Sec 3.1 Reading

Def 3.1.1: Image of a Function

The image of a function consists of all the values the function takes in its target space. If f is a function from X to Y, then

$$\text{image}(f) = \{ f(x) : x \text{ in } X \}$$
$$= \{ b \text{ in } Y : b = f(x), \text{ for some } x \text{ in } X \}$$

Ex 1

Describe the image of the L.T.

$$T(\vec{x}) = A\vec{x} \quad \text{from } \mathbb{R}^2 \text{ to } \mathbb{R}^3, \quad \text{where } A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

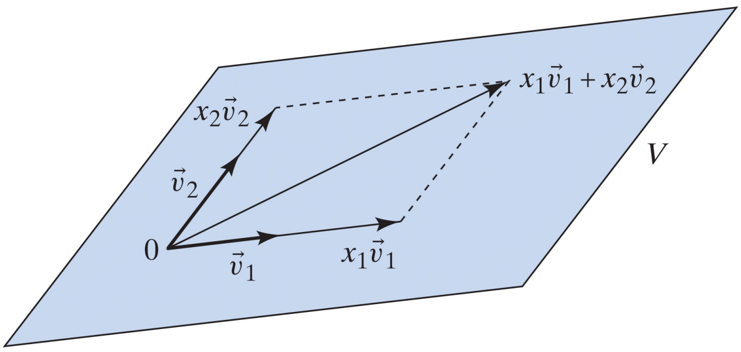
The image of T consists of all vectors of the form

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

that is, all linear combinations of the column vectors of A,

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Image(T) is the plane V spanned by $\vec{v}_1$ and $\vec{v}_2$.



Def 3.1.2: Span

Consider the vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$. The set of all linear combinations $c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ of the vectors $\vec{v}_1, \dots, \vec{v}_m$ is called their span :

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \{ c_1\vec{v}_1 + \dots + c_m\vec{v}_m \mid c_1, \dots, c_m \in \mathbb{R} \}$$

Theorem 3.1.3: Image of a L.T.

The image of a L.T. $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A. We denote the image of T by im(T) or im(A).

Proof:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$$

This shows that the image(T) consists of all linear comb. of the column vectors $\vec{v}_1, \dots, \vec{v}_m$ of matrix A.

Thus im(T) is the span of the vectors $\vec{v}_1, \dots, \vec{v}_m$

Theorem 3.1.4: Properties of the Image

The image of a L.T. $T: \mathbb{R}^m \to \mathbb{R}^n$ has the following properties.

(i) $\vec{0} \in \mathbb{R}^n$ is in the image of T.
(ii) The image of T is closed under addition. If $\vec{v}_1$ and $\vec{v}_2$ are in the image of T, then so is $\vec{v}_1 + \vec{v}_2$.
(iii) The image of T is closed under scalar multiplication. If $\vec{v} \in \text{img}\,T$ and $k \in \mathbb{R}$, then $k\vec{v} \in \text{img}\,T$.

Proof

(i) $\vec{0} = A\vec{0} = T(\vec{0})$
(ii) There exist vectors $\vec{w}_1$ and $\vec{w}_2$ in $\mathbb{R}^m$ s.t. $\vec{v}_1 = T(\vec{w}_1)$ and $\vec{v}_2 = T(\vec{w}_2)$. Then $\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) = T(\vec{w}_1 + \vec{w}_2)$, so that $\vec{v}_1 + \vec{v}_2$ is in the img T.
(iii) If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) = T(k\vec{w})$

Ex 2

Consider an $n \times n$ matrix A. Show that $\operatorname{im}(A^2)$ is a subset of $\operatorname{img} A$.

Let $\vec{b} = A^2 \vec{v} = AA\vec{v}$.

$\vec{b} = A(A\vec{v})$
$\quad = A\vec{w} \quad , \quad \vec{w} = A\vec{v}$

Thus $\vec{b}$ is in the $\operatorname{img} A$.

Def 3.1.5: Kernel

The kernel of a L.T. $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ consists of all zeroes of the transformation, that is, the solutions of the equation $T(\vec{x}) = A\vec{x} = \vec{0}$.

Note:

For a L.T. $T: \mathbb{R}^m \to \mathbb{R}^n$,

(i) $\operatorname{im} T = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^m \}$ is a subset of the target space $\mathbb{R}^n$ of T
(ii) $\ker T = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$ is a subset of the domain $\mathbb{R}^m$ of T.

Theorem 3.1.6: Properties of the Kernel

Consider a L.T. $T: \mathbb{R}^m \to \mathbb{R}^n$

(i) The zero vector $\vec{0}$ in $\mathbb{R}^m$ is in $\ker T$
(ii) The kernel is closed under addition.
(iii) The kernel is closed under scalar multiplication

Theorem 3.1.7: When is $Ker(A) = \{\vec{0}\}$?

(i) Consider an $n \times m$ matrix A. Then $Ker(A) = \{\vec{0}\}$ iff $rank(A) = m$.

(ii) Consider an $n \times m$ matrix A. If $Ker(A) = \{\vec{0}\}$, then $m \leq n$. Equivalently, if $m > n$, then there are nonzero vectors in $Ker(A)$.

(iii) For a square matrix A, we have $Ker(A) = \{\vec{0}\}$ iff A is invertible.

Note:

For an $n \times n$ matrix A, the following statements are equivalent.

(i) A is invertible
(ii) The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$
(iii) $rref(A) = I_n$
(iv) $rank(A) = n$
(v) $im(A) = \mathbb{R}^n$
(vi) $Ker(A) = \{\vec{0}\}$