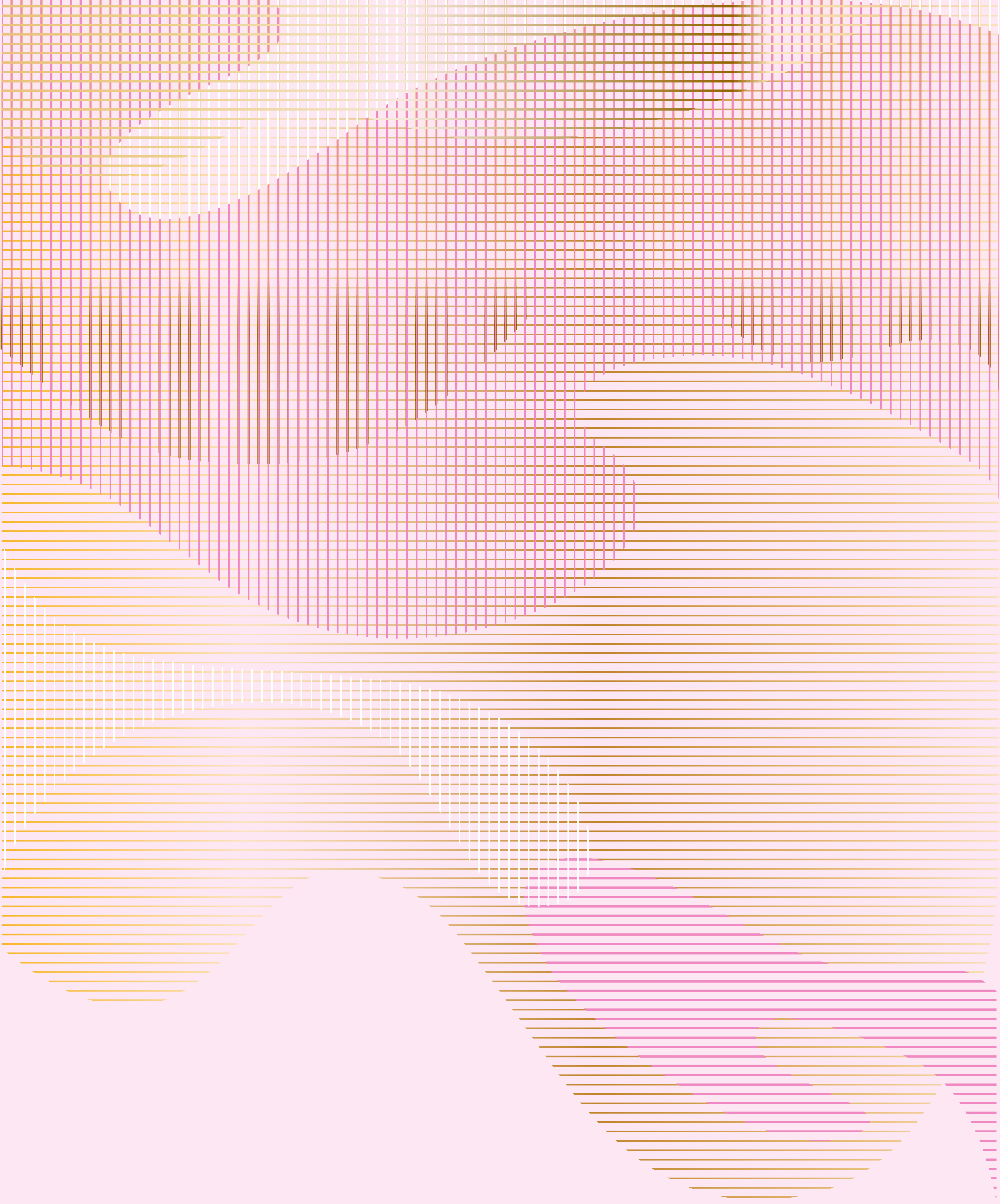


Theorem 4.1.1 :

For any functions f and g that are integrable on $[a,b]$ and any real number K,

(i) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(ii) $\int_a^b K f(x)\,dx = K\int_a^b f(x)\,dx$

Ex 1

$$\int \frac{\sin(2x)}{\sin(x)} + \pi 5^x\,dx = \int \frac{\sin(2x)}{\sin(x)}\,dx + \pi\int 5^x dx$$

$$= \int \frac{2\sin x\cos x}{\sin x}\,dx + \pi\int 5^x dx$$

$$= 2\int \cos x\,dx + \pi\int 5^x\,dx$$

$$= 2\sin x + \pi\frac{5^x}{\ln 5} + C$$

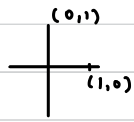
Ex 2

Find $\int_0^1 \left(x^2\sqrt{x} + \frac{1}{x^2+1}\right)dx = \int_0^1 x^{5/2}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx$

$$= \frac{2}{7}x^{7/2}\Big|_0^1 + \arctan x\Big|_0^1$$

$$= \frac{2}{7}(1)^{7/2} - 0 + \arctan 1 - \arctan 0$$

$$= \frac{2}{7} + \frac{\pi}{4} - 0$$



Theorem 4.24: Fundamental Theorem of Calculus (Part I)

Let $a, b \in \mathbb{R}$, $a < b$

If f is continuous on $[a,b]$ and F is any anti derivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Proof:

Suppose f is continuous on $[a,b]$ and F is any anti derivative of f.

WTS: $\int_a^b f(x)\,dx = F(b) - F(a)$

Let $P = \{x_i\}_{i=0}^n$ be any Riemann partition of $[a,b]$

Consider LHS

$$= \int_a^b f(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x, \quad \forall x_i^* \in [x_{i-1}, x_i]$$

Choose $x_i^* = c_i$

$$= \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^n \frac{F(x_i) - F(x_{i-1})}{\Delta x}\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^n F(x_i) - F(x_{i-1})$$

$$= \lim_{n\to\infty} \Big((\cancel{F(x_1)} - F(x_0)) + (\cancel{F(x_2)} - \cancel{F(x_1)}) + \dots + (\cancel{F(x_{n-1})} - \cancel{F(x_{n-2})}) + (F(x_n) - \cancel{F(x_{n-1})}) \Big)$$

$$= \lim_{n\to\infty} (F(x_n) - F(x_0))$$

$$= \lim_{n\to\infty} (F(b) - F(a)) = F(b) - F(a)$$

(Since $P = \{x_i\}_{i=0}^n$

Since F is anti derivative, then $F'(x) = f(x)$ $\forall x \in (a,b)$.

F is differentiable on $[a,b]$

F is continuity on $[a,b]$

F is cont. on each $[x_{i-1}, x_i] \subset [a,b]$

F is diff. on each $(x_{i-1}, x_i) \subset [a,b]$

Apply MVT (for F on $[x_{i-1}, x_i]$)

$\therefore$ By MVT, $\exists c_i \in (x_{i-1}, x_i)$

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$$

$$\Leftrightarrow F'(c_i)(x_i - x_{i-1}) = F(x_i) - F(x_{i-1})$$

$$\Leftrightarrow F'(c_i)\Delta x = F(x_i) - F(x_{i-1})$$

(By $P = \{x_i\}_{i=0}^n$