A22 Feb 26 Lec 2 Notes

**Def:**

Let $T: V \to W$ is a L.T.

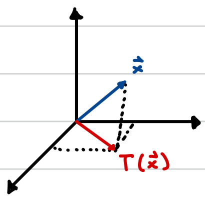
$$\operatorname{im} T = \operatorname{img} T := \{ T(\vec{v}) \mid \vec{v} \in V \} = \{ \vec{w} \in W \mid \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in V \}$$

$$\text{Kernel } T := \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0}_W \}$$

**Ex 1**

$$P: \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$




$$\operatorname{im} P = \{ P(\vec{x}) \mid \vec{x} \in \mathbb{R}^3 \}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \middle| \; x_1, x_2 \in \mathbb{R} \right\}$$

$= x_1 \; x_2$ plane

$$\ker P = \{ \vec{x} \in \mathbb{R}^3 \mid P(\vec{x}) = \vec{0}_{\mathbb{R}^3} \}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \middle| \; x_3 \in \mathbb{R} \right\}$$

$= x_3$ axis

**Ex 2**

$$G: P_3 \to P_3 \qquad P_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \text{ in } \mathbb{R} \}$$

$$p(x) \mapsto p'(x)$$

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\operatorname{im} G = \{ p'(x) \mid p(x) \in P_3 \}$$

$$= \{ a_1 + 2a_2 x + 3a_3 x^2 \mid a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= P_2 \subseteq P_3$$

$$\operatorname{Ker} G = \{ p(x) \in P_3 \mid p'(x) = 0 \}$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_1 + 2a_2 x + 3a_3 x^2 = 0 \}$$

$$= \{ a_0 \mid a_0 \in \mathbb{R} \}$$

$$= \mathbb{R} \subseteq P_3$$

$\therefore a_1 = 0$
$a_2 = 0$
$a_3 = 0$

## Ex 3

$$T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$
$$\vec{v} \longmapsto A\vec{v}$$

$$\begin{aligned} \operatorname{im} T_A &= \{ T_A(\vec{v}) \mid \vec{v} \in \mathbb{R}^3 \} \\ &= \{ A\vec{v} \mid \vec{v} \in \mathbb{R}^3 \} \\ &= \left\{ \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \middle| \; v_1, v_2, v_3 \text{ in } \mathbb{R} \right\} \\ &= \left\{ v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + v_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + v_3 \begin{pmatrix} 0 \\ 2 \end{pmatrix} \middle| \; v_1, v_2, v_3 \right\} \end{aligned}$$

= The set of all linear combination of columns of $A$.

$$\operatorname{im} A := \operatorname{im}(T_A) = \operatorname{col}(A)$$

$$\operatorname{Ker} T_A = \{ \vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \vec{0} \}$$

= the solution set to $A\vec{v} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 2/3 & 0 \end{array}\right]$$

$$v_1 = -\tfrac{2}{3} v_3$$
$$v_2 = -\tfrac{2}{3} v_3$$
$$v_3 = t$$

$$\therefore \operatorname{Ker} T_A = \left\{ \begin{pmatrix} -2/3 \\ -2/3 \\ 1 \end{pmatrix} t \;, \; t \in \mathbb{R} \right\}$$

## Def

Let $A_{n \times m}$ matrix, column space of $A$ is the set of all linear combinations of columns of $A$.

$$A = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \\ | & | & & | \end{bmatrix} \qquad \operatorname{col}(A) := \left\{ c_1 \vec{v_1} + c_2 \vec{v_2} + \dots + c_m \vec{v_m} \;\middle|\; \begin{array}{c} c_i \in \mathbb{R} \\ 1 \le i \le m \end{array} \right\}$$

**Def**

Given $A_{n\times m}$ matrix, **null space** of $A$ is $\text{Ker}(T_A)$, denoted by $\text{Nul}(A)$
i.e. $\text{Nul}(A)$ is solution set to $A\vec{x}=\vec{0}$.

**Remark**

$$T_A : \mathbb{R}^m \to \mathbb{R}^n$$
$$\vec{x} \mapsto A\vec{x}$$

$\text{Ker}(T_A) = \text{Ker}(A) = \text{Nul}(A)$
$\text{im}(T_A) = \text{img}\, T_A = \text{im}(A) = \text{col}(A)$

Suppose $T_A : \mathbb{R}^m \to \mathbb{R}^n$. $A$ is $n\times m$ matrix

$\text{Ker}\, T_A =$ Solution set to $A\vec{x} = \vec{0}$

$$\text{im}\, T_A = \left\{ c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_m\vec{v_m} \;\middle|\; \begin{matrix} c_i \in \mathbb{R} \\ 1 \le i \le m \end{matrix} \right\}$$

**Def**

Given $\vec{v_1}, \dots, \vec{v_m}$ in a vector space $V$, **span** of $\vec{v_1}, \dots, \vec{v_m}$ is the set of all linear combinations of $\vec{v_1}, \dots, \vec{v_m}$.

$$\text{span}(\vec{v_1}, \dots, \vec{v_m}) = \left\{ c_1\vec{v_1} + c_2\vec{v_2} + \dots + c_m\vec{v_m} \;\middle|\; \begin{matrix} c_i \in \mathbb{R} \\ 1 \le i \le m \end{matrix} \right\}$$

If $\text{Span}(\vec{v_1}, \dots, \vec{v_m}) = V$, we say $\vec{v_1}, \dots \vec{v_m}$ **spans** $V$.

We can also say $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_m}\}$ is a **spanning set** for $V$.

**Ex 4**

$V = \mathbb{R}^2 \qquad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\mathbb{R}^2 = \text{sp}(\vec{e_1}, \vec{e_2})$

The set $\{\vec{e_1}, \vec{e_2}\}$ is a spanning set for $\mathbb{R}^2$.

$\vec{e_1}$ and $\vec{e_2}$ spans $\mathbb{R}^2$

Why should we care about Ker T and img T ?

**Theorem**

Let $T: V \to W$ be a L.T.

(i) T is injective iff $\ker(T) = \{\vec{0}\}$

(ii) T is surjective iff $\operatorname{img} T = W$

Proof (i):

(⇒) if T is injective then $\underbrace{\ker T = \{\vec{0}\}}_{}$ Ker T is trivial

Suppose T is injective

$\forall \vec{w} \in W$ $\exists$ at most one $\vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$
i.e. $T(\vec{v_1}) = T(\vec{v_2})$ then $v_1 = v_2$

Pick $\vec{v} \in \ker T$, WTS $\vec{v} \in \{\vec{0}\}$

$\vec{v} \in \ker T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\} \Rightarrow T(\vec{v}) = \vec{0}$

on the other hand, $T(\vec{0}) = \vec{0}$

Since T is injective, $\vec{v} = \vec{0} \in \{\vec{0}\}$, thus $\ker T \subseteq \{\vec{0}\}$

WTS $\{\vec{0}\} \subseteq \ker T$
WTS $\vec{0} \in \ker T$

$T(\vec{0}) = \vec{0}$ since T is L.T. $\Rightarrow \vec{0} \in \ker T$

So $\{\vec{0}\} \subseteq \ker T$

Since $\{\vec{0}\} \subseteq \ker T$ and $\ker T \subseteq \{\vec{0}\}$, then $\ker T = \{\vec{0}\}$

Proof (i):

($\Leftarrow$) if $\operatorname{Ker} T = \{\vec{0}\}$ then $T$ is injective

Assume $\operatorname{Ker} T = \{\vec{0}\}$

$= \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$

Suppose $T(\vec{v_1}) = T(\vec{v_2})$ for some $\vec{v}_1, \vec{v}_2$ in $W$

$T(\vec{v_1}) = T(\vec{v_2}) \Rightarrow T(\vec{v_1}) - T(\vec{v_2}) = \vec{0}$

$\Rightarrow T(\vec{v_1} - \vec{v_2}) = \vec{0}$

$\Rightarrow \vec{v_1} - \vec{v_2} \in \operatorname{Ker} T = \{\vec{0}\}$

$\Rightarrow \vec{v_1} - \vec{v_2} = \vec{0}$

$\Rightarrow \vec{v_1} = \vec{v_2}$

$\Rightarrow T$ is injective

Proof (ii):

($\Rightarrow$) If $T$ is surjective then $\operatorname{img} T = W$

Suppose $T$ is surjective. WTS $\operatorname{img} T = W$

$\forall \vec{w} = W \quad \exists \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$

By def $\operatorname{im} T = \{T(\vec{v}) \mid \vec{v} \in V\} = \{\vec{w} \in W \mid T(\vec{v}) = \vec{w} \text{ for some } \vec{v}\} \subseteq W$

Thus $\operatorname{im} T \subseteq W$

Pick $\vec{w} \in W$. WTS $\vec{w} \in \operatorname{img} T$

Since $T$ is surjective $\exists \vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w} \in \{T(\vec{v}) \mid \vec{v} \in V\} = \operatorname{img} T$

Thus $W \subseteq \operatorname{img} T$

$\therefore$ Since $\operatorname{img} T \subseteq W$ and $W \subseteq \operatorname{img} T$, then $\operatorname{img} T = W$