B24 May 19 Lec 1 Notes

Remark:

We use I to denote the identity L.T.

If $v_1, \dots, v_n$ is any basis for a v.s. V, then

$$[I]_{v_1,\dots,v_n}^{v_1,\dots,v_n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

Definition:

A L.T. $T: V \to W$ is said to be invertible if there exists a L.T. $S: W \to V$ s.t. $ST = I_V$ and $TS = I_W$

The map S is called an inverse of T.

Remark:

If $T: V \to W$ s.t. $\exists S: W \to V$ with $ST = I_V$, it does not follow in general that $TS = I_W$.

Theorem:

Suppose that $T: V \to W$ is invertible, then its inverse is unique.

Proof:

Let S_1, S_2 be inverses of T. Then

$$S_1 T S_2 = S_1 (T S_2) = S_1 I_W = S_1$$

$$S_1 T S_2 = (S_1 T) S_2 = I_V S_2 = S_2$$

$\therefore S_1 = S_2$ ▨

The formula for rotation of $\mathbb{R}^2$ by θ degrees:

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$
$$R_\theta^{-1}(x,y) = (x\cos\theta + y\sin\theta,\ -x\sin\theta + y\cos\theta)$$

i.e. that $R_\theta \cdot R_\theta^{-1}(x,y) = (x,y)$, and $R_\theta^{-1} \cdot R_\theta(x,y) = (x,y)$

Definition:

We say a matrix A is invertible if there exists a matrix B s.t.

$$AB = BA = I$$

in which case B is said to be an inverse of A.

Theorem:

If a matrix A is invertible, then its inverse is unique.

Proof:

Same as for L.T.

e.g. The inverse of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can also verify that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = [R_\theta]_{(1,0),(0,1)}^{(1,0),(0,1)}$ and

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = [R_{-\theta}]_{(1,0),(0,1)}^{(1,0),(0,1)}$$

so that

$$[R_\theta]_{(1,0),(0,1)}^{(1,0),(0,1)} [R_\theta]_{(1,0),(0,1)}^{(1,0),(0,1)} = [R_\theta \cdot R_{-\theta}]_{(1,0),(0,1)}^{(1,0),(0,1)}$$

$$= [I]_{(1,0),(0,1)}^{(1,0),(0,1)}$$

Proposition:

If $T: V \to W$, $S: W \to U$ are invertible L.T.'s, then ST is invertible, and

$$(ST)^{-1} = T^{-1} S^{-1}$$

Remark:

In mathematics, two spaces being "isomorphic" means they are "structurally" the same

Proposition:

Let $T: V \to W$ be an isomorphism and $v_1, \dots, v_n \in V$. Then $v_1, \dots, v_2$ form a basis for V iff $Tv_1, \dots, Tv_n$ is a basis for W.

Theorem:

Let V, W be v.s with bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$. Then the L.T. $T: V \to W$ defined by $Tv_i = w_i$ for $1 \leq i \leq n$ is an isomorphism.

Proof:

Define $S: W \to V$ by $S(w_i) = v_i$ for $1 \leq i \leq n$, and verify $ST = I_V$, $TS = I_W$. For instance, given $v \in V$, there exists $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $v = \alpha_1 v_1 + \dots + \alpha_n v_n$, and so

$$\begin{aligned}
ST(v) &= ST(\alpha_1 v_1 + \dots + \alpha_n v_n) \\
&= S(\alpha_1 T(v_1) + \dots + \alpha_n T(v_n)) \\
&= S(\alpha_1 w_1 + \dots + a_n w_n) \\
&= \alpha_1 S(w_1) + \dots + \alpha_n S(w_n) \\
&= \alpha_1 v_1 + \dots + \alpha_n v_n \\
&= v
\end{aligned}$$

e.g. $T: \mathbb{P}_3^{\mathbb{R}} \to \mathbb{R}^4$ defined by

$$\begin{aligned}
T(1) &= (1,0,0,0) \\
T(x) &= (0,1,0,0) \\
T(x^2) &= (0,0,1,0) \\
T(x^3) &= (0,0,0,1)
\end{aligned}$$

is an isomorphism by the above theorem