Pre-lecture Video

$\mathbb{N}$ = set of natural numbers = $\{0, 1, 2, 3, \ldots\}$

Principle of Simple Induction (PSI)

We can prove $P(n)$ holds for $n \geq b$ by proving:

Basis: $P(b)$

Induction Step: For all $n \geq b$, if $P(n)$, then $P(n+1)$

Ex: ex 1.7, pg 30

$P(n)$: exact postage of n cents can be made using only 4-cent and 7-cent stamps.

Equivalent predicate:

$Q(n)$: there exists $k, l \in \mathbb{N}$ s.t. $4k + 7l = n$

Prove $\forall n \geq 18$, $Q(n)$.

Proof:

Basis: Let $n = 18$

Let $k = 1$, $l = 2$. Then $k, l \in \mathbb{N}$
and $4k + 7l = 4 \cdot 1 + 7 \cdot 2 = 18 = n$ as wanted

Induction Step: Let $n \geq 18$.

Suppose $Q(n)$ [I.H.]
i.e. there are $k, l \in \mathbb{N}$ s.t. $4k + 7l = n$
WTP: $Q(n+1)$ i.e. $\exists\ k', l' \in \mathbb{N}$ s.t. $4k' + 7l' = n+1$

Consider 2 cases: $l > 0$ and $l = 0$

Case 1: Suppose $l > 0$

Then let $k' = k + 2$, $l' = l - 1$

Then $l' \geq 0$, so $l' \in \mathbb{N}$

Also, $4k' + 7l' = 4(k+2) + 7(l-1)$
$= 4k + 8 + 7l - 7$
$= 4k + 7l + 1$
$= n+1$ [I.H.]

Proof (Continued...):

Case 2: Suppose $\ell = 0$

Since $n \geq 18$, we have

$$18 \leq n \overset{[I.H]}{=} 4k + \underbrace{7\ell}_{0} = 4k$$

Thus $k \geq 5$

Let $k' = k - 5 \geq 0$
$\ell' = \ell + 3$

Then $4k' + 7\ell' = 4(k-5) + 7(\ell+3)$
$= 4k - 20 + 7\ell + 21$
$= 4k + 7\ell + 1$
$= n+1$ [I.H.]

$\boxtimes$

Principle of Complete Induction (PCI)

We can prove $P(n)$ for all $n \geq b$ by proving

Basis: $P(b), P(b+1), \ldots, \underbrace{P(b+k-1)}_{k \text{ base cases}}$

I.S.: For $n \geq b+k$, if $P(j)$ holds whenever $b \leq j \leq n$, then $P(n)$

Ex: ex 1.12 pg 40

$Q(n): \exists k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n$

Use PCI to prove $\forall n \geq 18$, $Q(n)$

Base case:

For $n = 18$, let $k=1, \ell=2$. Then $4k + 7\ell = n$ as wanted.
For $n = 19$, $k=3$, $\ell=1$
For $n = 20$, $k=5$, $\ell=0$
For $n = 21$, $k=0$, $\ell=3$

I.S: Let $n \geq 22$

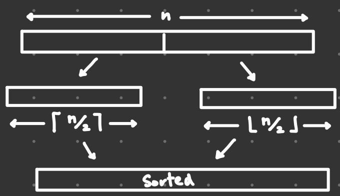
Suppose $Q(j)$ holds whenever $18 \leq j \leq n$ [I.H.]
WTP: $Q(n)$ holds i.e. $\exists k', \ell'$ s.t. $4k' + 7\ell' = n$
Since $n \geq 22$, we have $18 \leq n-4 < n$
By I.H., $Q(n-4)$ holds i.e. $\exists k, \ell \in \mathbb{N}$ s.t. $4k + 7\ell = n-4$
Let $k' = k+1$, $\ell' = \ell$
Then $4k' + 7\ell' = 4(k+1) + 7\ell = 4k + 7\ell + 4 = n - 4 + 4$ [I.H.]
$= n$ $\boxtimes$

Principle of Well-Ordering (PWO)

Every non-empty subset of $\mathbb{N}$ has a minimum element

Recurrences (CH 3) - Recursively / inductively defined functions

e.g. Merge sort



Let $f(n)$ be the number of item assignments needed for sorting n items with mergesort

$$f(n) = \begin{cases} 0 & \text{if } n=1, \\ f(\lceil n/2 \rceil) + f(\lfloor n/2 \rfloor) + 2n & \text{if } n>1 \end{cases}$$

Unwinding Recurrences

Let n be a large power of 2, i.e. $n = 2^k$ for some large $k \in \mathbb{N}$

since $n=2^k$ is even

$f(n) = 2f(\frac{n}{2}) + 2n$ 1st iteration

$= 2(2f(\frac{n}{4}) + 2\frac{n}{2}) + 2n$ 2nd iteration

$= 4f(\frac{n}{4}) + 4n$

$= 4[2f(\frac{n}{8}) + 2(\frac{n}{4})] + 4n$

$= 8f(\frac{n}{8}) + 6n$ 3rd iteration

$\vdots$

$= 2^i f(\frac{n}{2^i}) + 2in$ ith iteration

$\vdots$

$= 2^k f(\frac{n}{2^k}) + 2kn$ let $i=k$

$= 2^k f(\frac{n}{n}) + 2kn$ since $2^k = n$

$= 2^k f(1) + 2kn$ $f(1) = 0$

$= 2kn$

$= 2n \cdot \log_2 n$

$\approx n \cdot \log n$ # of assignments

Structural Induction (CH4)

2 uses:

- ↳ Define sets
- ↳ Prove properties of all elements in a set defined by structural induction

Ex: ex 4.1 pg 97

Define the set of all well-formed, fully parenthesized algebraic expressions with variables x, y, z and operators $+, -, \times, \div$.

e.g. x, $(x+y)$, $(((y-z)\times(z-x))+x)$

Remark:

$$\Sigma = \{\underbrace{x, y, z}_{\text{variables}}, \underbrace{+, -, \times, \div}_{\text{operators}}, \underbrace{(,)}_{\text{parentheses}}\} \qquad \mathcal{E} \subseteq \underbrace{\Sigma^*}_{\text{finite strings consisting of symbols from } \Sigma}$$

Definition: Let $\mathcal{E}$ be the smallest set s.t.

Base cases: $x, y, z \in \mathcal{E}$

Not variable x, but a string with the x in $\mathcal{E}$.

I.S.: If $e_1, e_2 \in \mathcal{E}$, then $(e_1+e_2), (e_1-e_2), (e_1\times e_2), (e_1\div e_2) \in \mathcal{E}$

Ex: ex 4.2 pg 100

For a string $e \in \Sigma^*$, we define

$vr(e)$ to the # of occurrences of variables in e.
$op(e)$ to the # of occurrences of operators in e.

For $e \in \Sigma^*$, we define a predicate

$P(e)$: $vr(e) = op(e) + 1$

Proof: Use structural induction to prove that $P(e)$ holds for $e \in \mathcal{E}$

Base cases: 3 cases: $e = x$, $e = y$, $e = z$

For $e = x$, $vr(e) = 1$, $op(e) = 0$

$\therefore vr(e) = op(e) + 1$

Similarly for $e = y$, $e = z$.

I.S.: Let $e_1, e_2 \in \mathcal{E}$

Suppose $P(e_1)$, $P(e_2)$ [I.H.]
i.e. $vr(e_1) = op(e_1) + 1$ and $vr(e_2) = op(e_2) + 1$
WTP: $P(e)$ holds for $e = (e_1+e_2)$, $e = (e_1-e_2)$, $e = (e_1\times e_2)$, $e = (e_1 \div e_2)$

In each case, we have

$$vr(e) = vr(e_1) + vr(e_2)$$
$$op(e) = op(e_1) + op(e_2) + 1$$

Proof (continued...):

I.S (continued...):

Thus $$\begin{aligned} vr(e) &= vr(e_1) + vr(e_2) \\ &= (op(e_1)+1) + (op(e_2)+1) \quad [\text{I.H.}] \\ &= (op(e_1) + op(e_2)) + 2 \\ &= (op(e) - 1) + 2 \\ &= op(e) + 1 \end{aligned}$$

$\blacksquare$

Ex:

Let S be the smallest set s.t.

Base Case: $0 \in S$

I.S.: if $n \in S$, then $n+1 \in S$

B36 Sept 15 Lec 1 Notes

1.

1. [**10 marks**] Consider the following recurrence defining a function $f : \mathbb{N} \to \mathbb{N}$.

$$f(n) = \begin{cases} 2 & \text{if } 0 \le n \le 2; \\ f(n-1) + f(n-2) + f(n-3) & \text{if } n > 2. \end{cases}$$

Identify all the faults in the following *bad* proof that proves $f(n) < 2^n$, for every integer $n \ge 2$. For each fault, circle it and briefly explain what is wrong.

PROOF:
We define a predicate P on integers greater than or equal to 2.
$P(n) : f(n) < 2^n$, for every integer $n \ge 2$. Quantifying predicate variable
BASIS: Let $n = 2$. More base cases
Then by definition of f, $f(n) = 2$.
Since $2 < 2^2$, therefore $f(n)$ holds as wanted. $2 \le n-1, n-2, n-3 < n$
INDUCTION STEP: Let $n > 2$. $n \ge 2$
Suppose $P(j)$ holds whenever $2 \le j < n$. [IH] $P(n) \to P(n+1)$
WTP: $P(n)$ holds.

✓ no stacking

$$\begin{array}{rcll} f(n) & < & 2^n & \leftarrow \text{started with WTP statement} \\ f(n-1)+f(n-2)+f(n-3) & < & 2^n & \text{[definition of } f\text{]} \\ 2^{n-1}+2^{n-2}+2^{n-3} & < & 2^n & \text{[by IH]} \\ \frac{7}{8}2^n & < & 2^n & \text{[factor out } 2^n\text{]} \\ \frac{7}{8} & < & 1 & \text{[divide by } 2^n\text{]} \end{array}$$

P(n)
Therefore ~~$f(n) < 2^n$~~ as wanted.
By Principle of Complete Induction, $f(n) < 2^n$ for every integer $n \ge 2$. ☐

2a.

$$a_n = \begin{cases} 2 & \text{if } n = 0; \\ 16 & \text{if } n = 1; \\ 7a_{n-2} + 6a_{n-1} + 3^n & \text{if } n > 1; \end{cases}$$

$\forall n \in \mathbb{N}$, $P(n)$: $a_n \le 3 \cdot 7^n$

Base case: $n = 0$, $n = 1$

$$\begin{aligned} a_2 &= 7a_0 + 6a_1 + 3^2 \\ &= 7 \cdot 2 + 6 \cdot 16 + 9 \quad = 119 \\ &\le 3 \cdot 7^2 \\ &= 149 \end{aligned}$$

Let $n > 1$
I.S.: Assume $P(n)$ holds for $0 \le j < n$ [I.H]

$$\begin{aligned} a_n &= 7a_{n-2} + 6a_{n-1} + 3^n \\ &\le 7(3 \cdot 7^{n-2}) + 6(3 \cdot 7^{n-1}) + 3^n \quad \text{[I.H]} \quad 0 \le n-2, n-1 < n \\ &= 21 \cdot \frac{7^n}{49} + 18 \cdot \frac{7^n}{7} + 3^n \\ &= \frac{3}{7} \cdot 7^n + \frac{18}{7} \cdot 7^n + 3^n \\ &= 7^n \left(\frac{3+18}{7}\right) + 3^n \\ &= 3 \cdot 7^n + 3^n \end{aligned}$$

$\therefore a_n \le 3 \cdot 7^n + 3^n$
$\therefore$ False!

2b. $Q(n)$: $a_n \le 3 \cdot 7^n - 3^n$