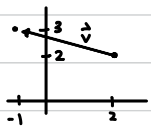
Tut 1

Warm-Up I

1. $\mathbb{R}^n$ - The set of all vectors with n components
2. Vector Addition in $\mathbb{R}^n$ - $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{pmatrix}$
3. Scalar Multiplication in $\mathbb{R}^n$ -
$$k\vec{v} = k\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{pmatrix}$$
4. Norm of a vector $\vec{v}$ in $\mathbb{R}^n$ -
$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$
5. Given $\vec{v}, \vec{w} \in \mathbb{R}^n$, the dot product $\vec{v}\cdot\vec{w}$ -
$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n$$
$$\vec{v}\cdot\vec{w} = \|\vec{v}\| \cos\theta \, \|\vec{w}\|$$

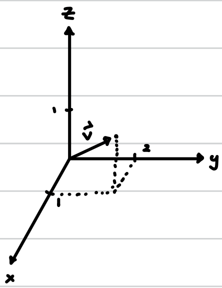
Warm-Up II

1a.



$\vec{v} = \langle -1-2, 3-2 \rangle$
$= \langle -3, 1 \rangle$

1b.



$\vec{v} = \langle 1-0, 2-0, 1-0 \rangle$
$= \langle 1, 2, 1 \rangle$

2a. $(1,-1), (2,0), (5,3)$

$\vec{v} = \langle 1-2, -1-0 \rangle$
$= \langle -1, -1 \rangle$

parametric equation of the line $= \langle 1,-1 \rangle + t\langle -1,-1 \rangle$

When $t=-4$, we get $\langle 5,3 \rangle$
When $t=-1$, we get $\langle 2,0 \rangle$
when $t=0$, we get $\langle 1,-1 \rangle$

Thus they are all on the same line.

26. $(1,0,2)$, $(4,7,-1)$, $(10,14,-5)$

$\vec{v} = \langle 1-4, 0-7, 2+1 \rangle$
$= \langle -3, -7, 3 \rangle$

parametric equation $= \langle 1,0,2 \rangle + t\langle -3,-7,3 \rangle$

$t=0 \Rightarrow \langle 1,0,2 \rangle$
$t=-1 \Rightarrow \langle 4,7,-1 \rangle$
$t=-3 \Rightarrow \langle 10,21,-7 \rangle$

$-3t+1=10 \qquad t=-3$
$-7t+0=14 \Rightarrow t=-2$
$3t+2=-5 \qquad t=-\frac{7}{3}$

Since all the t values are different, $(10,14,-5)$ is not on the parametric line consisting of $(1,0,2)$ and $(4,7,-1)$

A1. $2\vec{v}+\vec{u} = 2\langle 2,-1 \rangle + \langle 1,1 \rangle$
$= \langle 4+1, -2+1 \rangle$
$= \langle 5, -1 \rangle$

A2. $-\vec{v}+3\vec{u} = -\langle 2,-1 \rangle + 3\langle 1,1 \rangle$
$= \langle -2+3, 1+3 \rangle$
$= \langle 1, 4 \rangle$

A3. $(-5)\vec{v} = \langle -5(2), -5(-1) \rangle$
$= \langle -10, 5 \rangle$

B3. Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be in $\mathbb{R}^n$

$\vec{v}+\vec{0} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ By def of vector addition

$= \begin{pmatrix} v_1+0 \\ v_2+0 \\ \vdots \\ v_n+0 \end{pmatrix}$ By def of vector addition

$= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ 0 is additive identity over $\mathbb{R}$

$= \vec{v}$

B5. $r(\vec{v}+\vec{w}) = r\vec{v} + r\vec{w}$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ be in $\mathbb{R}^n$ and r be some scalar.

$r(\vec{v}+\vec{w}) = r\left(\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}\right)$ By def of vector addition

$= r\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}$ By def of vector addition

$= \begin{pmatrix} r(v_1+w_1) \\ r(v_2+w_2) \\ \vdots \\ r(v_n+w_n) \end{pmatrix}$ By def of scalar multiplication

$= \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \\ \vdots \\ rv_n + rw_n \end{pmatrix}$ By distributive property in $\mathbb{R}$

$= \begin{pmatrix} rv_1 \\ rv_2 \\ \vdots \\ rv_n \end{pmatrix} + \begin{pmatrix} rw_1 \\ rw_2 \\ \vdots \\ rw_n \end{pmatrix}$ By def of vector addition

$= r\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + r\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ By def of scalar multiplication

$= r\vec{v} + r\vec{w}$

B6. $(r+s)\vec{v} = r\vec{v} + s\vec{v}$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be in $\mathbb{R}^n$ and r, s be scalars.

$(r+s)\vec{v} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \\ \vdots \\ (r+s)v_n \end{pmatrix}$ By def of scalar multiplication

$= \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \\ \vdots \\ rv_n + sv_n \end{pmatrix}$ By distributive property over $\mathbb{R}$

$= \begin{pmatrix} rv_1 \\ rv_2 \\ \vdots \\ rv_n \end{pmatrix} + \begin{pmatrix} sv_1 \\ sv_2 \\ \vdots \\ sv_n \end{pmatrix}$ By def of vector addition

By def of scalar multiplication

$= r\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + s\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$= r\vec{v} + s\vec{v}$

C1. Dot product is commutative

$\vec{v} \cdot \vec{w} = \vec{w} \ \vec{v}$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ be in $\mathbb{R}^n$

$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$ By def of dot product

$= w_1 v_1 + w_2 v_2 + \dots + w_n v_n$ By commutative property over $\mathbb{R}$

$= \vec{w} \cdot \vec{v}$ By def of dot product

C2. $(r\vec{v}) \cdot \vec{w} = \vec{v} \cdot (r\vec{w})$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ be in $\mathbb{R}^n$ and r be some scalar.

$(r\vec{v}) \cdot \vec{w} = (rv_1)w_1 + (rv_2)w_2 + \dots + (rv_n)w_n$ By def of dot product

$= v_1(rw_1) + v_2(rw_2) + \dots + v_n(rw_n)$ By commutative property over $\mathbb{R}$

$= \vec{v} \cdot (r\vec{w})$ By def of dot product

C3. $\|r\vec{v}\| = r\|\vec{v}\|$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be in $\mathbb{R}^n$ and r be some scalar.

$\|r\vec{v}\| = \sqrt{(r\vec{v}) \cdot (r\vec{v})}$ By def of norm of vector

$= \sqrt{(rv_1)^2 + (rv_2)^2 + \dots + (rv_n)^2}$ By def of dot product

$= \sqrt{r^2 v_1^2 + r^2 v_2^2 + \dots + r^2 v_n^2}$ By distributive property of exponents

$= \sqrt{r^2 (v_1^2 + v_2^2 + \dots + v_n^2)}$ By distributive property over $\mathbb{R}$

$= \sqrt{r^2} \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$ By product property of square roots

$= r\|\vec{v}\|$ Def of norm of vector

C4. Prove that $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular iff $\|\vec{v}\| = \|\vec{w}\|$

Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$

Proof ($\rightarrow$): $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular $\rightarrow$ $\|\vec{v}\| = \|\vec{w}\|$

Since $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular,

$0 = (\vec{v}-\vec{w}) \cdot (\vec{v}+\vec{w})$

$0 = (v_1 - w_1)(v_1 + w_1) + (v_2 - w_2)(v_2 + w_2) + \ldots + (v_n - w_n)(v_n + w_n)$ By def of vector addition and dot product.

$0 = (v_1^2 - w_1^2) + (v_2^2 - w_2^2) + \ldots + (v_n^2 - w_n^2)$ Diff of squares

$w_1^2 + w_2^2 + \ldots + w_n^2 = v_1^2 + v_2^2 + \ldots + v_n^2$ Rearranging

$\|\vec{w}\|^2 = \|\vec{v}\|^2$ By Def of norm of vector

$\|\vec{w}\| = \|\vec{v}\|$

Proof ($\leftarrow$): $\|\vec{v}\| = \|\vec{w}\|$ $\rightarrow$ $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular

Backwards of ($\rightarrow$)

C5. Prove that the angle between two unit vectors in $\mathbb{R}^n$ is the arc cos of their dot product.

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be unit vectors in $\mathbb{R}^n$

Cool - Off:

1. Give example of 2 orthogonal vectors in $\mathbb{R}^5$

2 vectors are orthogonal if the dot product is 0

$\vec{u} = \langle 1, 1, 1, 1, 1 \rangle$
$\vec{v} = \langle -1, 1, -1, 1, 0 \rangle$

$\vec{u} \cdot \vec{v} = 1(-1) + 1(1) + 1(-1) + 1(1) + 1(0)$
$= 0$

2. Give example of 3 mutually orthogonal vectors in $\mathbb{R}^3$

$\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$, where $\vec{u} \cdot \vec{v} = 0$