B52 Oct 23 Lec 2 Notes

Covariance

Consider RVs X, Y with means & variances $\begin{cases} \mu_X = E(X), & \mu_Y = E(Y) \\ \sigma_X^2 = V(X), & \sigma_Y^2 = V(Y) \end{cases}$

The Covariance of X and Y is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

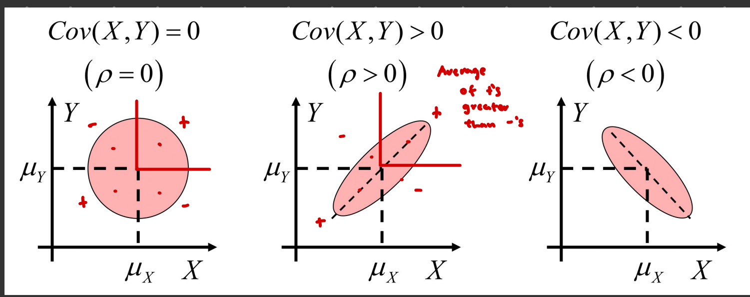
The correlation of X and Y is defined as

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}, \quad \rho_{X,Y} \in [-1, 1]$$

Covariance measures linear dependence between RVs X, Y.

↳ When $Cov(X,Y) > 0$, then on average $X\uparrow$ as $Y\uparrow$, and vice-versa.

↳ When $Cov(X,Y) < 0$, then on average $X\downarrow$ as $Y\uparrow$, and vice-versa.



Covariance Properties

(i) $Cov(X,X) = V(X)$

(ii) $Cov(X,Y) = E(XY) - \mu_X \mu_Y$

(iii) $V(X+Y) = V(X) + V(Y) + 2Cov(X,Y)$

(iv) If $X \perp Y \Rightarrow Cov(X,Y) = 0$

but $Cov(X,Y) = 0 \not\Rightarrow X \perp Y$

Ex 1:

Two processes run serially, with Geometric(.5) completion times. Find the variance of the time until both complete.

Recall $E(X_1 + X_2) = E(X_1) + E(X_2) = 2 + 2 = 4$

Total time : $X_1 + X_2$

$$V(X_1 + X_2) \overset{\text{if indep.}}{=} V(X_1) + V(X_2) = \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{\frac{1}{2}}{\frac{1}{4}} = 4 \Rightarrow sd = \sqrt{4} = 2$$

Assume completion times are correlated with $\rho = \frac{1}{2}$.

Since $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$Cov(X,Y) = \rho_{X,Y}\, \sigma_X \sigma_Y$

$= \rho_{X,Y} \sqrt{V(X)V(Y)}$

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\,Cor(X_1, X_2)$$
$$= 2 + 2 + 2 \cdot \rho \sqrt{V(X_1)V(X_2)} = 2 + 2 + 2\tfrac{1}{2}\sqrt{2 \cdot 2}$$
$$= 6$$