Ex 1

$$\int \frac{5x+11}{x^2+5x+6}\,dx = \int \frac{5x+11}{(x+2)(x+3)}\,dx$$

$$= \int \frac{4}{(x+2)} + \frac{1}{(x+3)}$$

$$= 4\ln|x+2| + \ln|x+3| + C$$

using rule 1, since $x+2$ and $x+3$ has a multiplicity of 1

PFD: $\frac{5x+11}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$

$$= \frac{A(x+2) + B(x+3)}{(x+3)(x+2)}$$

$$\Rightarrow 5x+11 = A(x+2) + B(x+3)$$

$$= Ax + 2A + Bx + 3B$$

$$= Ax + Bx + 2A + 3B$$

$$\Rightarrow 5 = A+B$$

$$11 = 2A+3B$$

$$\begin{bmatrix} 1 & 1 & \vdots & 5 \\ 2 & 3 & \vdots & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 4 \\ 0 & 1 & \vdots & 1 \end{bmatrix}$$

$A = 4$, $B = 1$

Remark:

In some situations there is a shortcut for solving for our unknowns

Ex: $5x+11 = A(x+2) + B(x+3)$

Choose $x = -2$; $-10+11 = A(0) + B(1)$

$1 = 0 + B$

$\therefore B = 1$

Choose $x = -3 \Rightarrow A = 4$

Ex 2

$$\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\,dx = \int \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)}\,dx$$

$$= \int \frac{2}{x} + \frac{-1}{x^2} + \frac{3x-2}{x^2+1}\,dx$$

PFD

PFD: $\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(x^2+1)}$

$$= \int \frac{2}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{3x}{x^2+1}\,dx - \int \frac{2}{x^2+1}\,dx$$

$$= 2\ln|x| + x^{-1} + \frac{3}{2}\ln|x^2+1| - 2\arctan x + C$$

$$Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 5x^3 - 3x^2 + 2x - 1$$

$$Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 = 5x^3 - 3x^2 + 2x - 1$$

$$(A+C)x^3 + (B+D)x^2 + Ax + B = 5x^3 - 3x^2 + 2x - 1$$

$$\Rightarrow A + C = 5$$

$$B + D = -3$$

$$A = 2$$

$$B = -1$$

$\Rightarrow 2 + C = 5 \Rightarrow C = 3$

$\Rightarrow -1 + D = -3 \Rightarrow D = -2$

$\Rightarrow$ $A = 2$ $C = 3$

$B = -1$ $D = -2$

Ex 3

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int \frac{(x^3-x^2-x+1)(x+1)+4x}{x^3-x^2-x+1}\,dx$$

$$= \int (x+1) + \frac{4x}{x^3-x^2-x+1}\,dx$$

PFD: $\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$(x-1)$:

	1	-1	-1	1
1		1	0	-1
	1	0	-1	0

$= (x-1)(x^2-1) = (x-1)^2(x+1)$

$\Rightarrow A(x+1)(x-1) + B(x+1) + C(x-1)^2 = 4x$

$Ax^2 - A + Bx + B + Cx^2 - 2Cx + C = 4x$

$(A+C)x^2 + (B-2C)x + (C+B-A) = 4x$

$\Rightarrow A+C=0$

$B-2C=4$

$C+B-A=0$

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 4 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \int (x+1) + \frac{1}{(x-1)} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)}\,dx$$

$$= \tfrac{1}{2}x^2 + x + \ln|x-1| - 2(x-1)^{-1} - \ln|x+1| + C$$