CH 7.1 Diagonalization

Def 7.1.1: Diagonalizable Matrices

Consider a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then A is said to be diagonalizable if the matrix B of T with respect to some basis is diagonal.

By theorem 3.4.4 and definition 3.4.5, matrix A is diagonizable iff A is similar to some diagonal matrix B, meaning that there exists an invertible matrix S s.t. $S^{-1}AS = B$ is diagonal.

To diagonalize a square matrix A means to find an invertible matrix S and a diagonal matrix B s.t. $S^{-1}AS = B$.

Theorem 3.4.7: When is the $\mathfrak{B}$-matrix of T diagonal?

Consider a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ be a basis of $\mathbb{R}^n$.

Then the $\mathfrak{B}$-matrix B of T is diagonal iff $T(\vec{v}_1) = c_1\vec{v}_1, \dots, T(\vec{v}_n) = c_n\vec{v}_n$ for some $c_i \in \mathbb{R}$.

From a geometrical point of view, this means that $T(\vec{v}_j)$ is parallel to $\vec{v}_j$ for all $j = 1, \dots, n$.

From theorem 3.4.7, we have:

$$B = \begin{array}{cc} \begin{array}{cccc} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{array} & \\ \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} & \begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{array} \end{array}$$

If we wish to diagonalize A, we need to find a basis of $\mathbb{R}^n$ consisting of vectors $\vec{v}$ s.t.

$$A\vec{v} = \lambda\vec{v}$$

Def 7.1.2: Eigenvectors, Eigenvalues, and Eigenbases

Consider a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$

A nonzero $\vec{v}$ in $\mathbb{R}^n$ is called an eigenvector of A (or T) if

$$A\vec{v} = \lambda\vec{v}$$

for some scalar λ. This λ is called the eigenvalue associated with eigenvector $\vec{v}$.

A basis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{R}^n$ is called an eigenbasis for A if the vectors $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors of A, meaning that $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda\vec{v}_n$ for $\lambda_i \in \mathbb{R}$.

A nonzero vector in $\mathbb{R}^n$ is an eigenvector of A if $A\vec{v}$ is parallel to $\vec{v}$.

If $\vec{v}$ is an eigenvector of matrix A, with an associated eigenvalue λ, then $\vec{v}$ is an eigenvector of $A^2, A^3, \dots$ as well, with

$$A^2\vec{v} = \lambda^2\vec{v}, \quad A^3\vec{v} = \lambda^3\vec{v}, \dots, A^m\vec{v} = \lambda^m\vec{v}$$

Proof by induction:

Base case:

$$A^1\vec{v} = A\vec{v} = \lambda\vec{v} = \lambda^1\vec{v} \quad \checkmark$$

IS:

$$\begin{aligned} A^{m+1}\vec{v} &= A(A^m\vec{v}) \\ &= A(\lambda^m\vec{v}) && \text{I.H} \\ &= \lambda^m A(\vec{v}) && \text{By linearity of } A \\ &= \lambda^m \cdot \lambda^1\vec{v} && \text{Base case} \\ &= \lambda^{m+1}\vec{v} \end{aligned}$$

▨

Theorem 7.1.3: Eigenbases and diagonalization

A is diagonalizable iff there exists an eigenbases for A.

If $\vec{v}_1, \dots, \vec{v}_n$ is an eigenbases for A, with $A\vec{v}_1 = \lambda_1\vec{v}_1, \dots, A\vec{v}_n = \lambda_n\vec{v}_n$, then the matrices

$$S = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

will diagonalize A, meaning that $S^{-1}AS = B$

Conversely, if the matrices S and B diagonalize A, then the column vectors of S will form an eigen bases for A, and the diagonal entries of B will be the associated eigen values.

Proof (7.1.3): ($\Rightarrow$)

Suppose there exists an eigen bases $\vec{v}_1, \dots, \vec{v}_n$ for A.

Then,

$$AS = A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \quad \text{def of } S$$

$$= \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= SB \quad \boxtimes$$

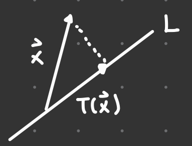
Ex 2:

Find all eigenvectors and eigenvalues of I_n.

Since $I_n\vec{v} = \vec{v} = 1\vec{v}$, $\forall \vec{v} \in \mathbb{R}^n$, all nonzero vectors of $\mathbb{R}^n$ are eigenvectors of I_n, with $\lambda = 1$.

Thus all bases for $\mathbb{R}^n$ are eigenbases for I_n

Clearly, I_n is diagonalizable.



Ex 3:

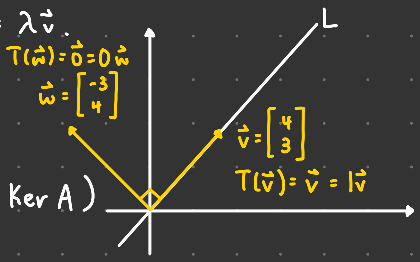
Consider the L.T. $T(\vec{x}) = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}\vec{x}$. $T(\vec{x})$ represents the orthogonal projection onto the line $L = \text{Span}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Describe the eigenvectors of A geometrically and find all eigenvalues of A.

We want to find any nonzero $\vec{v} \in \mathbb{R}^2$ s.t $T(\vec{v}) = A\vec{v} = \lambda\vec{v}$.

Any $\vec{v}$ parallel to L will give: $A\vec{v} = \vec{v} = 1\vec{v}$.

Any $\vec{w}$ pependicular to L will give: $A\vec{w} = \vec{0} = 0\vec{w}$. (Note: $\vec{w} \in \text{Ker } A$)



The eigenvalues are 1 and 0.

Ex 3 continued.. :

Thus $\mathcal{B} = (\vec{v}, \vec{w})$ will be an eigenbasis for A, and $\mathcal{B}$-matrix B of T will be the diagonal matrix.

$$B = \begin{array}{cc} T(\vec{v}) & T(\vec{w}) \\ \end{array}$$
$$B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]\begin{array}{c} \vec{v} \\ \vec{w} \end{array}$$

Thus the matrices $S = [\vec{v}\ \vec{w}] = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ will diagonalize A.

Note:

$S^{-1}AS = B$ is true here by thm 7.1.3.

Ex 4:

Let $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ be the rotation through an angle of 90° in Counterclockwise.

If $\vec{v}$ is any nonzero vector in $\mathbb{R}^2$, then $T(\vec{v}) = A\vec{v}$ fails to be parallel to $\vec{v}$. (it's perpendicular)

Thus there are no real eigenvectors and eigenvalues.

eigenvalues can also be complex

By def, 0 is an eigenvalue of A if there exists a nonzero vector $\vec{v}$ in $\mathbb{R}^n$ s.t. $A\vec{v} = 0\vec{v} = \vec{0}$.

Thus there exists a nonzero vector in Ker A. Thus 0 is an eigenvalue of A iff $\text{Ker } A \neq \{\vec{0}\}$, meaning that A is non-invertible.