CH 7.3 Finding Eigenvectors

Def 7.3.1: Eigenspaces

Consider an eigenvalue λ of an $n \times n$ matrix A. Then the kernel of the matrix $A - \lambda I_n$ is called the eigenspace associated with λ, denoted by E_λ:

$$E_\lambda = \ker(A - \lambda I_n) = \{ \vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda \vec{v} \}$$

Ex 3:

Find the eigenspaces of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$\lambda = 1, 0$

$$E_1 = \ker(A - I_2) = \ker \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$E_0 = \ker(A) = \text{span} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Since we can only find 2 L.I. eigenvectors, we cannot construct an eigenbasis.

$\therefore$ A is not diagonalizable.

Def 7.3.2: Geometric Multiplicity

Consider an eigenvalue λ of an $n \times n$ matrix A. The dimension of eigenspace $E_\lambda = \ker(A - \lambda I_n)$ is called the geometric multiplicity of eigenvalue λ, denoted $\text{gemu}(\lambda)$. Thus,

$$\text{gemu}(\lambda) = \text{nullity}(A - \lambda I_n) = n - \text{rank}(A - \lambda I_n)$$

Theorem 7.3.3: Eigenbases and gemu

(i) Consider an $n \times n$ matrix A. If we find a basis for each eigenspace of A and concatenate all these bases, then the resulting eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ will be L.I.

(ii) Matrix A is diagonalizable iff the gemu of the eigenvalues add up to n.

Proof (i): Done in lecture

Theorem 7.3.4: An nxn matrix with n distinct eigenvalues

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable. We can construct an eigenbasis by finding an eigenvector for each eigenvalue.

Theorem 7.3.5: The eigenvalues of similar matrices

Suppose A is simalar to B.

(i) char A = char B ; $f_A(\lambda) = f_B(\lambda)$

(ii) rank A = rank B ; nullity A = Nullity B

(iii) A and B have the same eigenvalues, with the same gemu and almu. However, the eigenvectors could be different.

(iv) det A = det B ; trace A = trace B

Proof (i).

If $B = S^{-1}AS$

$$\begin{aligned}\text{char } B &= \det(B - \lambda I_n) \\ &= \det(S^{-1}AS - \lambda I_n) \\ &= \det(S^{-1}AS - \lambda S^{-1}I_nS) \\ &= \det(S^{-1}(A - \lambda I_n)S) \\ &= \det(S^{-1})\det(A - \lambda I_n)\det S \\ &= \det(A - \lambda I_n) \\ &= \text{char } A\end{aligned}$$

prop of matrix multiplication
$\det(A^{-1})\det(A) = 1$

Theorem 7.3.6: almu vs gemu

If λ is an eigenvalue of a square matrix A, then

$$\text{gemu}(\lambda) \leq \text{almu}(\lambda)$$