CH 7.2 Finding Eigenvalues

By def 7.1.2, we have

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$A\vec{v} - \lambda I_n\vec{v} = \vec{0}$$

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

$$\therefore \ker(A - \lambda I_n) \neq \{\vec{0}\}$$

Theorem 7.2.1: Eigenvalues and determinants; characteristic equation

Consider if an $n \times n$ matrix A and a scalar λ. Then λ is an eigenvalue of A iff

$$\det(A - \lambda I_n) = 0$$

This is called the characteristic equation of matrix A.

Ex 1:

Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(3-\lambda) - 2 \cdot 4$$

$$= \lambda^2 - 4\lambda - 5$$

$$= (\lambda - 5)(\lambda + 1)$$
$$= 0$$

$\therefore \lambda_1 = 5$; $\lambda_2 = -1$ are the eigenvalues of A.

Ex 2:

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda)(4-\lambda)$$

$$= 0$$

$\therefore$ Eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$

Theorem 7.2.2: Eigenvalues of a triangular matrix

The Eigenvalues of a triangular matrix are its diagonal entries.

Ex 3:

Find the characteristic equation for $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det(A - \lambda I_2) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$= (a-\lambda)(d-\lambda)$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc)$$

$$= 0$$

Definition 7.2.3: Trace

The sum of the diagonal entries of a square matrix A is called the trace of A, denoted by $\operatorname{tr} A$.

Theorem 7.2.4: Characterization of a 2×2 matrix A

$$\det(A - \lambda I_2) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

If A is a 3×3 matrix, what does the Characteristic equation $\det(A-\lambda I_3)=0$ look like?

permutations of entry, and Laplace expansion

$$\det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) + (\text{a polynomial of degree} \le 1)$$

$$= (\lambda^2-(a_{11}+a_{12})\lambda + a_{11}a_{22})(a_{33}-\lambda) + \quad ''$$

$$= -\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 + \quad ''$$

$$= -\lambda^3 + (\operatorname{tr}A)\lambda^2 + \quad ''$$

$$= 0$$

Def 7.2.6: Algebraic multiplicity of an eigenvalue

We say that an eigenvalue λ_0 of a square matrix A has algebraic multiplicity K if λ_0 is a root of multiplicity K of the characteristic polynomial $f_A(\lambda)$, meaning that:

$$f_A(\lambda) = (\lambda_0-\lambda)^K g(\lambda)$$

for some polynomial $g(\lambda)$ with $g(\lambda_0)\neq 0$. We write $\operatorname{almu}(\lambda_0)=K$

Theorem 7.2.7: Number of eigenvalues

An $n\times n$ matrix has at most n real eigenvalues, even if they are counted with their algebraic multiplicities.

If n is odd, then an $n\times n$ matrix has at least one real eigenvalue.

Ex 4:

If n is even, $A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does not have any real eigenvalues.

$$f_A(\lambda) = \det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1$$

Theorem 7.2.8: Eigenvalues, determinant, and Trace

If an $n\times n$ matrix A has the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, listed with their algebraic multis, then

$$\det A = \lambda_1\lambda_2\cdots\lambda_n$$

and

$$\operatorname{tr} A = \lambda_1+\lambda_2+\ldots+\lambda_n$$