B24 May 7 Lec 2 Notes

Linear algebra is about linear transformations which take place on vector spaces.

Def:

A real vector space is a set V together with two operations $\begin{matrix} + : V \times V \to V \\ \cdot : \mathbb{R} \times V \to V \end{matrix}$ s.t.

(i) $v+w = w+v$, $\forall v,w \in V$ Commutative

(ii) $(u+v)+w = u+(v+w)$, $\forall v,w,u \in V$ Associative

(iii) $\exists\, 0 \in V$ s.t. $0+v = v$, $\forall v \in V$

(iv) $\forall v \in V$, $\exists\, w \in V$ s.t. $v+w = 0$

(v) $1 \cdot v = v$, $\forall v \in V$

(vi) $a(bv) = (ab)v$, $\forall v \in V$, $\forall a,b \in \mathbb{R}$

(vii) $k(u+v) = ku + kv$, $\forall u,v \in V$, $\forall k \in \mathbb{R}$

(viii) $(a+b)\cdot v = av + bv$, $\forall v \in V$, $\forall a,b \in \mathbb{R}$

Ex 1:

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R} \}$$

Def:

A complex v.s. is defined exactly as above but with $\mathbb{C}$ replaced with $\mathbb{R}$.

Ex 2:

$\mathbb{C}^n := \{ (z_1, \dots, z_n), z_1, \dots, z_n \}$ is a complex v.s.

Ex 3:

$$M^{\mathbb{F}}_{m \times n} := \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \;\middle|\; a_{ij} \in \mathbb{F},\ \text{for } 1 \le i \le m,\ 1 \le j \le n \right\}$$

Ex 4:

$$P_n := \{ a_n x^n + \dots + a_1 x + a_0 \mid a_n, \dots, a_0 \in \mathbb{F} \}$$

Ex 5:

$$C([0,1]) := \{ f : [0,1] \to \mathbb{F} \mid f \text{ is continuous} \}$$

Basis

Recall that $e_1 := (1,0,\dots 0)$

$\vdots$

$e_n := (0,\cdots,0,1)$

forms a basis for $\mathbb{R}^n$

Def:

Let $v_1,\dots,v_n \in V$ be vectors in a v.s. V. Then a linear combination of $v_1,\dots,v_n$ is an expression of the form

$$\alpha_1 v_1 + \dots + \alpha_n v_n \quad , \text{for some } \alpha_n \in \mathbb{F}$$

Def:

Vectors $v_1,\dots,v_n \in V$ are called a basis for V if for every $v \in V$, there exists unique $\alpha_1,\dots \alpha_n \in \mathbb{F}$ s.t.

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

in which case $\alpha_1,\dots,\alpha_n \in \mathbb{F}$ are called the coordinate of v.

Def:

Vectors $v_1,\dots,v_n \in V$ are called a spanning set if $\forall v \in V$, $\exists\, \alpha_1,\dots\alpha_n \in \mathbb{F}$ s.t.

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

Def:

Vectors $v_1,\dots,v_n \in V$ are said to be linearly independent if

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Theorem:

Let V be a v.s. Then $v_1,\dots,v_n \in V$ form a basis for V iff $v_1,\dots,v_n$ are L.I. and form a spanning set.

Theorem:

Let V be a v.s. with spanning set $v_1,\dots,v_n$. Then a subset of $\{v_1,\dots,v_n\}$ is a basis for V.

Def:

Let V, W be v.s. A function $T: V \to W$ is called a linear transformation if

(i) $T(u+v) = T(u) + T(v)$, $\forall u, v \in V$
(ii) $T(\alpha v) = \alpha T(v)$, $\forall v \in V$, $\forall \alpha \in \mathbb{F}$