CH 2.3 Reading

We can use theorem 1.3.10 to show that $T(\vec{x}) = B(A\vec{x})$ is linear.

$$\begin{aligned} T(\vec{v}+\vec{w}) = B(A(\vec{v}+\vec{w})) &= B(A\vec{v} + A\vec{w}) \\ &= B(A\vec{v}) + B(A\vec{w}) \\ &= T(\vec{v}) + T(\vec{w}) \end{aligned}$$

$$\begin{aligned} T(k\vec{v}) = B(A(k\vec{v})) &= B(kA(\vec{v})) \\ &= k\,B(A(\vec{v})) \\ &= k\,T(\vec{v}) \end{aligned}$$

Since T is linear, $T(\vec{e}_1) = B(A\vec{e}_1)$; $T(\vec{e}_2) = B(A\vec{e}_2)$; ... ; $T(\vec{e}_n) = B(A\vec{e}_n)$.

The matrix of $T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$.

Def 2.3.1: Matrix Multiplication

(a) Let B be an $n \times p$ matrix and A a $q \times m$ matrix. The product BA is defined iff $p = q$.
(b) If B is an $n \times p$ matrix and A a $p \times m$ matrix, then the product BA is defined as the matrix of the L.T. $T(\vec{x}) = B(A\vec{x})$
This means that $T(\vec{x}) = B(A(\vec{x})) = (BA)\vec{x}$, for all $\vec{x}$ in the vector space $\mathbb{R}^m$. The product BA is an $n \times m$ matrix.

Theorem 2.3.2: The columns of the matrix product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m$.
Then, the product BA is

$$BA = B\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_m \\ | & | & & | \end{bmatrix}$$

To find BA, we can multiply B by the columns of A and combine the resulting vectors.

Theorem 2.3.3: Matrix multiplication is noncommutative

$AB \neq BA$, in general However, at times it does happen that $AB = BA$; then we say that the matrices A and B commute.

Theorem 2.3.4: The entries of the matrix product

Let B be an $n \times p$ matrix and A a $p \times m$ matrix. The ijth entry of BA is the dot product of the ith row of B with the jth column of A.

$$BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{ip} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pm} \end{bmatrix}$$

is the $n \times m$ matrix whose ijth entry is

$$b_{i1} a_{1j} + b_{i2} a_{2j} + \ldots + b_{ip} a_{pj} = \sum_{k=1}^{p} b_{ik} a_{kj}$$

Theorem 2.3.5: Multiplying with the identity matrix

For an $n \times m$ matrix A,

$$AI_m = I_n A = A$$

Theorem 2.3.6: Matrix Multiplication is associative.

$$(AB)C = A(BC)$$

We can simply write ABC for the product $(AB)C = A(BC)$

Digram:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "C"'] \arrow[rr, bend left, "BC"] \arrow[rrr, bend left=50, "A(BC)"] \arrow[rrr, bend right=50, "(AB)C"'] & \mathbb{R}^q \arrow[r, "B"'] \arrow[rr, bend right, "AB"'] & \mathbb{R}^p \arrow[r, "A"'] & \mathbb{R}^n
\end{tikzcd}$$

Proof:

$$T(\vec{x}) = \big((AB)C\big)\vec{x} \quad ; \quad L(\vec{x}) = \big(A(BC)\big)\vec{x}$$

By def of matrix multiplication,

$$T(\vec{x}) = ((AB)C)\vec{x} = (AB)(C\vec{x}) = A(B(C\vec{x}))$$
$$L(\vec{x}) = (A(BC))\vec{x} = A((BC)\vec{x}) = A(B(C\vec{x}))$$

Thus the matrix of T and L are equivalent.

Theorem 2.3.7: Distributive property for matrices

If A and B are $n \times p$ matrices, and C and D are $p \times m$ matrices, then

$$A(C+D) = AC + AD, \text{ and}$$
$$(A+B)C = AC + BC$$

Theorem 2.3.8:

If A is an $n \times p$ matrix, B is a $p \times m$ matrix, and k is a scalar, then

$$(kA)B = A(kB) = k(AB)$$

Def 2.3.10: Regular transition matrices

A transition matrix is said to be positive if all its entries are positive.

A transition matrix is said to be regular if the matrix A^m is positive for some positive integer m.

Theorem 2.3.11: Equilibria for regular transition matrices.

Let A be a regular transition matrix of size $n \times m$.

(a) There exists exactly one distribution vector $\vec{x}$ in $\mathbb{R}^n$ s.t. $A\vec{x} = \vec{x}$. This is called the equilibrium distribution for A, denoted $\vec{x}_{equ}$. All the components of $\vec{x}_{equ}$ are positive.

(b) If $\vec{x}$ is any distribution vector in $\mathbb{R}^n$, then $\lim_{n \to \infty} (A^m \vec{x}) = \vec{x}_{equ}$.

(c) $\lim_{m \to \infty} A^m = \begin{bmatrix} | & & | \\ \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & & | \end{bmatrix}$, which is the matrix whose columns are all $\vec{x}_{equ}$.