B41 Oct 22 Lec 2 Notes

Ex 3: From previous lecture

Find the points on the surface defined by $x^2+2y^2+3z^2=1$ where the tangent plane is parallel to the plane defined by $3x-y+3z=1$.

Let $f = x^2+2y^2+3z^2$

Then $\nabla f = (2x, 4y, 6z)$

As the tangent plane is parallel to the plane $3x-y+3z=1$, we have

$$(2x, 4y, 6z) \parallel (3,-1,3)$$

$\Rightarrow$ $2x = 3r$, $4y = -r$, and $6z = 3r$

i.e. $x = \frac{3r}{2}$, $y = \frac{-r}{4}$, $z = \frac{r}{2}$

Because the points are on the surface $x^2+2y^2+3z^2=1$, we have

$$\left(\frac{3r}{2}\right)^2 + 2\left(\frac{-r}{4}\right)^2 + 3\left(\frac{r}{2}\right)^2 = 1$$

$$\left(\frac{9}{4} + \frac{2}{16} + \frac{3}{4}\right) r^2 = 1$$

$$\Rightarrow r = \pm \frac{2\sqrt{2}}{5}$$

Therefore the points are $\left(\frac{3\sqrt{2}}{5}, -\frac{\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$ and $\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$

Definition:

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $x_0 = (x_0, y_0)$. The linear approximation of f at the point x_0 is defined as

$$f(x,y) \approx L(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y-y_0)$$

OR

$$L(x) = f(x_0) + \nabla f(x_0)(x-x_0)$$

Ex 1.

Find linear approximation to the function $f(x,y) = \sin(xy)$ at $(1, \frac{\pi}{3})$

$f(1, \frac{\pi}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\frac{\partial f}{\partial x}(1, \frac{\pi}{3}) = (\cos\frac{\pi}{3})\frac{\pi}{3} = \frac{\pi}{6}$

$\frac{\partial f}{\partial y}(1, \frac{\pi}{3}) = (\cos\frac{\pi}{3})1 = \frac{1}{2}$

$$L(x,y) = f(1, \tfrac{\pi}{3}) + \left[\frac{\partial f}{\partial x}(1, \tfrac{\pi}{3})\right](x-1) + \left[\frac{\partial f}{\partial y}(1, \tfrac{\pi}{3})\right](y - \tfrac{\pi}{3})$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{6}(x-1) + \frac{1}{2}(y - \frac{\pi}{3})$$

$$= \frac{\pi}{6}x + \frac{1}{2}y - \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Definition:

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $x_0 = (x_0, y_0)$. The linear approximation of f at the point x_0 is defined as

$$f(x,y,z) \approx L(x,y,z) = f(x_0, y_0, z_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right](y - y_0) + \left[\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right](z - z_0)$$

OR

$$L(x) = f(x_0) + \nabla f(x_0)(x - x_0)$$

Theorem:

Assume that $\nabla f \neq 0$. Then $\nabla f(x)$ points in the direction along which f is increasing the fastest.