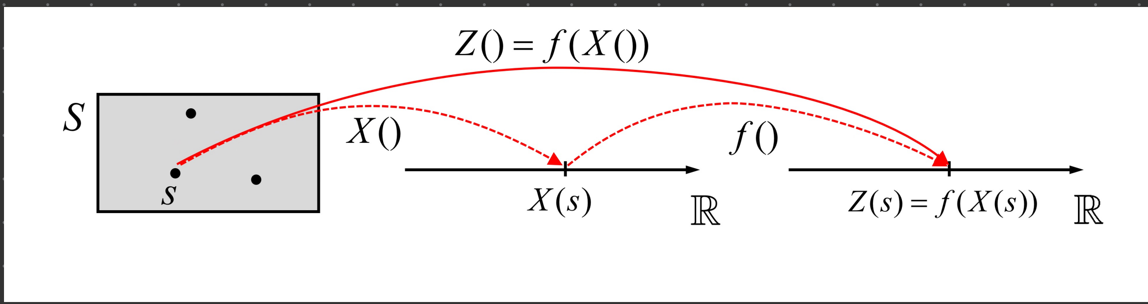
B52 Sept 29 Lec 1 Notes

A random variable is a function from the sample space (S) to the real line $(\mathbb{R})$

RVs used "in reverse" to describe events.

$$A = \{ s \in S : X(s) = b \} = \{ X = b \}$$

Real functions of RV are RVs. e.g. $Z = f(X)$ for RV X and real function $f()$.



Similarly for multivariate functions. e.g. if X, Y are RVs, then $Z = f(X, Y) = X + Y$ is also a RV.

Discrete RV

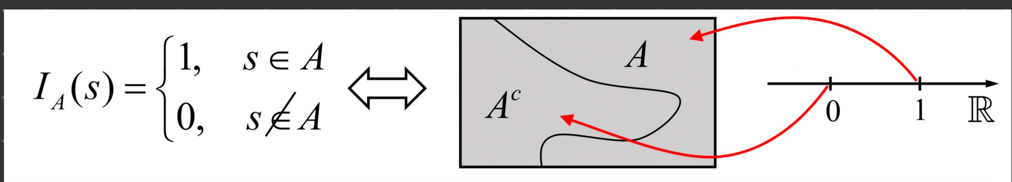
A RV X is called discrete if it can assume a finite $\{x_1, \dots, x_n\}$ or countably infinite $\{x_1, x_2, \dots\}$ number of values.

Discrete RVs partition Sample space in countable # of events.
e.g. for $X \in \{x_1, \dots, x_5\}$ define $A_i = \{X = x_i\}$, $\forall i = 1, \dots, 5$.

Indicator RVs

The indicator RV of event A, denoted by I_A, takes value 1 when A occurs and 0 otherwise.

Indicator RV partitions sample space in two events.



Ex 1:

Consider events A, B with indicator RVs I_A, I_B, and let $X = I_A \times I_B$.
Is X an indicator RV, and what is its event?

Yes, X is an IRV. $X(s) = I_A(s) \times I_B(s) = \begin{cases} 1 \times 1 = 1, & A \cap B \\ 1 \times 0 = 0, & A \cap B^c \\ 0 \times 1 = 0, & A^c \cap B \\ 0 \times 0 = 0, & A^c \cap B^c \end{cases}$

$I_{A \cap B}(s) = X(s)$

Ex 2 :

Consider events A, B with indicator RVs I_A, I_B

Express the indicator $I_{A\cup B}$ as a function of I_A, I_B.

$$I_{A\cup B}(s) = I_A(s) + I_B(s) - I_{A\cap B}(s)$$

$$= \begin{cases} 1 + 1 - 1\cdot 1 & , A\cap B \\ 0 + 1 - 0 & , A^c\cap B \\ 1 + 0 - 0 & , A\cap B^c \\ 0 + 0 - 0 & , A^c\cap B^c \end{cases}$$

$$= \begin{cases} 1 & , A\cup B \\ 1 & , A^c\cup B \\ 1 & , A\cup B^c \\ 0 & , A^c\cup B^c \end{cases}$$

Show $I_A - I_A \times I_B$ is an indicator RV & find its characteristic event.

$$I_A - I_A \times I_B = \begin{cases} 1 - 1\cdot 1 = 0 & , A\cap B \\ 1 - 1\cdot 0 = 1 & , A\cap B^c \\ 0 - 0\cdot 1 = 0 & , A^c\cap B \\ 0 - 0\cdot 0 = 0 & , A^c\cap B^c \end{cases}$$

$$= I_{A\cap B^c}$$

Another way:

$$I_A(s) \times I_A(s) I_B(s) = I_A(s) \times (1 - I_B(s))$$
$$= I_A(s) \times I_{B^c}(s)$$
$$= I_{(A\cap B^c)}(s)$$

RV Distribution

Events can also be defined by ranges of values of RVs. e.g. $\{X \in [a,b]\}$ or $\{X \geq 4\}$

The distribution of RV X is the collection of probabilities $P(X \in B)$ for all subsets B of the real line.

Cumulative Distributive Function

Distribution of any RV X is determined by its cumulative distribution function (CDF), defined as

$$F_X(x) = P(X \leq x) = P(\{s \in S : X(s) \leq x\}), \forall x \in \mathbb{R}$$

CDF gives probability of RV X being smaller or equal to x, for any value of X.

Use CDF to find probability of RV X being in any interval (a,b]

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a)$$

Proof:

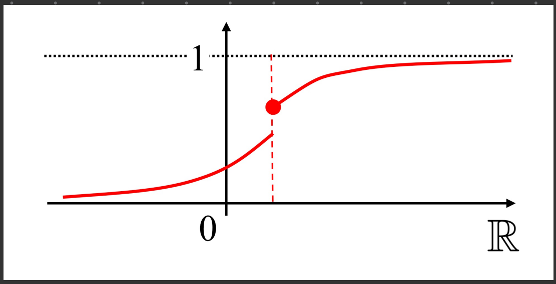
$$P(\{X \le b\}) = P(\{X \le b\} \cap \{X \le a\}) + P(\{X \le b\} \cap \{X \le a\}^c)$$
$$= P(X \le a) + P(\{a < X \le b\})$$
$$\Rightarrow F_X(b) = F_X(a) + P(\{a < X \le b\})$$

Every CDF must satisfy the following

(i) $F_X(-\infty) \equiv \lim_{x \to -\infty} F_X(x) = 0$

(ii) $F_X(\infty) \equiv \lim_{x \to \infty} F_X(x) = 1$

(iii) $\forall x_1 < x_2 \in \mathbb{R} \Rightarrow F_X(x_1) \le F_X(x_2)$



Discrete Distributions

Distribution of discrete RV $X \in \{x_1, x_2, \dots\}$ is determined by collection of all probabilities of the form

$$P(X = x_i) = P(\{s \in S : X(s) = x_i\}) = p_X(x_i), \ \forall i = 1, 2, \dots$$

Called probability mass function (PMF).

Properties of PMF:

(i) Sum to 1 : $\sum_{\forall i} p_X(x_i) = 1$

(ii) CDF given by : $F_X(x) = P(X \le x) = \sum_{x_i \le x} p_X(x_i), \ \forall x \in \mathbb{R}$.

Ex 3:

Roll 2 fair die & define RV X to be their absolute difference. Find the PMF of X.

a	0	1	2	3	4	5
$P(X = a)$	$6/36$	$10/36$	$8/36$	$6/36$	$4/36$	$2/36$