B24 July 28 Lec 1 Notes

Definition:

We call a function $f: V \to V$ (where V is an IPS) a rigid motion if

$$\|f(x) - f(y)\| = \|x - y\|, \ \forall x, y \in V$$

Theorem:

Let X be a real IPS, and $f: X \to X$ is a rigid motion, and define $T(x) := f(x) - f(0)$.
Then T is unitary (in particular, T is linear).

Lemma:

Let $T(x) := f(x) - f(0)$ be as above. Then:

(i) $\|Tx\| = \|x\|, \ \forall x \in X$
(ii) $\|Tx - Ty\| = \|x - y\|, \ \forall x, y \in X$
(iii) $\langle Tx, Ty \rangle = \langle x, y \rangle, \ \forall x, y \in X$

Proof: (i)

$$\|Tx\| = \|f(x) - f(0)\| = \|x - 0\| = \|x\| \quad \boxtimes$$

Proof: (ii)

$$\begin{aligned} \|Tx - Ty\| &= \|f(x) - f(0) - (f(y) - f(0))\| \\ &= \|f(x) - f(y)\| \\ &= \|x - y\| \quad \boxtimes \end{aligned}$$

Proof: (iii)

$$\begin{aligned} \|Tx - Ty\|^2 &= \|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, Ty \rangle \\ &= \|x\|^2 + \|y\|^2 - 2\langle Tx, Ty \rangle \end{aligned}$$

and

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$$

So

$$\|Tx - Ty\|^2 = \|x - y\|^2 \Rightarrow \langle x, y \rangle = \langle Tx, Ty \rangle \quad \boxtimes$$

Proof: of theorem

Since we already proved in the lemma that $\|Tx\| = \|x\|$, $\forall x \in X$, it suffices to prove T is a L.T.
Let $x, y \in X$ and $\alpha \in \mathbb{R}$.

$$\|T(x+\alpha y) - [T(x) + \alpha T(y)]\|^2 = \|[T(x+\alpha y) - T(x)] - \alpha T(y)\|^2$$

for $\mathbb{R}$-IPS, $\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle$

$$= \|T(x+\alpha y) - T(x)\|^2 + \|\alpha T(y)\|^2 - 2\langle T(x+\alpha y) - T(x), \alpha T(y)\rangle$$

$$= \|T(x+\alpha y) - T(x)\|^2 + |\alpha|^2 \|T(y)\|^2 - 2\alpha\langle T(x+\alpha y) - T(x), T(y)\rangle$$

$$= \|\cancel{x} + \alpha y - \cancel{x}\|^2 + |\alpha|^2\|y\|^2 - 2\alpha\langle Tx + \alpha y, T(y)\rangle + 2\alpha\langle T(x), T(y)\rangle$$

$$= \|\alpha y\|^2 + |\alpha|^2\|y\|^2 - 2\alpha\langle x + \alpha y, y\rangle + 2\alpha\langle x, y\rangle$$

$$= \|\alpha y\|^2 + |\alpha|^2\|y\|^2 - \cancel{2\alpha\langle x, y\rangle} \underbrace{-2\alpha\langle \alpha y, y\rangle}_{\substack{= -2\alpha^2\langle y, y\rangle \\ = -2\alpha^2\|y\|^2}} + \cancel{2\alpha\langle x, y\rangle}$$

$$= 2|\alpha|^2\|y\|^2 - 2\alpha^2\|y\|^2$$

$$= 0$$

Therefore $T(x+\alpha y) = T(x) + \alpha T(y)$ $\forall x, y \in X$, $\forall \alpha \in \mathbb{R}$

So T is linear $\boxtimes$

Theorem:

Let X be a $\mathbb{C}$-IPS, and $A: X \to X$ be a L.T. Then there exists an orthonormal basis $u_1, \dots, u_n$ for X s.t.

$$[A]_{u_1, \dots, u_n}^{u_1, \dots, u_n}$$

is upper triangular.

Corollary:

Any $n \times n$ complex matrix A is unitarily equivalent to an upper triangular matrix, i.e.

$$A = U \underset{\text{upper triangular}}{T} \overset{\text{unitary}}{U^*}$$

Proof:

We will induct on $n = \dim(X)$

Base case is trivial (All 1×1 matrices are upper triangular)

Assume (I.H) the result holds for n, we will show it holds for $n+1$.

Assume $\dim X = n+1$

Let u_1 be an eigenvector for A, with corresponding eigenvalue λ_1.

Roots of $\det(A - \lambda I)$ are exactly the eigenvalues of A.

Let $E = (\operatorname{span}(u_1))^{\perp}$.

Then $\dim(E) = n$, and let $v_2, \dots, v_{n+1}$ be an orthonormal basis for E.

Then:

$$[A]^{u_1, v_2, \dots, v_{n+1}}_{u_1, v_2, \dots, v_{n+1}} = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & \underbrace{A_1}_{n \times n \text{ matrix}} & \\ 0 & & & \end{array}\right]$$

A_1 defines a L.T. $A_1 : E \to E$, so since $\dim(E) = n$ inductive hypothesis applies to give us a basis $u_2, \dots, u_{n+1}$ s.t. $[A_1]^{u_2, \dots, u_{n+1}}_{u_2, \dots, u_{n+1}}$ is upper triangular.

Then:

$$[A]^{u_1, v_2, \dots, v_{n+1}}_{u_1, v_2, \dots, v_{n+1}} = \underbrace{\left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & [A_1]^{u_2, \dots, u_{n+1}}_{u_2, \dots, u_{n+1}} & \\ 0 & & & \end{array}\right]}_{\text{upper triangular}}$$

$\blacksquare$