List of Definitions for MATA22-Winter 2021

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1. VECTORS

1. **Definition.** *A real* ***column vector*** *is a* $n \times 1$ *matrix* $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, *where* $v_i \in \mathbb{R}$, $1 \leq i \leq n$.

2. **Definition.** *A real* ***row vector*** *is a* $1 \times n$ *matrix* $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, *where* $v_i \in \mathbb{R}$, $1 \leq i \leq n$.

[1]

3. **Definition.** *The Euclidean space* $\mathbb{R}^n$ *is*

$$\{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \mid v_i \in \mathbb{R}, 1 \leq i \leq n\}.$$

4. **Definition.** *Let* $\vec{v}$ *and* $\vec{w}$ *be (row or column) vectors with components* $v_1, v_2, \cdots, v_n$ *and* $w_1, w_2, \cdots, w_n$ *respectively. The* ***dot product*** *of* $\vec{v}$ *and* $\vec{w}$ *is a scalar denoted by* $\vec{v} \cdot \vec{w}$ *and is defined as*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

5. **Definition.** *Let* $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ *be in* $\mathbb{R}^n$. *Then* ***norm or magnitude or length*** *of* $\vec{v}$ *is denoted by* $\|\vec{v}\|$ *and is defined to be*

$$\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

6. **Definition.** *We say two vectors* $\vec{v}$ *and* $\vec{w}$ *are* ***parallel*** *if one is the scalar multiple of the other. That is if* $\vec{v} = k\vec{w}$ *for some* $k \in \mathbb{R}$ *or* $\vec{w} = k\vec{v}$ *for some* $k \in \mathbb{R}$.

7. **Definition.** *We say two vectors* $\vec{v}$ *and* $\vec{w}$ *are* ***perpendicular*** *or* ***orthogonal*** *if* $\vec{v} \cdot \vec{w} = 0$.

8. **Definition.** *Given vectors* $\vec{v}$ *and* $\vec{w}$, *the* ***angle*** *between* $\vec{v}$ *and* $\vec{w}$ *is define to be*[2]

$$\arccos(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}).$$

[1] There are other ways of representing row vectors such as $\langle v_1, \cdots, v_n \rangle$ or $(v_1, \cdots, v_n)$ that we saw in webwork questions. Our preference is to reserve the notation $(v_1, \cdots, v_n)$ for a point and $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ for a (row) vector

[2] this definition makes sense thanks to Cauchy Schwartz inequality

2. System of linear equations

9. **Definition.** *Let A be a matrix in REF form. An entry of A is called a* ***pivot entry*** *if it is the first non-zero entry of a row. A column of A is called a* ***pivot column*** *if it contains a pivot entry. If A is also in RREF, pivot entries are called* ***leading ones***

10. **Definition.** *Given a system of linear equations with m variables and n equations, with coefficient matrix A and augmented matrix $[A|\vec{b}]$*

(1) *a variable is called a* ***free variable*** *if its corresponding column in* $\mathrm{rref} A$ *does not contain a pivot (or a leading one).*
(2) *a variable is called a* ***leading variable or a basic variable or a dependent variable*** *if its corresponding column in* $\mathrm{rref} A$ *contains a pivot (or a leading one).*

3. Matrices

11. **Definition.** *The identity matrix I_n is an $n \times n$ matrix whose diagonal entries are 1 and off diagonal entries are* 0.

12. **Definition.** *The zero matrix $0_{n\times m}$ is an $n \times m$ matrix with all zero entries.*

Rank. The **rank** of a matrix A is the number of leading ones in $\mathrm{rref}(A)$.[3]

13. **Definition.** *Given an $n \times m$ matrix with rows $\vec{w}_1, \cdots, \vec{w}_n$ and a vector $\vec{x} \in \mathbb{R}^m$ then*

$$A\vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}.$$

4. Linear transformations: chapter 2 and 4

Vector Space. Let V be a set. Suppose an *addition operation* $+$ is defined on V, so that for every pair of elements v, w of V there is associated another element $v + w$ of V. Suppose also that a *scalar multiplication* by real numbers is defined on V, so that for every $c \in \mathbb{R}$ and $v \in V$ there is associated an element $c \cdot v$ in V. Then V is called a *vector space* if all the following are true:

VS-1: for all $u, v, w \in V$, $(u + v) + w = u + (v + w)$;
VS-2: for all $u, v \in V$, $u + v = v + u$;
VS-3: there is an element $\vec{0} \in V$ such that $\vec{0} + v = v$ for all $v \in V$;
VS-4: for all $v \in V$ there is a unique element $-v \in V$ such that $v + (-v) = \vec{0}$;
VS-5: for all $a \in \mathbb{R}$ and $v, w \in V$, $a \cdot (v + w) = a \cdot v + a \cdot w$;
VS-6: for all $a, b \in \mathbb{R}$ and for all $v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$;
VS-7: for all $a, b \in \mathbb{R}$ and for all $v \in V$, $a \cdot (b \cdot v) = (ab) \cdot v$;
VS-8: for all $v \in V$, $1 \cdot v = \vec{v}$.

An element of a vector space is called a *vector*.

Function. A **mapping** or **function** or **transformation** $X \xrightarrow{f} Y$ is a rule which assigns a unique element $f(x) \in Y$ to each element $x \in X$.

Source and Target. We call X the **domain** or **source** of f, and call Y the **codomain** or **target** of f.

[3] this definition makes sense thanks to uniqueness of RREF.

Image. The **image** of $f : X \to Y$ is defined to be the set $\mathrm{im}(f) = \{f(x) : x \in X\} = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}$. [4]

Surjective/injective. The function $: X \to Y f$ is **surjective** or **onto** if for all $y \in Y$, there exists some $x \in X$ such that $f(x) = y$. In this case, we also say "f is a **surjection.**"

The function f is **injective** or **one-to-one** if for all $y \in Y$ there exists at most one $x \in X$ such that $f(x) = y$. Equivalently: whenever $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$, it follows that $x_1 = x_2$. In this case, we also say "f is an **injection.**"

Invertible. We say the function f is **invertible** or **bijective** if for all $y \in Y$ there exists exactly one $x \in X$ such that $f(x) = y$.

Linear Transformation. Let V and W be vector spaces. A **linear transformation** from V to W is a mapping $V \xrightarrow{T} W$ that satisfies:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all vectors $\vec{x}, \vec{y} \in V$;
(2) $T(k\vec{x}) = kT(\vec{x})$ for all vectors $\vec{x} \in V$ and all scalars $k \in \mathbb{R}$.

5. Concepts in vector spaces

Linear combination. A **linear combination** of the finitely many vectors $\vec{v}_1, \ldots, \vec{v}_n$ in V is an expression of the form

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

where each c_i is a scalar.

[4] Note that the target space and the image of f are not necessarily the same. In previous courses, this distinction may not have been emphasized and you may have referred to both or either as the "range" of f.