# A37 Apr 5 Lec 1 Notes

**Theorem (pg. 626):** Direct Comparison Theorem

Let $\sum a_n, \sum b_n, \sum c_n$ be series.

(i) Convergent case

If $0 \le a_n \le b_n, \forall n \in \mathbb{N}$ and $\sum b_n$ conv., then $\sum a_n$ also converges.

(ii) Divergent case:

If $0 \le c_n \le a_n, \forall n \in \mathbb{N}$ and $\sum c_n$ div, then $\sum a_n$ also diverges.

**Proof (i):** Conv. case

Suppose $0 \le a_n \le b_n^{1}, \forall n \in \mathbb{N}$ and $\sum b_n$ conv.$^{2}$,

WTS $\sum a_n$ conv.

i.e. $\lim_{n\to\infty} S_n$ exists

i.e. $\{S_n\}$ converges use BMCT

Let $n \in \mathbb{N}$ be arbitrary

Note $S_{n+1} = \underbrace{S_n}_{\ge 0} + \underbrace{a_{n+1}}_{\ge 0}$ , by def $S_n, S_{n+1}$

$\ge S_n$

$\therefore \forall n \in \mathbb{N}, S_{n+1} \ge S_n$

i.e. $\{S_n\}$ is increasing.

Moreover $S_n \le \sum a_n$, by ① as $a_n \ge 0$

$\Rightarrow \sum a_n \le \sum b_n$

$\Rightarrow S_n \le \sum b_n$

$\therefore \forall n \in \mathbb{N}$ $S_n \le S$ for some $S \in \mathbb{R}$

i.e. $\{S_n\}$ is bounded above.

$\therefore$ By BMCT, $\{S_n\}$ conv.

i.e. $\lim_{n\to\infty} S_n$ exists

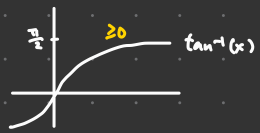
i.e. $\sum a_n$ conv. ▨

**Ex 1:**

Does $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^{1/2}+4^n}$ conv. or div.?

**Proof:**

(i) for any $n \in \mathbb{N}$, $a_n = \frac{\overset{\geq 0}{\tan^{-1}(n)}}{\underset{\geq 0}{n^{1/2}+4^n}} > 0$




(ii) Find a comparison

$$a_n = \frac{\tan^{-1}(n)}{n^{1/2}+4^n} \leq \frac{\frac{\pi}{2}}{4^n} =: b_n$$

Consider $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\pi}{2}\left(\frac{1}{4}\right)^n$

$\therefore$ By GS Test, $|r| < 1$ and $\sum b_n$ converges.

$\therefore$ By CT, $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^{1/2}+4^n}$ conv

**Def (pg 645):**

We say series $\sum a_n$

(i) absolutely converges if $\sum |a_n|$ conv.

(ii) conditionally converges if $\sum a_n$ conv. but $\sum |a_n|$ div.

**Ex 2:**

$\sum_{n=0}^{\infty} \frac{\sin(6n)}{4^n}$ AC, CC, or div?

Proof: "top-down" approach

Consider $\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \left|\frac{\sin(6n)}{4^n}\right|$

$$= \sum_{n=0}^{\infty} \frac{|\sin(6n)|}{4^n}$$

For $n \geq 0$, $|a_n| \geq 0$ ✓

$$|a_n| = \frac{|\sin(6n)|}{4^n}$$

$-1 \leq \sin 6n \leq 1$
$|\sin 6n| \leq 1$

$$\leq \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$$

Consider $\sum \left(\frac{1}{4}\right)^n$, $|r| = \frac{1}{4} < 1$.

$\therefore$ By GST, $\sum b_n$ conv

$\therefore$ By CT, $\sum |a_n|$ conv $\quad \therefore$ By def $\sum_{n=0}^{\infty} \frac{\sin(6n)}{4^n}$ AC.

Ex 3:

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \quad AC, CC, div ?$$

Theorem (pg 633): Ratio Test (RT)

Let $\sum a_n$ be series with $a_n \in \mathbb{R} - \{0\}$

Define $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| := L$

(i) If $L < 1 \Rightarrow \sum a_n$ AC

(ii) If $L > 1 \Rightarrow \sum a_n$ div

(iii) If $L = 1 \Rightarrow$ this test is inconclusive.