## Theorem 5.18 :

↳ $a^2 + u^2 \rightarrow$ Let $u = a\tan\theta$ , $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

↳ $a^2 - u^2 \rightarrow$ Let $u = a\sin\theta$ , $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , $|u| \le a$

↳ $u^2 - a^2 \rightarrow$ Let $u = a\sec\theta$ , $\theta \in \begin{cases} [0, \frac{\pi}{2}) & \text{, if } u \ge a \\ (\frac{\pi}{2}, \pi] & \text{, if } u \le -a \end{cases}$

## Ex 1

Can we trig sub? If so, which substitute?

(a) $\displaystyle\int_0^{\pi} \frac{3x+1}{\sqrt{x^2+9}}\,dx$ let $x = 3\tan\theta$ , $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(b) $\displaystyle\int \frac{(x+3)(4x^2-16)^{5/3}}{\sqrt{x}}\,dx$ , $|x| \ge 2$ let $2x = 4\sec\theta$ , $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

(c) $\displaystyle\int \frac{1}{\sqrt{2-(3x+1)^2}}\,dx$ , $\frac{-1-\sqrt{2}}{3} < x < \frac{\sqrt{2}-1}{3}$ let $3x+1 = \sqrt{2}\sin\theta$ , $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , $|x| \le \sqrt{2}$

## Ex 2

$\displaystyle\int \sqrt{4-x^2}\,dx$ , $|x| \le 2$

Let $x = 2\sin\theta$ , $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$dx = 2\cos\theta\, d\theta$

$= \displaystyle\int \sqrt{2^2 - 2^2\sin^2\theta} \cdot 2\cos\theta\, d\theta$

$= \displaystyle\int \sqrt{2^2(1-\sin^2\theta)} \cdot 2\cos\theta\, d\theta$

$= \displaystyle\int |2\cos\theta| \cdot 2\cos\theta\, d\theta$

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$= \displaystyle\int 4\cos^2\theta\, d\theta$

$= 4\displaystyle\int \frac{1+\cos(2\theta)}{2}\,d\theta$

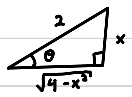
$= 2\displaystyle\int 1 + \cos(2\theta)\, d\theta$

$= 2(\theta + \frac{1}{2}\sin(2\theta)) + C$

$= 2\theta + 2\sin(2\theta) + C$

$= 2\theta + \frac{4\cos\theta\sin\theta}{2} + C$

$= 2\sin^{-1}(\frac{x}{2}) + 2\cos(\sin^{-1}(\frac{x}{2}))\frac{x}{2} + C$




$= 2\sin^{-1}(\frac{x}{2}) + \frac{2\sqrt{4-x^2}}{2} \cdot \frac{x}{2} + C$

$= 2\sin^{-1}(\frac{x}{2}) + \frac{x\sqrt{4-x^2}}{2} + C$

## Ex 3

$$\int \frac{x}{\sqrt{x^2+9}}\,dx$$

Let $x = 3\tan\theta$ , $\theta \in [-\pi/2, \pi/2]$

$dx = 3\sec^2\theta \, d\theta$

$$= \int \frac{3\tan\theta}{\sqrt{3^2\tan^2\theta + 3^2}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{3\tan\theta}{3\sqrt{\tan^2\theta+1}} \cdot 3\sec^2\theta\, d\theta$$

$$= \int \frac{3\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta$$

$$= \int 3\tan\theta \cdot \frac{\sec^2\theta}{|\sec\theta|}\, d\theta$$

## Ex 4

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{x}{\sqrt{-(x+1)^2+4}}\,dx$$

$$= \int \frac{x}{\sqrt{2^2-(x+1)^2}}\,dx$$

Let $x+1 = 2\sin\theta$ , $\theta \in [-\pi/2, \pi/2]$

$dx = 2\cos\theta\, d\theta$

$$= \int \frac{2\sin\theta - 1}{\sqrt{2^2 - 2^2\cos^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$= \int \frac{(2\sin\theta - 1)(2\cos\theta)}{2\sqrt{1-\cos^2\theta}}\, d\theta$$

$$= \int \frac{(2\sin\theta - 1)\cos\theta}{|\sin\theta|}\, d\theta$$

$$= \int \frac{2\sin\theta\cos\theta}{\sin\theta}\, d\theta - \int \frac{\cos\theta}{|\sin\theta|}\, d\theta$$

$$= \int 2\cos\theta\, d\theta - \int \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= 2\sin\theta - \ln|\sin\theta| + C$$

$$= 2\left(\frac{x+1}{2}\right) - \ln\left|\frac{x+1}{2}\right| + C$$

$$= x+1 - \ln\left|\frac{x+1}{2}\right| + C$$

An improper integral is an integral for which one or both of these conditions fail

## Ex 5

$\int_1^{\infty} \frac{\tan^{-1}(x)}{1+x^2}\,dx$ type I $\qquad \int_{\pi/2}^{\pi} \csc(x)\,dx$ type II

$\int_2^{3} \frac{8}{\sqrt{x-2}}\,dx$ type II $\qquad \int_{-\infty}^{1} \tan^{-1}(x)\,dx$ type I

$\int_{-1}^{1} \frac{1}{x^2}\,dx$ type II

$0 \in [-1,1]$ VA at $x=0$

## Ex 6

$$\int_1^{\infty} \frac{1}{(3x+1)^2}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{(3x+1)^2}\,dx \quad \text{By def of type I}$$

$$= \lim_{A\to\infty} -\frac{1}{3}(3x+1)^{-1}\Big|_1^A$$

$$= \lim_{A\to\infty} -\frac{1}{3}(3A+1)^{-1} + \frac{1}{3}(4)^{-1}$$

$$= \lim_{A\to\infty} \left(-\frac{1}{3(3A+1)} + \frac{1}{12}\right)$$

$$= 0 + \frac{1}{12} = \frac{1}{12} \qquad \therefore \int_1^{\infty} \frac{1}{(3x+1)^2} \text{ converges to } 1/12$$