B24 June 30 Lec 1 Notes

Definition:

Let V be a vector space over $\mathbb{F}$. An inner product on V is a function:

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$$

satisfying:

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\forall x, y \in V$ Conjugate symmetry
(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, $\forall \alpha, \beta \in \mathbb{F}$, $\forall x, y, z \in V$ Linearity
(iii) $\langle x, x \rangle \geq 0$, $\forall x \in V$ Non-negativity
(iv) $\langle x, x \rangle = 0$ iff $x = 0$ Non-degeneracy

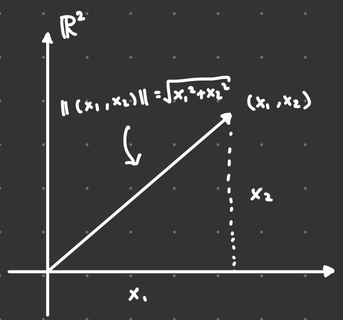
Given an inner product $\langle \cdot , \cdot \rangle$ on a v.s. V, we call $(V, \langle \cdot , \cdot \rangle)$ an inner product space, and define the norm of a vector $x \in V$ by:

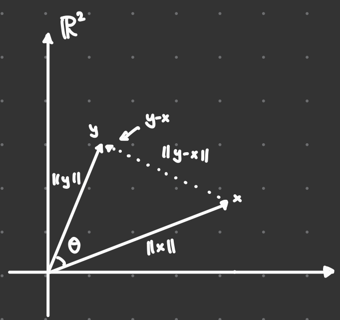
$$\|x\| := \sqrt{\langle x, x \rangle}$$

Ex 1:

$V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$, and $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := x_1 y_1 + \dots + x_n y_n$, so

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$$





Law of cosines:

$$\underbrace{\|y-x\|^2}_{\downarrow} = \|y\|^2 + \|x\|^2 - 2\|y\|\|x\| \cos\theta$$

$= |\langle y-x, y-x \rangle|$

$= \langle y, y \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle x, x \rangle$

$= \|y\|^2 - 2\langle x, y \rangle + \|x\|^2$

$\Rightarrow \dfrac{\langle x, y \rangle}{\|y\|\|x\|} = \cos\theta$

In particular, $x \perp y$ iff $\cos\theta = 0$ iff $\langle x, y \rangle = 0$. i.e. $\langle x, y \rangle$ measures how close two vectors are to being "orthogonal".

Ex 2:

$V = \mathbb{C}^n$, $\mathbb{F} = \mathbb{C}$ and $\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \rangle := z_1\bar{w_1} + \dots + z_n\bar{w_n}$, so

$$\|(z_1, \dots, z_n)\| = \sqrt{z_1\bar{z_1} + \dots + z_n\bar{z_n}}$$
$$= \sqrt{|z_1|^2 + \dots + |z_n|^2}$$

Ex 3:

$V = \mathbb{P}_n^{\mathbb{F}}$, and $\langle p, q \rangle := \int_0^1 p(t)\,\overline{q(t)}\,dt$

So e.g., $x \in \mathbb{P}_3^{\mathbb{R}}$, $x^2 \in \mathbb{P}_3^{\mathbb{R}}$, and

$$\langle x, x^2 \rangle := \int_0^1 x\,\overline{x^2}\,dx$$
$$= \int_0^1 x^3\,dx$$
$$= \left[\frac{x^4}{4}\right]_0^1 = 1/4$$

If $f : X \to \mathbb{C}$, then $\bar{f} : X \to \mathbb{C}$
$x \mapsto \overline{f(x)}$

So for instance if $p(z) = z^2 + 1 + i$
$\overline{p(z)} = \bar{z}^2 + 1 - i$

If $f : \mathbb{C} \to \mathbb{C}$, we can write $f :$ Real $(f) + i$ Imaginary(f), where $Re(f), Im(f) : \mathbb{C} \to \mathbb{R}$, and

$$\int f(x)\,dx := \int Re(f(x))\,dx + i\int Im(f(x))\,dx$$

So e.g. $z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy)$

So $\int z^2 := \int (x^2 - y^2) + i\int 2xy$

here $\|p(t)\| := \sqrt{\int_0^1 p(t)\,\overline{p(t)}\,dt} = \sqrt{\int_0^1 |p(t)|^2\,dt}$

Ex 4:

$V = C([0,1])$, $\langle f, g \rangle := \int_0^1 f(x)\,\overline{g(x)}\,dx$

Starting point for fourier series

Lemma:

Let V be an IPS and $x,y \in V$. Then $x=y$ iff $\langle x,z\rangle = \langle y,z\rangle$, $\forall z \in V$

Proof ($\Rightarrow$): obvious

Proof ($\Leftarrow$): If $\langle x,z\rangle = \langle y,z\rangle$, $\forall z \in V$, then

$\langle x-y, z\rangle = 0$, $\forall z \in V$,

So in particular,

if $z = x-y$

$\langle x-y, x-y\rangle = 0$

$\Rightarrow x-y=0$ By property (iv) of inner product

$\Rightarrow x=y$

Corollary:

If V, W are IPS and $A: V \to W$, $B: V \to W$ are L.T.'s, then if

$$\langle Ax, y\rangle = \langle Bx, y\rangle, \ \forall x \in V, \ \forall y \in W$$

then $A=B$.

This follows from above.