Feb 3 Lec 1 Notes

Def 4.1.1: Linear Spaces (or Vector spaces)

A linear space V is a set endowed with a rule for addition (if f and g are in V, then so is $f+g$) and a rule for scalar multiplication (if f is in V and k in $\mathbb{R}$, then kf is in V) such that these operations satisfy the following eight rules.

(i) $(f+g)+h = f+(g+h)$
(ii) $f+g = g+f$
(iii) There exists a neutral element n in V s.t. $f+n=f$, $\forall f$ in V.
This n is unique and denoted by 0.

Def:

Let V and W be vector spaces.

A linear transformation $T: V \to W$ is a map

(i) $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ — T preserves vector addition
(ii) $T(r\vec{v}) = rT(\vec{v})$ — T preserves scalar multiplication

Ex1

$V = C' = \{ f: \mathbb{R} \to \mathbb{R} \mid \text{the first derivative of } f \text{ is defined} \}$

$e^x \in V$, $2x \in V$

$$\begin{aligned} D: C' &\longrightarrow C' \\ f &\longmapsto f' \end{aligned}$$

To check (i), take f, g in C', $D(f+g) \stackrel{?}{=} D(f)+D(g)$

$$D(f+g) = (f+g)' = f'+g' = D(f)+D(g)$$

To check (ii), take $f \in C'$, $r \in \mathbb{R}$

$$D(rf) = (rf)' = rf' = rD(f)$$

(i) and (ii) hold so D is a linear transformation.

Theorem:

$T: V \to W$ is a linear transformation iff $T(\vec{v} + r\vec{w}) = T(\vec{v}) + rT(\vec{w})$
$\forall \vec{v}, \vec{w} \in V, \forall r \in \mathbb{R}$

Ex 2

$S: \mathbb{R}^2 \to \mathbb{R}^2$ is a map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{pmatrix} x_1 + x_2 + 1 \\ x_1 - x_2 \end{pmatrix}$$

Q: Is S a linear transformation?

To check (i), Take $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in $\mathbb{R}^2$

$$S(\vec{x} + \vec{y}) \overset{?}{=} S(\vec{x}) + S(\vec{y})$$

$$S(\vec{x} + \vec{y}) = S\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = S\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}\right) = \begin{pmatrix} (x_1 + y_1) + (x_2 + y_2) + 1 \\ (x_1 + y_1) - (x_2 + y_2) \end{pmatrix}$$

Different

$$S(\vec{x}) + S(\vec{y}) = S\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + S\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + 1 \\ x_1 - x_2 \end{pmatrix} + \begin{pmatrix} y_1 + y_2 + 1 \\ y_1 - y_2 \end{pmatrix}$$

$$= \begin{pmatrix} (x_1 + y_1) + (x_2 + y_2) + 2 \\ (x_1 + y_1) - (x_2 + y_2) \end{pmatrix}$$

Take $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$S(\vec{x} + \vec{y}) = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$S(\vec{x}) + S(\vec{y}) = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, so S is not a linear transformation

Proof :

Let $\vec{v}_1, \dots, \vec{v}_K$ in V. $T: V \to W$

WTS: $T(c_1\vec{v}_1 + \dots + c_K\vec{v}_K) = c_1T(\vec{v}_1) + \dots + c_K T(\vec{v}_K)$

Proof by Induction

Base case: $K=1$

$T(c_1\vec{v}_1) = c_1T(\vec{v}_1)$ holds

I.H: Assume the statement is true for $K-1$

I.S: WTS: $T(c_1\vec{v}_1 + \dots + c_{K-1}\vec{v}_{K-1} + c_K\vec{v}_K) = c_1T(\vec{v}_1) + \dots + cT(\vec{v}_{K-1}) + c_KT(\vec{v}_K)$

$= T(\underbrace{c_1\vec{v}_1 + \dots + c_{K-1}\vec{v}_{K-1}}_{\vec{w} \in V} + \underbrace{c_K\vec{v}_K}_{\vec{u} \in V})$ By (i) in def of L.T.

$= \underbrace{T(c_1\vec{v}_1 + \dots c_{K-1}\vec{v}_{K-1}) + T(c_K\vec{v}_K)}_{T(\vec{w}) + T(\vec{u})}$

$= c_1T(\vec{v}_1) + \dots + c_{K-1}T(\vec{v}_{K-1}) + c_KT(\vec{v}_K)$ By (ii) in def of L.T. //