Proof Practice

1. $f(n) = n^2 + n + 2,\ n \in \mathbb{N}$. Show that $f(n)$ is always even.

$f(n) = n(n+1) + 2$

$n(n+1)$ are consecutive integers, thus one is even, and the other is odd. The product of an even and odd integer is always even.

Let $n(n+1) = 2m\ ,\ m \in \mathbb{Z}$

$f(n) = 2m + 2 = 2(m+1) \Rightarrow 2(k)\ ,\ k \in \mathbb{Z}$.

Thus $f(n)$ is always even.

2. Prove that when the square of a positive odd integer is divided by 4 the remainder is always 1.

Let $2K+1$ be a positive odd integer for $K = 0, 1, 2, \ldots$

$(2K+1)^2 = 4K^2 + 4K + 1 = 4(K^2+K) + 1 \Rightarrow 4m + 1\ ,\ m \in \mathbb{Z}$

3. Show that $a^3 - a + 1$ is odd for all positive integer values of a.

$f(n) = n^3 - n + 1\ ,\ n \in \mathbb{N}$
$= n(n^2-1) + 1$
$= n(n-1)(n+1) + 1$
$\Rightarrow 2K + 1\ ,\ K \in \mathbb{Z}$

Thus $f(n)$ is odd.

One Even, Two odd
$= 2n(2n+1)(2n+1)$
$= (4n^2 + 2n)(2n+1)$
$= 8n^3 + 4n^2 + 4n^2 + 2n$
$= 8n^3 + 8n^2 + 2n$
$= 2(4n^3 + 4n^2 + n) \Rightarrow 2k\ ,\ k \in \mathbb{Z}$

One Odd, Two even
$= (2n+1)(2n)(2n)$
$= (2n+1)(4n^2)$
$= 8n^3 + 4n^2 \Rightarrow 2(4n^3 + 2n^2) \Rightarrow 2k,\ K \in \mathbb{Z}$

4. Prove that the square of a positive integer can never be of the form $3k+2$, $k \in \mathbb{N}$.

Proof by exhaustion

The number, let's say a, can take one of the following forms:

$$a = 3m, \quad a = 3m+1, \quad a = 3m+2, \quad m \in \mathbb{N}$$

Case 1: $a = 3m$

$$a^2 = 9m^2 = 3(3m^2) \Rightarrow 3k,\ k \in \mathbb{N}$$

Case 2: $a = 3m+1$

$$a^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1 \Rightarrow 3k+1,\ k \in \mathbb{N}$$

Case 3: $a = 3m+2$

$$a^2 = (3m+2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1 \Rightarrow 3k+1,\ k \in \mathbb{N}$$

$\therefore$ Squaring any integer only produces integers of the form $3k$ OR $3k+1$, $k \in \mathbb{N}$

$\therefore$ It is not possible to have a square number of the form $3k+2$, $k \in \mathbb{N}$

5. Disprove $|2x+1| \leq 5 \Rightarrow |x| \leq 2$ with a counter example.

$x = -3 \Rightarrow |2(-3)+1| \leq 5 \Rightarrow |-3| \leq 2$

$|-5| \leq 5 \Rightarrow |-3| \leq 2$

6. Prove by contradiction that for all real θ

$$\cos\theta + \sin\theta \leq \sqrt{2}$$

$\cos\theta + \sin\theta > \sqrt{2}$

$(\cos\theta + \sin\theta)^2 > (\sqrt{2})^2$

$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta > 2$

$1 + 2\sin\theta\cos\theta > 2$

$2\sin\theta\cos\theta > 1$

$\sin(2\theta) > 1$

But this is a contradiction as $\sin 2\theta \leq 1$.
Thus $\cos\theta + \sin\theta \leq \sqrt{2}$.

7. Prove by contradiction that if p and q are positive integers, then

$$\frac{p}{q} + \frac{q}{p} \geq 2$$

Contradiction: $\frac{p}{q} + \frac{q}{p} < 2$

$$= \frac{p^2 + q^2}{pq} < 2$$
$$= p^2 + q^2 < 2pq$$
$$= p^2 + q^2 - 2pq < 0$$
$$= (p-q)^2 < 0$$

This is a contradiction as a squared quantity is negative.
Thus $\frac{p}{q} + \frac{q}{p} \geq 2$

8. With out using induction, show that f(n) is a multiple of 8.

$$f(n) = 5^{2n} - 1 \quad , n \in \mathbb{N}$$

Solution 1:

$$5^{2n} - 1 = (5^1 - 1)(5^{2n-1} + 5^{2n-2} + \ldots + 5^{2(1)} + 5^0)$$
$$= 24\,(5^{2n-1} + 5^{2n-2} + \ldots + 5^2 + 5^0)$$
$$= 8\,(3)(5^{2n-1} + 5^{2n-2} + \ldots + 5^2 + 5^0)$$

Solution 2:

$f(n) = 5^{2n} - 1 = (5^n - 1)(5^n + 1)$
Since 5^n is always odd (ends in 5), $5^n - 1$ and $5^n + 1$ are both even.
Thus one of them is divisible by 4.

Let $5^n - 1 = 2a \quad a \in \mathbb{N}$
$5^n + 1 = 4b \quad b \in \mathbb{N}$

Then we have:

$$f(n) = (5^n - 1)(5^n + 1) = (2a)(4b)$$
$$= 8ab.$$

Thus f(n) is a multiple of 8.

9. Prove by contradiction that for all real x,

$$(13x+1)^2+3 > (5x-1)^2$$

Contradiction: $(13x+1)^2+3 \le (5x-1)^2$

$$169x^2+26x+1+3 \le 25x^2-10x+1$$
$$144x^2+16x+3 \le 0$$
$$144x^2+16x \le -3$$
$$(12x+\tfrac{3}{2})^2-\tfrac{9}{4}+3 \le 0$$
$$(12x+\tfrac{3}{2})^2+\tfrac{3}{4} \le 0$$

Thus contradiction.

10. It is given that $N = K^2-1$ and $K = 2^P-1$, $p \in \mathbb{N}$
use a direct proof to show that 2^{P+1} is a factor of N.

$$N = (2^P-1)^2-1$$
$$= (2^{2P}-2\cdot 2^P+1)-1$$
$$= 2^{2P}-2^{P+1}$$
$$= (2^{P+1})\left(\frac{2^{2P}}{2^{P+1}}-1\right)$$
$$= 2^{P+1}(2^{2P-P-1}-1)$$
$$= 2^{P+1}(2^{P-1}-1)$$

Thus 2^{P+1} is a factor of N.

11. Prove by exhaustion that if n is a positive integer that is not divisible by 3, then n^2-1 is divisible by 3.

Let n be in the forms:

$n = 3m+1$, $n = 3m+2$ $m \in \mathbb{Z}$

Therefore $3 \nmid n \rightarrow 3 \mid n^2-1$ is true by proof of exhaustion.

Case 1: $n = 3m+1$

$$n^2-1 = (3m+1)^2-1 = 9m^2+6m+1-1 = 3(3m^2+2m) \Rightarrow 3K, \ K \in \mathbb{Z}$$

Case 2: $n = 3m+2$

$$n^2-1 = (3m+2)^2-1 = 9m^2+12m+4-1 = 3(3m^2+4m+1) \Rightarrow 3K, \ K \in \mathbb{Z}$$

12. Prove that if we subtract 1 from a positive odd square number, the answer is always divisible by 8.

$\forall n \in \mathbb{N}$, n^2 is odd $\rightarrow$ $8 \mid n^2 - 1$

Let n be a positive arbitrary integer.

Suppose that n^2 is odd.

If n^2 is odd, then we know that n has to be odd.

By definition:

$$n = 2K+1 \quad , K \in \mathbb{Z}$$

Then :

$$n^2 - 1 = (2K+1)^2 - 1$$
$$= 4K^2 + 4K + 1 - 1$$
$$= 4K(K+1)$$

Since $K(K+1)$ is the product of two consecutive integers, then $K(K+1)$ is even.

Then :

$$4K(K+1) = 4(2m) \quad , m \in \mathbb{N}$$

Since n is a positive arbitrary integer.

13. Given that $K > 0$, use algebra to show that

$$\frac{K+1}{\sqrt{K}} \geq 2$$

Contradiction :

$$K+1 < 2\sqrt{K}$$
$$K+1 - 2\sqrt{K} < 0$$

let $u^2 = K$

$$u^2 - 2u + 1 < 0$$
$$(u-1)^2 < 0$$
$$(\sqrt{K}-1)^2 < 0$$

Thus $\frac{K+1}{\sqrt{K}} \geq 2$ is true.

14. Prove by the method of contradiction that there are no integers n and m which satisfy the following equation.

$$3n + 21m = 137$$

Contradiction: There ARE integers n and m that satisfy this.

$$3n + 21m = 137$$
$$3(n+7m) = 137$$
$$n+7m = \frac{137}{3}$$
$$n+7m = 45\tfrac{2}{3}$$

n is an integer and 7m is an integer. This implies that n+7m is an integer.

This is a contradiction as $45\frac{2}{3}$ is not an integer.
Thus $3n+21m = 137$ cannot be satisfied by any integers n, m.

15. Use proof by contradiction to show that if x then $\left|x+\frac{1}{x}\right| \geq 2$

Assume the contradiction : $\left|x+\frac{1}{x}\right| < 2$

$$\left|x+\frac{1}{x}\right|^2 < (2)^2$$
$$x^2 + 2 + x^{-2} < 4$$
$$x^2 + x^{-2} - 2 < 0$$
$$(x - \tfrac{1}{x})^2 < 0$$

But this is a contradiction as any quantity squared cannot be negative.
Thus $\left|x+\frac{1}{x}\right| \geq 2$.

16. Prove that the sum of two even consecutive powers of 2 is a multiple of 20.

Powers of 2: 1, 2, 4, 8, 16, 32, 64

Sum of 2 even consecutive powers of 2 = $2^{2n} + 2^{2n+2}$

$$= 2^{2n} + 4\cdot 2^{2n} = (2^{2n})(1+4) = (4^n)5 \Rightarrow 5\cdot 4K, \; K \in \mathbb{N}$$
$$= 20K$$

17. Prove that there are no integers a and b which satisfy the following equation.

$$a^2 - 8b = 7$$

Contradiction: $a^2 - 8b = 7$ for some integers a and b.

Then: $a^2 = 8b + 7$

$a^2 = 8b + 6 + 1$

$a^2 = 2(4b+3)+1 \Rightarrow a^2$ is odd $\Rightarrow$ a is odd

$a^2 = (2K+1)^2 = 8b+7$

$4K^2+4K+1 = 8b+7$

$4(K^2+K-2b) = 6$

$K^2+K-2b = \frac{3}{2}$

a^2 must be an integer but we have shown that it is $\frac{3}{2}$.
Thus we have a contradiction and $a^2-8b=7$ does not have integers a and b that satisfy it.

18. Show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}$, then $m^2 - n^2 \neq 102$

Case 1: m even, n even

$m = 2K$; $n = 2j$

$(2K)^2 - (2j)^2 \neq 102$

$4K^2 - 4j^2 \neq 102$

$2(2(K^2-j^2)) \neq 102$

$2(K^2-j^2) \neq 51 \Rightarrow 2i \neq 51$, $i \in \mathbb{N}$

Case 2: m even, n odd OR m odd, n even

$m = 2K$; $n = 2j+1$

$(2K)^2 - (2j+1)^2 \neq 102$

$4K^2 - (4j^2+4j+1) \neq 102$

$4K^2 - 4j^2 - 4j - 1 \neq 102$

$4(K^2-j^2-j) \neq 103$

$K^2-j^2-j \neq 25\frac{3}{4} \Rightarrow K^2-j^2-j$ is an integer and cannot be fraction.

Case 3: m odd, n odd

$m = 2K+1$; $n = 2j+1$

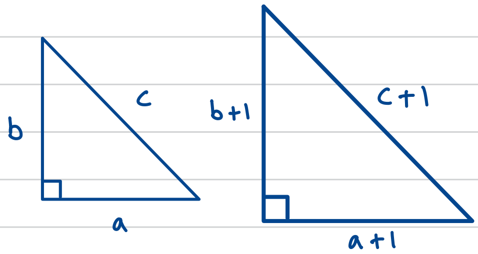
$(2K+1)^2 - (2j+1)^2 \neq 102$

$4K^2+4K+1 - (4j^2+4j+1) \neq 102$

$4K^2+4K - 4j^2 - 4j \neq 102$

$K^2+K-j^2-j \neq 25\frac{1}{2}$

19. Show that a, b, and c cannot all be integers.



$$(a+1)^2 + (b+1)^2 = (c+1)^2$$
$$a^2+2a+1 + b^2+2b+1 = c^2+2c+1$$
$$(2a+1)+(2b+1) = 2c+1$$
$$2(a+b)+1 = 2c \Rightarrow 2k+1 = 2c \quad , k \in \mathbb{N}$$

$2(a+b)+1$ is odd, but $2c$ is even.

Thus not all of a, b, and c are integers.

21. It is given that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x+y=1$.
Prove that:

$$x^2+y = y^2+x$$

$$\cancel{y = 1-x \Rightarrow x^2+(1-x) = (1-x)^2+x}$$
$$\cancel{\Rightarrow x^2+1-x = 1-2x+x^2+x}$$
$$\cancel{\Rightarrow 1 = 1}$$

Method 1:

$$\Rightarrow f(x,y) = x^2-y^2+y-x$$
$$\Rightarrow f(x,y) = (x^2-y^2)-(x-y)$$
$$\Rightarrow f(x,y) = (x-y)(x+y)-(x-y)$$
$$\Rightarrow f(x,y) = (x-y)(1)-(x-y)$$
$$\Rightarrow f(x,y) = 0$$

$$x^2-y^2+y-x = 0$$
$$x^2+y = y^2+x$$

Method 2

Case 1: $x = y = \frac{1}{2}$

$$x^2+y = \left(\tfrac{1}{2}\right)^2+\tfrac{1}{2} = \tfrac{1}{4}+\tfrac{1}{2} = \tfrac{3}{4}$$
$$y^2+x = \left(\tfrac{1}{2}\right)^2+\tfrac{1}{2} = \tfrac{3}{4}$$

Case 2: $x \neq y$, so $x-y \neq 0$

$$x+y=1$$
$$(x+y)(x-y) = 1(x-y)$$
$$x^2-y^2 = x-y$$
$$x^2+y = y^2+x$$

22. It is given that a and b are positive odd integers, with $a > b$. Show that if $a+b$ is a multiple of 4, then $a-b$ cannot be a multiple of 4.

$a+b = 4m \ , m \in \mathbb{N}$

Assume for contradiction that:

$a-b = 4n \ , n \in \mathbb{N}$

$a - (4m - a) = 4n$

$a - 4m + a = 4n$

$2a = 4n + 4m$

$a = 2(n+m) \Rightarrow a = 2K, \ K \in \mathbb{N}$

But this is a contradiction as a is even and odd.

Thus $a+b$ is a multiple of 4 $\rightarrow$ $a-b$ is not multiple of 4.