W11 Lecture 20 Notes

Examples:

1. Sketch $y = \frac{x}{\sqrt[3]{x^2-1}}$.

1) Domain

$$\text{Dom} f = (-\infty,-1) \cup (-1,1) \cup (1,\infty)$$

2) Intercepts

$y=0$, $x=0$ $\Rightarrow$ $(0,0)$

3) Symmetry

$f(-x) = \frac{-x}{\sqrt[3]{(-x)^2-1}} = -\frac{x}{\sqrt[3]{x^2-1}} = -f(x) \Rightarrow f(x)$ is odd $\Rightarrow$ Consider $[0,\infty)$

4) Asymptotes

VA: $x=-1$, $x=1$

$$\lim_{x\to 1^-} \frac{x}{\sqrt[3]{(x+1)(x-1)}} = \frac{1}{0^-} = -\infty$$

$$\lim_{x\to 1^+} \frac{x}{\sqrt[3]{(x+1)(x-1)}} = \frac{1}{0^+} = \infty$$

5) Slant Asymptotes

Right SA: $y = k_1 x + b_1$, where $k_1 = \lim_{x\to\infty} \frac{f(x)}{x}$, $b_1 = \lim_{x\to\infty} | f(x) - k_1 x |$

$$k_1 = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt[3]{x^2-1}} = \frac{1}{\infty} = 0 \quad \Rightarrow \text{No RSA}$$

$$b_1 = \lim_{x\to\infty} \frac{x}{\sqrt[3]{x^2-1}} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt[3]{\frac{x^2}{x^3} - \frac{1}{x^3}}} = \infty$$

Left SA: $y = k_2 x + b_2$, where $k_2 = \lim_{x\to-\infty} \frac{f(x)}{x}$, $b_2 = \lim_{x\to-\infty} | f(x) - k_2 x |$

No LSA b/c of symmetry.

No HA.

5) Derivatives

$$f'(x) = \frac{x^2-3}{(x^2-1)^{4/3}}$$

$$f''(x) = \frac{2x(9-x^2)}{9\sqrt[3]{(x^2-1)^2}}$$

6) Critical Points

$f'(x) = 0$: $x^2-3=0 \Rightarrow x=-\sqrt{3}$, $\boxed{x=\sqrt{3}} \in \text{Dom } f(x)$

$f'(x) = \text{DNE}$: $x=-1$, $x=1 \notin \text{Dom } f(x)$

Classification:

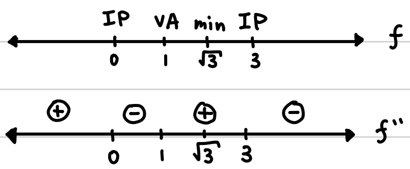
$$f''(\sqrt{3}) = \frac{2\sqrt{3}(9-3)}{9\sqrt[3]{(3-1)^7}} > 0 \Rightarrow$$ f(x) has a local min at $x=\sqrt{3}$

7) Points of Inflection

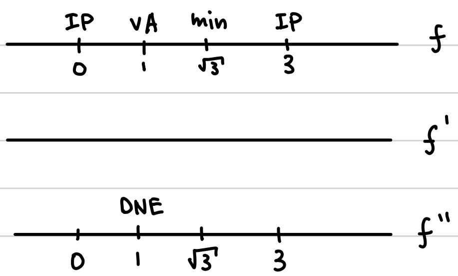
$f''(x) = 0$: $\boxed{x_1 = 0}$, $x=-3$, $\boxed{x_2 = 3}$

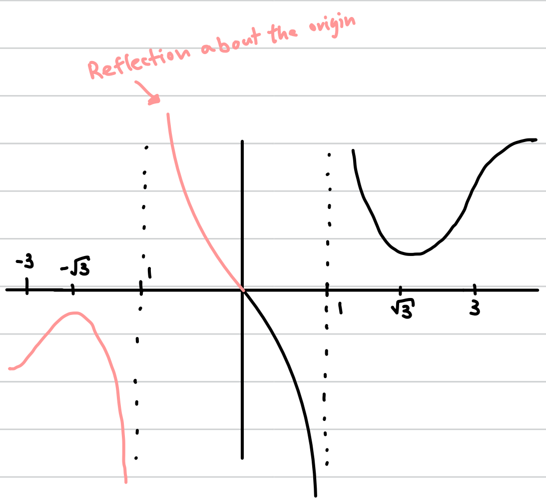
$f''(x) = \text{DNE}$: $x=-1$, $x=1 \notin \text{Dom } f(x)$

8) Concavity



9) Graph





2. Sketch $y = \sqrt{x^2+4x}$

1) **Domain**

$x^2 + 4x \geq 0$

$x(x+4) \geq 0$

$\oplus \quad\quad -4 \quad\quad \ominus \quad\quad 0 \quad\quad \oplus$

$\mathrm{Dom}\, f(x) = (-\infty, -4] \cup [0, \infty)$

2) **Intercepts**

$y = 0$:

$(0,0)$ and $(-4,0)$

3) **Symmetry**

$f(-x) \neq -f(x) \neq f(x)$

4) **Asymptotes**

a) VA: None

c) HA: None

b) SA:

RSA: $k_1 = \lim_{x\to\infty} \frac{\sqrt{x^2+4x}}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{1+4/x}}{x} = \lim_{x\to\infty} \sqrt{1+4/x} = 1$

$b_1 = \lim_{x\to\infty} \left|\sqrt{x^2+4x} - x\right| = \lim_{x\to\infty} \frac{(\sqrt{x^2+4x}-x)(\sqrt{x^2+4x}+x)}{(\sqrt{x^2+4x}+x)}$

$= \lim_{x\to\infty} \frac{4x}{|x|\sqrt{1+4/x}+x}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1+4/x}+1} = \frac{4}{2} = 2$

$\boxed{\text{RSA} = y = x + 2}$

LSA: $k_2 = \lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{x} = \lim_{x\to-\infty} \frac{|x|\sqrt{1+4/x}}{x} = \lim_{x\to-\infty} \frac{(-x)\sqrt{1+4/x}}{x} = -1$

$b_1 = \lim_{x\to-\infty} \left|\sqrt{x^2+4x} + x\right| = \lim_{x\to-\infty} \frac{(\sqrt{x^2+4x}+x)(\sqrt{x^2+4x}-x)}{(\sqrt{x^2+4x}-x)}$

$= \lim_{x\to-\infty} \frac{4x}{|x|\sqrt{1+4/x}-x}$

$= \lim_{x\to-\infty} \frac{4x}{(-x)\sqrt{1+4/x}-x}$

$= \lim_{x\to-\infty} \frac{4}{-\sqrt{1+4/x}-1} = \frac{4}{-2} = -2$

$\boxed{\text{LSA} = y = -x - 2}$

5) Derivatives

$$f'(x) = \frac{x+2}{\sqrt{x^2+4x}}$$

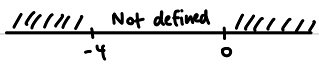
$$f''(x) = \frac{-4}{(x^2+4x)^{3/2}}$$

6) Critical Points

$f'(x) = 0$: $x+2=0 \Rightarrow x=-2 \notin \text{Dom } f(x)$

$f'(x) = \text{DNE}$: $x^2+4x=0 \Rightarrow x=0,\ x=-4 \in \text{Dom } f(x)$

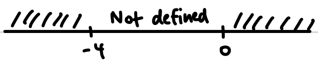
$f(x)$ has VA at $x=0$, $x=-4$.



7) Points of inflection

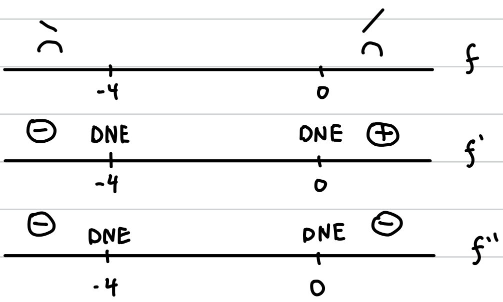
$f''(x) = 0$: No such points.

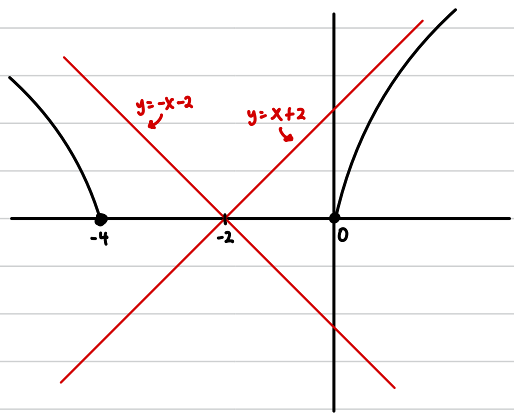
$f''(x) = \text{DNE}$. $x=0$, $x=-4$



Thus no IPs

8) Graph





L'Hospital's Rule for Indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Geometrical Motivation

Let $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ $\quad \lim\limits_{x\to c} f(x) = 0$; $\lim\limits_{x\to c} g(x) = 0$

Linear approx. of $f(x)$ and $g(x)$ in the neighbourhood of point $x=c$:

$$f(x) \approx f(c) + f'(c)(x-c)$$

$$g(x) \approx g(c) + g'(c)(x-c)$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} \approx \lim_{x\to c} \frac{\cancel{f(c)}^{0} + f'(c)\cancel{(x-c)}}{\cancel{g(c)}^{0} + g'(c)\cancel{(x-c)}} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)}$$

L'Hospital Rule for Indeterminate forms 0^∞, 1^∞ and ∞^0

1) If $\lim\limits_{x\to c} \ln(f(x)) = L$, then $\lim\limits_{x\to c} f(x) = e^L$.
2) If $\lim\limits_{x\to c} \ln(f(x)) = \infty$, then $\lim\limits_{x\to c} f(x) = \infty$.
3) If $\lim\limits_{x\to c} \ln(f(x)) = -\infty$, then $\lim\limits_{x\to c} f(x) = 0$.

Let $\lim\limits_{x\to c} [f(x)]^{g(x)}$ is of the form ∞^0

$$\lim_{x\to c} e^{\ln [f(x)]^{g(x)}} = \lim_{x\to c} e^{g(x)\ln(f(x))} = e^{\lim\limits_{x\to c} [g(x) \ln f(x)]} = e^L$$

$$\lim_{x\to c} [g(x)\cdot \ln f(x)] = \lim_{x\to c} \frac{\ln f(x)}{1/g(x)} = \dots L$$

Using L'Hospital

Examples:

1. $\lim\limits_{x\to 1^+} (x-1)^{\ln x}$

$$= \lim_{x\to 1^+} e^{\ln[(x-1)^{\ln x}]} = \lim_{x\to 1^+} e^{\ln x \cdot \ln(x-1)} = e^0 = 1$$

Aside:

$$\lim_{x\to 1^+} \frac{\ln(x-1)}{1/\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}} = \lim_{x\to 1^+} \frac{-x\ln^2 x}{x-1} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 1^+} -\frac{\ln^2 x + 2\ln x \cdot \frac{1}{x}\cdot x}{1}$$

$$= \lim_{x\to 1^+} -\ln^2 x - 2\ln x$$

$$= 0$$

2. Sketch the graph of $y = \ln(\sinh x)$

$$y = \ln\left(\frac{e^x - e^{-x}}{2}\right)$$

1) Domain

$$\frac{e^x - e^{-x}}{2} > 0$$

$\text{Dom } f(x) = (0, \infty)$

$$e^x > e^{-x}$$
$$e^{2x} > 1$$
$$2x > 0$$
$$x > 0$$

2) Intercepts

$$\frac{e^x - e^{-x}}{2} = 1$$

$$e^x - e^{-x} = 2$$
$$e^{2x} - 2e^x - 1 = 0$$

exclude $1 - \sqrt{2} < 0$

$$e^x = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2} \Rightarrow x = \ln(1 \pm \sqrt{2}) \Rightarrow x = \ln(2.4)$$

X-int = $(\ln 2.4, 0)$

Y-int = None

3) Symmetry

None

4) Asymptotes

VA:

$$\lim_{x \to 0} \left(\ln\left(\frac{e^x - e^{-x}}{2}\right)\right) = \pm\infty \Rightarrow \frac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = e^0 \Rightarrow \boxed{x = 0}$$

SA:

$$\text{RSA}: \quad K_1 = \lim_{x \to \infty} \frac{\ln\left(\frac{e^x - e^{-x}}{2}\right)}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x \to \infty} \frac{\frac{2}{e^x - e^{-x}}\left(\frac{e^x + e^{-x}}{2}\right)}{1}$$

$$= \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$= \lim_{x \to \infty} \frac{e^{2x} + 1}{e^{2x} - 1} = \lim_{x \to \infty} \frac{e^{2x}}{e^{2x}} = 1$$

RSA:

$$b_1 = \lim_{x\to\infty}\left[\ln \frac{e^x - e^{-x}}{2} - x\right] = \lim_{x\to\infty}\left(\ln \frac{e^x-e^{-x}}{2} - \ln e^x\right) = \lim_{x\to\infty} \ln\left(\frac{e^x - e^{-x}}{2e^x}\right)$$
$$= \lim_{x\to\infty} \ln\left(\frac{e^{2x}-1}{2e^{2x}}\right)$$
$$= \ln\left(\lim_{x\to\infty} \frac{e^{2x}-1}{2e^{2x}}\right) \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \ln\left(\lim_{x\to\infty}\left(\frac{2e^{2x}}{4e^{2x}}\right)\right) = \ln \tfrac{1}{2}$$
$$= -\ln 2$$

SA: $\boxed{y = x - \ln 2}$

LSA: None b/c of Dom $f(x) = (0, \infty)$

HA: None

5) Derivatives

$$f'(x) = \frac{e^{2x}+1}{e^{2x}-1}$$

$$f''(x) = \frac{-4e^{2x}}{(e^{2x}-1)^2}$$

6) Critical Points

$f'(x) = 0$: No such points

$f'(x) = \text{DNE}$: $e^{2x} - 1 = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0 \notin \text{Dom} f(x)$

7) Points of Inflection and Concavity

$f''(x) = 0$: No such points.

8) Graph