W 8 Lecture 16 Notes

Rules for Differentiation (Continued ...)

7. The Chain Rule: $\frac{d}{dx}\left(f[g(x)]\right) = f'(g(x))\cdot g'(x)$

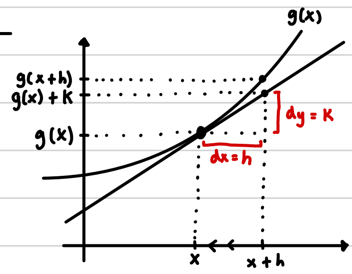
$$y'_x[u(x)] = \frac{dy}{\cancel{du}}\cdot\frac{\cancel{du}}{dx} = (y\circ u)'_x$$ ← derivative with respect to x.

$$\frac{d}{dx}\left(f[g(x)]\right) = \lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{h}\cdot\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$$

$$= \lim_{h\to 0}\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{K\to 0}\frac{f[g(x)+K]-f[g(x)]}{K}\cdot g'(x)$$

$$= f'[g(x)]\cdot g'(x)$$



As $h\to 0$, $K\to 0$
$g(x+h)\approx g(x)+K$

6. The Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\,g(x)-g'(x)f(x)}{(g(x))^2}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot g(x)^{-1}\right) = f'(x)\cdot g(x)^{-1} + \frac{d}{dx}\left(g(x)^{-1}\right)\cdot f(x)$$

$$= \frac{f'(x)}{g(x)} - f(x)\left(-1\,[g(x)]^{-2}\cdot g'(x)\right)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\cdot g'(x)}{g^2(x)} = \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)}$$

Implicit Differentiation

A function is defined **explicitly** if it can be described by expressing one variable in term of another.

A function is defined **implicitly** if it is described by a relation between x and y. Implicit functions **do not pass the vertical line test.**

Example:

1. $x^2+xy-y^3=4$. Find $\frac{dy}{dx}$.

$$x^2+xy(x)-y^3(x)=4$$
$$(x^2)'+(xy(x))'-(y^3(x))'=4'$$
$$2x+[x'\cdot y(x)+x\,y'(x)]-3y^2(x)\cdot y'(x)=0$$
$$2x+y(x)+x\,y'(x)-3y^2(x)\cdot y'(x)=0$$
$$y'(x)[x-3y^2(x)] = -2x-y(x)$$
$$y'(x) = \frac{-2x-y}{x-3y^2}$$

2. Find the equation of tangent line to the elliptic curve $y^2 = x^3 - 4x$ at $a = -1$.

$y = f(a) + f'(a)(x-a)$ OR $y(x) = f(x_0, y_0) + f'(x_0, y_0)(x - x_0)$

$a = 1 \Rightarrow y^2 = -1 + 4 = 3$

$\Rightarrow y = \pm\sqrt{3}$

$y'(-1, \sqrt{3}) = \frac{3(-1)^2 - 4}{2 \cdot \sqrt{3}} = \frac{-1}{2\sqrt{3}}$

$y'(-1, -\sqrt{3}) = \frac{3(-1)^2 - 4}{2(-\sqrt{3})} = \frac{1}{2\sqrt{3}}$

$f'(x_0, y_0) \Rightarrow 2y \cdot \frac{dy}{dx} = 3x^2 - 4$

$\frac{dy}{dx} = \frac{3x^2 - 4}{2y}$

$\ell_1: \quad y = \sqrt{3} - \frac{1}{2\sqrt{3}}(x+1)$

$\ell_2: \quad y = -\sqrt{3} + \frac{1}{2\sqrt{3}}(x+1)$

Rules for Differentiation (Continued...)

8. Derivative of the Exponential Function: $\frac{d}{dx}(a^x) = a^x \ln a$

$a^x = e^{x \cdot \ln a}$

$\frac{d}{dx}(a^x) = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$

$= e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$

Example:

3. Show that if $f(x) = e^x$ then $f'(x) = e^x$.

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = \lim_{h \to 0} \frac{e^x \cdot h}{h} = e^x$$

We know that $\lim_{h \to 0} (1+h)^{1/h} = e \Rightarrow (1+h)^{1/h} \approx e \Rightarrow (1+h) = e^h \Rightarrow e^h - 1 \approx h$