B41 Oct 25 Lec 1 Notes

Definition:

A function whose partial derivatives exist and are continuous, is said to be of class C^1.
We can define C^n for n times continuously differentiable

Ex 1: Does it always happen that $f_{xy} = f_{yx}$?

$$f(x,y) = \begin{cases} \dfrac{xy^3 - x^3y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy^3 - x^3y}{x^2+y^2}\right) = \frac{-x^4y - 4x^2y^3 + y^5}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy^3 - x^3y}{x^2+y^2}\right) = \frac{xy^4 + 4x^3y^2 - x^5}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^5}{h^4} - 0}{h} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0} \frac{-\frac{h^5}{h^4} - 0}{h} = -1$$

Theorem:

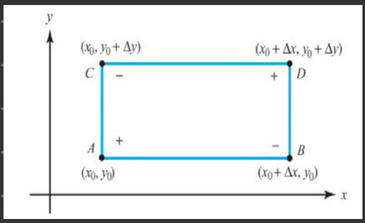
If $f(x,y)$ is of class C^2 (i.e. f is twice continuously differentiable), then the mixed partial derivatives are equal, that is,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

i.e. Suppose f is a real-valued function of two variables x, y and $f(x,y)$ is defined on an open subset U of $\mathbb{R}^2$.
Suppose further that both the second-order mixed partial derivatives $f_{xy}(x,y)$ and $f_{yx}(x,y)$ exist and are continuous on U. Then we have:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{on all of } U$$

Proof:



$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{\frac{\partial f}{\partial x}(x_0, y) - \frac{\partial f}{\partial x}(x_0, y_0)}{y - y_0}$$

$$= \lim_{y \to y_0} \frac{\lim_{x \to x_0} \frac{f(x,y) - f(x_0,y)}{x - x_0} - \lim_{x \to x_0} \frac{f(x,y) - f(x,y_0)}{x - x_0}}{y - y_0}$$

$$= \lim_{y \to y_0} \lim_{x \to x_0} \frac{[f(x,y) - f(x_0,y)] - [f(x,y) - f(x,y_0)]}{(y-y_0)(x-x_0)}$$

$$= \lim_{x \to x_0} \lim_{y \to y_0} \frac{[f(x,y) - f(x_0,y)] - [f(x,y) - f(x,y_0)]}{(y-y_0)(x-x_0)}$$

$$= \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Theorem:

Suppose f is a function of n variables defined on an open subset U of $\mathbb{R}^n$. Suppose all mixed partials with a certain number of differentiations in each input variable exist and are continuous on U. Then all the mixed partials are continuous and don't depend on the order of the differentiation.

Ex 2:

Let (x,y) be Cartesian coordinates in the plane and let (r,θ) be polar coordinates.

(i) If $z = f(x,y)$ is a function on the plane, express the partial derivatives $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(ii) Express $\frac{\partial^2 z}{\partial r^2}$ in Cartesian coordinates.

(i) By chain rule, using $x = r\cos\theta$, $y = r\sin\theta$

$$\left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

OR

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\frac{\partial z}{\partial x}\cos\theta + r\frac{\partial z}{\partial y}\sin\theta$$

(ii)
$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$
$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)$$
$$= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)\sin\theta$$
$$= \left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial x \partial y}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 z}{\partial x \partial y}\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin\theta\right)\sin\theta$$
$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$$
$$= \left(\frac{x^2}{r^2}\frac{\partial^2 z}{\partial x^2} + 2\frac{xy}{r^2}\frac{\partial^2 z}{\partial x \partial y} + \frac{y^2}{r^2}\frac{\partial^2 z}{\partial y^2}\right)$$
$$= \frac{1}{x^2+y^2}\left(x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial z}{\partial x}\frac{\partial y}{\partial r}$$
$$= \frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial x \partial y}\sin\theta$$

and similarly for $\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$

Definition: Heat Equation

If temperature $T = T(x, y, z, t)$, $\frac{\partial T}{\partial t} = a^2 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$

Definition: Wave Equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Definition: Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$