Theorem: Properties of Convergent Sequences

Let $a, b \in \mathbb{R}$. Let $\{a_n\}$, $\{b_n\}$ be sequences

If $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then

(i) $\{a_n + b_n\}$ converges to $a+b$.

i.e. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$

(ii) For any constant $k \in \mathbb{R}$, $\{k a_n\}$

Proof (i):

Suppose ① $a_n \to a$ as $n \to \infty$
② $b_n \to b$ as $n \to \infty$

WTS $\{a_n + b_n\}$ conv. to $a+b$

WTS $\forall \varepsilon > 0$, $\exists N > 0$ s.t. if $n > N$ then $|a_n + b_n - (a+b)| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

① $\Rightarrow \exists N_1 > 0$ s.t. if $n > N_1$ then $|a_n - a| < \underline{\varepsilon/2}$
② $\Rightarrow \exists N_2 > 0$ s.t. if $n > N_2$ then $|b_n - b| < \underline{\varepsilon/2}$

Choose $N = \underline{\max\{N_1, N_2\}} > 0$

Suppose $n > N$,

$$\begin{aligned} \text{then } |a_n + b_n - (a+b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \quad // \end{aligned}$$

Theorem Pg 596:

Let $\{a_n\}$ be a sequence.

If $\{a_n\}$ converges, then

(i) the limit is unique
(ii) $\{a_n\}$ is bounded
i.e $\{a_n\}$ is bounded above and bounded below
i.e $\exists C \in \mathbb{R}^+$ s.t. $|a_n| \leq C$, $\forall n \in \mathbb{N}$

Proof (i):

Suppose $\{a_n\}$ converges to both $\ell_1 \in \mathbb{R}$ and $\ell_2 \in \mathbb{R}$
① ②

WTS $\ell_1 = \ell_2$

① $\Rightarrow \exists N_1 > 0$ s.t. if $n > N_1$ then $|a_n - \ell_1| < \underline{\varepsilon/2}$

$\Leftrightarrow \ell_1 - \ell_2 = 0$

② $\Rightarrow \exists N_2 > 0$ s.t. if $n > N_2$ then $|a_n - \ell_2| < \underline{\varepsilon/2}$

$\Leftrightarrow$ WTS $\forall \varepsilon > 0$, $|\ell_1 - \ell_2| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary

$$
\begin{aligned}
|\ell_1 - \ell_2| &= |\ell_1 + 0 - \ell_2| \\
&= |\ell_1 + a_n - a_n - \ell_2| \\
&= |a_n - \ell_2 - (a_n - \ell_1)| \\
&\leq |a_n - \ell_2| + |a_n - \ell_1| && \text{By triangle inequality} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{provided } n > \max\{N_1, N_2\} \\
&= \varepsilon \; //
\end{aligned}
$$