B52 Nov 5 Lec 2 Notes

Conditional PDF

$f_{X|Y}(x|y)$ of X given $Y=y$ is proper PDF, i.e.

$$f_{X|Y}(x|y) \geq 0 \quad \& \quad \int_{\mathbb{R}} f_{X|Y}(x|y)\,dx = 1, \ \forall y$$

$f_{X|Y}$ can be integrated to find conditional probabilities:

$$P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx \quad , \text{ for } A \subseteq \mathbb{R}$$

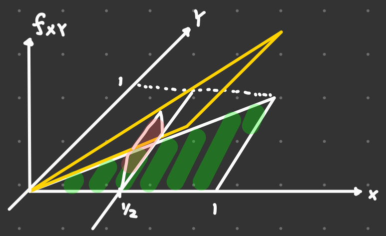
e.g. conditional CDF $F_{X|Y}(x|y) = P(X \leq x \mid Y=y) = \int_{-\infty}^{x} f_{X|Y}(t|y)\,dt$

$f_{X|Y}(x|y)$ can also be used to define joint PDFs

$$f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y|x) \quad , \ \forall x,y$$

Ex 1:

Let X,Y have joint pdf $f(x,y) = \begin{cases} 3x, & 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$



$f_{Y|X}(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{3x}{3x^2} = \dfrac{1}{x}, \ \forall\ 0 \leq y \leq x$

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \\ &= \int_0^x 3x\,dy \\ &= 3x^2 \end{aligned}$$

$\Rightarrow f_{Y|X}(y|\tfrac{1}{2}) = \dfrac{1}{\tfrac{1}{2}} = 2$

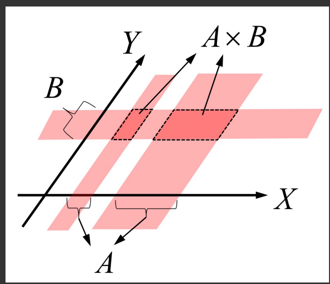
Independent RVs

RVs X,Y are independent (denoted $X \perp Y$) if $\forall A,B \subseteq \mathbb{R}$

$$P(X \in A, Y \in B) = P(X \in A)\,P(Y \in B)$$

Equivalent to CDF / PMF / PDF factorization

$$X \perp Y \iff \left\{ \begin{array}{ll} F_{X,Y}(x,y) = F_X(x)F_Y(y) & \text{(any)} \\ f_{X,Y}(x,y) = f_X(x)\, f_Y(y) & \text{(continuous)} \\ p_{X,Y}(x,y) = p_X(x)\, p_Y(y) & \text{(discrete)} \end{array} \right\}$$

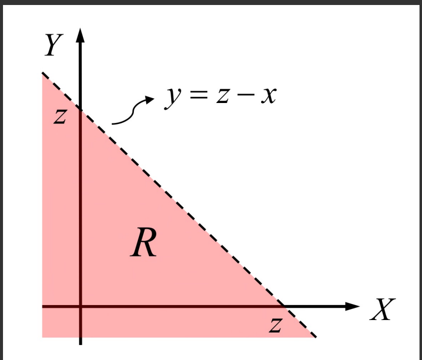


Sums of RVs

Sums of RVs $X_1 + X_2 + (+X_3 + \dots)$ is transformation of particular importance.

Consider RVs X, Y with joint PMF/PDF and let $Z = X+Y$; find CDF of Z as

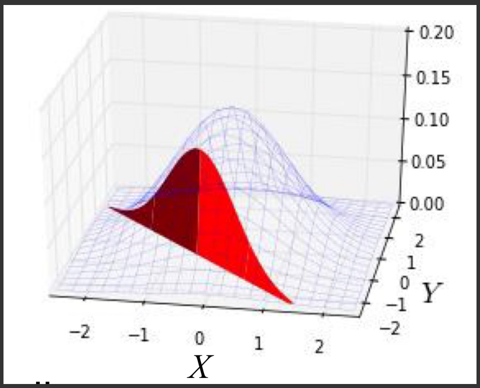
$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$
$$= P(Y \le z - X) = \begin{cases} \iint_R f_{X,Y}(x,y)\,dx\,dy \\ \sum\sum_R p_{X,Y}(x,y) \end{cases}$$



The Convolution Method finds PDF of Z directly and $Z = X+Y$, PDF of Z is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\,dy$$

Integral represents area of joint PDF slice along line $x+y=z \Leftrightarrow y = z-x$



Similarly for discrete RVs:

$$p_Z(z) = \sum_x p_{X,Y}(x, z-x) = \sum_y p_{X,Y}(z-y, y)$$

Ex 2:

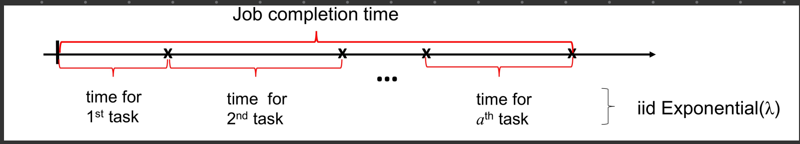
Let $Z = X+Y$ be the sum of two independent and identically distributed (IID) $\text{Exp}(\lambda)$ RVs X, Y. Find the PDF of Z.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})$$
$$= \lambda^2 e^{-\lambda(x+y)}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx$$
$$= \cancel{\int_{-\infty}^{0} 0\,dx}^{\,0} + \int_0^z \lambda^2 e^{-\lambda(x+z-x)}\,dx + \cancel{\int_z^{\infty} 0\,dx}^{\,0}$$
$$= \lambda^2 e^{-\lambda z} \int_0^z dx$$
$$= \lambda^2 e^{-\lambda z}(z-0)$$
$$= \lambda^2 z e^{-\lambda z}, \quad \forall z > 0$$

Gamma Distribution

Consider a job consisting of #a tasks, completed sequence according to independent $Exp(\lambda)$ times



Total job completion time (sum of IID exponentials) follows Gamma distribution.

PDF of Gamma Distribution is $f(x) = \begin{cases} \frac{1}{\Gamma(a)} \lambda^a x^{a-1} e^{-\lambda x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$

↳ Parameters $a, \lambda > 0$ where $a, \lambda \in \mathbb{R}$

↳ Denoted $X \sim Gamma(a, \lambda)$

Defined in terms of gamma function $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, dx$

↳ For $a > 1 \Rightarrow \Gamma(a) = (a-1)\Gamma(a-1)$

↳ For integer $a \Rightarrow \Gamma(a) = (a-1)!$

There is no closed-form CDF (no formula)

↳ Special case $a = 1 \Leftrightarrow Exp(\lambda)$

From Ex 2, $f_Z(z) = \lambda^2 z e^{-\lambda z} \sim Gamma(a = 2, \lambda)$