CH 6.3 Geo. int. of the Det.

Def 6.3.2: Rotation matrices

An orthogonal $n \times n$ matrix A with $\det A = 1$ is called a rotation matrix, and the L.T. $T(\vec{x}) = A\vec{x}$ is called a rotation.

Theorem 6.3.3: The determinant in terms of the columns

If A is an $n \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, then

$$|\det A| = \|\vec{v}_1\| \, \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_n^{\perp}\|$$

where $\vec{v}_k^{\perp}$ is the component of $\vec{v}_k$ perpendicular to span $(\vec{v}_1, \dots, \vec{v}_{k-1})$.

Theorem 6.3.4: Volume of a Parallelepiped in $\mathbb{R}^3$.

Consider a 3×3 matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$. Then the volume of the parallelopiped define by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ is $|\det A|$.

Theorem 6.3.6: Volume of a parallelopiped in $\mathbb{R}^n$.

Consider the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Then the m-volume of the m-parallelopiped defined by the vectors $\vec{v}_1, \dots, \vec{v}_m$ is

$$\sqrt{\det(A^T A)}$$

where A is the $n \times m$ matrix with columns $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

In particular, if $m = n$, this volume is

$$|\det A|$$

Det. as an Expansion Factor

Rotations preserve length of vectors and the angles between vectors.

Similarly, we can ask how a L.T. affect the area.

Theorem 6.3.7: Expansion Factor

Consider a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then $|\det A|$ is the expansion factor

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}$$

of T on parallelograms Ω.

Theorem 6.3.7 continued...:

Likewise, for a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, $|\det A|$ is the expansion factor of T on n-parallelopipeds:

$$V(A\vec{v}_1, \dots, A\vec{v}_n) = |\det A| \, V(\vec{v}_1, \dots, \vec{v}_n)$$

for all vectors $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$.

The expansion factor $|\det A^{-1}|$ is the reciprocal of the expansion factor $|\det A|$

$$|\det(A^{-1})| = \frac{1}{|\det A|}$$

The expansion factor $|\det AB|$ is the product of the expansion factors $|\det A|$ and $|\det B|$

Cramer's Rule

If a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible, we can express its inverse in terms of its determinant.

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

We can use Cramer's rule to find the solution to:

$$\left| \begin{matrix} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{matrix} \right| \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x} = A^{-1}\vec{b} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{\det A} \begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ a_{11}b_2 - a_{21}b_1 \end{bmatrix}$$

$$\Rightarrow \quad a_{22}b_1 - a_{12}b_2 = \det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$ replace 1st col of A by $\vec{b}$

$$a_{11}b_2 - a_{21}b_1 = \det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$ replace 2nd col of A by $\vec{b}$

$$\Rightarrow \quad x_1 = \frac{\det \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}{\det A}$$

$$x_2 = \frac{\det \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}{\det A}$$

Theorem 6.3.8: Cramer's Rule

Consider the linear system

$$A\vec{x} = \vec{b}$$

where A is an invertible $n\times n$ matrix. The components x_i of the solution vector $\vec{x}$ are

$$x_i = \frac{\det(A_{\vec{b},i})}{\det A}$$

where $A_{\vec{b},i}$ is the matrix obtained by replacing the ith col of A by $\vec{b}$.

Proof:

Write $A = [\vec{w}_1 \ \vec{w}_2 \cdots \vec{w}_i \cdots \vec{w}_n]$. If $\vec{x}$ is the solution of the system $A\vec{x} = \vec{b}$, then

$$\begin{aligned}
\det(A_{\vec{b},i}) &= \det[\vec{w}_1 \ \vec{w}_2 \cdots \vec{b} \cdots \vec{w}_n] \\
&= \det[\vec{w}_1 \ \vec{w}_2 \cdots A\vec{x} \cdots \vec{w}_n] \\
&= \det[\vec{w}_1 \ \vec{w}_2 \cdots (x_1\vec{w}_1 + x_2\vec{w}_2 + \dots + x_i\vec{w}_i + \dots + x_n\vec{w}_n) \cdots \vec{w}_n] \\
&= \det[\vec{w}_1 \ \vec{w}_2 \cdots x_i\vec{w}_i \cdots \vec{w}_n] \\
&= x_i \det[\vec{w}_1 \ \vec{w}_2 \cdots \vec{w}_i \cdots \vec{w}_n] \\
&= x_i \det A
\end{aligned}$$

Theorem 6.3.9: Adjoint and inverse of a matrix

Consider an invertible $n\times n$ matrix A. The classical adjoint $\operatorname{adj}(A)$ is the $n\times n$ matrix whose ijth entry is $(-1)^{i+j}\det(A_{ji})$. Then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A$$