B41 Oct 8 Lec 2 Notes

Theorem: Chain Rule

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that f maps U into V so that $g \circ f$ is defined. Let $a \in U$ and $b = f(a) \in V$. If f is differentiable at a and g is differentiable at b, then $g \circ f$ is differential at a and $D(g \circ f)(a) = (Dg(b))(Df(a))$.

Ex 1:

Let $x = t^2 - s^2$, $y = ts$, $u = \sin(x+y)$, $v = \cos(x-y)$

(i) Express (u,v) in terms of (t,s) and calculate $\frac{\partial(u,v)}{\partial(s,t)}$

$u = \sin(t^2 - s^2 + ts)$ $\quad$ $v = \cos(t^2 - s^2 - ts)$

$$\frac{\partial(u,v)}{\partial(s,t)} = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} = \begin{bmatrix} (-2s+t)\cos(t^2-s^2+ts) & (2t+s)\cos(t^2-s^2+ts) \\ -(-2s-t)\sin(t^2-s^2-ts) & -(2t-s)\sin(t^2-s^2-ts) \end{bmatrix}$$

(ii) Compute $\frac{\partial(x,y)}{\partial(s,t)}$, $\frac{\partial(u,v)}{\partial(x,y)}$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(s,t) = (x(s,t), y(s,t))$
$Df(s,t) = \frac{\partial(x,y)}{\partial(s,t)}$

$g : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g(x,y) = (u(x,y), v(x,y))$
$Dg(x,y) = \frac{\partial(u,v)}{\partial(x,y)}$

$g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $g \circ f(s,t) = (u(x(s,t), y(s,t)), v(x(s,t), y(s,t)))$
$D(g \circ f)(s,t) = \frac{\partial(u,v)}{\partial(s,t)}$

Verify chain rule: $\frac{\partial(u,v)}{\partial(s,t)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(s,t)}$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} -2s & 2t \\ t & s \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos(x+y) & \cos(x+y) \\ -\sin(x-y) & \sin(x-y) \end{bmatrix}$$