Thm. Rules of Vector Algebra

(i) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ Associative
(ii) $\vec{v}+\vec{w} = \vec{w}+\vec{v}$ Commutative
(iii) $\vec{v}+0 = \vec{v}$ $\quad \vec{0}$ is the additive identity
(iv) For each $\vec{v}$ in $\mathbb{R}^n$, there exists a unique $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v}+\vec{x}=\vec{0}$; namely, $\vec{x}=-\vec{v}$. $\quad -\vec{v}$ is the additive inverse
(v) $k(\vec{v}+\vec{w}) = k\vec{v}+k\vec{w}$
(vi) $(c+k)\vec{v} = c\vec{v}+k\vec{v}$
(vii) $c(k\vec{v}) = (ck)\vec{v}$
(viii) $1\vec{v} = \vec{v}$

Proof: (i)

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ be in $\mathbb{R}^n$

$$(\vec{u}+\vec{v})+\vec{w} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \quad \text{By def of vector addition}$$

$$= \begin{pmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ \vdots \\ (u_n+v_n)+w_n \end{pmatrix} \quad \text{By def of vector addition}$$

$$= \begin{pmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \\ \vdots \\ u_n+(v_n+w_n) \end{pmatrix} \quad \text{Since addition over } \mathbb{R} \text{ is associative}$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix} \quad \text{By def of vector addition}$$

$$= \vec{u}+(\vec{v}+\vec{w})$$

Proof: (iv)

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be an arbitrary vector in $\mathbb{R}^n$

Take $\vec{x}$ to be $-\vec{v}$.

$\vec{v} + \vec{x} = \vec{v} + (-\vec{v}) = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$ By def of $-\vec{v}$

$= \begin{pmatrix} v_1 - v_1 \\ v_2 - v_2 \\ \vdots \\ v_n - v_n \end{pmatrix}$ By def of vector addition

$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0}$

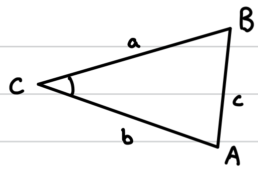
Note that $\vec{x}$ is unique since

$$\begin{aligned} \vec{v} + \vec{x} &= \vec{0} \\ \vec{v} &= -\vec{x} + \vec{0} \\ \vec{x} &= -\vec{v} \end{aligned}$$

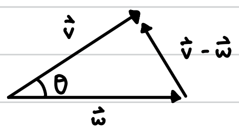
Dot Product

Law of Cosine

$$a^2 + b^2 = c^2 + 2ab\cos C$$



Geometric Representation of $\vec{v} \cdot \vec{w}$



$$\begin{aligned} \|\vec{v}\|^2 + \|\vec{w}\|^2 &= \|\vec{v} - \vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta \\ v_1^2 + v_2^2 + w_1^2 + w_2^2 &= (v_1 - w_1)^2 + (v_2 - w_2)^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta \\ v_1^2 + v_2^2 + w_1^2 + w_2^2 &= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta \\ 0 &= -2v_1w_1 - 2v_2w_2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta \\ v_1w_1 + v_2w_2 &= \|\vec{v}\|\|\vec{w}\|\cos\theta \\ \vec{v} \cdot \vec{w} &= \|\vec{v}\|\|\vec{w}\|\cos\theta \end{aligned}$$

Cauchy - Schwarz Inequality

$\forall\, \vec{v}, \vec{w}$ in $\mathbb{R}^n$, $|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|$

Thm. Rules for Dot Products

(i) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ Commutative

(ii) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ Distributive

(iii) $(K\vec{v}) \cdot \vec{w} = K(\vec{v} \cdot \vec{w})$ Homogeneity

(iv) $\vec{v} \cdot \vec{v} > 0$ for all nonzero $\vec{v}$ Positive Definite