B52 Oct 27 Lec 1 Notes

Continuous RVs

RV X is continuous if $P(X=x)=0$, $\forall x \in \mathbb{R}$ (i.e. PMF $=0$)

Continuous RVs assume uncountable # of values.

If area is an uncountable collection of lines, then we can use the definite integral

For CRV, we are interested in probabilities of intervals of values of x.

$$P(X \in (a,b]) = P(a < X \leq b), \ \forall a<b \in \mathbb{R}$$

	DRV	CRV
CDF	✓	✓
PMF	✓	×
PDF	×	✓

We can calculate interval probabilities in two ways

(i) Using CDF: $P(a<X\leq b) = F_X(b) - F_X(a)$

(ii) Using PDF

Probability Density Function (PDF)

For CRV X, its PDF is function $f_X(\cdot)$ s.t.

$$P(X \in [a,b]) = \int_a^b f_X(x)\,dx$$

Properties of PDF:

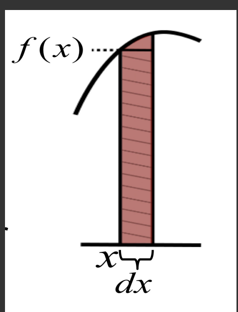
(i) $0 \leq f_X(x)$, $\forall x \in \mathbb{R}$ (can have $f(x) > 1$)

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow P(X \in \mathbb{R}) \to P(S)$

(iii) $F_X(x) = \int_{-\infty}^{x} f_X(u)\,du \Rightarrow f_X(x) = \frac{d}{dx} F_X(x) = F_X'(x)$ (PDF $\Leftrightarrow$ CDF)

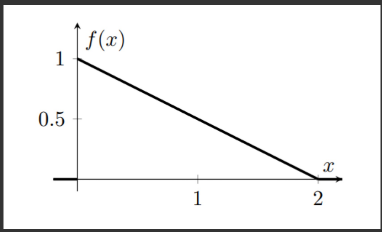
PDF gives rate at which probability accumulates around value x of RV X

$$P(x < X \leq x+dx) = F(x+dx) - F(x) \approx F'(x)\,dx = f(x)\,dx \ , \text{ for } dx \approx 0$$



Ex 1:

Find the CDF of X.



$$F_X(x) = P(X \le x) = \begin{cases} 0 & , x \le 0 \\ x\left(\frac{1+1-\frac{x}{2}}{2}\right) = x - \frac{x^2}{4} & , 0 < x \le 2 \\ \int_0^2 f(x)\,dx = 1 & , x > 2 \end{cases}$$

Ex 2:

Consider RV X with PDF $f(x) = \begin{cases} e^{-x} & , x \ge 0 \\ 0 & , x < 0 \end{cases}$

Find
$$\begin{aligned} P(2 < X < 4) &= \int_2^4 f(x)\,dx \\ &= \int_2^4 e^{-x}\,dx \\ &= \left[-e^{-x}\right]_2^4 \\ &= -e^{-4} + e^{-2} \end{aligned}$$

OR $= F_X(4) - F_X(2)$

$$\begin{aligned} F_X(x) = P(X \le x) = P(-\infty < X < x) = \int_{-\infty}^{x} f_X(u)\,du &= \int_{-\infty}^{0} 0\,du + \int_0^x e^{-u}\,du \\ &= \left[-e^{-u}\right]_{u=0}^{x} \\ &= -e^{-x} - -e^{-0} \\ &= 1 - e^{-x}, \quad x > 0 \end{aligned}$$

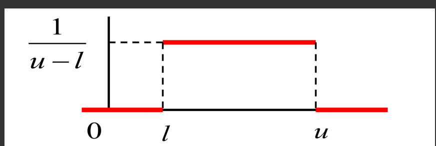
Uniform Distribution

Uniform RV X takes values in interval $[\ell, u]$, $\ell < u \in \mathbb{R}$, so that probability of any sub interval (a, b) is proportional to its length.

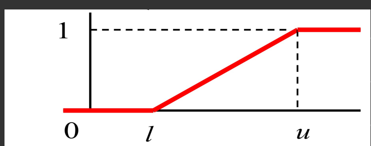
$$P(a < X < b) = \frac{b-a}{u-l}, \quad \forall\, l \le a \le b \le u$$

Denoted $X \sim \text{Uniform}(\ell, u)$

PDF: $f(x) = \begin{cases} \frac{1}{u-\ell} & , \ell \le x \le u \\ 0 & , \text{otherwise} \end{cases}$



CDF: $F(x) = \begin{cases} 0 & , x < \ell \\ \frac{x-\ell}{u-\ell} & , \ell \le x \le u \\ \ell & , x > u \end{cases}$



Ex 3:

Bus passes every 30 min and you arrive at bus stop at random time. Let RV X be your waiting time.

What is the distribution of X?

$X \sim \text{Uniform}(0,30) \Rightarrow F_X(x) = \frac{x-0}{30-0}$

What is the probability you wait more than 20 min?

$P(X>20) = P(20<X<30) = F_X(30) - F_X(20) = 1 - \frac{20-0}{30-0} = \frac{1}{3}$

What is the probability you wait more than 20 min, given that you have already waited for 10 min?

$$P(X>20 \mid X>10) = \frac{P(\{X>20\} \cap \{X>10\})}{P(X>10)} = \frac{P(\{X>20\})}{2/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

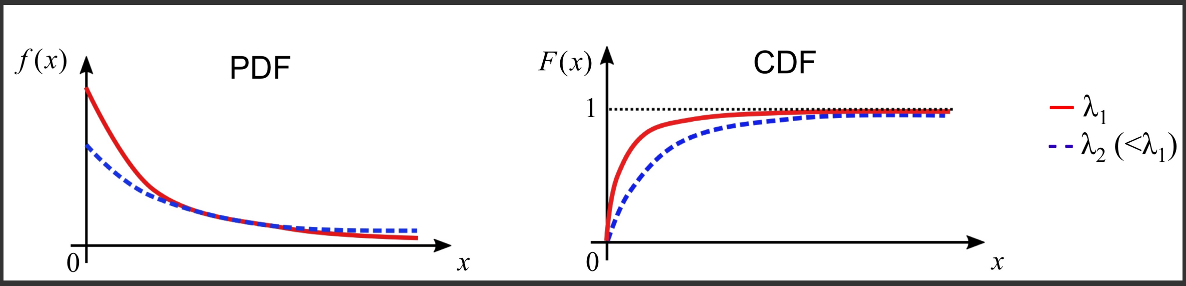
Exponential Distribution

Exponential RV X takes positive values according to PDF

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases} , \text{ for some } \lambda > 0$$

ERV typically measures time until something happens

Denoted by $X \sim \text{Exponential}(\lambda)$



Ex 4:

Service time X of request to web server follows Exponential(λ)

Find the CDF of the service time

$$\begin{aligned} F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)\,du &= \int_0^x \lambda e^{-\lambda u}\,du \\ &= \left[-e^{-\lambda u}\right]_0^x \\ &= -e^{-\lambda x} + 1 \quad , x > 0 \end{aligned}$$

Ex 4. Continued ..

Show that $P(X > x+y \mid X > y) = P(X > x)$ (i.e. Exponential is memoryless)

$$P(X > x+y \mid X > y) = \frac{P(\{X > x+y\} \cap \{X > y\})}{P(X > y)}$$

$$= \frac{P(X > x+y)}{P(X > y)}$$

$$= \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}}$$

$$= \frac{e^{-\lambda x} \cdot \cancel{e^{-\lambda y}}}{\cancel{e^{-\lambda y}}}$$

$$= e^{-\lambda x}$$

$$= P(X > x)$$