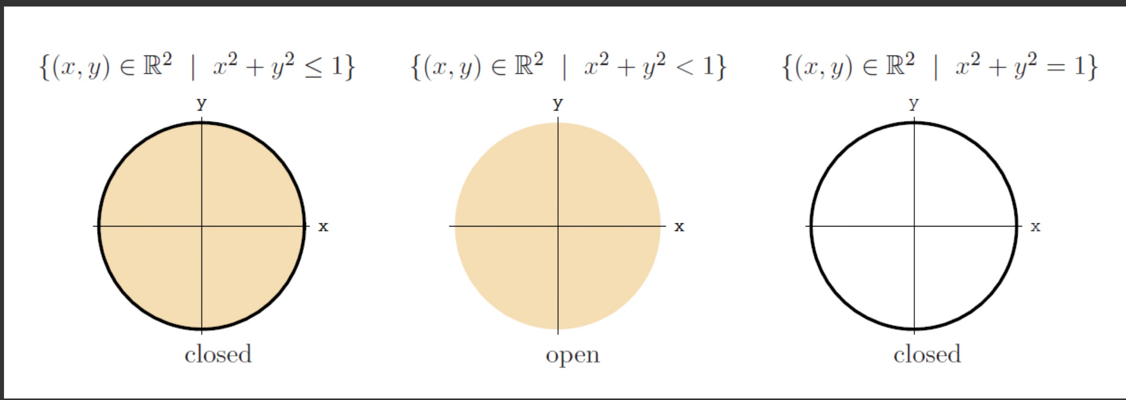
B41 Nov 8 Lec 1 Notes

Definition:

Let $U \subset \mathbb{R}^n$. A point $x_0 \in U$ is called an interior point of U if $D_r(x_0) \subset U$ for some r.

Points which are not interior points are called boundary points.

The set of boundary points is denoted ∂U. If every point in U is an interior point, U is said to be open. U is closed if $\mathbb{R}^n - U$ is open.



$U \subset \mathbb{R}^n$ is said to be bounded if U can be contained in an open ball, $D_M(0)$, for sufficiently large M, or, if $\|x\| < M$, for some $M \in \mathbb{R}$, $\forall x \in U$

A closed and bounded set in $\mathbb{R}^n$ is said to be compact.

Definition:

Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is defined on a set U in $\mathbb{R}^n$.

(i) A point $x_0 \in U$ is said to be a global (absolute) minimum of f on U if $f(x_0) \leq f(x)$ for all $x \in U$.

(ii) A point $x_0 \in U$ is said to be a global (absolute) maximum of f on U if $f(x_0) \geq f(x)$ for all $x \in U$.

(iii) If x_0 is either of these, it is a global (absolute) extremum.

Theorem: Extreme Value Theorem

Let D be a compact set in $\mathbb{R}^n$ and let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f assumes both a (global) maximum and a (global) minimum on D.

Procedure: Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous on a compact set D. To find the extrema:

(i) Find the critical points for f on the interior of D.

(ii) Find the critical points for f restricted to ∂D, the boundary.

(iii) Compute f at each of these critical points.

(iv) Compare and choose the largest and/or smallest.

Ex 1:

Find the global max and min value of $f(x,y) = x^2+y^2-2x+2y+5$ on the set $D = \{(x,y) \mid x^2+y^2 \le 4\}$.

Interior of D:

$$f_x = 2x-2 = 0 \Rightarrow x = 1$$
$$f_y = 2y+2 = 0 \Rightarrow y = -1$$

$(1,-1) \in$ interior of D is a critical point of f.

∂D: $x^2+y^2 = 4$

Let $x = 2\cos\theta$ and $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

$$g(\theta) = f(2\cos\theta, 2\sin\theta) = 4\cos^2\theta + 4\sin^2\theta - 4\cos\theta + 4\sin\theta + 5$$
$$= 4\sin\theta - 4\cos\theta + 9$$

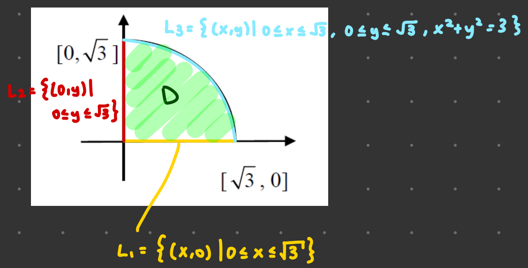
$$g'(\theta) = 4\cos\theta + 4\sin\theta = 1 \Rightarrow \tan\theta = -1$$
$$\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Therefore f has two critical points on ∂D: $(2\cos(3\pi/4), 2\sin(3\pi/4))$ and $(2\cos(7\pi/4), 2\sin(7\pi/4))$
$\Rightarrow (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$

$f(1,-1) = 3$ Global min on ∂D
$f(-\sqrt{2}, \sqrt{2}) = 9+4\sqrt{2}$ Global max on ∂D
$f(\sqrt{2}, -\sqrt{2}) = 9-4\sqrt{2}$

Ex 2:

Find global max and min value of $f(x,y) = xy^2$ on the set $D = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0, y \ge 0, x^2+y^2 \le 3\}$.



In D: $x>0, y>0$ and $x^2+y^2<3$
$f_x = y^2 = 0$, $f_y = 2xy = 0 \Rightarrow y^2 = 0, 2xy = 0$
$\Rightarrow y = 0$ and $0 < x < \sqrt{3}$
$\Rightarrow (x, 0)$ on L_1
Thus no critical points of f in D.

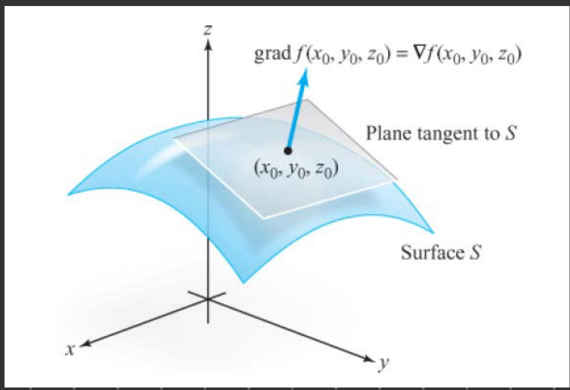
On ∂D: L_1 & L_2 & L_3
L_1 : $f(x,0) = 0$
L_2 : $f(0,y) = 0$
L_3 : $f(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta) = 3\sqrt{3}\sin^2\theta\cos\theta = g(\theta)$

$\Rightarrow g'(\theta) = 6\sqrt{3}\sin\theta\cos^2\theta - 3\sqrt{3}\sin^3\theta$
$= 3\sqrt{3}\sin\theta(2\cos^2\theta - \sin^2\theta)$ $\quad 0 \le \theta \le \frac{\pi}{2}$
$\Rightarrow \sin\theta = 0$, $2\cos^2\theta - \sin^2\theta = 0$
$\Rightarrow \theta = 0$, $3\cos^2\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$, $\sin\theta = \sqrt{\frac{2}{3}}$
$\Rightarrow y = 0$, critical point on $(\sqrt{3}\frac{1}{\sqrt{3}}, \sqrt{3}\sqrt{\frac{2}{3}}) = (1, \sqrt{2})$ on L_3

Therefore the global max value of f is $f(1,\sqrt{2}) = 2$ on the boundary L_3.
The global min value of f is 0 for all the points on L_1 & L_2.

Theorem:

Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ and $g: D \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let $x_0 \in D$ and $g(x_0) = C$. Let $S = \{x \in U \mid g(x) = C\}$. Assume that $\nabla g(x_0) \neq 0$. If x_0 is an extremum of f on S, then there is a real number λ s.t. $\nabla f(x_0) = \lambda \nabla g(x_0)$.



Theorem:

If f, when constrained to a surface S, has a max or min at x_0, then $\nabla f(x_0)$ is perpendicular to S.

Theorem: Lagrange Multiplier

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ be an extremum for function $f: \mathbb{R}^n \to \mathbb{R}$, subject to the constraint $g(x_1, \cdots, x_n) = C$. To find the coordinates of $a = (a_1, \dots, a_n)$, we solve the system

$$\nabla f(a) = \lambda \nabla g(a)$$
$$g(a) - C = 0$$

λ is called the lagrange multiplier.

Then we get a constrained critical point.

Procedure:

(i) Construct a new function $L: \mathbb{R}^{n+1} \to \mathbb{R}$ by $L(x, \lambda) = f(x) - \lambda(g(x) - C)$.

(ii) Finding all the critical points of L about λ and the constrained critical points of f.

(iii) Evaluating all the constrained critical points of f. The largest is the maximum value of f and the smallest is the min value of f.

The function L is called the Lagrange function or the Lagrangian.