CH 6.2 Prop. of Determinants

Theorem 6.2.1: Determinant of the transpose.

If A is a square matrix, then

$$\det(A^T) = \det A$$

Theorem 6.2.2: Linearity of the determinant in the rows and columns

Consider $T : \mathbb{R}^{1\times n} \to \mathbb{R}$,

$$T(\vec{x}) = \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_{i-1} & - \\ - & \vec{x} & - \\ - & \vec{v}_{i+1} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

$T(\vec{x})$ is a L.T. This property is referred to as linearity of the det. in the i^{th} row.

The det. is linear in all the columns.

Proof:

Observe that prod P is linear in all the rows and columns, since this product contains exactly one factor from each row and one from each column.

We can write $T(\vec{x}+k\vec{y}) = T(\vec{x}) + kT(\vec{y})$

$$\det \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{x}+k\vec{y} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix} = \det \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{x} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix} + k \cdot \det \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{y} & - \\ & \vdots & \\ - & \vec{v}_n & - \end{bmatrix}$$

Theorem 6.2.3: Elementary row operations and determinants

(i) If B is obtained by dividing a row of A by a scalar K, then

$$\det B = \left(\tfrac{1}{K}\right) \det A$$

(ii) If B is obtained from A by a row swap, then

$$\det B = -\det A$$

We say that the det. is alternating on the rows

(iii) If B is obtained from A by adding a multiple of a row of A to another row, then

$$\det B = \det A$$

Analogous results hold for elementary column operations.

Proof (i): Case with 2×2 matrix

$$B = \begin{bmatrix} a/K & b/K \\ c & d \end{bmatrix}$$

Then $\det B = \frac{a}{K} d - \frac{b}{K} c = \frac{1}{K} \det A$

$\blacksquare$

Proof (ii): Case with 2×2 matrix

$$B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Then $\det B = cb - da = -\det A$

$\blacksquare$

Proof (iii): Case with 2×2 matrix

$$B = \begin{bmatrix} a+Kc & b+Kd \\ c & d \end{bmatrix}$$

Then $\det B = (a+Kc)d - (b+Kd)c$

$= ad + Kcd - bc - Kcd$

$= \det A$ $\blacksquare$

Note: We can use thm 6.2.2 to prove the above 3 for an arbitrary size.

Suppose that when trying to find rref A, we swap rows s times and divide various rows by the scalars $k_1, k_2, \dots, k_r$. Then,

$$\det(\text{rref } A) = (-1)^s \frac{1}{k_1 k_2 \cdots k_r} \det A$$

$$\det A = (-1)^s k_1 k_2 \cdots k_r \det(\text{rref} A) \quad \text{By thm 6.2.3}$$

If A is invertible, then rref $A = I_n$, so $\det(\text{rref} A) = \det I_n = 1$, and

$$\det A = (-1)^s k_1 k_2 \cdots k_r \neq 0$$

Note that $\det A$ fails to be zero since $k_i \neq 0$.

If A is non invertible, then the last row of rref A contains all zeros, s.t. $\det(\text{rref } A) = 0 \Rightarrow \det A = 0$.

Theorem 6.2.4: Invertibility and determinant

A square matrix A is invertible iff $\det A \neq 0$.

Det. of a Product

Theorem 6.2.6: Determinants of products and powers

If A and B are $n \times n$ matrices and m is a positive integer, then

(i) $\det(AB) = (\det A)(\det B)$, and

(ii) $\det(A^m) = (\det A)^m$

Proof (i):

Consider A is invertible.

$$\text{rref}[A \mid AB] = [I_n \mid B]$$

Suppose that when trying to find rref, we swap rows s times and divide various by the scalars $k_1, k_2, \dots, k_r$. Then,

$$\det A = (-1)^s k_1 k_2 \cdots k_r$$

and

$$\det AB = (-1)^s k_1 k_2 \cdots k_r \det B$$
$$= (\det A)(\det B)$$

If A is not invertible, then neither is AB.

$$(\det A)(\det B) = 0 (\det B) = 0 = \det(AB)$$

Proof (ii):

We have

$$\det(A^m) = \det \underbrace{(A \cdot A \cdots A)}_{m \text{ times}} \overset{\text{by thm 6.26 (i)}}{=} \underbrace{(\det A)(\det A)\cdots(\det A)}_{m \text{ times}} = (\det A)^m$$

Def 3.4.5: Similar Matrices

Consider two $n \times n$ matrices A and B. We say that A is similar to B if there exists an invertible matrix S s.t.

$$AS = SB, \quad \text{or} \quad B = S^{-1}AS$$

Theorem 6.2.7: Determinants of Similar Matrices

If matrix A is similar to B, then $\det A = \det B$.

Theorem 6.2.8: Det. of an inverse

If A is an invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}$$

Proof:

$$1 = \det(I_n) = \det(AA^{-1}) = \det A \det A^{-1}$$

$$\det A^{-1} = \frac{1}{\det A}$$

$\boxtimes$

Minors and Laplace Expansion

Def 6.2.9: Minors

For an $n \times n$ matrix A, let A_{ij} be the matrix obtained by omitting the i^{th} row and the j^{th} column of A. The det. of the $(n-1)\times(n-1)$ matrix A_{ij} is called a minor of A.

Theorem 6.2.10: Laplace Expansion (Cofactor expansion)

Expansion down the jth column:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Expansion down the ith row:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Thm 6.2.10 justification: (down j^{th} column)

$$\det A = \sum (\text{sgn } P)(\text{prod } P)$$

$$= \sum_{i=1}^{n} \sum_{\substack{P \text{ contains} \\ a_{ij}}} (\text{sgn } P)(\text{prod } P)$$

$$= \sum_{i=1}^{n} \sum_{\substack{P \text{ contains} \\ a_{ij}}} (-1)^{i+j} a_{ij} (\text{sgn } P_{ij})(\text{prod } P_{ij})$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \sum_{\substack{P \text{ contains} \\ a_{ij}}} (\text{sgn } P_{ij})(\text{prod } P_{ij})$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

Def 6.2.11: Det. of a L.T.

Consider a L.T. $T: V \to V$, where V is a finite dimensional linear space. If $\mathfrak{B}$ is a basis of V and B is th $\mathfrak{B}$-matrix of T, then we define

$$\det T = \det B$$

This det. is independent of the basis $\mathfrak{B}$ we choose.