Sec 1.3 Reading

Theorem 1.3.1: Number of solutions of a linear system

A system of equations is said to be consistent if there is at least one solution; it is inconsistent if there are no solutions.

A linear system is inconsistent iff the reduced row-echelon form of its augmented matrix contains the row $[0 \ 0 \ \ldots \ 0 \mid 1]$.

If a linear system is consistent, then it has either
- ↳ infitely many solutions (if there is at least one free variable), or
- ↳ exactly one solution (if all the variables are leading)

Def 1.3.2: The rank of a Matrix

The rank of a matrix is the number of leading 1's in rref(A), denoted rank(A)

Example:

1. Consider a system of n linear equations with m variables, which has a coefficient matrix A of size $n \times m$. Show that:

   (a) The inequalities $\text{rank}(A) \leq n$ and $\text{rank}(A) \leq m$ hold.

   By def of rref, there is at most one leading 1 in each of the n rows and in each of the m columns of rref(A).

   (b) If the system is inconsitent, then $\text{rank}(A) < n$.

   If system is inconsistent, the rref of A will contain a row of the form: $[0 \ 0 \ \ldots \ 0 \mid 1]$. Then $\text{rank}(A) < n$.

   (c) If the system has exactly one solution, then $\text{rank}(A) = m$.

   Notice that

   $$\# \text{ free variables} = \text{total } \# \text{ of variables} - \# \text{ leading variables} = m - \text{rank}(A)$$

   If the system has one solution, then $0 = m - \text{rank}(A) \Rightarrow \text{rank}(A) = m$

(d) If the system has infinitely many solutions, then rank $(A) < m$.

If the system has infinitely many solutions, then there is at least one free variable, so that $m - \text{rank}(A) > 0$ and $\text{rank}(A) < m$.

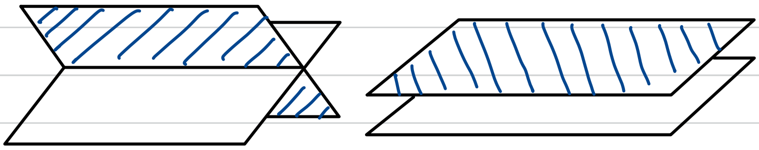
Theorem 1.3.3: # of equations vs # of unknowns

(a) If a linear system has exactly one solution, then there must be at least as many equations as there are variables ($m \le n$)

The contrapositive :

(b) A linear system with fewer equations than unknowns ($n < m$) has either no solutions or infinitely many solutions.

Illustration of (b) :



System of two linear equations with three unknowns cannot have a unique solution.

Theorem 1.3.4: Systems of n equations in n variables

A linear system of n equations in n variables has a unique solution iff the rank of its coefficient matrix A is n. In this case,

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else.

Def 1.3.5: Sums of Matrices

The sum of two matrices of the same size is defined entry by entry.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

Def 1.3.5: Scalar multiples of matrices

The product of a scalar with a matrix is defined entry by entry.

$$k \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

Def 1.3.7: The product $A\vec{x}$

If A is an $n \times m$ matrix with row vectors $\vec{w}_1, \dots, \vec{w}_n$, and $\vec{x}$ is a vector in $\mathbb{R}^m$ then

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

Note that the product $A\vec{x}$ is defined only if the number of columns of matrix A matches the number of components of vector $\vec{x}$.

$$\underbrace{\overset{n \times m}{A}\ \overset{m \times 1}{\vec{x}}}_{n \times 1}$$

Theorem 1.3.8: The product $A\vec{x}$ in terms of the columns of A

If the column vectors of $n \times m$ matrix A are $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{x}$ is a vector in $\mathbb{R}^m$ with components $x_1, \dots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$$

Proof:

We denote the rows of A by $\vec{w}_1, \dots, \vec{w}_n$ and the entries by a_{ij} It suffices to show that the ith component of $A\vec{x}$ is equal to the ith component of $x_1\vec{v}_1 + \dots + x_m\vec{v}_m$, for $i = 1, \dots, n$. Now

$$(\text{ith component of } A\vec{x}) = \vec{w}_i \cdot \vec{x} = a_{i1}x_1 + \dots + a_{im}x_m$$

$$= x_1(\text{ith component of } \vec{v}_1) + \dots + x_m(\text{ith comp. of } \vec{v}_m)$$

$$= \text{ith comp. of } x_1\vec{v}_1 + \dots + x_m\vec{v}_m \quad //$$

Def 1.3.9: Linear Combinations

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ if there exist scalars $x_1, \dots, x_m$ such that

$$\vec{b} = x_1\vec{v} + \dots + x_m\vec{v}_m$$

Example:

2. Is the vector $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of the vectors $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

According to the def of linear combinations, we need to find scalars x and y s.t. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x + 4y \\ 2x + 5y \\ 3x + 6y \end{bmatrix}$

Thus we have to solve the aug. matrix $M = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{bmatrix}$ and

$\mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$

The system is consistent, with $x = -1/3$ and $y = 1/3$. The vector $\vec{b}$ is a linear combination of $\vec{v}$ and $\vec{w}$, with $\vec{b} = -\frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}$.

Theorem 1.3.10: Algebraic rules for $A\vec{x}$

If A is an $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^m$, and k is scalar, then
(a) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$, and
(b) $A(k\vec{x}) = k(A\vec{x})$

Proof:

Denote the ith row of A by $\vec{w}_i$. Then

(ith comp. of $A(\vec{x}+\vec{y})$) $= \vec{w}_i \cdot (\vec{x}+\vec{y})$

ith comp of $A(\vec{x}+\vec{y}) = \begin{bmatrix} \vdots \\ - \ \vec{w}_i \ - \\ \vdots \end{bmatrix} (\vec{x}+\vec{y}) = \vec{w}_i \cdot (\vec{x}+\vec{y})$ (By def of product of $A\vec{x}$)

$= \vec{w}_i \cdot \vec{x} + \vec{w}_i \cdot \vec{y}$ (By dot product property)

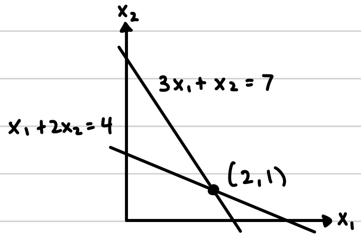
$=$ (ith comp. of $A\vec{x}$) + (ith comp. of $A\vec{y}$)

$=$ (ith comp. of $A\vec{x} + A\vec{y}$) //

Consider the linear system

$$\left| \begin{matrix} 3x_1 + x_2 = 7 \\ x_1 + 2x_2 = 4 \end{matrix} \right| \text{, with aug matrix } \left[\begin{array}{cc:c} 3 & 1 & 7 \\ 1 & 2 & 4 \end{array} \right]$$

We can interpret the solution of this system as the intersection of two lines.



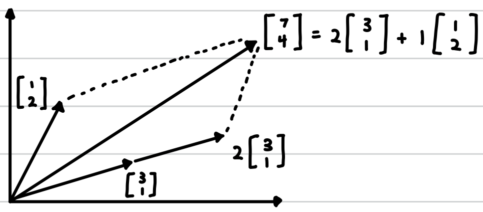
We can also write the system in vector form

$$\begin{bmatrix} 3x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Geometrically:

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$



Theorem 1.3.11: Matrix form of a linear system

We can write the linear system with augmented matrix $[A \,\vdots\, \vec{b}]$ in **matrix form** as

$$A\vec{x} = \vec{b}$$