A37 March 22 Lec 1 Notes

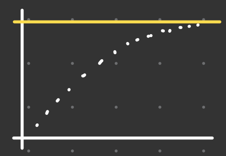
Theorem: Bounded Monotone Conv. Theorem (BMCT)

Let $\{a_n\}$ be a sequence

If $\{a_n\}$ is monotone and bounded, then $\{a_n\}$ converges.

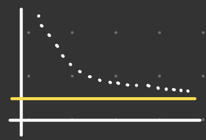
In particular,

(i) increasing and bounded above



OR

(ii) decreasing and bounded below



Proof (i):

Suppose $\{a_n\}$ is strictly increasing (1) and $\{a_n\}$ is bounded above. (2)

WTS $\{a_n\}$ converges.

WTS $\exists \ell \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ if $n > N$ then $|a_n - \ell| < \varepsilon$

Define $A = \{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$

Observe $A \neq \emptyset$ b/c $a_1 \in A$

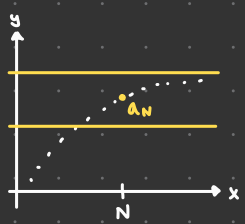
Moreover, A is bounded above by (2)

$\therefore$ By Completeness axiom, $\sup(A)$ exists. i.e. $\sup(A) \in \mathbb{R}$ $=: \alpha$

Choose $\ell = \underline{\ \alpha\ } \in \mathbb{R}$

Let $\varepsilon > 0$ be arbitrary.

Choose $N \in \mathbb{N}$ s.t. $\alpha - \varepsilon < a_N$



Suppose $n > N$,

Then

$$\alpha - \varepsilon < a_N < a_n \le \alpha \quad \text{since } \alpha = \sup(A)$$
since $n > N$
$$< \alpha + \varepsilon$$

$\Rightarrow \alpha - \varepsilon < a_n < \alpha + \varepsilon$ By transitivity of "<"

$\Rightarrow |a_n - \alpha| < \varepsilon$, as wanted. $\boxed{/\!/}$

Ex 1

Prove the sequence $\{a_n\}$ defined by

$$a_1 = \sqrt{6} \quad \text{and} \quad a_{n+1} = \sqrt{6 + a_n} \quad \text{if } n \geq 1$$

Converges.

$a_1 = \sqrt{6}$
$a_2 = \sqrt{6 + \sqrt{6}}$
$a_3 = \sqrt{6 + a_2} = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$

CLAIM: Prove $\{a_n\}$ is (i) bounded above
and
(ii) strictly increasing

Proof (i): $\exists M \in \mathbb{R}$ s.t. $a_n \leq M \quad \forall n \in \mathbb{N}$

Choose $M = \underline{\ 3\ } \in \mathbb{R}$

WTS $a_n < 3 \quad \forall n \in \mathbb{N}$

Base case: $n = 1$

$a_1 = \sqrt{6} < \sqrt{9}$ since $0 < 6 < 9$
$= 3$

Inductive Step: WTS $\forall k \in \mathbb{N}, (a_k < 3 \Rightarrow a_{k+1} < 3)$

Let $k \in \mathbb{N}$ be arbitrary.

Assume $a_k < 3$ (Induction Hypothesis)

WTS $a_{k+1} < 3$

Consider $a_{k+1} = \sqrt{6 + a_k}$ By def of $\{a_n\}$, $k \in \mathbb{N}$
$< \sqrt{6 + 3}$
$= 3$, as wanted

$\therefore$ By PMI, $\{a_n\}$ is strictly bounded above by 3. ▨

Proof (ii): WTS $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$

Let $n \in \mathbb{N}$ be arbitrary.

Consider $a_n^2 - a_{n+1}^2 = a_n^2 - (\sqrt{6+a_n})^2$ By def of $\{a_n\}$

$$= a_n^2 - a_n - 6$$

$$= \underbrace{(a_n - 3)}_{a_n<3 \Rightarrow <0}\underbrace{(a_n+2)}_{>0}$$

$$< 0$$

So $\forall n \in \mathbb{N}$,

$$a_n^2 - a_{n+1}^2 < 0$$

$$0 < a_n^2 < a_{n+1}^2$$

$$a_n < a_{n+1}$$, as wanted

$\therefore$ $\{a_n\}$ is strictly increasing.

$\therefore$ By BMCT, our sequence $\{a_n\}$ converges. $\blacksquare$

Intro to Series

Def (pg 606):

Given $\{a_n\}$ a sequence. The formal sum

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$$

is called a series or infinite series.

a_n is called the general term of $\sum a_n$

For each $n \in \mathbb{N}$, the finite sum

$$a_1 + a_2 + \dots + a_n = S_n$$

$$= n^{th} \text{ partial sum of } \sum a_n.$$

Observe $\sum a_n \approx S_n$

Def (pg 606-607):

Given $\sum a_n$. We say $\sum a_n$ converges if

$$\lim_{n\to\infty} S_n$$

exists.

We write $\sum a_n = \lim_{n\to\infty} S_n = s$

s is called the sum of $\sum a_n$.

If $\sum a_n$ does not converge, then $\sum a_n$ diverges.

Ex 2

Does $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ converge or diverge? Prove.

Proof: