B52 Nov 17 Lec 1 Notes

Normal Distribution

RV X follows Normal distribution with parameters mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if it has the following PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad \forall x \in \mathbb{R}$$

Case where $\mu = 0$ & $\sigma^2 = 1$ called standard Normal

No closed form CDF

Normal Approximation to Binomial

Theorem: De Moivre-Laplace Theorem

For RV $X \sim Bin(n,p)$, PMF of X converges to the PDF of Normal w/ the same mean (np) & variance (npq) as $n \to \infty$.

Normal Distribution

Properties:

(i) Linear functions of Normal RVs are Normal

(ii) Marginal distributions of multivariate Normal are Normal

(iii) Conditional distributions of multivariate Normal are Normal

Linear Functions

$E(x) = \mu$
$V(x) = \sigma^2$

Let $X \sim N(\mu, \sigma^2)$ and define the linear combination $Y = a + bX$ for real a, b. Verify that $Y \sim N(a + b\mu, b^2\sigma^2)$.

Since we have no CDF $\Rightarrow$ we have to use PDF:

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} \qquad y = h(x) = a + bx \Rightarrow \begin{cases} x = (y-a)/b = h^{-1}(y) \\ h'(x) = b \end{cases}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left\{-\frac{1}{2}\left(\frac{h^{-1}(y)-\mu}{\sigma}\right)^2\right.}{|b|}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 b^2}} \cdot \exp\left\{-\frac{1}{2}\left(\frac{\frac{y-a}{b} - \mu}{\sigma}\right)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 b^2}} \cdot \exp\left\{-\frac{1}{2}\left(\frac{y-(a+b\mu)}{\sigma \cdot b}\right)^2\right\}$$

$$\sim N(a + b\mu, \sigma^2 b^2)$$

Also, $E[Y] = E[a+bX] = a + bE(x) = a + b\mu$

$V[Y] = V[a+bX] = V[a] + V[bX] = b^2 V[X] = b^2\sigma^2$

Standardization

Ex 1:

For $X \sim N(5,4)$, find probability $P(X<-1)$.

Let $Z \sim N(0,1)$, $X \sim N(5,4)$. Then $X = 2Z + 5 \Rightarrow Z = \frac{X-5}{2}$

Z-transform: If $X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$

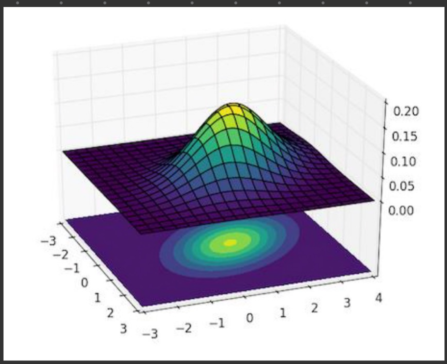
$$
\begin{aligned}
P(X<-1) &= P(X-5<-1-5) \\
&= P(\underbrace{\tfrac{X-5}{2}}_{\sim N(0,1)} < \tfrac{-1-5}{2}) \\
&= P(Z<-3) \\
&= 0.0013
\end{aligned}
$$

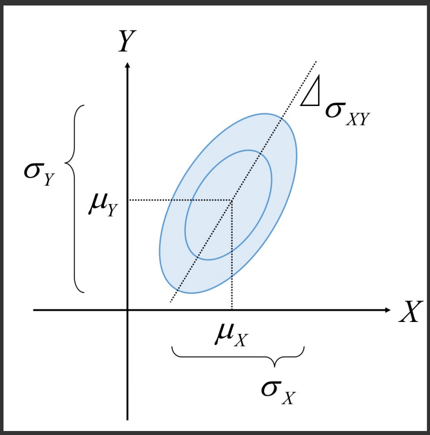
Bivariate Normal

Two RVs X, Y are jointly Normal if their PDF is

$$
f_{X,Y}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \begin{vmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{vmatrix}}} \cdot \exp\left\{ -\frac{1}{2} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}^T \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix}^{-1} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix} \right\}, \quad \forall x,y \in \mathbb{R}
$$

where
$$
\begin{cases}
\mu_X = E(X),\ \mu_Y = E(Y),\ \sigma_X^2 = V(X),\ \sigma_Y^2 = V(Y) \\
\sigma_{XY} = Cov(X,Y) = Corr(X,Y)\sqrt{V(X)\,V(Y)} = \rho_{XY}\,\sigma_X \sigma_Y
\end{cases}
$$





Marginal Distributions

$$
\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\mu = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{bmatrix} \right) \Rightarrow \begin{cases} X \sim N(\mu_X, \sigma_X^2) \\ Y \sim N(\mu_Y, \sigma_Y^2) \end{cases}
$$

Moreover, any linear combination of jointly Normal RVs is Normal, with parameters given by mean & variance of linear combination

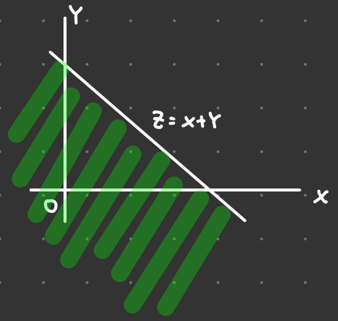
e.g. $W = aX + bY \Rightarrow W \sim N(E(W), V(W))$, where

$E(W) = aE(X) + bE(Y) = a\mu_X + b\mu_Y$

$V(W) = a^2 V(X) + b^2 V(Y) + 2ab\, Cov(X,Y) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}$

Ex 2:

Show that the sum of two i.i.d. standard Normal RVs is Normal. (With special case, where $X \perp Y$)



Let $Z = X+Y$, where $X \perp Y$ & $X, Y \sim N(0,1)$. WTS $Z \sim N(0,2)$

Find the PDF of Z using the convolution method

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x,z-x) f_Y(x,z-x)\,dx \qquad \text{By independence}$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left[-\frac{1}{2}(x^2+z^2+x^2-2zx)\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left[-\frac{1}{2}(z^2+2x^2-2zx)\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left[-\frac{1}{2}\left(\frac{x^2-zx+z^2/2}{1/2 = \sigma^2}\right)\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left[-\frac{1}{2}\left(\frac{x^2-2x\cdot\frac{z}{2}+\left(\frac{z}{2}\right)^2+\frac{z^2}{4}}{1/2}\right)\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\left[-\frac{1}{2}\frac{(x-z/2)^2}{1/2}\right]} e^{-z^2/4}\,dx$$

$$= \frac{1}{\sqrt{2\pi\cdot 2}} e^{-\frac{1}{2}\left(\frac{z-0}{2}\right)^2} \cdot \cancelto{1}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\cdot\frac{1}{2}}} e^{\left[-\frac{1}{2}\frac{(x-z/2)^2}{1/2}\right]}} dx$$

$$\sim N(0,2)$$