CH 6.1 Intro to Determinants

Let $A = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix}$.

A fails to be invertible if img A isn't all of $\mathbb{R}^3$, meaning that $\vec{u}, \vec{v}$, and $\vec{w}$ are contained in some plane V.
In this case, $\vec{v} \times \vec{w}$, being perpendicular to V, is perpendicular to $\vec{u}$, so that

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$$

Def 6.1.1: The determinant of a 3x3 matrix

If $A = [\vec{u} \ \vec{v} \ \vec{w}]$, then

$$\det A = \vec{u} \cdot (\vec{v} \times \vec{w})$$

A 3x3 matrix A is invertible iff $\det A \neq 0$

Theorem 6.1.2: Sarrus's rule

To find $\det A_{3\times3}$.

$$\left[\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right] \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$- \quad - \quad - \quad + \quad + \quad +$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Ex 5

Is $F(A) = \det A$ from the linear space $\mathbb{R}^{3\times3}$ to $\mathbb{R}$ a linear transformation?

No

$F(I_3 + I_3) = F(2I_3) = 8$

$F(I_3) + F(I_3) = 2 \neq 8$

Alternating property

Turns out that $\det B = -\det A$ can be obtained by swapping any two columns or any two rows.

$\det B = \det[\vec{u} \ \vec{w} \ \vec{v}] = \vec{u} \cdot (\vec{w} \times \vec{v}) = -\vec{u} \cdot (\vec{v} \times \vec{w}) = -\det[\vec{u} \ \vec{v} \ \vec{w}] = -\det A$

Determinant of an nxn matrix

We cannot generalize Sarrus's rule for an nxn matrix.

Note that each of the six terms in Sarrus's rule is the product of one entry from each row and column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

There are only n! permutations of these 'patterns' in an nxn matrix.

We have, $\det A = \sum \pm \text{prod } P$

The signs are related to the alternating property.

$$\det \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} = -\det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{31} & 0 & 0 \\ 0 & 0 & a_{23} \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{31} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \end{bmatrix} = a_{31} a_{12} a_{23} = a_{12} a_{23} a_{31}$$

Alteratively,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

0 inversion | 2 inversions | 2 inversions

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3 inversions | 1 inversion | 1 inversion

the number of inversions are the number of times one # out of 2 in a pattern is to the right and above the other.

Thus,

$$\det A = \sum (-1)^{(\#\text{ of inversions in } P)} \text{prod } P.$$

Theorem 6.1.4: Determinant of a triangular matrix.

The determinant of an upper or lower triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries