W8 Reading Notes

Video

Predicate Logic and Quantifiers

We want to talk about variables.

open → $G(x,y)$ = x is greater than y. Not a statement

closed $\begin{cases} G(2,1) = 2 \text{ is greater than } 1 & \text{True} \\ G(3,6) = 3 \text{ is greater than } 6 & \text{False} \end{cases}$

We also want to introduce quantifiers.

Universal $\forall x\, P(x)$: "For all x, x is P"

Existential $\exists x\, P(x)$: "For some x, x is P"

Negating Quantifiers

Define $\forall x, \exists x$ for a universe with elements $\{1, 2, \ldots, n\}$

$$\forall x\ P(x) \Leftrightarrow P(1) \wedge P(2) \wedge \ldots \wedge P(n)$$

$$\exists x\ P(x) \Leftrightarrow P(1) \vee P(2) \vee \ldots \vee P(n)$$

Examples:

1. Show that $\neg\forall x\,[P(x)] \Leftrightarrow \exists x\,[\neg P(x)]$

$$\underbrace{\neg\forall x\ P(x)}_{\text{"Not all dogs are brown."}} \Leftrightarrow \neg\left(P(1) \wedge P(2) \wedge \ldots \wedge P(n) \right)$$

$$\Leftrightarrow \neg P(1) \vee \neg P(2) \vee \ldots \vee \neg P(n)$$

$$\Leftrightarrow \underbrace{\exists x\,[\neg P(x)]}_{\text{"There is some dog that is'nt brown."}}$$

All Equivalences

$\forall_x P(x) \Leftrightarrow \neg \exists_x [\neg P(x)]$

$\exists_x P(x) \Leftrightarrow \neg \forall_x [\neg P(x)]$

$\neg \forall_x P(x) \Leftrightarrow \exists_x [\neg P(x)]$

$\neg \exists_x P(x) \Leftrightarrow \forall_x [\neg P(x)]$

$\neg \forall_x P(x)$		$\exists_x P(x)$
$- \forall + P$	change signs	$+\exists + P$
$+\exists - P$	and $\forall \Leftrightarrow \exists$	$-\forall - P$
$\exists_x \neg P(x)$		$\neg \forall_x [\neg P(x)]$

Examples:

2. Negate the following:

$$\forall_x \exists_y [P(x,y) \wedge Q(y)]$$

$\Leftrightarrow \neg [\forall_x \exists_y [P(x,y) \wedge Q(y)]]$

$\Leftrightarrow \exists_x \neg [\exists_y [P(x,y) \wedge Q(y)]]$

$\Leftrightarrow \exists_x \forall_y \neg [P(x,y) \wedge Q(y)]$

$\Leftrightarrow \exists_x \forall_y [\neg P(x,y) \vee \neg Q(y)]$

Reading

Predicate Logic is an extension of Propositional Logic. It adds the concept of predicates and quantifiers to better capture the meaning of statements that cannot be adequately expressed by propositional logic.

Consider the statement "x is greater than 3". It has 2 parts. "x" is the subject and "is greater than 3" is the predicate.

The statement "x is greater than 3" can be denoted by P(x) where x is the variable.

Universal Quantification — mathematical statements sometimes assert that a property is true for all the values of a variable in a particular domain (domain of discourse)

Existential Quantification — mathematical statements assert that there is an element with a certain property.

"Every person who is 18 years or older, is eligible to vote."

$\forall P(x) \leftrightarrow Q(x)$

$P(x)$ is the statement "x is 18 years or older"
$Q(x)$ is the Statement "x is eligible to vote."

Uniqueness Quantifier is denoted by $\exists!$

The notation $\exists! x P(x)$ states "There exists a unique X such that $P(x)$ is true."

Quantifiers with Restricted Domains:

1. Restriction of universal quantification is the same as the universal quantification of a conditional statement.
2. Restriction of an existential quantification is the same as the existential quantification of conjunction.

Definitions to Note:

1. **Binding variables** – a variable whose occurence is bound by a quantifier is called a bound variable. Variables not bound by any quantifiers are called free variables.
2. **Scope** – The part of the logical expression to which a quantifier is applied is called the scope of the quantifier.