Sec 2.1 Reading

Def 2.1.1: Linear Transformations[2]

A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if there exists an $n \times m$ matrix A s.t.

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x}$ in the vector space $\mathbb{R}^m$.

In linear algebra, the linear functions of m variables are those of the form

$$y = c_1 x_1 + c_2 x_2 + \dots + c_m x_m$$

A function with a constant term is called affine.

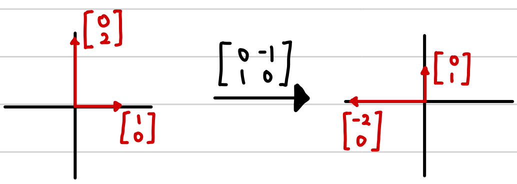
Ex 1

Consider the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\2\end{bmatrix}$. Show the effect of the linear transformation

$$T(\vec{x}) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\vec{x}$$

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$



(The L is rotated 90° counterclockwise)

Ex 2

Consider the linear transformation $T(\vec{x}) = A\vec{x}$, with $A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$. Find $T\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\0\\1\end{bmatrix}$.

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\4\\7\end{bmatrix}$$

1st col.

$$T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\6\\9\end{bmatrix}$$

3rd col.

Theorem 2.1.2: The columns of the matrix of a linear transformation

Consider a linear transformation T from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then, the matrix of T is

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}, \text{ where } \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith}$$

To justify Theorem 2.1.2.

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$$

Then

$$T(\vec{e}_i) = A\vec{e}_i = \begin{bmatrix} | & | & & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_i & \cdots & \vec{v}_m \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i$$

The standard vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ are often denoted by $\vec{i}, \vec{j}, \vec{k}$.

Ex 3

Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$.

(a) What is the relationship among $T(\vec{v})$, $T(\vec{w})$, and $T(\vec{v}+\vec{w})$, where $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^m$?

(b) What is the relationship between $T(\vec{v})$ and $T(k\vec{v})$, where $\vec{v}$ is a vector in $\mathbb{R}^m$ and k is a scalar?

Theorem 1.3.10: Linear Transformations

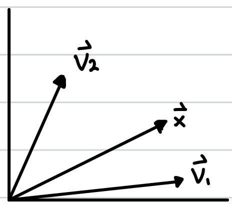
Let $A_{n \times m}$ be a matrix, $\vec{x}, \vec{y}$ in $\mathbb{R}^m$, $k \in \mathbb{R}$

(i) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$, $\forall\, \vec{x}, \vec{y}$ in $\mathbb{R}^m$

(ii) $A(k\vec{x}) = k\, A\vec{x}$, $\forall\, \vec{x} \in \mathbb{R}^m$, $\forall\, k \in \mathbb{R}$

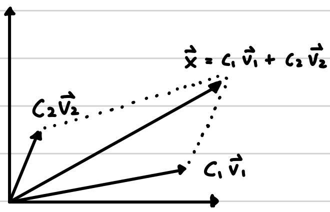
Ex 4

Consider a linear trans. T from $\mathbb{R}^2$ to $\mathbb{R}^3$ s.t. $T(\vec{v}_1) = \frac{1}{2}\vec{v}_1$ and $T(\vec{v}_2) = 2\vec{v}_2$. Sketch $T(\vec{x})$ for the given $\vec{x}$.



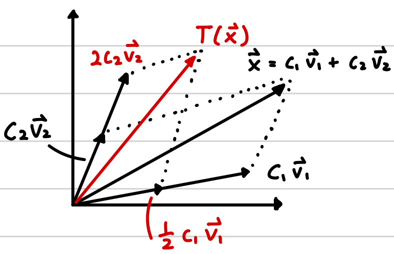
Using a parallelogram, we can represent $\vec{x}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$



By Theorem 2.1.3,

$$\begin{aligned} T(\vec{x}) &= T(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) \\ &= \tfrac{1}{2} c_1\vec{v}_1 + 2c_2\vec{v}_2 \end{aligned}$$



Def 2.1.4: Distribution vectors and transition matrices

A vector $\vec{x}$ in $\mathbb{R}^n$ is said to be a distribution vector if its components add up to 1 and all the components are positive or zero.

A square matrix A is said to be a transition matrix (or stochastic matrix) if all its columns are distribution vectors.

If A is a transition matrix and $\vec{x}$ is a distribution vector, then $A\vec{x}$ will be a distribution vector as well.