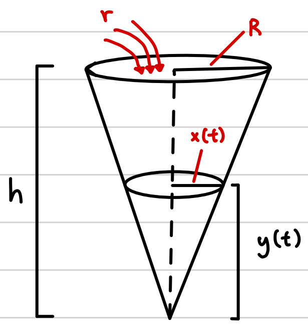
W12 Lecture 22 Notes

Related Rates

Examples:

1. Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of 10ft and a base radius of 5ft. How fast is the water level rising when the water is 6ft deep.

Notations, pics



r = rate with which the water runs into the tank.
R = radius of the base of the tank.
x = radius of the surface of the water in the tank.
$y(t)$ = depth of the water in the tank.
h = height of the tank.
$\frac{dy}{dt}$ = rate of change of water level.

Numerical Information

$r = \frac{dV}{dt} = 9\ ft^3/min$ $\quad R = 5ft \quad h = 10ft \quad$ level $= 6ft$

What we want to find

$\frac{dy(t)}{dt}$ when $y(t) = 6\ ft$

Equation that relates the variables

$V = \frac{1}{3}\pi x^2 y$

$V(t) = \frac{1}{3}\pi x^2(t)\, y(t)$

Differentiate with respect to t

$\frac{x(t)}{y(t)} = \frac{R}{h} \Rightarrow x(t) = \frac{R}{h}\cdot y(t) \Rightarrow x(t) = \frac{1}{2}y(t)$

$V(t) = \frac{1}{3}\pi \frac{1}{4} y^3(t) = \frac{\pi}{12} y^3(t)$

$\frac{dV(t)}{dt} = \frac{\pi}{12}\cdot 3y^2(t)\cdot \frac{dy(t)}{dt}$

Evaluate

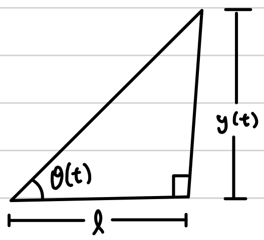
$$\frac{d\,y(t)}{dt} = \frac{12 \cdot \frac{dv}{dt}}{3\pi y^2} = \frac{dv(t)}{dt} \cdot \frac{4}{\pi y^2}\Big|_{y=6} = 9 \cdot \frac{4}{36\pi} = \frac{1}{\pi}$$

Formulate Answer

When the water is 6 ft deep, its level is rising at speed (rate) of $\frac{1}{\pi}$ ft/min.

2. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift off point. At the moment the range range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Picture



$\theta(t)$ = angle (in radians) that range finder makes with the ground

$y(t)$ = height of the balloon

ℓ : distance between range finder and lift off point.

Numerical Info

$\ell = 500$ ft $\qquad \frac{d\theta(t)}{dt} = 0.14 \frac{\text{rad}}{\text{min}} \qquad \theta_0 = \frac{\pi}{4}$

What do we need to find

$\frac{dy(t)}{dt}$ when $\theta = \frac{\pi}{4}$

Equation that relates the variables

$$\frac{y(t)}{\ell} = \tan\theta(t) \Rightarrow y(t) = \tan\theta(t) \cdot \ell$$

Differentiate with respect to t

$$\frac{d\,y(t)}{dt} = \ell \cdot \sec^2\theta(t) \cdot \frac{d\,\theta(t)}{dt} \Rightarrow \frac{dy(\theta)}{dt} = 500 \cdot \frac{1}{\cos^2\theta} \cdot \frac{d\theta(t)}{dt}\Big|_{\theta=\frac{\pi}{4}}$$

$$= 1000 \cdot 0.14 = 140 \text{ ft/min}$$

Formulate

At the moment when the elevation angle is $\frac{\pi}{4}$, the balloon is rising at the rate of 140 ft/min.

Integration

Examples:

1. $\int e^{2(x-3)}\,dx$

Substitution $= \int e^u \cdot \frac{du}{2} = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c$
$= \frac{1}{2}e^{2(x-3)} + C$

$d\,f(x) = f'(x)\cdot dx$
$f(x) = 2(x-3)$
$d(2(x-3)) = 2\,dx$

Let $u = 2(x-3)$
$du = 2\,dx$
$dx = \frac{du}{2}$

2. $\int \tan x\,dx$

$= \int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\int \frac{1}{u}\,du$
$= -\ln|u| + c$
$= -\ln|\cos x| + C$

Let $u = \cos x$
$du = -\sin x\,dx$
$-du = \sin x\,dx$

3. $\int 3\csc(\pi x)\cdot\cot(\pi x)\,dx$

Let $u = \pi x$
$du = \pi x\,dx$
$dx = \frac{du}{\pi}$

$= 3\int \csc u \cdot \cot u \cdot \frac{dt}{\pi}$

$= \frac{3}{\pi}\int \frac{1}{\sin u}\cdot\frac{\cos u}{\sin u}\,du$

Let $t = \sin t$
$dt = \cos t\,dt$

$= \frac{3}{\pi}\int \frac{\cos u}{\sin^2 u}\,du$

$= \frac{3}{\pi}\int \frac{dt}{t^2}$

$= -\frac{3}{\pi}t^{-1} + C = -\frac{3}{\pi}\cdot\frac{1}{\sin u} + C = -\frac{3}{\pi}\cdot\frac{1}{\sin \pi x} + C$
$= -\frac{3}{\pi}\csc \pi x + C$

4. $\int 5\sin(2x+1)\,dx$

Let $t = 2x+1$

$dt = 2\,dx$

$dx = \frac{dt}{2}$

$= \int 5\sin t \cdot \frac{dt}{2}$

$= \frac{5}{2}\int \sin t\,dt$

$= \frac{5}{2}(-\cos t) + C$

$= -\frac{5}{2}\cos(2x+1) + C$

5. $\int \frac{dx}{2+4x^2}$

$= \int \frac{dx}{2(1+2x^2)} = \frac{1}{2}\int \frac{dx}{1+2x^2} = \frac{1}{2}\int \frac{dx}{1+(\sqrt{2}x)^2}$

Let $t = \sqrt{2}x$

$dt = \sqrt{2}\,dx$

$dx = \frac{dt}{\sqrt{2}}$

$= \frac{1}{2}\int \frac{\frac{dt}{\sqrt{2}}}{1+t^2}$

$= \frac{1}{2\sqrt{2}}\int \frac{dt}{1+t^2}$

$= \frac{1}{2\sqrt{2}} \cdot \arctan t + C = \frac{1}{2\sqrt{2}} \cdot \arctan(\sqrt{2}x) + C$

6. $\int \frac{dx}{2\sqrt{x^2-1}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-1}}$

$= \frac{1}{2}\cosh^{-1}x + C$

7. $\int \frac{dx}{\sqrt{4-x^2}}$

$= \int \frac{dx}{2\sqrt{1-x^2/4}}$

$= \int \frac{dx}{2\sqrt{1-(\frac{x}{2})^2}} = \frac{1}{2}\int \frac{2dt}{\sqrt{1-t^2}}$

Let $t = \frac{x}{2}$

$dt = \frac{1}{2}dx$

$dx = 2dt$

$= \int \frac{dt}{\sqrt{1-t^2}}$

$= \arcsin t + C$

$= \arcsin\frac{x}{2} + C$