B52 Nov 24 Lec 1 Notes

Moments

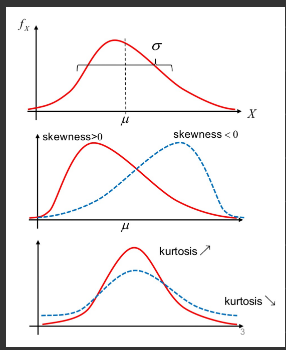
Moments of RV X are expected values of different powers of X.

r^{th} moment of X is defined as $E(X^r)$
↳ In particular, 1^{st} moment is the mean, $E(X) = \mu$.

r^{th} central moment of X is defined as $E[(X-\mu)^r]$
↳ In particular, 2^{nd} central moment is the variance, $E[(X-\mu)^2] = V(X)$

Note that $E[(X-\mu)^k] = E(X^k) + \dots + E(X^{k-1}) + \dots$

Moments of RV X describe different aspects of its distribution.
↳ Mean describes center
↳ Variance describes spread
↳ Skewness $E[(X-\mu)^3]$ describes symmetry
↳ Kurtosis $E[(X-\mu)^4]$ describes tail behaviour



Moment Generating Function

The Moment Generating Function (MGF) of RV X given by

$$m(t) = E(e^{tX})$$

$m(t)$ is well-defined when $m(t)$ is finite $\forall t < \varepsilon$, for some $\varepsilon > 0$.

MGF provides alternative way of characterizing a distribution

In particular, MGF allows calculation of all moments of X.

$$E(X^k) = m^{(k)}(0) = \frac{d^k}{dt^k} m(t) \Big|_{t=0}$$

Proof:

$$m(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} \Big(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!}\Big) f_X(x)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{-\infty}^{\infty} x^k f_X(x)\,dx$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \mu_k$$

$$= 1 + \frac{t}{1!}\mu + \frac{t^2}{2!}\mu_2 + \dots$$

$$m'(t) = \mu_1 + \frac{t}{1!}\mu_2 + \frac{t^2}{2!}\mu_3 + \dots$$
$$m''(t) = \mu_2 + \frac{t}{1!}\mu_3 + \frac{t^2}{2!}\mu_4 + \dots$$

$$\Rightarrow \quad m'(0) = \mu_1, \quad m''(0) = \mu_2, \quad \vdots \quad m^{(k)}(0) = \mu_k$$

Ex 1:

Find MGF of $X \sim \text{Exponential}$

$$m(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \int_0^{\infty} \lambda e^{-x(\lambda-t)}\,dx$$

$$= \lambda \frac{1}{(\lambda-t)} \int_0^{\infty} (\lambda-t) e^{-x(\lambda-t)}\,dx$$

$$= \frac{\lambda}{\lambda-t}, \quad \forall (\lambda-t)>0 \Rightarrow t<\lambda$$

Verify that $E(x) = 1/\lambda$

$$E(x) = m'(0) = \frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\Big|_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0}$$
$$= 1/\lambda$$

MGF Method

MGF uniquely characterizes the distribution of a RV

For RVs X, Y with MGFs $m_X(t)$, $m_Y(t)$

$$m_X(t) = m_Y(t) \Leftrightarrow X \sim Y$$

MGF can be used to find distribution of functions of RVs.

Let $Y = g(X_1, \dots, X_n)$ with $m_Y(t) = E(e^{tY}) = E(e^{t \cdot g(X_1, \dots, X_n)})$

If $m_Y(t)$ is MGF of some known distribution $\rightarrow$ Y follows that distribution.

MGF is particularly useful for linear functions of independent RVs

Let $Y = a_1 X_1 + \dots + a_n X_n$, where $X_1, \dots, X_n$ are independent with MGFs.
Then $m_Y(t) = m_{X_1}(a_1 t) \cdot \ldots \cdot m_{X_n}(a_n t) = \prod_{i=1}^{n} m_{X_i}(a_i t)$

Since $m_Y(t) = E[e^{tY}] = E[e^{t(a_1 X_1 + \dots + a_n X_n)}] = E[e^{a_1 t X_1} e^{a_2 t X_2} \cdot \ldots \cdot e^{a_n t X_n}]$
$$= E[e^{a_1 t X_1}] \cdot \ldots \cdot E[e^{a_n t X_n}]$$
$$= m_{X_1}(a_1 t) \cdot \ldots \cdot m_{X_n}(a_n t)$$

In particular, for i.i.d. $X_1, \dots, X_n$ and $Y = X_1 + \dots + X_n \Rightarrow m_Y(t) = (m_X(t))^n$

Ex 2:

Find MGF of Gamma(n, λ) distribution.

Let $Y \sim \text{Gamma}(n, \lambda) \Rightarrow f_Y(y) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}$, $y > 0$, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}$

$$m_Y(t) = E[e^{tY}] = \int_0^\infty e^{ty} \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y}\, dy$$

$$= \int_0^\infty \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-y(\lambda - t)}\, dy = \lambda^n \frac{1}{(\lambda-t)^n} \int_0^\infty \frac{(\lambda-t)^n}{\Gamma(n)} y^{n-1} e^{-y(\lambda-t)}\, dy$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^n , \quad t < \lambda$$

For i.i.d Exp(λ) RVs $X_1, \dots, X_n$, verify that $Y = X_1 + \dots + X_n$ follows Gamma(n, λ)

$$m_Y(t) = (m_X(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n \sim \text{Gamma}(n, \lambda)$$

Ex 3:

Let $X \sim \text{Uni}(\ell, u)$, and define $Y = aX + b$, for $a > 0$.

Show, using MGF, that $Y \sim \text{Uni}(a\ell + b, au + b)$.

$$m_X(t) = E[e^{tX}] = \int_\ell^u e^{tx} \frac{1}{u - \ell}\, dx$$

$$= \frac{1}{u-\ell} \left[e^{tx}/t \right]_\ell^u$$

$$= \frac{e^{tu} - e^{t\ell}}{t(u - \ell)}$$

$$m_Y(t) = E[e^{tY}] = E[e^{t(aX+b)}]$$

$$= e^{bt} E[e^{taX}]$$

$$= e^{bt} m_X(at)$$

$$= e^{bt} \left(\frac{e^{atu} - e^{at\ell}}{at(u-\ell)} \right)$$

$$= \frac{e^{t(au+b)} - e^{t(a\ell+b)}}{t[(au+b) - (a\ell+b)]}$$