Program Correctness 2 of 2

Pythonized Lemma 2.2 (PL 2.2)

For any integers a, b,

$$\text{if } a+1 < b, \text{ then } a < \left\lfloor \frac{a+b}{2} \right\rfloor < b$$

Slice $L[a:b]$ with length > 1

Midpoint $m = (a+b) \text{ div } 2$

Then slices $L[a:m]$, $L[m:b]$ are both nonempty and shorter than $L[a:b]$.

Definition:

Let L be a list of integers.
Let $L[p:q]$ be an nonempty slice. i.e., $0 \le p < q \le \text{len}(L)$

We say that $L[p:q]$ is **unimodal** iff there is a natural number m s.t.

(i) $p \le m < q$
(ii) $L[p:m+1]$ (strictly) increasing
(iii) $L[m:q]$ (strictly) decreasing

Furthermore, such a number m, if it exists, is called the **mode** of $L[p:q]$.

Since L is the same as $L[0:\text{len}(L)]$, we also say that L is unimodal if $L[0:\text{len}(L)]$ is unimodal.

Remark: Any increasing slice $L[p:q]$ is unimodal with mode $q-1$. Also, any decreasing slice $L[p:q]$ is unimodal with mode p.

$L[p:p+1]$ (i.e. any slice of length one) is both increasing and decreasing. So $L[p:p+1]$ is unimodal with mode p

Any smaller (non-empty) slice within an unimodal slice is also unimodal, though not necessarily with the same mode.

The maximum element of a unimodal slice occurs at its mode.

Ex 1: Binary Search

Pre: L is a list of integers, $0 \le p < q \le len(L)$, $L[p:q]$ is unimodal

Post: Return the max integer in $L[p:q]$

```
MAX(L, p, q)
1. low = p ; high = q
2. while low + 1 < high :
3.     mid = ⌊(low + high)/2⌋
4.     if L[mid-1] < L[mid]] :    low = mid
5.     else :   high = mid
6. return L[low]
```

	0	1	2	3	4	5	6
$L=$	0	2	3	5	7	6	4
$p=1$		low		mid			high
				low	mid		high
$q=6$					low	mid	high
					low	high	
					low	high	

Step 1: Find L.I.

(i) $p \le low < high \le q$

(ii) mode of $L[p:q]$ = mode of $L[low:high]$

Step 2: Prove L.I.

Basis: On entering the loop,

$low = p$, $high = q$ [Line 1]

$\therefore \; p \le \underbrace{low < high}_{\text{By Pre}} \le q$, as wanted for $LI(a)$.

Also, $L[p:q] = L[low:high]$

$\therefore$ mode of $L[p,q]$ = mode of $L[low:high]$

I.S.: Consider an arbitrary iteration.

Suppose L.I. before the iteration [I.H.]

WTP L.I. holds after the iteration.

There are 2 cases: $L[mid-1] < L[mid]$, $L[mid-1] > L[mid]$

Case 1: If $L[mid-1] < L[mid]$, then

$low' = mid = \left\lfloor \frac{low + high}{2} \right\rfloor$ and $high' = high$ [Lines 3,4]

$\therefore \; p \le low$ [I.H.]

$< mid = low'$ [PL 2.2]

$< high = high'$ [PL 2.2]

$\le q$ [I.H.]

Ex 1 continued ...

Step 2:

I.S:

Case 2: If $L[mid-1] > L[mid]$

$low' = low$; $high' = mid = \left\lfloor \frac{low+high}{2} \right\rfloor$

? ?

Step 3: Prove partial correctness

Suppose the loop terminates and consider the values of low and high on exit.
By LI(a), $low < high$ (or $low+1 \leq high$)
By exit condition, $low+1 \geq high$
$\therefore$ $low+1 = high$
By LI(b), mode of $L[p:q]$ = $L[low:high]$ = mode of $L[low:low+1]$
= low

$\therefore$ max int in $L[p:q]$ is $L[low]$ which is returned [line 6] as follows.

Step 4: Find an expression e to associate with each iteration.

$e = high - low$ ($high - low - 1$ also works)

Step 5: Prove (A) $e \geq 0$, (B) e decreasing

(A) $e = high - low \geq 0$ [LI(a)]

(B) Consider an arbitrary iteration

By line 2, $low + 1 < high$ (#)

2 cases:

Case 1: If $L[mid-1] < L[mid]$, then $e' = high' - low'$
$= high - mid$ [lines 3,4]
$< high - low$ [PL 2.2, lines 3]
$= e$, as wanted.

Case 2: If $L[mid-1] > L[mid]$

Similar to case 1.

Ex 2:

Pre: L is a list of integers, $0 \le p \le q \le len(L)$. $L[p:q]$ is unimodal.
Post: Return the max int in $L[p:q]$

```
MAX(L, p, q)
1. if p+1 == q :
2.      result = L[p]
3. else:
4.      mid = ⌊(p+q)/2⌋
5.      if L[mid-1] < L[mid]: result = MAX(L, mid, q)
6.      else: result = MAX(L, p, mid)
7. return result
```

For $n \in \mathbb{N}$, we define predicate $Q(n)$

$Q(n)$: If L is a list of integers, $0 \le p \le q \le len(L)$, $L[p:q]$ is unimodal, and $n = q-p$, then $MAX(L,p,q)$ returns the max int in $L[p:q]$.

Prove $Q(n)$ holds for all $n \ge 1$. (Then correctness follows).
We'll use PCI.

Base Case: Let $n=1$

Then $q-p=1$ (or $p+1=q$)

Thus $L[p:q]$ is a slice of length 1, and the max of $L[p,q]$ is $L[p]$.

By lines 1, 2, 6, $L[p]$ is returned as wanted.

I.S.: Let $n > 1$.

Suppose $Q(j)$ holds whenever $1 \le j < n$ [I.H.]

WTP $Q(n)$ holds also.

Since $q-p > 1$, lines 4-7 runs.
By line 4 and PL22, $p < mid < q$ *

There are 2 cases. $L[mid-1] < L[mid]$, $L[mid-1] > L[mid]$

Case 1: if $L[mid-1] < L[mid]$, then by line 5, $MAX(L, mid, q)$ is called and returned. By *, $1 \le q - mid < q-p = n$ $\therefore$ I.H. applies to $MAX(L, mid, q)$
By IH, $MAX(L, mid, q)$ returns the max int in $L[mid:q]$.
Since $L[mid-1] < L[mid]$, the mode of $L[p:q]$ = mode of $L[mid:q]$.
Thus max of $L[p:q]$ = max of $L[mid:q]$, which is returned as wanted.

Ex 2 continued ...

I.S:

Case 2: If $L[mid-1] > L[mid]$

Similar to case 1.