B24 June 4 Lec 2 Notes

Proposition:

Let V be a vector space with a finite basis. Then any two bases for V have the same # of elements.

Proof:

i^{th} standard basis element in $\mathbb{F}^n$

If $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are bases for V, then the map $T: V \to \mathbb{F}^n$ defined by $T(v_i) = e_i$ for $1 \le i \le n$ defines an isomorphism, and $S: V \to \mathbb{F}^m$ defined by $S(w_i) = e_i$ is an isomorphism.

So $ST^{-1}: \mathbb{F}^n \to \mathbb{F}^m$ is an isomorphism

$\Rightarrow ST^{-1}(e_1), \dots, ST^{-1}(e_n)$ is a basis for $\mathbb{F}^m$, so by result from last class $n = m$.

Definition:

The dimension of a vector space V is the # of elements in any basis for V. We say V is finite-dimensional if there is a basis for V with finitely many elements.

Ex 1:

$\dim(\mathbb{R}^3) = 3$ since $(1,0,0), (0,1,0), (0,0,1)$ form a basis for $\mathbb{R}^3$.

$\dim(\mathbb{C}^3) = 3$ since $(1,0,0), (0,1,0), (0,0,1)$ form a basis for $\mathbb{C}^3$.

($\mathbb{C}^3$ is considered as a complex v.s. above. $\mathbb{C}^3$ is also a real v.s with $\dim \mathbb{C}^3 = 6$)

$C([0,1])$ is not finite-dimensional.

Remark:

Any L.I. list of vectors in a v.s. V has $\le \dim(V)$ elements, and any spanning set in V has $\ge \dim(V)$ elements.

Proposition:

Suppose V is a finite-dimensional v.s. and $v_1, \dots, v_r \in V$ are L.I Then there exists $w_1, \dots, w_m \in V$ s.t. $v_1, \dots, v_r$ and $w_1, \dots, w_r$ form a basis for V.

Proof:

Suppose $v_1, \dots, v_r \in V$ are L.I. Let $w_1 \notin \text{span}(v_1, \dots, v_r)$.

Suppose that

$$\alpha_1 v_1 + \dots + \alpha_r v_r + \alpha_{r+1} w_1 = 0 \quad ★$$

Then $\alpha_{r+1} = 0$ since otherwise

$$w_1 = \frac{-\alpha_1}{\alpha_{r+1}} v_1 + \dots + \frac{-\alpha_r}{\alpha_{r+1}} v_r \in \text{span}(v_1, \dots, v_r)$$

So ★ $\Rightarrow \alpha_1 v_1 + \dots + \alpha_r v_r = 0$

$\Rightarrow \alpha_1 = \dots = \alpha_r = 0$ By L.I. of $v_1, \dots, v_r$

If $v_1, \dots, v_r, w_1$ spans V, we are done, otherwise we repeat.

This process must terminate when $m = \dim(V) - r$, since otherwise we would have a list of $r + (\dim(V) - r) + 1 = \dim(V) + 1$ L.I. vectors in V. This is a contradiction.

$\blacksquare$

Proposition:

Suppose V is a finite-dimensional v.s., and $W \subset V$ is a subspace. Then W is finite-dimensional with $\dim(W) \leq \dim(V)$ and if $\dim(W) = \dim(V)$ then $V = W$.

Remark:

If $v_1, \dots, v_n$ is a basis for V, in general no subset of $v_1, \dots, v_n$ is a basis for W.

Proof:

Let $w_1 \in W$, $w_1 \neq 0$. Then w_1 is L.I. so by previous proposition, and as in the previous proposition we can add vectors $w_2, \dots, w_n$ with $w_m \notin \text{span}(w_1, \dots, w_{m-1})$ so that $w_1, \dots, w_m$ are L.I. and this process must terminate after at most $m = \dim V$ since otherwise we obtain $\dim V + 1$ L.I. vectors in V, which is a contradiction.

So $w_1, \dots, w_m$ form a basis for W i.e. W is finite-dimensional. $\dim W \leq \dim V$ follows since $w_1, \dots, w_m$ are L.I. in V.

Continued...

Proof (continued...):

Now suppose $\dim W = \dim V$. We need to show $W = V$. If we suppose by way of contradiction $W \subsetneq V$, then there exists $v \in V \setminus W$ and so $w_1, \dots, w_m, v$ are L.I. in V. But this is a list of $\dim V + 1$ L.I. vectors in V, which is a contradiction.

Theorem:

Let $T: V \to W$ be a L.T. and suppose $b \in W$ and $x_0 \in V$ is a solution to the equation

$$Tx = b \quad \star$$

Then the set of solutions to $\star$ is:

$$\{x_0 + v : \underbrace{v \in \operatorname{Ker} T}_{\text{i.e. } T(v) = 0}\}$$

Proof:

If $v \in \operatorname{ker} T$, then

$$\begin{aligned} T(x_0 + v) &= T(x_0) + T(v) \\ &= b + 0 \\ &= b \end{aligned}$$

and if x_1 is a solution to $\star$, then $x_1 - x_0 \in \operatorname{Ker} T$ since

$$\begin{aligned} T(x_1 - x_0) &= T(x_1) - T(x_0) \\ &= b - b \\ &= 0 \end{aligned}$$

and $x_1 = x_0 + \underbrace{(x_1 - x_0)}_{\in \operatorname{Ker} T}$

So to find all solutions to $Tx = b$, it suffices to find a single solution and $\operatorname{Ker} T$

Remark:

We consider $m \times n$ matrices A as L.T. $\mathbb{R}^n \to \mathbb{R}^m$ by

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Definition:

The row space of A is defined as $\text{range}(A^T)$. The left null space of A is defined as $\text{Ker}(A^T)$.

Definition:

The rank of a L.T. $T: V \to W$ is defined by

$$\text{rank}(T) = \dim(\text{range}(T))$$

Remark:

If A is a matrix, we will denote by A_e the echelon form of A and by A_{re} the reduced echelon form of A.

Proposition:

The pivot columns of A (i.e. the columns of A for which A_e has a pivot) form a basis for $\text{range}(A)$

Proof:

Suppose A is $m \times n$. Let $A = [A_1 \cdots A_n]$. Then $\text{range}(A) = \text{span}(A_1, \dots, A_n)$

Also $\text{range}(A_{re})$ = span of columns of A_{re}

= Span of pivot columns of A_{re}

Note that $TA = A_{re}$ for invertible T. So if

$$A_{re} = [B_1 \cdots B_n], \text{ then}$$

$$A = T^{-1} A_{re} = [T^{-1} B_1 \cdots T^{-1} B_n]$$

So since pivot columns of A_{re} are L.I, the pivot columns of A are L.I.

Proof (continued...):

It remains to show the pivot columns of A span range(A).

Let $C_1, \dots, C_r$ be the pivot columns of A.

Then $TC_1, \dots, TC_r$ span range(A_{re})

So for any column A_j of A,

$$TA_j = \alpha_1 TC_1 + \dots + \alpha_r TC_r$$
$$= T(\alpha_1 C_1 + \dots + \alpha_r C_r)$$

Apply T^{-1}

$$\Rightarrow A_j = \alpha_1 C_1 + \dots + \alpha_r C_r$$

▨

Proposition:

The pivot rows of A_e (rows of A_e which have a pivot) form a basis for range(A^T).

Proof:

Pivot rows of A_e are L.I and if $A_e = EA$ for E invertible, then:

$$\text{range}(A_e^T) = \text{range}((EA)^T)$$
$$= \text{range}(A^T E^T)$$
$$= A^T \text{range}(E^T)$$
$$= A^T(\mathbb{R}^m)$$
$$= \text{range}(A^T)$$

Ex:

$$V \xrightarrow{T} W \xrightarrow{S} U$$

$\text{range}(ST) = \{u \in U \mid \exists v \in V \text{ with } ST(v) = u\}$

$S\,\text{range}(T) = S(\{w \in W \mid \exists v \in V \text{ with } Tv = w\})$

Since E^T is invertible, range$(E^T) = \mathbb{R}^m$

By definition of range.

▨

How to find Ker A ?

Well $\ker(A) = \ker(A_{re})$

Ex 1 :

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \quad A_e = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{re} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\text{Solving } A_{re} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

$\Rightarrow$ x_5 is free , x_4 is free

$x_3 = -x_4 - x_5/3$, x_2 is free

$x_1 = -x_2 - x_5/3$. i.e.

$$\left\{ x_5 \begin{pmatrix} -1/3 \\ 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R} \right\} = \ker A$$