B24 June 16 Lec 1 Notes

Remark:

$\lambda \in \mathbb{F}$ is an eigenvalue of A iff $\text{Ker}(A-\lambda I)$ is non-trivial (i.e. $\ker(A-\lambda I) \neq \{0\}$), in which case $\text{Ker}(A-\lambda I)$ is the eigenspace.

Since $A-\lambda I$ is invertible iff $\text{Ker}(A-\lambda I) = \{0\}$, we see the eigenvalues of A are precisely the solutions to

$$\det(A-\lambda I) = 0$$

Remark:

In general, if $A: V \to V$ is a L.T., the matrix representation $[A]$ of A depends on a choice of basis, but $\det(A)$ does not.

Indeed, if B and C are both matrices representing L.T. A, then there exists invertible Q with

$$B = QCQ^{-1}$$

$$\begin{aligned} \Rightarrow \det(B) &= \det(QCQ^{-1}) \\ &= \det(Q)\det(C)\det(Q^{-1}) \\ &= \det(Q)\det(C)\det(Q)^{-1} \\ &= \det(C) \end{aligned}$$

Definition:

$\det(A-\lambda I)$ is called the characteristic polynomial of A (here λ is the variable)

Remark:

The roots of characteristic polynomial of A are exactly the eigenvalues of A.

Remark:

$\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ will start making a substantial difference now.

e.g. Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbb{R}^2 \to \mathbb{R}^2$

$\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$.

Since $\lambda^2+1=0$ has no real solutions, $A: \mathbb{R}^2 \to \mathbb{R}^2$ has no eigenvalues

However if we consider $A: \mathbb{C}^2 \to \mathbb{C}^2$, then we have two solutions $\lambda = \pm i$.

Definition:

If λ_0 is an eigenvalue of A, the algebraic multiplicity of λ_0 is the largest positive integer k s.t. $(\lambda-\lambda_0)^k$ divides $\det(A-\lambda I)$.

e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (1-\lambda)^2(3-\lambda)$$

1 is an eigenvalue of A with almu 2, since

$$(1-\lambda)^2 \mid \det(A-\lambda I) \quad \text{but} \quad (1-\lambda)^3 \nmid \det(A-\lambda I$$

3 is an e.v of A with almu 1.

Remark:

Counting multiplicity means that if e.v. λ_0 has multiplicity m, it is counted m times towards the total # of eigenvalues.

e.g. if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\det(A-\lambda I) = (1-\lambda)^2(3-\lambda)$

then A has 2 distinct e.v's, but A has 3 e.vs (counting multiplicity) since the e.v. 1 has multiplicity 2.

Proposition:

Let $\dim V = n$ and $A : V \to V$ where $\mathbb{F} = \mathbb{C}$. Then A has n eigenvalues (counting multiplicity).

Proof:

$\det(A-\lambda I)$ is a complex polynomial of degree n, and therefore has n roots (counting multiplicity)

Definition:

The trace of a square matrix A is defined as the sum of its diagonal entries, and is denoted $\operatorname{tr}(A)$.

Theorem:

Let A be $n \times n$ over $\mathbb{C}$, and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues (counting multiplicities)

Then:

$$\text{(i) } tr(A) = \lambda_1 + \cdots + \lambda_n$$
$$\text{(ii) } \det(A) = \lambda_1 \cdot \ldots \cdot \lambda_n$$

Proof: (ii)

$\det(A - \lambda I) = (v_1 - \lambda) \cdot \ldots \cdot (v_n - \lambda)$ where $v_1, \dots, v_n \in \mathbb{C}$ are roots of $\det(A - \lambda I)$, hence each v_i is an e.v. λ_i of A, and $(\lambda_i - \lambda)$ appears multiplicity of λ_i many times in $(v_1 - \lambda) \cdot \ldots \cdot (v_n - \lambda)$. Plugging in $\lambda = 0$ to

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$$

yields the result.

Theorem:

Let A be an $n \times n$ matrix (over $\mathbb{R}$ or $\mathbb{C}$) Then there exists a diagonal matrix D and an invertible matrix S s.t.

$$A = SDS^{-1}$$

iff there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

Proof ($\Rightarrow$):

Assume there exists a diagonal matrix D and an invertible matrix S s.t.

$$A = SDS^{-1}$$

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

then $AS \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = SDS^{-1}S \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = SD \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = S\lambda_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_1 S \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

i.e. $S \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is an eigenvector of A, and similarly $Se_2, \dots, Se_n$ are e.v's of A. Since S is invertible, $Se_1, \dots, Se_n$ form a basis for $\mathbb{F}^n$. $\boxtimes$

Proof ($\Leftarrow$):

Assume there is a basis $v_1, \dots, v_n$ for $\mathbb{F}^n$ consisting of eigenvectors of A.

then $[A]^{v_1,\dots,v_n}_{v_1,\dots,v_n} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, where λ_i is the e.v. corresponding to the eigenvector v_i.

and $[A]^{e_1,\dots,e_n}_{e_1,\dots,e_n} = A$

So with $S = [I]^{v_1,\dots,v_n}_{e_1,\dots,e_n}$, we have

$$A = S \, [A]^{v_1,\dots,v_n}_{v_1,\dots,v_n} \, S^{-1}$$

$$= [I]^{\cancel{v_1,\dots,v_n}}_{e_1,\dots,e_n} \; [A]^{\cancel{v_1,\dots,v_n}}_{\cancel{v_1,\dots,v_n}} \; [I]^{e_1,\dots,e_n}_{\cancel{v_1,\dots,v_n}}$$

$$= [A]^{e_1,\dots,e_n}_{e_1,\dots,e_n}$$

$\boxtimes$