Jan 18 Lecture Notes

Def: Definite Integral

Let $a, b \in \mathbb{R}$, $a < b$. Suppose f is continuous on $[a,b]$.

Let $P = \{x_i\}_{i=0}^{n}$ be a Riemann partition of $[a,b]$

Then the definite integral of f on $[a,b]$ is

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided this limit exists.

Example

1. $\int_0^3 (x-5)\,dx$

(a) Compute with Riemann Sum

Given : $[a,b] = [0,3]$
$f(x) = x-5$

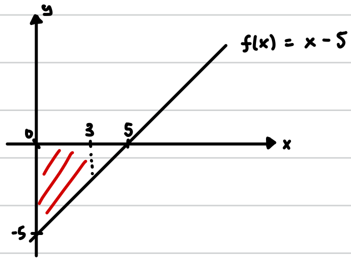
Define $\Delta x = \frac{3-0}{n} = \frac{3}{n} \Rightarrow x_i = a + i\,\Delta x$
$= \frac{3i}{n}$

Choose $x_i^* = x_i \Rightarrow f(x_i) = x_i - 5$
$= \frac{3i}{n} - 5$

$$\begin{aligned}
\int_0^3 (x-5)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n} - 5\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} \frac{3i}{n} - \sum_{i=1}^{n} 5\right) \\
&= \lim_{n\to\infty} \frac{3}{n}\left(\frac{3}{n}\sum_{i=1}^{n} i - 5\sum_{i=1}^{n} 1\right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{3}{n}\left(\frac{3}{n}\,\frac{n(n+1)}{2} - 5n\right) \\
&= 3\lim_{n\to\infty}\left(\frac{3(n+1)}{2n} - 5\right) \\
&= 3\lim_{n\to\infty}\left(\frac{3}{2}\left(1+\frac{1}{n}\right) - 5\right) \\
&= 3\left(\frac{3}{2} - 5\right) \\
&= \frac{-21}{2}
\end{aligned}$$

(b) Compute $\int_0^3 (x-5)\,dx$ geometrically



$$\int_0^3 (x-5)\,dx = 3(-2) + \frac{(-3)(3)}{2}$$

$$= -6 + \frac{-9}{2}$$

$$= \frac{-21}{2}$$

2. Express

$$\lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}}$$

as a definite integral

Choose $x_i^* = x_i$ (R_n, L_n will have a lot of $i-1$)

$$\lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6+\frac{5i}{n}}{\sqrt{4+\frac{5i}{n}}} \cdot \frac{5}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6+i\Delta x}{\sqrt{4+i\Delta x}} \Delta x$$

Choose $\Delta x = \frac{5}{n} \Rightarrow b-a = 5$

$$= \int_4^9 \frac{2+x}{\sqrt{x}}\,dx$$

$x_i = a + i\Delta x$
$= 4 + i\Delta x$
$a = 4$
$b = 9$

Def: Properties of Definite Integral

Let $a, b \in \mathbb{R}$, $a<b$. Suppose f, g are integrable on $[a,b]$. Then

(i) IF $f(x) \geq 0 \;\forall x \in [a,b]$ THEN $\int_a^b f(x)\,dx \geq 0$

IF $f(x) \leq 0 \;\forall x \in [a,b]$ THEN $\int_a^b f(x)\,dx \leq 0$

(ii) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(iii) For any $c \in \mathbb{R}$, $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$

(iv) $\int_a^a f(x)\,dx = 0$

(v) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(vi) The union interval property

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \text{For any constant } c \in (a,b)$$