B24 May 14 Lec 2 Notes

Notation:

Let V, W be v.s. We use the notation:

$$L(V,W) := \{ T : V \to W \mid T \text{ is a L.T.} \}$$

Proposition:

Let V, W be v.s. Then $L(V,W)$ is a v.s.

Proof:

We will demonstrate for instance, that:

$$\alpha \cdot (T+S) = \alpha \cdot T + \alpha \cdot S \quad , \forall T, S \in L(V,W) \text{ and } \forall \alpha \in \mathbb{F}$$

Indeed, let $v \in V$. Then,

$$\alpha \cdot (T+S)(v) := \alpha(T(v) + S(v))$$

$$= \alpha T(v) + \alpha S(v)$$

$$=: (\alpha T + \alpha S)(v)$$

$\square$

Remark:

If V, W have bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$, then $L(V,W)$ is essentially the same as $M_{m \times n}(\mathbb{F})$.

e.g. Let V be a v.s. with basis $v_1, \dots, v_n$. If $v \in V$ and $v = \alpha_1 v_1 + \dots + \alpha_n v_n$, we defined

$$[v]_{v_1, \dots, v_n} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

This can be thought of as an element of $L(V, \mathbb{F}^n)$: We are assining n scalars to each $v \in V$.

e.g. $3 + 5x^3 \in \mathbb{P}_3$

$$[3+5x^3]_{1, x, x^2, x^3} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

Recall if $T: V \to W$ is a L.T. and $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are bases for V, W respectively. Then,

$$[T]^{v_1,\dots,v_n}_{w_1,\dots,w_m} = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}$$

where

$$T(v_1) = B_{11} w_1 + \dots + B_{m1} w_m$$
$$\vdots$$
$$T(v_n) = B_{1n} w_1 + \dots + B_{mn} w_m$$

Now suppose $S: W \to U$ is a L.T., where U is a v.s. with basis $u_1, \dots, u_\ell$.

$$[S]^{w_1,\dots,w_m}_{u_1,\dots,u_\ell} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{\ell 1} & \cdots & \gamma_{\ell m} \end{bmatrix}$$

$$S(w_1) = \gamma_{11} u_1 + \dots + \gamma_{\ell 1} u_\ell$$
$$\vdots$$
$$S(w_m) = \gamma_{1m} u_1 + \dots + \gamma_{\ell m} u_\ell$$

So we have

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & W \arrow[r, "S"] & U \end{tikzcd}$$

What is $[ST]^{v_1,\dots,v_n}_{u_1,\dots,u_\ell}$?

Matrix multiplication is defined exactly as.

Bases for the domain

$$[ST]^{v_1,\dots,v_n}_{u_1,\dots,u_\ell} = [S]^{w_1,\dots,w_m}_{u_1,\dots,u_\ell} \, [T]^{v_1,\dots,v_n}_{w_1,\dots,w_m}$$

i.e. Matrix multi. corresponds to composition of the associated L.T.

e.g. $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$ $\qquad$ $\mathbb{R}^2 \xrightarrow{S} \mathbb{R}^2$

$(x,y) \mapsto (2x-y, 4y)$ $\qquad$ $(x,y) \mapsto (-y, x+y)$

So then $S \circ T((x,y)) = S((2x-y, 4y))$
$\qquad = (-4y, 2x+3y)$

So since $ST((1,0)) = (0,2) = 0(1,0) + 2(0,1)$

$ST((0,1)) = (-4,3) = -4(1,0) + 3(0,1)$

e.g. continued...

So $[ST]_{(1,0),(0,1)}^{(1,0),(0,1)} = \begin{bmatrix} 0 & -4 \\ 2 & 3 \end{bmatrix}$

Then $T((1,0)) = (2,0) = 2(1,0) + 0(0,1)$

$T((0,1)) = (-1,4) = -1(1,0) + 4(0,1)$

So $[T]_{(1,0),(0,1)}^{(1,0),(0,1)} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

Similarly,

$$[S]_{(1,0),(0,1)}^{(1,0),(0,1)} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Then,

$$\begin{bmatrix} 0 & -4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$[ST]_{(1,0),(0,1)}^{(1,0),(0,1)} = [S]_{(1,0),(0,1)}^{(1,0),(0,1)} [T]_{(1,0),(0,1)}^{(1,0),(0,1)}$$

Why does the identity below hold?

$$[ST]_{u_1,\dots,u_\ell}^{v_1,\dots,v_n} = [S]_{u_1,\dots,u_\ell}^{w_1,\dots,w_m} [T]_{w_1,\dots,w_m}^{v_1,\dots,v_n}$$

Proof:

$$ST(v_1) = S(\beta_{11} w_1 + \dots + \beta_{m1} w_m)$$

$$= \beta_{11} S(w_1) + \dots + \beta_{m1} S(w_m)$$

$$= \beta_{11}(\gamma_{11} u_1 + \dots + \gamma_{\ell 1} u_\ell) + \dots + \beta_{m1}(\gamma_{1m} u_1 + \dots + \gamma_{\ell m} u_\ell)$$

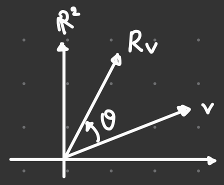
$$= (\beta_{11}\gamma_{11} + \dots + \beta_{m1}\gamma_{1m}) u_1 + \dots + (\beta_{11}\gamma_{\ell 1} + \dots + \beta_{m1}\gamma_{\ell m}) u_\ell$$

$$= \begin{bmatrix} \beta_{11}\gamma_{11} + \dots + \beta_{m1}\gamma_{1m} \\ \vdots \\ \beta_{11}\gamma_{\ell 1} + \dots + \beta_{m1}\gamma_{\ell m} \end{bmatrix}$$

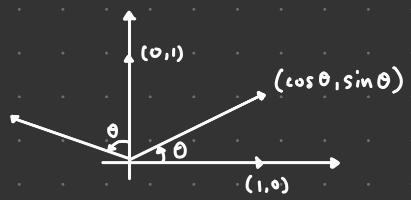
$$= \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{\ell 1} & \cdots & \gamma_{\ell m} \end{bmatrix} \begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix}$$

Ex 1:

What is the matrix representation of the L.T. of $\mathbb{R}^2$ defined as rotation of all vectors by an angle of θ?



It suffices to compute $R((1,0))$, $R((0,1))$



$$R((1,0)) = (\cos\theta, \sin\theta)$$
$$= \cos\theta \cdot (1,0) + \sin\theta \cdot (0,1)$$

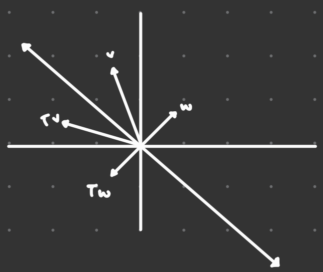
$$R((0,1)) = \left(\cos\left(\tfrac{\pi}{2}+\theta\right), \sin\left(\tfrac{\pi}{2}+\theta\right)\right)$$
$$= (-\sin\theta, \cos\theta)$$

So

$$[R]^{(1,0),(0,1)}_{(1,0),(0,1)} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ex 2:

Consider the mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as the reflection in the line $y = -\frac{2}{3}x$.



We have to recognize T as a composition $R^{-1}UR$ where R is rotation by the angle $y = -\frac{2}{3}x$ makes with the x-axis, and U is the reflection in the x-axis.

We have already seen that

$$[R]^{(1,0),(0,1)}_{(1,0),(0,1)} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad [R^{-1}]^{(1,0),(0,1)}_{(1,0),(0,1)} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We know reflection in the x-axis is given by:

$$[U]^{(1,0),(0,1)}_{(1,0),(0,1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since $T = R^{-1}UR$, $[T]^{(1,0),(0,1)}_{(1,0),(0,1)} = [R^{-1}]^{(1,0),(0,1)}_{(1,0),(0,1)} \; [U]^{(1,0),(0,1)}_{(1,0),(0,1)} \; [R]^{(1,0),(0,1)}_{(1,0),(0,1)}$

Ex 2 continued...:

$$[T]_{(1,0),(0,1)}^{(1,0),(0,1)} = \begin{bmatrix} 3/\sqrt{13} & 2/\sqrt{13} \\ -2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3/\sqrt{13} & -2/\sqrt{13} \\ 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$$

$$= \begin{bmatrix} 5/13 & -12/13 \\ -12/13 & -5/13 \end{bmatrix}$$