A22 Apr 9 Lec 2 Notes

Theorem:

Let $T: V \to V$ L.T. Suppose λ is an eigenvector for T.

$E_\lambda = \{ \vec{v} \in V \mid T(\vec{v}) = \lambda \vec{v} \}$ is a subspace of V.

Proof:

Case: $T: \mathbb{R}^n \to \mathbb{R}^n$ is a L.T.
$\vec{x} \mapsto A\vec{x}$

(i) E_λ is nonempty

$T(\vec{0}) = \lambda \cdot \vec{0} = \vec{0}$. Thus $\vec{0} \in E_\lambda$ ✓

(ii) E_λ is closed under addition

Let $\vec{x}, \vec{y} \in E_\lambda$

$T(\vec{x}) = \lambda \vec{x}$
$T(\vec{y}) = \lambda \vec{y}$

$$\begin{aligned} T(\vec{x}+\vec{y}) &= \lambda(\vec{x}+\vec{y}) \\ &= \lambda\vec{x} + \lambda\vec{y} \\ &= T(\vec{x}) + T(\vec{y}) \end{aligned}$$

(iii) E_λ is closed under scalar multiplication

Let $\vec{x} \in E_\lambda$ and $c \in \mathbb{R}$

$$\begin{aligned} T(c\vec{x}) &= \lambda(c\vec{x}) \\ &= c(\lambda\vec{x}) \\ &= c\,T(\vec{x}) \end{aligned}$$

▨

Theorem: $T: \mathbb{R}^n \to \mathbb{R}^n$. The following are equivalent. (TFAE)
$\vec{x} \mapsto A\vec{x}$

(i) ⇙ ⇑
(ii) ⇒ (iii)

(i) $\mathbb{R}^n$ has an eigenbasis for T
(ii) $\mathbb{R}^n$ has a basis $\mathcal{B}$ s.t. $[T]_\mathcal{B}$ is a diagonal
(iii) $\exists$ invertible matrix P and a diagonal matrix D s.t. $P^{-1}AP = D$

Proof (i ⇒ ii):

Suppose $\mathbb{R}^n$ has an eigenbasis for T. $\mathcal{B} = (\vec{b}_1, \cdots, \vec{b}_n)$ i.e. $\mathcal{B}$ is a basis for $\mathbb{R}^n$ and $T(\vec{b}_i) = \lambda_i \vec{b}_i$, $\forall\, 1 \le i \le n$.

Proof ($i \Rightarrow ii$) continued... :

WTS $[T]_B$ is diagonal

$$[T]_B = \begin{bmatrix} | & & | \\ [T(\vec{b_1})]_{\mathcal{B}} & \cdots & [T(\vec{b_n})]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_i & \\ 0 & & & & \ddots \; \lambda_n \end{bmatrix}$$

$$T(\vec{b_i}) = 0\vec{b_1} + \ldots + \lambda_i \vec{b_i} + 0\vec{b_{i+1}} + \ldots + 0\vec{b_n}$$ $\boxtimes$

Proof ($ii \Rightarrow iii$).

Assume (ii)

WTS (iii)

WTS A is similar to a diagonal matrix.

Let $D = [T]_{\mathcal{B}}$, $P = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_n} \\ | & & | \end{bmatrix}$

Note:

P is invertible since it's columns are L.I.
D is diagonal by (ii)

$$P^{-1}AP = D \Leftrightarrow AP = PD$$

LHS

$$A \begin{bmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{bmatrix}$$

RHS

$$\begin{bmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_i & \\ 0 & & & & \ddots \; d_n \end{bmatrix}$$

Let's show $T(\vec{b_i}) = d_i \vec{b_i}$

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T(\vec{b_1})]_{\mathcal{B}} & \cdots & [T(\vec{b_n})]_{\mathcal{B}} \\ | & & | \end{bmatrix} = \begin{bmatrix} d_1 & & & & 0 \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_i & \\ 0 & & & & \ddots \; d_n \end{bmatrix}$$

i^{th} col : $[T(\vec{b_i})]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ d_i \\ \vdots \\ 0 \end{pmatrix} = d_i \vec{e_i} = d_i [\vec{b_i}]_{\mathcal{B}}$

$\therefore$ d_i is an eigenvalue $T(\vec{b_i}) = d_i \vec{b_i}$

LHS: $A \begin{bmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{b_1} & \cdots & A\vec{b_n} \\ | & & | \end{bmatrix}$ $\quad \therefore$ LHS = RHS $\boxtimes$

RHS: $\begin{bmatrix} | & | & & | \\ d_1\vec{b_1} & d_2\vec{b_2} & \cdots & d_n\vec{b_n} \\ | & | & & | \end{bmatrix}$

Proof (iii ⇒ i):

Assume (iii)

WTS (i)

Let $P = \begin{bmatrix} | & & | \\ \vec{b_1} & \cdots & \vec{b_n} \\ | & & | \end{bmatrix}_{n \times n}$. Since P is invertible, $\{\vec{b_1}, \dots, \vec{b_n}\}$ L.I. $\dim \mathbb{R}^n = n$

$B = \{\vec{b_1}, \dots, \vec{b_n}\}$ is a basis for $\mathbb{R}^n$.

WTS B is an eigenbasis for T

WTS $T(\vec{b_i}) = \lambda_i \vec{b_i}$, $\forall\ 1 \le i \le n$

By (iii),

$$P^{-1}AP = D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

$$AP = PD$$

$$A \begin{bmatrix} | & | & & | \\ \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ d_1\vec{b_1} & d_2\vec{b_2} & \cdots & d_n\vec{b_n} \\ | & | & & | \end{bmatrix}$$

$\forall\ 1 \le i \le n$, $A\vec{b_i} = d_i \vec{b_i}$ so d_i's are eigenvalues of T and $\vec{b_i}$'s are corresponding eigenvectors.

$\therefore B = (\vec{b_1}, \dots, \vec{b_n})$ is an eigenbasis for T.

▨

Theorem:

Let $T: V \to V$ be a L.T.

Eigenvectors corresponding to distinct eigenvalues are L.I.

i.e. $I = \{\vec{v_1}, \dots, \vec{v_r}\} \subset V$, s.t. $T(\vec{v_i}) = \lambda_i \vec{v_i}$, $\lambda_i \ne \lambda_j$ $\forall\ 1 \le i \ne j \le r$

Then I is L.I.

Ex 1:

$A_{3\times3}$, $\operatorname{char} A = (\lambda-1)(\lambda-3)(\lambda-5)$.

Is A diagonalizable?

$\lambda = 1, 3, 5$ are eigenvalues.

Let $\overset{\neq 0}{\vec{v}_1} \in E_1$, $\overset{\neq 0}{\vec{v}_2} \in E_3$, $\overset{\neq 0}{\vec{v}_3} \in E_5$. By thm , $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is L.I.

$\Rightarrow \mathcal{B}$ is an eigenbasis for $\mathbb{R}^n$
$\Rightarrow [T]_{\mathcal{B}}$ is diagonal
$\Rightarrow A$ (or T_A) is diagonalizable.

Proof:

WTS $\{\vec{v}_1, \cdots, \vec{v}_r\}$ are L.I.

Suppose $\{\vec{v}_1, \cdots, \vec{v}_r\}$ are L.D. , by contradiction

There exists a redundant vector in I.

Let $\vec{v}_k$ be the first redundant vector in $\{\vec{v}_1, \cdots, \vec{v}_r\}$

$\vec{v}_k = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_{k-1} \vec{v}_{k-1}$

Since $\vec{v}_k$ is the FIRST redundant vector, then $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k-1}\}$ is L.I.

$\lambda_k \vec{v}_k = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2 + \dots + \lambda_{k-1} c_{k-1} \vec{v}_{k-1}$

$T(\lambda_k \vec{v}_k) = T(\lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2 + \dots + \lambda_{k-1} c_{k-1} \vec{v}_{k-1})$

$\lambda_k \vec{v}_k = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \dots + \lambda_{k-1} c_{k-1} \vec{v}_{k-1}$

$\vec{0} = c_1(\lambda_k - \lambda_1)\vec{v}_1 + c_2(\lambda_k - \lambda_2)\vec{v}_2 + \dots + c_{k-1}(\lambda_k - \lambda_{k-1}) \vec{v}_{k-1}$

$c_i \neq 0, \forall i$

Since λ_i's are distinct, this is a nontrivial linear relation on $\{\vec{v}_1, \cdots, \vec{v}_{k-1}\}$

Thus $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ is L.I. by contradiction. ▨

Theorem:

$T: V \to V$ L.T., with eigenvalues $\lambda_1, \cdots, \lambda_i$

Let $\mathcal{B}_1, \ldots, \mathcal{B}_K$ be bases for $E_1, \ldots, E_K$

$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_K$ is a L.I. set in V.

Proof:

Suppose $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s)$ is L.D.

Let $\vec{v}_m$ is the first redundant vector in $(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_s)$

Note $\{\vec{v}_1, \ldots, \vec{v}_{m-1}\}$ is L.I.

$* \ \vec{v}_m = c_1\vec{v}_1 + \ldots + c_{m-1}\vec{v}_{m-1}$

$\vec{v}_1, \cdots, \vec{v}_m$ cannot all be in the same $\mathcal{B}_i$

$\exists \ c_k, 1 \le k \le m-1$ s.t. $\lambda_k \neq \lambda_m$.

Multiply $*$ by λ_m

$\lambda_m \vec{v}_m = c_1 \lambda_m \vec{v}_1 + \ldots + c_{m-1} \lambda_m \vec{v}_{m-1}$ (i)

Take T from both sides of $*$

$\lambda_m \vec{v}_m = c_1\lambda_1 \vec{v}_1 + \ldots + c_{m-1}\lambda_{m-1} \vec{v}_{m-1}$ (ii)

(i) - (ii)

$\vec{0} = c_1(\lambda_m - \lambda_1)\vec{v}_1 + \ldots + c_k \underbrace{(\lambda_m - \lambda_k)}_{\neq 0}\vec{v}_k + \ldots + c_{m-1}(\lambda_m - \lambda_{m-1})\vec{v}_{m-1}$

Since $c_i \neq 0$ and $\lambda_m - \lambda_k \neq 0$, this is a notrivial relation

Thus $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s)$ is L.I. by contradiction.

▨

Summary:

Let $T: V \to V$ L.T. eigenvalue $\lambda_1, \cdots, \lambda_m$ with alg. multi $r_1, \cdots, r_r$ and geo multi $m_1, \cdots, m_r$ with eigenspace $E_1, \cdots, E_r$ and bases $\mathcal{B}_1, \cdots, \mathcal{B}_r$

$T: V \to V$ has an eigenbases iff $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ a basis for V

iff $\dim E_1 + \dim E_2 + \ldots + \dim E_r = \dim V = n$

iff $m_1 + m_2 + \cdots + m_r = n$