Lecture 11 Notes

Examples:

1. Write δ-ε proof that shows that $f(x) = 2x+1$ is continuous at $x=5$.

1st approach:

a) $\text{Dom} f(x) = (-\infty, \infty)$; $x=5$ is in the $\text{Dom} f(x)$

b) $\lim_{x\to 5} (2x+1) = 11$. Show $\forall \varepsilon>0\ \exists \delta>0$ st $0<|x-5|<\delta \Rightarrow |2x+1-11|<\varepsilon$

Aside:

$$|2x+1-11| = |2x-10| = 2|x-5| < 2\delta$$

Stipulate $\delta = \frac{\varepsilon}{2}$

Proof:

Given $\varepsilon>0$, choose $\delta = \frac{\varepsilon}{2}$. Suppose x satisfies $0<|x-5|<\delta$, then

$$|2x+1-11| = 2|x-5| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

QED

c) $f(5) = 2\cdot 5 + 1 = 11$

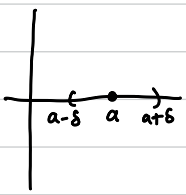
$\lim_{x\to 5} 2x+1 = f(5) = 11$

Therefore $f(x)$ is continuous at 5.

2nd approach:

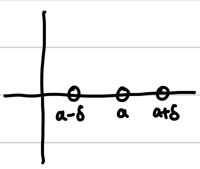
$f(x)$ is continuous at any point a in the $\text{Dom} f(x)$ if $\lim_{x\to a} f(x) = f(a)$.

We need to show: $\forall \varepsilon>0\ \exists \delta>0$ st $|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$



Aside:

$$|2x+1-(2a+1)| = |2x-2a| = 2|x-a| < 2\delta \Rightarrow \delta = \frac{\varepsilon}{2}$$



Proof:

Given $\varepsilon>0$, choose $\delta = \frac{\varepsilon}{2}$. Then if $|x-a|<\delta$, we have:

$|2x+1-(2a+1)| = 2|x-a| < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$

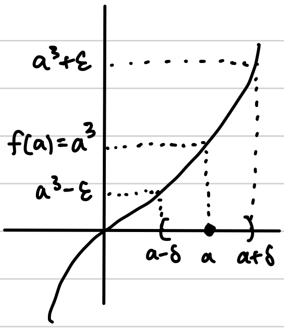
QED

Triangle Inequality ($\triangle$) : $|A+B| \leq |A| + |B|$

Reverse Inequality (∇) : $|A-B| \geq |A| - |B|$

Exercise:

2. Prove that $f(x) = x^3$ is continuous on its domain.



$f(x) = x^3$ is continuous on its domain means that $\lim_{x\to a} x^3 = a^3$, $\forall a \in \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |x^3 - a^3| < \varepsilon$

Aside:

Triangle Inequality

To estimate use ($\triangle$):

$|x^2+xa+a^2| \leq |x^2| + |xa| + |a^2|$

$|x| = |x-a+a| \leq |x-a| + |a|$

Let $\delta \leq 1 \Rightarrow |x-a| < 1$

$|x| \leq |x-a| + |a| < 1 + |a|$

$$
\begin{aligned}
|x^3 - a^3| &= |(x-a)(x^2+xa+a^2)| \\
&= |x-a|\,|x^2+xa+a^2| \\
&< \delta \cdot (|x|^2 + |x||a| + |a|^2) \\
&< \delta(|1+|a||^2 + |1+|a|| \cdot |a| + |a|^2) \\
&< \delta(|1+|a||^2 + |1+|a||^2 + |1+|a||^2) = \delta \cdot 3(1+|a|)^2 = \varepsilon
\end{aligned}
$$

Since $|a| < 1 + |a|$.

In the text book:

$= 1 + 2|a| + |a|^2 + |a| + |a|^2 + |a|^2$

$= 3|a|^2 + 3|a| + 1$

$|x^3 - a^3| < \delta(3|a|^2 + 3|a| + 1)$

Proof:

Given $\varepsilon > 0$, choose $\delta = \min\left\{1, \frac{\varepsilon}{3(1+|a|)^2}\right\}$ OR $\delta = \min\left\{1, \frac{\varepsilon}{3|a|^2 + 3|a| + 1}\right\}$

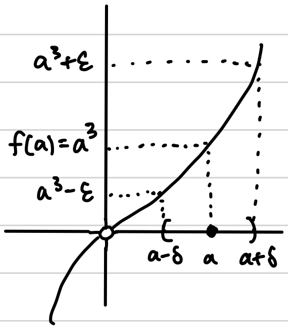
If $|x-a| < \delta$ we have:

$$|x^3 - a^3| = |x-a|\,|x^2 + xa + a^2| < \delta\,(|x^2| + |x||a| + |a^2|) \ldots\ldots < \delta(1+|a|)^2 = \frac{\varepsilon \cdot \cancel{3(1+|a|)^2}}{\cancel{3(1+|a|)^2}}$$

QED

3. Prove that a) $f(x)$ is continuous at any a, except $a=0$. ($f(x)=x^3$)

b) $f(x)$ is continuous at $x=0$.



a) $f(x)$ is continuous at $\forall a \in \mathbb{R} - \{0\}$ (excluding set $\{0\}$) if $\lim_{x \to a} x^3 = a^3$

Aside: $\delta \leq |a|$ ← guarantees that the δ interval is to the right of the origin.

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |x^3 - a^3| < \varepsilon$

$$|x^3 - a^3| = |x-a|\,|x^2+ax+a^2| = |x-a|\,\big|\,|x|^2 + |a||x| + |a|^2\big|$$

$$< \delta\,(4|a|^2 + |a|2|a| + |a|^2)$$

$|x| = |x-a+a| \leq \underbrace{|x-a|}_{\delta \leq |a|} + |a| < 2|a|$ $\xrightarrow{\text{sub } x<2|a|}$ $< \delta\,(4|a|^2 + 3|a|^2) = \delta(7|a|^2) = \varepsilon$

Proof:

Given $\varepsilon > 0$, choose $\delta = \min\{|a|, \frac{\varepsilon}{7|a|^2}\}$. If $|x-a| < \delta$, then

$$|x^3 - a^3| = |x-a|\,|x^2+ax+a^2| = \ldots < \delta(7|a|^2) = \frac{\varepsilon}{\cancel{7|a|^2}} \cdot \cancel{7|a|^2} = \varepsilon$$

QED

b) $f(x)$ is continuous at $x=0$ if $\lim_{x \to 0} x^3 = 0$

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $|x| < \delta \Rightarrow |x^3| < \varepsilon$, but $|x| < \delta$, so stipulate $\delta = \sqrt[3]{\varepsilon}$.

Proof:

Given $\varepsilon > 0$, choose $\delta = \sqrt[3]{\varepsilon}$. Suppose $|x| < \delta$, then $|x^3| < \delta^3 = (\sqrt[3]{\varepsilon})^3 = \varepsilon$

QED

4. Prove $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

1st approach:

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |x-2| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$

Aside:

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{2|x|} = |2-x| \cdot \frac{1}{2} \cdot \frac{1}{|x|}$$
$$< \delta \cdot \frac{1}{2} \cdot 1 = \varepsilon$$

Let $\delta \leq 1$.

$|2-x| = |x-2| < 1$

$-1 < x-2 < 1$

$1 < x < 3$

$1 > \frac{1}{x} > \frac{1}{3}$

$\left|\frac{1}{x}\right| < \max\{1, 1/3\} = 1$

We are looking for upper bound for $\left|\frac{1}{x}\right|$ or lower bound for $|x|$.

Proof:

Guess $\delta = 2\varepsilon$

Given $\varepsilon > 0$, choose $\delta = 2\varepsilon$. If $0 < |x-2| < \delta$ then:

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2|x|} < \delta \frac{1}{2} = \frac{2\varepsilon}{2} = \varepsilon$$

QED

2nd approach:

$$|2| - |x| \leq |2-x| = |x-2| < \delta$$

Let $\delta \leq 1$

$|2| - |x| < 1$

$|x| > 1$

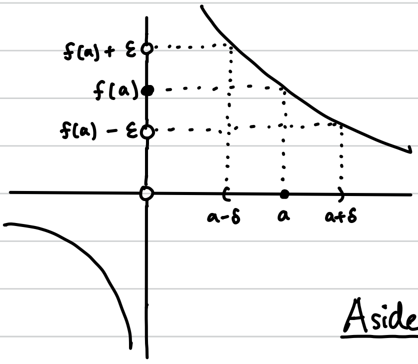
$\frac{1}{|x|} < 1$ ← bound is also 1 here.

Proof:

Same as 1st approach.

QED

5. Prove that $f(x) = \frac{1}{x}$ is continuous at $x = a$, $a \neq 0$.



We need to show that

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a}, \quad \forall a \in \mathbb{R},\ a \neq 0$$

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } |x-a| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{a} \right| < \varepsilon$$

Aside:

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \left| \frac{a-x}{xa} \right| = \frac{|a-x|}{|a||x|} < \frac{\delta}{|a|} \cdot \frac{1}{|x|} < \frac{\delta}{|a|} \cdot \frac{2}{|a|} = \frac{2}{|a|^2}\delta = \varepsilon$$

Guess $\delta = \frac{\varepsilon a^2}{2}$

Let $\delta \leq \frac{|a|}{2}$

$|a| - |x| \leq |a-x| = |x-a| < \frac{|a|}{2}$

$|a| - |x| < \frac{|a|}{2}$

$\frac{|a|}{2} < |x|$

$\frac{1}{|x|} < \frac{2}{|a|}$

Proof:

Given $\varepsilon > 0$, choose $\delta = \min\left\{ \frac{|a|}{2}, \frac{\varepsilon a^2}{2} \right\}$. If x satisfies $|x-a| < \delta$, then we have

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|a-x|}{|a||x|} \cdots < \frac{\delta}{|a|} \cdot \frac{1}{|x|} < \frac{\delta}{|a|} \cdot \frac{2}{|a|} = \frac{2\delta}{a^2} = \frac{2\varepsilon \cancel{a^2}}{\cancel{2a^2}} = \varepsilon$$

QED

3 Important Consequences of Continuity

- All continuous functions attain their min and max values on closed interval $[a,b]$
- if $c \in (a,b)$ then $f(c) < f(b)$; $f(a) < f(c)$
- All continuous functions on closed intervals are bounded from above and from below.

The Extreme Value Theorem

If $f(x)$ is continuous on closed interval $[a,b]$, then there exist some values M and m in the interval $[a,b]$ such that $f(M)$ is the maximum value of $f(x)$ on $[a,b]$ and $f(m)$ is the minimum value of $f(x)$ on $[a,b]$.

The Intermediate Value Theorem

If $f(x)$ is continuous on closed interval $[a,b]$, then for any K strictly between $f(a)$ and $f(b)$ there exists at least one $c \in (a,b)$ such that $f(c) = K$.

Upper and Lower Bounds — Infinum and Supremum

Let $S \subset \mathbb{R}$

Set S is bounded from above if $\forall x \in S \; \exists M \in \mathbb{R}$ s.t. $x \le M$

M - upper bound for S; $[M, \infty)$ — set of upperbounds

a least upper bound for S (lub S) is called supremum S (sup S)

lub S = Sup S

Set S is bounded from below if $\forall x \in S \; \exists m \in \mathbb{R}$ s.t. $x > m$

m - lower bound for S; $(-\infty, m]$ is the set of lower bounds

a greatest lower bound for S (glb S) is called infinum S (Inf S)

glb S = Inf S

Largest element of S (max S) exists if Sup $S \in S$

Smallest element of S (min S) exists if Inf $S \in S$

Set S	Max S	Min S	Sup S	Inf S
$(0, \sqrt{2})$	No	No	$\sqrt{2}$	0
$(-\infty, 3]$	3	No	3	No
$[-1, 4)$	No	-1	4	-1
$\{1, 3, \ldots (2n-1) \ldots\}$	No	1	No	1

Least Upper Bound Axiom:

Every nonempty set of real numbers is bounded from above has a supremum.

$S_1 = (-10, 0] \rightarrow S_1$ has largest element, Sup S_1 = Max $S_1 = 0$

$[0, \infty)$ is the set of upper bounds for S_1.

$S_2 = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots -\frac{1}{n}, \ldots\} \rightarrow S_2$ does not have the largest element, so there are no Max S_2.

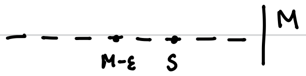
$\rightarrow S_2$ is bounded above by 0. Sup $S_2 = 0$

$[0, \infty)$ is the set of upperbounds for S_2.

Theorem for Supremum

If M is $\sup S$ and $\varepsilon > 0$, then there exist at least one number s in S such that $M - \varepsilon < s \le M$

Proof:



① $\sup S = M \Rightarrow s \le M$
② Show that $M - \varepsilon < s$

From the contrary supposition, assume that there is no such s in S.

If there is no $M - \varepsilon < s$, then all $x \in S$ are less or equal to $M - \varepsilon$.
But if $x \le M - \varepsilon$, then $M - \varepsilon$ is $\sup S$ (not M!) which contradicts the supposition.
since we know that $\sup S = M$

Therefore, there is such s that $M - \varepsilon < s$

QED

Examples:

6. Let $S = \{0, 1, 2, 3, 4\}$ $\varepsilon_1 = 0.1$, $\varepsilon_2 = 2$

$\sup S = 4$ $\quad 4 - 0.1 < s \le 4 \Rightarrow s = 4$
$\quad 4 - 2 < s \le 4 \Rightarrow s = 3$

Greatest Lower Bound Axiom:

Every nonempty set of real numbers is bounded from below has an infinum.

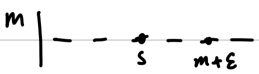
$S_1 = [-10, 0) \rightarrow S_1$ has -10 as a smallest element, $\inf S_1 = \min S_1 = -10$
$(-\infty, -10]$ is the set of lower bounds for S_1.

$S_2 = \{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\} \rightarrow S_2$ does not have the smallest element, so there are no $\min S_2$.
$\rightarrow S_2$ is bounded below by 0. $\inf S_2 = 0$.
$(-\infty, 0]$ is the set of lower bounds for S_2.

Theorem for Infinum

If m is $\text{Inf } S$ and $\varepsilon > 0$, then there is at least one number s in S such that $m \le s < m + \varepsilon$.

Proof:



① $\text{Inf } S = m \Rightarrow s \ge m$

② Show that $m + \varepsilon > s$

From the contrary supposition, assume that there is no such s in S.

If there is no $m + \varepsilon > s$, then all $x \in S$ are greater than or equal to $m + \varepsilon$.
But if $x \ge m + \varepsilon$, then $m + \varepsilon$ is $\text{Inf } S$ (not m!) which contradicts the supposition.

Therefore, there is such s that $m + \varepsilon > s$.

QED