B41 Oct 4 Lec 1 Notes

Theorem: Properties of continuous functions

(i) Let $f(x)$ and $g(x)$ be continuous real valued at x_0, and let c be a constant. Then cf, $f \pm g$, fg, and f/g $(g(x_0) \neq 0)$ are continuous at x_0.

(ii) Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$. f is continuous at x_0 iff each of the real valued function $f_1, f_2, \cdots, f_m$ is continuous at x_0.

(iii) Let $g : U_1 \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : U_2 \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(U_1) \subset U_2$, so that $f \circ g$ is defined on U_1. If g is continuous at x_0 and f is continous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

Definition:

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$. f is continuous at x_0 if given any $\varepsilon > 0$, there is a $\delta > 0$ s.t. $\|f(x) - f(x_0)\| < \varepsilon$ if $\|x - x_0\| < \delta$.

Differentiation

The directional derivative of f at a in direction v, denoted by $D_v(f(a)) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t\|v\|}$

When $v = e_i$, $i = 1, 2, \cdots, n$, $D_{e_i}(f(a))$ is denoted by $\frac{\partial f}{\partial x_i}(a)$ and is called the partial derivative of f with respect to x_i at a, $i = 1, 2, \cdots, n$

Then,

$$\frac{\partial f}{\partial x_i}(a) = D_{e_i}(f(a)) = \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t}$$

$$= \lim_{t \to 0} \frac{f(a_1, a_2, \cdots, a_i + t, \cdots a_n) - f(a_1, a_2, \cdots, a_i, \cdots, a_n)}{t}$$

Definition:

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. f is differentiable at $a \in U$ if the partial derivatives of f exist at a and if

$$\lim_{x \to a} \frac{\|f(x) - f(a) - Df(a)(x - a)\|}{\|x - a\|} = 0$$

where

$$Df(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right), \quad i = 1, 2, \cdots, m \text{ and } j = 1, 2, \cdots, n,$$

is the derivative (Jacobian matrix) of f at a given by

$$Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

Ex 1:

Calculate $Df(a)$ where $f(x,y,z) = (\underbrace{x^2 + y\sin z}_{f_1(x,y,z)}, \underbrace{xe^y}_{f_2(x,y,z)}, \underbrace{z\cos x}_{f_3(x,y,z)})$ at $a = (1,1,1)$

Note $f: \mathbb{R}^3 \to \mathbb{R}^3 \Rightarrow Df(a)$ is 3×3

$$D_f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & \sin z & y\cos z \\ e^y & xe^y & 0 \\ -z\sin x & 0 & \cos x \end{bmatrix}$$

$$Df(a) = \begin{bmatrix} 2 & \sin 1 & \cos 1 \\ e & e & 0 \\ -\sin 1 & 0 & \cos 1 \end{bmatrix}$$

Let $m=1$. $f: U \subset \mathbb{R}^n \to \mathbb{R}$. $Df(a)$ is the $1\times n$ matrix.

$$Df(a) = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots, \frac{\partial f}{\partial x_n}(a) \right)$$

It is called the gradient of f at a and denoted as $\nabla f(a)$

Ex 2:

Is the function $f(x,y) = x^{1/3} y^{1/3}$ differentiable at $(0,0)$.

$$\frac{\partial f}{\partial x} = \frac{1}{3} x^{-2/3} y^{1/3} \qquad \frac{\partial f}{\partial y} = \frac{1}{3} x^{1/3} y^{-2/3}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to0} \frac{0-0}{t} = 0$$

Similar for $\frac{\partial f}{\partial y}(0,0) = 0$

By def of differentiation,

$$\lim_{(x,y)\to(0,0)} \frac{\| f(x,y) - f(0,0) - Df(0,0)((x,y)-(0,0)) \|}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{x^{1/3} y^{1/3}}{\sqrt{x^2+y^2}}$$

$$= \text{DNE}$$

Theorem:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ of f all exist and are continuous in a neighborhood $a \in U$. Then f is differentiable at $a \in U$.