Q5

2. Use 2nd derivative test for $f(x) = e^x(x^2 - x - 19)$

$f(x) = e^x(x^2 - x - 19)$

$f'(x) = e^x(2x-1) + e^x(x^2 - x - 19)$
$= e^x(x^2 + x - 20)$

$f''(x) = e^x(2x+1) + e^x(x^2 + x - 20)$
$= e^x(x^2 + 3x - 19)$

$f'(x) = 0 = e^x(x^2 + x - 20)$
$= e^x(x+5)(x-4)$

$x = -5, 4$

$f''(4) = 491 > 0$
$f''(-5) = -0.06 < 0$

3. Find intervals of $f(x) = x^4 - 6x^3 - 9$ which are concave up or down.

$f(x) = x^4 - 6x^3 - 9$
$f'(x) = 4x^3 - 18x^2$
$f''(x) = 12x^2 - 36x$
$= 12x(x-3)$

$0 = 12x(x-3)$
$x_{c_1} = 0$
$x_{c_2} = 3$

4. Same for Q3. $f(x) = \dfrac{\sqrt{x}}{x-4}$.

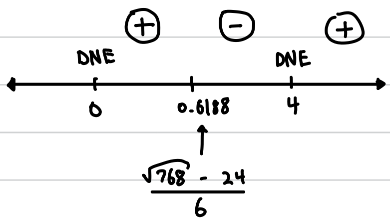
$$f(x) = \frac{\sqrt{x}}{x-4}$$

$$f'(x) = \frac{(x-4)\frac{1}{2}x^{-\frac{1}{2}} - \sqrt{x}}{(x-4)^2} = \frac{\frac{x}{2\sqrt{x}} - \frac{4}{2\sqrt{x}} - \sqrt{x}}{(x-4)^2} = \frac{\frac{1}{2}\sqrt{x} - \sqrt{x} - \frac{2}{\sqrt{x}}}{(x-4)^2}$$

$$= \frac{-\frac{1}{2}\sqrt{x} - \frac{2}{\sqrt{x}}}{(x-4)^2} = \frac{-\frac{1}{2}x - 2}{\sqrt{x}(x-4)^2}$$

$$f''(x) = \left(\frac{3x^2 + 24x - 16}{4x\sqrt{x}(x-4)^3}\right)$$

$x_{c_1} = 0.6188$
$x_{c_2} = 0$
$x_{c_3} = 4$



5. Same for Q3. $f(x) = \dfrac{x}{\ln(5x)}$

$$f'(x) = \frac{\ln(5x)\cdot 1 - x\left(\frac{1}{5x}\right)\cdot 5}{(\ln(5x))^2} = \frac{\ln 5x - 1}{(\ln(5x))^2} = \frac{1}{\ln 5x} - \frac{1}{(\ln 5x)^2}$$

$$f''(x) = -1\left(\ln(5x)\right)^{-2}\cdot\frac{5}{5x} - (-2)\left(\ln 5x\right)^{-3}\cdot\frac{5}{5x}$$

$$= \frac{-1}{(\ln 5x)^2\cdot x} + \frac{2}{(\ln 5x)^3\cdot x}$$

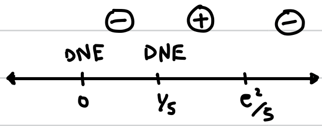
$$= \frac{2-\ln 5x}{(\ln 5x)^3\cdot x}$$

$f'(x) = 0$: $2 - \ln 5x = 0$

$2 = \ln 5x$

$e^2 = 5x$

$x = \frac{e^2}{5}$



$f'(x) = \text{DNE}$: $\ln 5x = 0$ $x = 0$

$5x = e^0$

$x = \frac{1}{5}$

8. $\displaystyle\lim_{x\to 0}\frac{2^x - 1}{4^x - 1}$

$$\overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{2^x\ln 2}{4^x\ln 4} = \frac{\ln 2}{\ln 4}\lim_{x\to 0}\left(\frac{2}{4}\right)^x = \frac{\ln 2}{\ln 4} = \log_4 2 = \frac{1}{2}$$

With L'H Rule:

$$\lim_{x\to 0} e^{\ln\frac{2^x-1}{4^x-1}} = e^{\lim_{x\to 0}\left[\ln(2^x-1) - \ln(4^x-1)\right]}$$

$= ??$

9. $\displaystyle\lim_{x\to\infty}\frac{x+9^x}{9-8^x}$

$$\overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \frac{1+9^x\ln 9}{-8^x\ln 8} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} -\frac{9^x\ln 9\ln 9}{8^x\ln 8\ln 8}$$

$$= -\frac{\ln^2 9}{\ln^2 8}\left(\frac{9}{8}\right)^x$$

$$= -\infty$$

10. $\lim_{x\to\infty} \left(4e^x - \frac{20e^x}{x+5}\right)$

$= \lim_{x\to\infty} \frac{4e^x(x+5) - 20e^x}{x+5}$

$= \lim_{x\to\infty} \frac{4xe^x + \cancel{20e^x - 20e^x}}{x+5} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{4e^x + 4xe^x}{1} = \lim_{x\to\infty} 4e^x(1+x)$

$= \infty$

11. $\lim_{x\to\infty} \frac{2xe^{-5x}}{x^2+6x+1}$

$= \frac{\frac{2x}{e^{5x}}}{x^2+6x+1}$

$= \frac{2x}{(x^2+6x+1)e^{5x}}$

$\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \frac{2}{e^{5x}(2x+6) + 5e^{5x}(x^2+6x+1)}$

12. $\lim_{x\to\infty} \frac{4\ln x}{2\ln(2x+1)}$

$= 2 \cdot \lim_{x\to\infty} \frac{\ln x}{\ln(2x+1)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} 2 \cdot \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{2}{2x+1}}$

$= \lim_{x\to\infty} \frac{2x+1}{x}$

$= \lim_{x\to\infty} \frac{2x}{x} + \lim_{x\to\infty} \frac{1}{x}$

$= 2 + 0$

13. $\lim_{x\to 0^+} \frac{\log_5 x}{\log_5 4x}$

$\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \frac{\frac{1}{x\ln 5}}{\frac{4}{4x\ln 5}}$

$= 1$

14. $\lim_{x\to 0} \frac{5-5\cos x}{4\tan x}$

$\overset{\frac{0}{0}}{\underset{L'H}{=}} \frac{5\sin x}{4\sec^2 x} = 0$

15. $\lim_{x\to 0} \frac{4x\cos x}{5-5e^x}$

$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{4\cos x + 4x(-\sin x)}{-5e^x} = \frac{4(1) + 4(0)(0)}{-5(1)} = -\frac{4}{5}$

16. Give formulas for the volume V and the surface area S of a cone whose height h is three times its radius r.

$$V = \pi r^2 \left(\frac{3r}{3}\right) \qquad (h = 3r)$$

$$SA = \pi r \left(r + \sqrt{(3r)^2 + r^2}\right)$$

17. Write down an equation that relates the two quantities described. Then use implicit differentiation to obtain a relationship between the rates at which the quantities change over time.

The surface area S and height h of a cylinder with a fixed radius of 12 units.

$$S(t) = 2\pi r^2 + 2\pi r h(t)$$

$$\frac{dS(t)}{dt} = 2\pi r \cdot \frac{dh(t)}{dt}$$

$$\left.\frac{dS(t)}{dt}\right|_{r=12} = 24\pi \cdot \frac{dh(t)}{dt}$$

18. Same as 17. The volume V and height h of a cone with a fixed radius of 8 units.

$$V(t) = \frac{\pi}{3} r^2 h(t)$$

$$\frac{dv(t)}{dt} = \left(\frac{\pi}{3} r^2\right) \frac{dh(t)}{dt}$$

$$\left.\frac{dv(t)}{dt}\right|_{r=8} = \frac{64\pi}{3} \cdot \frac{dh(t)}{dt}$$

19. Suppose the sides of a cube are expanding at a rate of 7 inches/min. How fast is the volume of the cube changing at the moment that the cube has a side length of 10 inches?

$$V(t) = a^3(t)$$

$$\frac{dv(t)}{dt} = 3a^2(t) \cdot \frac{da(t)}{dt}$$

$$\frac{dv(t)}{dt} = 3(10)^2 \cdot 7$$

$$= 2100$$

20. Consider a large helium balloon is being inflated at the rate of 120 in^3/s.

How fast is the radius of the balloon increasing at the instant that the balloon has a radius of 10 in?

$$V(t) = \frac{4}{3}\pi\, r^3(t)$$

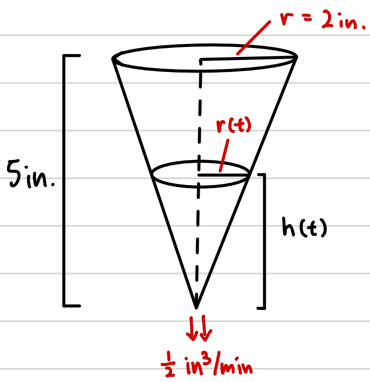
$$\frac{dV(t)}{dt} = \left(\frac{4}{3}\pi\right)(3)\cdot r^2(t)\cdot\frac{dr(t)}{dt}$$

$$\frac{dV(t)}{dt} = 4\pi\cdot r^2(t)\cdot\frac{dr(t)}{dt}$$

$$120\ in^3/s = 4\pi(10\,in)^2\cdot\frac{dr(t)}{dt}$$

$$\frac{dr(t)}{dt} = \frac{120}{400\pi} = \frac{3}{10\pi}\ in/s$$

21. Riley is holding an ice cream cone on a hot summer day. The cone has a small hole in the bottom. Ice cream is dripping through the hole at a rate of half a cubic inch per minute and a height of 5 inches.



$$\frac{r(t)}{h(t)} = \frac{2\,in.}{5\,in.}$$

$$r(t) = \frac{2}{5}\cdot h(t)$$

$$V = \frac{\pi}{3} r^2 h$$

$$V(t) = \frac{\pi}{3}\, r^2(t)\cdot h(t)$$

$$V(t) = \frac{\pi}{3}\left(\frac{2}{5}h(t)\right)^2\cdot h(t) = \frac{4\pi}{75}h^3(t)$$

$$\frac{dV(t)}{dt} = \frac{4\pi}{25}\cdot h^2(t)\cdot\frac{dh(t)}{dt}$$

$$\frac{dh(t)}{dt} = \frac{-\frac{1}{2}\ in^3/min}{\frac{4\pi}{25}(2\,in)^2}$$

$$\frac{dh(t)}{dt} = \frac{-\frac{1}{2}}{\frac{16\pi}{25}}\ in./min$$

$$= -\frac{25}{32\pi}$$