Week 5 Reading Notes

Two events E and F are independent if $P(E|F) = P(E)$.

Theorem (The Product Rule)

If E and F are two events in an experiment then the probability that both E and F occur is:

$$P(E \cap F) = P(E \cap F) = P(F) \cdot P(E|F) = P(E) \cdot P(F|E)$$

If E and F are two independent events, then $P(E \cap F) = P(E) \cdot P(F)$

Examples:

1. Suppose we roll two standard dice, one blue and one grey. What is the probability that the sum of the numbers showing face up is 8, given that both of the numbers are even?

A = event that both numbers are even
B = event that sum of the numbers are 8.

$P(A) = \frac{9}{36}$ $\quad P(B) = \frac{5}{36}$

$P(A \cap B) = \{(2,6), (4,4), (6,2)\} = \frac{3}{36}$

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$

2. Suppose there is a noisy communication channel in which either a 0 or a 1 is sent with the following properties:
   - Probability a 0 is sent is 0.4
   - $P(1 \text{ sent}) = 0.6$
   - $P(1 \text{ changed to a } 0) = 0.1$
   - $P(0 \text{ changed to a } 1) = 0.2$

   Suppose that a 1 is received. What is the probability that a 1 was sent.

   Let A denote that a 1 was received and B the event that a 1 was sent.

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)} = \frac{0.6 \cdot (1-0.1)}{0.62}$$

$$\boxed{\approx 0.87}$$

$$P(A) = \underbrace{P(0 \text{ sent}) \cdot P(1 \text{ received} \mid 0 \text{ sent})}_{\text{transmission error}} + \underbrace{P(1 \text{ sent}) \cdot P(1 \text{ received} \mid 1 \text{ sent})}_{\text{no transmission error}}$$

$$P(A) = 0.4(0.2) + 0.6(1-0.1) = 0.62$$

Theorem (Bayes' Rule)

Let A and B be events in the same sample space. If neither P(A) nor P(B) are zero, then:

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}$$

Theorem (Total Probability)

Let a sample space S be a disjoint union of events $E_1, E_2, \ldots, E_n$ with positive probabilities and let $A \subseteq S$. Then:

$$P(A) = \sum_{i=1}^{n} P(A|E_i)\cdot P(E_i)$$

Definition

Suppose the possible outcomes of an experiment are real numbers $a_1, a_2, \ldots, a_n$ which each occur with probabilities $p_1, p_2, \ldots, p_n$. The expected value of the process is :

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + \ldots + a_n p_n$$

An expected value is the value of the outcome a random experiment will have on average.

Example:

3. Suppose that 500,000 people pay \$5 each to play a lottery game with the following prizes:
   - One \$1,000,000 grand prize
   - Ten \$1,000 second prizes
   - A thousand \$500 third prizes
   - Ten thousand \$10 fourth prizes.

What is the expected winnings or value of a ticket?

We assume that each of the 500,000 tickets have the same chance of winning So $p_k = \frac{1}{500{,}000}$ for each $k = 1, 2, 3, \ldots, 500{,}000$.

Value of each ticket :

$1{,}000{,}000 - 5 = 999{,}995$

$1000 - 5 = 995$

$500 - 5 = 495$

$10 - 5 = 5$

$0 - 5 = -5$

$$E(\text{winnings}) = \sum_{k=1}^{500{,}000} a_k \cdot p_k$$

$$= \frac{1}{500{,}000}\sum_{k=1}^{500{,}000} a_k$$

$$= \frac{1}{500{,}000}\left(999{,}995 + 10\cdot 995 + 1000\cdot 495 + 10{,}000\cdot 5 + 488{,}989(-5)\right)$$

$$\boxed{= -\$1.78}$$