B24 May 12 Lec 1 Notes

Representation of linear transformations as matrices

Let V, W be vector spaces with bases $(v_1, \dots, v_n)$, $(w_1, \dots, w_m)$ respectively.

Any $v \in V$ can be expressed uniquely

$$\vec{v} = \alpha_1 v_1 + \dots + \alpha_n v_n \quad \text{, where} \quad \alpha_1, \dots, \alpha_n \in \mathbb{F}$$

i.e. $\alpha_1, \dots, \alpha_n$ completely encode V

Def:

We call $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ the coordinate vector of v with respect to the basis $(v_1, \dots, v_n)$, and write:

$$[v]_{v_1, \dots, v_n} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Remark: The coordinate vector depends on a choice of basis.

Given a L.T. $T: V \to W$, we have:

For any $v \in V$,

$$T(v) = T(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 T(v_1) + \dots + \alpha_n T(n)$$ By linearity

In other words, T is completely determined by $T(v_1), \dots, T(v_n)$

$T(v_1), \dots, T(v_n)$ are elements of W, and so there exists $(B_{ij})_{i=1}^{m}{}_{j=1}^{n}$ s.t.

$$T(v_1) = B_{11} w_1 + \dots + B_{m1} w_m$$
$$\vdots$$
$$T(v_n) = B_{1n} w_1 + \dots + B_{mn} w_m$$

So $T(v) = \alpha_1 T(v_1) + \dots + \alpha_n T(v_n)$

$$= \alpha_1 (B_{11} w_1 + \dots + B_{m1} w_m) + \dots + \alpha_n (B_{1n} w_1 + \dots + B_{mn} w_m)$$

$$= (\alpha_1 B_{11} + \dots + \alpha_n B_{1n}) w_1 + \dots + (\alpha_1 B_{m1} + \dots + \alpha_n B_{mn}) w_m$$

$$[T(v)]_{w_1, \dots, w_m} = \begin{bmatrix} \alpha_1 B_{11} + \dots + \alpha_n B_{1n} \\ \vdots \\ \alpha_1 B_{m1} + \dots + \alpha_n B_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Def:

The matrix $\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}$ is called the matrix of $T: V \to W$ with respect to the bases

$(v_1, \ldots, v_n)$, $(w_1, \ldots, w_m)$ and we write:

$$[T]^{v_1, \ldots, v_n}_{w_1, \ldots, w_m} = \begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}$$

Thus we have the identity

$$[Tv]_{w_1, \ldots, w_m} = [T]^{v_1, \ldots, v_n}_{w_1, \ldots, w_m} [v]_{v_1, \ldots, v_n}$$

Ex 1:

$\frac{d}{dx} P_3 \to P_3$ is a L.T.

Recall $\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$

They are completely determined by: $\frac{d}{dx}(1) = 0$, $\frac{d}{dx}(x) = 1$

So,

$$\frac{d}{dx}(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_0 \frac{d}{dx}(1) + a_1 \frac{d}{dx}(x) + a_2 \frac{d}{dx}(x^2) + a_3 \frac{d}{dx}(x^3)$$

Ex 2:

Consider $\frac{d}{dx} P_3 \to P_3$. Then

$$\frac{d}{dx}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0(1) + 0(x) + 0(x^2) + 0(x^3)$$

$$\frac{d}{dx}(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1(1) + 0(x) + 0(x^2) + 0(x^3)$$

$$\frac{d}{dx}(x^2) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \ldots$$

$$\frac{d}{dx}(x^3) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \ldots$$

Ex 2 continued:

So $$\left[\frac{d}{dx}\right]_{1,x,x^2,x^3}^{1,x,x^2,x^3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_{11} & \cdots & B_{14} \\ \vdots & & \vdots \\ B_{41} & \cdots & B_{44} \end{bmatrix}$$

e.g.

$$\left[\frac{d}{dx}(3x^2+4x^3)\right]_{1,x,x^2,x^3} = \left[\frac{d}{dx}\right]_{1,x,x^2,x^3}^{1,x,x^2,x^3}$$

$$= \left[3x^2+4x^3\right]$$

$$\begin{bmatrix} 0 \\ 6 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$$