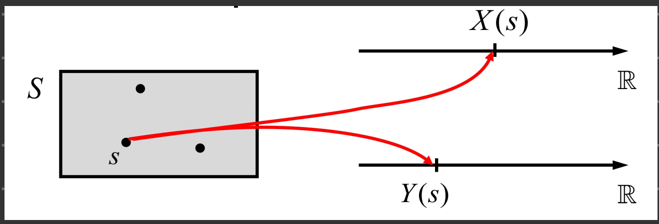
B52 Oct 6 Lec 1 Notes

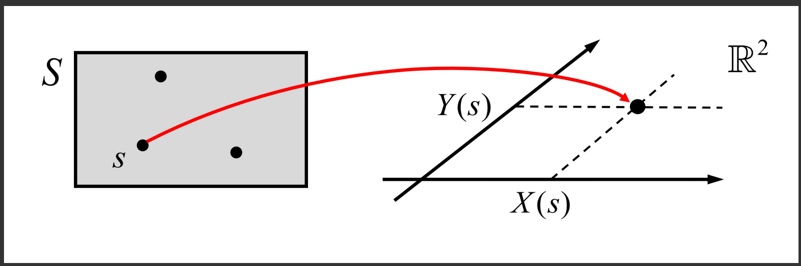
Multivariate Distribution

Can have multiple RV's defined in random experiment.

e.g. roll two 6-sided dice and let
↳ X = value of 1^{st} die
↳ Y = value of 2^{nd} die

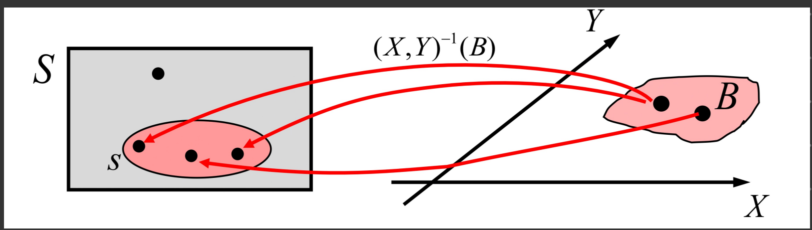


Describe pairs of RV values as coordinates in 2D space.



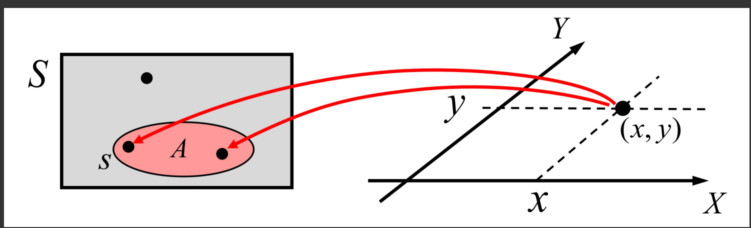
For any RVs X, Y their joint (bivariate) distribution is the collection of all probabilities of the form

$$P((X,Y) \in B) = P(\{ s \in S : (X(s), Y(s)) \in B \}) \ , \ \forall B \subseteq \mathbb{R}^2$$



Joint PMF

For discrete RVs X, Y, interested in specific combinations of values.



Their joint (Bivariate) PMF is defined as $p_{X,Y}(x,y) = P(X=x, Y=y) = P(\{X=x\} \cap \{Y=y\})$

Ex 1:

S					
1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Possible values of $(X_{min}, X_{max}) = \{(x,y) : 1 \le x \le y \le 6\}$

$$p_{X_{min}, X_{max}}(x,y) = P(\{X_{min}(s) = x\} \cap \{X_{max}(s) = y\})$$

$$= \begin{cases} 1/36, & 1 \le x = y \le 6 \\ 2/36, & 1 \le x < y \le 6 \\ 0, & \text{otherwise} \end{cases}$$

$$\sum_{x,y} p_{X_{min}, X_{max}}(x,y) = 6 \cdot \tfrac{1}{36} + 15 \cdot \tfrac{2}{36}$$
$$= 1$$

Multinomial Distribution

Consider n independent trials, each resulting in one of k categories with corresponding probabilities $p_1, \dots, p_k$.

Joint distribution of result category counts $x_1, \dots, x_k$.

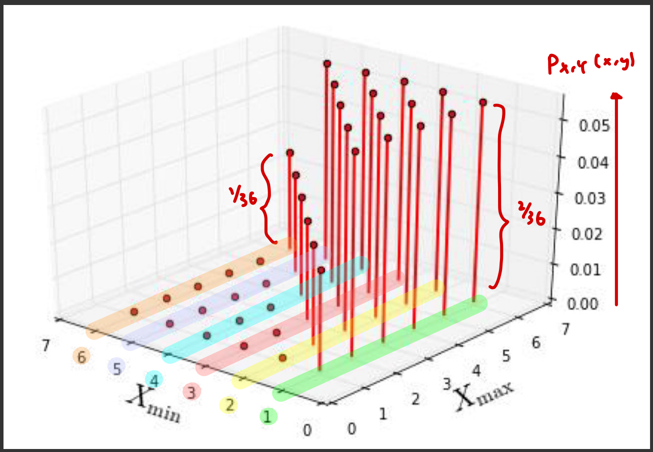
$$p(x_1, \dots, x_k) = \frac{n!}{x_1!\, x_2! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}, \quad \text{for } \begin{cases} x_1, \dots, x_k \ge 0 \\ x_1 + \dots + x_k = n \end{cases}$$

Binomial distribution is a special case of the multinomial distribution when $k=2$.

Marginal PMF

Can get individual (aka marginal) distribution from joint PMF

$$p_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y p_{X,Y}(x,y)$$



$$p_{X_{min}}(x) = P(X_{min} = x) = \begin{cases} \frac{1}{36} + 5 \cdot \frac{2}{36}, & x = 1 \\ \frac{1}{36} + 4 \cdot \frac{2}{36}, & x = 2 \\ \frac{1}{36} + 3 \cdot \frac{2}{36}, & x = 3 \\ \frac{1}{36} + 2 \cdot \frac{2}{36}, & x = 4 \\ \frac{1}{36} + 1 \cdot \frac{2}{36}, & x = 5 \\ \frac{1}{36}, & x = 6 \\ 0, & \text{otherwise} \end{cases}$$

Ex 2:

Define RV's X_W, X_D, X_L to be the # of wins / draws / losses in $n=10$ rounds

$\Rightarrow X_L = n - X_W - X_D$

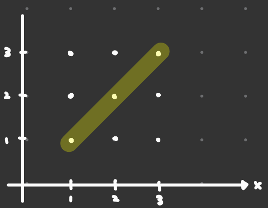
$P_W = P_L = P_D = \frac{1}{3}$

$$P_{X_W,X_D}(x_W, x_D) = \frac{n!}{x_W!\, x_D!\, (n-x_W-x_D)!} \cdot P_W^{x_W} \cdot P_D^{x_D} \cdot (1-P_W-P_D)^{(n-x_W-x_D)}$$

$$\begin{aligned}
P_{X_W}(x_W) &= \sum_{x_D=0}^{n-x_W} P_{X_W,X_D}(x_W,x_D) \\
&= \sum_{x_D=0}^{n-x_W} \frac{n!}{x_W!\, x_D!\, (n-x_W-x_D)!} \cdot P_W^{x_W} \cdot P_D^{x_D} \cdot (1-P_W-P_D)^{(n-x_W-x_D)} \\
&= \frac{n!}{x_W!\,(n-x_W)!} P_W^{x_W} \sum_{x_D=0}^{\overbrace{n-x_W}^{m}} \frac{(n-x_W)!}{x_D!\,(n-x_W-x_D)!} \cdot P_D^{x_D} (1-P_W-P_D)^{n-x_W-x_D} \\
&= \binom{n}{x_W} p_W^{x_W} \cdot \sum_{x_D=0}^{m} \binom{m}{x_D} p_D^{x_D} (1-p_W-p_D)^{m-x_D} \\
&= \binom{n}{x_W} P_W^{x_W} (1-p_W)^{n-x_W} \qquad \forall x_W \in \{0,\dots,n\} \\
&= \text{Binomial}(n, p_W)
\end{aligned}$$

Ex 3:

$$\begin{aligned}
P_X(x) &= \sum_{y=1}^{\infty} P_{X,Y}(x,y) \\
&= \sum_{y=1}^{\infty} p^2 q^{x+y-2} \\
&= p^2 q^{x-1} \sum_{x=1}^{\infty} q^{y-1} \\
&= p^2 q^{x-1} \underbrace{\sum_{i=0}^{\infty} q^i}_{\text{converges since } q<1} \\
&= p^2 q^{x-1} \cdot \frac{1}{1-q} \\
&= \underbrace{p\, q^{x-1}}_{\text{Geometric }(p)}, \quad \forall x \in \mathbb{N}
\end{aligned}$$



Find $P(X=Y) = P(\{s \in S : X(s) = Y(s)\})$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} p_{X,Y}(k,k) \\
&= \sum_{k=1}^{\infty} p^2 q^{k+k-2} \\
&= p^2 \sum_{k=1}^{\infty} q^{2(k-1)} \\
&= p^2 \sum_{i=0}^{\infty} (q^2)^i = p^2 \cdot \frac{1}{1-q^2} = \frac{p^2}{(1-q)(1+q)} = \frac{p}{1+q} = \frac{p}{2-p}
\end{aligned}$$