O S O

G T G T

5 E 5

N O N O

O S O

G T G T

5 E 5

N O N O

O S O

G T G T

5 E 5

A22 Mar 19 Lec 2 Notes

$\mathbb{R}$ is a field

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$q(x) = x^2 + 1$

Let i be a root for $q(x)$: $i^2 + 1 = 0$

$i^2 = -1$

$\mathbb{C} := \{ a + bi \mid a, b \in \mathbb{R} \}$

$\forall r \in \mathbb{R},\ r = r + 0i \in \mathbb{C}$ i.e $\subseteq \mathbb{C}$

Def : Arithmetic on $\mathbb{C}$

$z_1 = a_1 + b_1 i$; $z_2 = a_2 + b_2 i$, $a_1, b_1 \in \mathbb{R}$

$z_1 + z_2 := (a_1 + a_2) + (b_1 + b_2)i$

$i^2 = (\sqrt{-1})^2 = -1$

$$\begin{aligned} z_1 \cdot z_2 &:= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \end{aligned}$$

$\forall z = a + bi \in \mathbb{C}$,

$-z := (-a) + (-b)i \in \mathbb{C}$

$$\begin{aligned} z + (-z) &= (a + bi) + ((-a) + (-bi)) \\ &= (a - a) + (b - b)i \\ &= 0 + 0i \\ &= 0_F \end{aligned}$$

Ex 1

$z_1 = 1 + i$, $z_2 = 3 - i$

$$\begin{aligned} z_1 + z_2 &= (1+3) + (1-1)i \\ &= 4 + 0i \end{aligned}$$

$$\begin{aligned} z_1 \cdot z_2 &= (1+i)(3-i) \\ &= 3 + 3i - i - i^2 \\ &= (3+1) + (3-1)i \\ &= 4 + 2i \end{aligned}$$

Def: Absolute value

$z = a+bi \in \mathbb{C}$

$|z| := \sqrt{a^2 + b^2}$

Def: Complex conjugate

$z = a+bi \in \mathbb{C}$

$\bar{z} := a - bi$

$$\begin{aligned} z \cdot \bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + bai - (bi)^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

Def: Multiplicative Invese

$z \cdot \bar{z} = a^2 + b^2 \in \mathbb{R}$

$\left(\frac{1}{a^2+b^2}\right) z \cdot \bar{z} = a^2 + b^2 \left(\frac{1}{a^2+b^2}\right)$

$\frac{z \cdot \bar{z}}{z \cdot \bar{z}} = 1$

$z \cdot \left(\frac{\bar{z}}{z\bar{z}}\right) = 1$

Def: Division

$z_1, \; z_2 \neq 0 \in \mathbb{C}$

$\frac{z_1}{z_2} := z_1\left(\frac{1}{z_2}\right)$

Ex 2

$z_1 = 3+2i \;, \; z_2 = -1+i \;\; \neq 0$

$$\begin{aligned} \frac{z_1}{z_2} &= (3+2i)\left(\frac{1}{-1+i}\right) \\ &= (3+2i)\left(\frac{(-1-i)}{(-1+i)(-1-i)}\right) \\ &= (3+2i)\left(\frac{(-1-i)}{(-1)^2 - (i)^2}\right) \\ &= (3+2i) \cdot \frac{(-1-i)}{2} \\ &= \frac{1}{2}(-3-3i-2i+2) \quad = -\frac{1}{2} - \frac{5}{2}i \end{aligned}$$

Complex Numbers & Geometry

Theorem:

$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$ is a real vector space.

Proof:

Addition:

$z_1 = a_1 + b_1 i$; $z_2 = a_2 + b_2 i$, $a_1, b_1 \in \mathbb{R}$

$z_1 + z_2 := (a_1 + a_2) + (b_1 + b_2)i$

Scalar multiplication:

$z = a + bi \in \mathbb{C}$

$$\begin{aligned} rz &= r(a+bi) \quad , r \in \mathbb{R} \\ &= ra + rbi \end{aligned}$$

Additive invese:

Choose a $z = a + bi \in \mathbb{C}$,

Let $x = -z := (-a) + (-b)i \in \mathbb{C}$

$$\begin{aligned} z + (-z) &= (a + bi) + ((-a) + (-bi)) \\ &= (a - a) + (b - b)i \\ &= 0 + 0i \\ &= 0_F \end{aligned}$$

And more...

$\blacksquare$

Ex 3

Can we find a basis for $\mathbb{C}$?

$\dim(\mathbb{C}) = ?$

$\mathbb{C} = \{ a(1) + bi \mid a, b \in \mathbb{R} \}$

$= \operatorname{span}(1, i) \Rightarrow \{1, i\}$ is a spanning set for $\mathbb{C}$

$\underbrace{a(1) + bi}_{\in \mathbb{C}} = \underbrace{0 + 0i}_{\in \mathbb{C}} \Rightarrow a = 0$ and $b = 0 \Rightarrow \{1, i\}$ is L.I.

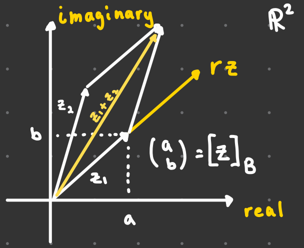
$\therefore \{1, i\}$ is a basis for $\mathbb{C}$, $\dim \mathbb{C} = 2$

$B = (1, i)$ ordered basis

$T_B : \mathbb{C} \longrightarrow \mathbb{R}^2$

$z = a + bi \longmapsto [z]_B = \begin{pmatrix} a \\ b \end{pmatrix}$

T_B is an isomorphism



$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$

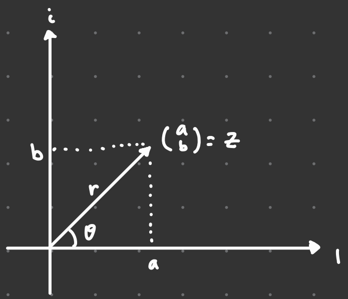
$T_B(z_1 + z_2) = T_B(z_1) + T_B(z_2)$

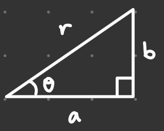
$T_B(r z_1) = r T_B(z_1)$

Polar Form

$z = a + bi$

$B = (1, i)$





$|z| = r$

$a = r\cos\theta$

$b = r\sin\theta$

$-\pi \leq \theta \leq \pi$ principle argument of z i.e. $\arg(z)$

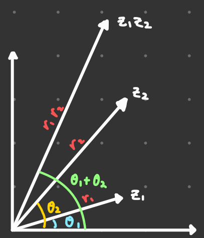
$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + \sin\theta\, i) \end{aligned}$$

Def: Euler's Formula

$$\cos\theta + \sin\theta\, i = e^{i\theta}$$

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$

$$\begin{aligned} z_1 \cdot z_2 &= r_1 r_2 e^{i\theta_1} \cdot e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \in \mathbb{C} \end{aligned}$$



Ex 4

$T: \mathbb{C} \longrightarrow \mathbb{C}$

$a+bi \longmapsto -b+ai \longleftarrow i(a+bi)$

Is T linear?

WTS $T((a_1+b_1i) + r(a_2+b_2i)) = T(a_1+b_1i) + rT(a_2+b_2i)$

LHS $i((a_1+b_1i) + r(a_2+b_2i)) = i(a_1+b_1i) + ri(a_2+b_2i)$
$= T(a_1+b_1i) + rT(a_2+b_2i)$

$\boxed{/\!/}$

$T': \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} -b \\ a \end{pmatrix}$

$$A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ rotation through $\frac{\pi}{2}$

$\therefore$ Multiplication by i corresponds to rotation in $\mathbb{R}^2$