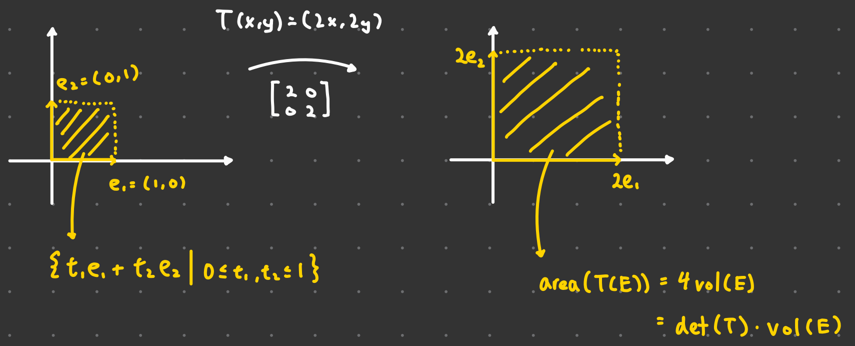
B24 June 11 Lec 2 Notes

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a L.T. , and $E \subset \mathbb{R}^n$, then $\text{vol}(T(E)) = \underbrace{\det(T)}_{\text{is not dependent on } E} \text{vol}(E)$

i.e. $\frac{\text{vol}(T(E))}{\text{vol}(E)}$ is constant (over all E) and does not depend on E.

e.g.

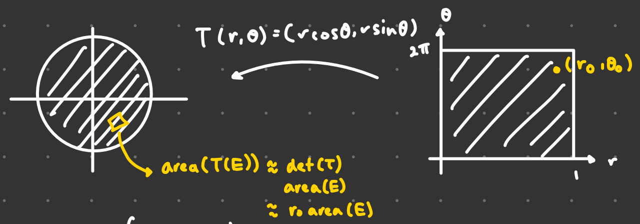


e.g.

$$\text{Area}(\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\})$$

$$= \int dxdy$$

$$= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta$$



and near (r_0, θ_0),

$$T(r,\theta) \approx T(r_0, \theta_0) + \underbrace{\begin{bmatrix} \cos\theta_0 & -r_0 \sin\theta_0 \\ \sin\theta_0 & r_0 \cos\theta_0 \end{bmatrix}}_{\text{derivative of } T \text{ at } (r_0, \theta_0)} \begin{bmatrix} r - r_0 \\ \theta - \theta_0 \end{bmatrix}$$

$$\det \begin{bmatrix} \cos\theta_0 & -r_0 \sin\theta_0 \\ \sin\theta_0 & r_0 \cos\theta_0 \end{bmatrix} = r_0 \cos^2\theta_0 + r_0 \sin^2\theta_0 = r_0$$

Theorem:

There exists a unique function

$$\mathrm{Det}: \underbrace{\mathbb{F}^n \times \mathbb{F}^n \times \dots \times \mathbb{F}^n}_{n} \to \mathbb{F}$$

such that

(i) Linearity

$$\mathrm{Det}(v_1, \dots, \alpha u_k + \beta v_k, \dots, v_n) = \alpha \,\mathrm{Det}(v_1, \dots, u_k, \dots, v_n) + \beta \,\mathrm{Det}(v_1, \dots, v_k, \dots, v_n)$$

(ii) Antisymmetry

$$\mathrm{Det}(v_1, \dots, \underset{k}{v_j}, \dots, \underset{j}{v_k}, \dots, v_n) = (-1)\,\mathrm{Det}(v_1, \dots, \underset{k}{v_k}, \dots, \underset{j}{v_j}, \dots, v_n)$$

(iii) Normalization

$$\mathrm{Det}(e_1, \dots, e_n) = 1$$

Remark:

If $A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$,

then $\det(A) := \det\left(\begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{n1} \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{1n} \\ \vdots \\ \alpha_{nn} \end{bmatrix} \right)$.

Proposition:

If A is a square matrix, then:

(i) if A has a 0 column, $\det(A) = 0$

(ii) if A has two identical columns, $\det(A) = 0$.

(iii) if columns of A are linearly dependent, then $\det(A) = 0$

Proof (i):

Write $A = [v_1 \cdots v_n]$

$$\begin{aligned} \det(v_1, \dots, \underset{j}{0}, \dots, v_n) &= \det(v_1, \dots, \underset{j}{0 \cdot 0}, \dots, v_n) \\ &= 0 \cdot \det(v_1, \dots, \underset{j}{0}, \dots, v_n) \quad \text{By linearity} \\ &= 0 \end{aligned}$$

$\boxtimes$

Proof (ii):

If $v_j = v_k$, then

$$\det(v_1, \dots, \underset{j}{v_j}, \dots, \underset{k}{v_k}, \dots, v_n) = -\det(v_1, \dots, \underset{j}{v_k}, \dots, \underset{k}{v_j}, \dots, v_n) \quad \text{Anti-symmetry}$$

$$= -\det(v_1, \dots, \underset{j}{v_j}, \dots, \underset{k}{v_k}, \dots, v_n) \quad \text{Since } v_j = v_k$$

$$\Rightarrow \det(v_1, \dots, \underset{j}{v_j}, \dots, \underset{k}{v_k}, \dots, v_n) = 0$$

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Proof (iii):

Now suppose $v_1, \dots, v_n$ are L.D., so the without loss of generality, $v_1 = \alpha_2 v_2 + \dots + \alpha_n v_n$. Then

$$\det(v_1, \dots, v_n) = \det(\alpha_2 v_2 + \dots + \alpha_n v_n, v_2, v_n)$$

$$= \alpha_2 \underbrace{\det(v_2, v_2, \dots, v_n)}_{=0 \text{ by (ii)}} + \dots + \alpha_n \underbrace{(v_n, v_2, \dots, v_n)}_{=0 \text{ by (ii)}} \quad \text{By Linearity}$$

$$= 0$$

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Remark:

We know columns of A are L.I. iff rank(A) is full iff A is invertible. So A is not invertible $\Rightarrow \det A = 0$.

Lemma:

If E is an elementary matrix, A is a square (of the same size as E), then:

$$\det(AE) = \det(A)\det(E)$$

Proof:

(A row operation would be EA instead.)

AE corresponds to performing an elementary column operation on A

(i) Interchanging columns

(ii) Replacing a column with its sum with a scalar multiple α of another column.

(iii) Multiplying column by non-zero scalar α.

Proof (Continued...):

For (i), $\det(AE) = -\det(A)$ by anti-symmetry, and

$$E = \begin{array}{c} \\ j \\ \\ k \\ \\ \end{array}\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ 0 \cdots & 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ & & & \ddots & & & \\ 0 \cdots 0 & 1 & 0 \cdots 0 & 0 & 0 \cdots 0 \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$
$$\qquad\qquad \begin{array}{cc} j & k \end{array}$$

and $\det(E) = -1$ i.e.

$$\det(AE) = \det(A)\det(E) \quad \text{in case (i)}$$

In case (ii),

$$\det(AE) = \det(v_1, \ldots, \underbrace{v_j + \alpha v_k}_{j}, \ldots, v_n)$$

$$= \det(v_1, \ldots, v_n) + \alpha \det(v_1, \ldots, \underset{j}{v_k}, \ldots, \underset{k}{v_k}, \ldots, v_n) \quad \text{By linearity}$$

$$= \det(v_1, \ldots, v_n) + 0$$

$$= \det(A)$$

and $E = \begin{array}{c} \\ j \\ \\ k \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ 0 \cdots & 1 & 0 & \cdots & 0 \\ & 0 \\ \vdots \\ 0 & \ddots & & \\ 0 \cdots 0 & \alpha & 0 \cdots 0 & 1 & 0 \cdots 0 \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix}$
$\qquad\qquad\qquad j \qquad k$

and $\det E = 1$, so also in case (ii) we have $\det(AE) = \det A \det E$.

Case (iii) is similar.

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Proposition:

Let A be a square matrix. Then A is invertible iff $\det A \neq 0$.

Proof ($\Leftarrow$): Already done earlier

Proof ($\Rightarrow$):

Suppose A is invertible. Then $A = E_n \cdots E_1$ where each E_i is an elementary matrix. So

$$\det(A) = \det(E_n \cdots E_2 E_1)$$
$$= \det(E_n \cdots E_2)\det(E_1) \quad \text{By Lemma}$$
$$= \det(E_n) \cdots \det(E_1) \quad \text{By Lemma}$$
$$\neq 0 \quad \text{Since each } \det(E_i) \neq 0$$

Theorem 1:

For a square matrix A:

$$\det(A) = \det(A^T)$$

Proof:

If A is not invertible, then $\det(A) = 0$, and since $\text{rank}(A) = \text{rank}(A^T)$, $\text{rank}(A^T) \neq n \Rightarrow A^T$ is not invertible $\Rightarrow \det A^T = 0$

If A is invertible, $A = E_n \cdots E_1$ where each E_i is an elementary matrix. So:

$$\det(A) = \det(E_n) \cdots \det(E_1) \quad \text{By Lemma}$$

So $A^T = (E_n \cdots E_1)^T = E_1^T \cdots E_n^T$

So $\det(A^T) = \det(E_1^T) \cdots \det(E_n^T)$

So we can verify directly that if E is an elementary matrix, then

$$\det(E) = \det(E^T)$$

So

$$\begin{aligned} \det(A^T) &= \det(E_1^T) \cdots \det(E_n^T) \\ &= \det(E_1) \cdots \det(E_n) \\ &= \det(A) \end{aligned}$$

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Theorem 2:

For $n \times n$ matrices A, B:

$$\det(AB) = \det(A)\det(B)$$

Proof:

Case 1: Assume B is not invertible.

Then $\det(B) = 0$, so $\det(A)\det(B) = 0$.
It suffices to show AB is not invertible. Since B is not invertible, $\text{Ker}\, B$ is non-trivial so since

$$\text{Ker}(B) \subseteq \text{Ker}(AB)$$

$\text{Ker}(AB)$ is non-trivial, hence AB is not invertible.

Proof (continued...):

Case 2: Assume B is invertible

So $B = E_n \cdots E_1$ where each E_i is an elementary matrix.

Then

$$\begin{aligned} \det(AB) &= \det(AE_n \cdots E_1) \\ &= \det(AE_n \cdots E_2)\det(E_1) \\ &= \det(A)\det(E_n)\cdots\det(E_1) \\ &= \det(A)\det(E_n \cdots E_1) \\ &= \det(A)\det(B) \end{aligned}$$

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Definition:

Let V be a v.s. over $\mathbb{F}$. Let $A: V \to V$ be a L.T., we say $\lambda \in \mathbb{F}$ is an eigenvalue of A if there exists a non-zero $v \in V$ with $Av = \lambda v$, in which case we call v an eigenvector, and

$$\{ v \in V \mid Av = \lambda v \}$$

the eigenspace.

The set of eigenvalues of A is called the spectrum of A, denoted $\sigma(A)$.

Remark:

Why find eigenvalues? It trivializes computations, e.g. if $\dim(V) = n$, and A has distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding $v_1, \dots, v_n$, then:

$$[A]_{v_1,\dots,v_n}^{v_1,\dots,v_n} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

So e.g. computing $[A]_{v_1,\dots,v_n}^{v_1,\dots,v_n} x$ involves about n operations, whereas computing $[A]_{w_1,\dots,w_n}^{w_1,\dots,w_n} x$ involves about n^2 operations.

e.g.