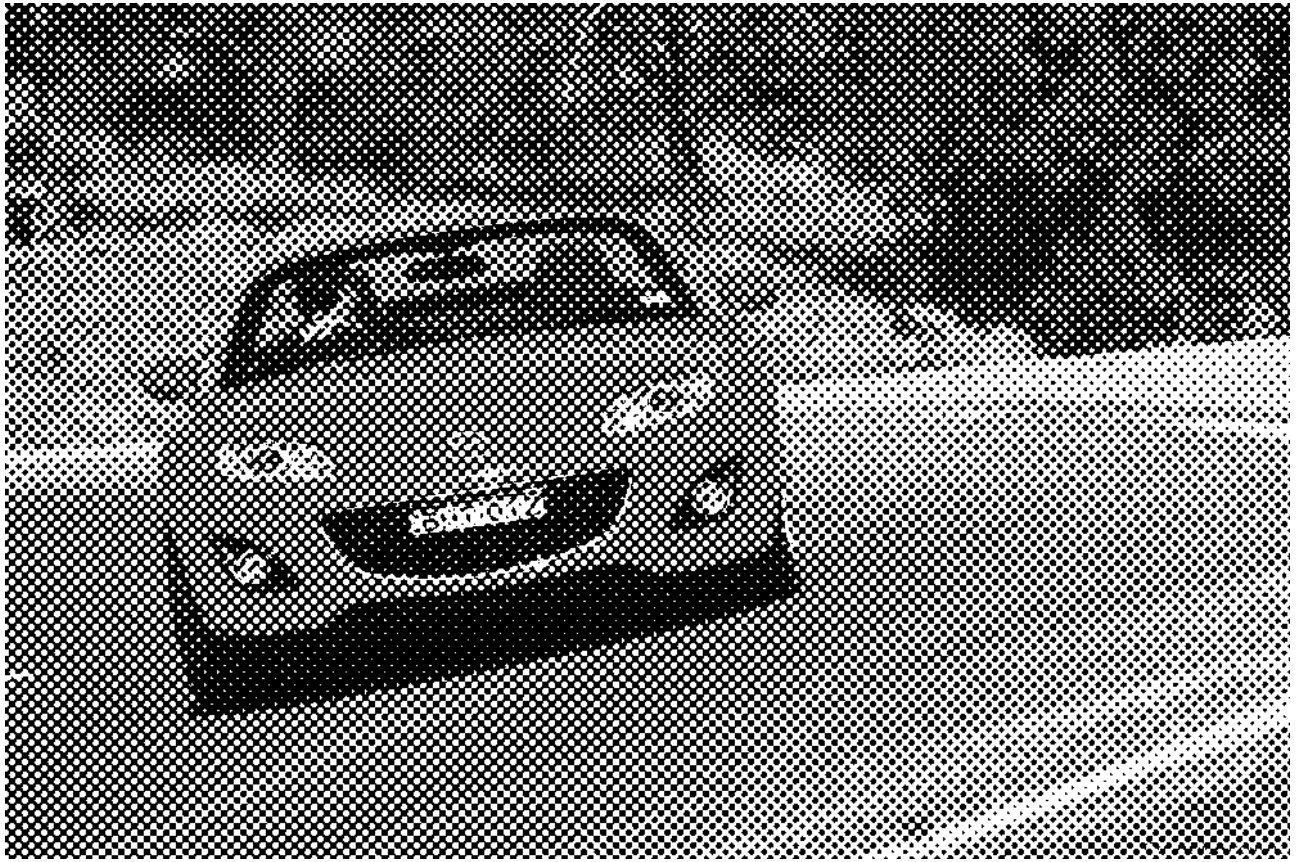
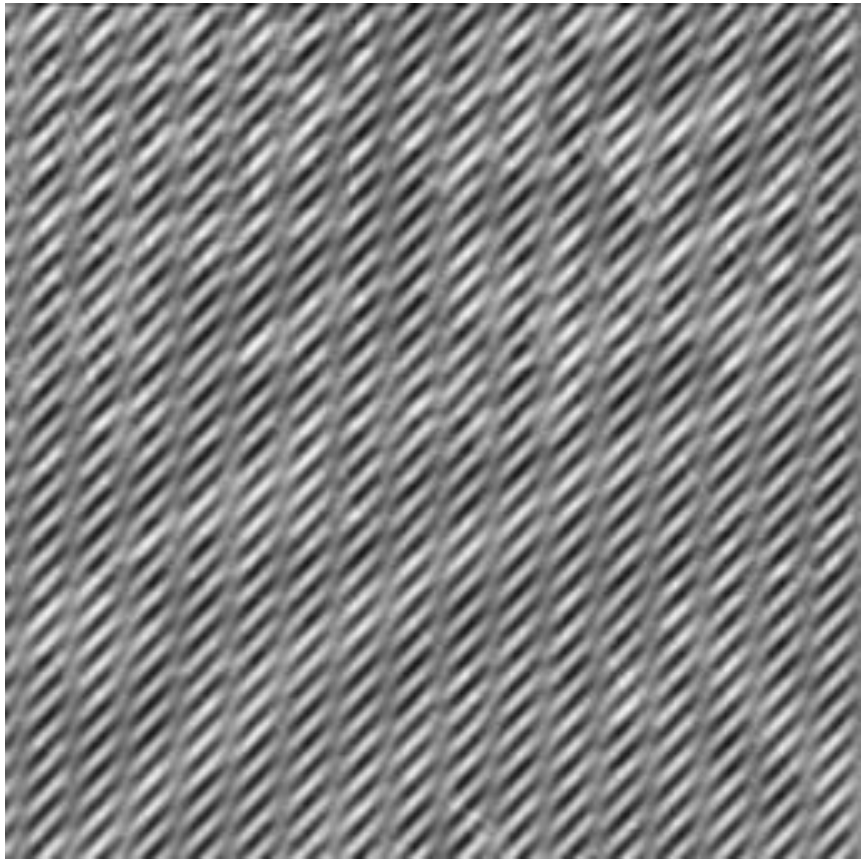
Case Study #1 : Investigation Evidence

Do the image enhancement to get the good and usable images. Explain why and how?