A22 Mar 17 Lec 1 Notes

Def :

Suppose $(\vec{b}_1, \dots, \vec{b}_m)$ is a basis for a subspace W of $\mathbb{R}^n$.

$\forall \vec{v} \in W \quad [\vec{v}]_\varepsilon = r_1\vec{b}_1 + \dots + r_m\vec{b}_m$

$$\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \\ | & | & & | \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = [\vec{v}]_\varepsilon$$

theorem 1.3.8

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_m \\ | & | & & | \end{bmatrix}}_{S_{B\to\varepsilon}}[\vec{v}]_B = [\vec{v}]_\varepsilon$$

Ex 1

Suppose $W = \text{span}\left(\underset{\vec{b}_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}\right) \underset{s.s}{\subseteq} \mathbb{R}^3$

$(\vec{b}_1, \vec{b}_2)$ is an ordered basis for W.

$\vec{v} \in W, \quad [\vec{v}]_B = \begin{pmatrix}1\\2\end{pmatrix}$

$$[\vec{v}]_\varepsilon = \begin{bmatrix}1 & 1\\0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix}$$

Ex 2

$\text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \mathbb{R}^3$

$B = (\vec{b}_1, \vec{b}_2, \vec{b}_3) \qquad \vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix} \in \mathbb{R}^3 \qquad [\vec{v}]_B = ?$

$S_{B\to\varepsilon}[\vec{v}]_B = [\vec{v}]_\varepsilon = \begin{pmatrix}1\\2\\3\end{pmatrix}$

Note: $S_{B\to\varepsilon}$ is always invertible

$S^{-1}_{B\to\varepsilon} S_{B\to\varepsilon}[\vec{v}]_B = S^{-1}_{B\to\varepsilon}[\vec{v}]_\varepsilon$

$I\,[\vec{v}]_B = S^{-1}_{B\to\varepsilon}[\vec{v}]_\varepsilon$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0\\ 0 & 1 & 0 & 0 & 1 & -1\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right]$$

$S^{-1}_{B\to\varepsilon} = S_{\varepsilon\to B}$

Coordinate Isomorphism

Def:

Let V be a v.s. $\dim V = n$.

Let $B = (\vec{b_1}, \dots, \vec{b_n})$ be an ordered basis for V.

$$\begin{aligned} T_B : V &\longrightarrow \mathbb{R}^n \\ \vec{v} &\longmapsto [\vec{v}]_B = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \end{aligned}$$

Theorem:

(i) T_B is a L.T.
(ii) T_B is an isomorphism between V and $\mathbb{R}^n$

Proof (i):

Pick $\vec{v_1}, \vec{v_2} \in V$, $\lambda \in \mathbb{R}$

WTS $T_B(\vec{v_1} + \lambda \vec{v_2}) = T_B(\vec{v_1}) + \lambda T(\vec{v_2})$

Suppose $[\vec{v_1}]_B = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$, $[\vec{v_2}]_B = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$

$$\begin{aligned} T_B(\vec{v_1} + \lambda \vec{v_2}) &= [\vec{v_1} + \lambda \vec{v_2}]_B \\ &= (r_1\vec{b_1} + r_2\vec{b_2} + \dots + r_n\vec{b_n}) + \lambda(s_1\vec{b_1} + \dots + s_n\vec{b_n}) \\ &= (r_1 + \lambda s_1)\vec{b_1} + \dots + (r_n + \lambda s_n)\vec{b_n} \\ &= \begin{pmatrix} r_1 + \lambda s_1 \\ r_2 + \lambda s_2 \\ \vdots \\ r_n + \lambda s_n \end{pmatrix} \\ &= \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} + \lambda \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \\ &= [\vec{v_1}]_B + \lambda [\vec{v_2}]_B \\ &= T_B(\vec{v_1}) + \lambda T_B(\vec{v_2}) \end{aligned}$$

$\therefore T_B$ is a L.T. $\square$

Proof (ii):

What is $\text{Ker}\, T_B$?

$$\text{Ker}\, T_B = \{ \vec{v} \in V \mid T_B(\vec{v}) = \vec{0} \}$$

$\vec{v} \in \text{Ker}\, T_B$,

$$[\vec{v}]_B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vec{v} = 0\vec{b}_1 + 0\vec{b}_2 + \dots + 0\vec{b}_n = \vec{0}_V$$

So $\text{Ker}\, T_B = \{\vec{0}\}$. $\Rightarrow T_B$ is injective.

What is $\text{img}\, T_B$?

$$\text{img}\, T_B = \{ T_B(\vec{v}) \mid \vec{v} \in V \} = \{ [\vec{v}]_B \mid \vec{v} \in V \} \underset{s.s}{\subseteq} \mathbb{R}^n$$

Pick $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \in \mathbb{R}^n$

$$\vec{v} = r_1\vec{b}_1 + \dots + r_n\vec{b}_n \in V$$

$$[\vec{v}]_B = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \Rightarrow T_B(\vec{v}) = \vec{x}$$

i.e. $\vec{x} \in \text{img}\, T_B$

So $\text{img}\, T_B = \mathbb{R}^n$. T_B is surjective.

$\therefore T_B$ is bijective $\Rightarrow T_B$ is an isomorphism.

Ex 3

$$P_n = \{ a_0 + a_1 x + \dots + a_n x^n \mid a_i \in \mathbb{R} \}, \quad B = (1, x, x^2, \dots, x^n)$$

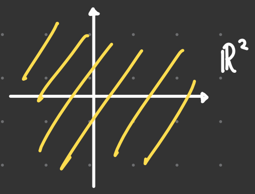
$$[p(x) = a_0 + a_1 x + \dots + a_n x^n]_B$$

$$T_B = P_n \xrightarrow{\cong} \mathbb{R}^{n+1}$$
$$p(x) \longmapsto [p(x)]_B$$

Ex 4

$W \underset{s.s}{\subseteq} \mathbb{R}^3$

W is the plane $x + y + z = 0$



$x = y = z$
$y = t$
$z = t$

$B = \left(\vec{b_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{b_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$ basis for W

$$T_B = W \longrightarrow \mathbb{R}^2$$
$$\vec{w} \longmapsto [\vec{w}]_B$$

$\vec{w} \cong \mathbb{R}^2$, T_B is an isomorphism

Ex 5

$V = M_{2x2}$ $\quad$ $\dim V = 4$

$B = (\vec{b_1}, \dots, \vec{b_4})$ $\quad$ $T_B : V \xrightarrow{\cong} \mathbb{R}^4$

$M_{2x2} \cong \mathbb{R}^4$