Feb 5 Lec 2 Notes

Ex1

$D: C^2 \to C^1$ $\quad$ C^2: all functions whose 2nd deriv exist

$f \mapsto f'$

D is a L.T $\quad$ C^1: all functions whose 1st deriv exist

Ex2

$T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad$ Q: Is T a L.T?

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 + 5x_3 \\ x_2 \end{pmatrix}$$

We have to check:

$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ $\quad$ By theorem 1.3.10

$T(r\vec{x}) = rT(\vec{x})$

$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$ $\quad$ Any vector can be written as a linear combination of the standard vectors

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3) = \underbrace{x_1}_{\mathbb{R}}\underbrace{T(\vec{e}_1)}_{\in\mathbb{R}^2} + \underbrace{x_2}_{\mathbb{R}}\underbrace{T(\vec{e}_2)}_{\in\mathbb{R}^2} + \underbrace{x_3}_{\mathbb{R}}\underbrace{T(\vec{e}_3)}_{\in\mathbb{R}^2}$$

$T(\vec{e}_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$T(\vec{e}_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$T(\vec{e}_3) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}$

$$= \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ By theorem 1.3.8

$$= \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

Theorem:

$T: \mathbb{R}^m \to \mathbb{R}^n$ is a L.T iff $\exists$ a matrix $A \in M_{n\times m}(\mathbb{R})$ s.t. $T(\vec{x}) = A\vec{x}$.
Moreover,

$$A_{n\times m} = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix},$$ where $\vec{e}_i$'s are standard vectors in $\mathbb{R}^m$.

Proof:

Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a L.T.

Q Is there a matrix A s.t. $T(\vec{x}) = A\vec{x}$?

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

Let $$\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_m\vec{e}_m$$

$$T(\vec{x}) = T(x_1\vec{e}_1 + \dots + x_m\vec{e}_m) = x_1\underbrace{T(\vec{e}_1)}_{\in\mathbb{R}^n} + \dots + x_m\underbrace{T(\vec{e}_m)}_{\mathbb{R}^n}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}_{n\times m}}_{A \text{ (Standard matrix of } T)} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

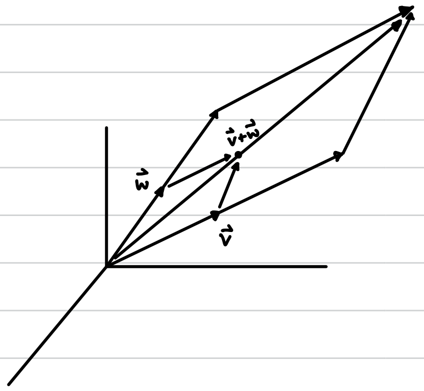
$$= A\vec{x}$$

The other direction is proved in theorem 1.3.10 //

Sec 2.2 Linear Transformation and Geometry

Scaling

Ex 3



$$T: \mathbb{R}^3 \to \mathbb{R}^3$$
$$\vec{v} \mapsto k\vec{v}$$

Q: Is T a L.T ?

$$\forall x, y \text{ in } \mathbb{R}^3 \quad T(\vec{x} + r\vec{y}) = k(\vec{x} + r\vec{y})$$
$$= k\vec{x} + kr\vec{y}$$
$$= k\vec{x} + r(k\vec{y}) = T(\vec{x}) + rT(\vec{y})$$

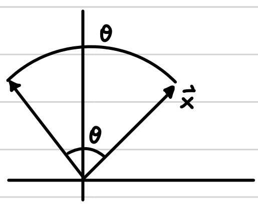
Since T is a L.T. $\exists A_{3\times3}$ s.t. $T(\vec{v}) = A\vec{v}$

$$A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}_{3\times3}$$

$$= \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 5 \\ 12 \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 12 \end{bmatrix} = \begin{bmatrix} 2k \\ 5k \\ 12k \end{bmatrix}$$

Rotation



$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \mapsto$ rotate $\vec{x}$ counterclockwise through θ

Q: Is R_θ a L.T. ?

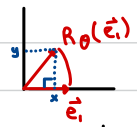
$$R_\theta(\vec{v} + \vec{w}) = R_\theta(\vec{v}) + R_\theta(\vec{w})$$
$$R_\theta(r\vec{v}) = r R_\theta(\vec{v})$$

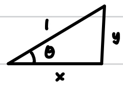
$\exists A \in M_{2\times2}(\mathbb{R})$ s.t. $R_\theta(\vec{x}) = A\vec{x}$

$$A_{2\times 2} = \begin{bmatrix} | & | \\ R_\theta(\vec{e}_1) & R_\theta(\vec{e}_2) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

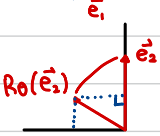
$$R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

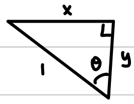




$\cos\theta = \frac{x}{1} = x$
$\sin y = \frac{y}{1} = y$

$$R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

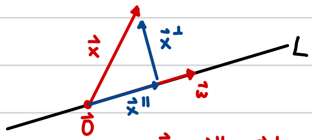




$\sin\theta = x$
$\cos\theta = y$

Orthogonal Projection

$\mathbb{R}^2 \to \mathbb{R}^2$



$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ orthogonal projection of $\vec{x}$ with respect to L

Define $\text{proj}_{\vec{w}}\, \vec{x} := \vec{x}^{\parallel}$

orthogonal projection of $\vec{x}$ onto $\vec{w}$

Q: How to compute $\text{proj}_{\vec{w}}\, \vec{x}$?

$\vec{x}^{\parallel} = K\vec{w}$ since $\vec{x}^{\parallel}$ is parallel to $\vec{w}$

$\vec{x}^{\parallel} \cdot \vec{x}^{\perp} = 0$

$(K\vec{w}) \cdot (\vec{x} - K\vec{w}) = 0$

$(K\vec{w}) \cdot \vec{x} - (K\vec{w}) \cdot (K\vec{w}) = 0$

$K(\vec{w} \cdot \vec{x}) - K^2(\vec{w} \cdot \vec{w}) = 0$

$K((\vec{w} \cdot \vec{x}) - K(\vec{w} \cdot \vec{w})) = 0$

$\Downarrow$ $K \neq 0$

$(\vec{w} \cdot \vec{x}) - K(\vec{w} \cdot \vec{w}) = 0$

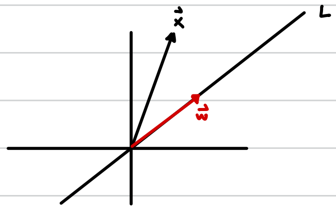
$$K = \frac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}}$$

$$\text{proj}_{\vec{w}}\, \vec{x} = \vec{x}^{\parallel} = K\vec{w}$$

$$= \left(\frac{\vec{w} \cdot \vec{x}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

Ex 4

$\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$\text{proj}_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \mapsto \text{proj}_{\vec{w}} \vec{x}$

$\exists\ A_{2\times2}$ matrix s.t. $\text{proj}_{\vec{w}} \vec{x} = A\vec{x}$

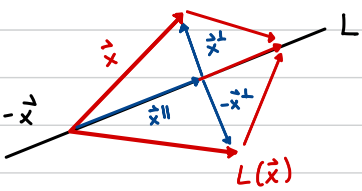
$$A = \begin{bmatrix} | & | \\ \text{proj}_{\vec{w}} \vec{e}_1 & \text{proj}_{\vec{w}} \vec{e}_2 \\ | & | \end{bmatrix}_{2\times2}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

Reflection

$L : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \mapsto 2\,\text{proj}_{\vec{w}} \vec{x} - \vec{x}$



$L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$

$\vec{x} + L(\vec{x}) = \vec{x}^{\parallel} + \vec{x}^{\perp} + \vec{x}^{\parallel} - \vec{x}^{\perp}$

$= 2\vec{x}^{\parallel}$

$L(\vec{x}) = 2\vec{x}^{\parallel} - \vec{x}$

Ex 5

$\vec{w} = \binom{1}{2}$

$L : \mathbb{R}^2 \to \mathbb{R}^2$

$\vec{x} \mapsto 2\,\text{proj}_{\vec{w}} \vec{x} - \vec{x}$

$\exists\ A_{2\times2}$ matrix s.t. $L(\vec{x}) = A\vec{x}$

$$A = \begin{bmatrix} | & | \\ L(\vec{e}_1) & L(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}$$

$$L\begin{bmatrix} 3 \\ 2 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$