B52 Sept 24 Lec 2 Notes

Mutual Independence

↳ Generalize independence to $n \geq 3$ events:

↳ Finite collection of events $A_1, A_2, \cdots, A_n$ is called mutually independent if for any sub-collection $A_{k_1}, A_{k_2}, \cdots, A_{k_m}$

$$P(A_{k_1} \cap A_{k_2} \cap \cdots \cap A_{k_m}) = P(A_{k_1})P(A_{k_2}) \cdots P(A_{k_m}) \Leftrightarrow P\left(\bigcap_{i=1}^{m} A_{k_i}\right) = \prod_{i=1}^{m} P(A_{k_i})$$

Conditioning on multiple events

↳ For events A, B, C, conditional probability of A given B and C, denoted by $P(A|B,C)$, is

$$P(A|B,C) = P(A|B \cap C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}, \text{ for } P(B \cap C) > 0$$

Ex 1:

$$P(\text{"only one event"}) = P(\text{"only A"} \cup \text{"only B"} \cup \text{"only C"})$$

$$= P((A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C))$$

$$= P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C)$$ Since they are disjoint

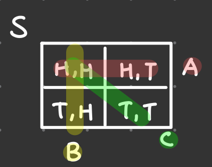
$$= P(A) \cdot P(B^c) \cdot P(C^c) + P(A^c) \cdot P(B) \cdot P(C^c) + P(A^c) \cdot P(B^c) \cdot P(C)$$

Mutual vs Pairwise Independence

↳ Pairwise independence does not imply mutual independence

$$P(A_i \cap A_j) = P(A_i)P(A_j), \ \forall i < j = 1, \cdots, n \not\Rightarrow A_1, A_2, \cdots, A_n \text{ mutually independent}$$

Ex 2:



(i) Consider pair A, B: $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)\ P(B)$

Similarly for $\{A, C\}$, $\{B, C\}$

(ii) $P(A \cap B \cap C) = \frac{1}{4} \neq P(A) \cdot P(B) \cdot P(C) = (\frac{1}{2})^3 = \frac{1}{8} \Rightarrow$ Not mutually independent

Multiplication Rule of Probability

↳ Probability of intersection of events $A_1, A_2, \cdots, A_n$ can be broken down as

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \times P(A_2 \mid A_1) \times P(A_3 \mid A_2, A_1) \times \cdots \times P(A_n \mid A_{n-1}, A_{n-2}, \cdots, A_1)$$

Ex 3:

$$\begin{aligned} P(R1 \cap B2 \cap R3) &= P(R1) \times P(B2 \mid R1) \times P(R3 \mid R1 \cap B2) \\ &= \tfrac{2}{5} \times \tfrac{3}{6} \times \tfrac{3}{7} \end{aligned}$$

Conditional Independence

↳ Events A, B are called conditionally independent given C when

$$P(A \cap B \mid C) = P(A \mid C) P(B \mid C)$$