Def:

Let V be a vector space. A nonempty subset W of V is called a subspace provided that

(i) $\vec{0}_V \in W$
(ii) $\forall \vec{w}_1, \vec{w}_2 \in W \quad \vec{w}_1 + \vec{w}_2 \in W$
(iii) $\forall \vec{w} \in W \ \forall r \in \mathbb{R} \quad r\vec{w} \in W$

Ex 1

Is $\{\vec{0}_V\} \subseteq V$ a subspace of V ?

The three conditions above are satisfied.

Notation:

$W \underset{s.s}{\subseteq} V$ W is a subspace of V

$W \subseteq V$ W is a subset of V

Theorem:

Let $T: V \to W$ be a L.T.

(i) $\operatorname{Ker} T \underset{s.s}{\subseteq} V$

(ii) $\operatorname{im} T \underset{s.s}{\subseteq} W$

Proof (ii):

1. Prove img T is non empty

   Pick $v \in \mathbb{R}^n$.

   Then $T(v) = w$ is in $im(T)$.

   $\therefore$ img T is non-empty

2. Prove img T is closed under addition

   Suppose $\vec{w}_1$ and $\vec{w}_2$ are vectors in img T.

   WTS: $\vec{w}_1 + \vec{w}_2$ is in img T

   By def of image,

   $\exists \ \vec{v}_1, \vec{v}_2$ s.t. $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$

   $\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2)$

   $= T(\vec{v}_1 + \vec{v}_2)$

   $= T(\vec{v}_3)$ where $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

   $\therefore$ $\vec{w}_1 + \vec{w}_2$ is also in img T

3. Prove img T is closed under scalar multiplication

   $\vec{w}_1$ in img T. $\exists \ T(v) = \vec{w}_1$, $r\vec{w}_1 = rT(v) = T(rv)$

(2) $\forall \vec{w}_1, \vec{w}_2 \in \text{Ker } T$
$\vec{w}_1 + \vec{w}_2 \in \text{Ker } T$

(0) $\text{Ker } T \neq \{ \} \subseteq V$
(1) $\vec{0}_v \in \text{Ker } T$

(3) $\forall \vec{w} \in \text{Ker } T \ \forall r \in \mathbb{R} \ \ r\vec{w} \in W$

Proof (i): WTS $\text{Ker } T \underset{s.s}{\subseteq} V$. $\Rightarrow$

$\vec{0} \in \text{Ker } T$ since $T(\vec{0}_v) = \vec{0}_w$ and $\text{ker } T = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0}_w \} \subseteq V$

$\therefore$ $\vec{0}_v \in \text{Ker } T$. This satisfies (1) and (0)

Pick $\vec{w}_1, \vec{w}_2$ in $\text{Ker } T$

Is $\vec{w}_1 + \vec{w}_2 \in \text{Ker } T$?

$$
\begin{aligned}
T(\vec{w}_1 + \vec{w}_2) &= T(\vec{w}_1) + T(\vec{w}_2) && \text{L.T. Property} \\
&= \vec{0}_w + \vec{0}_w && \text{Since } \vec{w}_1, \vec{w}_2 \in \text{Ker } T \Rightarrow T(\vec{w}_1) = \vec{0}_w,\ T(\vec{w}_2) = \vec{0}_w \\
&= \vec{0}_w
\end{aligned}
$$

$\therefore$ $\vec{w}_1 + \vec{w}_2 \in \text{Ker } T$. This satisfies (2).

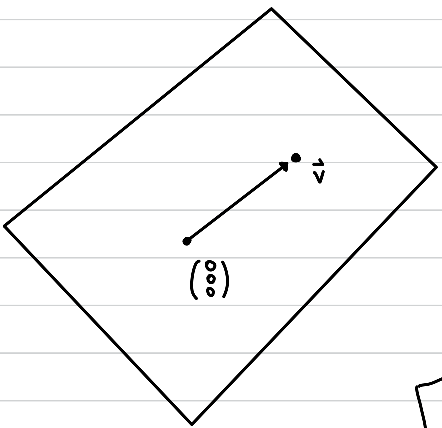
Pick $\vec{w} \in \text{Ker } T$, pick $r \in \mathbb{R}$

$$
\begin{aligned}
T(r\vec{w}) &= r\,T(\vec{w}) \\
&= r\,\vec{0}_w && \text{Since } \vec{w} \in \text{ker } T \Rightarrow T(\vec{w}) = \vec{0} \\
&= \vec{0}_w
\end{aligned}
$$

$\therefore$ $r\vec{w} \in \text{Ker } T$ satisfies (3)

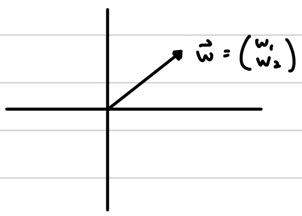
$\therefore$ $\text{Ker } T \underset{s.s}{\subseteq} V$

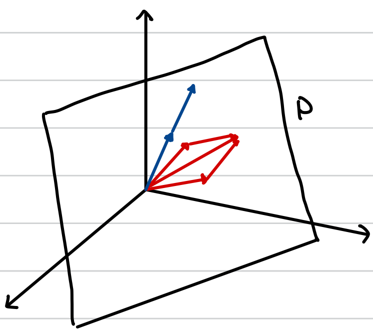
Intuition on subspace



$\mathbb{R}^3$

$\mathbb{R}^2$





P

P itself is a vector space. $P \subseteq \mathbb{R}^3$. It is a vector space within another vector space.

Theorem

Let W be a subspace of a vector space V.

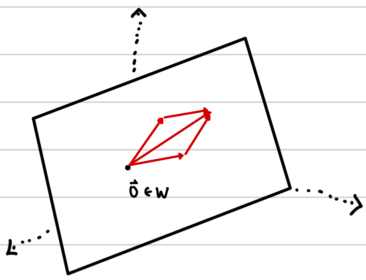
Then W is a vector space together with vector addition and scalar multiplication of V.

Ex 2

$V = \mathbb{R}^3$

W = plane given by $x+y+z=0$

Is $W \underset{s.s}{\subseteq} V$?



$x+y+z=0$

$$\begin{array}{cccc} & t & s & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{pmatrix} -t-s \\ t \\ s \end{pmatrix} \quad t, s \in \mathbb{R}$$

$$= \left\{ t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Theorem:

Let $\vec{v}_1, \dots, \vec{v}_k$ in V. Then $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) \underset{s.s}{\subseteq} V$.

Proof :

(0) $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k) \subseteq V$

$\vec{v}_1, \dots, \vec{v}_k \in V$

Take $\vec{w} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k) = r\,\vec{w} = r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_k\vec{v}_k \in V$

Thus $\text{span}(\vec{v}_1, \dots, \vec{v}_0) \subseteq V$

Proof continued :

(1) WTS $\vec{0}_V \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

$$\vec{0}_V = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_k \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$$

(2) Pick $\vec{w}_1, \vec{w}_2$ in $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$

$\vec{w}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$, $\vec{w}_2 = r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_k\vec{v}_k$ some c_i, r_i in $\mathbb{R}$

WTS $\vec{w}_1 + \vec{w}_2 \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

$$\vec{w}_1 + \vec{w}_2 = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k + r_1\vec{v}_1 + r_2\vec{v}_2 + \dots + r_k\vec{v}_k$$

$$= \underbrace{(c_1 + r_1)}_{\in \mathbb{R}}\vec{v}_1 + \underbrace{(c_2 + r_2)}_{\in \mathbb{R}}\vec{v}_2 + \dots + \underbrace{(c_k + r_k)}_{\in \mathbb{R}}\vec{v}_k$$ Distributive property

$\in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

$\therefore \vec{w}_1 + \vec{w}_2 \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

(3) Pick $\vec{w} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$. Pick $r \in \mathbb{R}$

WTS $r\vec{w} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$

$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ some $c_1, \dots, c_k$ in $\mathbb{R}$

$$r\vec{w} = r(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k)$$

$$= \underbrace{rc_1}_{\in \mathbb{R}}\vec{v}_1 + \underbrace{rc_2}_{\in \mathbb{R}}\vec{v}_2 + \dots + \underbrace{rc_k}_{\in \mathbb{R}}\vec{v}_k$$

$\in \text{span}(\vec{v}_1, \dots, \vec{v}_k)$ By def of span