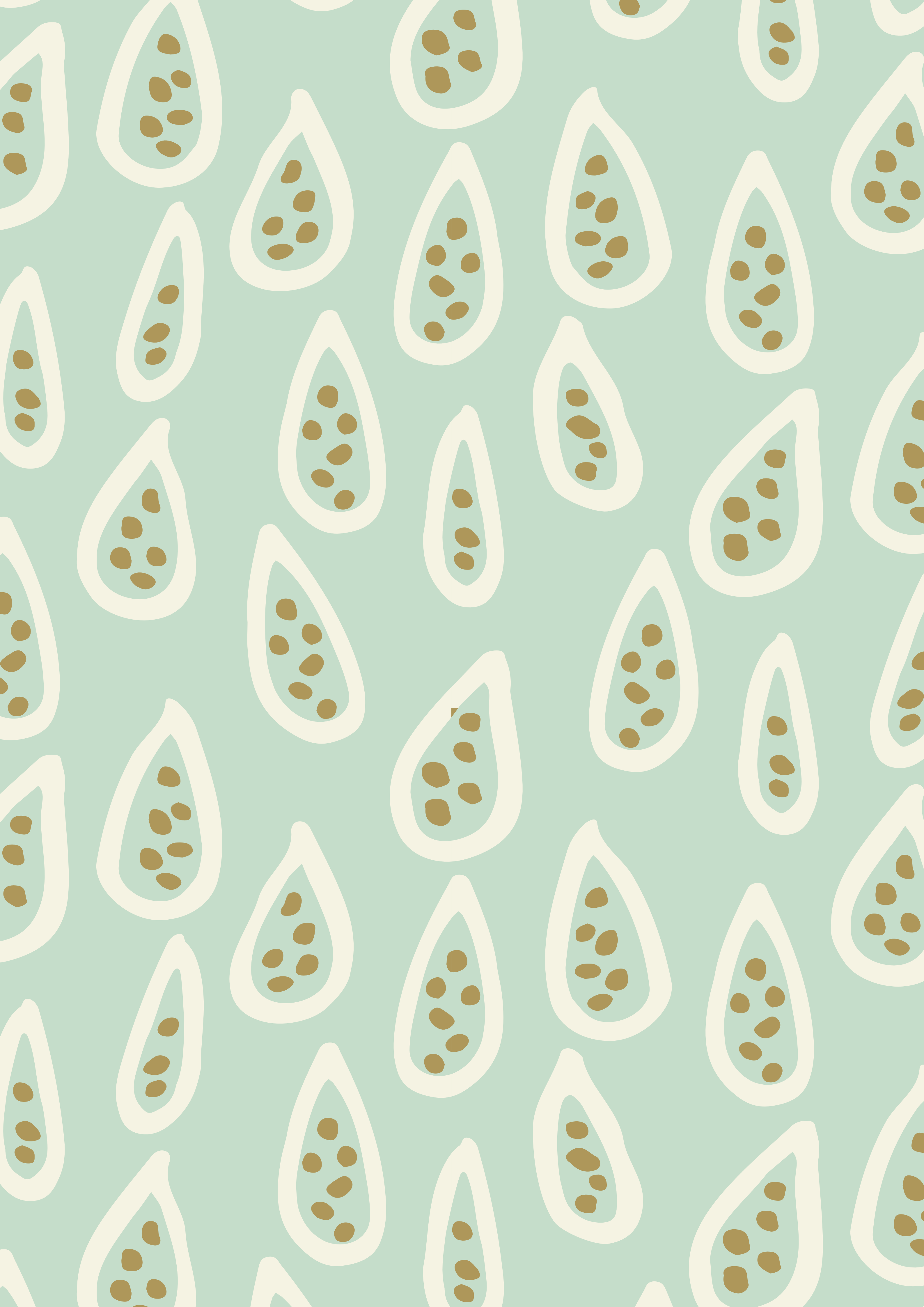

B24 May 28 Lec 2 Notes

Definition:

A system of linear equations or a linear system is a collection of equations:

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &+ b_m \end{aligned} \quad \bigstar$$

Where $x_1, \dots, x_n$ are unknowns and $a_{ij} \in \mathbb{F}$ and $b_j \in \mathbb{F}$ for $1 \le i \le m$, $1 \le j \le n$

Remark:

$\bigstar$ can be represented as:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. solving $\bigstar$ means finding all vectors $(x_1, \dots, x_n) \in \mathbb{F}^n$ which map to $(b_1, \dots, b_m) \in \mathbb{F}^m$ under the L.T. defined by

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

In order to solve the system $\bigstar$, we use the following operations.

(i) Interchange the two rows of the augmented matrix.
(ii) Multiply a row by a non-zero scalar
(iii) Replace the k^{th} row by its sum with a scalar multiple of the j^{th} row.

Remark:

Operations (i), (ii), (iii) do not change the set of solutions to $\bigstar$.

Remark:

Operation (i) corresponds to multiplying augmented matrix on the left by:

$$\begin{array}{cc} & \\ j & \\ k & \\ & \end{array} \begin{bmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{bmatrix}$$

$$\qquad\qquad\qquad j \qquad\qquad\qquad k$$

Remark:

Operation (ii) corresponds to multiplying augmented matrix on the left by:

$$\begin{array}{cc} & \begin{array}{c} j \end{array} \\ j & \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \alpha & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \end{array}$$

Operation (iii) corresponds to multiplying augmented matrix on the left by:

$$\begin{array}{cc} \begin{array}{c} \\ j \\ \\ k \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ 0 \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\ & 0 & \ddots & & & & \\ & \vdots & & & & & \\ 0 \cdots 0 & \alpha & 0 & \cdots 0 & 1 & 0 \cdots & 0 \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \\ & \begin{array}{c} \qquad\quad j \qquad\qquad k \end{array} \end{array}$$

Remark:

Why do operations (i), (ii), (iii) not change the set of solutions $(x_1, \dots, x_n)$? Well, if

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

and B is an invertible matrix, then

$$B \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = B \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

multiply by B^{-1}:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Definition:

A matrix is said to be in echelon form if:

(i) All zero rows are below all non-zero entries.
(ii) For any non-zero row, its first non-zero entry (leading / pivot entry or pivot) is strictly the right of the first non-zero entry in any previous row.

Definition:

A matrix is said to be in reduced echelon form if it is in echelon form and:

(i) All pivot entries are 1.
(ii) Any entry above a pivot is 0.

Definition:

A linear system $Ax=b$ is said to be consistent if there exists a solution x.

Otherwise $Ax=b$ is said to be inconsistent.

Theorem:

Suppose $Ax=b$ has a solution. Then this solution is unique iff the echelon form of the coefficient matrix has a pivot in every column.

Proof ($\Rightarrow$):

If a column is missing a pivot, this corresponds to a free variable, i.e. solution is not unique.

Proof ($\Leftarrow$):

The unique solution is read off as in above example.

Theorem:

The linear system $Ax=b$ is consistent for all b iff the echelon form of the coefficient matrix has a pivot in every row.

Proof ($\Leftarrow$):

Suppose the coefficient has a pivot in every row. Then the augmented matrix cannot have a row of the form

$$(0 \cdots 0 \mid c), \quad c \neq 0$$

hence by previous remark, $Ax=b$ is consistent for any b. $\square$

Proof ($\Rightarrow$):

Suppose (contrapositive) the echelon form of the coefficient matrix A_e, has a row with no pivot, so the last row of A_e is

$$(0 \cdots 0)$$

Note that $EA = A_e$ for E a product of matrices coming from operations (i) - (iii), in particular E is invertible. Note furthermore that

$$A_e x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

does not have a solution x, hence

$$E^{-1} A_e x = Ax = E \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

has no solution x, i.e.

$$Ax = E \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ is inconsistent}$$

$\boxtimes$

Theorem:

$Ax = b$ has a unique solution iff the reduced echelon form of A has a pivot in every row and every column.

Proposition:

Suppose $(v_{11}, \dots, v_{1n})$
$\vdots$
$(v_{m1}, \dots, v_{mn})$ $\in \mathbb{F}^n$

Let $A = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{m1} & \cdots & v_{mn} \end{bmatrix}$. Then:

(i) $(v_{11}, \dots, v_{1n}), \dots, (v_{m1}, \dots, v_{mn})$ are L.I. iff the echelon form of A^T has a pivot in every column.

(ii) $(v_{11}, \dots, v_{1n}), \dots, (v_{m1}, \dots, v_{mn})$ are a spanning set for $\mathbb{F}^n$ iff the echelon form of A^T has a pivot in every row.

(iii) $(v_{11}, \dots, v_{1n}), \dots, (v_{m1}, \dots, v_{mn})$ form a basis for $\mathbb{F}^n$ iff the echelon form of A^T has a pivot in every row and column.

Proof:

$(v_{11}, \dots, v_{1n}), \dots, (v_{m1}, \dots, v_{mn})$ are LI iff the only solution to

$$x_1 (v_{11}, \dots, v_{1n}) + \dots + x_m (v_{m1}, \dots, v_{mn}) = A^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

iff (prev. theorem) the echelon form of A^T has a pivot in every column.

This proves (i) - (iii).