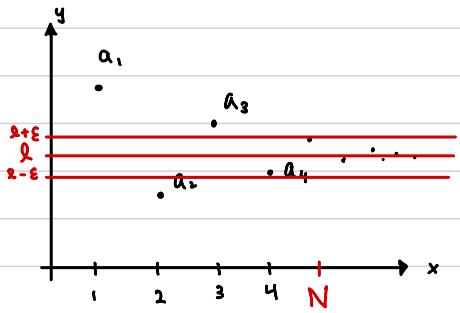
Def:

An infinite sequence of real numbers is a function whose domain is $\mathbb{N}$

Denoted $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or $a_n = f(n)$

a_n is the general term of our sequence

e.g.



What happens as $n \to \infty$?

Def:

Given $\{a_n\}$ a sequence we say $\{a_n\}$ converges to some $\ell \in \mathbb{R}$ iff

$\exists \ell \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ s.t. all $n \in N$,

if $n > N$ then $|a_n - \ell| < \varepsilon$

Denoted $\lim_{n\to\infty} a_n = \ell$ or $a \to \ell$ as $n \to \infty$

We say $\{a_n\}$ diverges if $\{a_n\}$ does not converge.

Terminology: ℓ = limit of our sequence $\{a_n\}$

Ex 1

Prove $\left\{ \frac{(-1)^n \sin n}{n^2+1} \right\}$ convergent to 0

WTS $\forall \varepsilon > 0, \exists N > 0$ s.t. if $n > N$ then $|a_n - \ell| < \varepsilon$

Proof:

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{1/\sqrt{\varepsilon}} > 0$

Suppose $n > N$,

$$|a_n - 0| = \left| \frac{(-1)^n \sin n}{n^2+1} - 0 \right|$$

$$= \frac{|(-1)^n \sin n|}{|n^2+1|}$$

$$= \frac{|\sin n|}{n^2+1}$$

$$\leq \frac{1}{n^2+1},$$

max number b/c $-1 \leq \sin(n) \leq 1$ so $|\sin(n)| \leq 1$

$$\leq \frac{1}{n^2}$$

given $n > N > 0$:
$\Rightarrow n^2 > N^2 > 0$
$\Rightarrow 1/n^2 < 1/N^2$

$$< \frac{1}{N^2} = \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2}$$ as wanted //

Ex 2

Prove $a_n = \frac{n^2-2}{n^2+2n+2}$ converges

Choose $\ell = \underline{\quad 1 \quad} \in \mathbb{R}$ $\qquad \lim_{n\to\infty} a_n = 1$

Let $\varepsilon > 0$ be arbitrary.

Choose $N = \underline{\quad 2/\varepsilon \quad} > 0$

Suppose $n > N$,

$$
\begin{aligned}
|a_n - 1| &= \left| \frac{n^2-2}{n^2+2n+2} - 1 \right| \\
&= \left| \frac{-2n-4}{n^2+2n+2} \right| \\
&= 2\,\frac{|n+2|}{|n^2+2n+2|} \\
&= \frac{2(n+2)}{n^2+2n+2} && \text{since } n > N > 0 \\
&\le \frac{2(n+2)}{n^2+2n} && \text{drop } +2 \text{ in denominator} \\
&= \frac{2}{n} \\
&< \frac{2}{N} = \frac{2}{\frac{2}{\varepsilon}} = \varepsilon && \text{as wanted } //
\end{aligned}
$$

Ex 3

Prove $\{n^2\}$ is divergent.

WTS $n^2 \to \infty$ as $n \to \infty$

WTS $\forall M>0, \exists N>0, n>N \Rightarrow a_n > M$

Proof:

Let $M>0$ be arbitrary.

Choose $N = \underline{\sqrt{M}} > 0$

Suppose $n > N$,

$$\begin{aligned} a_n &= n^2 \\ &> N^2 \quad \text{Since } n > N > 0 \\ &= (\sqrt{M})^2 = M \quad \text{as wanted} \ // \end{aligned}$$

Ex 4

Prove $\{1+(-1)^n\}$ diverges.

Proof:

Assume $\{(-1)^n+1\}$ converge to some $\ell \in \mathbb{R}$ by contradiction

We have $\forall \varepsilon > 0, \exists N > 0$, if $n > N$, then $|a_n - \ell| < \varepsilon$

Let $\varepsilon = 1 > 0$

We have $\exists N > 0$, if $n > N$ then $|1+(-1)^n - \ell| < 1$

Case 1: n is odd

$$\begin{aligned} \text{If } n > N \Rightarrow & \ |1+(-1)^n - \ell| < 1 \\ \Leftrightarrow & \ |1-1-\ell| < 1 \\ \Leftrightarrow & \ |-\ell| < 1 \\ \Leftrightarrow & \ -1 < \ell < 1 \end{aligned}$$

Case 2: n is even

$$\begin{aligned} \text{If } n > N \Rightarrow & \ |1+(-1)^n - \ell| < 1 \\ \Leftrightarrow & \ |2-\ell| < 1 \\ \Leftrightarrow & \ -1 < \ell - 2 < 1 \\ \Leftrightarrow & \ 1 < \ell < 3 \end{aligned}$$

So $\exists \ell \in \mathbb{R}$ s.t. $\ell \in (-1,1) \cap (1,3) = \phi$

This is a contradiction, thus a_n must diverge. //