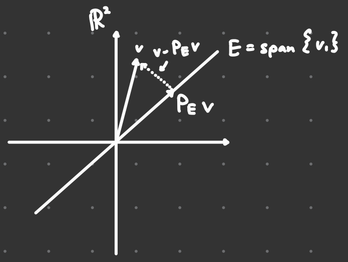
B24 July 14 Lec 1 Notes



We would like to define, in V, $E \subset V$ as a subspace, $v \in V$, the projection $P_E v$ as the unique vector s.t. :

(i) $P_E v \in E$
(ii) $v - P_E v \perp E$

Proposition:

Let V be IPS, $E \subset V$ s.s. and $v_1, \dots, v_n$ is an orthogonal basis for E. Then the vector:

$$w := \frac{\langle v, v_1 \rangle}{\| v_1 \|^2} v_1 + \dots + \frac{\langle v, v_n \rangle}{\| v_n \|^2} v_n$$

Satisfies $w \in E$ and $v - w \perp E$

Proof:

$w \in E$ since w is a linear combination of $v_1, \dots, v_n \in E$.

To show $v - w \perp E$, it suffices to show $v - w \perp v_k$ for $1 \leq k \leq n$

$$\langle v - w, v_k \rangle = \langle v - \left(\frac{\langle v, v_1 \rangle}{\| v_1 \|^2} v_1 + \dots + \frac{\langle v, v_n \rangle}{\| v_n \|^2} v_n \right), v_k \rangle$$

$$= \langle v, v_k \rangle - \underbrace{\langle \frac{\langle v, v_1 \rangle}{\| v_1 \|^2} v_1 + \dots + \frac{\langle v, v_n \rangle}{\| v_n \|^2} v_n, v_k \rangle}$$

$$= \langle \frac{\langle v, v_k \rangle}{\| v_k \|^2} v_k, v_k \rangle$$

$$= \frac{\langle v, v_k \rangle}{\cancel{\| v_k \|^2}} \cdot \cancel{\| v_k \|^2}$$

$$= \langle v, v_k \rangle - \langle v, v_k \rangle$$
$$= 0$$ ☒

Theorem:

Let $E \subset V$ be a subspace, V is an IPS, $v \in V$, and let $w \in E$ be s.t. $v - w \perp E$. Then, if $x \in E$, we have

$$\| v - w \| \leq \| v - x \|$$

and if $x \in E$ s.t.

$$\| v - w \| = \| v - x \|$$

then $x = w$.

Proof:

$$v - x = v - w + \underbrace{w - x}_{\in E}$$

So $v-w \perp w-x$, thus the pythagorean theorem gives

$$\|v-x\|^2 = \|v-w\|^2 + \|w-x\|^2$$
$$\geq \|v-w\|^2$$

And if $\|v-w\| = \|v-x\|$, then

$$\cancel{\|v-x\|^2} = \cancel{\|v-x\|^2} + \|w-x\|^2$$

$$\Rightarrow 0 = \|w-x\|^2 \Rightarrow w = x \quad \boxtimes$$

Corollary:

Let $E \subset V$ be a subspace, V is IPS, $v \in V$. Then there is at most one vector $w \in E$ s.t. $v - w \perp E$.

Definition:

If V is a finite-dimensional IPS, and $E \subset V$ s.s., and $v \in V$, then $P_E v$ is defined as the unique vector satisfying:

(i) $P_E v \in E$
(ii) $v - P_E v \perp E$

Proposition:

$P_E : V \to V$ is a L.T.

Proof:

Consider the formula:

$$P_E v := \frac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \frac{\langle v, v_n \rangle}{\|v_n\|^2} v_n$$

(where $v_1, \dots, v_n$ is an orthogonal basis for E.)

Question:

How do we find orthogonal bases?

$$\text{Basis} \xrightarrow{\text{Gram-Schmidt orthogonalization algorithm}} \text{Orthogonal basis}$$

Algorithm:

Suppose $x_1, \dots, x_n \in V$ are L.I.

Step 1: Let $v_1 := x_1$, and $E_1 := \operatorname{span}\{x_1\}$

Step r+1: (after step r)

Let $v_{r+1} := x_{r+1} - P_{E_r}(x_{r+1})$

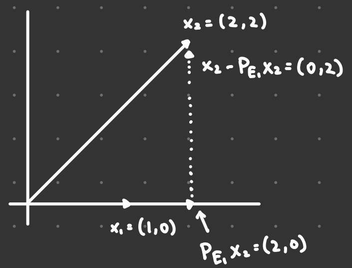
$$= x_{r+1} - \sum_{k=1}^{r} \frac{\langle x_{r+1}, v_k \rangle}{\|v_k\|^2} v_k$$

and $E_{r+1} := \operatorname{span}\{x_1, \dots, x_{r+1}\} = \operatorname{span}\{v_1, \dots, v_{r+1}\}$

Then $v_1, \dots, v_n$ is an orthogonal system with

$$\operatorname{span}\{x_1, \dots, x_n\} = \operatorname{span}\{v_1, \dots, v_n\}$$

Ex 1:



$v_1 = x_1 = (1,0)$

$E_1 = \operatorname{span}\{x_1\} = \mathbb{R}$

$v_2 = x_2 - P_{E_1}x_2 = (0,2)$

$\operatorname{Span}(x_1, x_2) = \operatorname{span}(v_1, v_2)$ and v_1, v_2 is an orthogonal system.

Ex 2:

$x_1 = (1,1,1)$, $x_2 = (0,1,2)$, $x_3 = (1,0,2) \in \mathbb{R}^3$

Step 1: $v_1 := x_1 = (1,1,1)$, $E_1 = \operatorname{span}\{(1,1,1)\}$

Step 2: $v_2 := x_2 - P_{E_1}x_2$

$$= (0,1,2) - \frac{\langle (0,1,2), (1,1,1) \rangle}{\|(1,1,1)\|^2} (1,1,1)$$

$$= (-1,0,1)$$

$E_2 := \operatorname{span}\{(1,1,1), (0,1,2)\} = \operatorname{span}\{(1,1,1), (-1,0,1)\}$

Step 3: $v_3 := x_3 - \left[\frac{\langle x_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle x_3, v_2 \rangle}{\|v_2\|^2} v_2 \right]$

$$= (1,0,2) - \left[\frac{\langle (1,0,2), (1,1,1) \rangle}{\|(1,1,1)\|^2} (1,1,1) + \frac{\langle (1,0,2), (-1,0,1) \rangle}{\|(-1,0,1)\|^2} (-1,0,1) \right]$$