B52 Dec 3 Lec 2 Notes

Remark: For i.i.d $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, sample variance is independent of sample mean ($S_n^2 \perp \bar{X}_n$)

Proof:

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 = g((x_1-\bar{x}_n), \dots, (x_n-\bar{x}_n))$$

If $\bar{x}_n \perp (x_i - \bar{x}_n)$, $\forall i \Rightarrow \bar{x}_n \perp S_n^2 = g((x_1-\bar{x}_n), \dots, (x_n-\bar{x}_n))$

$\bar{X}_n$, $(X_i - \bar{X}_n)$ are linear functions of i.i.d. Normals $\Rightarrow$ They follow normal

$\Rightarrow$ They are independent iff $\mathrm{Cov}(\bar{X}_n, (x_i - \bar{x}_n)) = 0$

$$\begin{aligned}
\mathrm{Cov}(\bar{x}_n, x_i - \bar{x}_n) &= \mathrm{Cov}(\bar{x}_n, x_i) - \mathrm{Cov}(\bar{x}_n, \bar{x}_n) \\
&= \mathrm{Cov}\left(\frac{1}{n}\sum_{j=1}^{n} x_j, x_i\right) - \frac{\sigma^2}{n} \\
&= \frac{1}{n}\underbrace{\sum_{j=1}^{n}\mathrm{Cov}(x_j, x_i)}_{= \begin{cases}\sigma^2, & j=i \\ 0, & j\neq i\end{cases}} - \frac{1}{n}\sigma^2 \\
&= \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0 \quad \blacksquare
\end{aligned}$$