Lec 1 Notes and Appendix A

Def: A matrix is an array of numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}_{n \times m} = (a_{ij})_{n \times m}$$

Notation

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Def: Vector

A matrix with 1 column is called a column vector.
A matrix with 1 row is called a row vector.

$$\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{Col vector}$$

entries / components

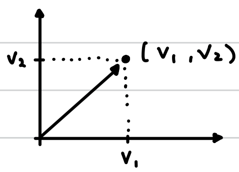
$$\mathbb{R}^n = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| \begin{matrix} v_i \in \mathbb{R} \\ 1 \leq i \leq n \end{matrix} \right\} = \text{the set of all vectors with } n \text{ rows}$$

Real Euclidian Space

Geometric Representation of Vectors

$\mathbb{R}^2$ Cartesian Coordinate Plane

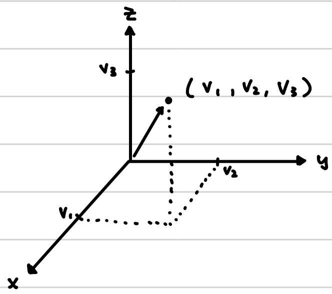
$\vec{v} \in \mathbb{R}^2$



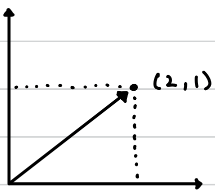
$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad v_1, v_2 \in \mathbb{R}$$

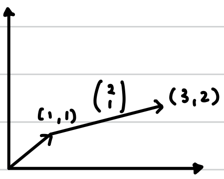
$\mathbb{R}^3$

$\vec{v} \in \mathbb{R}^3 \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad v_1, v_2, v_3 \in \mathbb{R}$



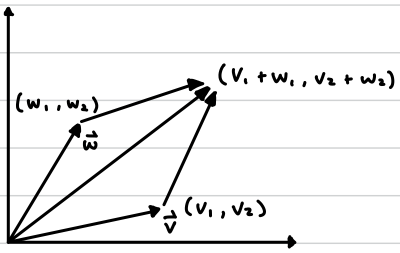
We can translate vectors.





$3-1=2$
$2-1=1$ $\Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Adding vectors in $\mathbb{R}^2$ can be represented by a parallelogram.



Def: We say that two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are parallel if one of them is a scalar multiple of the other.

Dot Product, Length, Orthogonality

Def: The dot product of $\vec{v}$ and $\vec{w}$ is:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

Geometrically:

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cos\theta \, \|\vec{w}\|$$

where θ is the angle enclosed by $\vec{v}$ and $\vec{w}$.

Note that the dot product of two vectors is a scalar.

Thm. Rules for dot products

(i) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
(ii) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
(iii) $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$
(iv) $\vec{v} \cdot \vec{v} > 0$ for all nonzero $\vec{v}$

Since $\vec{v}$ is nonzero, at least one of the components v_i is nonzero, so that v_i^2 is positive. Then $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \dots + v_n^2$ is positive as well.

The length of a vector in $\mathbb{R}^2$ is $\sqrt{x_1^2 + x_2^2}$

$$\vec{x} \cdot \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 = \|\vec{x}\|^2$$

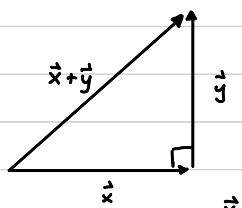
$$\therefore \|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

Def: The length $\|\vec{x}\|$ of a vector $\vec{x}$ in $\mathbb{R}^n$ is

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$$

Def: A vector $\vec{u}$ in $\mathbb{R}^n$ is called a unit vector if $\|\vec{u}\| = 1$

Consider two perpendicular vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$.



By pythagoras theorem,

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$
$$(\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y}$$
$$\vec{x} \cdot \vec{x} + 2(\vec{x} \cdot \vec{y}) + \vec{y} \cdot \vec{y} = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y}$$
$$\vec{x} \cdot \vec{y} = 0$$

Def: Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are called perpendicular / orthogonal if $\vec{v} \cdot \vec{w} = 0$

Cross Product

Def: Cross Product in $\mathbb{R}^3$ Properties:

(i) $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$

(ii) $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \sin\theta \|\vec{w}\|$, where θ is the angle between $\vec{v}$ and $\vec{w}$, with $0 \le \theta \le \pi$.
The magnitude of the vector $\vec{v} \times \vec{w}$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

(iii) The direction of $\vec{v} \times \vec{w}$ follows the right-hand rule.

Thm. Properties of the cross product

(i) $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$: The cross product is anticommutative

(ii) $(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w})$

(iii) $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$

(iv) $\vec{v} \times \vec{w} = \vec{0}$ iff $\vec{v}$ is parallel to $\vec{w}$

(v) $\vec{v} \times \vec{v} = \vec{0}$

(vi) $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$
$\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3$, $\vec{e}_3 \times \vec{e}_2 = -\vec{e}_1$, $\vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$

Express cross product in components

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = (v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3) \times (w_1\vec{e}_1 + w_2\vec{e}_2 + w_3\vec{e}_3)$$
$$= (v_2w_3 - v_3w_2)\vec{e}_1 + (v_3w_1 - v_1w_3)\vec{e}_2 + (v_1w_2 - v_2w_1)\vec{e}_3$$
$$= \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

Thm. The cross product in components

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$