B24 June 18 Lec 2 Notes

Theorem:

Let $A: V \to V$ be a L.T and $\lambda_1, \dots, \lambda_r$ distinct eigenvalues for A, with corresponding eigenvectors $v_1, \dots, v_r$. Then $v_1, \dots, v_r$ are L.I.

Proof: We will induct on r.

When $r=1$, the statement is obvious. (Any single non-zero vector is L.I.)

IH: Assume the statement holds for $r-1$.

IS: Suppose $A: V \to V$ be a L.T. and $\lambda_1, \dots, \lambda_r$ distinct eigenvalues for A, with corresponding eigenvectors $v_1, \dots, v_r$. Let $\alpha_1, \dots, \alpha_r \in \mathbb{F}$ with

$$\alpha_1 v_1 + \dots + \alpha_r v_r = 0 \quad ★$$

Apply $(A - \lambda_r I)$ to ★:

$$\alpha_1(\underbrace{Av_1}_{\lambda_1 v_1} - \lambda_r v_1) + \dots + \alpha_{r-1}(\underbrace{Av_{r-1}}_{\lambda_{r-1} v_{r-1}} - \lambda_r v_{r-1}) + \underbrace{\alpha_r(\underbrace{Av_r}_{\lambda_r v_r} - \lambda_r v_r)}_{=0} = 0$$

$$\Rightarrow \quad \alpha_1 \underbrace{(\lambda_1 - \lambda_r)}_{\neq 0} v_1 + \dots + \alpha_{r-1} \underbrace{(\lambda_{r-1} - \lambda_r)}_{\neq 0} v_{r-1} = 0$$

$$\overset{\text{I.H}}{\Rightarrow} \quad \alpha_1(\lambda_1 - \lambda_r) = \dots = \alpha_{r-1}(\lambda_{r-1} - \lambda_r) = 0$$

$$\Rightarrow \quad \alpha_1 = \dots = \alpha_{r-1} = 0$$

$$\Rightarrow \quad \alpha_r v_r = 0$$

$$\Rightarrow \quad \alpha_r = 0$$

Definition:

A square matrix A is said to be diagonalizable if there exists invertible S, and diagonal matrix D s.t. $A = SDS^{-1}$.

Corollary:

Let $A: V \to V$ be a L.T. and $\dim V = n$. Then if A has n distinct eigenvalues, A is diagonalizable.

Proof:

Suppose A has n distinct eigenvalues $\lambda_1, \dots \lambda_n$ with corresponding eigenvectors $v_1, \dots, v_n$. Then by prev. thm, $v_1, \dots, v_n$ are L.I. $\Rightarrow$ $v_1, \dots, v_n$ are a basis for V $\Rightarrow$ A is diagonalizable by the theorem proved last lecture:

> **Theorem:** From B24 June 16 Lec 1 Notes
>
> Let A be an $n \times n$ matrix (over $\mathbb{R}$ or $\mathbb{C}$) Then there exists a diagonal matrix D and an invertible matrix S s.t.
>
> $$A = SDS^{-1}$$
>
> iff there is a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

$\boxtimes$

Remark:

The above is redundant for $\mathbb{F} = \mathbb{C}$. Recall:

> **Proposition:** From B24 June 16 Lec 1 Notes
>
> Let $\dim V = n$ and $A: V \to V$ where $\mathbb{F} = \mathbb{C}$. Then A has n eigenvalues (counting multiplicity).

e.g. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

$\Rightarrow$ $\lambda x = x + y$

$\lambda y = y$ $\quad$ $x, y \neq 0$. eigenvectors are non-zero

$\Rightarrow \lambda = 1$

A has only one eigenvalue $\lambda = 1$, and $A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

So $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ $y = 0$, x free

i.e. $\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \middle| x \in \mathbb{C} \right\} = \operatorname{Ker}(A - I)$

So $\operatorname{Ker}(A - I)$ is 1-dimensional.

i.e. There is no basis for $\mathbb{C}^2$ consisting of eigenvectors for A, i.e. A is not diagonalizable.

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$ i.e. the eigenvalue 1 has multiplicity 2.

i.e. $\underbrace{\dim(\operatorname{Ker}(A - I))}_{=1} < \underbrace{\text{multiplicity}(\lambda = 1)}_{=2}$

Definition:

Let $V_1, \dots, V_p$ be subspaces of V. We say $V_1, \dots, V_p$ form a basis for V if for any $v \in V$ there exist unique $v_i \in V_i$ $(1 \le i \le p)$ with

$$v = v_1 + \dots + v_p$$

We say $V_1, \dots, V_p$ are L.I. if $v_i \in V_i$ $(1 \le i \le p)$ are s.t.

$$v_1 + \dots + v_p = 0$$

then $v_1 = \dots = v_p = 0$

We say $V_1, \dots, V_p$ are spanning / generating if for any $v \in V$ there exist $v_i \in V_i$ $(1 \le i \le p)$ with

$$v = v_1 + \dots + v_p$$

Remark:

If $\lambda_1, \dots, \lambda_r$ are distinct e.v's of A, we have proven is $\operatorname{Ker}(A - \lambda_1 I), \dots, \operatorname{Ker}(A - \lambda_r I)$ are L.I. Recall:

> **Theorem:**
>
> Let $A: V \to V$ be a L.T and $\lambda_1, \dots, \lambda_r$ distinct eigenvalues for A, with corresponding eigenvectors $v_1, \dots, v_r$. Then $v_1, \dots, v_r$ are L.I.

and furthermore if $r = \dim V$, then

$$\operatorname{Ker}(A - \lambda_1 I), \dots, \operatorname{Ker}(A - \lambda_r I)$$

form a basis for V.

Lemma:

Suppose $V_1, \dots, V_p \subset V$ are L.I. subspaces and let $\mathcal{B}_1 \subset V_1, \dots, \mathcal{B}_p \subset V_p$ be L.I. sets of vectors. Then $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_1$ are L.I.

Proof:

Let $\mathcal{B}_1 = \{ v_1^1, \dots, v_{n_1}^1 \}$

$\vdots$

$\mathcal{B}_p = \{ v_1^p, \dots, v_n^p \}$

then if

$$\underbrace{\alpha_1^1 v_1^1 + \dots + \alpha_{n_1}^1 v_{n_1}^1}_{\in V_1} + \dots + \underbrace{\alpha_1^p v_1^p + \dots + \alpha_{n_p}^p v_{n_p}^p}_{\in V_p} = 0$$

$$\Rightarrow \alpha_1^1 v_1^1 + \dots + \alpha_{n_1}^1 v_{n_1}^1 = 0$$

$$\vdots \qquad \text{L.I. of } V_1, \dots, V_p$$

$$\alpha_1^p v_1^p + \dots + \alpha_{n_p}^p v_{n_p}^p = 0$$

$$\Rightarrow \alpha_1^1 = \dots = \alpha_{n_1}^1 = 0$$

$$\vdots \qquad \text{L.I. of } \mathcal{B}_1, \dots, \mathcal{B}_p$$

$$\alpha_1^p = \dots = \alpha_{n_p}^p = 0$$

Theorem:

Let $A: V \to V$ be a L.T., $\dim V = n$. Suppose A has n eigenvalues (counting multiplicity). Then A is diagonalizable iff for each eigenvalue λ of A, the multiplicity of λ coincides with $\dim(\operatorname{Ker}(A - \lambda I))$.

Proof ($\Rightarrow$):

Suppose A is diagonal, let λ be an eigenvalue of A. Then multiplicity of λ is the # of times λ appears in A, and $\dim(\operatorname{Ker}(A-\lambda I))$ is the # of columns in which λ appears.

Suppose A is diagonalizable, i.e. there exist invertible S and diagonal D with

$$A = SDS^{-1}$$

So,

$$\det(A - \lambda I) = \det(S)\det(A - \lambda I)\det(S^{-1})$$

$$= \det(S(A - \lambda I)S^{-1})$$

$$= \det(SAS^{-1} - S\lambda I S^{-1})$$

$$= \det(D - \lambda I)$$

So i.e. A, D have the same eigenvalues and the same multiplicity for each e.v. and ...

Proof (Continued...)

Proof (continued...):

v is an eigenvector of D for λ iff $S^{-1}v$ is an eigenvector of A for λ:

$$\begin{aligned} Dv = \lambda v &\Leftrightarrow SAS^{-1}v = \lambda v \\ &\Leftrightarrow AS^{-1}v = S^{-1}\lambda v \\ &\Leftrightarrow A(S^{-1}v) = \lambda(S^{-1}v) \end{aligned}$$

i.e. $\dim(\mathrm{Ker}(A-\lambda I)) = \dim(\mathrm{Ker}(D-\lambda I))$

Since $\mathrm{Ker}(A-\lambda I) = S^{-1}\,\mathrm{Ker}(D-\lambda I)$ Basis is the same after a transformation with an invertible matrix.

Thus we can use ★

Proof ($\Leftarrow$):

Suppose for each eigenvalue λ of A, the multiplicity of λ coincides with $\dim(\mathrm{Ker}(A-\lambda I))$.

Let $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of A. Then

$$\mathrm{Ker}(A-\lambda_1 I), \dots, \mathrm{Ker}(A-\lambda_r I)$$

are L.I. by ▲.

Choose a basis B_1 for $\mathrm{Ker}(A-\lambda_1 I)$

$\vdots$

B_r for $\mathrm{Ker}(A-\lambda_r I)$

Then by prev. lemma,

$$B_1 \cup \dots \cup B_r \text{ is L.I. in } V$$

Since the multiplicity of λ coincides with $\dim(\mathrm{Ker}(A-\lambda I))$,

$$\begin{aligned} |B_1| + \dots + |B_r| &= \sum_{i=1}^{r} \mathrm{multi}(\lambda_i) \\ &= n \end{aligned}$$

i.e. $B_1 \cup \dots \cup B_r$ is L.I. in V and has $\dim V$ elements $\Rightarrow$ $B_1 \cup \dots \cup B_r$ is a basis for V $\Rightarrow$

A is diagonalizable. $\blacksquare$

Ex 1:

$$A = \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \quad \text{Then} \quad \det(A-\lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)^2 - 16$$
$$= \lambda^2 - 2\lambda - 15$$
$$= (\lambda-5)(\lambda+3)$$

i.e. A has 2 distinct eigenvalues: $5, -3$.

$\mathrm{Ker}(A-5I) = ?$

$$A - 5I = \begin{bmatrix} -4 & 2 \\ 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ y is free, $x - \frac{1}{2}y = 0$

$\Rightarrow$ $x = \frac{1}{2}y$

$\mathrm{Ker}(A-5I) = \{ x \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \mid x \in \mathbb{F} \}$

Similarly,

$\mathrm{Ker}(A-3I) = \{ x \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mid x \in \mathbb{F} \}$

So $[A]^{(1/2,1),(1,-2)}_{(1/2,1),(1,-2)} = \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}}_{D}$

What is S ?

$$[I]^{(1/2,1),(1,-2)}_{(1,0),(0,1)} = \begin{bmatrix} 1/2 & 1 \\ 1 & -2 \end{bmatrix}$$

Since $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Similarly

$$[I]^{(1,0),(0,1)}_{(1/2,1),(1,-2)} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & -1/4 \end{bmatrix}$$

So,

$$= [A]^{(1/2,1),(1,-2)}_{(1/2,1),(1,-2)}$$

$$= \underbrace{[I]^{(1,0),(0,1)}_{(1/2,1),(1,-2)}}_{S^{-1}} \underbrace{[A]^{(1,0),(0,1)}_{(1,0),(0,1)}}_{A} \underbrace{[I]^{(1/2,1),(1,-2)}_{(1,0),(0,1)}}_{S}$$

$$= \begin{bmatrix} 1 & 1/2 \\ 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$$

Ex 2:

$\phi_0 = 0$, $\phi_1 = 1$

$\phi_{n+2} = \phi_{n+1} + \phi_n$, $\forall n \geq 0$

Can we find a non-recursive formula for ϕ_n ?

Note that

$$\phi_{n+2} = \phi_{n+1} + \phi_n$$

implies

$$\begin{bmatrix} \phi_{n+2} \\ \phi_{n+1} \end{bmatrix} = \begin{bmatrix} \phi_{n+1} + \phi_n \\ \phi_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{n+1} \\ \phi_n \end{bmatrix}$$

Let's call $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

So

$$\begin{aligned} \begin{bmatrix} \phi_{n+2} \\ \phi_{n+1} \end{bmatrix} &= A \begin{bmatrix} \phi_{n+1} \\ \phi_n \end{bmatrix} \\ &= A^2 \begin{bmatrix} \phi_n \\ \phi_{n-1} \end{bmatrix} \\ &= \dots = A^{n+1} \begin{bmatrix} \phi_1 \\ \phi_0 \end{bmatrix} \\ &= A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

A^n is difficult to compute. We could diagonalize A instead.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$= \lambda^2 - \lambda - 1 = 0$$

$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$

$\mathrm{Ker}\left(A - \left(\frac{1+\sqrt{5}}{2}\right) I\right) = \left\{ x \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

$\mathrm{Ker}\left(A - \left(\frac{1-\sqrt{5}}{2}\right) I\right) = \left\{ x \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

So $v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

$$S \begin{bmatrix} \phi_{n+1} \\ \phi_n \end{bmatrix} = S A^n S^{-1} S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Ex 2 (continued..):

$$= \begin{bmatrix} \frac{\phi_{n+1}}{\sqrt{5}} + \frac{-1+\sqrt{5}}{2\sqrt{5}}\phi_n \\ \frac{-\phi_{n+1}}{\sqrt{5}} + \frac{1+\sqrt{5}}{2\sqrt{5}}\phi_n \end{bmatrix} = \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n 1/\sqrt{5} \\ -\left(\frac{1-\sqrt{5}}{2}\right)^n 1/\sqrt{5} \end{bmatrix}$$

$$\Rightarrow \phi_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \frac{1}{\sqrt{5}} - \left(\frac{1-\sqrt{5}}{2}\right)^n \frac{1}{\sqrt{5}}$$