Theorem

Let $B \subseteq V$.

B is a basis iff $\forall \vec{v} \in V$ can be written as a l.c. of vectors in B in a unique way.

Proof ($\Leftarrow$):

Suppose every vector $\vec{v}$ in V can be written as a unique l.c. of vectors in $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$.

WTS B is a basis for V

WTS B is L.I. and $\text{span}(B) = V$

<u>WTS $\text{span}(B) = V$:</u>

$\text{span}(B) \subseteq V$ by def of span

$V \overset{??}{\subseteq} \text{span}(B)$

Let $\vec{v} \in V$. WTS $\vec{v} \in \text{span}(B)$

By hyp $\exists r_1, \ldots, r_n$ in $\mathbb{R}$ s.t. $\vec{v} = r_1\vec{b}_1 + \ldots + r_n\vec{b}_n$

i.e. $\vec{v} \in \text{span}(B)$

<u>WTS B is L.I:</u>

Suppose $r_1\vec{b}_1 + \ldots + r_n\vec{b}_n = \vec{0}$. WTS $r_1 = r_2 = \ldots = r_n = 0$

Note $\vec{0} = 0\vec{b}_1 + 0\vec{b}_2 + \ldots + 0\vec{b}_n$

Since $\vec{0}$ can be written as a l.c. of $\vec{b}_i$ uniquely.

Then it must be the case that $r_1 = r_2 = \ldots = r_n = 0$ //

Theorem 3.3.7: Rank-nullity Theorem

For any $n \times m$ matrix A,

$$\dim(\ker A) + \dim(\operatorname{im} A) = m$$
$$\dim(\operatorname{Nul} A) + \dim(\operatorname{col} A) = m$$

$\ker T_A = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$=$ solution set to $A\vec{x} = \vec{0}$

e.g. $[A \mid \vec{0}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\ker T_A = \left\{ t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \end{pmatrix},\ t, s \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right)$$

$I = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $\ker T_A$

By Theorem 3.2.5, I is also L.I.

I is a basis $\ker T_A = \operatorname{Nul} A$

$\dim \ker T_A = \dim \operatorname{Nul} A = 2$ # of non pivot cols in rref(A)

$\operatorname{img} T_A = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^m\} = \operatorname{span}(\vec{a}_1, \ldots, \vec{a}_m) = \operatorname{col} A$

$\{\vec{a}_1, \ldots, \vec{a}_m\}$ is a spanning set for $\operatorname{img} T_A$

but $\{\vec{a}_1, \ldots, \vec{a}_m\}$ may not be a basis for $\operatorname{img} T_A$

e.g. $[A \mid \vec{0}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Theorem 3.8

$t=1,\ s=0 \quad \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \quad \begin{bmatrix} | & | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \\ | & | & | & | \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \Rightarrow \vec{a}_2 = 2\vec{a}_1$

$t=0,\ s=1 \quad \begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \end{pmatrix} \quad A\begin{pmatrix} 1 \\ 0 \\ -5 \\ 1 \end{pmatrix} = \vec{0} \quad \vec{a}_4 = -\vec{a}_1 + 5\vec{a}_3$

$\text{span}(\vec{a}_1, \vec{a}_3) = \text{span}(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$ $\therefore$ $\vec{a}_2$ and $\vec{a}_4$ is redundant

$\{\vec{a}_1, \vec{a}_2\}$ is a basis for T_A

$\left[\begin{array}{cc|c} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{0} \\ | & | & | \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow r\vec{a}_1 + s\vec{a}_2 = \vec{0} \Rightarrow r=s=0$

$\dim \text{img}\, T_A = 2 =$ # of pivot col's in rref (T_A).

$\therefore \dim \ker T_A + \dim \text{img}\, T_A = m$

$\begin{pmatrix} \text{\# of non-pivot} \\ \text{cols of rref}(A) \end{pmatrix} + \begin{pmatrix} \text{\# of pivot cols} \\ \text{of rref}(A) \end{pmatrix} =$ total # of cols of A

Theorem : General Rank Nullity Theorem

Suppose V and W are v.s. and $T: V \to W$ be a L.T.

$$\dim \ker T + \dim \text{img}\, T = \dim V$$

Theorem

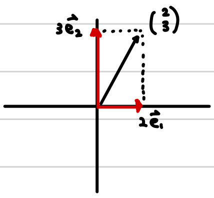
Let $B \subseteq V$.

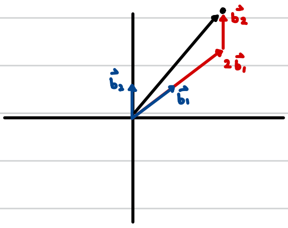
B is a basis iff $\forall \vec{v} \in V$ can be written as a l.c. of vectors in B in a unique way.

Ex 1

$B = \{\vec{e}_1, \vec{e}_2\}$ $\qquad$ $B = \{\binom{0}{1}, \binom{1}{1}\}$

$\binom{2}{3} = 2\vec{e}_1 + 3\vec{e}_2$ $\qquad$ $\binom{2}{3} = 2\binom{1}{1} + 1\binom{0}{1}$





Let V be a v.s. . $\dim V = n$. Let $B = (\vec{b}_1, \dots, \vec{b}_n)$ be an ordered basis for V.

e.g. $\mathbb{R}^2$ $\{\vec{e}_1, \vec{e}_2\} = \{\vec{e}_2, \vec{e}_1\}$

$\vec{v} = \binom{2}{3} = 2\vec{e}_1 + 3\vec{e}_2$

$\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ be an ordered basis

$\forall \vec{v} \in V$, $\vec{v} = r_1 \vec{b}_1 + \dots + r_n \vec{b}_n$

Def :

B coordinate of $\vec{v}$ in V to be

$$[\vec{v}]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

e.g. $V = P_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_1, a_2, a_0 \in \mathbb{R}\}$

$B_1 = (1, x, x^2)$, $\quad p(x) = 2x + 3x^2 \in P_2$

$= \underset{r_1}{0(1)} + \underset{r_2}{2x} + \underset{r_3}{3x^2}$

$$[p(x)]_{B_1} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$B_2 = (x, 1, x^2)$, $p(x) = 2x + 0(1) + 3x^2$

$$[p(x)]_{B_2} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

e.g. continued...

$\{1, 1+x, 1+x+x^2\}$ is a basis for P_2

$r_1 + r_2(1+x) + r_3(1+x+x^2) = \vec{0}$

$(r_1+r_2+r_3) + (r_2+r_3)x + r_3x^2 = 0 + 0x + 0x^2$

$$\begin{cases} r_1+r_2+r_3=0 \\ r_2+r_3=0 \\ r_3=0 \end{cases} \Rightarrow \begin{matrix} r_1=0 \\ r_2=0 \\ r_3=0 \end{matrix}$$

$\{1, 1+x, 1+x+x^2\}$ is a L.I

$$B_3 = (\underset{\vec{b}_1}{1}, \underset{\vec{b}_2}{1+x}, \underset{\vec{b}_3}{1+x+x^2})$$

$$[p(x)]_{B_3} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

$2x+3x^2 = r_1(1) + r_2(1+x) + r_3(1+x+x^2)$

$2x+3x^2 = (r_1+r_2+r_3)(1) + (r_2+r_3)x + r_3x^2$

$$\begin{cases} r_1+r_2+r_3=0 \\ r_2+r_3=2 \\ r_3=3 \end{cases} \Rightarrow \begin{matrix} r_1=-2 \\ r_2=-1 \\ r_3=3 \end{matrix}$$

Ex 2

$V=\mathbb{R}^2 \qquad \vec{v} = \binom{2}{3} \in \mathbb{R}^2$

$\mathcal{E} = (\vec{e}_1, \vec{e}_2) \qquad \vec{v} = [\vec{v}]_{\mathcal{E}} = \binom{2}{3}$ $\qquad \vec{v}$ of a coordinate with respect to $\mathcal{E}$.

$B = \left(\binom{1}{0}, \binom{1}{1}\right) \qquad [\vec{v}]_B = \binom{-1}{3}$

$\vec{v} = r_1\vec{b}_1 + r_2\vec{b}_2$

$\binom{2}{3} = r_1\binom{1}{0} + r_2\binom{1}{1}$

$$\underbrace{\begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix}}_{S} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \vec{v}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{matrix} r_1=-1 \\ r_2=3 \end{matrix}$$

must get unique solution

$S[\vec{v}]_B = [\vec{v}]_{\mathcal{E}} \qquad T_S : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \longmapsto S\vec{x}$