B41 Sept 27 Lec 1 Notes

Theorem:

$D_r(x_0)$ is an open set

Proof:

Let y be a point in $D_r(x_0)$. Then $\|y-x_0\| < r$

Set $s = r - \|y-x_0\| > 0$

Let $D_s(y) = \{x \in \mathbb{R}^n \mid \|x-y\| < s\}$

For any $x \in D_s(y)$,

$$\begin{aligned}\|x-x_0\| &= \|x-y+y-x_0\| \\ &\leq \|x-y\| + \|y-x_0\| \\ &< s + \|y-x_0\| = r\end{aligned}$$

Therefore, $x \in D_r(x_0)$. That is $D_s(y) \subset D_r(x_0)$

Let $A \subset \mathbb{R}^n$. A point x_0 in $\mathbb{R}^n$ is a boundary point of A if every disk or ball of x_0 contains at least one point in A and at least one point not in A.

Definition:

Let $a = (a_1, a_2, \cdots, a_s)$ be a point in $\mathbb{R}^n$ and $x = (x_1, x_2, \cdots, x_n)$ be any point in $\mathbb{R}^n$.
Let $f: \mathbb{R}^n \to \mathbb{R}$. $L \in \mathbb{R}$ is called the limit of f as x approaches a if $f(x)$ can be made arbitrarly close to L by taking x sufficiently close to a.

We denote it as $\lim_{x \to a} f(x) = L$ or $\lim_{(x_1, x_2, \cdots, x_n) \to (a_1, \cdots, a_n)} f(x_1, x_2, \cdots x_n) = L$

i.e. $f(x) \to L$ as $x \to a$

More formally,

Definition:

If given any $\varepsilon > 0$, there is a $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $0 < \|x-a\| < \delta$.

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. when } 0 < \|x-a\| < \delta, \text{ we have } |f(x) - L| < \varepsilon$$

Ex 1:

Show that $\lim_{(x,y)\to(0,0)} xy = 0$

For any $\varepsilon > 0$, there is a $\delta > 0$ s.t. if $0 < \|(x,y)-(0,0)\| < \delta$, we have $|xy - 0| < \varepsilon$.

$$(x \pm y)^2 = x^2 + 2xy + y^2 \geq 0 \quad \Rightarrow \mp 2xy \leq x^2+y^2$$
$$\Rightarrow \mp xy \leq (x^2+y^2)/2$$
$$\Rightarrow |xy| \leq (x^2+y^2)/2 < \delta^2/2$$

Set $\delta^2 = 2\varepsilon \Rightarrow \delta = \sqrt{2\varepsilon}$

Proof:

For any $\varepsilon > 0$, set $\delta = \sqrt{2\varepsilon} > 0$. If $0 < \|(x,y)\| = \sqrt{x^2+y^2} < \delta$

$$|f(x,y)| = |xy| \leq (x^2+y^2)/2 < \delta^2/2 = \varepsilon$$

$\blacksquare$

Ex 2:

Prove that $\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+y^2} = 0$

For any $\varepsilon > 0$, there is a $\delta > 0$ s.t. if $0 < \sqrt{x^2+y^2} < \delta$, we have $\left|\frac{2x^2y^2}{x^2+y^2}\right| < \varepsilon$

$x^2 \leq x^2+y^2$ or $y^2 \leq x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1 \Rightarrow 0 \leq \frac{2x^2y^2}{x^2+y^2} \leq 2y^2(1) \leq 2x^2+2y^2 < 2\delta^2$

Set $\delta^2 = \varepsilon/2 \Rightarrow \delta = \sqrt{\frac{\varepsilon}{2}}$

Proof:

For any $\varepsilon > 0$, set $\delta = \sqrt{\frac{\varepsilon}{2}} > 0$. If $0 < \|(x,y)\| = \sqrt{x^2+y^2} < \delta$

$$\left|\frac{2x^2y^2}{x^2+y^2}\right| \leq |2y^2|\left|\frac{x^2}{x^2+y^2}\right| \leq 2y^2 \leq 2(x^2+y^2) < 2\delta^2 = \varepsilon$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^2+y^2} = 0$

$\blacksquare$

If $x \to a$ along path C, $f(x) \to L$, while $x \to a$ along path D, $f(x) \to M$, where $L \neq M$, then we may conclude that $\lim_{x\to a} f(x) \neq L$

Ex 3:

Find $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

If we approach $(0,0)$ along x-axis, $y=0$. $\frac{x^2-y^2}{x^2+y^2} \Rightarrow \frac{x^2}{x^2} = 1$

If we approach $(0,0)$ along y-axis, $x=0$. $\frac{x^2-y^2}{x^2+y^2} \Rightarrow \frac{-y^2}{y^2} = -1$

So limit DNE.

Ex. 4:

Find $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

Let $x = r\cos\theta$ and $y = r\sin\theta$

Then $r^2 = x^2+y^2$

$(x,y)\to(0,0) \Leftrightarrow r\to 0^+$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2}$$
$$= 1$$