Jan 22 Lec2 Notes

Def: Row Echelon Form (ref)

A matrix is in REF if

(i) All zero rows are in the bottom
(ii) The first nonzero entry in each row is to the right of the first nonzero entry of all rows above it.
(iii) All entries below each first non-zero entry is zero.

Def: Reduced Row Echelon Form (rref)

A matrix A is in rref if

(i) A is in ref
(ii) Every leading entry in A is 1.
(iii) Leading ones are the only nonzero in their column.

Example

1. Use Guass-Jordan Elimination

$$\left|\begin{array}{ccc:c} 2 & 3 & -1 & 1 \\ 0 & 1 & -3 & 2 \\ 4 & 5 & -2 & 1 \end{array}\right| \quad R_1 \leftrightarrow \tfrac{1}{2}R_1 \quad \sim$$

"Row equivalent to"

$$\left|\begin{array}{ccc:c} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & -3 & 2 \\ 4 & 5 & -2 & 1 \end{array}\right| \quad R_3 \leftrightarrow R_3 - 4R_1 \quad \sim$$

$$\left|\begin{array}{ccc:c} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & -3 & 2 \\ 0 & -1 & 0 & -1 \end{array}\right| \quad R_3 \leftrightarrow R_3 + R_2 \quad \sim$$

$$\left|\begin{array}{ccc:c} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & -3 & 1 \end{array}\right| \quad R_3 \leftrightarrow (-\tfrac{1}{3})R_3 \quad \sim$$

$$\left|\begin{array}{ccc:c} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right| \quad R_2 \leftrightarrow R_2 + 3R_3 \quad \sim$$

(In REF)

$$\left|\begin{array}{ccc:c} 1 & 3/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/3 \end{array}\right| \quad R_1 \leftrightarrow R_1 + (-\tfrac{1}{2})R_3 \quad \sim$$

$$\left|\begin{array}{ccc:c} 1 & 3/2 & 0 & -7/6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right| \quad R_1 \leftrightarrow R_1 + (-\tfrac{3}{2})R_2 \quad \sim$$

$$\left|\begin{array}{ccc:c} 1 & 0 & 0 & -\frac{8}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right|$$

(In RREF)

2. Find solution set of $\begin{cases} 2x_1 + 4x_2 + 10x_3 = 2 \\ x_1 + 3x_2 + 7x_3 = 0 \\ 3x_1 + 6x_2 + 15x_3 = 3 \end{cases}$

$$\left|\begin{array}{ccc:c} 2 & 4 & 10 & 2 \\ 1 & 3 & 7 & 0 \\ 3 & 6 & 15 & 3 \end{array}\right| \cdots \rightarrow \left|\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

$\begin{cases} x_1 + x_3 = 3 \\ x_2 + 2x_3 = -1 \\ 0 = 0 \end{cases}$ x_1, x_2 are leading / basic / pivot variables

x_3 is a free / non-leading variable

$\begin{cases} x_1 = 3 - x_3 \\ x_2 = -1 - 2x_3 \end{cases}$ Any value for x_3 is a solution

Set $s := x_3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3-s \\ -1-2s \\ s \end{pmatrix} = s\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

Solution set: $\left\{ s\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$

Theorem: Types of Solutions of a System of Linear Equation

Let S be a system of linear equations and let $[\underset{n\times m}{A} \mid \underset{n\times(m+1)}{\vec{b}}]$ be the augmented matrix corresponding to S.

(i) No solution or inconsistent system iff there is a leading 1 in the aug col ($\vec{b}$) of $\text{rref}([A\mid\vec{b}])$.

$$\text{rref}([A\mid\vec{b}]) = \left[\begin{array}{cccc:c} & & & & \\ & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

(ii) Consistent iff all leading ones in the coefficient part of $\text{rref}([A\mid\vec{b}])$

(a) A unique solution iff
1. No leading one in aug. column
2. All variables are leading variables

(b) Infinitely many solutions iff
1. no leading one in aug. column.
2. there is at least one free variable