Midterm Practice

1. You have 3 different-colored bottles, each with a distinct cap. In how many ways can these caps be put on the bottles such that none of the caps are on the correct bottles?

$3!$ ways to arrange caps

$C_1 \quad C_2 \quad C_3$
$C_1 \quad C_3 \quad C_2$
$C_2 \quad C_1 \quad C_3$
$C_2 \quad C_3 \quad C_1$
$C_3 \quad C_1 \quad C_2$
$C_3 \quad C_2 \quad C_1$

$\left.\begin{array}{l} A : 3 \text{ caps same} \\ B = 2 \text{ caps same} \end{array}\right] \Rightarrow 1$

$C = 1 \text{ caps same} = C(3,1) = 3$

$D = 0 \text{ caps Same} = 3! - 1 - 3 = 2$

2. How many 5-digit numbers without repetition of digits can be formed using the digits 0, 2, 4, 6, 8?

$4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96$

3. If there are 25 railway stations on a railway line, how many types of single second-class tickets must be printed, so as to enable a passenger to travel from one station to another?

There are 25 distinct stations: $S_1, S_2, \dots S_{25}$

From one station, there are 24 other stations to go to.

Thus # of tickets $= 25 \times 24 = 600$

Order matters here thus we don't account for repeated pairs by dividing by 2.

4. In general, if Anna had 12 different ornaments and would like to place k of them on the mantle (with $1 < k \leq 12$) and if Lisa has 13 different ornaments and would like to place $k-1$ of them on a mantle, for what values of k does Anna have more choices in the possible number of ways to place all of her ornaments.

$$P(12, k) > P(13, k-1)$$

$$\frac{12!}{(12-k)!} > \frac{13!}{(13-(k-1))!}$$

$$\frac{(14-k)!}{(12-k)!} > 13$$

$$(14-k)(13-k) > 13$$

$$182 - 27k + k^2 > 13$$

$$k^2 - 27k + 169 > 0$$

$$(k - 17.14)(k - 9.86) > 0$$

$$k = 9.86$$

5. From the name below, how many different names can be created such that the relative order of consonants and vowels does not change.

HUNG WOEI NEOH

6 vowels $\frac{6!}{2!2!}$

6 consonants $\frac{6!}{2!2!}$

$$\# ways = \left(\frac{6!}{2!2!}\right)^2 = 32400$$

6. Lisa has 4 different dog ornaments and 6 different cat ornaments that she wants to place on her mantle. All of the dog ornaments should be consecutive and the cat ornaments should also be consecutive.

2 possibilities to place dog or cats first. Then we can arrange 6! ways and 4! ways.

Thus $\# ways = 2 \cdot 6! \cdot 4! = 34560$

7. How many ways are there to arrange 5 red, 5 blue, and 5 green balls in a row such that no two blue balls lie next to each other.

Arrange the red and green balls first, which can be done in $C(10,5)$ ways. The blue balls can then only be placed one at either end of the row, or in a space between two of the red and green balls. There are 11 places where they can be put, and this can be done in $C(11,5)$ ways.

— = where red and green balls can be placed

Δ _ Δ _ Δ _ Δ _ Δ _ Δ _ Δ _ Δ _ Δ _ Δ _ Δ

Δ = where blue balls can be placed

Thus there are $C(11,5) \cdot C(10,5)$ ways.

8. How many positive integers less than 10000 are there such that they each have only 3 and/or 7 as their digits.

1 digit # $= 2^1$

2 digit # $= 2^2 = 4$

3 digit # $= 2^3 = 8$

4 digit # $= 2^4 = 16$

$\# ways = 16 + 8 + 4 + 2 = 30$

9. With 4 colors, how many different ways can you color in the 16 squares in a 4×4 grid such that each of the nine 2×2 grids inside the 4×4 grid contains each of the four colors?

1	2	1	2
3	4	3	4
1	2	1	2
3	4	3	4

10. Six distinct friends want to form a club. They decide that there will be 1 president, 1 secretary, and 4 ordinary members. How many different ways can they organize this club?

6 ways to choose 1 president, 5 ways to choose a secretary, and the rest are ordinary.

Thus $6 \times 5 \times 1 = 30 = P(6,2)$

11. Given two identical standard deck of cards, how many different permutations are there?

Since the decks of cards are identical, there are 2 identical cards of each type.

$\#$ of permutations $= \dfrac{(52+52)!}{(2!)^{52}}$

12. How many distinct words of any (nonzero length) can be formed using the letters of KEPLER at most one each?

$\#$ of 6 letter permutations $= \dfrac{6!}{2!} = 360$

$\#$ of 5 letter permutations $= C(5,2) \cdot P(4,3) + P(5,5) = 360$

$\#$ of 4 letter permutations $= C(4,2) \cdot P(4,2) + P(5,4) = 192$

$\#$ of 3 letter permutations $= C(3,2) \cdot P(4,1) + P(5,3) = 72$

$\#$ of 2 letter permutations $= C(2,2) \cdot P(4,0) + P(5,2) = 21$

$\#$ of 1 letter permutations $= 5$

$\#$ total ways $= 360 + 360 + 192 + 72 + 21 = 1010$

We can split this into 2 mutually exclusive cases

= When word contains 2 E's
First we get # of combinations that we can have with 2 E's, then multiply that with the permutation of the remaining words.

= When word contains 1 E
Do a simple permutation as there are no duplicates

13. Let us call a 6-digit number cool if each of its digits is no less than the preceding digit. How many cool 6-digit numbers are there?

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 = 6$$

$$C(6+9-1, 6) = C(14,6) = 3003$$

14. The digits 1, 7, 8, and 5 copies of the digit 5 are all arranged to form an 8-digit integer. How many different integers can be formed.

Permutations with indistinguishable objects.

$$P(8;5) = \frac{8!}{5!} = 336$$

15. If $P(n,5) = 20P(n,3)$, what is n?

$$\frac{n!}{(n-5)!} = 20\frac{n!}{(n-3)!}$$

$$\frac{(n-3)!}{(n-5)!} = 20$$

$$20 = \frac{(n-3)(n-4)\cancel{(n-5)!}}{\cancel{(n-5)!}}$$

$$20 = n^2 - 7n + 12$$

$$0 = n^2 - 7n - 8$$

$$0 = (n-8)(n+1)$$

$$n = 8, -1$$

$$n = 8$$

16. Three flags colored yellow, red, and blue are prepared for sending signals. Each signal consists of one, two, or three flags where repetition in flag color is allowed.

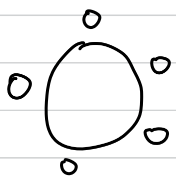
1 flag = 3

2 flags = $3^2 = 9$

3 flags = $3^3 = 27$

ways = $3 + 9 + 27 = 39$

17. How many ways can three dogs and two cats be seated around a circular table? Consider two animals of the same type to be identical, and configurations that can be rotated to match are also considered identical.



$$\frac{5!}{3!2!} = \frac{5\times4\times3\times2}{3\times2\times2\times5} = 2$$

18. Among 5 girls in a group, exactly two of them are wearing red shirts. How many ways are there to seat all 5 girls in a row such that the two girls wearing red shirts are not adjacent to each other?

Total # of ways to seat 5 girls in a row $= 5! = 120$

\# of ways to seat the girls such that two girls with red skirts sit together: $2 \cdot (5-1)!$

\# ways the red skirts can sit

\# ways girls with red skirts are not sitting together: $120 - 2 \cdot 24 = 72$

19. 10 people including A, B, and C are waiting in a line. How many distinct line-ups are there such that A, B, and C are not all adjacent?

\# ways A, B, C grouped as one : $8!$

\# ways A, B, C can be ordered : $3!$

\# ways total for 10 ppl inline : $10!$

$$= 10! - 8! \cdot 3!$$
$$= 3386880$$

20. 3 boys and 2 girls are about to be seated at a round table. If the 2 girls want to sit next to each other, find the # of ways seating these boys and girls.

\# ways to arrange 3 boys and a group of girls $= \frac{4!}{4} = 6$

Since the 2 girls can switch, #ways $= 2 \cdot 6 = 12$

21. Mary has enrolled in 6 courses: C, P, M, E, F, B. She has one text book for each course and wants to place them on a shelf. How many ways can she arrange the textbooks so that the E textbook is placed at any position to the left of the F textbook.

Solution #1

There are $6! = 720$ ways to arrange the textbooks on the shelf. There are $\frac{6!}{2} = 360$ possible arrangements where the F book is placed before the E text book.

Solution #2

There are $C(6,2) = 15$ ways to place the two textbooks so that the E comes before F. Now there are $4! = 24$ ways to place the other 4 textbooks. Thus there are $15 \times 24 = 360$ possible arrangements.

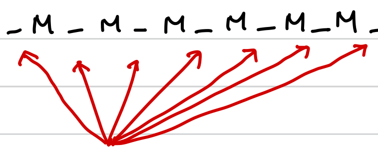
22. How many ways can the numbers 1,2,3,4,5,6 be arranged such that the numbers 1 and 2 are not adjacent to each other, and the numbers 5 and 6 are not adjacent to each other.

There are $6! = 720$ ways to arrange the numbers. There are $2 \times 5! = 240$ ways to arrange the numbers so that 1 and 2 are adjacent, and the same # ways for 5 and 6 adjacent. There are $2^2 \times 4! = 96$ ways to have 1 adjacent to 2 AND 5 adjacent to 6. By the principle of inclusion and exclusion there are $720 - 240 - 240 + 96 = 336$ such arrangements.

23. A total of 10 people -- 4 women and 6 men are standing in a line. If no two women may be adjacent to each other, how many distinct line-ups are there?

First observe there are 6! ways to arrange 6 men in a line.

To arrange the women, place 4 women among the 7 places in the line-up of 6 men. This gives P(7,4) ways to place the women.

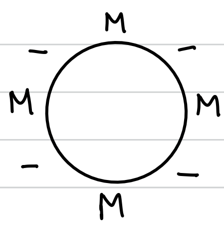


P(7,4) ways to place the women.

$\#$ ways $= 6! \cdot C(7,4) = 604{,}800$

24. Similar to Q23, but 4 M, 3 W, and they are sitting on a round table.

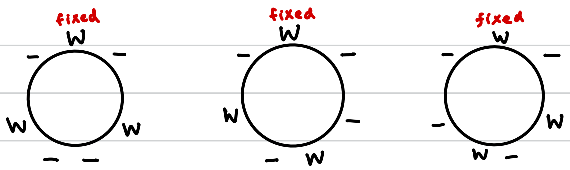
Solution #1



$3! \cdot P(4,3) = 144$

Solution #2

If we fix one of the W seat, there are 3 ways to assign seats to the other W, as shown below:



Since the other 2 W can switch seats, there are $3 \times 2 = 6$ ways to seat the 3 W. There are $4! = 24$ ways to seat the 4M.

Therefore $6 \cdot 4! = 144$ ways

25. Lisa has 4 different dog ornaments and 6 different cat ornaments. She is arranging the ornaments on her mantle and wants all of the dog ornaments to be consecutive, with no restriction on the cat ornaments. If there are N different ways the 10 ornaments be arranged in such manner, what is N?

$7! \cdot 4! = 120960$

26. Consider the two sets $A = \{1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4\}$ and a function $f: A \to B$. Find the number of functions f that satisfy $f(1) + f(2) = 2$.

Case 1 : $f(1) = 0$, $f(2) = 2$.

For the elements of A other than 1 and 2, any of the 5 elements of B can be mapped. Thus we have $5^2 = 25$ possibilities.

Case 2 : $f(1) = 1$, $f(2) = 1$

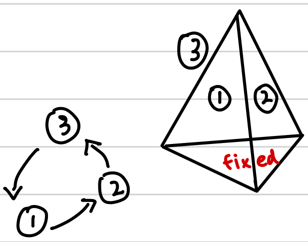
The same argument can be made as in case 1

Case 3 : $f(1) = 2$, $f(2) = 0$

The same argument can be made as in case 1

Therefore the # of functions is $3 \cdot 25 = 75$

27. Find the # of ways to paint the faces of a regular tetrahedron using 4 different colors. All colors must be used.



First, paint one face with any color, and fix it as a base to avoid double counting when the regular tetrahedron is rotated. Then the other 3 faces are 3 identical equilateral triangles arranged in a circular way. There are only $\frac{3!}{3} = 2$ ways to paint them in different colors.

28. There are 6 people and they each have their own desk. How many ways are there for them to occupy a seat at the various desks, such that at most 1 of them is in the correct spot?

Some dearangement problem!

29. How many different ways are there to color a 3x3 grid with green, red, and blue paints, using each color 3 times?

3G, 3R, 3B

_ _ _ _ _ _ _ _ _ $\frac{9!}{3!3!3!} = 1680$

30. How many arrangements are there for the letters of BANANA such that no two N's appear in adjacent positions?

$$\frac{6!}{3!\,2!} - \frac{5!}{3!} = 40$$

Total permutations of "BANANA"

Total permutations where "NN" is grouped up.

31. 9 different books are to be arranged on a bookshelf. 4 of these books are labeled A, 2 are labeled B, and 3 labeled C. How many possible permutations are there if the C books must be separated from one another.

WRONG: $9! - 7! \cdot 3!$ ← This only accounts for 3 books as one, but not 2 adjacent books.

CORRECT: _ x _ x _ x _ x _ x _ x _

x = A or B books = 6! ways to order them
_ = C books : $C(7,3) \cdot 3! = P(7,3)$

\# ways = $6! \cdot C(7,3) \cdot 3! = 151200$

32. There are 9 children. How many ways are there to group these 9 children into 2, 3 and 4?

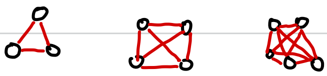
\# ways to choose 2 children out of 9 is $C(9,2) = 36$. # ways to choose 3 children out of 7 is $C(7,3) = 35$. Finally, there will be $C(4,4) = 1$ ways for the last 4 children.

By rule of product, # ways = $36 \cdot 35 \cdot 1 = 1260$.

33. At a party, everyone shook hands with everybody else. There were 66 handshakes. How many people were at the party?

Observation:

There are 12 people there.



3ppl 4ppl 5ppl

3 6 4+3+2+1=9 shakes

Solution:

Handshakes = $C(n,2)$, where n is the # of people in the party.

$C(n,2)$ chooses 2 people out of n and order doesn't matter. Thus the 2 ppl would perform a handshake.

$$C(n,2) = 66$$
$$\frac{n!}{2!(n-2)!} = 66$$
$$n(n-1) = 66$$
$$(n+11)(n-12) = 0$$

34. We are trying to divide 5 European countries and 5 African countries into 5 groups of 2. How many ways are there to do this under the restriction that at least one group must have only European countries?

# of ways to divide 5+5=10 countries into 5 groups of each:

$$\frac{\binom{10}{2}\times\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}\times\binom{2}{2}}{5!} = \frac{45\cdot 28\cdot 15\cdot 6\cdot 1}{120} = 945$$

We subtract the above with the # of groupings where all 5 groups have 1 European and 1 African country. This is equivalent to # ways to match each of the 5 European countries with one African country:

$$5! = 120$$

Thus # ways = $945 - 120 = 825$

34. Find the number of rectangles in a 10×12 chess board.

In an $a \times b$ board, there are $a+1$ horizontal lines and $b+1$ vertical lines. Since a rectangle has 2 horizontal lines and 2 vertical lines, then the # of rectangles would be:

$$C(11,2) \cdot C(13,2) = 4290$$

35. There are 2 distinct boxes, 10 identical red balls, 10 identical yellow balls, and 10 identical blue balls. How many ways are there to sort the 30 balls into the two boxes so that each box has 15?

$$x_1 + x_2 + x_3 = 15 \qquad 0 \le x_1, x_2, x_3 \le 10$$

$$C(15+3-1, 3-1) = C(17,2)$$

$$x_1 + x_2 + x_3 = 4 \qquad x_1, x_2, x_3 \ge 11$$

$$C(4+3-1, 2) = C(6,2)$$

$$\#\text{ways} = C(17,2) - 3C(6,2) = 91$$

36. PizzaHot makes 7 kinds of pizza, 3 of which are on sale every day, 7 days a week. According to their policy, any kinds of pizza that are on sale on a same day can never be on sale again on the same day again during the rest of that calendar week. Let X be the number of all the possible sale strategies during a calendar week. What is the remainder of X upon division by 1000?

Let a, b, c, d, e, f, g be the 7 kinds of pizza. There

37. Three squares are chosen at random on a chess board. Find the probability that they lie on any diagonal.

choose 3 squares from diagonal

$$E = \left[\left(C(3,3) + C(4,3) + C(5,3) + C(6,3) + C(7,3)\right) \times 2 + C(8,3)\right] \times 2$$

$$S = C(64,3)$$

$$\frac{E}{S} = \frac{392}{41664}$$

38. A and B are the only candidates who contest in an election. They secure 11 and 7 votes, respectively. In how many ways can this happen if it is known that A stayed ahead of B throughout the counting process of votes?

$$= \binom{p+q-1}{p-1} - \binom{p+q-1}{p}$$

$$= \binom{17}{10} - \binom{17}{11} = 7072$$

39. Winston must choose 4 classes for his final semester of school. He must take at least 1 science class and at least 1 arts class. If his school offers 4 (distinct) science classes, 3 (distinct) arts classes and 3 other (distinct) classes, how many different choices for classes does he have?

There are 10 classes offered. There are $C(10,4) = 210$ total choices. However, not all of them include at least 1 science class and at least 1 arts class. There are $C(6,4) = 15$ choices with no science classes, and $C(7,4) = 35$ choices with no arts classes.

Thus # outcomes $= 210 - 15 - 35 = 160$

40. How many 6 digit integers contain exactly four different digits?

Let the numbers be a, a, b, b, c, d. There are $C(10,4)$ ways to select the 4 numbers. There are $C(4,2)$ ways to select which ones will be repeated. There are $\frac{6!}{2!2!}$ ways to arrange them.

$$C(10,4) \cdot C(4,2) \cdot \frac{6!}{2!\,2!} = 226800$$

Now let the numbers be a, a, a, b, c, d.

$$C(10,4) \cdot C(4,1) \cdot \frac{6!}{3!} = 100800$$

Adding the two cases give 327600.

10% of these start with 0, so $\frac{9}{10} \cdot 327600 = 294800$

41. You randomly choose a treasure chest to open, and then randomly choose a coin from that treasure chest. If the coin you choose is gold, then what is the probability that you chose chest A?

100 gold coins CHEST A	50 gold + 50 silver CHEST B

A = Chose chest A

B = Coin is gold

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A')P(A') + P(B|A) \cdot P(A)}$$

$$= \frac{1 \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{4}+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{4}{6} = \frac{2}{3}$$

42. 10% of the world population are infected by a virus. A test kit will test positive for 90% of the infected people and test negative for 70% of the non-infected. If the test kit showed a positive result, what would be the probability that the tested subject was truly zombie?

A = test kit shows positive
B = subject was actually infected

$$P(B|A) = \frac{P(A|B)\cdot P(B)}{P(A|B')\cdot P(B') + P(A|B)\cdot P(B)}$$

$$= \frac{0.9 \cdot 0.1}{0.3 \cdot 0.9 + 0.9 \cdot 0.1}$$

$$= \frac{1}{4}$$

43. A disease test is advertised as being 99% accurate: if you have the disease, you will test positive 99% of the time. If you don't have the disease, you will test negative 99% of the time.

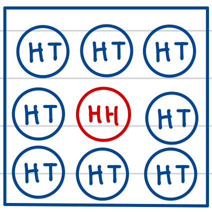
If 1% of all the people have this disease and you test positive, what is the probability that you actually have the disease?

A = You test positive
B: You actually have the disease.

$$P(B|A) = \frac{0.99 \cdot 0.01}{0.01 \cdot 0.99 + 0.99 \cdot 0.01}$$

$$= 0.5$$

44. Zeb's coin box contains 8 fair, standard coins (heads and tails) and 1 coin which has heads on both sides. He selects a coin randomly and flips it 4 times, getting all heads. If he flips this coin again, what is the probability it will be heads?



F = Having a fair coin
H_4 = Flipping 4 Heads

$$P(F|H_4) = \frac{P(F)\cdot P(H_4|F)}{P(F)\cdot P(H_4|F) + P(\bar{F})\cdot P(H_4|\bar{F})} = \frac{\frac{8}{9}\cdot\frac{1}{16}}{\frac{8}{9}\cdot\frac{1}{16} + \frac{1}{9}\cdot 1} = \frac{1}{3}$$

Since the probability of having a fair coin given that 4 heads were flipped, the probability that the next flip is heads:

$$\underbrace{\frac{1}{3}}_{P(F|H_4)}\cdot\underbrace{\frac{1}{2}}_{P(H|F)} + \underbrace{\frac{2}{3}}_{1-P(F|H_4)}\cdot\underbrace{1}_{P(H|\bar{F})} = \frac{5}{6}$$

45. There are 10 boxes containing blue and red balls.

The number of blue balls in the nth box is given by $B(n) = 2^n$
The number of red balls in the nth box is given by $R(n) = 1024 - B(n)$

A box is picked at random, and a ball is chosen randomly from that box. If the ball is blue, what is the probability that the 10th box was picked?

A = Ball is blue
B = 10th box was picked

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

$$= \frac{\frac{2^{10}}{2^{10} + 0} \cdot \frac{1}{10}}{\sum_{i=1}^{10} \frac{2^i}{1024 - 2^i + 2^i}\left(\frac{1}{10}\right)}$$

$$= 0.500488$$