A37 March 25 Lec 2 Notes

Ex 1

Does $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ converge or diverge? Prove.

Proof:

$$= \sum_{n=1}^{\infty} (\ln(n+1) - \ln(n))$$

$$= (\ln \cancel{2} - \ln 1) + (\ln \cancel{3} - \cancel{\ln 2}) + \cancel{\ldots} + (\ln \cancel{(n)} - \ln \cancel{(n-1)}) + (\ln(n+1) - \cancel{\ln(n)})$$

$$= -\ln 1 + \ln(n+1)$$ telescoping series

$$= \ln(n+1)$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty$ limit DNE

$\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges by def

$\blacksquare$

Def (pg 609):

Let $a, r \in \mathbb{R}$, $a \neq 0$. A series of the form

$$a + ar + ar^2 + \ldots + ar^n + \ldots = \sum_{n=0}^{\infty} ar^n$$

is called a geometric series. The number r is called the ratio of the GS.

e.g. $1 - \frac{1}{e} + \frac{1}{e^2} - \frac{1}{e^3} + \ldots \Rightarrow r = -\frac{1}{e}$ is a GS

e.g. $1 + \frac{2}{e} + \frac{3}{e^2} + \frac{4}{e^3} + \ldots = \sum_{n=0}^{\infty} \frac{n+1}{e^n}$ is not a GS

Ex 2

For what values of r do the geometric series $\sum_{n=0}^{\infty} ar^n$ converge? For what values of r does it diverge?

Proof:

$$S_n = a + ar + ar^2 + \ldots + ar^n$$
$$rS_n = ar + ar^2 + \ldots + ar^n + ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$
$$S_n(1-r) = a(1-r^{n+1})$$

$$\therefore S_n = \frac{a}{1-r}(1-r^{n+1}) \quad \text{if } r \neq 1$$

if $r = 1 \Rightarrow S_n = \underbrace{a(1) + a(1) + \ldots + a(1)}_{n+1 \text{ times}}$

$= (n+1)a$

Case 1: $r = 1$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (n+1)a = \pm\infty$

$\therefore \lim$ DNE

Proof continued....

Case 2: $r \neq 1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1-r}\left(1-r^{n+1}\right)$$

$$= \frac{a}{1-r}\left(1-\lim_{n\to\infty} r^{n+1}\right) \quad \text{by limit laws}$$

$$= \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{DNE} & \text{if } |r|>1 \text{ or } r=-1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } |r|<1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r\leq -1 \end{cases} \quad \text{oscillates}$$

$\therefore$ Our GS $\sum ar^n$ converges with sum $= \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$

$\boxtimes$

Theorem (pg 609): GS Test

$\sum ar^n$ converges with sum $= \frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$

Ex 3:

Does $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{4^n}{7^{n+2}}$ converge or diverge?

Proof:

$$\sum_{n=2}^{\infty} (-1)^n \frac{4^n}{7^{n+2}} = \frac{4^2}{7^4} - \frac{4^3}{7^5} + \frac{4^4}{7^6} - \dots$$

$r = -4/7$

By GS Test, $|r| = \frac{4}{7} < 1$

$\therefore$ Our series converges with sum $= \frac{a}{1-r}$

$$= \frac{\frac{4^2}{7^2}}{1-(-4/7)}$$

$$= \frac{4^2}{11(7^3)}$$

$\boxtimes$

Theorem (pg 608): Properties of Convergent series.

Let $\sum a_n$ and $\sum b_n$ be series.

If $\sum a_n = s$ and $\sum b_n = t$ for some $s, t \in \mathbb{R}$, then

(i) for any $c \in \mathbb{R}$, $\sum c a_n$ converges with sum cs
(ii) $\sum (a_n \pm b_n)$ converges with sum $s \pm t$
(iii) $\lim_{n\to\infty} a_n = 0$ (vanishing condition)

Proof (iii):

Suppose $\sum a_n = s$ i.e. $\sum a_n = \lim_{n\to\infty} S_n = S$

Consider
$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (a_n + 0) \\
&= \lim_{n\to\infty} \left(a_n + (a_1 + a_2 + \dots + a_{n-1}) - (a_1 + a_2 + \dots + a_{n-1}) \right) \\
&= \lim_{n\to\infty} \left((a_1 + \dots + a_n) - (a_1 + a_2 + \dots + a_{n-1}) \right) \\
&= \lim_{n\to\infty} (S_n - S_{n-1}) \\
&= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{\underbrace{n-1}_{\to\infty}} \\
&= S - S = 0
\end{aligned}$$

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Theorem (pg 618): Divergence Test

Let $\sum a_n$ be a series

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Proof:

This is the contrapositive of the vanishing condition.

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Ex 4

Does $\sum_{n=1}^{\infty} \frac{\sqrt{5n^4+2n}}{3n^2+7n}$ converge or diverge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{5n^4+2n}}{3n^2+7n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^4(5+2/n^3)}}{n^2(3+7/n)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\sqrt{(5+2/n^3)}}{\cancel{n^2}(3+7/n)}$$

$$= \frac{\sqrt{5+0}}{3+0} = \frac{\sqrt{5}}{3} \neq 0$$

$\therefore$ By div test, our series diverges

Ex 5

Does $\sum_{n=0}^{\infty} \frac{2^n - 3^{n+1}}{4^n} = \sum_{n=0}^{\infty} \frac{2^n}{4^n} - \frac{3^{n+1}}{4^n}$ converge or diverge?

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n - \sum_{n=0}^{\infty} 3\left(\tfrac{3}{4}\right)^n$$

$r = \frac{1}{2}$ $\quad$ $r = 3/4$

$|r| = \frac{1}{2} < 1$ $\quad$ $|r| = 3/4 < 1$

$\therefore$ Convergent by GS test with sums

$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$ $\quad$ $\frac{a}{1-r} = \frac{3}{1-3/4} = 12$

$\therefore$ Our series converges with sum $2 - 12 = -10$