B41 Nov 12 Lec 2 Notes

Ex 1: Recall from Ex 1 from previous lecture.

$f(x,y) = x^2 + y^2 - 2x + 2y + 5$ on the set $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 4\}$

$\partial D: x^2+y^2=4$

Let $g(x,y) = x^2+y^2=4$

Let $L(x,y,\lambda) = x^2 + y^2 - 2x + 2y + 5 - \lambda(x^2+y^2-4)$

$$\begin{aligned}
L_x &= 2x - 2 - 2\lambda x = 0 &\Rightarrow 2x(1-\lambda) = 2 \\
L_y &= 2y - 2 - 2\lambda y = 0 &\Rightarrow 2y(1-\lambda) = -2 \\
L_\lambda &= x^2+y^2-4 = 0 &\Rightarrow x^2+y^2 = 4
\end{aligned}$$

$\Rightarrow x = -y$

$2x^2 = 4$, $x = \pm\sqrt{2}$ and $y = \mp\sqrt{2}$

Thus f has two critical points on ∂D. $(-\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$

Ex 2:

Find the shortest distance from the point $(1,1,1)$ to the plane $x+y-z=5$.

The distance from $(1,1,1)$ to (x,y,z) on the plane is:

$$d = \sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}$$

It suffices to find min value of $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ subject to the constraint that (x,y,z) is on the plane $x+y-z=5$.

i.e. $g(x,y,z) = x+y+z-5$

$$\begin{aligned}
L(x,y,z,\lambda) &= f(x,y,z) - \lambda g(x,y,z) \\
&= (x-1)^2 + (y-1)^2 + (z-1)^2 + \lambda(x+y+z-5)
\end{aligned}$$

$\nabla L = \nabla f - \lambda \nabla g = 0$

$$\begin{aligned}
L_x &= 2(x-1) - \lambda = 0 &\Rightarrow x = \lambda/2 + 1 \\
L_y &= 2(y-1) - \lambda = 0 &\Rightarrow y = \lambda/2 + 1 \\
L_z &= 2(z-1) + \lambda = 0 &\Rightarrow z = -\lambda/2 + 1
\end{aligned}$$

Plugging into $x+y+z-5=0$, we get $\lambda = 8/3$.

Thus the constrained critical point is: $x = 7/3$, $y = 7/3$, $z = -1/3$

Thus $d = 4/\sqrt{3}$

Ex 3:

A rectangular box without a lid is to be made from $12 m^2$ of cardboard. Find the maximum volume of such a box.

Let x = length , y = width , z = height

The volume of the box is $V = xyz$

The surface of the box is $S = 2xz + 2yz + xy$, where $S = 12 m^2$

To find max Volume,

Set $L(x,y,z,\lambda) = xyz - \lambda(2xz + 2yz + xy - 12)$

$$\begin{aligned}
L_x &= yz - \lambda(2z+y) = 0 &&\Rightarrow yz = \lambda(2z+y) &&\Rightarrow xyz = \lambda(2z+y)x \\
L_y &= xz - \lambda(2z+x) = 0 &&\Rightarrow xz = \lambda(2z+x) &&\Rightarrow yxz = \lambda(2z+x)y \\
L_z &= xy - \lambda(2x+2y) = 0 &&\Rightarrow xy = \lambda(2x+2y) &&\Rightarrow zxy = \lambda(2x+2y)z \\
L_\lambda &= -(2xz + 2yz + xy - 12) = 0 &&\Rightarrow 2xz + 2yz + xy = 12
\end{aligned}$$

$xz = yz \Rightarrow x = y$

$xy = 2xz \Rightarrow y = 2z$

$4z^2 + 4z^2 + 4z^2 = 12 \Rightarrow z = \pm 1$

So $z = 1$ since $z \geq 0$

Thus $x = y = 2$

Thus constrained critical point is $(2,2,1)$

The max volume of the box is $V_{max} = xyz = 4m^3$

Theorem:

If there are multiple constraints $g_1(x) = C_1,\ g_2(x) = C_2, \dots, g_k(x) = C_k$, we may construct the lagrange function:

$$L(x, \lambda_1, \lambda_2, \dots, \lambda_k) = f(x) - \lambda_1(g_1(x) - C_1) - \lambda_2(g_2(x) - C_2) - \dots - \lambda_k(g_k(x) - C_k)$$

and then find all the critical points of h about λ and the constrained critical points of f.

Ex 4:

Find extrema values of $f(x,y,z) = 3x - y - 3z$ subject to the constraints $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

$$L(x,y,z,\lambda,\mu) = 3x - y - 3z - \lambda(x+y-z) - \mu(x^2 + 2z^2 - 1)$$

$L_x = 3 - \lambda - 2\mu x = 0 \quad \Rightarrow 4 - 2\mu x = 0 \quad \Rightarrow x = 2/\mu \quad \Rightarrow x = \pm\sqrt{6}/3$

$L_y = -1 - \lambda = 0 \quad \Rightarrow \lambda = -1$

$L_z = -3 + \lambda - 4\mu z = 0 \quad \Rightarrow -4 - 4\mu z = 0 \quad \Rightarrow z = -1/\mu \quad \Rightarrow z = \mp\frac{\sqrt{6}}{6}$

$L_\lambda = -(x + y - z) = 0 \quad \Rightarrow y = z - x$

$L_\mu = -(x^2 + 2z^2 - 1) = 0 \quad \Rightarrow (2/\mu)^2 + 2(-1/\mu)^2 = 1 \quad \Rightarrow \mu = \pm\sqrt{6}$

Two constrained critical points : $\overset{x_0}{\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right)}, \overset{x_1}{\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right)}$. Thus $\overset{max}{f(x_1)} = 2\sqrt{6}$, $\overset{min}{f(x_2)} = -2\sqrt{6}$.