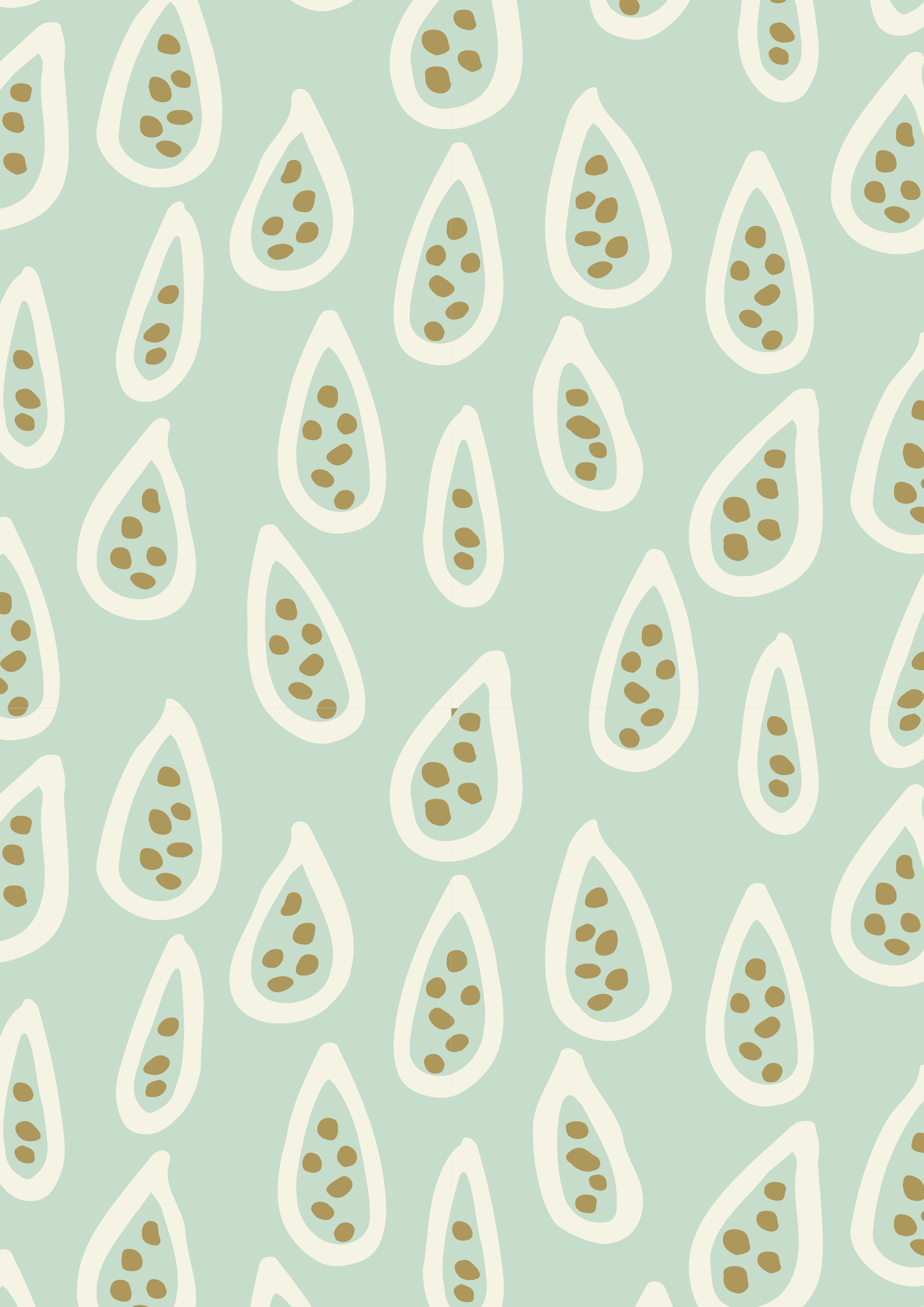

B24 June 9 Lec 1 Notes

Theorem:

Let A be an $m \times n$ matrix. Then,

(i) $\dim(\ker(A)) + \operatorname{rank}(A) = n$
(ii) $\dim(\ker(A^T)) + \operatorname{rank}(A^T) = m$

Proof:

Consider ref A_{re} of A. Then $\operatorname{rank}(A)$ is the # of pivot columns of A, and $\dim(\ker(A))$ is the # of pivotless columns of A (corresponding to free variables), i.e.

$$\dim(\ker(A)) + \operatorname{rank}(A) = \text{\# of columns of } A = n$$

Theorem: Rank Theorem

For a matrix A,

$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

Proof:

(i) $\operatorname{rank}(A)$ = # of pivot columns in A_{re}
= # of pivots in A_{re}
(ii) $\operatorname{rank}(A^T)$ = # of pivot rows in A_{re}
= # of pivots in A_{re}

i.e. $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

Theorem:

Let A be an $m \times n$ matrix. Then

$$Ax = b$$

has a solution for all $b \in \mathbb{R}^m$ iff the only solution to

$$A^T x = 0$$

is $x = 0$.

Proof (⇒):

If $Ax=b$ has a solution for all $b\in\mathbb{R}^m$, then $\text{range}(A)=\mathbb{R}^m$ i.e. $\text{rank}(A)=m$, so by the rank theorem, $\text{rank}(A^T)=m$. Since

$$\text{rank}(A^T)+\dim(\ker(A^T))=m$$

So $\text{rank}(A^T)=m \Rightarrow \dim(\ker(A^T))=0$
$\Rightarrow A^Tx=0$ has the only solution $x=0$.

$\blacksquare$

Proof (⇐):

Suppose the only solution to $A^Tx=0$ is $x=0$. Then $\dim(\ker(A^T))=0$, and:

$$\text{rank}(A^T)+\dim(\ker(A^T))=m$$

$\Rightarrow \text{rank}(A^T)=m=\text{rank}(A)$ (by rank thm) so $\dim(\text{range}(A))=m$, i.e. $\text{range}(A)=\mathbb{R}^m$, i.e. $Ax=b$ has a solution for all $b\in\mathbb{R}^m$.

$\blacksquare$

Recall if $T: V\to W$ is a L.T. and

basis $v_1,\dots,v_n$ basis $w_1,\dots,w_m$

$T(v_1)=\alpha_{11}w_1+\dots+\alpha_{m1}w_m$

$\vdots$

$T(v_n)=\alpha_{1n}w_1+\dots+\alpha_{mn}w_n$

Then $$[T]^{v_1,\dots,v_n}_{w_1,\dots,w_n}=\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1n}\\ \vdots & \ddots & \vdots\\ \alpha_{m1} & \cdots & \alpha_{mn}\end{bmatrix}$$

Usually if $V=W$, we consider $\{w_1,\dots,w_n\}=\{v_1,\dots,v_n\}$, i.e.

$$[T]^{v_1,\dots,v_n}_{v_1,\dots,v_n}$$

Suppose we have two bases $v_1,\dots,v_n$ and $w_1,\dots,w_n$ for V, and $T: V\to V$. How do

$$[T]^{v_1,\dots,v_n}_{v_1,\dots,v_n},\ [T]^{w_1,\dots,w_n}_{w_1,\dots,w_n}$$

relate?

Recall we proved that if

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \arrow[r, "S"] \arrow[d] & U \arrow[d] \\ \text{bases } v_1,\dots,v_n & \text{bases } w_1,\dots,w_m & \text{bases } u_1,\dots,u_k \end{tikzcd}$$

then

$$[ST]^{v_1,\dots,v_n}_{u_1,\dots,u_k} = [S]^{\cancel{w_1,\dots,w_m}}_{u_1,\dots,u_k} [T]^{v_1,\dots,v_n}_{\cancel{w_1,\dots,w_m}}$$

Suppose we have two bases $v_1,\dots,v_n$ and $w_1,\dots,w_n$ for V, and $T: V \to V$. How do

$$[T]^{v_1,\dots,v_n}_{v_1,\dots,v_n}, \quad [T]^{w_1,\dots,w_n}_{w_1,\dots,w_n}$$

relate?

We consider $I: V \to V$ denote the identity, and consider

$$[I]^{w_1,\dots,w_n}_{v_1,\dots,v_n}, \quad [I]^{v_1,\dots,v_n}_{w_1,\dots,w_n}$$

Then

$$= [I]^{\cancel{v_1,\dots,v_n}}_{w_1,\dots,w_n} [T]^{\cancel{v_1,\dots,v_n}}_{\cancel{v_1,\dots,v_n}} [I]^{w_1,\dots,w_n}_{\cancel{v_1,\dots,v_n}}$$

$$= [ITI]^{w_1,\dots,w_n}_{w_1,\dots,w_n}$$

$$= [T]^{w_1,\dots,w_n}_{w_1,\dots,w_n}$$

Note furthermore that

$$[I]^{w_1,\dots,w_n}_{v_1,\dots,v_n} = \left([I]^{v_1,\dots,v_n}_{w_1,\dots,w_n}\right)^{-1}$$

Since

$$[I]^{w_1,\dots,w_n}_{v_1,\dots,v_n} [I]^{v_1,\dots,v_n}_{w_1,\dots,w_n} = [II]^{v_1,\dots,v_n}_{v_1,\dots,v_n}$$

$$= [I]^{v_1,\dots,v_n}_{v_1,\dots,v_n}$$

$$= \text{Identity matrix}$$

and similarly for other way.

Terminology:

$[I]^{v_1,\dots,v_n}_{w_1,\dots,w_n}$ is called a **change of basis matrix**.

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (-y+x, 3x)$

Let $e_1 = (1,0)$, $e_2 = (0,1)$

$$[T]^{e_1,e_2}_{e_1,e_2} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$$

Now consider the basis $v_1 = (1,1)$, $v_2 = (1,-1)$ for $\mathbb{R}^2$.

$$I(1,0) = (1,0) = \tfrac{1}{2}(1,1) + \tfrac{1}{2}(1,-1)$$
$$I(0,1) = (0,1) = \tfrac{1}{2}(1,1) + (-\tfrac{1}{2})(1,-1)$$

So

$$[I]^{e_1,e_2}_{v_1,v_2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Similarly,

$$[I]^{v_1,v_2}_{e_1,e_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So,

$$[T]^{v_1,v_2}_{v_1,v_2} = [I]^{e_1,e_2}_{v_1,v_2}\,[T]^{e_1,e_2}_{e_1,e_2}\,[I]^{v_1,v_2}_{e_1,e_2}$$
$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Definition:

Two matrices and A, B are said to be similar if $\exists$ an invertible matrix Q with

$$A = Q^{-1} B Q$$

Remark:

We just showed if A, B represent the same L.T. with respect to different bases, then A, B are similar.