Sec 2.4 Reading

Def 2.4.1: Invertible Functions

A function T from X to Y is called invertible if the equation $T(x) = y$ has a unique solution x in X for each y in Y.

In this case, the inverse T^{-1} from Y to X is defined by

$$T^{-1}(y) = (\text{the unique } x \text{ in } X \text{ s.t } T(x) = y)$$

To put it differently, the equation

$$x = T^{-1}(y) \quad \text{means that} \quad y = T(x)$$

Note that

$$T^{-1}(T(x)) = x \quad \text{and} \quad T(T^{-1}(y)) = y$$

for all x in X and for all y in Y.

Conversely, if L is a function from Y to X s.t.

$$L(T(x)) = x \quad \text{and} \quad T(L(y)) = y$$

for all x in X and for all y in Y, then T is invertible and $T^{-1} = L$.

If a function T is invertible, then so is T^{-1} and $(T^{-1})^{-1} = T$.

Def 2.4.2: Invertible Matrices

A square matrix A is said to be invertible if the L.T. $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case, the matrix of T^{-1} is denoted by A^{-1}. If the L.T. $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

Theorem 2.4.3: Invertibility

An $n \times n$ matrix A is invertible iff

$$\text{rref}(A) = I_n$$
OR
$$\text{rank}(A) = n$$

Theorem 2.4.4: Invertibility and Linear systems

Let A be an $n \times n$ matrix.

(a) Consider a vector $\vec{b}$ in $\mathbb{R}^n$. If A is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If A is noninvertible, then the system $A\vec{x} = \vec{b}$ has infinitely many solutions or none.

(b) Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If A is invertible, then this is the only solution. If A is noninvertible, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

Theorem 2.4.5: Finding the inverse of a matrix

To find the inverse of an $n \times n$ matrix A, form the $n \times (2n)$ matrix $[A \,\vdots\, I_n]$ and compute $\text{rref}[A \,\vdots\, I_n]$.

- If $\text{rref}[A \,\vdots\, I_n]$ is of the form $[I_n \,\vdots\, B]$, then A is invertible, and $A^{-1} = B$.
- If $\text{rref}[A \,\vdots\, I_n]$ is of another form (i.e., its left half fails to be I_n), then A is not invertible. Note that the left half of $\text{rref}[A \,\vdots\, I_n]$ is $\text{rref}(A)$.

Theorem 2.4.6: Multiplying with the Inverse

For an invertible $n \times n$ matrix A,

$$A^{-1}A = I_n \quad \text{and} \quad AA^{-1} = I_n$$

Theorem 2.4.7: The Inverse of a product of matrices

If A and B are invertible $n \times n$ matrices, then BA is invertible as well, and

$$(BA)^{-1} = A^{-1} B^{-1}$$

Justification:

$$\vec{y} = BA\vec{x}$$

$$B^{-1}\vec{y} = B^{-1}BA\vec{x} = I_n A\vec{x} = A\vec{x}$$

$$A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x} = I_n\vec{x} = \vec{x}$$

Verification

$$= BA\,A^{-1}B^{-1} = B\,I_n\,B^{-1} = BB^{-1} = I_n$$

Theorem 2.4.8: A criterion for invertibility

Let A and B be two $n \times n$ matrices s.t.

$$BA = I_n$$

Then

(a) A and B are both invertible,
(b) $A^{-1} = B$ and $B^{-1} = A$, and
(c) $AB = I_n$

Proof:

To demonstrate that A is invertible, it suffices to show that $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.

$A\vec{x} = \vec{0} \Rightarrow BA\vec{x} = B\vec{0} = \vec{0} \Rightarrow \vec{x} = I_n\vec{x} = BA\vec{x} = \vec{0}$ Therefore A is invertible

$BA = I_n \Rightarrow BAA^{-1} = I_nA^{-1} \Rightarrow B = A^{-1} \Rightarrow B^{-1} = (A^{-1})^{-1} = A$

$AB = AA^{-1} = I_n$

Ex 2

Suppose A, B, C are three $n \times n$ matrices s.t. $ABC = I_n$. Show that B is invertible, and express B^{-1} in terms of A and C.

$ABC = (AB)C = I_n \Rightarrow$

$\Rightarrow C(AB) = I_n$ By thm 2.48c

$\Rightarrow (CA)B = I_n$ Matrix multiplication is associative

$\Rightarrow (CA)BB^{-1} = I_n B^{-1}$ Multiply B^{-1} from right on both sides

$\Rightarrow CA = B^{-1}$

By thm 2.4.8, B is invertible.

Ex 3

For an arbitrary 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute the product $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

When is A invertible? What is A^{-1}?

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc) I_2$$

If $ad-bc \neq 0$, we can write:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = I_2$$

By thm 2.4.8, A is invertible.

Then:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 2.4.9: Inverse and determinant of a 2×2 matrix

(a) The 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible iff $ad - bc \neq 0$

$$\det(A) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

(b) If A is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 2.4.10: Geometrical Interpretation of the determinant of a 2×2 matrix

If $A = [\vec{v} \;\; \vec{w}]$ is a 2×2 matrix with nonzero columns $\vec{v}$ and $\vec{w}$, then

$$\det A = \det[\vec{v} \;\; \vec{w}] = \|\vec{v}\| \sin\theta \, \|\vec{w}\|,$$

where θ is the oriented angle from $\vec{v}$ to $\vec{w}$, with $-\pi < \theta \leq \pi$. It follows that

- $|\det A| = \|\vec{v}\| \, |\sin\theta| \, \|\vec{w}\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\det A = 0$ if $\vec{v}$ and $\vec{w}$ are parallel, meaning that $\theta = 0$ or $\theta = \pi$
- $\det A > 0$ if $0 < \theta < \pi$, and
- $\det A < 0$ if $-\pi < \theta < 0$