# A22 Apr 7 Lec 1 Notes

**Def:**

Given $A$ $n\times n$ matrix, $\det(A-\lambda I)$ is a polynomial of degree $n$ in variable $\lambda$ is called a characteristic polynomial of $A$.

**Ex 1:**

$$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$
$$\vec{x} \mapsto A\vec{x}$$

Can I find an eigenbasis for $T$?

Step 1: Find all eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} \qquad \underline{C_3 \leftrightarrow C_3 + C_1}$$

$$= \begin{vmatrix} 2-\lambda & 1 & 2-\lambda \\ -1 & -\lambda & 0 \\ 1 & 3 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & -\lambda & 0 \\ 1 & 3 & 1 \end{vmatrix} \qquad \underline{R_3 \leftrightarrow R_3 - R_1}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & -\lambda & 0 \\ -1-\lambda & 2 & 0 \end{vmatrix}$$

$$= (2-\lambda)\Big(1\big((-1)(2) + \lambda(-1-\lambda)\big)\Big)$$

$$= (2-\lambda)(-2-\lambda+\lambda^2)$$

$$= (2-\lambda)(\lambda-2)(\lambda+1)$$

$\text{char}(A) = -(\lambda-2)^2(\lambda+1)^1$

(Characteristic poly)

$\lambda = 2$, $\lambda = -1$

with alg. multiplicity of 2.

**Def:**

Algebraic multiplicity of an eigenvalue $\lambda_0$ is the # of times $(\lambda-\lambda_0)$ shows up in $\text{char}(A)$.

Ex 8 continued... from last class

Q : Find all possible eigenvectors for $\lambda = 2$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 2$: $\text{Nul}(A - 2I) = \text{span}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigenspace $E_2 = \text{span}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ geo. multi. of $\lambda = 2$ is 1
alg. multi. of $\lambda = 2$ is 2

$\lambda = 1$: $\text{Nul}(A - (-1)I) = \text{Nul}(A + I)$

$$= \text{Nul}\begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \text{Span}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

$E_{-1} = \text{span}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ geo. multi. of $\lambda = -1$ is 1
alg. multi. of $\lambda = -1$ is 1

$\therefore$ There is no eigen basis as there is not enough L.I. eigen vectors.

$\therefore$ A is not diagonalizable.

Explanation :

If $\mathcal{B} = (\vec{b_1}, \vec{b_2}, \vec{b_3})$ is an eigen basis

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & | \\ [T(\vec{b_1})]_{\mathcal{B}} & [T(\vec{b_2})]_{\mathcal{B}} & [T(\vec{b_3})]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$T(\vec{b_1}) = \lambda_1 \vec{b_1} + 0\vec{b_2} + 0\vec{b_3}$$
$$T(\vec{b_2}) = 0\vec{b_1} + \lambda\vec{b_2} + 0\vec{b_3}$$
$$T(\vec{b_3}) = 0\vec{b_1} + 0\vec{b_2} + \lambda\vec{b_3}$$

$\therefore \nexists$ eigenbasis for $\mathbb{R}^3$

Def :

Given an eigenvalue $\lambda$ for $T: V \to V$, eigenspace for $\lambda$ is

$$E_\lambda = \{ \vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v} \}$$

Note. $E_\lambda$ contains 0, but set of all eigenvectors for eigenvalue $\lambda$ doesn't.

**Def:**

Given an eigenvalue $\lambda$, geometric multiplicity of $\lambda$ is $\dim(E_\lambda)$.

**Ex 2:**

$$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$$
$$\vec{x} \mapsto A\vec{x}$$

Find an invertible matrix $S$ and a diagonal matrix $D$ s.t. $S^{-1}AS = D$

If $\mathcal{B} = (\vec{b_1}, \vec{b_2}, \vec{b_3})$ eigenbasis

$$[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$$S_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} | & | & | \\ \vec{b_1} & \vec{b_2} & \vec{b_3} \\ | & | & | \end{bmatrix} \qquad S^{-1}_{\mathcal{B}\to\mathcal{E}} A S_{\mathcal{B}\to\mathcal{E}} = [T]_{\mathcal{B}}$$

Step 1: Find eigenvalues, $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[(-5-\lambda)(4-\lambda) + 18\right]$$
$$= (1-\lambda)(\lambda^2 + \lambda - 2)$$
$$= (1-\lambda)(\lambda+2)(\lambda-1)$$

$$\text{char } A = -(\lambda-1)^2(\lambda+2)$$

$\lambda = 1$: Alg. mult is 2

$\lambda = -2$. Alg. mult is 1.

$E_1$: $\text{Nul}(A - (1)I)$

$$A - I = \begin{bmatrix} 0 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$ geo. mult. of $\lambda = 1$ is $\dim(E_1) = 2$.

$E_{-2}$: $$A + 2I = \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nul}(A - (-2)I) = \text{span}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ geo. mult. of $\lambda = -1$ is $\dim(E_{-2}) = 1$

Ex 2 continued...:

$$\mathcal{B} = \{ \underbrace{\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\in E_1}, \underbrace{\vec{b}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\in E_2} \}$$ an eigenbasis for $T$.

Want to find $[T]_{\mathcal{B}}$

$T(\vec{b}_1) = 1\vec{b}_1$
$T(\vec{b}_2) = 1\vec{b}_2$
$T(\vec{b}_3) = 2\vec{b}_2$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$S^{-1}AS = [T]_{\mathcal{B}}$$