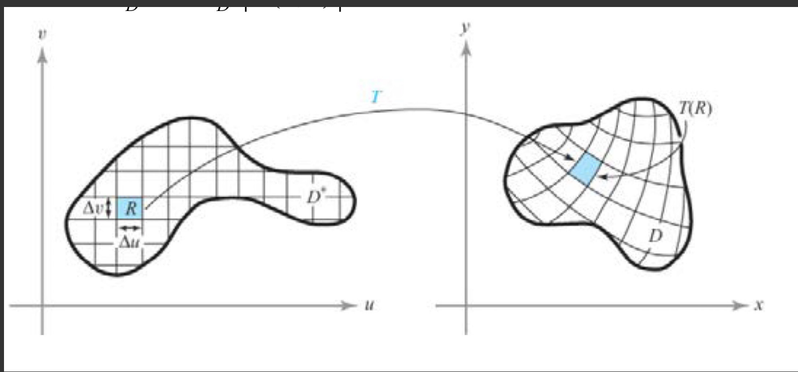
# B41 Nov 29 Lec 1 Notes

**Definition:** Jacobian Determinant

Let $T: D^* \subset \mathbb{R}^2 \to D \subset \mathbb{R}^2$ be a $C^1$ transformation given by $T(u,v) = (x(u,v), y(u,v))$.

The Jacobian determinant of $T$, written $\frac{\partial(x,y)}{\partial(u,v)}$ is the determinant of the derivative matrix $DT(u,v)$ of $T$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$




$$A(D) = \iint_D dA$$

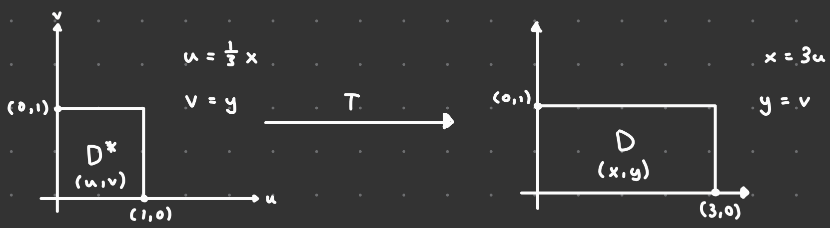
$$= \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

**Ex 1:**

$T(u,v) = (3u, v)$

$(x,y) = T(u,v) = (3u, v)$

$(u,v) = T^{-1}(x,y) = (\frac{1}{3}x, y)$




$$A(D) = \iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

$A(D^*) = 1$ $\qquad$ $A(D) = 3$

$$A(D^*) = \iint_{D^*} dA = \int_0^1 \int_0^1 du\,dv = 1$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

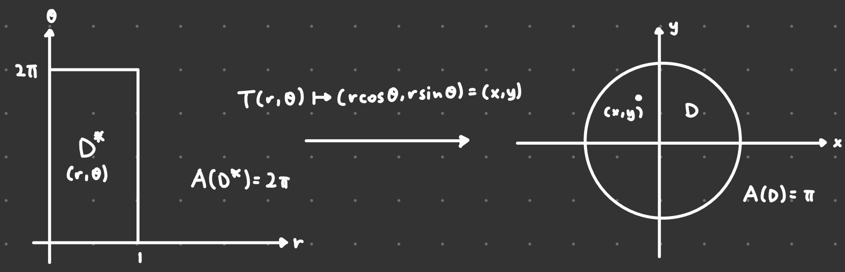
$$A(D) = \iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

$$= \int_0^1 \int_0^1 |3| \, du\,dv$$

$$= 3$$

## Ex 2:

$$T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$




$$(r,\theta) \in D^* = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$A(D^*) = \iint_{D^*} dA = \int_0^1\int_0^{2\pi} d\theta\, dr = 2\pi$$

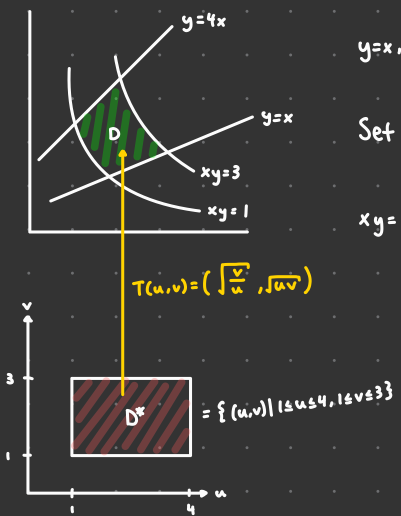
$$A(D) = \iint_D dA = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA^*$$

$$= \int_0^1\int_0^{2\pi} |r|\, d\theta\, dr$$

$$= \pi$$

## Ex 3:

Find the area of D where D is the region bounded by $y=x$, $y=4x$ and $xy=1$, $xy=3$.




$y=x,\ y=4x \Rightarrow \frac{y}{x} = 1\ ,\ \frac{y}{x} = 4$

Set $u = \frac{y}{x} \Rightarrow 1 \le u \le 4$

$xy=1\ ,\ xy=3 \Rightarrow$ Set $v = xy \Rightarrow 1 \le v \le 3$

$uv = \frac{y}{x} \cdot xy = y^2 \Rightarrow y = \sqrt{uv}$

$\frac{v}{u} = \frac{xy}{\frac{y}{x}} = x^2 \Rightarrow x = \sqrt{\frac{v}{u}}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-\sqrt{v}}{2u^{3/2}} & \frac{1}{2\sqrt{uv}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix}$$

$$= -\frac{1}{2u}$$

$$A(D) = \iint_D dA = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA^*$$

$$= \int_1^3\int_1^4 \left|-\frac{1}{2u}\right| du\, dv$$

$$= \ln 4$$

## Theorem:

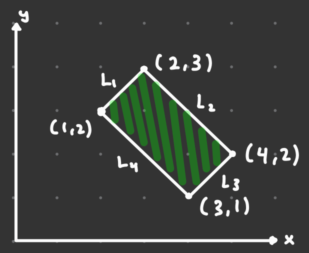
Let $D$ and $D^*$ be elementary regions in $\mathbb{R}^2$ and let $T: D^* \to D$ be one-to-one $C'$ transformation given by $T(u,v) = (x(u,v), y(u,v))$ with $D = T(D^*)$. Then for any integrable function $f: D \to \mathbb{R}$, we have

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

## Ex 4:

Let $D$ be the region in xy-plane enclosed by the parallelogram with the points $(1,2)$, $(2,3)$, $(3,1)$ and $(4,2)$.

(i) Find a mapping $T$ from the uv-plane to the xy-plane and a rectangle $D^* = [a,b] \times [c,d]$ (where $a,b,c,d \in \mathbb{R}$) in the uv-plane s.t. the image of $D^*$ under $T$ is $D$.




$L_1$: $y - 1 = \frac{3-2}{2-1}(x-1)$ $\Rightarrow x - y = -1$ $\quad L_2$: $x + 2y = 8$

$L_3$: $x - y = 2$ $\quad L_4$: $x + 2y = 5$

Set $u = x + 2y$, $v = x - y$

$$D^* = \{(u,v) \mid 5 \le u \le 8,\ -1 \le v \le 2\}$$
$$= [5,8] \times [-1,2]$$

(ii) Use the transformation to evaluate $\iint_D (2y^2 - x^2 - xy)\,dx\,dy$

$$\begin{matrix} u = x + 2y \\ v = x - y \end{matrix} \Rightarrow \begin{matrix} u + 2v = 3x \\ u - v = 3y \end{matrix} \Rightarrow \begin{matrix} x = \frac{1}{3}(u+2v) \\ y = \frac{1}{3}(u-v) \end{matrix}$$

$T$ from $D^*$ in the uv-plane to $D$ in the xy-plane is

$$T(u,v) = \left( \tfrac{1}{3}(u+2v), \tfrac{1}{3}(u-v) \right)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

$$\iint_D (2y^2 - x^2 - xy)\,dx\,dy = \iint_{D^*} \left[ \tfrac{2}{9}(u-v)^2 - \tfrac{1}{9}(u+2v)^2 - \tfrac{1}{9}(u+2v)(u-v) \right] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint_{D^*} \tfrac{1}{9}[-9uv] \left| -\tfrac{1}{3} \right| du\,dv$$

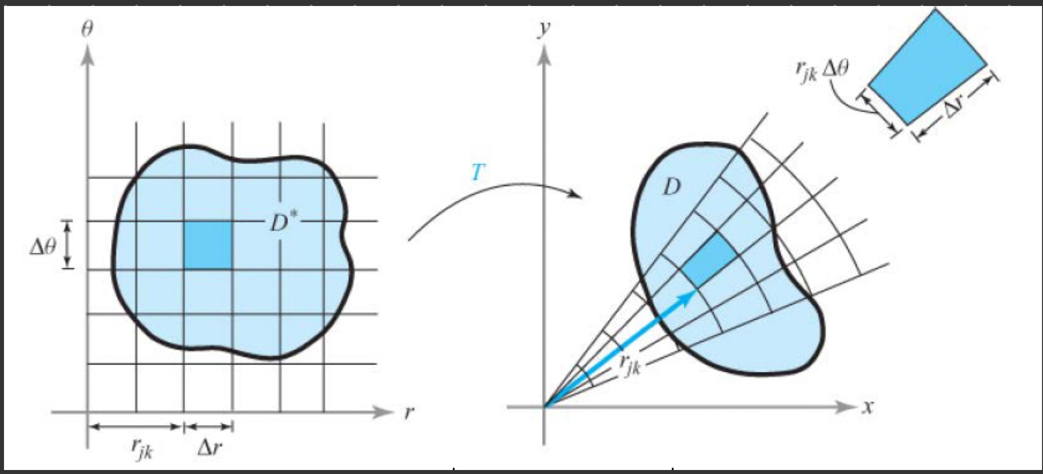
$$= -\frac{1}{3} \int_{-1}^{2} \int_{5}^{8} uv\,du\,dv$$

$$= -\frac{39}{4}$$

# Double Integrals over Polar Rectangular Regions

Change of variable to Polar coordinates:

$$x = r\cos\theta,\ y = r\sin\theta,\quad a \le r \le b,\ \alpha \le \theta \le \beta$$




The Jacobian of T is: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r$$

Let $f$ be continuous on the region in the xy-plane $R = \{(r,\theta) \mid 0 \le h_1(\theta) \le r \le h_2(\theta),\ \alpha \le \theta \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(r,\theta), y(r,\theta)) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(x(r,\theta), y(r,\theta))\,|r|\,dr\,d\theta$$

Ex 5:

Find the volume of the region beneath the surface $z = xy + 10$ and above the annular region $D = \{(x,y) \mid 4 \le x^2 + y^2 \le 16\}$ on the xy-plane.

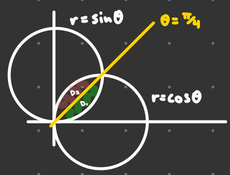
Converting this into polar coordinates yields $D^* = \{2 \le r \le 4,\ 0 \le \theta \le 2\pi\}$

and $z = xy + 10$
$= r^2 \sin\theta\cos\theta + 10$
$= \frac{1}{2} r^2 \sin(2\theta) + 10$

$$V = \iint_D (xy + 10)\,dx\,dy$$
$$= \int_0^{2\pi}\int_2^4 \left(\frac{1}{2}r^2\sin(2\theta) + 10\right) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$
$$= \int_0^{2\pi}\int_2^4 \left(\frac{1}{2}r^2\sin(2\theta) + 10\right)|r|\,dr\,d\theta$$
$$= 120\pi$$

## Ex 6:

Compute the area with both circles $r=\cos\theta$ and $r=\sin\theta$.




$r=\cos\theta \Rightarrow r^2 = r\cos\theta \Rightarrow x^2+y^2=x \Rightarrow (x-\frac{1}{2})^2+y^2 = \frac{1}{4}$

$r=\sin\theta \Rightarrow r^2 = r\sin\theta \Rightarrow x^2+y^2=y \Rightarrow x^2+(y-\frac{1}{2})^2 = \frac{1}{4}$

Let $D = D_1 \cup D_2$ where $D_1 = \{(r,\theta) \mid 0 \le r \le \sin\theta,\ 0 \le \theta \le \pi/4\}$

$D_2 = \{(r,\theta) \mid 0 \le r \le \cos\theta,\ \pi/4 \le \theta \le \pi/2\}$

$$A = \iint_D dA = 2\iint_{D_1} dA$$ Since $D_1 = D_2$

$$= 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Change of variables in triple integrals is similar.

## Ex 7:

Evaluate $\iiint_W xz\,dV$ where $W$ is a parallelopiped bounded by the planes $y=x$, $y=x+2$, $z=x$, $z=x+3$, $z=0$, and $z=4$.

Note that $W$ is bounded by 3 pairs of parallel planes.

$$y-x=0,\ y-x=2$$
$$z-x=0,\ z-x=3$$
$$z=0 \text{ and } z=4$$

Then set $u=y-x$, $v=z-x$, $w=z \Rightarrow x=w-v$, $y=u-v+w$, $z=w$

The new region of integration is $B=\{(u,v,w) \mid 0 \le u \le 2,\ 0 \le v \le 3,\ 0 \le w \le 4\}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = 1$$

$$\iiint_W xz\,dx\,dy\,dz = \iiint_B (w-v)w\,dw\,dv\,du$$

$$= \int_0^2\int_0^3\int_0^4 (w-v)w\,dw\,dv\,du$$

$$= 56$$

## Change of Variable to Cylindrical Coordinates

$$x = r\cos\theta \qquad a \le r \le b, \; \alpha \le \theta \le \beta$$
$$y = r\sin\theta$$
$$z = z$$

The Jacobian is: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

**Ex 8:**

Evaluate $\iiint_W (x^2+y^2)\,dv$ where $W$ is bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy plane.

On the xy-plane, $z = 0$, $\Rightarrow 4 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 4$

Therefore $D = \{(x,y) \mid x^2+y^2 \le 4\}$
$W = \{(x,y,z) \mid (x,y) \in D,\ 0 \le z \le 4 - x^2 - y^2\}$

Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Then $D^* = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$

$W^* = \{(r,\theta,z) \mid (r,\theta) \in D^*,\ 0 \le z \le 4 - r^2\}$

$$\iiint_W (x^2+y^2)\,dv = \iiint_{W^*} r^2 |r| \, dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^3 \, dz\,dr\,d\theta$$
$$= \frac{32}{3}\pi$$

## Change of Variable to Spherical Coordinates

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

The Jacobian is: $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$

$$= -\rho^2\sin\phi$$