B24 July 21 Lec 1 Notes

Recall from last class that for a matrix A, $A^* := \overline{A}^T$.

Proposition:

Let A be an $m \times n$ matrix and $x \in \mathbb{F}^n$, $y \in \mathbb{F}^n$

Then:

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

Proof:

$\langle Ax, y \rangle = y^* Ax$ By def of $\langle \cdot , \cdot \rangle$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ i.e. $\langle [x_1 \; \dots \; x_m], \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \rangle = x_1\bar{y}_1 + \dots + x_m \bar{y}_m = \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix} [x_1 \; \cdots \; x_m]$

$= (A^* y)^* x$ For matrices B, C s.t. BC is defined, we have

$$(BC)^* = (\overline{BC})^T = \overline{C}^T \overline{B}^T = \overline{C}^T \overline{B}^T = C^* B^*$$

and $(A^*)^* = (\overline{\overline{A}^T})^T = A$

$= \langle x, A^* y \rangle$ By def of $\langle \cdot , \cdot \rangle$

$\boxtimes$

Proposition:

Let V, W be finite-dimensional IPS and $A : V \to W$ a L.T., then there is a unique L.T. $A^* : W \to V$ satisfying

$$\langle Av, w \rangle = \langle v, A^* w \rangle, \; \forall v \in V, \forall w \in W$$

Proof:

Let $v_1, \dots, v_n$ and $w_1, \dots, w_n$ be orthonormal bases for V, W (respectively).

Consider $[A]_{w_1, \dots, w_n}^{v_1, \dots, v_n}$

Let A^* be defined by the matrix

$$\left[[A]_{w_1, \dots, w_n}^{v_1, \dots, v_n} \right]^*$$

i.e. if $w \in W$, then $w = \sum_{i=1}^{m} B_i w_i$, and the coordinates of A with respect to the basis $v_1, \dots, v_n$ are

$$\left[[A]_{w_1, \dots, w_n}^{v_1, \dots, v_n} \right]^* \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix}$$

Proof (Continued...):

Let $v \in V$, say $v = \sum_{i=1}^{n} \alpha_i v_i$

Let $w \in W$, say $w = \sum_{j=1}^{m} B_j w_j$. Let $[A]_{w_1, \dots, w_n}^{v_1, \dots, v_n} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$

$$\langle Av, w \rangle = \langle A\left(\sum_{i=1}^{n} \alpha_i v_i\right), \sum_{j=1}^{m} B_j w_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} \langle \underbrace{A v_i}_{A_{1i} w_1 + \dots + A_{mi} w_m}, w_j \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} \sum_{k=1}^{m} A_{ki} \underbrace{\langle w_k, w_j \rangle}_{= \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}}$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} A_{ji}$$

Similarly,

$$\langle x, A^* y \rangle = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} \langle v_i, \underbrace{A^* w_j}_{\sum_{k=1}^{n} \overline{A_{jk}} v_k = \overline{A_{j1}} v_1 + \dots + \overline{A_{jn}} v_n} \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} \sum_{k=1}^{n} \overline{\overline{A_{jk}}} \underbrace{\langle v_i, v_k \rangle}_{= \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}}$$

$$= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} \overline{B_j} A_{ji}$$

This shows that $\langle x, A^* y \rangle = \langle Ax, y \rangle$

If $B : W \to V$ is a L.T. satisfying

$$\langle x, By \rangle = \langle Ax, y \rangle,$$

then

$$\langle x, By \rangle = \langle x, A^* y \rangle \quad , \forall x \in V$$

$$\Rightarrow \quad By = A^* y$$

$$\Rightarrow \quad B = A^* \quad \blacksquare$$

Proposition:

Let $A, B : V \to W$, then:

(i) $(A+B)^* = A^* + B^*$
(ii) $(\alpha A)^* = \bar{\alpha} A^*$
(iii) $(AB)^* = B^* A^*$
(iv) $(A^*)^* = A$
(v) $\langle y, Ax \rangle = \langle A^* y, x \rangle$

Proof: of (iii)

Let $v \in V$, $w \in W$

$$\begin{aligned}\langle ABv, w \rangle &= \langle A(Bv), w \rangle \\ &= \langle Bv, A^* w \rangle \\ &= \langle v, B^* A^* w \rangle \\ &= \langle v, (B^* A^*) w \rangle\end{aligned}$$

By uniqueness of adjoint, $B^* A^* = (AB)^*$ ☒

Theorem:

Let V, W be IPS and $A : V \to W$ a L.T. Then:

(i) $\operatorname{Ker}(A^*) = (\operatorname{Ran} A)^\perp$
(ii) $\operatorname{Ker}(A) = (\operatorname{Ran} A^*)^\perp$
(iii) $\operatorname{Ran}(A) = (\operatorname{Ker} A^*)^\perp$
(iv) $\operatorname{Ran}(A^*) = (\operatorname{Ker} A)^\perp$

Proof:

(ii) follows from (i) (replace A^* with A)
(iii) follows from (i) (take $\perp$ of both sides)
(iv) follows from (iii) (replace A with A^*)

Suffices to prove (i):

$(\operatorname{Ran} A)^\perp \subseteq \operatorname{Ker}(A^*)$: if $y \in (\operatorname{Ran} A)^\perp$, then

$$\langle Ax, y \rangle = 0 = \langle x, A^* y \rangle, \quad \forall x \in V$$

$$\Rightarrow \qquad A^* y = 0$$

Proof (continued...):

$\ker(A^*) \subseteq (\operatorname{Ran} A)^\perp$: if $y \in \ker(A^*)$, then

$$\langle x, A^* y\rangle = 0 = \langle Ax, y\rangle, \quad \forall x \in V$$

$$\Rightarrow \quad y \perp \text{range}(A)$$

$$\Rightarrow \quad y \in (\operatorname{Ran} A)^\perp$$

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