A37 Mar 29 Lec 1 Notes

Theorem 7.28: Integral Test (IT)

If $f(x)$ is positive, continuous, and dereasing on $[1,\infty)$ and $f(n) = a_n$, $\forall n \in \mathbb{N}$, then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \iff \int_1^\infty a(x)\,dx \text{ conv.}$$

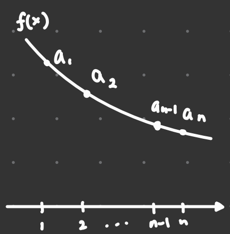
Proof:

$(\Rightarrow)$:

Suppose $f(x) > 0$ on $[1,\infty)$
$f(x)$ decreasing on $[1,\infty)$
$a_n = f(n)$ $\forall n \in \mathbb{N}$
f is cont. on $[1,\infty)$

WTS $\sum_{n=1}^{\infty} a_n$ converges $\iff \int_1^\infty f(x)\,dx$ converges

e.g.



Let $n \in \mathbb{N}$ be arbitrary

$$\int_1^n f(x)\,dx \le a_1(1) + a_2(1) + \dots + a_{n-1}(1) \quad \text{Left hand riemann sum}$$

$$= a_1 + a_2 + \dots + a_{n-1} \quad **$$

and $a_2(1) + a_3(1) + \dots + a_n(1) \le \int_1^n f(x)\,dx$ $*$ Right hand riemann sum

$(\Leftarrow)$:

Suppose $\int_1^\infty f(x)\,dx$ converges.

WTS $\sum_{n=1}^{\infty} a_n$ converges

$* \Rightarrow a_1 + a_2 + \dots + a_n \le a_1 + \int_1^n f(x)\,dx$

$S_n \le a_1 + \int_1^n f(x)\,dx$

Proof Continued...

$$S_n \le \underbrace{a_1 + \overbrace{\int_1^n f(x)\,dx}^{<\int_1^\infty f(x)\,dx}}_{>0}$$

$$< \underbrace{a_1 + \int_1^\infty f(x)\,dx}_{=: M}$$

So $\forall n \in \mathbb{N}$, $S_n < M$ for some $M \in \mathbb{R}^+$

i.e. $\{S_n\}$ is bounded above.

Moreover, $S_{n+1} = S_n + a_{n+1}$ By def of S_{n+1}

$> S_n$

$\therefore \forall n \in \mathbb{N}$, $S_{n+1} > S_n$

i.e. $\{S_n\}$ is increasing

$\therefore$ By BMCT, $\{S_n\}$ converges.

i.e. $\lim_{n\to\infty} S_n$ exists

i.e. $\sum_{n=1}^{\infty} a_n$ Converges.

Ex 1

$\sum_{n=3}^{\infty} ne^{-n}$ converge or diverge?

Proof:

Let $a_n = f(n) = \frac{n}{e^n}$ $\forall n \in \mathbb{N}, n \ge 3$

So $f(x) = xe^{-x}$ on $[3, \infty)$

On $[3, \infty)$, $f(x) = xe^{-x} > 0$

On $[3,\infty)$, $f'(x) = e^{-x} + -e^{-x}x$

$= \underset{+}{e^{-x}}\underset{-}{(1-x)}$ on $[3,\infty)$

< 0

$\therefore$ $f(x)$ is decreasing on $[3,\infty)$

Consider $\int_3^\infty f(x)\,dx = \int_3^\infty xe^{-x}\,dx$

$= \lim_{A\to\infty} \int_3^A xe^{-x}\,dx$ type I

$= 4/e^3$

$\therefore \int_3^\infty xe^{-x}\,dx$ converges

$\therefore$ By IT, $\sum_{n=3}^{\infty} ne^{-n}$ converges

$\blacksquare$

Def (pg 619): p-Series

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where $p \in \mathbb{R}^+$ is a p-series

The number p is called the p-value

e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$ ✓ $p=1$

e.g. $\sum_{n=2}^{\infty} \frac{1}{n^{7.8+\pi}}$ ✓ $p = 7.8+\pi$

e.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ✓ $p=\frac{1}{2}$

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ ✗

Proof:

Let $a_n = f(n) \quad \forall n \in \mathbb{N}$
$= n^{-p}$

So $f(x) = x^{-p}$ on $[1,\infty)$

On $[1,\infty)$, $f(x) = \frac{1}{x^p} > 0$

On $[1,\infty)$, $f'(x) = -px^{-p-1} < 0$

$$\int_1^{\infty} x^{-p}\,dx = \lim_{A\to\infty} \int_1^A x^{-p}\,dx = \begin{cases} \text{conv} & \text{if } p>1 \\ \text{div} & \text{if } 0<p\leq 1 \end{cases}$$

$\therefore$ By IT, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\rightarrow$ conv if $p>1$; $\rightarrow$ div if $0<p\leq 1$