Theorem

Let V be a vector space, and $\dim V = K$.

(i) $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K$ are L.I. $\Rightarrow \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K\}$ is a basis for V.

(ii) $\operatorname{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K) = V \Rightarrow \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K\}$ is a basis for V.

Proof (i):

Assume $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K\}$ is L.I. WTS $\{\vec{v}_1, \dots, \vec{v}_K\}$ is a basis

WTS $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_K) = V$

Proof by Contradiction

Suppose $\exists \vec{v} \in V$ s.t. $\vec{v} \notin \operatorname{span}(\vec{v}_1, \dots, \vec{v}_K)$

Consider $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K, \vec{v}\}$ has no redundant vector

So $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K, \vec{v}\}$ is L.I.

Since $\dim V = K$, V has a basis, which is also a spanning set for V, with K vectors. We must have $|\{\vec{v}_1, \dots, \vec{v}_K, \vec{v}\}| \leq |B| = K$.

$K+1 \not\leq K$

Contradiction!

Thm 3.3.1

So $\vec{v}$ does not exist.

$\therefore V = \operatorname{span}(\vec{v}_1, \dots, \vec{v}_K)$

Proof (ii) :

Suppose $\operatorname{span}(\vec{v}_1, \dots, \vec{v}_k) = V$. WTS $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a basis for V.

Proof by Contradiction.

Suppose $\{\vec{v}_1, \dots, \vec{v}_k\}$ is not L.I.

$\exists i$ s.t. $\vec{v}_i$ is redundant , $\vec{v}_i \in \operatorname{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1})$

$\operatorname{span}(\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k)$

So $\underbrace{\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}}_{k-1 \text{ vectors}}$ is a spanning set for V. Since $\vec{v}_i$ is redundant

Since $\dim V = k$, V has a basis with k elements.

$$|\{\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k\}| \geq |B| \qquad \text{Theorem 3.3.1}$$

$$k-1 \geq k$$

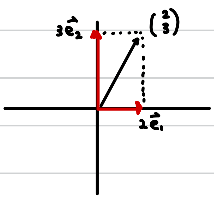
$\times$

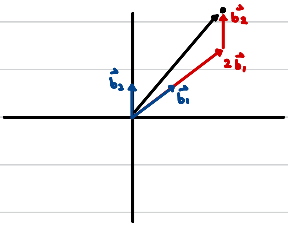
contradiction!

Ex 1

$B = \{\vec{e}_1, \vec{e}_2\}$ $\qquad$ $B = \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$

$\begin{pmatrix}2\\3\end{pmatrix} = 2\vec{e}_1 + 3\vec{e}_3$ $\qquad$ $\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}$





Ex 2

$$S = \left\{ \overset{\vec{s}_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \overset{\vec{s}_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}, \overset{\vec{s}_3}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\}$$

$\text{span}(S) = \mathbb{R}^2$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \vec{v} = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{v} = 1\vec{s}_1 + 2\vec{s}_2 + 1\vec{s}_3$$
$$= 2\vec{s}_1 + 1\vec{s}_2 + 0\vec{s}_3$$

Theorem

Let $B \subseteq V$.

B is a basis iff $\forall \vec{v} \in V$ can be written as a l.c. of vectors in B in a unique way.

Proof ($\Rightarrow$):

Suppose B is a basis. WTS $\forall \vec{v} \in V$ is a unique l.c. of vectors in B.

Since B is a basis for V, $\text{span}(B) = V$.

So $\forall \vec{v} \in V$, $\vec{v} \in \text{span}(B)$

$\forall \vec{v} \in V$, $\exists c_1, \dots, c_n \in \mathbb{R}$, $\vec{v} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$, $\vec{b}_i$ in B.

To see that $c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ is unique,

Suppose $\vec{v} = r_1\vec{b}_1 + \dots + r_n\vec{b}_n$, $\vec{r}_i \in \mathbb{R}$

WTS $r_1 = c_1, \dots, r_n = c_n$

$$\vec{v} = r_1\vec{b}_1 + \dots + r_n\vec{b}_n = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$$

$$(r_1 - c_1)\vec{b}_1 + \dots + (r_n - c_n)\vec{b}_n = \vec{0}$$

Since B is L.I., $r_1 - c_1 = 0, \dots, r_n - c_n = 0$

$\therefore r_1 = c_1, \dots, r_n = c_n$ //