

We can use theorem 1.3.10 to show that  $T(\vec{x}) = B(A\vec{x})$  is linear.

Since T is linear,  $T(\vec{e_1}) = B(A\vec{e_1})$ ;  $T(\vec{e_2}) = B(A\vec{e_2})$ ; ...;  $T(\vec{e_n}) = B(A\vec{e_n})$ .

The matrix of  $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$ .

Def 2.3.1 Matrix Multiplication

- (a) Let B be an  $n \times p$  matrix and A a  $q \times m$  matrix. The product BA is defined iff p=q.
- (b) If B is an  $n \times p$  matrix and A a  $p \times m$  matrix, then the product BA is defined as the matrix of the L.T.  $T(\vec{x}) = B(A\vec{x})$ . This means that  $T(\vec{x}) = B(A(\vec{x})) = (BA)\vec{x}$ , for all  $\vec{x}$  in the vector space  $R^m$ . The product BA is an  $n \times m$  matrix.

Theorem 2.3.2: The columns of the matrix product

Let B be an  $n \times p$  matrix and A a  $p \times m$  matrix with columns  $\vec{v}_1, \vec{v}_2, ... \vec{v}_m$ . Then , the product BA is

$$BA = B \begin{bmatrix} 1 & 1 & 1 \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ B \vec{v_1} & B \vec{v_2} & \dots & B \vec{v_m} \end{bmatrix}$$

To find BA, we can multiply B by the columns of A and combine the resulting vectors.

Theorem 2.3.3 Matrix multiplication is noncommutative

AB \* BA, in general However, at times it does happen that AB = BA; then we say that the matrices A and B commute.

Theorem 2.3.4: The entries of the matrix product

Let B be an nxp matrix and A a pxm matrix. The ijth entry of BA is the dot product of the ith row of B with the jth column of A.

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

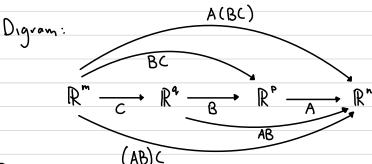
is the nxm matrix whose ijth entry is

Theorem 2.3.5: Multiplying with the identity matrix

For an nxm matrix A,

Theorem 2.3.6: Matrix Multiplication is associative.

We can simply write ABC for the product (AB) C = A(BC)



Proof:

$$T(\vec{x}) = ((AB)C)\vec{x}$$
;  $L(\vec{x}) = (A(BC))\vec{x}$ 

By def of matrix multiplication,

$$T(\hat{x}) = ((AB)(\hat{x}) = (AB)((\hat{x}) = A(B((\hat{x})))$$

$$L(\hat{x}) = (A(BC))\hat{x} = A((BC)\hat{x}) = A(B((\hat{x}))$$

Thus the matrix of Tand L are equivalent.

Theorem 2.3.7: Distributive property for matrices

If A and B are nxp matrices, and C and D are pxm matrices, then

$$A(C+D) = AC+AD$$
, and  $(A+B)C = AC+BC$ 

Theorem 2.38:

If A is an nxp matrix, B is a pxm matrix, and K is a scalar, then (KA)B = A(KB) = K(AB)

## Def 2.3.10: Regular transition matrices

A transition matrix is said to be positive if all its entries are positive.

A transition matrix is said to be regular if the matrix Amis positive for some positive integer m.

Theorem 2.3.11: Equilibria for regular transition matrices.

Let A be a regular transition matrix of size nxm.

- (a) There exists exactly one distribution vector  $\vec{x}$  in  $\mathbb{R}^n$  s.t.  $A\vec{x}=\vec{x}$ . This is called the equilibrium distribution for A, denoted  $\vec{x}$  equ. All the components of  $\vec{x}$  equ. are positive.
- (b) If  $\vec{x}$  is any distribution vector in  $\mathbb{R}^n$ , then  $n \neq \infty$  ( $A^m \vec{x}$ )  $= \vec{x} \in \mathbb{R}^n$
- (c)  $\lim_{m\to\infty} A^m = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ , which is the matrix whose columns are