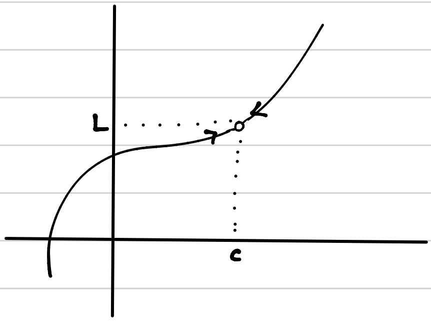
Quiz 2 Review Seminar

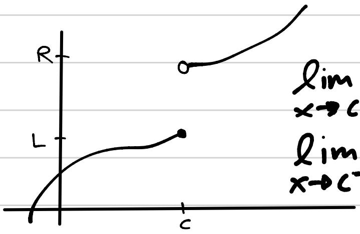
Coverage: 1.1, 1.2, 1.3

<u>limits</u> → limits are foundation of everything in calculus



$\lim_{x \to c} f(x) = L$

"the limit of $f(x)$ as x approaches c is L."

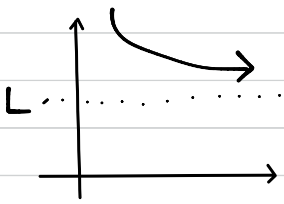


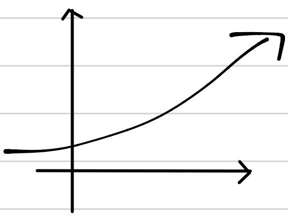
$\lim_{x \to c^+} f(x) = R$

$\lim_{x \to c^-} f(x) = L$

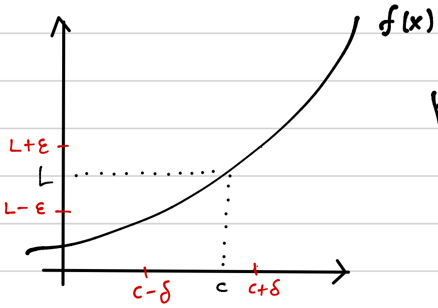
Limit at $\pm\infty$

$\lim_{x \to \pm\infty} f(x) = L$ or $\pm\infty$ or undefined



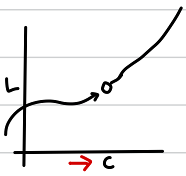


Formal Definition of limits

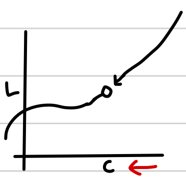


$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\underbrace{x \in (c-\delta, c) \cup (c, c+\delta)}_{0 < |x-c| < \delta} \Rightarrow \underbrace{f(x) \in (L-\varepsilon, L+\varepsilon)}_{|f(x)-L| < \varepsilon}$

Algebraic

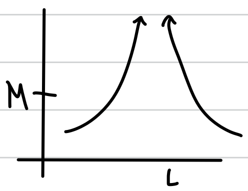


$\lim_{x \to c^-} f(x) = L$ $\quad \forall \varepsilon > 0 \ \exists \delta > 0$ st $x \in (c-\delta, c) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$



$\lim_{x \to c^+} f(x) = L$ $\quad \forall \varepsilon > 0 \ \exists \delta > 0$ st $x \in (c, c+\delta) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$

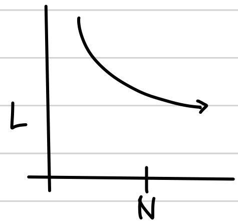
Infinite Limit



$$\lim_{x \to c} f(x) = \infty$$

$$\forall M>0 \ \exists \delta>0 \ \text{s t} \ x \in (c-\delta, c) \cup (c, c+\delta) \Rightarrow f(x) \in (M, \infty)$$

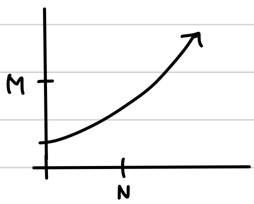
Limit at Infinity



$$\lim_{x \to \infty} f(x) = L$$

$$\forall \varepsilon>0 \ \exists N>0 \ \text{s t} \ x \in (N, \infty) \Rightarrow f(x) \in (L-\varepsilon, L+\varepsilon)$$

Infinite Limit at Infinity



$$\lim_{x \to \infty} f(x) = \infty$$

$$\forall M>0 \ \exists N>0 \ \text{s t} \ x \in (N, \infty) \Rightarrow f(x) \in (M, \infty)$$

Mathematical Logic with Quantifiers

$P \Rightarrow Q$ → If P is true, then Q follows

$\forall, \exists, \exists!, \leq, >$

Examples

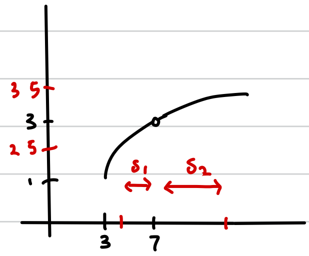
1 Write the following statement using mathematical symbol and/or logic

"For any r and s belonging to the set of real numbers, there exists a real number p, q such that if the difference between r and s is greater than q, then the sum of r and s is less than 2 raised to the power p"

$$\boxed{\forall r, s \in \mathbb{R}, \exists p, q \in \mathbb{R} \mid r-s>q \Rightarrow r+s<2^p}$$

Examples

2. For $\lim_{x\to 7}(\sqrt{x-3}+1)=3$, find the largest δ that works for the given $\varepsilon=\frac{1}{2}$



$\forall\varepsilon>0\ \exists\delta>0$ st $x\in(7-\delta,7)\cup(7,7+\delta)\Rightarrow\sqrt{x-3}+1\in(3-\varepsilon,3+\varepsilon)$

$$|\sqrt{x-3}+1-3|<0.5$$
$$|\sqrt{x-3}-2|<0.5$$
$$-0.5<\sqrt{x-3}-2<0.5$$
$$\left.\begin{array}{l}5.25<x<9.25\\ 7-\delta<x<7+\delta\end{array}\right]\Rightarrow\begin{array}{l}\delta_1=7-5.25=1.75\\ \delta_2=9.25-7=2.75\end{array}$$

Take $\delta=\min\{\delta_1,\delta_2\}$

Thus $\delta=\min\{1.75, 2.75\}$
$\boxed{\delta=1.75}$

3. Prove Uniqueness of Limit

If $\lim_{x\to c^-}f(x)=L$ and $\lim_{x\to c^-}f(x)=M$

Then $L=M$

Proof: Suppose to the contrary that there exists $\lim_{x\to c^-}f(x)=L$ and $\lim_{x\to c^-}f(x)=M$ and $L>M$

Let's choose $\varepsilon=\frac{k}{2}$ With the choice of ε intervals do not overlap

$\lim_{x\to c^-}f(x)=L$ means $\forall\varepsilon>0\ \exists\delta_1>0$ st $x\in(c-\delta_1,c)\Rightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$
$\lim_{x\to c^-}f(x)=M$ means $\forall\varepsilon>0\ \exists\delta_2>0$ st $x\in(c-\delta_2,c)\Rightarrow f(x)\in(L-\varepsilon,L+\varepsilon)$

Let's take $\min\{\delta_1,\delta_2\}=\delta$

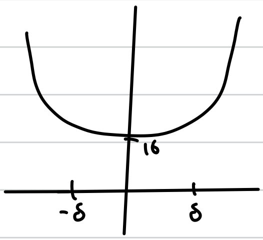
For such $\delta>0$, $f(x)\in(L-\varepsilon,L+\varepsilon)$ and $f(x)\in(M-\varepsilon,M+\varepsilon)$ but this is not possible if intervals do not overlap

$\boxed{\text{The initial supposition is wrong and } L=M}$

Examples

4 Give $\varepsilon-\delta$ definition for $\lim_{x\to 0} x^2+16=16$ and interpret geometrically

$\forall \varepsilon>0\ \exists \delta>0$ st $x\in(0-\delta,0)\cup(0,0+\delta) \Rightarrow x^2+16\in(16-\varepsilon,16+\varepsilon)$



5 Prove $\lim_{x\to 1^+}\sqrt{2x-2}=0$

$\forall \varepsilon>0\ \exists \delta>0$ st $x\in(1,1+\delta) \Rightarrow \sqrt{2x-2}\in(0-\varepsilon,0+\varepsilon)$

Aside

because we know that $|x-1|<\delta$

$$|f(x)-L| = \left|\sqrt{2x-2}-0\right| = \left|\sqrt{2(x-1)}\right| < \sqrt{2\delta} = \varepsilon$$

$$\delta = \frac{\varepsilon^2}{2}$$

Proof

Given $\varepsilon>0$, choose $\delta=\frac{\varepsilon^2}{2}$ Then if $0<x-1<\delta$, then

$$|f(x)-L| = \underbrace{\left|\sqrt{2x-2}\right| = \left|\sqrt{2(x-1)}\right|}_{|f(x)-L|} < \underbrace{\sqrt{2\delta} = \sqrt{2\left(\frac{\varepsilon^2}{2}\right)} = \varepsilon}_{\varepsilon}$$

Basically that shows that $|f(x)-L|<\varepsilon$ given a $\delta=\frac{\varepsilon^2}{2}$

6 Prove $\lim_{x\to 5}(2x-8)=2$

$\forall \varepsilon>0\ \exists \delta>0$ st $0<|x-5|<\delta \Rightarrow |2x-8-2|<\varepsilon$

Aside $|f(x)-L| = |2x-8-2| = |2x-10| = |2(x-5)| < 2\delta$

$$\delta = \varepsilon/2$$

Proof Given $\varepsilon>0$, choose $\delta=\varepsilon/2$ Then if $0<|x-5|<\delta$, then

$$|2x-8-2| = 2|x-5| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$$

7 $\lim_{x\to 1^+} \sqrt{2x+2} = 2$

$\forall \varepsilon>0\ \exists \delta>0$ st $x\in(1,1+\delta) \Rightarrow |f(x)-L|<\varepsilon$

$1<x<1+\delta$

$0<x-1<\delta$

Aside

$$|f(x)-L| = |\sqrt{2x+2}-2| = \left|\sqrt{2x+2}-2 \cdot \left(\frac{\sqrt{2x+2}-2}{\sqrt{2x+2}-2}\right)\right|$$

$$= \left|\frac{2x+2-4}{\sqrt{2x+2}+2}\right| = \left|\frac{2x-2}{\sqrt{2x+2}+2}\right| = \left|\frac{2(x-1)}{\sqrt{2x+2}+2}\right|$$

$$= 2|x-1|\left|\frac{1}{\sqrt{2x+2}+2}\right|$$

we know that this is $<2\delta$

we have to bound this because x is unpredictable

$$< \frac{2}{2+\sqrt{2}}\delta = \varepsilon$$

$$\delta = \frac{\varepsilon(2+\sqrt{2})}{2}$$

Helper Assumption

$|x-1|<\delta \to$ Restrict ≤ 1

because there are 2 variables

$|x-1|<1$

$-1<x-1<1$

$0<x<2$

$0<2x<4$

$2<2x+2<6$

$\sqrt{2}<\sqrt{2x+2}<\sqrt{6}$

$\sqrt{2}+2<\sqrt{2x+2}+2<\sqrt{6}+2$

$\frac{1}{\sqrt{6}+2} < \frac{1}{\sqrt{2x+2}+2} < \frac{1}{\sqrt{2}+2}$ Upper bound

Proof Given $\varepsilon>0$, choose $\delta=\min\{1, \frac{(2+\sqrt{2})\varepsilon}{2}\}$ If $1<x<1+\delta$, then

$$|f(x)-L| = |\sqrt{2x+2}-2| = \frac{2|x-1|}{\sqrt{2x+2}+2} < 2\delta\frac{1}{2+\sqrt{2}} = \frac{2}{2+\sqrt{2}}\left(\frac{2+\sqrt{2}}{2}\right)\varepsilon$$
$$= \varepsilon$$

8 $\lim_{x\to -1}(3x^2-x+5)=9$

Aside $|f(x)-L| = |3x^2-x+5-9| = |3x^2-x-4|$

$= |(3x-4)(x+1)|$

have to bound this

we know $|x+1|<\delta$

$< 10\delta = \varepsilon$

$\delta = \frac{\varepsilon}{10}$

Restrict $\delta\le 1$

$-1<x+1<1$

$-2<x<0$

$-6<3x<0$

$-10<3x-4<-4$

$4<|3x-4|<10$ upper bound

Proof Given $\varepsilon>0$, choose $\min\{1,\frac{\varepsilon}{10}\}$

If $0<|x+1|<\delta$

then $|f(x)-L| = |3x^2-x+5-9|$

$= |3x-4||x+1| < 10\delta = 10\left(\frac{\varepsilon}{10}\right) = \varepsilon$

9 $\lim_{x\to\infty} \frac{3x+5}{x+2} = 3$

N-ε

$\forall \varepsilon>0\ \exists N>0$ st $x>N \Rightarrow |f(x)-L|<\varepsilon$

Aside

$|f(x)-L| = \left|\frac{3x+5}{x+2} - 3\right|$

$= \left|\frac{-1}{x+2}\right| = \left|\frac{1}{x+2}\right|$

We Know

$x>N$

We want

$x+2$

Assuming that $x>N$, $x+2>N+2$

$x>N \Leftrightarrow x+2>N+2$

$\frac{1}{x+2} < \frac{1}{N+2} = \varepsilon \Rightarrow (N+2)\varepsilon = 1$

$|f(x)-L| = \left|\frac{1}{x+2}\right| < \varepsilon$

$N\varepsilon + 2\varepsilon = 1$

$N = \frac{1}{\varepsilon} - 2$

Proof Given $\varepsilon>0$. choose $N = \frac{1}{\varepsilon}-2$, if $x>N$, then

$|f(x)-L| = \left|\frac{3x+5}{x+2}-3\right| = \left|\frac{1}{x+2}\right| < \left|\frac{1}{N+2}\right| = \left|\frac{1}{\frac{1}{\varepsilon}-2+2}\right|$

$= \varepsilon$

10 Prove $\lim_{x\to -2} \frac{1}{x+2} = \infty$

δ-M

$\forall M>0\ \exists \delta>0$ st $|x-c|<\delta \Rightarrow f(x)>M$

Aside

Since $f(x)>M$

$x+2 < \delta$

$\frac{1}{x+2} > \frac{1}{\delta} = M$

$\delta = \frac{1}{M}$

Proof Given $M>0$, choose $\delta = \frac{1}{M}$ If $0<x+2<\delta$, then

$\frac{1}{x+2} > \frac{1}{\delta} = \frac{1}{1/M} = M$

11 Prove $\lim_{x\to\infty} 3x-5 = \infty$

M-N

$\forall M>0\ \exists N>0$ st $x>N \Rightarrow f(x)>M$

Aside

$x>N$

$3x-5 > 3N-5 = M$

$N = \frac{M+5}{3}$

Proof $\forall M>0$, choose $N = \frac{M+5}{3}$. If $x>N$. then

$3x-5 > 3N-5 = 3\left(\frac{M+5}{3}\right)-5 = M$