B41 Nov 1 Lec 1 Notes

Theorem: Mean Value Theorem

Let f be differentiable on its domain, including the line joining a and b, then

$$f(b) - f(a) = Df(x)(b-a)$$

for some x on the joining a to b.

If $Df(x) = 0$, $\forall x$, then f is a constant.

Ex 1:

Let $f(x,y) = x^2 + y^2$. Verify that the MVT for $a = [1,1]$ and $b = [2,3]$.

$Df(x,y) = \nabla f(x,y) = (2x, 2y)$

$$\begin{aligned} Df(x,y)(b-a) &= (2x, 2y)\cdot(2-1, 3-1) \\ &= 2x + 4y \end{aligned}$$

$f(b) - f(a) = 13 - 2 = 11$

The line connecting a and b is $\frac{y-1}{x-1} = \frac{3-1}{2-1} = 2$, i.e. $y = 2x - 1$

Thus $2x + 8x - 4 = 11$

$10x = 15 \Rightarrow x = 3/2$, $y = 2$

So $z = (3/2, 2)$

$\nabla f(3/2, 2) = (3, 4)$

$11 = f(b) - f(a) = \nabla f(3/2, 2)\cdot(2-1, 3-1) = (3,4)\cdot(1,2)$

Definition:

A quadratic form is a degree 2 homogenous polynomial function.

$$f(x) = f(x_1, x_2, \cdots, x_n) = \sum_{i \le j,\, i,j=1}^{n} u_{ij} x_i x_j \text{, where not all } u_{ij} \text{ are zero.}$$

$f(x)$ can be written as

$$f(x) = x^T A x = [x_1, x_2, \cdots, x_n] \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Ex 2:

Find a change of variables that will reduce the quadratic form

$$f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 6x_1x_3 + 3x_2^2 + 2x_2x_3 + x_3^2$$

to a sum of squares, and express the quadratic form in terms of the new variables.

$$f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 6x_1x_3 + 3x_2^2 + 2x_2x_3 + x_3^2$$

$$= [x_1, x_2, x_3]\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x^T A x$$

We want $at_1^2 + bt_2^2 + ct_3^2$

$$= [t_1, t_2, t_3]\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

So we have to diagonalize A.

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 5-\lambda & 5-\lambda & 5-\lambda \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & -2 & -2-\lambda \end{vmatrix} = \dots = (5-\lambda)(2-\lambda)(-2-\lambda)$$

Then find eigenvectors: $\lambda_1 = 5$, $v_1 = [1,1,1]$

$\lambda_2 = 2$, $v_2 = [1,-2,1]$

$\lambda_3 = -2$, $v_3 = [-1,0,1]$

Normalize the eigenvectors: $q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $q_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Let $C = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$ an orthogonal matrix

Then $C^TAC = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D$

Substituting $x = Ct$ into $f(x)$, where $t = [t_1, t_2, t_3]$ into the function to obtain

$$f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 6x_1x_3 + 3x_2^2 + 2x_2x_3 + x_3^2$$

$$= x^T A x$$

$$= (Ct)^T A (Ct)$$

$$= t^T C^T A C t$$

$$= t^T D t$$

$$= 5t_1^2 + t_2^2 - 2t_3^2$$

Theorem: The Principle Axis Theorem

If A is a symmetric $n \times n$ matrix, then there is an orthogonal matrix C that transforms the quadratic form $x^T A x$ into a quadratic form.

$$t^T D t = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \dots + \lambda_n t_n^2$$ with no product terms, where $x = Ct$

Definition:

A quadratic form $f(x) = x^T A x$ is said to be

(i) Positive definite if $f(x) > 0$ for $x \neq 0$
(ii) Negative definite if $f(x) < 0$ for $x \neq 0$
(iii) Indefinite if $f(x)$ has both positive and negative values.

Theorem:

If A is a symmetric matrix, then

(i) $f(x) = x^T A x$ is positive definite iff all eigenvalues of A are positive.
(ii) $f(x) = x^T A x$ is negative definite iff all eigenvalues of A are negative.
(iii) $f(x) = x^T A x$ is indefinite iff A has at least one positive eigenvalue and at least one negative eigenvalue.

Theorem:

The quadratic form given by a symmetric matrix A is positive definite iff the determinant of A_k, every $k \times k$ submatrix containing the first k rows and the first k columns is positive.

Corollary:

The quadratic form given by a symmetric $n \times n$ matrix A is negative definite iff the sign of $\det(A_k)$ is given by $(-1)^k$.

Ex 3:

Let $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$

$A_1 = [2]$, $\det A_1 > 0$

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\det A_2 > 0$

$A_3 = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$, $\det A_3 > 0$

$\therefore$ By the theorem above, A is positive definite.