A22 Mar 24 Lec | Notes

Determinant is a function

$$\begin{aligned} M_{n\times n}(F) &\longrightarrow F \\ A &\longmapsto \det(A) \end{aligned}$$

Recall from TUT 5

$A \in M_{2\times 2}(\mathbb{R}) = \mathbb{R}^{2\times 2}$

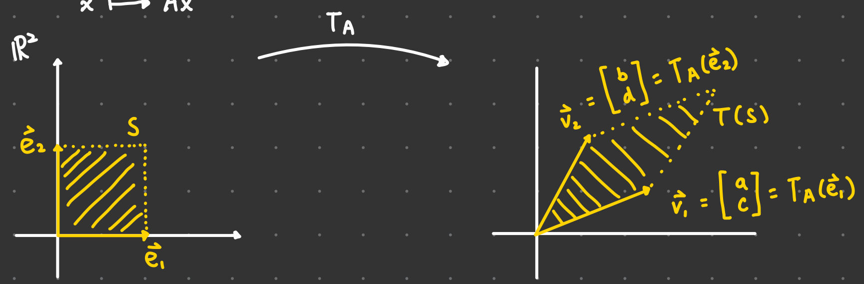
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det A = ad - bc$

We proved that A is invertible iff $\det A \neq 0$

$|\det A|$ is a factor by which $T_A(\vec{x}) = A\vec{x}$ changes area.

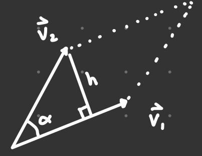
$$\begin{aligned} T_A : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \vec{x} &\longmapsto A\vec{x} \end{aligned}$$



$S = \{ r_1\vec{e}_1 + r_2\vec{e}_2 \mid 0 \le r_1, r_2 \le 1$ $\qquad T(S) = \{ r_1 T_A(\vec{e}_1) + r_2 T_A(\vec{e}_2) \mid 0 \le r_1, r_2 \le 1 \}$

Area of $S = 1$

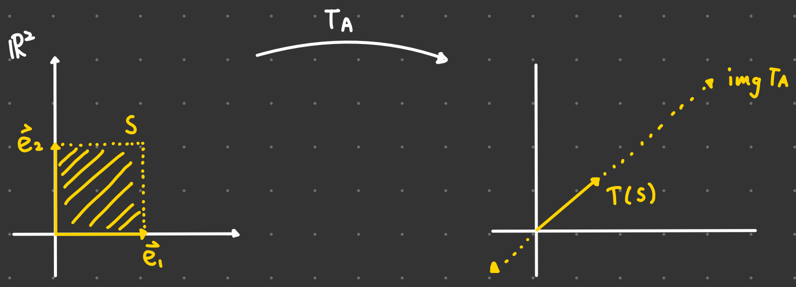
What is the area of $T(S)$?



$$\begin{aligned} (\text{Area of } T(S))^2 &= \|\vec{v}_1\|^2 h^2 \\ &= \|\vec{v}_1\|^2 \sin^2\alpha \, \|\vec{v}_2\|^2 \\ &= \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 (1 - \cos^2\alpha) \\ &= \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 - \|\vec{v}_1\|^2 \|\vec{v}_2\|^2 \cos^2\alpha \\ &= (\vec{v}_1 \cdot \vec{v}_1)(\vec{v}_2 \cdot \vec{v}_2) - (\vec{v}_1 \cdot \vec{v}_2)^2 \\ &= (a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 \\ &= (ad - bc)^2 \end{aligned}$$

$\therefore$ Area of $T(S) = |ad - bc|$

What if A is not invertible?



$$\det A = \text{area of } T(S)$$
$$= 0$$

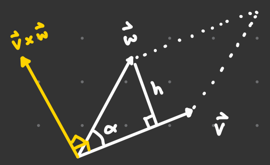
Cross Product

$$\begin{array}{ccc} \vec{v} & & \vec{w} \\ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} & \times & \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \end{array} = (v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3) \times (w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3)$$
$$= (v_2w_3 - v_3w_2)\vec{e}_1 + (v_3w_1 - v_1w_3)\vec{e}_2 + (v_1w_2 - v_2w_1)\vec{e}_3$$
$$:= \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

$\vec{v}\times\vec{w} \perp \vec{v}$ and $\vec{v}\times\vec{w} \perp \vec{w}$

$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\,\|\vec{w}\|\sin\theta$

Observation:



$$\text{Area} = \|\vec{v}\|\,h$$
$$= \|\vec{v}\|\|\vec{w}\|\sin\theta \qquad h = \sin\theta\,\|\vec{w}\|$$
$$= \|\vec{v}\times\vec{w}\|$$

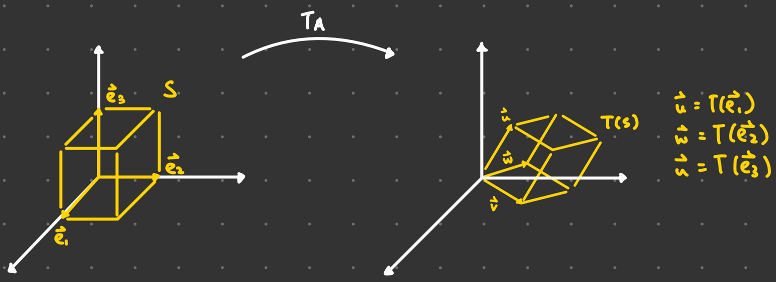
$$\vec{v}\times\vec{w} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

$$= (v_2w_3 - v_3w_2)\vec{e}_1 + (v_3w_1 - v_1w_3)\vec{e}_2 + (v_1w_2 - v_2w_1)\vec{e}_3$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{e}_1 + (-1)\begin{vmatrix} v_1 & -v_3 \\ w_1 & w_3 \end{vmatrix}\vec{e}_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{e}_3$$

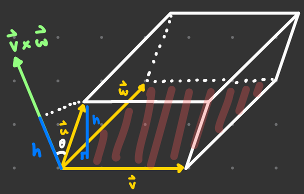
Ex 1

Consider $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. How does T change value?



$\text{Vol}(S) = |\det A| \, \text{vol}\, T(S)$

What is vol $T(S)$?



$$\begin{aligned}
\text{Volume} &= (\text{area of base})\, h \\
&= \|\vec{v} \times \vec{w}\| \, \|\vec{u}\| \, |\cos\theta| \\
&= \frac{\cancel{\|\vec{v} \times \vec{w}\|} \, \|\vec{u}\| \, |\cos\theta| \, \|\vec{v} \times \vec{w}\|}{\cancel{\|\vec{v} \times \vec{w}\|}}
\end{aligned}$$

$h = \|\vec{u}\| \, |\cos\theta|$

$$= |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

Def:

Let $A = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$

$$\det A := \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{pmatrix} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \\ -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \\ \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{pmatrix}$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & w_1 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_3 \end{vmatrix}$$