Ex 1

Given $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$

Can we say : $w = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) \underset{s.s}{\subseteq} \mathbb{R}^3$?

$$A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \qquad A\vec{x} = \vec{0}$$

$$[A|\vec{0}] = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right] \sim \begin{array}{c} \quad\quad\quad\quad t \quad\quad \\ \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x_1 = -t \\ x_2 = -t \\ x_3 = t \end{array}$$

$$\text{Nul}(A) = \left\{ \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} , t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} , t \in \mathbb{R} \right\} = \text{span} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Pick $\vec{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$$A \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \vec{0}$$

$$-\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$$

$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \Rightarrow \vec{v}_3 \in \text{span}(\vec{v}_1, \vec{v}_2)$$

$$\therefore \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{span}(\vec{v}_1, \vec{v}_2)$$

Theorem:

$$\vec{w} \in \text{span}(\vec{v}_1, \dots, \vec{v}_k) \text{ iff } \text{span}(\vec{v}_1, \dots, \vec{v}_k, \vec{w}) = \text{span}(\vec{v}_1, \dots, \vec{v}_k)$$

Theorem:

$I = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$

$\vec{v}_i$ is redundent $\exists i$ iff there is a nontrivial relation on I.

Proof ($\Rightarrow$):

Suppose $\vec{v}_i$ is redundent for some i.

$\Rightarrow \vec{v}_i \in \text{span}(\vec{v}_1, \dots, \vec{v}_{i-1})$

$\Rightarrow \vec{v}_i = c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1}$ some $c_1, \dots, c_{i-1}$ in $\mathbb{R}$

$\Rightarrow c_1\vec{v}_1 + \dots + c_{i-1}\vec{v}_{i-1} - \vec{v}_i = \vec{0}$

$\Rightarrow$ coefficient of $\vec{v}_i$ is -1. Thus this is a nontrivial relation on I. //

Proof ($\Leftarrow$):

Assume there is a nontrivial relation on I. (WTS $\exists i \in \{1, \dots, k\}$ s.t. $\vec{v}_i$ is redundant)

$\exists\, c_1, \dots, c_k$ not all zero s.t.

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$

Suppose c_i is the right most nonzero coefficient

$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_i\vec{v}_i + 0\vec{v}_{i+1} + \dots + c_k\vec{v}_k = \vec{0}$

$(c_i \neq 0)$

$c_i\vec{v}_i = -c_1\vec{v}_1 - c_2\vec{v}_2 - \dots - c_{i-1}\vec{v}_{i-1}$

$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \frac{c_2}{c_i}\vec{v}_2 - \dots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}$

$\Rightarrow \vec{v}_i \in \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1})$

$\therefore \vec{v}_i$ is redundent //

Theorem

$I = \{\vec{v}_1, \dots, \vec{v}_k\}$ is L.I. iff every relation I is trivial

Ex 2

$$\left\{ \underset{\vec{v}_1}{\begin{pmatrix}1\\-1\\2\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}0\\1\\3\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}1\\0\\5\end{pmatrix}}, \underset{\vec{v}_4}{\begin{pmatrix}1\\1\\8\end{pmatrix}}, \underset{\vec{v}_5}{\begin{pmatrix}2\\-1\\7\end{pmatrix}} \right\} \subseteq \mathbb{R}^3$$

$$A\vec{x} = \vec{0}$$

$$A = \begin{bmatrix} | & | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \\ | & | & | & | & | \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccccc} & & t & s & k \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x_1 = -t-s-2k \\ x_2 = -t-2s-k \\ x_3 = t \\ x_4 = s \\ x_5 = k \end{array}$$

$$\text{Nul}(A) = \left\{ t\underset{\vec{w}_1}{\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix}} + s\underset{\vec{w}_2}{\begin{pmatrix}-1\\-2\\0\\1\\0\end{pmatrix}} + k\underset{\vec{w}_3}{\begin{pmatrix}-2\\-1\\0\\0\\1\end{pmatrix}} \;\middle|\; t,s,k \in \mathbb{R} \right\}$$

$$= \text{span}(\vec{w}_1, \vec{w}_2, \vec{w}_3)$$

Since nul(A) is the solution to $A\vec{x} = \vec{0}$, we can plug certain solutions from nul(A) into $A\vec{x} = 0$ to get this linear relation

When $t=1;\ s=0;\ k=0 \Rightarrow -\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0} \Rightarrow \vec{v}_3 = \vec{v}_1 + \vec{v}_2$

When $t=0;\ s=1;\ k=0 \Rightarrow -\vec{v}_1 - 2\vec{v}_2 + \vec{v}_4 = \vec{0} \Rightarrow \vec{v}_4 = \vec{v}_1 + 2\vec{v}_2$

When $t=0;\ s=0;\ s=1 \Rightarrow -2\vec{v}_1 - \vec{v}_2 + \vec{v}_5 = \vec{0} \Rightarrow \vec{v}_5 = 2\vec{v}_1 + \vec{v}_2$

$\therefore\ \vec{v}_3, \vec{v}_4, \vec{v}_5$ is redundant

$\vec{v}_3 \in \text{span}(\vec{v}_1, \vec{v}_2)$; $\vec{v}_4 \in \text{span}(\vec{v}_1, \vec{v}_2)$; $\vec{v}_5 \in \text{span}(\vec{v}_1, \vec{v}_2)$

$w = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5) = \text{span}(\vec{v}_1, \vec{v}_2)$

Def:

Let V be a vector space. A basis of V is a subset B of V provided that

(i) $\text{Span}(B) = V$ or B is a spanning set for V
(ii) B is linearly independent.

Ex 3

$V = \mathbb{R}^2$

$\{ \vec{e}_1, \vec{e}_2 \} \subseteq \mathbb{R}^2$

(i) $\operatorname{span}(\vec{e}_1, \vec{e}_2) = \mathbb{R}^2$ ✓

(ii) $\{ \vec{e}_1, \vec{e}_2 \}$ is L.I. ✓

$c_1 \vec{e}_1 + c_2 \vec{e}_2 = \vec{0}$

$$\begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$ Only trivial solutions

$\Rightarrow c_1 = c_2 = 0$

$\therefore \{ \vec{e}_1, \vec{e}_2 \}$ is a basis for $\mathbb{R}^2$.
$\vec{e}_1, \vec{e}_2$ can be called standard basis

Ex 4

$V = \mathbb{R}^2$

$B = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

Does B span $\mathbb{R}^2$: $\operatorname{span}(\vec{v}_1, \vec{v}_2) = \mathbb{R}^2$

We know that: $\operatorname{sp}(\vec{v}_1, \vec{v}_2) \overset{\checkmark}{\subseteq} \mathbb{R}^2$. I
But we don't know: $\mathbb{R}^2 \overset{?}{\subseteq} \operatorname{sp}(\vec{v}_1, \vec{v}_2)$

Pick $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ W.T.S $\vec{b} \in \operatorname{span}(\vec{v}_1, \vec{v}_2)$
WTS $\vec{b} = c_1 \vec{v}_1 + c_2 \vec{v}_2$ for some c_1 and c_2
WTS $\left[\begin{array}{cc:c} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{b} \\ | & | & | \end{array}\right]$ consistent

$$\left[\begin{array}{cc:c} 1 & 1 & b_1 \\ 1 & 0 & b_2 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 1 & b_1 \\ 0 & -1 & b_2 - b_1 \end{array}\right]$$

$\therefore \vec{b} \in \operatorname{span}(\vec{v}_1, \vec{v}_2) \Rightarrow \mathbb{R}^2 \subseteq \operatorname{Span}(\vec{v}_1, \vec{v}_2)$ II
$\therefore$ By I and II, $\operatorname{Span}(\vec{v}_1, \vec{v}_2) = \mathbb{R}^2$

Ex 4 continued ...

Is B L.I. ?

WTS $c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ where $c_1 = c_2 = 0$

Suppose $\overset{A}{\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$

$\left[A \mid \vec{0}\right] \sim \left[\begin{array}{cc:c} 1 & 1 & \vec{0} \\ 0 & -1 & \end{array}\right]$ has a unique solution. Thus $c_1 = c_2 = 0$.

$\therefore$ B is L.I.

Ex 5

$\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ basis for $\mathbb{R}^2$

$\{5\vec{e}_1, 21\vec{e}_2\}$ basis for $\mathbb{R}^2$

Ex 6

$V = P^2$ $\quad P^2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \text{ in } \mathbb{R}\}$

$B = \{1, x, x^2\}$ is a basis for P^2.

Does B span P^2 ?

Pick $p(x) = a_0 + a_1x + a_2x^2 \in P^2$

$p(x) \in \text{span}(1, x, x^2)$

$P^2 \overset{\checkmark}{\subseteq} \text{span}(1, x, x^2)$

$\text{Span}(1, x, x^2) \overset{\checkmark}{\subseteq} P^2$ By def of span

$\therefore \quad \text{span}(1, x, x^2) = P^2$.

Is B a L.I ?

Suppose $c_0 + c_1x + c_2x^2 = \vec{0} = 0 + 0x + 0x^2$

$\Rightarrow c_0 = 0, c_1 = 0, c_2 = 0$

Thus B is L.I.

Theorem 3.3.1:

Number of vectors in any linearly independent set in a vector space V is less than or equal to the number of vectors in any spanning set of V.

Theorem

Any basis for a vector space has the same number of vectors

Proof:

Suppose B_1, B_2 are bases for vectorspace V.

B_1 is a spanning set for V
B_2 is L.I set in V $\Rightarrow |B_2| \leq |B_1|$ ← # of elements in the set

B_2 is a spanning set for V
B_1 is a L.I. set in V $\Rightarrow |B_1| \leq |B_2|$

So $|B_1| = |B_2|$

Def:

Given a vector space V, dimension of V is the # of vectors in any basis of V. Denoted by $\dim(V)$.