A22 Apr 2 Lec 2 Notes

Recall TUT 9:

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \mathcal{E} = (\vec{e}_1, \vec{e}_2) \text{ basis for } \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} \qquad B = \left(\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$

$$\forall \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

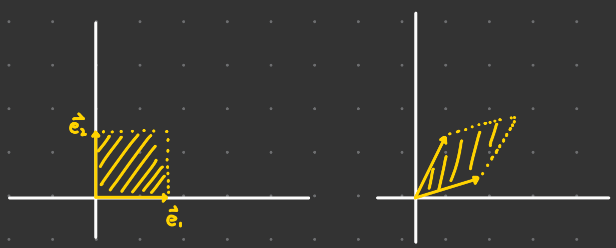
$$T(\vec{v}) = T(x\vec{e}_1 + y\vec{e}_2) = xT(\vec{e}_1) + yT(\vec{e}_2)$$

$$= \underbrace{\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}}_{[T]_{\mathcal{E}}} \begin{pmatrix} x \\ y \end{pmatrix}$$

with respect to $\mathcal{E}$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T([\vec{v}]_{\mathcal{E}}) = A[\vec{v}]_{\mathcal{E}}$$



Given $[\vec{v}]_B$, want to find $[T(\vec{v})]_B$?

$$\vec{v} = r\vec{b}_1 + s\vec{b}_2 \quad , r,s \in \mathbb{R}$$

$$[\vec{v}]_B = \begin{pmatrix} r \\ s \end{pmatrix} \qquad [T(\vec{v})]_B = ?$$

$$T(\vec{v}) = T(r_1\vec{b}_1 + s\vec{b}_2)$$

$$= r_1 T(\vec{b}_1) + sT(\vec{b}_2)$$

$$[T(\vec{v})]_B = [rT(\vec{b}_1) + sT(\vec{b}_2)]_B$$

Recall: $T_B: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a L.T.

$$\vec{v} \longmapsto [\vec{v}]_B$$

$$[T(\vec{v})]_B = r_1[T(\vec{b}_1)]_B + s[T(\vec{b}_2)]_B$$

$$= \underbrace{\begin{bmatrix} | & | \\ [T(\vec{b}_1)]_B & [T(\vec{b}_2)]_B \\ | & | \end{bmatrix}}_{B\text{-matrix of } T = [T]_B} \begin{pmatrix} r \\ s \end{pmatrix}$$

with respect to B

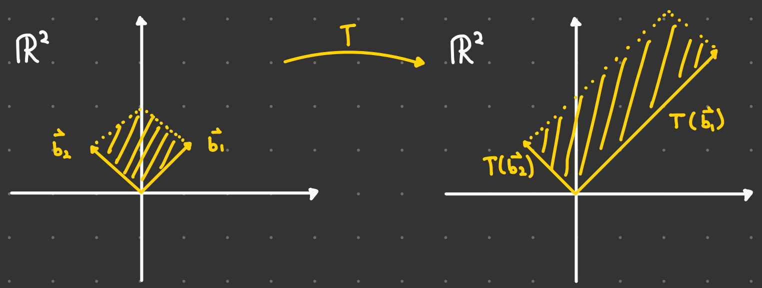
Ex 1

$$B = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$$[T]_B = \begin{bmatrix} | & | \\ \left[T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_B & \left[T\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_B \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad T(\vec{b}_1) = 3\vec{b}_1$$

$$T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad T(\vec{b}_2) = 1\vec{b}_2$$



TUT 9:

What is the connection between $[T]_\mathcal{E} = A$ and $[T]_\mathcal{B} = B$?

$$A[\vec{v}]_\mathcal{E} = [T(\vec{v})]_\mathcal{E} \qquad B[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B}$$

Given $[\vec{v}]_\mathcal{B}$, can I use A to find $[T(\vec{v})]_\mathcal{B}$

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix}$$ Change of basis matrix

$$S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_\mathcal{B} = [\vec{v}]_\mathcal{E}$$

$$S_{\mathcal{B}\to\mathcal{E}}^{-1} S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_\mathcal{B} = S_{\mathcal{B}\to\mathcal{E}}^{-1}[\vec{v}]_\mathcal{E}$$ $S_{\mathcal{B}\to\mathcal{E}}$ must be invertible

$$[\vec{v}]_\mathcal{B} = S_{\mathcal{B}\to\mathcal{E}}^{-1}[\vec{v}]_\mathcal{E}$$

Let $S_{\mathcal{E}\to\mathcal{B}} := S_{\mathcal{B}\to\mathcal{E}}^{-1}$

Note:

$$\underbrace{S_{\mathcal{B}\to\mathcal{E}}^{-1} \underbrace{A \underbrace{S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_\mathcal{B}}_{[\vec{v}]_\mathcal{E}}}_{[T(\vec{v})]_\mathcal{E}}}_{[T(\vec{v})]_\mathcal{B}}$$

So

$$S_{\mathcal{B}\to\mathcal{E}}^{-1} A S_{\mathcal{B}\to\mathcal{E}}[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B}$$

$$B[\vec{v}]_\mathcal{B} = [T(\vec{v})]_\mathcal{B}$$

$$\Rightarrow S_{\mathcal{B}\to\mathcal{E}}^{-1} A S_{\mathcal{B}\to\mathcal{E}} = B$$

Ex 1 continued...

$$S_{B\to\varepsilon} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B = S^{-1}AS \Rightarrow SBS^{-1} = A$$

Ex 2:

$$A^{2021} = (SBS^{-1})^{2021}$$

$$= (SB\overbrace{\cancel{S^{-1})(S}}^{I_n}BS^{-1}) \dots (SBS^{-1})$$

$$= SB^{2021}S^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{2021} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$=$$

$$B^{2021} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}^{2021} = \begin{bmatrix} 3^{2021} & 0 \\ 0 & 1^{2021} \end{bmatrix}$$

Try: $B^2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

Note: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is diagonalizable

Def:

Let A be an $n\times n$ matrix. Let $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$

An $n\times n$ matrix is diagonalizible if there exists a basis $\mathcal{B}$ for $\mathbb{R}^n$ s.t. $[T_A]_{\mathcal{B}}$ is diagonal.

Def:

Let $T: V \to V$. An eigenvector of T is a nonzero vector $\vec{v}$ s.t. $T(\vec{v}) = \lambda\vec{v}$, $\lambda \in \mathbb{R}$.

If $\vec{v}$ is an eigenvector, $T(\vec{v}) = \lambda\vec{v}$, λ is called corresponding eigenvalue to $\vec{v}$.

Def:

An eigenbasis for a L.T. $T: V \to V$ is a basis for V consisting of eigenvectors for T

Ex 3:

$B = \left(\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$

(i) B is a basis for $\mathbb{R}^2$

$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ii) B consists of eigen vectors for T

Ex 4:

$$\begin{aligned} id: V &\longrightarrow V \\ \vec{v} &\longmapsto \vec{v} \end{aligned}$$

$T(\vec{v}) = 1\vec{v}$ eigen value $\lambda = 1$

Every vector is an eigenvector

An eigen basis : $(\vec{e}_1, \dots, \vec{e}_n)$

: Any basis for $\mathbb{R}^n$ is an eigenbasis

Ex 5:

$$\begin{aligned} 0: V &\longrightarrow V \\ \vec{v} &\longmapsto \vec{0} \end{aligned}$$

$T(\vec{v}) = 0\vec{v}$

Every $\vec{v} \in V$ is an eigen vector v

Ex 6:

$$\begin{aligned} T: \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \vec{x} &\longmapsto \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x} \end{aligned}$$

$T(k\vec{e}_1) = kT(\vec{e}_1) = k(3\vec{e}_1) = 3(k\vec{e}_1) \Rightarrow \forall\ 0 \neq \vec{v} \in \text{span}(\vec{e}_1)$ is an eigenvector for $\lambda = 3$

* same for $\text{span}(\vec{e}_2)$, $\lambda = 2$; $\text{span}(\vec{e}_3)$, $\lambda = 0$

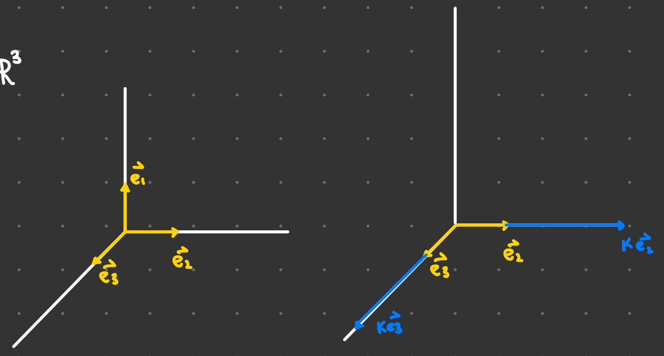
$T(\vec{e}_1) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3\vec{e}_1 \quad \Rightarrow \vec{e}_1$ is an eigen vector , $\lambda = 3$

$T(\vec{e}_2) = 4\vec{e}_2 \quad \Rightarrow \vec{e}_2$ is an eigen vector , $\lambda = 2$

$T(\vec{e}_3) = 0\,\vec{e}_3 \quad \Rightarrow \vec{e}_3$ is an eigen vector , $\lambda = 0$ $\Rightarrow$ T is not injective

$\forall\ \vec{v} \in \ker T$, $T(\vec{v}) = 0\vec{v}$, $\vec{v}$ eigenvector for $\lambda = 0$

$\mathbb{R}^3$



Ex 7:

$$T: C^\infty \longrightarrow C^\infty$$
$$f \longmapsto f'$$

C^∞ : set of smooth functions

: set of functions with derivative of any order

$(e^x)' = 1e^x$

$(e^{kx})' = ke^{kx}$

e^{kx} is an eigen vector for $\lambda = k$

$(2^x)' = \ln 2 \cdot 2^x$

$(c)' = 0 = 0 \cdot c$, $c \in \mathbb{R}$

$c \in \mathbb{R}$ is an eigen vector for $\lambda = 0$

Theorem :

(i) $\vec{v} \neq \vec{0}$ is an eigen vector of $T: V \to V$ iff $\vec{v} \in \ker(T - \lambda \, id)$ for some scalar λ.

Proof (i):

Given $T: V \to V$

Suppose $\vec{v} \neq \vec{0}$ is an eigenvector, $T(\vec{v}) = \lambda \vec{v}$, $\lambda \in \mathbb{R}$

$$T(\vec{v}) = \lambda \vec{v} \Leftrightarrow T(\vec{v}) - \lambda \vec{v} = \vec{0}$$

$$\Leftrightarrow T(\vec{v}) - \lambda \, id(\vec{v}) = \vec{0}$$

$id: \vec{v} \mapsto \vec{v}$

$$\Leftrightarrow (T - \lambda \, id)(\vec{v}) = \vec{0}$$

Define $T - \lambda \, id : V \to V$

$\vec{v} \mapsto T(\vec{v}) - \lambda \, id(\vec{v})$

$$\Leftrightarrow \vec{v} \in \ker(T - \lambda \, id)$$

(ii) $\lambda \in \mathbb{R}$ is an eigenvalue of $T: V \to V$ iff $\mathrm{Ker}(T - \lambda\, id) \neq \{\vec{0}\}$

Special Case: $T: \mathbb{R}^n \to \mathbb{R}^n$
$\vec{x} \mapsto A\vec{x}$

$\mathrm{Ker}(T - \lambda\, id) = \mathrm{Nul}(A - \lambda\, id)$

$(T - \lambda\, id)(\vec{v}) = \vec{0} \Rightarrow T(\vec{v}) - \lambda\, id(\vec{v}) = \vec{0}$
$A\vec{v} - \lambda I(\vec{v}) = \vec{0}$
$(A - \lambda I)(\vec{v}) = \vec{0}$

$0 \neq \vec{v}$ is an $\underset{\text{eigen vector for } A}{\text{eigen vector for } T}$ iff $\vec{v} \in \mathrm{Nul}(A - \lambda I)$

$\lambda \in \mathbb{R}$ is an $\underset{\text{eigen value for } A}{\text{eigen value for } T}$ iff $\mathrm{Nul}(A - \lambda I) \neq \{\vec{0}\}$

Ex 8:

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$
$\vec{x} \longmapsto A\vec{x}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

Q: Is $\lambda = 2$ an eigen value for A?

$\mathrm{Nul}(A - \lambda I) \overset{?}{=} \{0\}$ iff $A - \lambda I$ not invertible $\Rightarrow$ iff $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 2-2 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 3 & 1-2 \end{vmatrix} \end{array}$$

$$= (-1) \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= (-1)(1-1) = 0$$ $\therefore$ $\lambda = 2$ is an eigen value

Q: Is $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ an eigenvector for A?

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \overset{?}{\in} \mathrm{Nul}(A - \lambda I)$$

Instead, we calculate $A\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ Since eigen vectors map to multiple of itself.

$= \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ is an eigen vector for $\lambda = 2$.

Ex 8 continued...

Q: Find all possible eigenvectors for $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}(A - 2I) = \text{span}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Find all eigenvalues of λ:
i.e. $\forall \lambda \in \mathbb{R}$ s.t. $\det(A - \lambda I) = 0$

$$\det \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} =$$ Next class