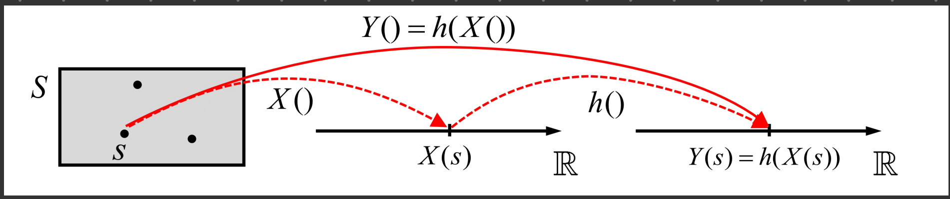
B52 Oct 29 Lec 2 Notes

Transformations

Assume RV X follows certain distribution & RV $Y = h(X)$ defined as some function h of X (transformation / change of RV)

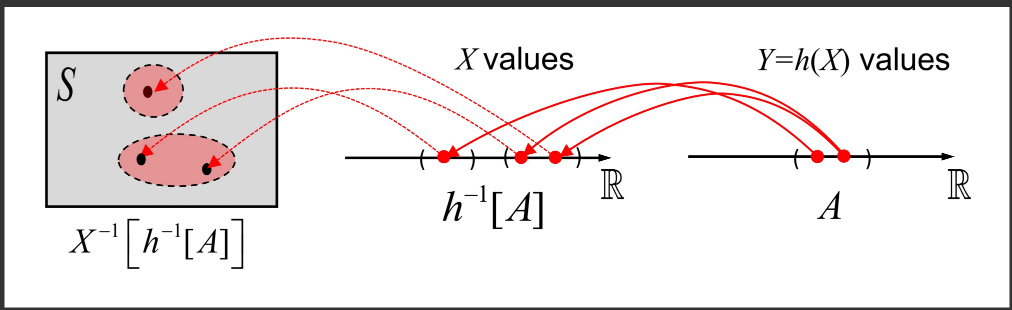


How can we find distribution of Y based on that of X?
- ↳ General Method
- ↳ CDF Method
- ↳ PDF Method (Continuous RVs)

Change or RV

Let X RV with known distribution $P(X \in B)$, $\forall B \subseteq \mathbb{R}$ and $Y = h(X)$

Probability $P(Y \in A)$ is equal to $P(X \in h^{-1}[A])$
↳ $h^{-1}[A] = \{ x \in \mathbb{R} : h(x) \in A \}$ is inverse image of A.



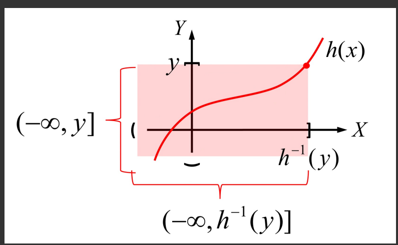
CDF Method

Restrict attention to half-lines of the form $A = (-\infty, y]$, $y \in \mathbb{R}$.

For continuous one-to-one function h, inverse image is also half-line ⇒ can use CDF of X to find CDF & PDF of Y.

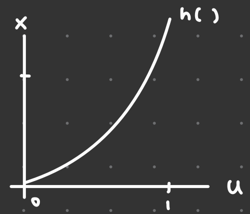
↳ e.g. if h, is strictly increasing, then

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X \in h^{-1}(-\infty, y]) \\ &= P(X \in (-\infty, h^{-1}(y)]) \\ &= P(X \leq h^{-1}(y)) = F_X(h^{-1}(y)) \end{aligned}$$



Ex 1:

Let RV $U \sim \text{Uniform}(0,1)$ and find CDF of $X = \overbrace{-\log(1-U)}^{=h(U)}$.



$$
\begin{aligned}
F_X(x) &= P(X \le x) \\
&= P(h(U) \le x) \\
&= P(-\log(1-U) \le x) \\
&= P(e^{\log(1-u)} \ge e^{-x}) \\
&= P(1-u \ge e^{-x}) \\
&= P(u \le 1-e^{-x}) \\
&= F_u(1-e^{-x}) \\
&= 1-e^{-x}
\end{aligned}
$$
, which is the CDF of an $\text{Exp}(\lambda=1)$ dist.

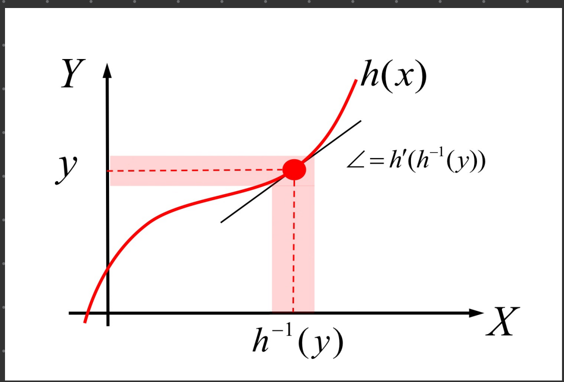
PDF Method

Assume $Y = h(X)$ for continuous one-to-one function h, where PDF $f_X(x)$ is known but there is no closed-form CDF.

PDF of Y is given by

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|}$$

Interval around y maps to interval around $h^{-1}(y)$ scaled by derivative at $(y, h^{-1}(y))$, and so does PDF.



$$
\begin{aligned}
P(Y \in A) \approx F_X(y)\cdot dy &= P(X \in h^{-1}(A)) \\
&\approx f_X(h^{-1}(y))\cdot dx \Rightarrow \\
\Rightarrow f_y(y) &= \frac{f_X(h^{-1}(y))}{\left(\frac{dy}{dx}\right)} \\
&= \frac{f_X(h^{-1}(y))}{h'(h^{-1}(y))}
\end{aligned}
$$

Ex 2:

For $U \sim \text{Uniform}(0,1)$, verify that $X \overset{h(u) = -\log(1-u)}{=} -\log_e(1-U)$ follows Exponential (1) using the PDF method.

$$
\begin{aligned}
f_X(x) &= \frac{f_u(h^{-1}(x))}{|h'(h^{-1}(x))|} \\
&= \frac{1}{\left|\frac{1}{1-(1-e^{-x})}\right|} \\
&= e^{-x}
\end{aligned}
$$

$f_u(h^{-1}(x)) = 1$ since $f_U(u) = \begin{cases} 1 \\ 0 \end{cases}$

$h(u) = -\log(1-u) \Rightarrow \begin{cases} h^{-1}(x) = 1-e^{-x} \\ h'(u) = \frac{1}{1-u} \end{cases}$

Let $x = -\log(1-u) \Rightarrow$
$\Rightarrow e^{\log(1-u)} = e^{-x}$
$\Rightarrow u = 1-e^{-x}$
$= h^{-1}(x)$

When $\lambda = 1$, we have e^{-x}, $\forall x > 0$.
Thus PDF of $\text{Exp}(\lambda=1)$ ∎