B24 July 9 Lec 2 Notes

Definition:

Let V be a v.s. A norm on V is a function

$$\|\cdot\| : V \to [0, \infty)$$

s.t.

(i) $\|\alpha v\| = |\alpha| \, \|v\|$, $\forall \alpha \in \mathbb{F}$, $\forall v \in V$
(ii) $\|u + v\| \leq \|u\| + \|v\|$, $\forall u, v \in V$
(iii) $\|u\| \geq 0$, $\forall u \in V$
(iv) $\|u\| = 0$ iff $u = 0$

A normed space is a v.s. together with a norm

e.g.

Let $1 \leq p \leq \infty$ and for $(x_1, \cdots, x_n) \in \mathbb{F}^n$,

$$\overbrace{\|(x_1, \ldots, x_n)\|_p}^{\text{called "}L^p\text{ norm"}} := \left[|x_1|^p + \ldots + |x_n|^p \right]^{1/p}$$

So when $p = 2$,

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$$
$$= \sqrt{\langle (x_1, \ldots, x_n), (x_1, \ldots, x_n) \rangle}$$

But when $p \neq 2$, $\|\cdot\|_p$ defines a norm on $\mathbb{F}^n$ which does not arise from an inner product, i.e, there is no inner product $\langle \cdot, \cdot \rangle$ on V s.t.

$$\|v\|_p = \sqrt{\langle v, v \rangle} \quad \text{for all } v \in V$$

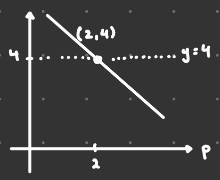
We will show $\|\cdot\|_p$ can not arise from an inner product by showing the parallelogram identity fails for $p \neq 2$:

$$\|u + v\|_p^2 + \|u - v\|_p^2 \neq 2\left(\|u\|_p^2 + \|v\|_p^2\right) \qquad p \neq 2$$

Consider $u = (1, 0, \ldots, 0)$, $v = (0, 1, 0, \ldots, 0)$

$$= \left[1^p + 1^p\right]^{2/p} + \left[1^p + |-1|^p\right]^{2/p} \qquad 2(1+1) = 4$$
$$= 2^{2/p} + 2^{2/p}$$
$$= 2^{1 + 2/p}$$

The claim follows if we can show $2^{1+2/p}$ is a strictly decreasing function of p, since $2^{1+2/2} = 4$



$$\frac{d}{dp}\left[2^{1+2/p}\right] = 2^{1+2/p} \cdot \underbrace{2\log 2}_{>0} \cdot \underbrace{-1/p^2}_{<0} < 0$$

for all $p \geq 1$. $\square$

If $V = C([0,1])$, or $V = P_n^{\mathbb{F}}$, and $f \in V_p$

$$\|f\|_p := \left[\int_0^1 |f(x)|^p \, dx\right]^{1/p}$$ this is a norm

When $p=1$, $\|f\|_1 = \int_0^1 |f(x)| \, dx$ is an average value of $|f|$ on $[0,1]$

As $p \to \infty$, $\|f\|_p$ is a weighted average of $|f|$ on $[0,1]$ where larger values are weighed more heavily.

So we define

$$\|f\|_\infty := \sup_{x \in [0,1]} \{ |f(x)| : x \in [0,1] \}$$

Definition:

If V is an IPS, we say $u, v \in V$ are orthogonal and write $u \perp v$ if:

$$\langle u, v \rangle = 0$$

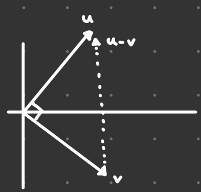
Remark:

In $\mathbb{R}^n$, u, v meet at a right iff $\langle u, v \rangle = 0$

Proposition: Pythagorean Theorem

If V is IPS, and $u, v \in V$ s.t. $u \perp v$, then:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



$$\Rightarrow \|u-v\|^2 = \|u\|^2 + \|-v\|^2$$
$$= \|u\|^2 + \|v\|^2$$

Proof:

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \cancel{\langle v,u\rangle}^{0} + \cancel{\langle u,v\rangle}^{0} + \langle v,v\rangle \quad \text{By orthogonality} \\ &= \|u\|^2 + \|v\|^2\end{aligned}$$

$\boxtimes$

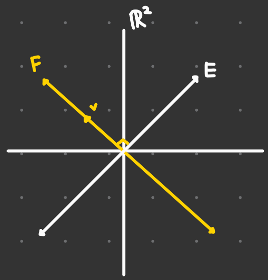
e.g.

If $V = C([0,1])$, pythagorean theorem says if $f, g \in C([0,1])$ and $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$, then:

$$\int_0^1 |f(x)|^2\,dx + \int_0^1 |g(x)|^2\,dx = \int_0^1 |f(x)+g(x)|^2\,dx$$

Definition:

If $E \subset V$ is a subspace, and $v \in V$, we say v is orthogonal to E if $\langle v, e\rangle = 0$, $\forall e \in E$.
If $F \subset V$ is also a subspace, we say F is orthogonal to E if $\langle e, f\rangle = 0$, $\forall e \in E$, $\forall f \in F$.



Lemma:

Let $v \in V$ and $E \subset V$ be a subspace spanned by $v_1, \dots, v_r$. Then $v \perp E$ iff $v \perp v_k$ for $1 \le k \le r$.

Proof ($\Rightarrow$): definition of $v \perp E$

Proof ($\Leftarrow$):

If $w \in E$, then $\exists\, \alpha_i \in \mathbb{F}$ for $1 \le i \le r$ with

$$w = \alpha_1 v_1 + \dots + \alpha_r v_r$$

and

$$\begin{aligned}\langle v, w\rangle &= \langle v, \alpha_1 v_1 + \dots + \alpha_r v_r\rangle \\ &= \langle v, \alpha_1 v_1\rangle + \dots + \langle v, \alpha_r v_r\rangle \\ &= \overline{\alpha}_1 \cancel{\langle v, v_1\rangle}^{0} + \dots + \overline{\alpha}_r \cancel{\langle v, v_r\rangle}^{0} \\ &= 0\end{aligned}$$

$\boxtimes$

By IPS axiom (ii), we have

$$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$$

Consider $\langle x, \alpha y\rangle$,

$$\begin{aligned}
\langle x, \alpha y\rangle &= \overline{\langle \alpha y, x\rangle} && \text{By axiom (i)} \\
&= \overline{\alpha \langle y, x\rangle} && \text{By axiom (ii)} \\
&= \bar{\alpha}\, \overline{\langle y, x\rangle} && \overline{zw} = \bar{z}\cdot\bar{w} \\
&= \bar{\alpha} \langle x, y\rangle && \text{By axiom (i)}
\end{aligned}$$

Definition:

We say vectors $v_1, \dots, v_n \in V$ are orthogonal or form an orthogonal system if $\langle v_i, v_j\rangle = 0$ for $i \neq j$.
If, in addition, $\|v_i\| = 1$ for $1 \le i \le n$, we say $v_1, \dots, v_n$ are orthonormal or form an orthonormal system

Lemma:

If $v_1, \dots, v_n \in V$ are orthogonal, and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, then:

$$\left\| \sum_{k=1}^{n} \alpha_k v_k \right\|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \|v_k\|^2$$

Proof:

$$\begin{aligned}
\left\| \sum_{n=1}^{n} \alpha_k v_k \right\|^2 &= \left\langle \sum_{k=1}^{n} \alpha_k v_k, \sum_{\ell=1}^{n} \alpha_\ell v_\ell \right\rangle \\
&= \sum_{k=1}^{n} \left\langle \alpha_k v_k, \sum_{\ell=1}^{n} \alpha_\ell v_\ell \right\rangle \\
&= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \langle \alpha_k v_k, \alpha_\ell v_\ell \rangle \\
&= \sum_{k=1}^{n} \sum_{\ell=1}^{n} \alpha_k \overline{\alpha_\ell} \underbrace{\langle v_k, v_\ell\rangle}_{=0 \text{ if } k \neq \ell} \\
&= \sum_{k=1}^{n} \alpha_k \overline{\alpha_k} \langle v_k, v_k\rangle \\
&= \sum_{k=1}^{n} |\alpha_k|^2 \|v_k\|^2
\end{aligned}$$

$\blacksquare$

Corollary:

If $v_1, \dots, v_n \in V$ are orthogonal and non-zero, then $v_1, \dots, v_n$ are L.I.

Proof:

Suppose $\alpha_1 v_1 + \dots + \alpha_n v_n = 0$

Then $\| \alpha_1 v_1 + \dots + \alpha_n v_n \|^2 = 0$

and by previous lemma,

$$\| \alpha_1 v_1 + \dots + \alpha_n v_n \|^2 = |\alpha_1|^2 \|v_1\|^2 + \dots + |\alpha_n|^2 \|v_n\|^2$$
$$= 0$$

$\Rightarrow |\alpha_1|^2 \|v_1\|^2 = \dots = |\alpha_n|^2 \|v_n\|^2 = 0$ each $|\alpha_k|^2 \|v_k\|^2 \geq 0$

$\Rightarrow |\alpha_1|^2 = \dots = |\alpha_n|^2 = 0$ $v_1, \dots, v_n \neq 0$

$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$

$\blacksquare$

Definition:

If $v_1, \dots, v_n \in V$ form a basis for V, and $v_1, \dots, v_n$ are orthogonal, then we call $v_1, \dots, v_n$ an orthogonal basis.

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Remark:

For any basis $v_1, \dots, v_n$ for V. and $v \in V$ we know there exist coordinates $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t.

$$v = \alpha_1 v_1 + \dots + \alpha_n v_n$$

but in particular, finding $\alpha_1, \dots, \alpha_n$ is not trivial.

If $v_1, \dots, v_n$ is an orthogonal basis, then finding $\alpha_1, \dots, \alpha_n$ is much easier.

Let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$$\langle v, v_1 \rangle = \langle \alpha_1 v_1 + \dots + \alpha_n v_n, v \rangle$$
$$= \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \underbrace{\langle v_2, v_1 \rangle}_{=0} + \dots + \alpha_n \underbrace{\langle v_n, v_1 \rangle}_{=0}$$
$$= \alpha_1 \|v_1\|^2$$

$$\Rightarrow \alpha_1 = \frac{\langle v, v_1 \rangle}{\|v_1\|^2}$$

i.e. $v = \dfrac{\langle v, v_1 \rangle}{\|v_1\|^2} v_1 + \dots + \dfrac{\langle v, v_n \rangle}{\|v_n\|^2} v_n$

Similarly $\alpha_k = \dfrac{\langle v, v_k \rangle}{\|v_k\|^2}$ for $1 \leq k \leq n$

Ex 1:

$(1,1),(1,-1)$ form an orthogonal basis for $\mathbb{R}^2$ since $\langle(1,1),(1,-1)\rangle = 1+(-1) = 0$

So

$$(x,y) = \frac{\langle (x,y),(1,1)\rangle}{\|(1,1)\|^2}(1,1) + \frac{\langle (x,y),(1,-1)\rangle}{\|(1,-1)\|^2}(1,-1)$$

$$= \frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)$$

$$= \left(\frac{x+y}{2}+\frac{x-y}{2}, \frac{x+y}{2}-\frac{x-y}{2}\right)$$

OR we could try to solve for

$$x = \alpha_1 + \alpha_2$$
$$y = \alpha_1 - \alpha_2$$

$(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})$ form an orthonormal basis since $\langle(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})\rangle = 0$ and

$\|(1/\sqrt{2}, 1/\sqrt{2})\| = \|(1/\sqrt{2}, -1/\sqrt{2})\| = 1$

and

$$(x,y) = \langle (x,y), (1/\sqrt{2}, 1/\sqrt{2})\rangle (1/\sqrt{2}, 1/\sqrt{2}) + \langle (x,y), (1/\sqrt{2}, -1/\sqrt{2})\rangle (1/\sqrt{2}, -1/\sqrt{2})$$

e.g.

"Fourier Series" is the study of the extent a formula such as

$$v = \sum \frac{\langle v, v_k\rangle}{\|v_k\|^2} v_k$$

holds when $v \in C([0,1])$, and (v_k) is a collection of orthogonal functions. (usually cosines and sines of different periods.