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W10 Lecture 18 Notes

Application of Differentiation

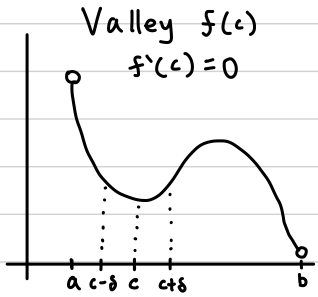
Local Relative Extrema

Let $f(x)$ be a function which is defined on open interval (a,b)

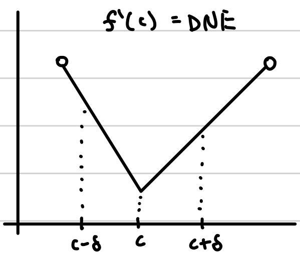
Definition 1

A function $f(x)$ has a local minimum (or relative minimum) at point $x=c$ if there exists some $\delta > 0$ such that $f(c) \le f(x)$ for all $x \in (c-\delta, c+\delta)$,

A function $f(x)$ has a local maximum (or relative maximum) at point $x=c$ if there exists some $\delta > 0$ such that $f(c) \ge f(x)$ for all $x \in (c-\delta, c+\delta)$



Local extrema at either peak or valley. These points are called critical.



Definition 2

A critical number (point) of a function $f(x)$ is the point $x=c$ in the domain of $f(x)$ such that either $f'(c) = 0$ or $f'(c) = DNE$

Examples.

1. Find critical points of $f(x) = (x-1)^{2/3}$

   1) $\text{Dom } f(x) = (-\infty, \infty)$
   2) $f'(x) = \dfrac{2}{3\sqrt[3]{x-1}}$
   3) Check where $f'(x) = 0$. $f'(x) \neq 0$ at any values of x.
   4) Check where $f'(x) = DNE$. $f'(x) = \infty$ at $x = 1 \in \text{Dom } f(x) \Rightarrow x_c = 1$ is critical point

Examples:

2. Find critical points of $f(x) = x^3 - 27x + 4$

1) Dom $f(x)$: $(-\infty, \infty)$
2) $f'(x) = 3x^2 - 27$
3) $f'(x) = 0$

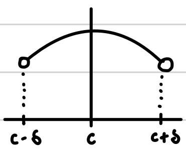
   $3x^2 - 27 = 0$ $\qquad$ $x_1, x_2 \in$ Dom f $\Rightarrow$ $x_{c_1} = 3$, $x_{c_2} = -3$ are critical points

   $x = \pm 3$
4) $f'(x) =$ DNE. No such points.

Theorem Fermat's Theorem for Local Extrema

If $f(x)$ has a local extremum at an interior point c and $f'(x)$ exists, then $f'(c) = 0$.

Proof: (For local max)



Given: $f(x)$ has a local max at $x=c$.

$f'(c)$ does exist.

Prove: $f'(c) = 0$

$$\left.\begin{aligned} &x \in [c, c+\delta) \\ &f(x) - f(c) \le 0 \\ &x - c > 0 \end{aligned}\right] \quad f'(c^+) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$$

$$\left.\begin{aligned} &x \in (c-\delta, c] \\ &f(x) - f(c) \le 0 \\ &x - c < 0 \end{aligned}\right] \quad f'(c^-) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0$$

We know that $f'(x)$ at $x=c$ does exist.

This means that $f'(c^+) = f'(c^-) = 0$

QED

The inverse of this theorem is not true.

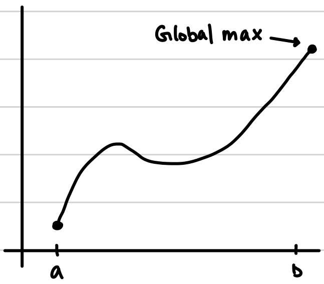
Global (Absolute) Extrema

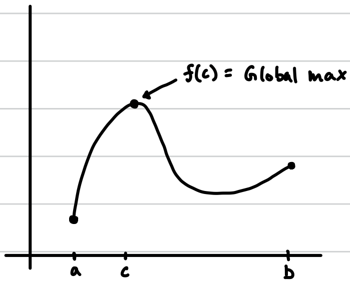
Let $f(x)$ be a function which is defined on closed interval $[a,b]$

Definition 3

A function $f(x)$ has a global maximum (or absolute maximum) at point $x=c$ if $f(c) \geq f(x)$ for all x in the domain of $f(x)$.

A function $f(x)$ has a global minimum (or absolute minimum) at point $x=c$ if $f(c) \leq f(x)$ for all x in the domain of $f(x)$.





Global extrema can be either at end points or critical points of $[a,b]$

Theorem The Extreme Value Theorem

If function $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum value and an absolute min. value at some numbers in $[a,b]$.

EVT doesn't work for discontinuous functions.

The Closed Interval Method

1) Find the values of $f(x)$ at the critical points of $f(x)$ in (a,b).
2) Find the values $f(a)$ and $f(b)$ at endpoints of the interval
3) The largest value from step 1 and 2 is the abs. max value.
   The smallest value from step 1 and 2 is the abs. min value.

Examples:

3. Find abs min and max of the function $f(x) = x^3 - 27x + 1$ on $[-1, 6]$

$f(x)$ is continuous on $\mathbb{R}$ and is defined on $[-1, 6]$, so by EVT it attains its global extrema on this interval.

① Critical points of $f(x)$ on $(-1, 6)$

$\underline{f'(x) = 0}$ $f'(x) = 3x^2 - 27 \Rightarrow x = \pm 3$

$\underline{f'(x) = DNE}$ No such points.

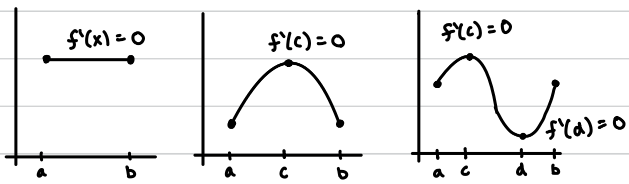
$-3 \notin (-1, 6)$; $x_1 = 3 \in (-1, 6) \Rightarrow x_c = 3$

② $f(3) = -53$
$f(-1) = 27$
$f(6) = 53$

③ $f(x)$ attains global max at $x = 6$
$f(x)$ attains global min at $x = 3$

Theorem Rolle's Theorem

If function $f(x)$ differentiable on open interval (a, b) and continuous on closed interval $[a, b]$ with $f(a) = f(b)$ then there exists at least one number $c \in (a, b)$ such that $f'(c) = 0$.



Examples:

4. Let $f(x) = x^2 - x - 2$, $x \in [-1, 2]$. Use Rolle's Thm. to show that there exists point $c \in [-1, 2]$ with a horizontal tangent.

Conditions for Rolle's Theorem:

↳ $f(x)$ is diff. on $(-1, 2)$ b/c all polynomials are diff. on $\mathbb{R}$.

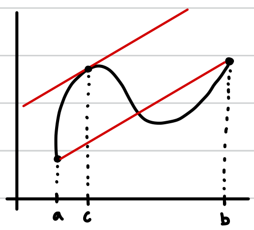
↳ $f(x)$ is cont. on $(-1, 2)$ b/c all polynomials are cont. on $\mathbb{R}$.

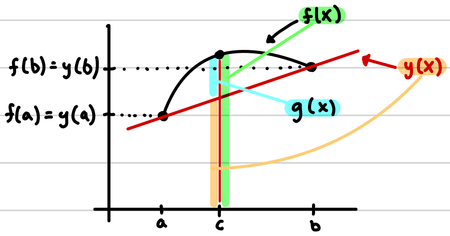
↳ $f(-1) = 1 + 1 - 2 = 0$
$f(2) = 4 - 2 - 2 = 0$
$f(-1) = f(2) = 0$

By Rolle's Theorem
$\exists\, c \in (-1, 2)$ s.t. $f'(c) = 0$
So, $f(x)$ has a horizontal tangent at $x = c$.
$f'(x) = 2x - 1 \Rightarrow x = \frac{1}{2} \Rightarrow c = \frac{1}{2}$

Theorem Lagrange's Theorem (Mean Value Theorem)

If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on open interval (a,b) then there exists at least one number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$





Proof:

$g(x) = f(x) - y(x)$
$g(x)$ is cont. on $[a,b]$
$g(x)$ is diff. on (a,b)
$g(a) = f(a) - y(a) = 0$
$g(b) = f(b) - y(b) = 0$

By Rolle's Thm, $\exists\, c \in (a,b)$ where $g'(c) = 0$

Point - Point Equation of this secant line:

$$\frac{y - y(a)}{y(b) - y(a)} = \frac{x-a}{b-a} \Rightarrow y(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

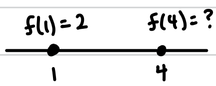
From $g(x) = f(x) - y(x)$ we have $f(x) = g(x) + y(x)$

$$y'(c) = 0 + \frac{f(b)-f(a)}{b-a} \qquad \Rightarrow \quad f'(c) = g'(c) + y'(c) = \frac{f(b)-f(a)}{b-a}$$
$$g'(c) = 0$$

QED

Examples:

5. Suppose that $f(x)$ is differentiable on $(1,4)$ and is continuous on $[1,4]$. Given that $2 \le f'(x) \le 3$ for all $x \in (1,4)$ and $f(1) = 2$. What is the least and greatest value that $f(x)$ can take on at 4?



$\left.\begin{array}{l} f(x) \text{ is diff. on } (1,4) \\ f(x) \text{ is cont. on } [1,4] \end{array}\right\}$ By MVT, $\exists\, c \in (1,4)$ s.t. $f'(c) = \frac{f(4) - f(1)}{4-1}$

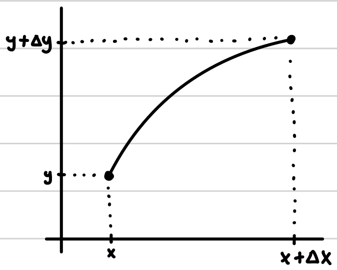
$$3f'(c) = f(4) - 2$$
$$f(4) = 3f'(c) + 2$$
$$3(2) + 2 \le f(4) \le 3(3) + 2$$
$$8 \le f(4) \le 11$$

Increasing / Decreasing Test

a) If $f'(x) > 0$ for all $x \in (a,b)$, then f is decreasing on that interval.
b) If $f'(x) < 0$ for all $x \in (a,b)$, then f is increasing on that interval.
c) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on that interval.

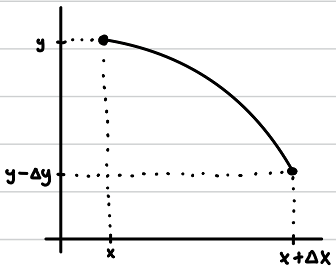
Proof:

a)



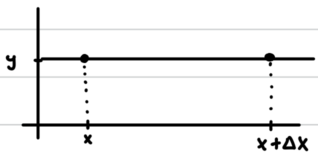
$$f'(x) = \lim_{\Delta x \to 0} \frac{(y+\Delta y) - y}{(x+\Delta x) - x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} > 0$$

b)



$$f'(x) = \lim_{\Delta x \to 0} \frac{(y-\Delta y) - y}{(x+\Delta x) - x} = \lim_{\Delta x \to 0} \frac{-\Delta y}{\Delta x} < 0$$

c)



$$f'(x) = \lim_{\Delta x \to 0} \frac{y - y}{(x+\Delta x) - x} = 0$$

The 1st Derivative Test

a) If $f'(x)$ changes from positive to negative at c, then $f(x)$ has local max at c.
b) If $f'(x)$ changes from negative to positive at c, then $f(x)$ has local min at c.
c) If $f'(x)$ does not change sign at c, then $f(x)$ has no local extremum at c.

Examples:

6. Sketch the graph of the function

a) $f(x) = x^2 - 2x + 1$

① $\text{Dom } f(x) = (-\infty, \infty)$

② x, y-intercepts

$f(x) = 0 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow x = 1$

$x = 0 \Rightarrow f(x) = 0$

x-int = $(1, 0)$

y-int = $(0, 1)$

③ Symmetry

$f(-x) = x^2 + 2x + 1$

$f(-x) \neq f(x) \neq -f(x)$

neither even nor odd.

④ Asymptotes

None

⑤ Derivatives

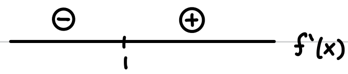
$f'(x) = 2x - 2$

⑥ Critical points

$f'(x) = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x_c = 1 \in \text{Dom } f$

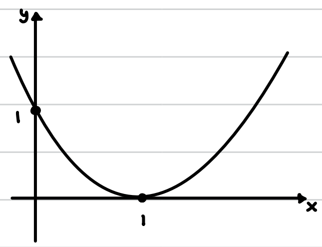
$f'(x) = \text{DNE} \Rightarrow$ No such points

⑦



⑧ Concavity

⑨ Graph



6b. $f(x) = \frac{x+1}{x-1}$

① $\text{Dom} f = (-\infty, 1) \cup (1, \infty)$

② x, y - intercepts

$f(x) = 0 \Rightarrow x = -1 \Rightarrow$ $\boxed{x\text{-int}: (-1, 0)}$

$x = 0 \Rightarrow f(x) = -1 \Rightarrow$ $\boxed{y\text{-int}: (0, -1)}$

③ Symmetry

$f(-x) \neq f(x) \neq -f(x)$

No symmetry

④ Asymptotes

VA: $x = a$

If $\lim_{x \to a} f(x) = \pm\infty$

Thus: $\lim_{x \to 1} \frac{x+1}{x-1} = \infty$

VA: $\boxed{x = 1}$

SA.

If $f(x) = \frac{P_n(x)}{Q_m(x)}$ and $n > m$ then $f(x)$ has SA

$f(x) = \frac{P_1(x)}{Q_1(x)}$, thus no SA

HA: $y = b$

$\lim_{x \to \pm\infty} f(x) = b$

$\lim_{x \to \pm\infty} \frac{x+1}{x-1} = \lim_{x \to \pm\infty} \frac{x/x + 1/x}{x/x - 1/x} = 1$

HA: $\boxed{y = 1}$

Aside:

$1 = \frac{x+1}{x-1}$ $\therefore$ No intersection between $y = 1$ (HA) and $f(x)$.

$x - 1 = x + 1$

$-1 \neq 1$

⑤ Derivatives

$f'(x) = \frac{-2}{(x-1)^2}$

⑥ Critical Points

$f'(x) = 0$. No such points

$f'(x) = \text{DNE}$: $x = 1 \notin \text{Dom} f$

⑦

	╲	VA	╲	$f(x)$
	⊖	DNE (1)	⊖	$f'(x)$

⑧ Concavity

⑨