# A22 Mar 31 Lec 1 Notes

Ex 1:

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}$$

$$\det A = 2 \underbrace{\begin{vmatrix} 1 & 1 & 0 & 2 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{vmatrix}}_{A'} \qquad A' \overset{2R_1}{\sim} A$$

$$\det A = 2 \det A'$$

$$= 2 \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 2 & -3 \end{vmatrix}$$

$$= 2 \left( 1 \begin{vmatrix} 0 & 2 & -4 \\ 1 & 3 & 2 \\ -2 & 2 & -3 \end{vmatrix} - 0 + 0 - 0 \right)$$

$$= 2(-1) \begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ -2 & 2 & -3 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 8 & 1 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 2 & -4 \\ 8 & 1 \end{vmatrix}$$

$$= (-2)(2) \begin{vmatrix} 1 & -2 \\ 8 & 1 \end{vmatrix}$$

$$= (-4) \begin{vmatrix} 1 & -2 \\ 0 & 17 \end{vmatrix}$$

$$= (-4)(17) = -68$$

Equivalently,

$$A \sim \begin{bmatrix} 2 & 2 & 0 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \quad \text{REF}(A)$$

$$\det A = -(2)(1)(2)(17)$$

2nd row was switched during $\sim$

## Theorem:

Let $A \in M_{n\times n}(\mathbb{R})$.

$A$ is invertible iff $\det A \neq 0$

**Proof:**

Row reduce $A$ into rref(A)

$$A \sim A_1 \sim A_2 \cdots \sim A_k = \text{rref}(A)$$

$$\forall i, \ \det A_i = \pm k_i \det A_{i-1}, \quad \text{some } k_i \in \mathbb{R} \setminus \{0\}$$

$$\det \text{rref} A = \pm k \ \det A$$

det rref A and det A are either both zero or both nonzero

$\det \text{rref} A = 0$ iff rref has a zero in a pivot position

$\det A \neq 0$ iff $\det \text{rref} A \neq 0$

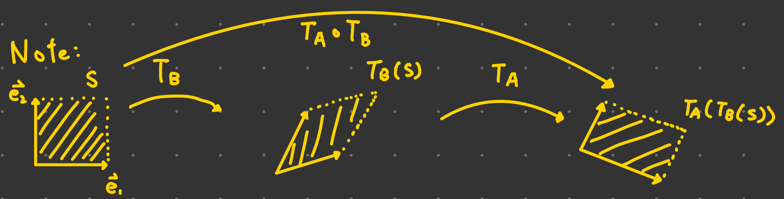
Thus A is not invertible

$\blacksquare$

## Theorem:

If $A, B$ are $n\times n$ matrices, $\det(AB) = \det(A)\det(B)$

Note:




area $T_B(S) = \det B \cdot \text{area } S$

$$\text{Area } T_A(T_B(S)) = \det A \cdot \text{area } T_B(S)$$
$$= \det A \cdot \det B \cdot \text{area of } S$$
$$= \det A \cdot \det B \cdot 1$$

**Proof:**

Case 1: Suppose A is diagonal

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{bmatrix}, \quad B = (b_{ij})$$

$$\det AB = \det \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix}$$

$$= a_{11} \det \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} \cdots a_{nn} \det B$$

$$= \det A \det B$$

**Case 2:** $A$ is invertible

$A$ is invertible $\Rightarrow A \sim I_n$

Apply row reduction to $A$ without scaling rows.

$$A \sim \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} =: D$$

$D = EA$ , $E$ is a product of elementary matrices

$\det D = (-1)^r \det A$ , $r$ is the # of row switches

$\det A = (-1)^r \det D$

$EAB = (EA)B$

$= DB \Rightarrow \det(DB) = (-1)^r \det AB$

$\Rightarrow \det AB = (-1)^r \det DB$

$\Rightarrow \det AB = (-1)^r \det D \det B$ from case 1

$= \det A \det B$ $\det A = (-1)^r \det D$

**Case 3:** $A$ is not invertible

$\Rightarrow$ show $AB$ is not invertible

$\Rightarrow \det A = 0$

$$\Rightarrow \underbrace{\det AB}_{0} = \underbrace{\det A}_{0} \det B$$

$\blacksquare$