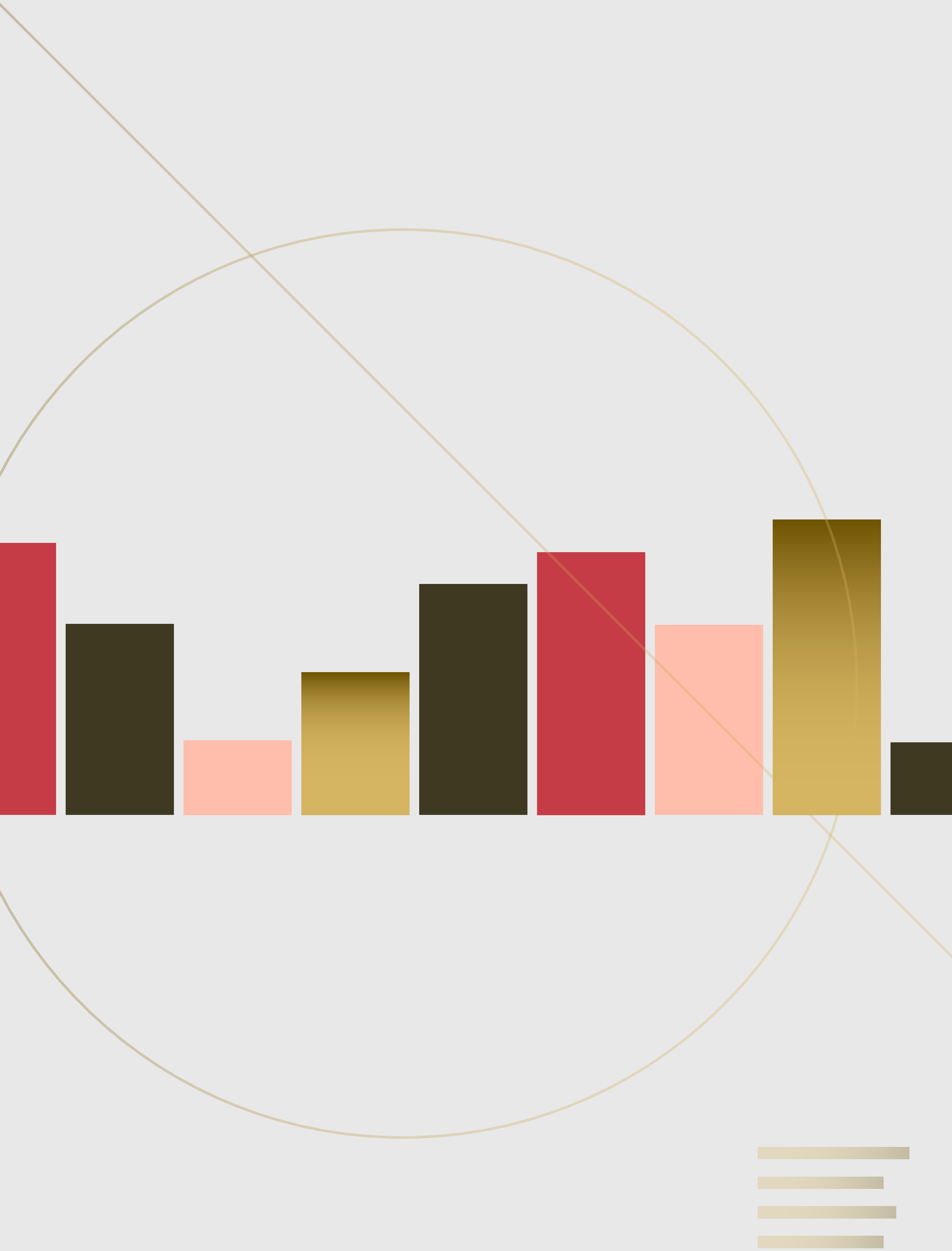


Jan 25 Lec 1 Notes

Def: Darboux Integral

Let $a, b \in \mathbb{R}$, $a < b$. Let $P = \{x_i\}_{i=0}^{n}$ be any partition of $[a,b]$. Suppose f is bounded on $[a,b]$.

For each $i = 1, \dots, n$ define

- $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$
- $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$

Then

$U(f,P)$ = Upper darboux sum of f for P on $[a,b]$

$$= \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

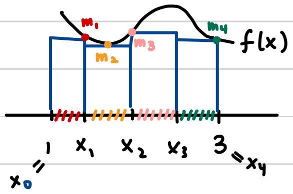
$L(f,P)$ = Lower darboux sum of f for P on $[a,b]$

$$= \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

$$= m_1(x_1 - x_0) + m_2(x_2 - x_1) + \dots + m_n(x_n - x_{n-1})$$

Example

1. Draw $L(f,P)$ for $f(x)$ below



2. Consider $f(x) = \begin{cases} 1 \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{cases}$

Compute $U(f,P)$ for any partition P of $[0,1]$.

$M_i = \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \}$

$f(x) = 0, 1,$ because $Q, \mathbb{I}$ are dense in $\mathbb{R}$

$M_i = \sup \{0, 1\} = 1$

So $U(f,P) = \sum_{i=0}^{n} M_i (x_i - x_{i-1})$ By def of $U(f,P)$

$= \sum_{i=1}^{n} x_i - x_{i-1}$

$= (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \ldots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})$ By def of Σ-not.

$= x_n - x_0$

$= 1 - 0 = 1$

Def: Darboux definition of the Definite Integral

Let $a, b \in \mathbb{R}$, $a < b$. Suppose f is bounded on $[a,b]$. We say that f is integrable on $[a,b]$ iff

$\sup \{ L(f,P) \mid P \text{ any partition of } [a,b] \} =$

$= \inf \{ U(f,P) \mid P \text{ any partition of } [a,b] \}$

Example:

3. Consider $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Prove $\int_0^1 f(x)\,dx$ DNE i.e $f(x)$ is not integrable on $[0,1]$.

Proof: WTS $\sup\{L(f,P) \mid P \text{ any partition of } [0,1]\} \neq \inf\{U(f,P) \mid P \text{ any partition of } [0,1]\}$

Let $P = \{x_i\}_{i=0}^{n}$ be an arbitrary partition of $[0,1]$.

From Ex 2, we have $U(f,P) = 1$

For $i = 1, 2, \dots, n$

$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ by def of m_i

$m_i = \inf\{0,1\} = 0$

So $L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ By def of $L(f,P)$

$= \sum_{i=1}^{n} 0$

$= 0$

Since P is arbitrary, $L(f,P) = 0$ and $U(f,P) = 1$ $\forall$ Partition P in $[0,1]$

$\therefore = \inf\{U(f,P) \mid P \text{ any partition of } [0,1]\}$
$= \inf\{1\}$

$= \sup\{L(f,P) \mid P \text{ any partition of } [0,1]\}$
$= \sup\{0\}$

$\therefore$ $1 \neq 0$, f is not integrable on $[0,1]$

Riemann def of the definite integral is a special case of Darboux definition of definite integral

If f is integrable on $[a,b]$ then

$$L(f,P) \le \int_a^b f(x)\,dx \le U(f,P) \quad \forall \text{ Partition } P \text{ of } [a,b]$$

Def: ε - Reformulation of the Definite Integral

Let $a,b \in \mathbb{R}$, $a<b$. Suppose f bounded on $[a,b]$. Where f is integrable on $[a,b]$ iff

$$\forall \varepsilon>0,\ \exists P \text{ of } [a,b] \text{ s.t. } U(f,P) - L(f,P) < \varepsilon$$