Feb 11 Lec 2 Notes

5 techniques of integration

(i) By inspection
(ii) Substitution Rule
(iii) Integration by parts
(iv) Partial Fraction Decomposition
(v) Trig substitution

Theorem 5.1: Substitution Rule

If f and g are functions s.t. $f(g(x))g'(x)$ is integrable, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{Indefinite form}$$

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad \text{Definite form}$$

Let $u = g(x)$
$du = g'(x)\,dx$

Proof: Definite Form

Suppose $f, g, f(g(x))g'(x)$ are cont. on $[a,b]$.

WTS: $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Let $u = g(x)$
$du = g'(x)\,dx$

Let F is any antiderivative of f on $[a,b]$

Claim: $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a,b]$

Check: Let $x \in [a,b]$ be arbitrary

$$\begin{aligned}(F(g(x)))' &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x) \quad \text{because } F' = f\end{aligned}$$

Consider $$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big|_a^b \quad \text{FTOC Part 1}$$

$$= F(g(b)) - F(g(a))$$

Consider $\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)}$, $\forall u \in [g(a), g(b)]$ By FTOC Part I

$$= F(g(b)) - F(g(a))$$

$\therefore$ LHS = RHS //

Ex 1

Evaluate $\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du$ Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(\ln x)^3 + C$$

Ex 2

$\int_0^1 \sqrt{2-x}\,dx = \int_2^1 -\sqrt{u}\,du$ Let $u = -x + 2$

$du = -1\,dx$

$$= -\frac{2}{3}u^{3/2}\Big|_2^1$$

$$= -\frac{2}{3}(1 - 2\sqrt{2})$$