B41 Oct 18 Lec 1 Notes

Theorem:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of f all exist and are continuous in a neighbourhood $a \in U$. Then f is differentiable at $a \in U$.

Ex 1:

Let f be a differentiable function. Verify that $w = f(x^2-y^2, y^2-x^2)$ is a solution to the differential equation

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0$$

Let $u = x^2-y^2$, $v = y^2-x^2$

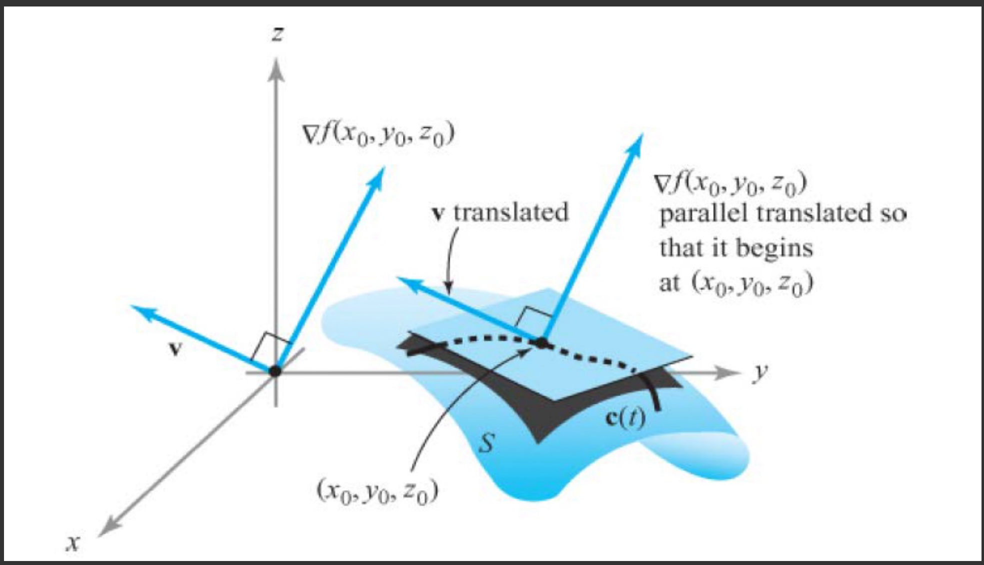
$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial w}{\partial u}(2x) + \frac{\partial w}{\partial v}(-2x)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial w}{\partial u}(-2y) + \frac{\partial w}{\partial v}(2y)$$

Thus $y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0$

Theorem:

Let $f: \mathbb{R}^3 \to \mathbb{R}$ have continuous partial derivatives and let $x_0 = (x_0, y_0, z_0)$ lie on the level surface S defined by $f(x,y,z) = k$, for k a constant. Then $\nabla f(x_0)$ is normal to the level surface S.



Proof:

Let $c(t) = (x(t), y(t), z(t))$ be any differential curve pass through x_0 at $t = t_0$ on the level surface S.
Then

$$f(x(t), y(t), z(t)) = k$$

Proof: (continued...)

Differentiating both sides of the equation, by the chain rule,

$$\left.\frac{df}{dt}\right|_{t=t_0} = \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)_{t=t_0} = \left.\frac{d(k)}{dt}\right|_{t=t_0} = 0$$

That is $\nabla f(x_0) \cdot c'(t_0) = 0$

$\therefore$ $\nabla f(x_0)$ is perpendicular to the tangent line of any curves that pass through x_0 on the surface.

Definition:

Let S be the surface consisting of those (x,y,z) s.t. $f(x,y,z) = k$, for k a constant. Let f be differentiable at $x_0 = (x_0, y_0, z_0)$. The tangent plane of S at x_0 in $\mathbb{R}^3$ is defined by the equation

$$\nabla f(x_0) \cdot (x - x_0) = 0$$

That is, $\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$

Ex 2:

Find the tangent plane of the graph $e^x + z\sin y = 2$ at the point $(0, \pi/2, 1)$

Let $f(x,y,z) = k$, where $f(x,y,z) = e^x + z\sin y$ and $k = 2$.

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (e^x, z\cos y, \sin y)$

So $\nabla f(0, \frac{\pi}{2}, 1) = (1, 0, 1)$

Tangent plane : $(1,0,1) \cdot (x - 0, y - \pi/2, z - 1) = 0$

Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differential function at (x_0, y_0). Then the tangent plane of the graph of f at the point $(x_0, y_0, f(x_0, y_0))$ is given by

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

OR

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right] \cdot [x - x_0, y - y_0]$$

$$= f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

Ex 3:

Find the points on the surface defined by $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane defined by $3x - y + 3z = 1$.

Let $f = x^2 + 2y^2 + 3z^2$

Then $\nabla f = (2x, 4y, 6z)$

(Continued in next lecture)