Theorem 3.3.1:

Consider vectors $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ in a subspace V of $\mathbb{R}^n$. If the vectors $\vec{v}_1, \dots, \vec{v}_p$ are L.D., and the vectors $\vec{w}_1, \dots, \vec{w}_q$ span V, then $q \geq p$.

Proof:

Consider the matrices

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} | & & | \\ \vec{w}_1 & \cdots & \vec{w}_q \\ | & & | \end{bmatrix}$$

Note that $\operatorname{img} B = V$, since the vectors $\vec{w}_1, \dots, \vec{w}_q$ span V.

The vectors $\vec{v}_1, \dots, \vec{v}_p$ are in $\operatorname{img} B$, so

$$\vec{v}_1 = B\vec{u}_1, \quad \dots, \quad \vec{v}_p = B\vec{u}_p$$

for some vectors $\vec{u}_1, \dots, \vec{u}_p$ in $\mathbb{R}^q$.

Then,

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_p \\ | & & | \end{bmatrix} = B \overset{C}{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_p \\ | & & | \end{bmatrix}}$$

$$\text{or} \quad A = BC$$

$\ker C$ is a subset of $\ker A$ (if $C\vec{x} = \vec{0}$, then $A\vec{x} = BC\vec{x} = \vec{0}$)
But $\ker A = \{\vec{0}\}$ since $\vec{v}_1, \dots, \vec{v}_p$ are L.I. Therefore $\ker C = \{\vec{0}\}$ too.
Theorem 3.1.7b now tells us that the $q \times p$ matrix C has at least as many rows as it has columns, that is, $q \geq p$, as claimed.

Theorem 3.3.2: Number of Vectors in a basis

All bases of a subspace V of $\mathbb{R}^n$ consist of the same number of vectors.

Theorem 3.3.4: Indepent vectors and spanning vectors in a subspace of $\mathbb{R}^n$.

Consider a subspace V of $\mathbb{R}^n$ with $\dim(V) = m$.

(a) We can find at most m L.I. vectors in V.
(b) We need at least m vectors to span V.
(c) If m vectors in V are linearly independent, then they form a basis of V.
(d) If m vectors in V span V, then they form a basis of V.

Theorem 3.3.5: Using rref to construct a basis of the Image

To construct a basis of $\text{im} A$, pick the column vectors of A that correspond to the columns of rref A containing the leading 1's.

Theorem 3.3.6: Dimension of the image

For any matrix A,

$$\dim(\text{im } A) = \text{rank}(A)$$

Theorem 3.3.8: Finding bases of the kernel and image by Inspection

Suppose you are able to spot the redundant columns of a matrix A. Express each redundant column as a l.c. of the preceding columns, $\vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$; Write a corresponding relation, $-c_1\vec{v}_1 - \dots - c_{i-1}\vec{v}_{i-1} + \vec{v}_i = \vec{0}$; and generate the vector

$$\begin{bmatrix} -c_1 \\ \vdots \\ -c_{i-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

in $\ker A$. The vectors so constructed form a basis of $\ker A$.

Theorem 3.3.9: Bases of $\mathbb{R}^n$

The vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$ iff the matrix

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

is invertible.

Summary 3.3.10: Various characterizations of invertible matrices

For an $n \times n$ matrix A, the following statements are equivalent.

(i) A is invertible
(ii) The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, for all $\vec{b}$ in $\mathbb{R}^n$
(iii) $\text{rref}(A) = I_n$
(iv) $\text{rank}(A) = n$
(v) $\text{im}(A) = \mathbb{R}^n$
(vi) $\text{Ker}(A) = \{\vec{0}\}$
(vii) The column vectors of A form a basis of $\mathbb{R}^n$
(viii) $\text{col}(A)$ spans $\mathbb{R}^n$
(ix) The column vectors of A are L.I.