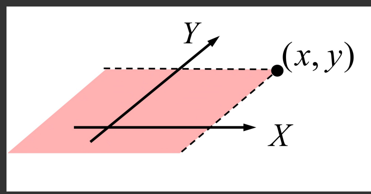
B52 Nov 3 Lec 1 Notes

Joint CDF

For arbitrary RVs X,Y, joint CDF provides probabilities of type



$$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$$
$$= P(\{X \leq x\} \cap \{Y \leq y\})$$

Joint CDF describes joint distributions of a RVs. We can use this to find marginal CDFs.

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = F_{X,Y}(x,\infty)$$

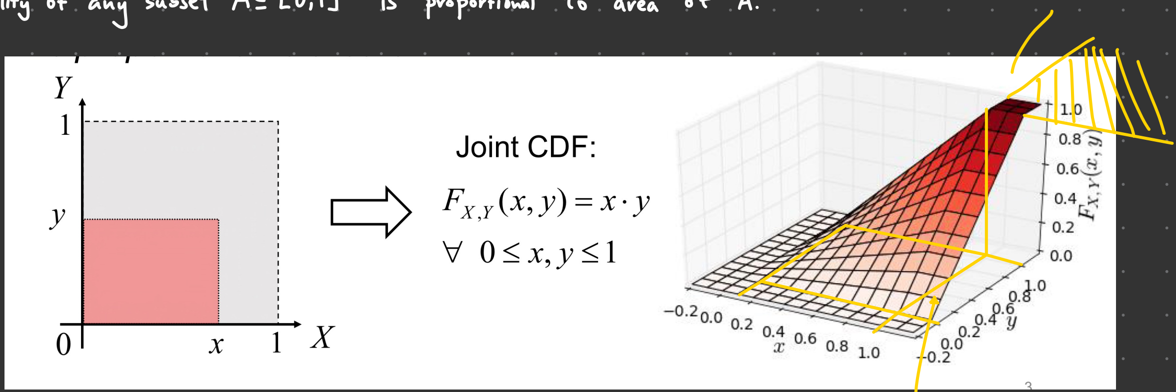
$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = F_{X,Y}(\infty,y)$$

Ex 1: (2D Uniform)

Let X,Y be 2D uniform RVs over unit square:

Probability of any subset $A \subseteq [0,1]^2$ is proportional to area of A.

If $x,y > 1$, $F_{X,Y}(x,y) = 1$



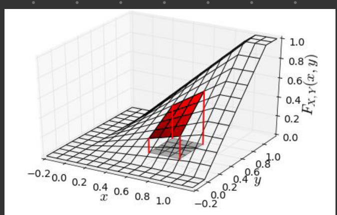
unit square

Ex 2: (Continued from Ex 1)

Find $P(0.5 \leq X \leq 0.8, 0.3 \leq Y \leq 0.7)$

$= (0.3) \times (0.4)$

$= 0.12$

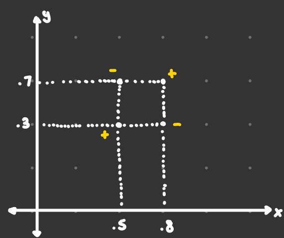


OR

$= F_{XY}(0.8, 0.7) - F_{XY}(0.5, 0.7)$

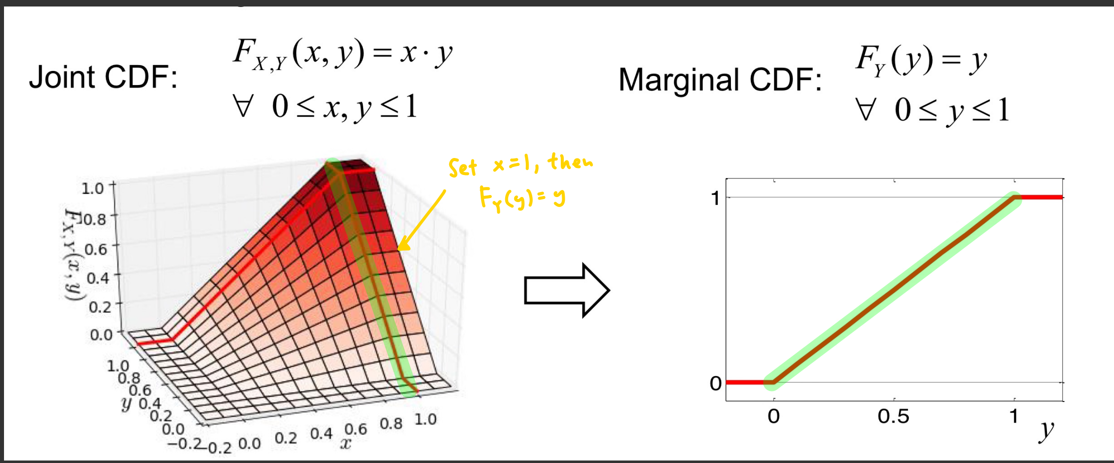
$\quad - F_{XY}(0.8, 0.3) + F_{XY}(0.5, 0.3)$

$= 0.12$



Ex 3: (Continued from Ex 1)

Find marginal CDF of Y.



Joint CDF:

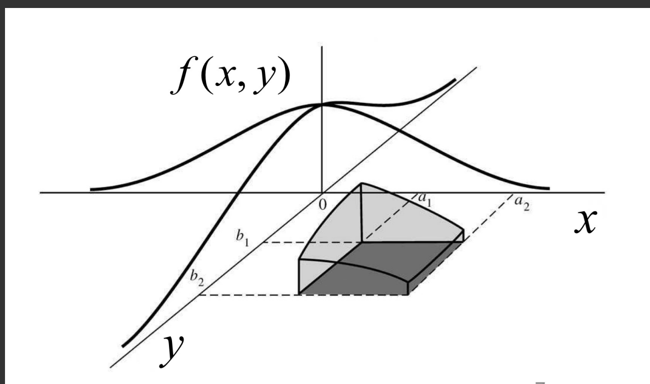
Joint CDF is useful for finding probabilities of rectangular areas of values.

Need different tool for calculating probabilities of arbitrary regions.

For continuous RVs X, Y, the joint PDF is a function $f_{X,Y}(x,y)$ s.t.

$$P((X,Y) \in R) = \iint_R f_{X,Y}(x,y)\, dx\, dy$$

We can think of probabilities as volume contained under function $f_{X,Y}$ and over region R.



Properties of joint PDFs:

(i) $f_{X,Y}(x,y) \ge 0$

(ii) $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\, dx\, dy = 1$

Relationship between joint PDF and joint CDF

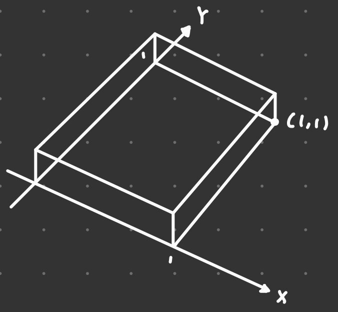
(i) $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\, ds\, dt$, $\forall x, y \in \mathbb{R}$

(ii) $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$, $\forall x, y$ where derivatives exists

Ex 4: (2D Uniform)

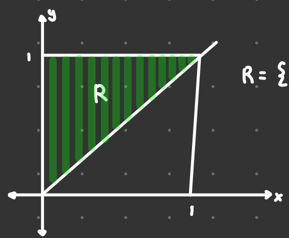
Consider RVs X, Y with joint CDF $F_{X,Y}(x,y) = x \cdot y$, $0 \le x, y \le 1$.

(i) Find their joint PDF:



$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} x \cdot y\right)$$
$$= \frac{\partial}{\partial x}(x)$$
$$= 1 \ , \ \forall x,y \in (0,1)$$

(ii) Calculate $P(X<Y)$



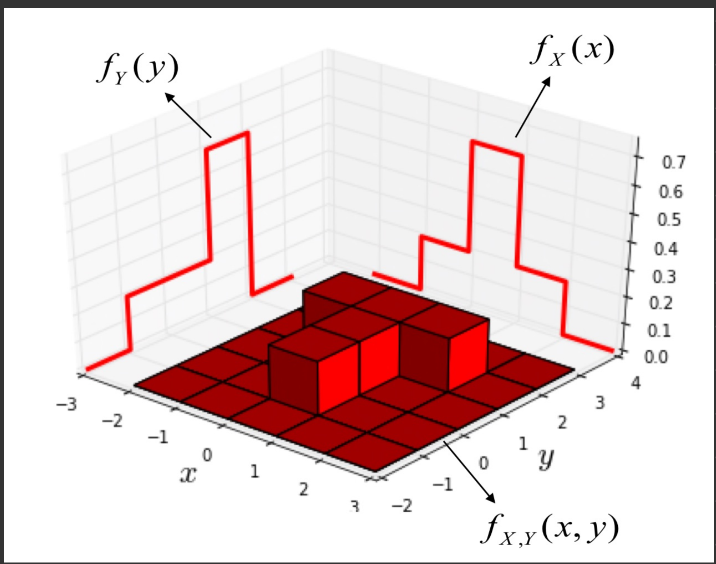
$R = \{(x,y) : 0 \le x,y \le 1 \ , \ y > x\}$

$$P(X<Y) = \iint_R f_{XY}(x,y)\,dx\,dy$$
$$= \int_0^1 \left[\int_x^1 f_{XY}(x,y)\,dy \right] dx$$
$$= \int_0^1 \int_x^1 1 \,dy\,dx$$
$$= \int_0^1 [y]_x^1 \,dx$$
$$= \int_0^1 (1-x)\,dx$$
$$= 1 - \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

Marginal PDF

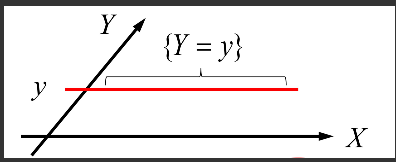
For RVs X, Y, with joint PDF $f_{X,Y}$, the marginal PDF of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$



Conditional PDF

With continuous RVs X, Y, can condition on lower dimensional space with 0 probability
e.g. condition on $\{Y=y\}$, which is a line.



Conditioning on $\{Y=y\}$ restricts effective "sample space" from real plane to real line $\{Y=y\}$.

$$\begin{cases} (X, Y) \text{ values live in real plane } \mathbb{R}^2 \\ (X, Y=y) \text{ values live in real plane } \mathbb{R} \end{cases}$$

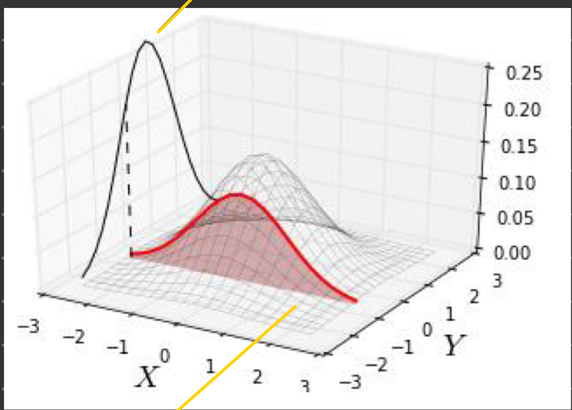
Let continuous RVs X, Y have joint PDF $f_{X,Y}(x,y)$

Conditional PDF of X given $Y=y$ defined as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \text{ for } f_Y(y) > 0$$

$f_{X|Y}(x|y)$ cuts slice of $f_{X,Y}(x,y)$ at $Y=y$ and scales it by $1/f_Y(y)$ so that it integrates to 1.

marginal PDF of Y, $f_Y(y)$ = slice area



slice of joint PDF
$f_{X,Y}(x,y)$, along $Y=y$