Webwork 12

1. The matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

has two real eigenvalues, one of multiplicity 1 and one of multiplicity 2.
Find the eigenvalues and a basis for each eigenspace.

$$\text{char}(A) = \det(A - \lambda I)$$

$$= \det \begin{bmatrix} -\lambda & -1 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$$= -\lambda\left((-1-\lambda)(-\lambda)\right)$$

$$= -\lambda^2(\lambda+1)$$

$\lambda = -1$ with alg. multi. of 1.
$\lambda = 0$ with alg. multi. of 2.

$\lambda_2 = 0$

$$E_0 = \text{Nul } A$$

$$= \begin{array}{c} \begin{array}{ccc} t & & s \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

$$\vec{x} = \begin{bmatrix} t \\ 0 \\ s \end{bmatrix}$$

$$E_0 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s \ , \ t, s \in \mathbb{R} \right\}$$

$\lambda_1 = -1$

$$E_{-1} = \text{Nul}(A - (-1)I)$$

$$= \text{Nul} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \text{Nul} \begin{array}{c} \begin{array}{ccc} & & t \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

$$\vec{x} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} \qquad \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \end{aligned}$$

$$\text{Solution space} = E_{-1} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

2. Find char A, $A = \begin{bmatrix} 3 & 4 & 0 \\ 0 & -3 & 3 \\ -5 & -3 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 & 0 \\ 0 & -3-\lambda & 3 \\ -5 & -3 & -\lambda \end{bmatrix}$$

$$\text{char}\,A = \det(A - \lambda I) = \det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 3-\lambda & 4 & 0 \\ 0 & -3-\lambda & 3 \\ -5 & -3 & -\lambda \end{bmatrix} \end{array}$$

$$= 3-\lambda \begin{vmatrix} -3-\lambda & 3 \\ -3 & -\lambda \end{vmatrix} - 4 \begin{vmatrix} 0 & 3 \\ -5 & -x \end{vmatrix}$$

$$= 3-\lambda\left((-3-\lambda)(-\lambda) + 9\right) - 4(15)$$

3. The matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 3 & 0 \\ 2 & 2 & -1 \end{bmatrix}$$

has one real eigenvalue. Find λ, it's multiplicity, and $\dim E_\lambda$.

$$\text{char}\,A = \begin{vmatrix} 1-\lambda & 3 & -1 \\ 0 & 3-\lambda & 0 \\ 2 & 2 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} + 2\begin{vmatrix} 3 & -1 \\ 3-\lambda & 0 \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)(-1-\lambda) + 2(3-\lambda)$$

$$= (3-\lambda)\left[(1-\lambda)(-1-\lambda) + 2\right]$$

$$= (3-\lambda)\left[-1 + \lambda^2 + 2\right]$$

$$= (3-\lambda)(\lambda^2 + 1)$$

$\lambda = 3$

$$E_3 = \text{Nul}(A - 3I) = \text{Nul}\begin{bmatrix} -2 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & 2 & -4 \end{bmatrix}$$

$$= \text{Nul}\begin{array}{c} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} & & t \end{array} \end{array}$$

$$E_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t \,,\ t \in \mathbb{R} \right\}$$

$\dim E_3 = 1$

4. If $\vec{v}_1$ and $\vec{v}_2$ are L.I. eigenvectors, then they correspond to distinct eigenvalues.

False

5a.

$$A = \begin{bmatrix} -1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 3 & -5 & -4 & -16 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A\vec{v}_1 = \begin{bmatrix} 1+0 \\ 0+0 \\ -3+4 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = (-1)\vec{v}_1$$

6. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ be eigenvectors of the matrix A which corresponds to the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 2$, respectively, and let $\vec{x} = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$

Express $\vec{x}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ and find $A\vec{x}$.

$$\left[\begin{array}{ccc|c} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{x} \\ | & | & | & | \end{array}\right] = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = (-1)\vec{v}_1 + (-2)\vec{v}_2 + \vec{v}_3$$

$$S^{-1}AS = D$$

$$S^{-1}AS = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = SDS^{-1}$$

$$A\vec{x} = SDS^{-1}\vec{x}$$

$$= \begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix}$$

7. Suppose $A_{n \times n}$ and $\vec{v}$ is an eigenvector of A with $\lambda = 7$. $\vec{v}$ is the eigenvector of the following matrices. Find the associated eigenvalues.

(a). A^5

$A\vec{v} = 7\vec{v}$

$A^5\vec{v} = \lambda\vec{v}$

$\lambda = 7^5$

(b) A^{-1}

$\lambda = \frac{1}{7}$, going backwards

(c) $A - 3I_n$

$\lambda = 7 - 3 = 4$

(d) $-4A$

$\lambda = 7 \cdot (-4) = -28$