Jan 29 Lec 2 Notes

The following statements are equivalent:

(i) The system of linear equations is consistent
(ii) $\exists \vec{x}$ in $\mathbb{R}^m$ s.t. $A\vec{x} = \vec{b}$
(iii) $\vec{b}$ is a linear combination of columns of A.

Example:

1. $\vec{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$

Is $\vec{d}$ a linear comb. of $\vec{b}$ and $\vec{c}$?

$$A = \begin{bmatrix} | & | \\ \vec{b} & \vec{c} \\ | & | \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Is the system of linear equations with coefficient matrix A and aug. column $\vec{b}$ consistent?

$$[A \mid \vec{b}] = \left[\begin{array}{cc|c} 4 & 2 & 8 \\ 1 & 3 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{matrix} x_1 = 1 \\ x_2 = 2 \end{matrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is a solution}$$

$$\Rightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ 3 \end{pmatrix} x_2 = \vec{d}$$

$$\Rightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix}(1) + \begin{pmatrix} 2 \\ 3 \end{pmatrix}(2) = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$$

2. $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

Q. Is $\vec{b}$ a linear comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3$?

Q. Is $\vec{b}$ a linear comb. of columns of A?

Q. Does $A\vec{x} = \vec{b}$ have a solution?

Q Is the system of linear equations $[A \mid \vec{b}]$ consistent?

Same thing

$$A: \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 2 & 0 & 4 & 2 \\ 3 & -1 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_3 = 1 \\ x_2 + 6x_3 = 5 \end{cases}$$

x_3 is a free variable
x_1, x_2 is basic (dependent)

$$\begin{cases} x_1 = 1 - 2s \\ x_2 = 5 - 6s \\ x_3 = s \end{cases} \quad s \in \mathbb{R}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-2s \\ 5-6s \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix}$$

Solution set: $\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ -6 \\ 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$

Def:

$f : X \to Y$ is surjective provided that

$\forall y \in Y, \exists x \in X, f(x) = y$

Theorem:

$f : X \to Y$ is surjective iff $\operatorname{img} f = Y$

Example:

3. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

$T_A : \underset{\text{Domain}}{\mathbb{R}^3} \longrightarrow \underset{\text{Codomain}}{\mathbb{R}^2}$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longmapsto A\vec{v} = \begin{pmatrix} 2v_1 \\ 3v_2 \end{pmatrix}$

$$A \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 3v_2 \end{pmatrix}$$

Is $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ injective?

No, $T_A(\vec{v}) = T_A(\vec{w})$, but $\vec{v} \neq \vec{w}$

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = A\vec{w}$$

Is T_A surjective?

$\forall \vec{w} \in \mathbb{R}^2 \; \exists \vec{v} \in \mathbb{R}^3$ s.t. $T_A(\vec{v}) = \vec{w}$

$\forall \vec{w} \in \mathbb{R}^2$, $A\vec{v} = \vec{w}$ is consistent

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & w_1 \\ 0 & 3 & 0 & w_2 \end{array}\right], \text{ consistent } \forall \vec{w} \in \mathbb{R}^2$$

Theorem 1.3.10:

Let $A_{n \times m}$ be a matrix, $\vec{x}, \vec{y}$ in $\mathbb{R}^m$, $k \in \mathbb{R}$

(i) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$, $\forall\ \vec{x}, \vec{y}$ in $\mathbb{R}^m$

(ii) $A(k\vec{x}) = k\ A\vec{x}$, $\forall\ \vec{x} \in \mathbb{R}^m$, $\forall\ k \in \mathbb{R}$

Proof: wts Theorem 1.3.10 (ii)

Let $k \in \mathbb{R}$ and $\vec{x}$ in $\mathbb{R}^m$.

Suppose $A = \begin{bmatrix} — \vec{w}_1 — \\ \vdots \\ — \vec{w}_n — \end{bmatrix}$

$$A(k\vec{x}) = \begin{bmatrix} — \vec{w}_1 — \\ — \vec{w}_2 — \\ \vdots \\ — \vec{w}_n — \end{bmatrix} (k\vec{x}) \quad \text{By def 1.3.7}$$

$$= \begin{bmatrix} \vec{w}_1 \cdot (k\vec{x}) \\ \vec{w}_2 \cdot (k\vec{x}) \\ \vdots \\ \vec{w}_n \cdot (k\vec{x}) \end{bmatrix}$$

$$= \begin{bmatrix} k(\vec{w}_1 \cdot \vec{x}) \\ k(\vec{w}_2 \cdot \vec{x}) \\ \vdots \\ k(\vec{w}_n \cdot \vec{x}) \end{bmatrix}$$

$$= k \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix} = k\ A\vec{x}$$

Given $A_{n \times m}$, an $n \times m$ matrix, define

$$T_A : \mathbb{R}^m \to \mathbb{R}^n$$
$$\vec{x} \mapsto A\vec{x}$$

T_A is a function, or a map, or a transformation

$$T_A(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) \quad \text{Theorem 1.3.10}$$
$$= A\vec{x} + A\vec{y}$$
$$= T_A(\vec{x}) + T_A(\vec{y})$$

$$T_A(k\vec{x}) = A(k\vec{x}) \quad \text{Theorem 1.3.10}$$
$$= k\,A\vec{x}$$
$$= k\,T_A(\vec{x})$$

$$M_{n \times m} := \left\{ \text{all } n \times m \text{ matrices with real value} \right\} = \left\{ (a_{ij})_{n \times m} \,\middle|\, a_{ij} \in \mathbb{R} \right\}$$

The set of $n \times m$ matrices is a vector space