W1 Jan 11 Lecture Notes

Sigma Notation

Let $m, k, n \in \mathbb{Z}^{\geq 0}$ s.t. $m \leq k \leq n$.

If a_k a real valued function of k, then:

$$a_m + a_{m+1} + \ldots + a_k + \ldots + a_n = \sum_{k=m}^{n} a_k$$

final value (n), general term (a_k), index (k), initial value (m)

Properties

(i) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(ii) For any $c \in \mathbb{R}$

$$\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$$

Proof:

If a_k is a real valued function of k, then $\forall c \in \mathbb{R}$, $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$.

Suppose $a_k \in \mathbb{R}$

We want to show $\forall c \in \mathbb{R}$, $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$

Let $c \in \mathbb{R}$ be an arbitrary number

Consider $\sum_{k=1}^{n} c\, a_k$:

$$= c a_1 + c a_2 + \ldots + c a_n \quad \text{by def of } \Sigma \text{ not}$$
$$= c(a_1 + a_2 + \ldots + a_n)$$
$$= c \cdot \sum_{k=1}^{n} a_k$$

QED

Examples

1. Evaluate $\sum_{i=1}^{203} (2i - 1)$

$$\sum_{i=1}^{203} (2i-1) = \sum_{i=1}^{203} 2i - \sum_{i=1}^{203} 1$$
$$= 2 \cdot \sum_{i=1}^{203} i - \sum_{i=1}^{203} 1$$
$$= 2 \cdot \frac{203(203+1)}{2} - 203$$
$$= 203^2$$

Formulas:

$\sum_{i=1}^{n} 1 = n$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{5}{n^4}(k^3+1)$

$$
\begin{aligned}
= \lim_{n\to\infty} \frac{5}{n^4}\cdot\sum_{k=1}^{n}(k^3+1) &= 5\cdot\lim_{n\to\infty}\frac{1}{n^4}\cdot\left(\sum_{k=1}^{n}k^3+\sum_{k=1}^{n}1\right)\\
&= 5\cdot\lim_{n\to\infty}\frac{1}{n^4}\cdot\left(\frac{n^2(n+1)^2}{4}+n\right)\\
&= 5\cdot\left(\lim_{n\to\infty}\frac{(n+1)^2}{4n^2}+\frac{1}{n^3}\right)\\
&= 5\cdot\left(\frac{1}{4}+0\right)\\
&= \frac{5}{4}
\end{aligned}
$$

W1 Jan 14 Lecture Notes

Def: Let $a,b\in\mathbb{R}$ s.t. $a<b$. A partition, P, of the interval $[a,b]$ is a finite collection of points in $[a,b]$ s.t. one point is a and one point is b.

Example:

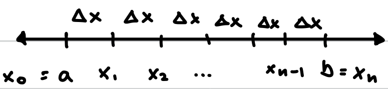
Consider $I=[0,1]$. Then $P=\{0,\frac{1}{5},\frac{1}{2},1\}$ is a partition of I

Example: Riemann Partition

Consider $I=[a,b]$, $a,b\in\mathbb{R}$, $a<b$

Then $P=\{x_0,x_1,\dots,x_n\}$

$=\{x_i\}_{i=0}^{n}$

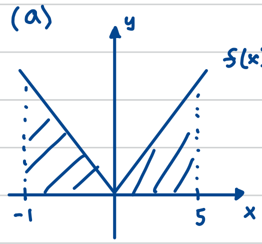


s.t. $x_i=a+i\Delta x$ where $\Delta x=\frac{b-a}{n}$

Examples:

1. What is the exact area A of the following regions:

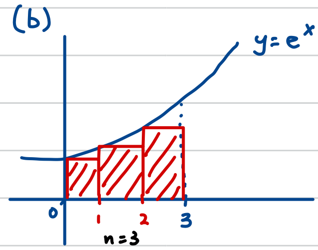
(a)



Area $=\frac{1}{2}(1)(1)+\frac{1}{2}(5)(5)$

$=13$

(b)



Given: $[0,3]$

Riemann Partition with $n=3$

$A\approx f(x_0)\Delta x+f(x_1)\Delta x+f(x_2)\Delta x$

$\approx\sum_{i=1}^{3}f(x_{i-1})\Delta x$

Left Riemann Sum for f on $[a,b]$ (L_n)

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

Right Riemann Sum for f on $[a,b]$ (R_n)

$$\sum_{i=1}^{n} f(x_i)\,\Delta x$$

Generic Riemann Sum of f on $[a,b]$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad \text{, where } x_i^* \in [x_{i-1}, x_i]$$

x_i is called the sample point