B24 July 7 Lec 1 Notes

Theorem: Cauchy-Schwarz Inequality

If V is an IPS, then $\forall x, y \in V$:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|$$

Proof:

First we assume $\langle x, y\rangle \in \mathbb{R}$.

Then for all $t \in \mathbb{R}$:

$$0 \leq \|x-ty\|^2 := \langle x-ty, x-ty\rangle$$

$$= \langle x, x\rangle - \langle ty, x\rangle - \langle x, ty\rangle + t^2\langle y, y\rangle$$

$$= \|x\|^2 - 2t\langle x, y\rangle + t^2\|y\|^2 := p(t)$$

$p(t)$ is a quadratic polynomial (in t), and we can check (set $p'(t) = 0$) that $p(t)$ has a minimum at $t = \frac{\langle x, y\rangle}{\|y\|^2}$.

Substituting $t = \frac{\langle x, y\rangle}{\|y\|^2}$ into

$$0 \leq \|x\|^2 - 2t\langle x, y\rangle + t^2\|y\|^2$$

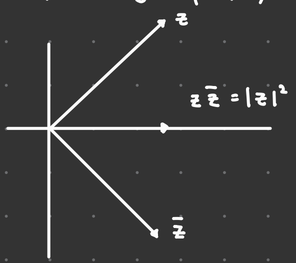
we get

$$0 \leq \|x\|^2 - \frac{2(\langle x, y\rangle)^2}{\|y\|^2} + \frac{(\langle x, y\rangle)^2}{\|y\|^2}$$

$$= \|x\|^2 - \frac{(\langle x, y\rangle)^2}{\|y\|^2}$$

$$\Rightarrow (\langle x, y\rangle)^2 \leq \|x\|^2 \|y\|^2$$

If $\langle x, y\rangle \notin \mathbb{R}$, Let $\alpha \in \mathbb{C} \setminus \{0\}$ s.t. $\alpha\langle x, y\rangle \in \mathbb{R}$



Proof (continued...):

By previous part:

$$|\alpha||\langle x,y\rangle| = |\langle \alpha x, y\rangle|$$
$$\leq \|\alpha x\| \, \|y\|$$
$$= |\alpha| \, \|x\| \, \|y\|$$

$$\Rightarrow |\langle x,y\rangle| \leq \|x\| \, \|y\|$$

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e.g. If $V = \mathbb{R}^n$, CS-ineq. says:

$$|x_1y_1 + \dots + x_ny_n| \leq \sqrt{x_1^2 + \dots + x_n^2}\,\sqrt{y_1^2 + \dots + y_n^2}$$

If $V = C([0,1])$, CS-ineq. says

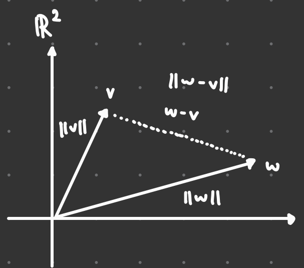
$$\left|\int_0^1 f(x)\overline{g(x)}\,dx\right| \leq \sqrt{\int_0^1 |f(x)|^2\,dx}\,\sqrt{\int_0^1 |g(x)|^2\,dx}$$

Theorem: Triangle Inequality

Let V be an IPS, $x,y \in V$

Then

$$\|x+y\| \leq \|x\| + \|y\|$$



Δ-ineq. $\Rightarrow \|v + w - v\| \leq \|v\| + \|w-v\|$

i.e. $\|w\| \leq \|v\| + \|w-v\|$

Proof:

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$
$$= \langle x,x\rangle + \underbrace{\langle y,x\rangle}_{=\overline{\langle x,y\rangle}} + \langle x,y\rangle + \langle y,y\rangle$$
$$= \|x\|^2 + 2\,\mathrm{Real}\langle x,y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$
$$= (\|x\| + \|y\|)^2$$

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i.e. $\mathrm{Real}(x+iy) = x$

i.e. $x+iy + \overline{x+iy} = x+iy + x-iy$
$= 2x$
$= 2\,\mathrm{Re}(x+iy)$

So,
$\langle x,y\rangle + \overline{\langle x,y\rangle} = 2\,\mathrm{Real}\langle x,y\rangle$

$|x+iy| = \sqrt{x^2+y^2}$
$\geq \sqrt{x^2}$
$= |x|$
$\geq x = \mathrm{Real}(x+iy)$

Lemma: Popularization Identity

Let V be a real IPS. Then for $x, y \in V$:

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

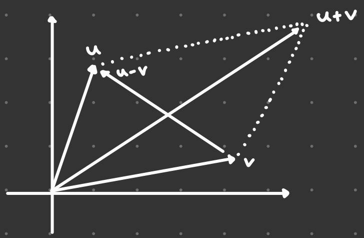
Lemma: Popularization Identity

Let V be a complex IPS. Then for $x, y \in V$:

$$\langle x, y \rangle = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \| x + \alpha y \|^2$$

Proposition: Parallelogram Identity

Let V be an IPS, and $u, v \in V$. Then:



$$\| u + v \|^2 + \| u - v \|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Proof:

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle \\
&= \left[\langle u, u \rangle - \cancel{\langle v, u \rangle} + \cancel{\langle u, v \rangle} + \langle v, v \rangle\right] + \left[\langle u, u \rangle - \cancel{\langle v, u \rangle} - \cancel{\langle u, v \rangle} + \langle v, v \rangle\right] \\
&= 2\|u\|^2 + 2\|v\|^2
\end{aligned}$$

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