B41 Nov 22 Lec 1 Notes

Triple Integrals in Rectangular box

Let $B = [a,b] \times [c,d] \times [e,f]$ be a compact box in $\mathbb{R}^3$. Let $f: B \to \mathbb{R}$ be a continuous function. Proceding as in double integrals, we partition the three sides of B into n equal parts and form the Riemann sum

$$\sum_{i,j,k=0}^{n} f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

The limit of the riemann sum, if it exists, is the triple integral of f over B.

$$\iiint_B f(x,y,z)\,dV = \lim_{n\to\infty} \sum_{i,j,k=0}^{n} f(x_i^*, y_j^*, z_k^*)\,\Delta V$$

$$= \int_p^q \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Ex 1:

The density of a box solid B decreases linearly and is given by $f(x,y,z) = 2-z$ where $B = \{(x,y,z) \mid 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1\}$. Find the mass of the box.

$$m = \iiint_B (2-z)\,dV$$

$$= \int_0^3 \int_0^2 \int_0^1 (2-z)\,dz\,dy\,dx$$

$$= 9$$

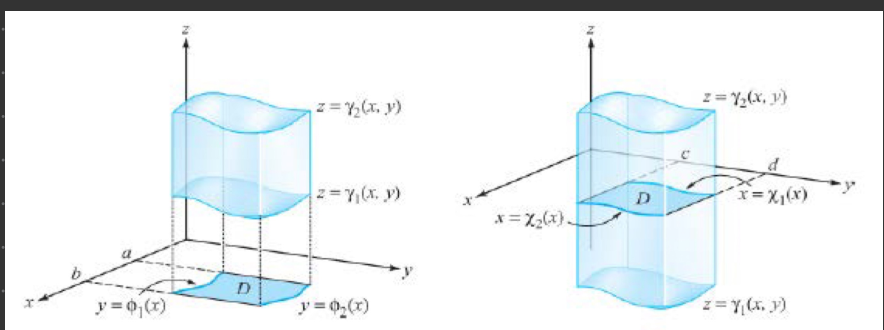
Type I Triple Integral

Let W be a region in $\mathbb{R}^3$.

$W = \{(x,y,z) \mid (x,y) \in D$ is an x-simple or y-simple in xy plane, $\gamma_1(x,y) \le z \le \gamma_2(x,y)$, $\gamma_i(x,y)$ is cont. in D, $i=1,2\}$

Let $f: W \to \mathbb{R}$ be integrable. Then $\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz \right) dA$

The volume of the solid bounded by W is $V = \iiint_W 1\,dV$



Ex 2:

Find the volume of the region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$.

The region W: $0 \le z \le 4 - x^2 - y$

The surface meets xy plane by $4 - x^2 - y = 0$ i.e. $y = 4 - x^2$

Therefore $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 4 - x^2\}$

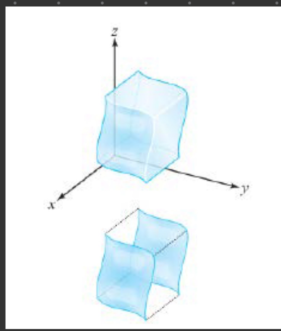
$$V = \iiint_W 1\, dV$$

$$= \iint_D \left(\int_0^{4-x^2-y} 1\, dz \right) dA$$

$$= \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\, dy\, dx$$

$$= \int_0^2 \left[4(4-x^2) - x^2(4-x^2) - \tfrac{1}{2}(4-x^2)^2 \right] dx$$

$$= \frac{1}{2} \int_0^2 \left[16 - 8x^2 + x^4 \right] dx = \frac{128}{15}$$



Type II Triple Integral

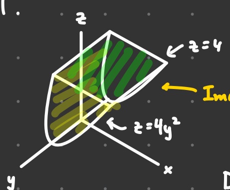
Let W be a region in $\mathbb{R}^3$.

$W = \{(x,y,z) \mid (y,z) \in D$ is an y-simple or z-simple in yz plane, $u_1(y,z) \le x \le u_2(y,z)$, $u_i(y,z)$ is cont. in D, $i = 1,2\}$

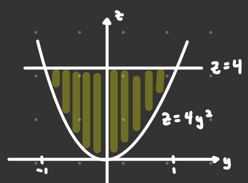
Let $f: W \to \mathbb{R}$ be integrable. Then $\iiint_W f(x,y,z)\, dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx \right) dA$

Ex 3:

Evaluate $\iiint_W x^2 y^2\, dV$, where W is bounded by $z = 4y^2$ above and $z = 4$, and on the ends by planes $x = -1$, $x = 1$.



Imagine an upside down toblerone with curved edeges



$D = \{(y,z) \mid 4y^2 \le z \le 4,\ -1 \le y \le 1\}$

$W = \{(x,y,z) \mid -1 \le x \le 1,\ 4y^2 \le z \le 4,\ -1 \le y \le 1\}$

$$\iiint_W x^2 y^2\, dV = \iint_D \left(\int_{-1}^{1} x^2 y^2\, dx \right) dA$$

$$= \frac{2}{3} \iint_D y^2\, dA$$

$$= \frac{2}{3} \int_{-1}^{1} \int_{4y^2}^{4} y^2\, dz\, dy$$

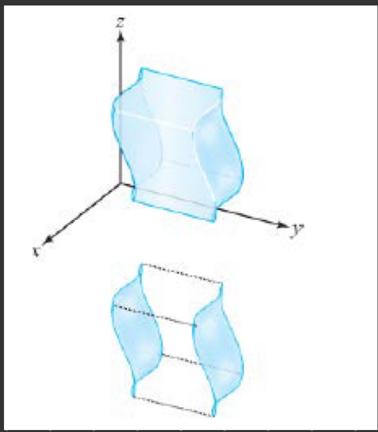
$$= \frac{32}{45}$$

Type III Triple Integral

Let W be a region in $\mathbb{R}^3$.

$W = \{(x,y,z) \mid (x,z) \in D$ is an x-simple or z-simple in xz plane, $v_1(x,z) \le y \le v_2(x,z)$, $v_i(x,z)$ is cont. in D, $i=1,2\}$

Let $f: W \to \mathbb{R}$ be integrable. Then $\iiint_W f(x,y,z)\,dV = \iint_D \left(\int_{v_1(x,z)}^{v_2(x,z)} f(x,y,z)\,dy \right) dA$



Ex 4

Evaluate $\iiint_W \sqrt{x^2+z^2}\,dV$ where W is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$.

The region $x^2+y^2 \le y \le 4$

Its projection D onto xz plane bounded by $x^2+z^2=4$.

Therefore $D = \{(x,z) \mid x^2+z^2 \le 4\}$
$= \{(x,z) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le z \le \sqrt{4-x^2}\}$

$$\iiint_W \sqrt{x^2+z^2}\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA$$

Let $x = r\cos\theta$, $z = r\sin\theta$.

note the extra r because we need the jacobian determinant

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+z^2}\,(4-x^2-z^2)\,dz\,dx = \int_0^{2\pi} \int_0^{2} (4-r^2)\,r^2\,dr\,d\theta$$

$$= \frac{128\pi}{15}$$