B52 Oct 8 Lec 2 Notes

Conditional PMF

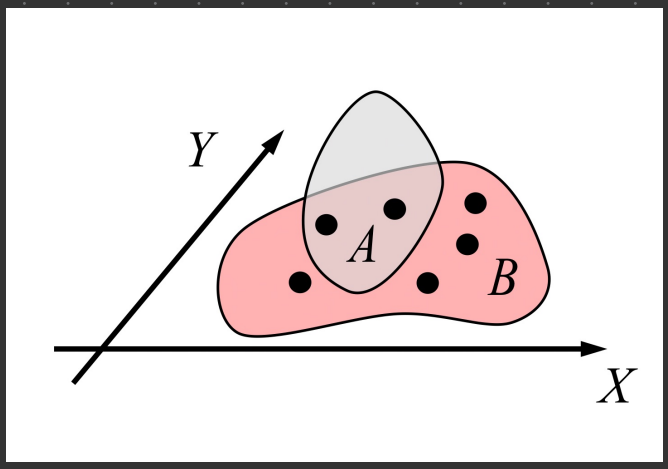
For discrete RVs X, Y, conditional probabilities can be found as

$$P((X,Y)\in A \mid (X,Y)\in B) = \sum_{(x,y)\in A} P(X=x, Y=y \mid (X,Y)\in B)$$

where

$$P(X=x, Y=y \mid (X,Y)\in B) = \frac{P(\{X=x, Y=y\}\cap\{(X,Y)\in B\})}{P((X,Y)\in B)}$$

is a conditional PMF of X, Y given $(X,Y)\in B$



Conditional PMF of X given $Y=y$

$$P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Joint 2D PMF

Marginal PMF

Conditional PMF $P_{X|Y}(x,y)$ must be proper PMF, i.e.

$$P_{X|Y}(x|y) \geq 0 \quad \& \quad \sum_{\text{All } x} P_{X|Y}(x|y) = 1, \forall y$$

If $X \perp Y$, then conditional = marginal PMF

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{P_X(x)P_Y(y)}{P_Y(y)} = P_X(x)$$

Ex 1:

You are given a random number $N \sim Geom(.5)$ of multiple choice (A-D) questions which you answer at random. Find the probability that you get at least one answer correct

Let X be the # correct answers. $\Rightarrow X|N=n \sim Binom(n, p=1/4)$

$$P(X \geq 1) = 1 - P(X=0) \quad \text{complement}$$

$$= 1 - \sum_{n=1}^{\infty} P(X=0 \cap N=n) \quad \text{law of total probability}$$

$$= 1 - \sum_{n=1}^{\infty} \underbrace{P(X=0|N=n)}_{\text{Binom}} \cdot \underbrace{P(N=n)}_{\text{Geom}}$$

$$= 1 - \sum_{n=1}^{\infty} \binom{n}{0} p^0 q^{n-0} (p \cdot q^{n-1})$$

$$= 1 - \sum_{n=1}^{\infty} 1 \cdot 1 \cdot (3/4)^n \left(\tfrac{1}{2} \cdot \tfrac{1}{2}^{n-1}\right)$$

$$= 1 - \sum_{n=1}^{\infty} (3/4)^n (1/2)^n$$

$$= 1 - \sum_{n=1}^{\infty} (3/8)^n$$

$$= 1 - 3/8 \sum_{n=0}^{\infty} (3/8)^n$$

$$= 1 - 3/8 \left(\frac{1}{1 - 3/8}\right) = 3/5$$

Independent PMF

Discrete RVs X, Y are called independent if joint PMF factorizes as

$$P_{X,Y}(x,y) = P(X=x, Y=y) = P(X=x)P(Y=y) = P_X(x) P_Y(y), \ \forall x,y$$

RVs cannot be independent if their joint range of values is not rectangular