B52 Oct 20 Lec 1 Notes

Expected Value

RV X measures some quantity of random experiment

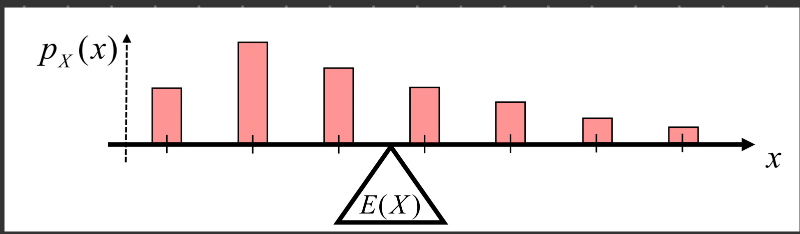
For discrete RV X with PMF $p_X(x)$, expected value is

$$E(X) = \sum_x x\, p_X(x)$$

Consider RV X, and assume you repeat experiment #n times, getting values $(c_1, \dots, c_n)$. Average of #n values is

$$\frac{1}{n}\sum_i c_i = \sum_x x \frac{\#\text{ times } c_i = x}{n} \approx E(X)$$

Expected value of RV X is center of gravity of distribution



Ex 1:

$X \sim \text{Geometric}(p) \Rightarrow p_X(x) = p \cdot q^{x-1}, \ \forall x \in \mathbb{N}_+$

$$E(X) = \sum_{x=1}^{\infty} x \cdot p_X(x) = \sum_{x=1}^{\infty} x \cdot p \cdot q^{x-1}$$

$$= \sum_{y=0}^{\infty} (y+1)\, p \cdot q^y \qquad y = x-1$$

$$= \sum_{y=0}^{\infty} y \cdot p \cdot q^y + \sum_{y=0}^{\infty} p \cdot q^y$$

$$= q\left(\sum_{y=1}^{\infty} y \cdot p \cdot q^{y-1}\right) + \frac{p}{1-q}$$

$$= q\left(\sum_{y=1}^{\infty} y \cdot p \cdot q^{y-1}\right) + 1$$

$$= q \cdot E(X) + 1$$

$$\Rightarrow E(X) = \frac{1}{1-q} = \frac{1}{p}$$

Expected Value – Indicator RVs

Consider indicator RV $I_A(s) = \begin{cases} 1 & , s \in A \\ 0 & , s \notin A \end{cases}$, for some $A \subseteq S$

Theorem: $E(I_A) = P(A)$

Proof:

$$E(I_A) = \sum_{x=0,1} x \cdot P_{I_A}(x) = 0 \cdot P_{I_A}(0) + 1 \cdot P_{I_A}(1)$$
$$= P_{I_A}(1)$$
$$= P(I_A = 1)$$
$$= P(A)$$

$\boxtimes$

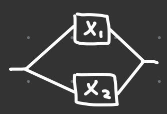
Consider discrete RV X with known distribution, and assume we want $E(Y) = E(g(X))$.

$$E(Y) = E(g(X)) = \sum_x g(x)\, p_X(x)$$

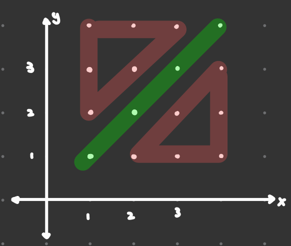
For multivariate function $Z = g(x,y)$ of discrete RVs X, Y

$$E[Z] = E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)$$

Ex 2:



Let X_1, X_2 be indep. Geom(½) $\Rightarrow p_{X_1,X_2}(x,y) = P_{X_1}(x) \cdot P_{X_2}(y)$ Since independent
$= p \cdot q^{x-1} \cdot p \cdot q^{y-1}$
$= (\frac{1}{2})^{x+y}$



(completion time for both processes in parallel) : $g(x_1, x_2) = \max(x_1, x_2)$

Want to find
$$E[\max(X_1, X_2)] = \sum_{x,y=1}^{\infty} \max(x,y) \cdot p_{X_1,X_2}(x,y)$$
$$= \sum_{x,y=1}^{\infty} \max(x,y) \left(\tfrac{1}{2}\right)^{x+y}$$
$$= \underbrace{\sum_{x=1}^{\infty} x \left(\tfrac{1}{2}\right)^{2x}}_{y=x} + 2\sum_{x=2}^{\infty} \sum_{y=1}^{x-1} x \cdot \left(\tfrac{1}{2}\right)^{x+y}$$
$$= \frac{4}{9} + \frac{20}{9} = \frac{8}{3}$$

Properties of Expected Values

Linearity of expectations: for any RVs X, Y

$$E(aX + bY) = aE(X) + bE(Y)$$

Proof: $E[aX+bY] = \sum_x \sum_y (ax+by)\, p_{X,Y}(x,y) = \sum_x \sum_y a \cdot x\, p_{X,Y}(x,y) + \sum_x \sum_y b \cdot y\, p_{X,Y}(x,y)$
$$= a \cdot \sum_x x \sum_y p_{X,Y}(x,y) + b \sum_y y \sum_x p_{X,Y}(x,y)$$
$$= a \sum_x x \cdot p_X(x) + b \sum_y y\, p_Y(y) = aE(X) + bE(Y)$$ $\boxtimes$

Moreover, if RVs are independent ($X \perp Y$), then

$$E[X \times Y] = E[X] \times E[Y]$$

Proof:

$$\begin{aligned} E(X \times Y) &= \sum_x \sum_y p_{X,Y}(x,y) \\ &= \sum_x x \sum_y y \, p_X(x) \, p_Y(y) \\ &= E(x) \, E(y) \end{aligned}$$

$\blacksquare$

Ex 3:

$\left(\text{Completion time of two processes serially}\right)$: $X_1 + X_2 \Rightarrow E[X_1 + X_2] = E[X_1] + E[X_2] = 2 + 2 = 4$ By linearity of $E(x)$

Ex 4: Matching Problem

X = # of ppl who get their keys $= \sum_{i=1}^{n} I_i$, where $I_i = \begin{cases} 1, & \text{if person } i \text{ gets their keys} \\ 0, & \text{o/w} \end{cases}$

$$E[x] = \sum x \cdot p_X(x) \Rightarrow E\left(\sum_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} E[I_i]$$

$$= \sum_{i=1}^{n} P(I_i = 1)$$

$$= \sum_{i=1}^{n} P(i^{th} \text{ person get their keys})$$

$$= \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$$

Variance

Variance is a measure of spread defined as

$$\begin{aligned} V(x) &= E\left((X - E(x))^2\right) \\ &= E(X^2) - \mu^2 \end{aligned}$$ where $\mu = E(x)$

Variance is denoted by σ^2

Ex 5:

$$\begin{aligned} E((X-\mu)^2) &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \quad \text{By linearity} \\ &= E[X^2] - 2\mu^2 + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$