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W9 Lecture 17 Notes

Rules for Differentiation (Continued...)

8. Derivatives of Log functions

a) If $f(x) = \log_b x$ for any $b>0$, $b \neq 1$ then $f'(x) = \frac{1}{x \ln b}$

b) If $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$

c) If $f(x) = \ln|x|$ then $f'(x) = \frac{1}{x}$

<u>Proof:</u>

b) $f(x) = \ln x$

$$e^{f(x)} = e^{\ln x}$$

$$f'(x) \cdot e^{f(x)} = 1$$

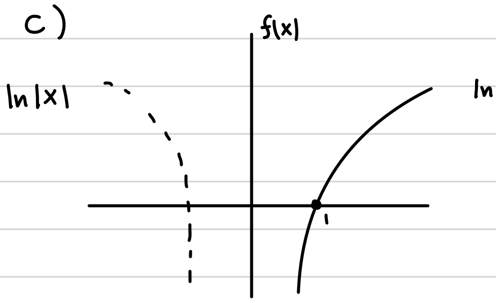
$$f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

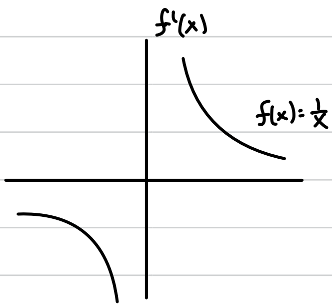
a) $x = b^{\log_b x}$

$$1 = b^{\log_b x} \cdot \ln b \cdot (\log_b x)'$$

$$(\log_b x)' = \frac{1}{x \ln b}$$

c)





9. Log Differentiation

$$[\ln f(x)]' = \frac{1}{f(x)} \cdot f'(x)$$

Example:

1. Find $f'(x)$ if $f(x) = \frac{(x-1)^3 (x+6)^6}{(x-2)^2 (x^2+2x+6)^5}$.

$$|f(x)| = \left| \frac{(x-1)^3 \cdot (x+6)^6}{(x-2)^2 \cdot (x^2+2x+6)^5} \right|$$ ← Take abs because we want to stay in domain of $\ln x$

$$\ln|f(x)| = \ln|x-1|^3 + \ln|x+6|^6 - \ln|x-2|^2 - \ln|x^2+2x+6|^5$$

$$\ln|f(x)| = 3\ln|x-1| + 6\ln|x+6| - 2\ln|x-2| - 5\ln|x^2+2x+6|$$

$$\frac{f'(x)}{f(x)} = \frac{3}{x-1} + \frac{6}{x+6} - \frac{2}{x-2} - \frac{5(2x+2)}{x^2+2x+6}$$

$$f'(x) = f(x) \cdot \left(\frac{3}{x-1} + \frac{6}{x+6} - \frac{2}{x-2} - \frac{5(2x+2)}{x^2+2x+6} \right)$$

2. Find $f'(x)$ if $f(x) = x^{x^x}$

$$(\ln y)' = (x^x \ln x)'$$

$$\frac{y'}{y} = (x^x)' \ln x + x^x (\ln x)'$$

$$y' = x^{x^x}\left[x^x(\ln x + 1)\cdot \ln x + x^x \cdot \frac{1}{x}\right]$$

3. $f(x) = \begin{cases} x^2, & x < 1 \\ 1 + \ln x, & x \geq 1 \end{cases}$

a) Is $f(x)$ continuos on $\mathbb{R}$?
b) Is $f(x)$ differentiable on $\mathbb{R}$?
c) $f'(x) = ?$

a) For $f(x)$ to be continuous, we need:

$$\lim_{x \to 1^-} x^2 = \lim_{x \to 1^+} (1 + \ln x) = f(x)\Big|_{x=1} = 1$$

$f(x)$ is continuous on $\mathbb{R}$.

b) $f(x)$ is differentiable at $x=1$ if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(1^-) = f'(1^+)$

$$f'(1^-) = 2x\big|_{x=1} = 2$$

$$f'(1^+) = \frac{1}{x}\Big|_{x=1} = 1$$

$f'(1^-) \neq f'(1^+)$, thus $f(x)$ is not differentiable at $x = 1$.

c) $f'(x) = \begin{cases} 2x, & x < 1 \\ \text{DNE}, & x = 1 \\ \frac{1}{x}, & x > 1 \end{cases}$

Derivatives of Trig Functions

Each trig function is differentiable on its domain.

a) $\frac{d}{dx}(\sin x) = \cos x$

b) $\frac{d}{dx}(\cos x) = -\sin x$

c) $\frac{d}{dx}(\tan x) = \sec^2 x$

d) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

e) $\frac{d}{dx}(\sec x) = \sec x \tan x$

f) $\frac{d}{dx}(\cot x) = -\csc^2 x$

Prove $\frac{d}{dx}(\cos x) = -\sin x$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cosh - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x (\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= 0 - \sin x = -\sin x$$

QED

Derivatives of Inverse Trig Functions

a) $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

b) $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$

c) $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

d) $\frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$

e) $\frac{d}{dx}(\operatorname{arc\,sec} x) = \frac{1}{|x|\sqrt{x^2-1}}$

f) $\frac{d}{dx}(\operatorname{arc\,csc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$

Prove $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

$$y = \arctan x \Leftrightarrow x = \tan y$$

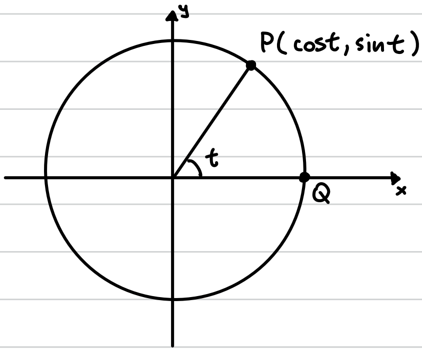
$$1 = \sec^2 y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Euclidean Geometry (Euclid's 5 postulates)

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Given any straight line and a point not on it, there exists one and only one straight line which passes through the point and never intersects the first line, no matter how far they are extended.

Hyperbolic Functions

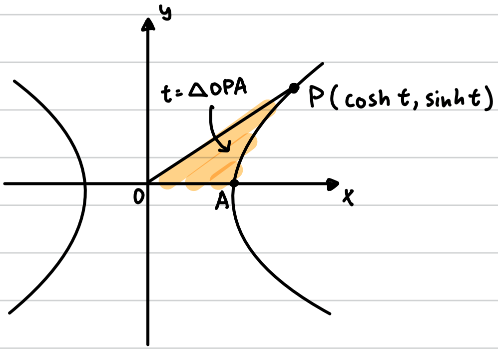


cost, sint are circular functions

$$x^2 + y^2 = 1$$

$x = \cos t$, $y = \sin t$ ← Parametrization of unit circles

$$\cos^2 t + \sin^2 t = 1$$



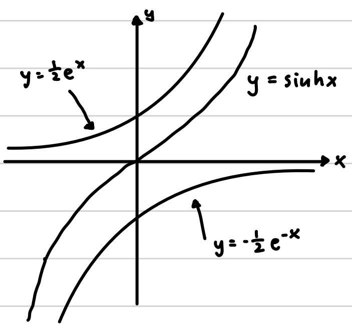
Any point with coord $\left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right)$

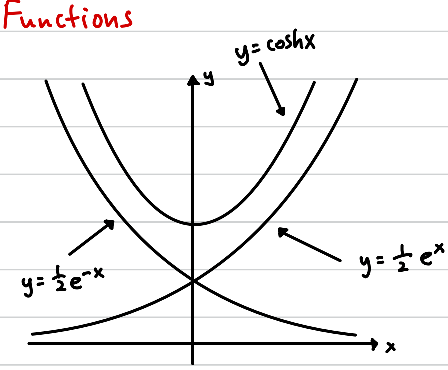
t - parameter = area of OPA

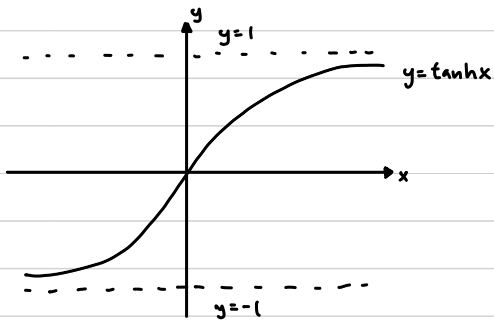
$\cosh t = \frac{e^t + e^{-t}}{2}$, $\sinh t = \frac{e^t - e^{-t}}{2}$

$$\cosh^2 t - \sinh^2 t = 1$$

Graphs of Hyperbolic Functions







Derivatives of Hyperbolic Functions

a) $\frac{d}{dx}(\sinh x) = \cosh x$

b) $\frac{d}{dx}(\cosh x) = \sinh x$

c) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

d) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$

e) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$

f) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

Prove $\frac{d}{dx}(\cosh x) = \sinh x$

$$\frac{d}{dx}\left(\frac{e^x + e^{-x}}{2} \right) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

Derivatives of Inverse Hyberbolic Function

a) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}$

b) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$ x is not an angle.

c) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

Prove $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{x^2+1}}$

$$(x)' = (\sinh(\sinh^{-1}x))'$$
$$1 = \cosh x \cdot \sinh^{-1}x \cdot (\sinh^{-1}x)'$$
$$(\sinh^{-1}x)' = \frac{1}{\cosh x \cdot \sinh^{-1}x}$$
$$= \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1}x)}}$$
$$= \frac{1}{\sqrt{1+x^2}}$$

Prove $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

$$y = \sinh^{-1}x \Leftrightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \Leftrightarrow x = \frac{e^y - e^{-y}}{2}$$
$$2x = e^y - e^{-y} \Leftrightarrow e^y - e^{-y} - 2x = 0$$
$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$
$$= x + \sqrt{x^2+1}$$
$$\ln e^y = \ln(x+\sqrt{x^2+1})$$
$$y = \sinh^{-1}x = \ln(x+\sqrt{x^2+1})$$

$e^y > 0$
$\Rightarrow x \pm \sqrt{x^2+1} > 0$
if $x < \sqrt{x^2+1}$, then
$\ominus$ needs to be excluded

Examples:

4. Find $\frac{d}{dx}(\tanh(1+e^{2x}))$

$$= \operatorname{sech}^2(1+e^{2x}) \cdot (1+e^{2x})'$$
$$= \operatorname{sech}^2(1+e^{2x}) \cdot 2e^{2x}$$

5. Find $\frac{d}{dx}\left(\sqrt[4]{\frac{1+\tanh x}{1-\tanh x}}\right)$

$$\frac{1+\tanh x}{1-\tanh x} = \frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}} = \frac{\cosh x + \sinh x}{\cosh x - \sinh x} = \frac{e^x + \cancel{e^{-x}} + e^x - \cancel{e^{-x}}}{\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}}$$
$$= \frac{2e^x}{2e^{-x}} = e^{2x}$$

$$y = \sqrt[4]{e^{2x}} = e^{\frac{x}{2}}$$

$$y' = \frac{1}{2}e^{x/2}$$