Jan 28 Lecture Notes

Def: ε-Reformulation of Darboux Def

Let $a, b \in \mathbb{R}$ $a < b$ Suppose f is bounded on $[a,b]$
We say f is integrable on $[a,b]$.

$\int_a^b f(x)\,dx$ exists iff $\forall \varepsilon > 0$, $\exists$ a $P = \{x_i\}_{i=0}^n$ of $[a,b]$ s.t.

$U(f,P) - L(f,P) < \varepsilon$

Example:

1. Let $g(x) = \begin{cases} 213 & ; \text{ if } x \in \mathbb{Q} \\ 0 & ; \text{ if } x \notin \mathbb{Q} \end{cases}$

Use reformulation of integrability to prove g is not integrable on $[0,1)$.

Proof: wts our def above is false

$\Rightarrow \neg(\forall \varepsilon > 0, \exists P \text{ of } [0,1] \text{ s.t. } U(g,P) - L(g,P) < \varepsilon)$

$\Rightarrow \exists \varepsilon > 0, \forall P \text{ of } [0,1] \text{ s.t. } U(g,P) - L(g,P) \geq \varepsilon)$

Choose $\varepsilon = \underline{\ 213\ } > 0$

Let $P = \{x_i\}_{i=0}^n$ be an arbitrary partition of $[0,1]$

For $i = 1, \dots, n$

$m_i = \inf\{g(x) \mid x \in [x_{i-1}, x_i]\}$ By def of m_i

$= \inf\{213, 0\} = 0$

$M_i = \sup\{g(x) \mid x \in [x_{i-1}, x_i]\}$ By def of M_i

$= \sup\{213, 0\} = 213$

Thus

$$U(G,P) - L(G,P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1})$$

$$= 213 \sum_{i=1}^n (x_i - x_{i-1}) - \sum_{i=1}^n 0$$

$$= 213\left((\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \ldots + (x_n - \cancel{x_{n-1}})\right) - \sum_{i=1}^{n} 0$$

$$= 213(x_n - x_0) - 0$$

$$= 213(1-0) = 213 \geq 213 = \varepsilon \text{ , as wanted}$$

$$\therefore \int_0^1 g(x)\,dx \text{ DNE } //$$

Def: Indefinite Integral

An indefinite integral of a continuous function $f(x)$, denoted $\int f(x)\,dx$, is an infinite family of anti derivatives of $f(x)$

$$\int f(x)\,dx = F(x) + C$$