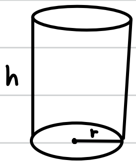
W12 Lecture 21 Notes

Optimization

Examples:

1. A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of metal to manufacture the can.

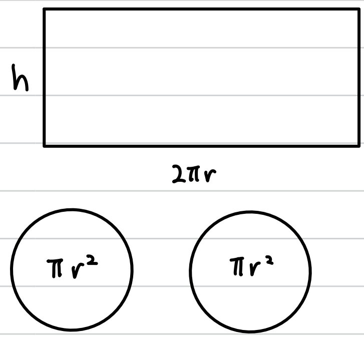
Notations, pics:



r = radius of the bottom.

h = height of the cylinder.

A = surface area of the can.



Formulate in math terms:

$$A = 2\pi r h + 2\pi r^2$$

Find min of $A(r, h)$

Solve the problem:

To eliminate the h variable we can use the fact that volume of the can $V = 1000\ cm^3$ is equal to the volume of a cylinder $\pi r^2 h$.

$$1000 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r \cdot \frac{1000}{\pi r} + 2\pi r^2 = \frac{2000}{r} + 2\pi r^2 = A(r)$$

If $\mathrm{Dom}\, A(r) = (0, \infty)$ min of $A(r)$ occurs at the critical point.

To find critical point:

$$A'(r) = 0 \Rightarrow A'(r) = 0 = -2000\, r^{-2} + 4\pi r$$

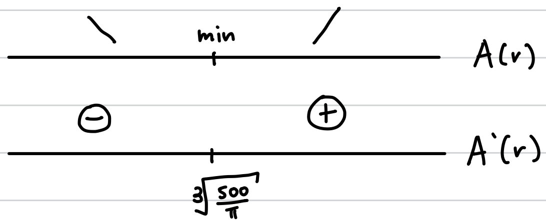
$$\Rightarrow 4\pi r^3 = 2000$$

$$\Rightarrow r_c = \sqrt[3]{\frac{500}{\pi}} \in \mathrm{Dom}\, A(r)$$

$$A'(r) = \text{DNE} \Rightarrow r = 0 \notin \mathrm{Dom}\, A(r)$$

Classify the critical points

a) Use 1st derivative test to investigate $A(r)$ to the right and left of r_c.



$A(r)$ has global min at $r_c = \sqrt[3]{\frac{500}{\pi}}$

b) $\lim\limits_{r \to 0} A(r) = \infty$ $\qquad$ $\lim\limits_{r \to \infty} A(r) = \infty$

There must be a minimum value of $A(r)$ which occurs at critical point.

Formulate the Answer

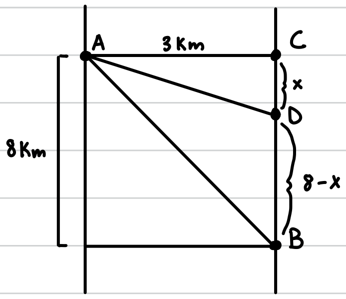
$$h = \frac{1000}{\pi r_c^2} = \frac{1000}{\pi \left(\frac{2000}{\pi}\right)^{2/3}} = 2 \cdot \sqrt[3]{\frac{500}{\pi}}$$
$$= 2r$$

To minimize the cost of the can, the radius should be $\sqrt[3]{\frac{500}{\pi}}$ cm and the height of the can should be equal to twice the radius. The minimal surface area of the can is:

$$A = 2\pi \cdot \left(\frac{500}{\pi}\right)^{2/3} + 2\pi \cdot r \cdot 2r = 2\pi \cdot \left(\frac{500}{\pi}\right)^{2/3} + 4\pi \cdot \left(\frac{500}{\pi}\right)^{2/3}$$
$$= 6\pi \cdot \left(\frac{500}{\pi}\right)^{2/3}$$

2. A man launches his boat from point A on the bank of a river, 3km wide and wants to reach point B, 8 km downstream on the opposite bank as quickly as possible. He could row his bat directly across the river to point C and then run to B, or he could row directly to B or he can row to some point D between C and B and run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that speed of the water is negligible compared with the speed at which the man rows).

Notations, picture



Let $|CD| = x$

$|AD| = \sqrt{9 - x^2}$

$|DB| = 8 - x$

$t = \dfrac{\text{Distance}}{\text{speed}}$

Formulate in math terms

$$T(x) = \{\text{Rowing time}\} + \{\text{Running time}\}$$

$$T(x) = \frac{\sqrt{9 + x^2}}{6} + \frac{8 - x}{8}$$

Solve the problem

Confined by question itself.

$$\text{Dom } T(x) = [0, 8]$$

$T'(x) = 0$: $\quad T'(x) = \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{2x}{\sqrt{9 + x^2}} - \frac{1}{8} = \frac{8x - 6\sqrt{9 + x^2}}{6 \cdot 8\sqrt{9 + x^2}} = 0 \Rightarrow$

$T'(x) = \text{DNE}$: No such points.

$$\Rightarrow 8x = 6\sqrt{9 + x^2}$$
$$16x^2 = 9(9 + x^2)$$
$$7x^2 = 81$$
$$x = \pm \frac{9}{\sqrt{7}}$$

Test the critical point and endpoints of the Domain

$T(0) \approx 1.5$

$T(9/\sqrt{7}) = 1 + \frac{7}{8\sqrt{7}} \approx 1.3$

$T(8) \approx 1.4$

$T(x)$ attains its global min. at $x = \frac{9}{\sqrt{7}}$.

Formulate the answer

The man should land the boat at a point $9/\sqrt{7}$ Km downstream from his standing point and run from this point to point B.