Sec 2.2 Reading

Scalings

For any positive constant k, the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k, since

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}$$

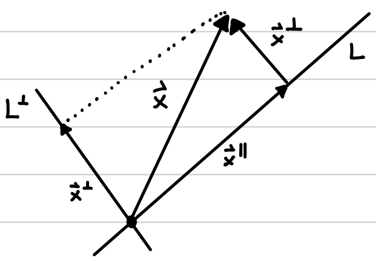
There is a dilation if $k > 1$ and a contraction if $0 < k < 1$.

Orthogonal Projections

Consider a line L in the plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L.



$T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the orthogonal projection of $\vec{x}$ onto L, often denoted by $\text{proj}_L(\vec{x})$.

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}$$

To find a formula for $\vec{x}^{\parallel}$:

Let $\vec{w}$ be a nonzero vector parallel to L. Since $\vec{x}^{\parallel}$ is parallel to $\vec{w}$, we can write

$$\vec{x}^{\parallel} = k\vec{w}$$

Since $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w}$ is perpendicular to line L, we have:

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0$$

It follows that

$$\operatorname{proj}_L(\vec{x}) = \vec{x}^{\parallel} = k\vec{w} = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

Is the transformation $T(\vec{x}) = \operatorname{proj}_L(\vec{x})$ linear?

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

unit vector $\vec{u}$

Then

$$\operatorname{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2x_1 + u_1u_2x_2 \\ u_1u_2x_1 + u_2^2x_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\vec{x}$$

Therefore $T(\vec{x}) = \operatorname{proj}_L(\vec{x})$ is a linear transformation with matrix $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$.

Ex1

Find the matrix P of the orthogonal projection onto the line L spanned by $\vec{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$P = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Def 2.2.1 Orthogonal Projections

Consider a line L in the coordinate plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L.

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the orthogonal projection of $\vec{x}$ onto L, often denoted by $\text{proj}_L(\vec{x})$. If $\vec{w}$ is a nonzero vector parallel to L, then

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

In particular, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector parallel to L, then

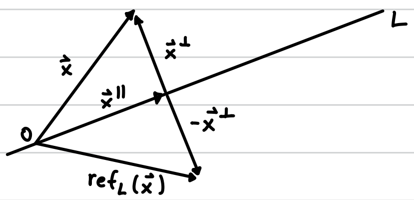
$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}$$

The transformation $T(\vec{x}) = \text{proj}_L(\vec{x})$ is linear, with matrix

$$P = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$$

Reflections

Consider a line L in the coordinate plane, running through the origin. Let $\vec{x}$ be a vector in $\mathbb{R}^2$.



We can see that

$$ref_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$$

Adding up $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ and $ref_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$, we find that $\vec{x} + ref_L(\vec{x}) = 2\vec{x}^{\parallel} = 2\,proj_L(\vec{x})$, so

$$ref_L(\vec{x}) = 2proj_L(\vec{x}) - \vec{x} = 2P\vec{x} - \vec{x} = (2P - I_2)\vec{x}$$

where P is the matrix representing the orthogonal projection onto the line L.

Thus the matrix S of the reflection is

$$S = 2P - I_2 = \begin{bmatrix} 2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Turns out that matrix S is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

Any matrix of the above form represents a reflection about a line.

The column vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b \\ -a \end{bmatrix}$ of a reflection matrix are unit vectors. They are reflections of the standard vector $\begin{bmatrix} a \\ b \end{bmatrix} = ref_L(\vec{e}_1)$ and $\begin{bmatrix} b \\ -a \end{bmatrix} = ref_L(\vec{e}_2)$ by theorem 2.1.2.

Reflection preserves length. The dot product $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab + b(-a) = 0$.

Def 2.2.2: Reflections

Consider a line L in the coordinate plane, running through the origin.
Let $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ be a vector in $\mathbb{R}^2$.

The linear transformation $T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ is called the reflection of $\vec{x}$ about L, often denoted by $ref_L(\vec{x})$:

$$ref_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$$

We have a formula relating $ref_L(\vec{x})$ to $proj_L(\vec{x})$:

$$ref_L(\vec{x}) = 2\,proj_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$$

The matrix of T is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2+b^2=1$. Conversely, any matrix of this form represents a reflection about a line.

Orthogonal Projections and Reflections in Space

Let L be a line in coordinate space, running through the origin Any vector in $\mathbb{R}^3$ can be written uniquely as $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel}$ is parallel to L, and $\vec{x}^{\perp}$ is perpendicular to L. We define:

$$proj_L(\vec{x}) = \vec{x}^{\parallel}$$

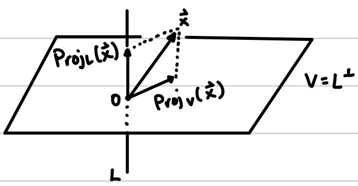
and we have the formula

$$proj_L(\vec{x}) = \vec{x}^{\parallel} = (\vec{x}\cdot\vec{u})\vec{u}$$

Where $\vec{u}$ is a unit vector parallel to L (By def 2.2.1)

Let $L^{\perp} = V$ be the plane through the origin perpendicular to L

Note that the vector $\vec{x}^{\perp}$ will be parallel to $L^{\perp} = V$. Then we have the formulas:



$$proj_V(\vec{x}) = \vec{x} - proj_L(\vec{x}) = \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$$

$$ref_L(\vec{x}) = proj_L(\vec{x}) - proj_V(\vec{x}) = 2\,proj_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$$

$$ref_V(\vec{x}) = proj_V(\vec{x}) - proj_L(\vec{x}) = -ref_L(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u}$$

Ex 3

Let V be the plane defined by $2x_1 + x_2 - 2x_3 = 0$ and let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$.
Find $\mathrm{ref}_V(\vec{x})$

The vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to plane V.

Thus:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Using the formula for $\mathrm{ref}_V(\vec{x})$:

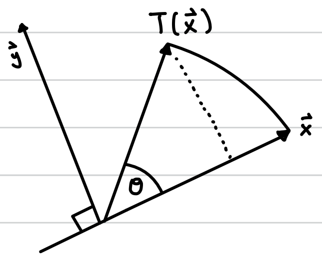
$$\mathrm{ref}_V(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u} = \begin{bmatrix} 5 \\ 4 \\ -9 \end{bmatrix} - \frac{2}{9}\left(\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\right)\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 4 \\ -9 \end{bmatrix} - \frac{2}{9}\left(10 + 4 + 4\right)\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 4 \\ -9 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$$

Rotations

Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through a fixed angle θ in the counter clockwise direction.



The auxiliary vector $\vec{y}$ is obtained by rotating $\vec{x}$ through $\pi/2$.

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $\vec{y} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

Then.

$$T(\vec{x}) = (\cos\theta)\vec{x} + (\sin\theta)\vec{y} = (\cos\theta)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (\sin\theta)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\theta)x_1 - (\sin\theta)x_2 \\ (\sin\theta)x_1 + (\cos\theta)x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\vec{x}$$

Theorem 2.2.3: Rotations

The matrix of a counter clockwise rotation in $\mathbb{R}^2$ through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The matrix of T is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2+b^2=1$. Conversely, any matrix of this form represents a rotation.

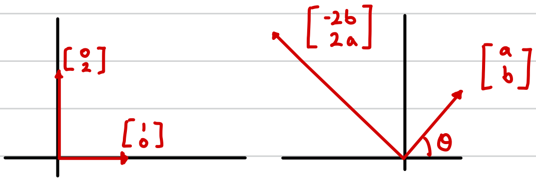
Rotations Combined with a Scaling

Ex 5

Examine how the L.T.

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \vec{x}$$

affect the diagram below.



In polar coordinates, this is a rotation through the polar angle θ of vector $\begin{bmatrix} a \\ b \end{bmatrix}$ with a scaling by the magnitude $r = \sqrt{a^2+b^2}$. We can write the vector in polar coordinates as:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}$$

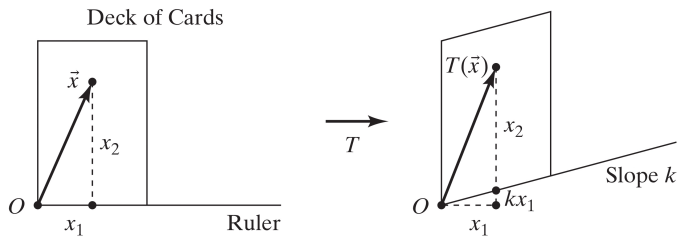
$$= r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Theorem 2.2.4: Rotations Combined with a Scaling

A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling. More precisely, if r and θ are the polar coordinates of vector $\begin{bmatrix} a \\ b \end{bmatrix}$, then $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation through θ combined with a scaling by r.

Shear

Vertical Shear

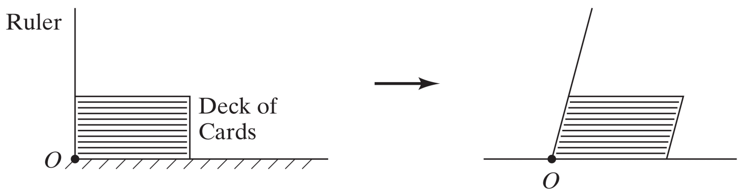


$$T(\vec{x}) = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\vec{x}$$

Horizontal Shear



Analogous to above.

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Theorem 2.2.5: Horizontal and Vertical Shears

The matrix of a horizontal shear is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a vertical shear is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is an arbitrary constant.