Proof : FTOC Part II

Suppose :

(i) f is continuous on $[a,b]$

(ii) F is defined as $F(x) = \int_a^x f(t)\,dt$ where any $x \in [a,b]$.

WTS :

$\forall x \in [a,b]$, $F'(x) = f(x)$

Let $x \in [a,b]$ be arbitrary

Case 1 : $x \in (a,b)$

Consider $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$ By def of derivative of F

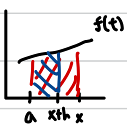
$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

Case 1: $h \geq 0$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt + \int_a^x f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

Case 2 : $h < 0$



$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

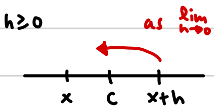
$$= \lim_{h \to 0} \frac{-\left(\int_a^x f(t)\,dt - \int_a^{x+h} f(t)\,dt \right)}{h}$$

$$= \lim_{h \to 0} \frac{-\int_{x+h}^x f(t)\,dt}{h}$$

MVT : $\int_a^b f(t)\,dt = f(c)(b-a)$

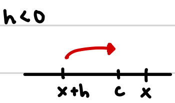
$$f(c) = \frac{\int_a^b f(t)\,dt}{(b-a)}$$

So $F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$



$$= \lim_{h \to 0} \frac{1}{h+0} \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h \to 0} \frac{1}{(x+h)-x} \int_x^{x+h} f(t)\,dt$$



$h \to 0$

$\Leftrightarrow x+h \to x$

$\Leftrightarrow |c-x| \to 0$

$\Leftrightarrow c \to x$

$= \lim_{h \to 0} f(c)$ By def of MVT, $\exists c$ between x and $x+h$

$= \lim_{c \to x} f(c)$

$= f(x)$ we can substitute x into c b/c f is continuous

Case 2: $x = a$ and $x = b$

WTS $F'_+(a) = f(a)$, $F'_-(b) = f(b)$

By an analogous argument to the prior case (just replace x with the appropriate end point and replace the 2-sided with the corresponding 1-sided limit) we get our desired result.

$\therefore$ By case 1 and case 2,

$$F'(x) = f(x), \ \forall x \in [a,b]$$

So F' exists (2-sided) $\forall x \in [a,b]$.

Thus F is diff on (a,b) which implies F is continuous on $[a,b]$ //

Ex 1

Let $H(x) = \int_{x^2}^{e^x} (\tan^{-1}(t) + t^2)\, dt$. Find $H'(x)$.

continuous on its domain

t^2 is a polynomial does cont. on $\operatorname{dom}(t^2) = \mathbb{R}$

But f is a sum function so continuous on the common points of cont.
Thus f is continuous

In particular, f is continuous on $[x^2, e^x] \subset \mathbb{R}$

Define $F(x) = \int_c^x f(t)\, dt$, any constant $c \in (x^2, e^x)$

So $H'(x) = \frac{d}{dx}\left(\int_{x^2}^{e^x} f(t)\, dt \right)$

$$= \frac{d}{dx}\left(\int_{x^2}^{c} f(t)\, dt + \int_c^{e^x} f(t)\, dt \right)$$

$$= \frac{d}{dx}\left(-\int_c^{x^2} f(t)\, dt \right) + \frac{d}{dx} \int_c^{e^x} f(t)\, dt$$

$$= -2x\left(\tan^{-1}(x^2) + x^4\right) + e^x\left(\tan^{-1}(e^x) + (e^x)^2\right)$$