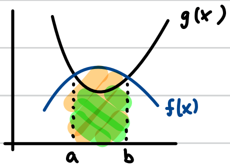
Jan 21 Lecture Notes

Def: Properties of Definite Integral (continued...)

(vii) If $f(x) \le g(x)$ $\forall x \in [a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



(viii) Integral Inequality

If $\exists m, M \in \mathbb{R}$ s.t. $m \le f(x) \le M$ $\forall x \in [a,b]$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Examples:

1. Let $a, b, c, \in \mathbb{R}$ s.t. $a<b$. Use Riemann def of the definite integral to prove:

If f and g are integrable on $[a,b]$, then $\int_a^b (f(x) + c\,g(x))\,dx = \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$

Proof:

Let $P = \{x_i\}_{i=0}^n$ be a Riemann partition of $[a,b]$

Suppose f and g are Riemann integrable on $[a,b]$

$\left(\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x \quad x_i^* \in [x_{i-1}, x_i] \right)$ exists

Consider RHS $= \int_a^b f(x)\,dx + c\int_a^b g(x)\,dx$

$= \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x + c \lim_{n\to\infty} \sum_{i=1}^n g(x_i^*)\Delta x$ By Riemann def

$= \lim_{n\to\infty} \left(\sum_{i=1}^n f(x_i^*)\Delta x + c \sum_{i=1}^n g(x_i^*)\Delta x \right)$ By limit laws

$= \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x + c\,g(x_i^*)\Delta x$ By Σ-property

$= \lim_{n\to\infty} \sum_{i=1}^n \left(f(x_i^*) + c\,g(x_i^*) \right)\Delta x$ By factoring

$= \int_a^b f(x) + c\,g(x)\,dx$ $f(x) + c\,g(x)$ is continuous on $[a,b]$ (common points of continuity)

If f is Riemann integrable on $[a,b]$ if f continuous on $[a,b]$ or if f has a finite # of finite jump discontinuities.

Def: Darboux Integral

Let $a, b \in \mathbb{R}$, $a < b$. Let $P = \{x_i\}_{i=0}^{n}$ be any partition of $[a,b]$. Suppose f is bounded on $[a,b]$.

For each $i = 1, \dots, n$ define

- $m_i = \inf\{ f(x) \mid x \in [x_{i-1}, x_i]\}$
- $M_i = \sup\{ f(x) \mid x \in [x_{i-1}, x_i]\}$

Then

$$U(f,P) = \text{Upper darboux sum of } f \text{ for } P \text{ on } [a,b]$$
$$= \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(f,P) = \text{Lower darboux sum of } f \text{ for } P \text{ on } [a,b]$$
$$= \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$