Def:

Let $A_{n\times m} = a_{ij}$, $B_{n\times m} = b_{ij}$, $k \in \mathbb{R}$

$A + B := (a_{ij} + b_{ij})_{n\times m}$ $\quad$ $A - B := A + (-B)$

$kA := (k\, a_{ij})_{n\times m}$ $\quad$ $0_{n\times m} := (0)_{n\times m}$

$-A := (-a_{ij})_{n\times m}$ $\quad$ $A + 0_{n\times m} = A$

Def: Matrix Vector multiplication

Let $A_{n\times m} = \begin{bmatrix} — \vec{w}_1 — \\ — \vec{w}_2 — \\ \vdots \\ — \vec{w}_n — \end{bmatrix}$ and $\vec{x}$ in $\mathbb{R}^m$

$$A\vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}_{n\times 1}$$

Theorem (1.3.8):

If $A_{n\times m} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ then

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$$

Proof:

Note $A\vec{x}$ and $* = x_1\vec{v} + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ are both in $\mathbb{R}^n$.
To show $A\vec{x} = *$ it is enough to show

i^{th} comp. of $A\vec{x}$ = i^{th} comp. of $*$

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} — \vec{w}_1 — \\ — \vec{w}_2 — \\ \vdots \\ — \vec{w}_n — \end{bmatrix}$

$$\text{ith comp of } A\vec{x} = \vec{w_i} \cdot \vec{x}$$
$$= a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m$$

$$\text{ith comp of } * = x_1\vec{v_1} + x_2\vec{v_2} + \dots + x_m\vec{v_m}$$
$$= x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots\ a_{i1} \\ a_{n1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots\ a_{i2} \\ a_{n2} \end{pmatrix} + \dots + x_m\begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots\ a_{im} \\ a_{nm} \end{pmatrix}$$

$$= x_1 a_{i1} + x_2 a_{i2} + \dots + x_m a_{im}$$
$$= a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m$$

Thus ith comp $A\vec{x}$ = ith comp $*$ //

Example:

1. $$* \begin{cases} 2x_1 + 4x_2 + 10x_3 = 2 \\ x_1 + 3x_2 + 7x_3 = 0 \\ 3x_1 + 6x_2 + 15x_3 = 3 \end{cases}$$

$$A\vec{x} = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 3 & 7 \\ 3 & 6 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

We can represent $*$ by $A\vec{x} = \vec{b}$ (matrix-vector equation / matrix form of a linear system)

$$A\vec{x} = x_1\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + x_2\begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} + x_3\begin{pmatrix} 10 \\ 7 \\ 15 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

Def: Linear Combination

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of vectors $\vec{v_1}, \dots, \vec{v_m}$ in $\mathbb{R}^n$ if there exists scalars $c_1, \dots, c_m$ in $\mathbb{R}$ s.t.

$$\vec{b} = c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_m\vec{v_m}$$