Induction

$\mathbb{N}$ = set of natural numbers = $\{0, 1, 2, 3, \dots\}$

Principle of Simple Induction (PSI)

We can prove $P(n)$ holds for $n \geq b$ by proving:

Basis: $P(b)$

Induction Step: For all $n \geq b$, if $P(n)$, then $P(n+1)$

Ex: ex 1.7, pg 30

$P(n)$: exact postage of n cents can be made using only 4-cent and 7-cent stamps.

Equivalent predicate:

$Q(n)$: there exists $k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n$

Prove $\forall n \geq 18$, $Q(n)$.

Proof:

Basis: Let $n = 18$

Let $k = 1$, $\ell = 2$. Then $k, \ell \in \mathbb{N}$
and $4k + 7\ell = 4 \cdot 1 + 7 \cdot 2 = 18 = n$ as wanted

Induction Step: Let $n \geq 18$.

Suppose $Q(n)$ [I.H.]
i.e. there are $k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n$
WTP: $Q(n+1)$ i.e. $\exists\, k', \ell' \in \mathbb{N}$ s.t. $4k' + 7\ell' = n+1$

Consider 2 cases: $\ell > 0$ and $\ell = 0$

Case 1: Suppose $\ell > 0$

Then let $k' = k + 2$, $\ell' = \ell - 1$

Then $\ell' \geq 0$, so $\ell' \in \mathbb{N}$

Also,
$$\begin{aligned} 4k' + 7\ell' &= 4(k+2) + 7(\ell - 1) \\ &= 4k + 8 + 7\ell - 7 \\ &= 4k + 7\ell + 1 \\ &= n + 1 \quad \text{[I.H.]} \end{aligned}$$

Proof (continued...):

Case 2: Suppose $\ell = 0$

Since $n \geq 18$, we have

$$18 \leq n = 4k + \overset{[I.H]}{\underbrace{7\ell}_{0}} = 4k$$

Thus $k \geq 5$

Let $k' = k - 5 \geq 0$
$\ell' = \ell + 3$

Then $$\begin{aligned} 4k' + 7\ell' &= 4(k-5) + 7(\ell+3) \\ &= 4k - 20 + 7\ell + 21 \\ &= 4k + 7\ell + 1 \\ &= n + 1 \quad [I.H.] \end{aligned}$$

∎

Principle of Complete Induction (PCI)

We can prove $P(n)$ for all $n \geq b$ by proving

k base cases

Basis: $P(b), P(b+1), \dots, P(b+k-1)$

I.S.: For $n \geq b+k$, if $P(j)$ holds whenever $b \leq j \leq n$, then $P(n)$

Ex: ex 1.12 pg 40

$Q(n)$: $\exists k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n$

Use PCI to prove $\forall n \geq 18$, $Q(n)$

Base case:

For $n = 18$, let $k = 1, \ell = 2$. Then $4k + 7\ell = n$ as wanted.
For $n = 19$, $k = 3$, $\ell = 1$
For $n = 20$, $k = 5$, $\ell = 0$
For $n = 21$, $k = 0$, $\ell = 3$

I.S: Let $n \geq 22$

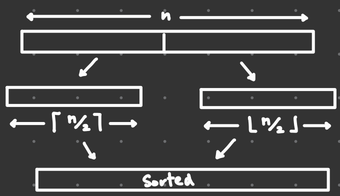
Suppose $Q(j)$ holds whenever $18 \leq j \leq n$ [I.H.]
WTP: $Q(n)$ holds i.e. $\exists k', \ell'$ s.t. $4k' + 7\ell' = n$
Since $n \geq 22$, we have $18 \leq n - 4 < n$
By I.H., $Q(n-4)$ holds i.e. $\exists k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n - 4$
Let $k' = k + 1$, $\ell' = \ell$
Then $4k' + 7\ell' = 4(k+1) + 7\ell = 4k + 7\ell + 4 = n - 4 + 4$ [I.H.]
$= n$ ∎

Principle of Well-Ordering (PWO)

Every non-empty subset of $\mathbb{N}$ has a minimum element

Recurrences (CH 3) - Recursively / inductively defined functions

e.g. Merge sort



Let $f(n)$ be the number of item assignments needed for sorting n items with mergesort

$$f(n) = \begin{cases} 0 & \text{if } n=1, \\ f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor) + 2n & \text{if } n>1 \end{cases}$$

Unwinding Recurrences

Let n be a large power of 2, i.e. $n = 2^k$ for some large $k \in \mathbb{N}$

since $n = 2^k$ is even

$$\begin{aligned}
f(n) &= 2f(\tfrac{n}{2}) + 2n && \text{1st iteration} \\
&= 2(2f(\tfrac{n}{4}) + 2\tfrac{n}{2}) + 2n && \text{2nd iteration} \\
&= 4f(\tfrac{n}{4}) + 4n \\
&= 4[2f(\tfrac{n}{8}) + 2(\tfrac{n}{4})] + 4n \\
&= 8f(\tfrac{n}{8}) + 6n && \text{3rd iteration} \\
&\vdots \\
&= 2^i f(\tfrac{n}{2^i}) + 2in && \text{ith iteration} \\
&\vdots \\
&= 2^k f(\tfrac{n}{2^k}) + 2kn && \text{let } i=k \\
&= 2^k f(\tfrac{n}{n}) + 2kn && \text{since } 2^k = n \\
&= 2^k f(1) + 2kn && f(1)=0 \\
&= 2kn \\
&= 2n \cdot \log_2 n \\
&\approx n \cdot \log n && \text{\# of assignments}
\end{aligned}$$

Structural Induction (CH4)

2 uses:

- ↳ Define sets
- ↳ Prove properties of all elements in a set defined by structural induction

Ex: ex 4.1 pg 97

Define the set of all well-formed, fully parenthesized algebraic expressions with variables x, y, z and operators +, -, ×, ÷.

e.g. x, $(x+y)$, $(((y-z)\times(z-x))+x)$

Remark:

$$\Sigma = \{\underbrace{x, y, z}_{\text{variables}}, \underbrace{+, -, \times, \div}_{\text{operators}}, \underbrace{(,)}_{\text{parentheses}}\} \qquad \underbrace{\mathcal{E} \subseteq \Sigma^*}$$

finite strings consisting of symbols from Σ

Definition: Let $\mathcal{E}$ be the smallest set s.t.

Base cases: $x, y, z \in \mathcal{E}$

Not variable x, but a string with the x in $\mathcal{E}$.

I.S.: If $e_1, e_2 \in \mathcal{E}$, then $(e_1+e_2), (e_1-e_2), (e_1\times e_2), (e_1\div e_2) \in \mathcal{E}$

Ex: ex 4.2 pg 100

For a string $e \in \Sigma^*$, we define

$vr(e)$ to the # of occurrences of variables in e.
$op(e)$ to the # of occurrences of operators in e.

For $e \in \Sigma^*$, we define a predicate

$P(e)$. $vr(e) = op(e) + 1$

Proof: Use structural induction to prove that $P(e)$ holds for $e \in \mathcal{E}$

Base cases: 3 cases: $e=x$, $e=y$, $e=z$

For $e=x$, $vr(e)=1$, $op(e)=0$

$\therefore vr(e) = op(e)+1$

Similarly for $e=y$, $e=z$.

I.S.: Let $e_1, e_2 \in \mathcal{E}$

Suppose $P(e_1)$, $P(e_2)$ [I.H.]

i.e. $vr(e_1) = op(e_1)+1$ and $vr(e_2) = op(e_2)+1$

WTP: $P(e)$ holds for $e=(e_1+e_2)$, $e=(e_1-e_2)$, $e=(e_1\times e_2)$, $e=(e_1\div e_2)$

In each case, we have

$$vr(e) = vr(e_1) + vr(e_2)$$
$$op(e) = op(e_1) + op(e_2) + 1$$

Proof (continued...):

I.S (continued...):

Thus $vr(e) = vr(e_1) + vr(e_2)$

$= (op(e_1)+1) + (op(e_2)+1)$ [I.H.]

$= (op(e_1) + op(e_2)) + 2$

$= (op(e) - 1) + 2$

$= op(e) + 1$

$\blacksquare$

Ex:

Let S be the smallest set s.t.

Base Case: $0 \in S$

I.S.: if $n \in S$, then $n+1 \in S$