W6 Lecture 13 Notes

Addition Law $\lim_{x \to c}[f(x) \pm g(x)] = L \pm M$

Proof:

We need to show that $\lim_{x \to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x \to c} g(x) = L+M$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-c| < \delta \Rightarrow |f(x)+g(x)-(L+M)| < \varepsilon$

$\lim_{x\to c} f(x) = L$: $\forall \varepsilon_1 > 0 \ \exists \delta_1 > 0$ s.t. $0 < |x-c| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$

$\lim_{x\to c} g(x) = M$: $\forall \varepsilon_2 > 0 \ \exists \delta_2 > 0$ s.t. $0 < |x-c| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon_2$

Let $\varepsilon_1 = \frac{\varepsilon}{2}$; $\varepsilon_2 = \frac{\varepsilon}{2}$

Choose $\delta = \min\{\delta_1, \delta_2\}$, then both $g(x)$ and $f(x)$ are within $\frac{\varepsilon}{2}$ of M and L.

If $0<|x-c|<\delta$ we have $|f(x)-L| < \frac{\varepsilon}{2}$ and $|g(x)-M| < \frac{\varepsilon}{2}$

$$-\frac{\varepsilon}{2} < f(x) - L < \frac{\varepsilon}{2}$$
$$-\frac{\varepsilon}{2} < g(x) - M < \frac{\varepsilon}{2}$$
$$\Downarrow$$
$$-\varepsilon < f(x)+g(x)-(L+M) < \varepsilon \Rightarrow |f(x)+g(x)-(L+M)| < \varepsilon$$

So we have shown that for all $\varepsilon > 0$ there exists $\delta > 0$ such that when $0 < |x-c| \ \delta$, $|f(x)+g(x) - (L+M)| < \varepsilon$.

QED

Product Law $\lim_{x\to c} [f(x)\cdot g(x)] = LM$

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then $\lim_{x\to c} f(x)\cdot g(x) = LM$

Aside:

$$|f(x)g(x) - LM| = |f(x)g(x) - LM + Lg(x) - Lg(x)| = |g(x)(f(x)-L) + L(g(x)-M)|$$
$$\le \underbrace{|g(x)||f(x)-L|}_{\frac{\varepsilon}{2}} + \underbrace{|L||g(x)-M|}_{\frac{\varepsilon}{2}}$$

Since $\frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

For $|L||g(x)-M| < \frac{\varepsilon}{2}$

Let's choose δ_1 such that $\forall \varepsilon>0\ \exists \delta_1>0$

s.t. $0<|x-c|<\delta_1 \Rightarrow |g-M| < \frac{\varepsilon}{2|L|}$

To get this: $\frac{\cancel{|L|}\,\varepsilon}{2\,\cancel{|L|}}$

$$|g-M| < \frac{\varepsilon}{2|L|} \Rightarrow \frac{\varepsilon}{2|L|} > |g(x)-L| \ge |g(x)| - |M|$$

$$\frac{\varepsilon}{2|L|} > |g| - |M| \Rightarrow |g| < |M| + \frac{\varepsilon}{2|L|}$$
$$|g| < |M| + 10$$

since ε is any number.

Let's choose δ_2 such that $\forall \varepsilon>0\ \exists \delta_2>0$

s.t. $0<|x-c|<\delta_2 \Rightarrow |g| < |M| + 10$

For $|g(x)||f(x)-L| = \frac{\varepsilon}{2}$

Let's choose δ_3 such that $\forall \varepsilon>0\ \exists \delta_3>0$

s.t. $0<|x-c|<\delta_3 \Rightarrow |f-L| < \frac{\varepsilon}{2(|M|+10)}$

Proof:

Given $\varepsilon>0$, choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then if $0<|x-c|<\delta$, we have:

$$|fg - LM| = \dots \le |g||f-L| + |L||g-M| < \cancel{(|M|+10)}\frac{\varepsilon}{2\cancel{(|M|+10)}} + \frac{\cancel{|L|}\,\varepsilon}{2\,\cancel{|L|}}$$
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

QED

Reciprocal Law $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$, $M \neq 0$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-c| < \delta \Rightarrow \left| \frac{1}{g} - \frac{1}{M} \right| < \varepsilon$

Aside:

$$\left| \frac{1}{g} - \frac{1}{M} \right| = \left| \frac{M-g}{gM} \right| = |M-g| \cdot \frac{1}{|g|} \quad \frac{1}{|M|}$$

$$< \frac{M^2}{2}\varepsilon \cdot \frac{2}{|M|} \cdot \frac{1}{M}$$

Proof:

Given $\varepsilon > 0$, choose $\delta = \min\{\delta_1, \delta_2\}$, then if $0 < |x-c| < \delta$ we have:

$$\left| \frac{1}{g} - \frac{1}{M} \right| = |M-g| \cdot \frac{1}{|g|} \cdot \frac{1}{|M|} < \frac{M^2}{2}\varepsilon \cdot \frac{2}{|M|} \cdot \frac{1}{M}$$

$$= \varepsilon$$

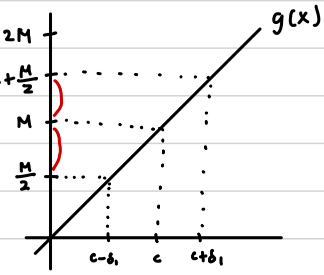
QED

Let's choose δ_1 such that if $0 < |x-c| < \delta_1$, then $|g-M| = |M-g| < \frac{|M|}{2}$

If $|M-g| < \frac{|M|}{2}$,
then $|g| > \frac{|M|}{2}$

$\Downarrow$

$$\frac{1}{|g(x)|} < \frac{2}{|M|}$$



Let's choose δ_2 such that if $0 < |x-c| < \delta_2$, then $\underbrace{|g-M| < \frac{M^2}{2}\varepsilon}$

such that we will have ε here

Theorem 6 Continuity of power function $f(x) = x^k$ where k is a positive integer.

Power function $f(x) = x^k$ is continuous on $(-\infty, \infty)$

<u>Proof:</u>

We need to show that $\lim_{x \to c} x^k = c^k$ for $k \in \mathbb{Z}$, $c \in \mathbb{R}$ and $x \in (-\infty, \infty)$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |x^k - c^k| < \varepsilon$

$$\lim_{x \to c} x^k = \underbrace{\lim_{x \to c} x \cdot \lim_{x \to c} x \cdots \lim_{x \to c} x}_{k \text{ times}} \quad \text{(product law)}$$

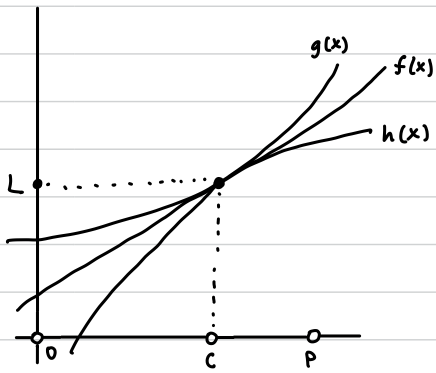
$$\lim_{x \to c} x = c, \Rightarrow \lim_{x \to c} x^k = c^k$$

QED

We can also prove continuity of $f(x) = x^{-k}$ and $f(x) = x^{n/k}$ with the same strategy.

Theorem 7 Squeeze Theorem

Let $p > 0$. Suppose that, for all x such that $0 < |x - c| < p$, $h(x) \le f(x) \le g(x)$. If $\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$ then $\lim_{x \to c} f(x) = L$.



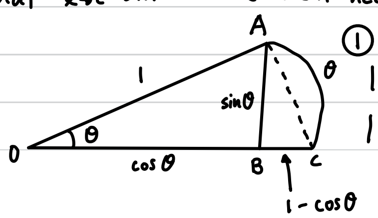
Theorem 8 Continuity of Trig functions

All trig functions are continuous on their domains

<u>Proof:</u>

We need to show that $\lim_{x \to c} \sin x = \sin c$ for $c \in \mathbb{R}$ and $x \in (-\infty, \infty)$

To prove that $\lim_{x \to c} \sin x = \sin c$ we'll need to prove that $\lim_{\theta \to 0} \sin \theta = 0$ and that $\lim_{\theta \to 0} \cos \theta = 1$



①

$|\text{Arc } AC| = \theta$

$|AB| = \sin\theta \Rightarrow 0 < |AB| < |\text{arc } AC|$

$\lim_{\theta \to 0} 0 = 0$, $\lim_{\theta \to 0} |\text{arc } AC| = 0$

By squeeze theorem,

$\lim_{\theta \to 0} |AB| = \lim_{\theta \to 0} \sin\theta = 0$

Continued Proof:

② Consider $\triangle BAC$

$$|AC| = \sqrt{|AB|^2 + |BC|^2} = \sqrt{\sin^2\theta + (1-\cos\theta)^2}$$
$$= \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos^2\theta}$$
$$= \sqrt{2 - 2\cos\theta}$$

$$0 \le |AC| \le \theta$$
$$0 \le \sqrt{2 - 2\cos\theta} \le \theta$$
$$0 \le 2 - 2\cos\theta \le \theta^2$$
$$0 \le 1 - \cos\theta \le \frac{\theta^2}{2}$$
$$-1 \le -\cos\theta \le \frac{\theta^2}{2} - 1$$
$$1 \ge \cos\theta \ge 1 - \frac{\theta^2}{2}$$

$$\lim_{\theta\to 0} 1 = 1, \quad \lim_{\theta \to 0} 1 - \frac{\theta^2}{2} = 1$$

By the Squeeze theorem,

$$\lim_{\theta\to 0} \cos\theta = 1$$

③ To show that $\lim_{x\to c} \sin x = \sin c$:

Let $x = h + c$

$$\lim_{\substack{(x\to c)\\ h\to 0}} \sin(h+c) = \lim_{h\to 0} (\sin h \cos c + \cos h \sin c)$$
$$= \left(\lim_{h\to 0} \sin h\right)\left(\lim_{h\to 0} \cos c\right) + \left(\lim_{h\to 0} \cos h\right)\left(\lim_{h\to 0} \sin c\right)$$
$$= 0 \cdot \cos c + 1 \cdot \sin c = \sin c$$

QED

Theorem 9 Continuity of Exponential Functions

All exponential and logarithmic functions are continuous on their domains

We define e to be the number that $(1+h)^{1/h}$ approaches as h approaches 0:
$\lim_{h\to 0} (1+h)^{1/h} = e$.

$$(1+h)^{1/h} \approx e \Rightarrow 1 + h \approx e^h \Rightarrow 1 \approx \frac{e^h - 1}{h}$$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Proof:

We have to show that $\lim_{x\to c} e^x = e^c$.

Let $x = c + h$.

$$\lim_{x\to c} e^x = \lim_{h\to 0} e^{c+h} = \lim_{h\to 0} e^c(e^h - 1 + 1) = e^c \lim_{h\to 0}\left(\frac{e^h - 1}{h} h + 1\right)$$
$$= \left(\lim_{h\to 0} e^c\right)\left(\lim_{h\to 0} \frac{e^h-1}{h}\left(\lim_{h\to 0} h\right) + \lim_{h\to 0} 1\right) = e^c[1 \cdot 0 + 1]$$
$$= e^c$$

QED

Remarkable Limits

$$\lim_{x\to0}\frac{\sin x}{x}=1\ ;\quad \lim_{x\to0}\frac{1-\cos x}{x}=0\ ;\quad \lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k\ ;\quad \lim_{x\to0}(1+x)^{1/x}=e$$

1) Prove that $\lim_{x\to0}\frac{\sin x}{x}=1$ using the squeeze theorem

Area of $\triangle OAC = \frac{\sin\theta}{2}$

Area of sector $OAC = \frac{\theta}{2}$

area   angle

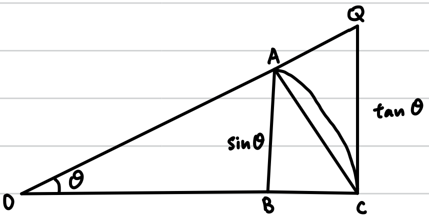
$\pi r^2 \to 2\pi$

$x \to \theta$

$x = \frac{\pi r^2\theta}{2\pi} = \frac{\theta}{2}r^2$

Proportions $r=1$

Area of $\triangle OQC = \frac{\tan\theta}{2}$



Area $\triangle OAC \le$ Area sector $OAC \le$ Area $\triangle OQC$

$$\frac{\sin\theta}{2} \le \frac{\theta}{2} \le \frac{\tan\theta}{2}$$

$$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

By Squeeze Theorem,

$$\lim_{\theta\to0}1=1\ ,\quad \lim_{\theta\to0}\cos\theta=1 \qquad\qquad \lim_{\theta\to0}\frac{\sin\theta}{\theta}=1$$

2) Prove that $\lim_{x\to0}\frac{1-\cos x}{x}=0$

$$\lim_{x\to0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to0}\frac{\sin^2x}{x(1+\cos x)}=\lim_{x\to0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right)$$

$$=\left(\lim_{x\to0}\frac{\sin x}{x}\right)\left(\lim_{x\to0}\frac{\sin x}{1+\cos x}\right)$$

$$=1\cdot\frac{\sin 0}{1+\cos 0}=0$$

QED

Theorem 10

Polynomial, rational, root, trig, exponential, log, are continuous on their domains.

Theorem 11

If f and g are continuous at c, then the following functions are also continuous at c:

1) $f \pm g$
2) $f \cdot g$
3) $\frac{f}{g}$ if $g(c) \neq 0$

Theorem 12 Limits whose Denominators Approach Zero from the Right or the left.

a) If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^+}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$

b) If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^-}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$

Proof: (a)

$\forall M>0 \ \exists \delta>0$ s.t. $0<|x-c|<\delta \Rightarrow \frac{f(x)}{g(x)} > M$

$\lim_{x\to c} f(x) = 1$ means that $\forall \varepsilon_1>0 \ \exists \delta_1$ s.t. $0<|x-c|<\delta_1 \Rightarrow |f(x)-1|<\varepsilon_1$

$-\varepsilon_1 < f(x)-1 < \varepsilon_1$

$1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to c} g(x) = 0$ means that $\forall \varepsilon_2>0 \ \exists \delta_2>0$ s.t. $0<|x-c|<\delta_2 \Rightarrow |g(x)|<\varepsilon_2$

$-\varepsilon_2 < g(x) < \varepsilon_2$

Proof:

Let $\delta = \min\{\delta_1, \delta_2\}$ then for $0<|x-c|<\delta$ we have.

$\frac{f(x)}{g(x)} > \frac{1-\varepsilon_1}{\varepsilon_2}$. Let $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \frac{1}{2M}$ then $\frac{f(x)}{g(x)} > \frac{1/2}{1/2M} = M$

QED

Theorem 13 Limits whose Denominators become Infinite

a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

b) If $\lim_{x\to-\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{-\infty}$, then $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = 0$

Proof:

We need to show that $\forall \varepsilon>0 \ \exists N>0$ s.t. $x>N \Rightarrow \left|\frac{f(x)}{g(x)}\right| < \varepsilon$

$\lim_{x\to\infty} f(x) = 1$ means that $\forall \varepsilon_1>0 \ \exists N_1>0$ s.t. $x>N_1 \Rightarrow |f(x)-1| < \varepsilon_1$

$1-\varepsilon_1 < f(x) < 1+\varepsilon_1$

$\lim_{x\to\infty} g(x) = \infty$ means that $\forall M>0 \ \exists N_2>0$ s.t. $x>N_2 \Rightarrow g(x)>M$

Proof:

Given $M>0$, choose $N = \max\{N_1, N_2\}$. We have:

$\left|\frac{f(x)}{g(x)}\right| < \frac{1+\varepsilon_1}{M}$. Let $\varepsilon_1 = 1$, $M = \frac{2}{\varepsilon}$. Then $\left|\frac{f(x)}{g(x)}\right| < \frac{2}{2}\varepsilon = \varepsilon$

QED