B24 July 23 Lec 2 Notes

Definition:

A L.T. $U : X \to Y$ (X, Y are IPS) is called an isometry if

$$\|Ux\| = \|x\|, \quad \forall x \in X$$

In other words, it is called a length-preserving L.T.

e.g.

Rotations are isometries, as are reflections.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ is not an isometry
$(x,y) \mapsto (2x, 2y) = 2(x,y)$

Theorem:

A L.T. is an isometry iff

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \quad \forall x, y \in X$$

Proof ($\Leftarrow$):

$$\|Ux\| = \sqrt{\langle Ux, Ux \rangle}$$
$$= \sqrt{\langle x, x \rangle} \qquad \text{By assumption}$$
$$= \|x\| \quad \square$$

Proof ($\Rightarrow$):

Recall the polarization identities.

Assume that X, Y are real IPS. Then:

$$\begin{aligned} \langle Ux, Uy \rangle &= \tfrac{1}{4}\left(\|Ux + Uy\|^2 - \|Ux - Uy\|^2\right) \\ &= \tfrac{1}{4}\left(\|U(x+y)\|^2 - \|U(x-y)\|^2\right) && \text{linearity} \\ &= \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) && \text{def of isometry} \\ &= \langle x, y \rangle && \text{polarization identity} \end{aligned}$$

The case that X, Y are $\mathbb{C}$-IPS is similar using complex polarization identity. $\square$

Lemma:

A L.T. $U: X \to Y$ is an isometry iff

$$U^* U = \underbrace{I_X}$$ Identity transformation on x

Proof ($\Leftarrow$):

Assume $U^* U = I_X$.

$$\begin{aligned} \|x\|^2 &= \langle x, x \rangle \\ &= \langle U^* U x, x \rangle \quad && U^* U = I \\ &= \langle Ux, Ux \rangle \\ &= \|Ux\| \end{aligned}$$

$\forall x \in X$, so U is an isometry. $\blacksquare$

Proof ($\Rightarrow$):

Assume $U: X \to Y$ is an isometry.

$$\begin{aligned} \langle U^* U x, y \rangle &= \langle Ux, Uy \rangle \\ &= \langle x, y \rangle \quad && \text{By previous theorem} \end{aligned}$$

$\Rightarrow U^* U x = x$

$\Rightarrow U^* U = I_X$ $\blacksquare$

Definition:

An isometric $U: X \to Y$ is called **unitary** if U is invertible.

e.g. $\mathbb{R}^2 \to \mathbb{R}^3$ is an isometry since
$(x,y) \mapsto (x,y,0)$

$$\begin{aligned} \|(x,y,0)\| &= \sqrt{x^2 + y^2 + 0^2} \\ &= \sqrt{x^2 + y^2} \\ &= \|(x,y)\| \end{aligned}$$

but not invertible ($\dim \mathbb{R}^3 \neq \dim \mathbb{R}^2$)

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and $U^* U = I_2$, but $U^* U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

Proposition:

An isometry $U: X \to Y$ is unitary iff $\dim X = \dim Y$.

Proof ($\Rightarrow$):

If $U: X \to Y$ is unitary, then in particular, U is invertible, so $\dim X = \dim Y$.

Proof ($\Leftarrow$):

Assume $\dim X = \dim Y$. Since U is an isometry, by prev. lemma, $U^* U = I_X$. So $\ker(U) = \{0\}$, so U is invertible.

Proposition:

Let U be a $m \times n$ matrix. Then U is an isometry iff the columns of U form an orthonormal system in $\mathbb{F}^m$.

Proof ($\Rightarrow$):

Assume U is an isometry.

Ue_i = i^{th} column of U.

and $\langle Ue_i, Ue_j \rangle = \langle e_i, e_j \rangle$ Since U is isometry

$$= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

i.e. columns of U form an orthonormal system in $\mathbb{F}^n$. $\blacksquare$

Proof ($\Leftarrow$):

Assume the columns of U form an orthonormal system in $\mathbb{F}^m$.

Let U_i denote the i^{th} column of U.

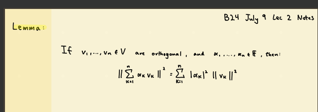
Let $x \in \mathbb{F}^n$, and let $x = \sum_{i=1}^{n} \alpha_i e_i$.

$$\|Ux\|^2 = \left\|U\left(\sum_{i=1}^{n} \alpha_i e_i\right)\right\|^2$$

$$= \left\|\sum_{i=1}^{n} \alpha_i U(e_i)\right\|^2$$

$$= \left\|\sum_{i=1}^{n} \alpha_i U_i\right\|^2$$

$$= \sum_{i=1}^{n} |\alpha_i|^2 \|U_i\|^2$$

Recall from 

Lemma:

B24 July 9 Lec 2 Notes

If $v_1, \dots, v_n \in V$ are orthogonal, and $\alpha_1, \dots, \alpha_n \in \mathbb{F}$, then:

$$\left\|\sum_{k=1}^{n} \alpha_k v_k\right\|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \|v_k\|^2$$

and $U_1, \dots, U_n$ form an orthogonal system

$$= \sum_{i=1}^{n} |\alpha_i|^2 = \left\|\sum_{i=1}^{n} \alpha_i e_i\right\|^2 = \|x\|^2$$

i.e. $\|Ux\| = \|x\|$. $\blacksquare$

e.g. rotation in $\mathbb{R}^2$ is given by

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

and $\begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix}, \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$ form an orthonormal system in $\mathbb{R}^2$ since

$$\left\langle \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix}, \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix} \right\rangle = \cos\alpha\sin\alpha - \sin\alpha\cos\alpha$$
$$= 0$$

and $\left\| \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} \right\|^2 = 1$, and $\left\| \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix} \right\|^2 = 1$

Proposition:

Let U be a unitary matrix Then,

(i) $|\det U| = 1$
(ii) if λ is an eigenvalue of U, then $|\lambda| = 1$

Lemma:

If $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$ and $\overline{A} = \begin{bmatrix} \overline{A_{11}} & \cdots & \overline{A_{1n}} \\ \vdots & & \vdots \\ \overline{A_{n1}} & \cdots & \overline{A_{nn}} \end{bmatrix}$, then $\det(\overline{A}) = \overline{\det(A)}$.

Proof: Of lemma above.

Cofactor expansion.

Proof: (i)

Since U is unitary, $U^*U = I$, so

$$\underbrace{\det(U^*)}_{= \det(\overline{U^T}) \underset{\text{By lemma}}{=} \overline{\det(U^T)} = \overline{\det(U)}} \det(U) = 1$$

So

$$\overline{\det(U)} \cdot \det(U) = |\det(U)|^2 = 1$$

$\Rightarrow |\det(U)| = 1$

$\boxtimes$

Proof: (ii)

Assume λ is an eigenvalue of $\mathcal{U}$, with eigenvector v.

Then

$$\begin{aligned} \|v\| &= \|\mathcal{U}v\| \quad \mathcal{U} \text{ is an isometry} \\ &= \|\lambda v\| \\ &= |\lambda|\,\|v\| \end{aligned}$$

$$\overset{v \neq 0}{\Rightarrow} |\lambda| = 1$$

$\boxtimes$

Definition:

Linear transformations A, B are called unitarily equivalent if there exists a unitary L.T. $\mathcal{U}$ s.t.

$$A = \mathcal{U} B \mathcal{U}^*$$

Remark:

In particular, unitarily equivalent $\Rightarrow$ similar

Proposition:

A $n \times n$ matrix A is unitarily equivalent to a diagonal matrix iff there is an orthonormal basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

Proof ($\Rightarrow$)**:**

Assume A is unitarily equivalent to a diagonal matrix, i.e.

$$A = \mathcal{U} D \mathcal{U}^*$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues (i.e. diagonal entries) of D Then:

$$\begin{aligned} A\mathcal{U}e_i &= \mathcal{U} D \mathcal{U}^* \mathcal{U} e_i \\ &= \mathcal{U} D e_i \\ &= \mathcal{U} \lambda_i e_i \\ &= \lambda_i (\mathcal{U} e_i) \end{aligned}$$

i.e. $\mathcal{U}e_1, \dots, \mathcal{U}e_n$ are eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$, and since $\mathcal{U}$ is invertible $\mathcal{U}e_1, \dots, \mathcal{U}e_n$ forms a basis for $\mathbb{F}^n$.

And $\|\mathcal{U}e_i\| = \|e_i\| = 1$ and

Proof (continued...):

$$\langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

$\Rightarrow Ue_1, \dots, Ue_n$ forms an orthonormal basis for $\mathbb{F}^n$. ☒

Proof ($\Leftarrow$):

Assume there is an orthonormal basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

Let $u_1, \dots, u_n$ be an orthonormal basis of eigenvectors of A with corresponding e.v. $\lambda_1, \dots, \lambda_n$.

Let $D = [A]^{u_1,\dots,u_n}_{u_1,\dots,u_n} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

By the change of basis formula,

$$A = [A]^{e_1,\dots,e_n}_{e_1,\dots,e_n}$$

$$= \underbrace{[I]^{e_1,\dots,e_n}_{u_1,\dots,u_n}}\ \underbrace{[A]^{u_1,\dots,u_n}_{u_1,\dots,u_n}}_{D \text{ diagonal}}\ [I]^{u_1,\dots,u_n}_{e_1,\dots,e_n}$$

is isometry since the columns of $[I]^{u_1,\dots,u_n}_{e_1,\dots,e_n}$ are exactly $u_1,\dots,u_n$ which we assumed from an orthogonal system and hence unitary.

i.e. $A = UDU^*$

☒

Definition:

We call a function $f: V \to V$ (where V is an IPS) a rigid motion if

$$\|f(x) - f(y)\| = \|x - y\|, \quad \forall x, y \in V$$

Remark:

We are not assuming f above is a L.T.

Ex 1:

If $U: V \to V$ is unitary, then

$$\|U(x) - U(y)\| = \|U(x-y)\| = \|x-y\|, \quad \forall x, y \in V$$

i.e. U is rigid motion

Ex 2:

If $a \in V$ and $f_a : V \to V$ is defined by $f_a(v) := v + a$, then f_a is a rigid motion, since

$$\begin{aligned} \|f_a(x) - f_a(y)\| &= \|x + a - (y + a)\| \\ &= \|x - y\| \end{aligned}$$

But f_a is not a L.T. since $f_a(0) = a \neq 0$.

Theorem:

Let X be a real IPS, and $f: X \to X$ is a rigid motion, and define $T(x) := f(x) - f(0)$.
Then T is unitary (in particular, T is linear).