Feb 12 Lec 2 Notes

Theorem:

$f: X \to Y$ is invertible iff f is injective and onto

Def:

Let X be a set $id_x : X \to X$ is called identity.
$x \mapsto x$

Theorem:

$f: X \to Y$ is invertible iff there exists a function $g: Y \to X$ s.t.
$f \circ g = id$ and $g \circ f = id$.

g if it exists is called the inverse of f and denoted by f^{-1}

Def:

Let $T: V \to W$ be a L.T. We say T is an isomorphism if there exists $S: W \to V$ s.t. $S \circ T = id_V$ and $T \circ S = id_W$.

Such an S, if it exists, is called the inverse of T and is denoted by T^{-1}.

If there exists an isomorphism between V and W, we say V is isomorphic to W, $V \cong W$

Ex 1

$T: \mathbb{R}^3 \to \mathbb{R}^3$ is a L.T. Is T invertible?
$\vec{v} \mapsto 5\vec{v}$

$S: \mathbb{R}^3 \to \mathbb{R}^3$
$\vec{v} \mapsto \frac{1}{5}\vec{v}$

Claim: $S = T^{-1}$.

i.e. $S \circ T = id_{\mathbb{R}^3}$ $\quad T \circ S = id_{\mathbb{R}^3}$

$$S(T(\vec{v})) = S(5\vec{v}) \qquad \forall \vec{v} \in \mathbb{R}^3$$
$$= \tfrac{1}{5}(5\vec{v})$$
$$= \vec{v}$$

$$T(S(\vec{v})) = T(\tfrac{1}{5}\vec{v})$$
$$= 5(\tfrac{1}{5}\vec{v})$$
$$= \vec{v}$$

Thus claim is proved. $S = T^{-1}$, $S \circ T$ is invertible.

T is an isomorphism, $\mathbb{R}^3 \cong \mathbb{R}^3$

Ex 2

$$T: \mathbb{R}^3 \to P_2 \qquad P_2 = \left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; \begin{matrix} a_i \in \mathbb{R} \\ 1 \le i \le 2 \end{matrix} \right\}$$
$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \mapsto a_0 + a_1 x + a_2 x^2$$

Is T L.T ?

Let $\vec{v}, \vec{w} \in \mathbb{R}^3$

$$T(\vec{v} + \vec{w}) = T\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$$

$$= (v_1 + w_1) + (v_2 + w_2)x + (v_3 + w_3)x^2$$
$$= (v_1 + v_2 x + v_3 x^2) + (w_1 + w_2 x + w_3 x^2)$$
$$= T(\vec{v}) + T(\vec{w})$$

Thus $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$

Is T invertible ?

$$S: P_2 \longrightarrow \mathbb{R}^3$$
$$a_0 + a_1 x + a_2 x^2 \longmapsto \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

We WTS $T \circ S = id$ and $S \circ T = id$

$\forall\ a_0 + a_1 x + a_2 x^2$ in P_2

$$T \circ S(a_0 + a_1 x + a_2 x^2) = T(S(a_0 + a_1 x + a_2 x^2))$$
$$= T\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$
$$= a_0 + a_1 x + a_2 x^2$$

$$\therefore T \circ S = id_{P_2}$$

Works analogously for $S \circ T = id_{P_2}$

Thus T is an isomorphism. $P_2 \cong \mathbb{R}^3$.

Ex 3

Suppose $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$ is an invertible L.T. i.e. there exists

$S: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ s.t. $T \circ S = id_{\mathbb{R}^n}$, $S \circ T = id_{\mathbb{R}^m}$

Is S a L.T ?

Yes. Then $\exists$ a matrix $B_{m \times n}$

What is the relation between A and B ?

Since T is invertible, then T is injective and onto.

↳ T is injective: $\forall\ \vec{w} \in \mathbb{R}^n$ there is at most one vector $\vec{v} \in \mathbb{R}^n$ s.t. $T(\vec{v}) = \vec{w}$.
  iff $\forall\ \vec{w} \in \mathbb{R}^n$ $\exists$ at most one $\vec{v} \in \mathbb{R}^m$ s.t. $A\vec{v} = \vec{w}$
  iff rref (A) has pivot in every column
  iff rank $(A) = m$

↳ T is onto: if $\forall\ \vec{w} \in \mathbb{R}^n$ $\exists\ \vec{v} \in \mathbb{R}^m$ s.t. $T(\vec{v}) = \vec{w}$
  iff $\forall\ \vec{w} \in \mathbb{R}^n\ \exists\ \vec{v} \in \mathbb{R}^m$ s.t. $A\vec{v} = \vec{w}$
  iff rref (A) has pivot in every row.
  iff rank $(A) = n$

$\therefore A$ is a square matrix i.e. $n = m$. rref$(A) = I_m$

$T : \mathbb{R}^m \to \mathbb{R}^n$
$\vec{x} \mapsto A\vec{x}$

$T^{-1} : \mathbb{R}^n \to \mathbb{R}^m$
$\vec{x} \mapsto B\vec{x}$

$$
\begin{aligned}
T \circ T^{-1}(\vec{v}) &= T(T^{-1}(\vec{v})) \quad \forall \vec{v} \in \mathbb{R}^n \\
&= T(B\vec{v}) \\
&= A(B\vec{v}) \\
&= (AB)\vec{v} \\
&= id(\vec{v}) \\
&= \vec{v} \qquad \Rightarrow (AB)\vec{v} = \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n
\end{aligned}
$$

$$
\begin{aligned}
T^{-1} \circ T(\vec{v}) &= T^{-1}(T(\vec{v})) \quad \forall \vec{v} \in \mathbb{R}^n \\
&= T(A\vec{v}) \\
&= B(A\vec{v}) \\
&= (BA)\vec{v} \\
&= id(\vec{v}) \\
&= \vec{v} \qquad \Rightarrow (BA)\vec{v} = \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n
\end{aligned}
$$

Claim: $AB = BA = I_n$

Proof:

We know $(AB)\vec{v} = \vec{v} \quad \forall \vec{v} \in \mathbb{R}^n$

WTS: $AB = I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}_n = \begin{bmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & & | \end{bmatrix}$

Take $\vec{v} = \vec{e}_1 \Rightarrow (AB)\vec{e}_1 =$ 1st column of $AB = \vec{e}_1$
Take $\vec{v} = \vec{e}_2 \Rightarrow (AB)\vec{e}_1 =$ 2nd column of $AB = \vec{e}_2$

Thus $AB = I_n$. Analagously we have $BA = I_n$

Theorem

Suppose $T : V \to W$ is an invertible L.T. Then $T^{-1} : W \to V$ is also linear.

Proof: