Regular Languages 3 of 3

Closure Properties of Regular Languages

(i) Regex Method (Structural Induction)

(ii) FSA Method

(a) Suppose $L = \mathcal{L}(M)$ for some DFSA M

(b) Construct an NFSA (or DFSA) that accepts $f(L)$

↳ Also works for $f(L_1, L_2)$

Ex 1:

Suppose $M = (Q, \Sigma, \delta, s, F)$ accepts L, where M is a DFSA. Find an FSA $\bar{M}$ s.t. $\bar{M}$ accepts $\bar{L}$. (This proves closure of regular languages under complementation.)

Let $\bar{M} = (Q, \Sigma, \delta, s, \bar{F})$, $\bar{F} = Q - F = \{ q \in Q , q \notin F \}$

M accepts $x \iff \delta^*(s, x) \in F \iff \delta^*(s, x) \notin \bar{F}$
$\iff \bar{M}$ rejects x

Closure Under Union:

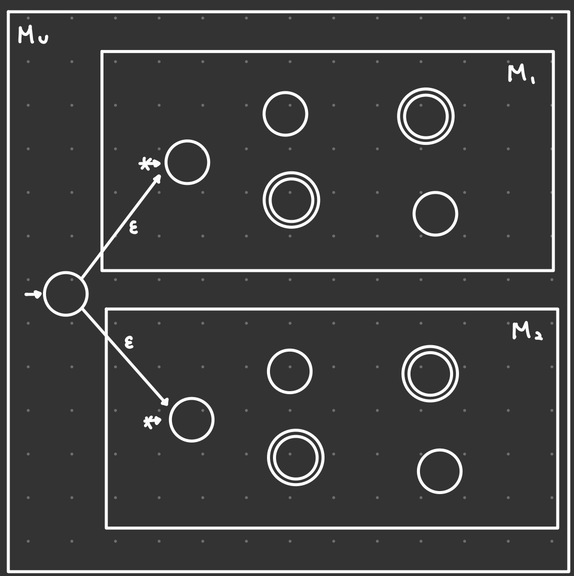
Regex Method:

Suppose $L_1 = \mathcal{L}(R_1)$ and $L_2 = \mathcal{L}(R_2)$, where R_1, R_2 are regexes.

Then $L_1 \cup L_2 = \underline{\mathcal{L}(R_1 + R_2)}$

FSA Method:

Suppose $L_1 = \mathcal{L}(M_1)$, $L_2 = \mathcal{L}(M_2)$, where M_1, M_2 are DFSAs. Construct FSA $M_\cup$ s.t. $\mathcal{L}(M_\cup) = L_1 \cup L_2$



Closure Under Intersection: Suppose L_1, L_2 are regular. Prove that $L_1 \cap L_2$ is regular.

Regex Method: Not easy to do

Different Method: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

FSA Method (Cartesian Product Construction):

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$, $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$, M_1, M_2 are DFSAs

Construct DFSA $M_n = (Q_n, \Sigma, \delta_n, s_n, F_n)$

Let $Q_n = Q_1 \times Q_2 = \{(q_1, q_2), q_1 \in Q_1, q_2 \in Q_2\}$

$s_n = (s_1, s_2)$

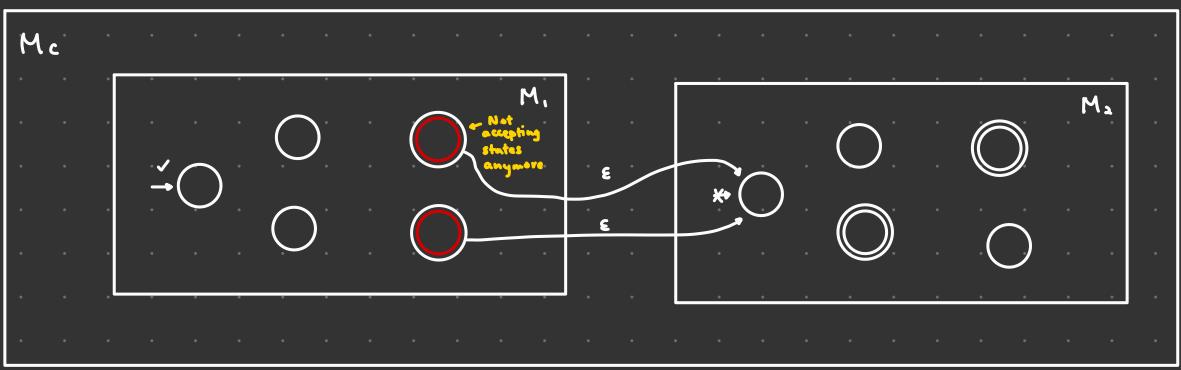
$F_n = F_1 \times F_2$

$\delta_n((q_1,q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$

Closure Under Concatenation:

Regex Method: Suppose $L_1 = \mathcal{L}(R_1)$, $L_2 = \mathcal{L}(R_2)$. Then $L_1 \cdot L_2 = \mathcal{L}(R_1 R_2)$

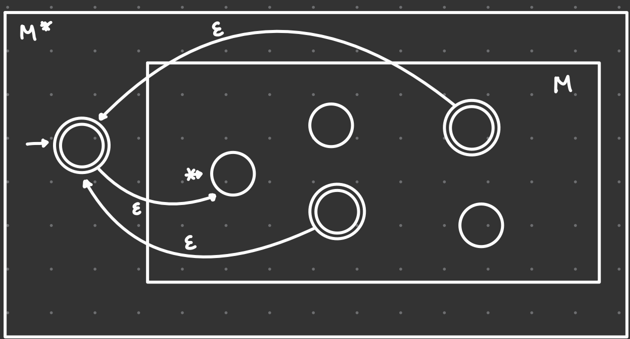
FSA Method: Suppose $L_1 = \mathcal{L}(M_1)$, $L_2 = \mathcal{L}(M_2)$, M_1, M_2 are DFSAs.
Construct NFSA M_c s.t. $\mathcal{L}(M_c) = L_1 \cdot L_2$



Closure Under Star: Suppose L is regular. Prove L^* is also regular.

Regex Method: Easy

FSA Method: Suppose $L = \mathcal{L}(\overset{\text{DFSA}}{M})$. Construct NFSA M^* s.t. $\mathcal{L}(M^*) = L^*$.



Proving Nonregularity (Sec 7.7)

Definition:

For arbitrary strings x, y, define $\#_y(x)$ to be

$$\left| \{ (u,v) \} : x = uyv \} \right|$$

$\#_y(x)$ is the # of places where y appears in x.

e.g. $x = 111$, $y = 1$, $\#_y(x) = \#_1(111) = 3$

Pumping Lemma (PL)

Let L be a regular language. Then there's a # $n > 0$ s.t. every x in L with length at least n satisfies the following property.

There exist strings u, v, w s.t.

(i) $x = uvw$

(ii) $v \neq \varepsilon$

(iii) $|uv| \leq n$

(iv) for every $k \in \mathbb{N}$, $uv^k w \in L$

e.g. $x = \underbrace{a_1 \quad a_2 \quad \cdots \quad a_n}_{n} \quad \cdots \quad a_m$

$\underbrace{\quad}_{u}\underbrace{\quad}_{v}\underbrace{\qquad\qquad\qquad}_{w}$

DFSA M, $\mathcal{L}(M) = L$

Let $n =$ # states in M

$$x = \overbrace{a_1 \quad a_2 \cdots a_i}^{u} \overbrace{\cdots a_j}^{v} \overbrace{\cdots a_n \quad \cdots \quad a_m}^{w}$$

$\underbrace{q_0 \quad q_1 \quad q_2 \quad q_i \quad q_j \quad q_n}_{n+1 \text{ states}} \qquad q_m \qquad q_i = \delta(s, \text{first } i \text{ symbols of } x)$

$q_i = q_j$, $i < j$

$q_i = q_j$ by the pigeonhole principle.

For any $k \in \mathbb{N}$, v^k will get us to the same state k many times since $q_i = q_j$.

Ex 2:

Let $\Sigma = \{0,1\}$. Let $L = \{ x \in \Sigma^* : \#_0(x) = \#_1(x) \}$

Prove that L is not regular. Use PL.

Every PL proof has these lines.

By way of contradiction, suppose L is regular.

Let n be as in PL.

Let $x = 0^n 1^n$

Then $|x| = 2n \geq n$ and $x \in L$ def of L

By PL, there are u, v, w, s.t.

(i) $x = uvw$

(ii) $v \neq \varepsilon$

(iii) $|uv| \leq n$

(iv) for every $k \in \mathbb{N}$, $uv^k w \in L$

By (i) and (iv), $uv = 0^j$, where $0 \leq j \leq n$
By (ii), $v = 0^j$ where $0 < j \leq n$
By (iv), $uv^2 w \in L$, but $uv^2 w = 0^{n+j} 1^n \notin L$

$\therefore$ Contradiction

$\boxtimes$

Ex 3:

Let $\Sigma = \{0,1\}$. Let $L' = \{ 0^n 1^n : n \in \mathbb{N} \}$

L' is not regular

$L' = L \cap \mathcal{L}(0^* 1^*)$

From Ex 2

$\therefore$ L from Ex 2 is also not regular by closure of intersection.