B24 June 2 Lec 1 Notes

Corollary:

If $v_1, \ldots, v_m \in \mathbb{F}^n$ are linearly independent, then $m \leq n$.

Proof:

# of pivots in any matrix $\leq$ # of rows, so forming $A = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$ and considering A^T, then

A^T has m columns and n rows, so if $v_1, \ldots, v_m$ are L.I., then the echelon

form of A^T has a pivot in every column.

So we have $m \leq n$. $\boxtimes$

Corollary:

If $v_1, \ldots, v_m \in \mathbb{F}^n$ are spanning, then $m \geq n$.

Proof:

Consider A^T as previous proof. By previous result, A^T has a pivot in every row. i.e. A^T has n pivots (A^T is $n \times m$). Since:

$$\text{\# of pivots in any matrix} \leq \text{\# of columns}$$

so

$$n \leq m.$$

$\boxtimes$

Corollary:

Let $v_1, \ldots, v_m \in \mathbb{F}^n$ be a basis. Then $m = n$.

Proof:

$v_1, \ldots, v_m$ is L.I. $\Rightarrow m \leq n$
$v_1, \ldots, v_m$ is spanning $\Rightarrow m \geq n$

Therefore $m = n$.

$\boxtimes$

Proposition:

A matrix A is invertible iff the echelon form of A has a pivot in every row and column

Proof:

Recall

> **Theorem:** From B24 June 2 Lec 1 Notes
>
> Let $T: V \to W$ be a L.T. Then T is invertible iff for any $w \in W$ the equation $Tx = w$ has an unique solution $x \in V$.

Corollary:

An invertible matrix must be square.

Proof:

If # of rows > # of columns, then echelon form can not have a pivot in every row.
Therefore # of rows $\leq$ # of columns.

Similarly we can prove # of columns $\leq$ # of rows.

Algorithm:

In order to find the inverse A^{-1} of an invertible matrix A, we:

(i) We form an augmented matrix $[A \mid I]$
(ii) Perform row operations so that A becomes I.

Then the right hand side of the resulting augmented matrix is A^{-1}.

Proof:

Each row operation corresponds to multiplying on the left by an invertible matrix, so:

$$E_n E_{n-1} \cdots E_1 A = I \quad \bigstar$$

Where each $E_1, \ldots, E_n$ is an invertible matrix

$$E_n E_{n-1} \cdots E_1 = A^{-1} \qquad \text{Multiply } \bigstar \text{ by } A^{-1}$$

$$E_n E_{n-1} \cdots E_1 I = A^{-1}$$

$\square$

Remark:

If A is not invertible, then it does not row reduce to I.

Corollary:

Any invertible matrix is a product of elementary matrices (i.e. matrices corresponding to the three elementary row operations).