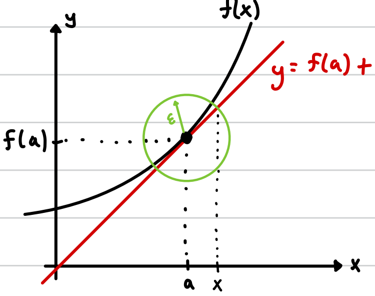
W7 Lecture 15 Notes

Local Linearity



In the neighborhood of point a where $(x-a)^2 + (y-f(a))^2 < \varepsilon$ the value of function f can be approximated by the value of y:

$$f(a+\Delta x) \approx f(a) + f'(a)\Delta x$$
$$L(x) = f(a) + f'(a)\Delta x$$

Example:

1. Find linearization of x^2 near $a=2$.

$f(2) = 4$

$$f'(a) = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = 2a$$

$f'(2) = 2\cdot 2 = 4$

$L(x) = 4 + 4(x-2)$, $\Delta x = x-2$

2. Find the approximation to the value of $\sqrt{16.1}$

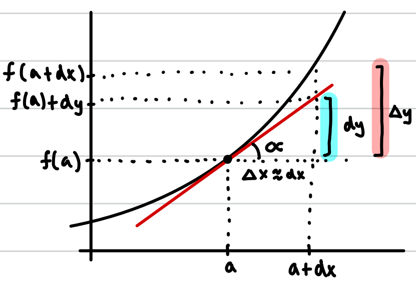
Let $f(x) = \sqrt{x}$ $a = 16$ $\Delta x = 0.1$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})\cdot(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h - x}{h(\sqrt{x+h}) + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$f(a+\Delta x) = f(16+0.1) = \sqrt{16} + \frac{1}{2\sqrt{16}}\cdot 0.1 = 4.0125$$

Differentials and Leibniz Notation



Increments Δx and Δy are small finite changes in x and y.
Differentials dx and dy are infinitesimally small changes in x and y.

As dx approaches 0,

$$f(a+dx) \approx f(a) + dy$$
$$f(a) + f'(a)dx \approx f(a) + dy$$
$$dy \approx f'(a)\,dx$$
$$dy = f'(x)\,dx$$ – Differential y

$\tan\alpha = m = f'(a)$
$\Rightarrow f'(a) = \frac{dy}{dx}$
$\Rightarrow dy = f'(a)\,dx$
$\Rightarrow dy = f'(x)\,dx$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

If $\Delta x \to 0$, then $\Delta x = dx$

Examples:

3. For the given function $f(x) = \frac{1}{x+2}$ find Δy and dy when $x=1$ and $\Delta x = 0.01$.

$$\Delta y = y(x+\Delta x) - y(x)$$
$$\Delta y = y(1+0.01) - y(1)$$
$$= \frac{1}{1.01+2} - \frac{1}{3}$$
$$= -0.0011074$$

$$\boxed{\begin{aligned} dy &= \left(-\tfrac{1}{9}\right)\cdot 0.01 \\ &= -0.0011111 \end{aligned}}$$

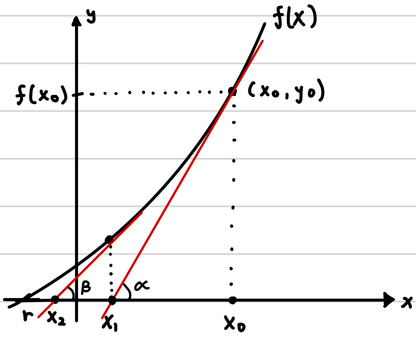
we can say this when we are close to the point

$$dy = f'(1)\,dx \;, \quad \underbrace{dx \approx \Delta x}_{} = 0.01$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+\cancel{2}-\cancel{x}-\cancel{h}-\cancel{2}}{(x+h+2)(x+2)\cancel{h}} = \frac{1}{(x+2)^2} \Rightarrow f'(1) = -\frac{1}{9}$$

Newton's Method



Newton's method is iterative method for generating a sequence of approximations to a solution of the equation $f(x) = 0$.

Definition

We say that sequence of approximations $x_1, x_2, \dots x_n \dots$ converges to the solution r if $\forall \varepsilon > 0\ \exists N > 0,\ N \in \mathbb{Z}$ s.t. $|x_n - r| < \varepsilon$ for any $n \geq N$.

Iterations

1st: $f'(x_0) = \tan\alpha = \frac{y_0 - 0}{x_0 - x_1} \Rightarrow f'(x_0)(x_0 - x_1) = y_0 = f(x_0) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

2nd: $f'(x_1) = \tan\beta = \frac{y_1 - 0}{x_1 - x_2} \Rightarrow f'(x_1)(x_1 - x_2) = y_1 = f(x_1) \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

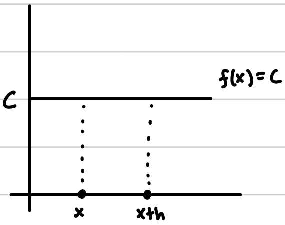
(n+1)th: $f'(x_n) = \tan\theta_n = \frac{y_n - 0}{x_n - x_{n+1}} \Rightarrow f'(x_n)(x_n - x_{n+1}) = y_n = f(x_n) \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Iterative Formula of the Newton Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rules for Differentiation

1. Derivative of constant function: $\frac{dc}{dx} = 0$



$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h}$$

← c-c=0. It is not approaching 0. h is.

$$= 0$$

QED

2. Power Rule. $f(x) = x^n$, $\frac{d}{dx}(x^n) = nx^{n-1}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + n \cdot a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \ldots + b^n$$

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2} x^{n-2} h^2 + \ldots + h^n) - x^n}{h}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \ldots + h^{n-1}\right)$$

$$= nx^{n-1}$$

QED

Examples:

4. $f_1(x) = x^3 + 3$ $\quad f_1'(x) = 3x^2$
$f_2(x) = x^3 + 100$ $\quad f_2'(x) = 3x^2$
$f_3(x) = x^3 + C$ $\quad f_3'(x) = 3x^2$

$3x^2$ is the derivative of x^3+c

x^3+c is the anti derivative of $3x^2$

$x^3 + c = \int 3x^2 \, dx$ Indefinite integral of $3x^2$ with respect to x

3. Constant Multiple Rule: $\frac{d}{dx}(rf(x)) = r \cdot \frac{d}{dx}(f(x))$

Let $f(x)$ be differentiable on $\mathbb{R}$, and $r \in \mathbb{R}$

$$\frac{d}{dx}(rf(x)) = \lim_{h\to 0} \frac{rf(x+h) - rf(x)}{h} = \lim_{h\to 0} r \cdot \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} r \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= rf'(x)$$

QED

4. The Algebraic Sum Rule: $f(x) \pm g(x) = f'(x) \pm g'(x)$

Let $f(x)$ and $g(x)$ be differentiable on $\mathbb{R}$, $x \in \mathbb{R}$

$$\frac{d}{dx}(f+g) = \lim_{h \to 0} \frac{(f(x+h) \pm g(x+h)) - (f(x) \pm g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \pm g'(x)$$

5. The Product Rule: $f(x) \cdot g(x) = f'(x) g(x) + f(x) g'(x)$

Let $f(x)$ and $g(x)$ be differentiable on $\mathbb{R}$, $x \in \mathbb{R}$

$$\frac{d}{dx}(f(x) \cdot g(x)) = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)] g(x+h) + [g(x+h) - g(x)] \cdot f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h \to 0} f(x)$$

$$= f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

QED