B24 May 26 Lec 1 Notes

Ex 1:

(i) $M_{2\times 2}^{\mathbb{F}} \to \mathbb{F}^4$ is an isomorphism

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a,b,c,d)$

(ii) Let V, W be v.s. with bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ respectively. Recall
$L(V,W) := \{ T: V \to W \mid T \text{ is a L.T.} \}$

Then,

$$L(V, W) \to M_{m\times n}^{F}$$
$$T \mapsto [T]_{w_1,\dots,w_m}^{v_1,\dots,v_n}$$

is an isomorphism. In particular, there is a 1-1 correspondence between $L(V,W)$ and $M_{m\times n}^{\mathbb{F}}$.

The mapping / isomorphism $L(V,W)$ depends on $v_1, \dots, v_n$ and $w_1, \dots, w_m$.

What is the inverse of $L(V,W)$?

If $B \in M_{m\times n}^{F}$ and $v \in V$, then $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{F}$ s.t. $v = \alpha_1 v_1 + \dots + \alpha_n v_n$.

Then $B \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ is an $m\times 1$ matrix, call it $\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$.

We define $\Psi(B): V \to W$ by $\Psi(B)(v) = B_1 w_1 + \dots + B_m w_m$

We can check that $\Psi: M_{m\times n}^{\mathbb{F}} \to L(V,W)$ is a L.T. and is inverse to

$[\cdot]: L(V,W) \to M_{m\times n}^{\mathbb{F}}$

Theorem:

Let $T: V \to W$ be a L.T. Then T is invertible iff for any $w \in W$ the equation $Tx = w$ has an unique solution $x \in V$.

Proof ($\Rightarrow$):

$T^{-1}w$ is a solution to $Tx = w$ since $T(T^{-1}w) = (TT^{-1})w = Iw = w$
And if x^* is also a solution to $Tx = w$, then

$$\begin{aligned} & Tx^* = w = T(T^{-1}w) = w \\ \Rightarrow\ & T^{-1}Tx^* = T^{-1}w \\ \Rightarrow\ & x^* = T^{-1}w \end{aligned}$$

Proof ($\Leftarrow$):

Define $S: W \to V$ by Sw is defined as the unique solution x to $Tx = w$, for any $w \in W$.

Then $TSw = T(Sw) = w$, i.e. $TS = I_W$

and STv by def is the unique solution to $Tx = Tv$, which is $x = v$
i.e. $STv = v$, so $ST = I_V$

Definition:

Let V be a v.s. A subset $V_0 \in V$ is called a subspace of V if:

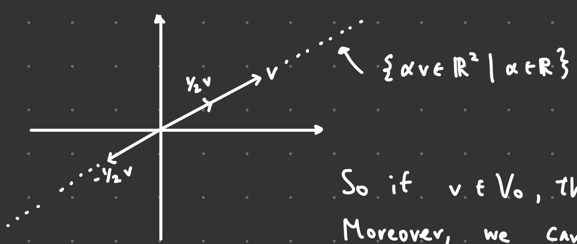
(i) $v \in V_0 \Rightarrow \alpha v \in V_0$, $\forall \alpha \in \mathbb{F}$, and

(ii) $u, v \in V_0 \Rightarrow u + v \in V_0$

Remark: if $V_0 \in V$ is a subspace, V_0 is also a v.s.

Ex 2:

Let $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$. What are the subspaces?
Let $V_0 \subset \mathbb{R}^2$ be a subspace, and $v \in V_0$.



So if $v \in V_0$, then the line passing through v is also a subset of V_0.
Moreover, we can check $\{\alpha v \in \mathbb{R}^2 \mid \alpha \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.
The only subspaces of $\mathbb{R}^2$ are $\{(0,0)\}$, lines through the origin, and $\mathbb{R}^2$.

Ex 3:

$V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$. Here subspaces are $\{0\}$, lines through 0 vector, planes through 0 vector, $\mathbb{R}^3$.

Definition:

If $v_1, \dots, v_n \in V$, the linear span of $v_1, \dots, v_n$ is defined as:

$$L(v_1, \dots, v_n) := \{ \alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{F} \}$$

Definition:

Given a L.T. $T: V \to W$. The nul space or Kernel of T is defined by:

$$\mathrm{Nul}(T) = \mathrm{Ker}(T) := \{ v \in V : Tv = 0 \}$$

Definition:

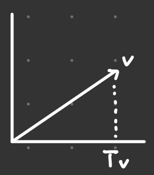
Given a L.T. $T: V \to W$. The range or image of T is defined by:

$$\mathrm{Ran}(T) = \mathrm{img}(T) := \{ w \in W \mid \exists v \in V \text{ with } Tv = w \}$$

Remark: Kernel, image, and linear span are all subspaces.

Ex 4:

Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (x, 0)$



$\mathrm{Ker}\, T = \{ (0, y) \mid y \in \mathbb{R} \}$

$\mathrm{Img}\, T = \{ (x, 0) \mid x \in \mathbb{R} \}$

Ex 5:

What are the Kernel, ranges of projection onto the line $y = {}^{-2x}/_{3}$.