Def 3.2.1: Subspaces of $\mathbb{R}^n$

A subset W of the vector space $\mathbb{R}^n$ is called a (linear) subspace of $\mathbb{R}^n$ if it has the following three properties:

(i) W contains the zero vector in $\mathbb{R}^n$
(ii) W is closed under addition.
(iii) W is closed under scalar multiplication.

Theorem 3.2.2: Image and kernel are subspaces

If $T(\vec{x}) = A\vec{x}$ is a L.T. from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

(i) $\ker T = \ker A$ is a subspace of $\mathbb{R}^m$, and
(ii) $\operatorname{img} T = \operatorname{img} A$ is a subspace of $\mathbb{R}^n$

Ex 1

Is $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$?

W consists of all vectors in the first quadrant of the xy plane.

W is closed under addition

W is not closed under scalar multiplication with a negative scalar.

Thus W is not a subspace of $\mathbb{R}^2$.

Ex 2

Consider the plane V in $\mathbb{R}^3$ given by the equation $x_1 + 2x_2 + 3x_3 = 0$

(a) Find a matrix A s.t. $V = \ker(A)$

the equation $x_1 + 2x_2 + 3x_3 = 0$ is equivalent to $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Thus $V = \ker \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(b) Find a matrix B s.t. $V = \operatorname{img}(B)$

Ex 2b continued:

Since the img of a matrix is the span of its columns, we need to describe V as the span of some vectors. For the plane V, any two non parallel vectors will do.

Choose $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Thus $V = \text{im}\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex 3

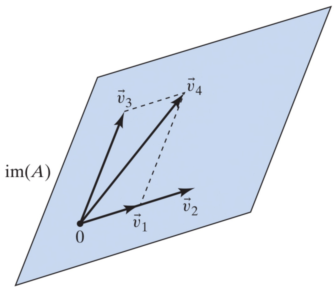
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Find vectors in $\mathbb{R}^3$ that span img A. What is the smallest number of vectors needed to span img A?

From theorem 3.1.3, img A is spanned by the four column vectors of A.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$



We observe that $\vec{v}_2 = 2\vec{v}_1$ and $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$.
Then we have,

$$\text{im}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$$

$$= \underbrace{\text{span}(\vec{v}_1, \vec{v}_3)}_{\text{Basis}}$$

im A cannot be spanned by one vector alone.

If a vector $\vec{v}$ is in span $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, then

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\ &= c_1\vec{v}_1 + c_2(2\vec{v}_1) + c_3\vec{v}_3 + c_4(\vec{v}_1 + \vec{v}_3) \\ &= (c_1 + 2c_2 + c_4)\vec{v}_1 + (c_3 + c_4)\vec{v}_3 \end{aligned}$$

Thus $\vec{v}$ is in span $(\vec{v}_1, \vec{v}_3)$

Def 3.2.3: Redundant vectors ; linear independence ; basis

Consider vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$.

Note: $\vec{v}_1$ is redundant if it is $\vec{0}$.

(a) We say that a vector $\vec{v}_i$ in the list $\vec{v}_1, \dots, \vec{v}_m$ is redundant if $\vec{v}_i$ is a linear combination of the preceding vectors $\vec{v}_1, \dots, \vec{v}_{i-1}$.

(b) The vectors $\vec{v}_1, \dots, \vec{v}_m$ are called linearly independent if none of them is redundant. Otherwise, the vectors are called linearly dependent (meaning that at least one of them is redundant)

(c) We say that the vectors $\vec{v}_1, \dots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$ form a basis of V if they span V and are linearly independent.

Theorem 3.2.4: Basis of the image

To construct a basis of the image of a matrix A, list all the column vectors of A, and omit the redundant vectors from this list.

Ex 4

Are the following vectors in $\mathbb{R}^7$ linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

$\vec{v}_1$ is non redundant since $\vec{v}_1$ is non zero

$\vec{v}_2$ is non redundant since $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

$0 \neq k1$, $\forall k \in \mathbb{R}$ at 4th component

$\vec{v}_3$ is non redundant b/c $\vec{v}_3$ is not a linear comb. of $\vec{v}_1$ and $\vec{v}_2$

$\vec{v}_4$ is non redundant as $\vec{v}_4$ is not a linear comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Thus the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, and $\vec{v}_4$ are linearly independent.

Theorem 3.2.5: Linear Independence and zero components

Consider vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. If $\vec{v}_1$ is nonzero, and if each of the vectors $\vec{v}_i$ (for $i \geq 2$) has a nonzero entry in a component where all the preceding vectors $\vec{v}_1, \dots, \vec{v}_{i-1}$ have a 0, then the vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

Ex 5

Are the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ linearly independent?

$\vec{v}_1$ is nonredundant because $\vec{v}_1$ is nonzero.

$\vec{v}_2$ is nonredundant because $\vec{v}_2$ is not a scalar multiple of $\vec{v}_1$.

To see if $\vec{v}_3$ is nonredundant, we need to see if $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

In other words, whether $\vec{v}_3 = c_1 \vec{v}_1 + c_2 \vec{v}_2$.

$$M = \left[\begin{array}{cc:c} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}\right] \Rightarrow \text{rref}(M) = \left[\begin{array}{cc:c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = 1, \ c_2 = 2$

$\Rightarrow \vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$

$\Rightarrow \vec{v}_3$ is redundant

Thus $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are linearly dependent.

The linear relation of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ is:

$$\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$$

Def 3.2.6: Linear Relations

Consider the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. An equation of the form

$$c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$$

is called a linear relation among the vectors $\vec{v}_1, \dots, \vec{v}_m$. There is always the trivial relation, with $c_1 = \dots = c_m = 0$. Nontrivial relations (where at least one coefficient c_i is nonzero) may or may not exist among the vectors $\vec{v}_1, \dots, \vec{v}_m$.

Theorem 3.2.7: Relations and linear dependence

The vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ are linearly dependent iff there are nontrivial relations among them.

Ex 6

Suppose the column vectors of an $n \times m$ matrix A are linearly independent. Find the Kernel of matrix A.

We need to solve:

$$A\vec{x} = \vec{0} \quad \text{or} \quad \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0} \quad \text{or} \quad x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}$$

We see that finding $\ker A$ is the same as finding the relations among column vectors of A.

By Theorem 3.2.7, there is only the trivial relation with $x_1 = \dots = x_m = 0$.
Thus $\ker A = \{\vec{0}\}$

Theorem 3.2.8: Kernel and Relations

The vectors in the kernel of an $n \times m$ matrix A correspond to the linear relations among the column vectors $\vec{v}_1, \dots, \vec{v}_m$ of A: The equation

$$A\vec{x} = \vec{0} \quad \text{means that} \quad x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}$$

In particular, the column vectors of A are linearly independent iff $\ker A = \{\vec{0}\}$, or, equivalently, if $\text{rank}(A) = m$. This condition implies that $m \le n$.

Thus we can find at most n linearly independent vectors in $\mathbb{R}^n$.

Summary 3.2.9: Various characterizations of linear independence

For a list $\vec{v}_1, \dots, \vec{v}_m$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

(i) Vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.
(ii) None of the vectors $\vec{v}_1, \dots, \vec{v}_m$ is redundant, meaning that none of them is a linear combination of preceding vectors.
(iii) None of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$ in the list.
(iv) There is only the trivial relation among the vectors $\vec{v}_1, \dots, \vec{v}_m$, meaning that the equation $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ has only the solution $c_1 = \dots = c_m = 0$
(v) $\ker \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \{\vec{0}\}$

(vi) $\text{rank} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = m$

Theorem 3.2.10: Basis and Unique Representation

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a basis of V iff every vector $\vec{v}$ in V can be expressed uniquely as a linear combination

$$\vec{v} = c_1 \vec{v}_1 + \ldots + c_m \vec{v}_m$$