B41 Nov 5 Lec 2 Notes

Definition:

A local (relative) minimum point of $f: U \subset \mathbb{R}^n \to R$ is a point $x_0 \in U$ s.t. $f(x_0) \leq f(x)$ $\forall x$ in some neighbourhood of x_0. $f(x_0)$ is the corresponding local (relative) minimum value.

A local (relative) maximum point of $f: U \subset \mathbb{R}^n \to R$ is a point $x_0 \in U$ s.t. $f(x_0) \geq f(x)$ $\forall x$ in some neighbourhood of x_0. $f(x_0)$ is the corresponding local (relative) maximum value.

If x_0 is either of these, it is a local (relative) extremum and $f(x_0)$ is the local (relative) extremum value.

A point x_0 is a critical point of f if either f is not differentiable at x_0, or if it is, $Df(x_0) = 0$.

Definition: First derivative Test

If $U \subset \mathbb{R}^n$ is open, $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable and x_0 is a local extremum, then x_0 is a critical point s.t. all the partials of f vanish at x_0.

$$\frac{\partial f}{\partial x_1}(x_0) = 0, \quad \frac{\partial f}{\partial x_2}(x_0) = 0, \ldots, \frac{\partial f}{\partial x_n}(x_0) = 0$$

Ex 1:

Find the critical points of $f(x,y) = xy(x-2)(y+3)$

$f_x = 2y(x-1)(y+3) = 0$ (1) $\Rightarrow y=0, x=0$, or $y=-3$
$f_y = x(x-2)(2y+3) = 0$ (2)

$\Rightarrow y = 0$	$\Rightarrow x = 1$	$\Rightarrow y = -3$
$\Rightarrow 3x(x-2) = 0$ (2)	$\Rightarrow -(2y+3) = 0$	$\Rightarrow -3x(x-2) = 0$
$\Rightarrow x=0$ and $x=2$	$\Rightarrow y = -3/2$	$\Rightarrow x=0$ and $x=2$
$\Rightarrow$ critical points $(0,0)$ and $(2,0)$	$\Rightarrow (1, -3/2)$	$\Rightarrow (0,-3)$ and $(2,-3)$

Thus there are five critical points: $(0,0), (2,0), (1, -3/2), (0,-3), (2,-3)$

Remark: Not all critical points are extreme values. A critical point that is not a local extremum is called a saddle point.

Ex 2:

Find extreme values of $f(x,y) = x^2 - y^2$

$f_x = 2x$ and $f_y = -2y$

Thus $f_x(x,y) = f_y(x,y) = 0$ at $(0,0)$

Ex 2: Continued...

However, for points on the x-axis, we have $y=0$, $f(x,0)=x^2 \geq 0 = f(0,0)$
for points on the y-axis, we have $x=0$, $f(0,y)=-y^2 \leq 0 = f(0,0)$

This means that on every disk containing $(0,0)$, $f(x,y)=x^2-y^2$ takes on both positive values and negative values.

Thus $f(0,0)=0$ is neither min nor max value for f.

Thus $(0,0)$ is a saddle point.

Definition: Hessian

If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is of class C^3, then the Hessian of f at x_0 is the quadratic function of h given by

$$Hf(x_0)(h) = \frac{1}{2!}\sum_{i_1,i_2=1}^{n} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0)$$

$$= \frac{1}{2} h^T \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x_0) \end{bmatrix} h \quad \text{, where } h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

Definition: Second Derivative Test

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^2, $x_0 \in$ an open disk $\subset U$ be a critical point of f.
If $Hf(x_0)$ is positive definite, then $(x_0, f(x_0))$ is local (relative) minimum of f.

Similarly,

$Hf(x_0)$ is negative definite $\Rightarrow$ $(x_0, f(x_0))$ is local (relative) maximum of f
$Hf(x_0)$ is neither, but $\det(Hf(x_0)) \neq 0$ $\Rightarrow$ $(x_0, f(x_0))$ is of saddle type.
$\det(Hf(x_0)) = 0$ $\Rightarrow$ degenerate type.

Ex 3:

Let $f(x,y,z) = x^2+y^2+z^2-2xyz$. Find and classify all critical points of f.

$f_x = 2x - 2yz = 0 \Rightarrow x = yz$ (1)
$f_y = 2y - 2xz = 0 \Rightarrow y = xz$ (2)
$f_z = 2z - 2xy = 0 \Rightarrow z = xy$ (3)

Any one of x, or y, or $z = 0$ $\Rightarrow$ All $= 0$
Thus $(0,0,0)$ is a critical point.

Let $(x,y,z) \neq (0,0,0)$, (1) & (2) $\Rightarrow$ $x = xz^2 \Rightarrow 1 = z^2 \Rightarrow z = \pm 1$
When $z=1$: (2) $\Rightarrow x=y$
(3) $\Rightarrow xy = 1$
$\Rightarrow 1 = x^2 = y^2 \Rightarrow x = \pm 1$ & $y = \pm 1$
Thus critical points: $(1,1,1)$, $(-1,-1,1)$

Ex 3: Continued...

Similarly for $z=-1$, we have $(-1,1,-1), (1,-1,-1)$ as critical points.

Thus 5 critical points : $(0,0,0), (1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)$

$$Hf(x,y,z) = \begin{bmatrix} 2 & -2z & -2y \\ -2z & 2 & -2x \\ -2y & -2x & 2 \end{bmatrix}$$

$$Hf(0,0,0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ positive definite $\Rightarrow (0,0,0)$ strict local min.

$$Hf(1,1,1) = \begin{bmatrix} 2 & -2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$ $|2| > 0$, $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$, thus we have to find eigenvalues.

$$|Hf(1,1,1) - \lambda I| = \begin{vmatrix} 2-\lambda & -2 & -2 \\ -2 & 2-\lambda & -2 \\ -2 & -2 & 2-\lambda \end{vmatrix}$$ $\Rightarrow$ eigenvalues $= -2, 4, 4$ $\Rightarrow$ indefinite $\Rightarrow$ saddle type

Similar workings for the rest.