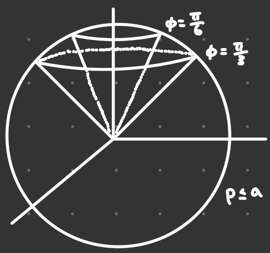
B41 Dec 3 Lec 2 Notes

Ex 1:

Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$.



$B = \{(\rho,\theta,\phi) \mid \theta \in [0,2\pi],\ \phi \in [\pi/6, \pi/3],\ \rho \in [0,a]\}$

$$\iiint_V dv = \iiint_B \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_{\pi/6}^{\pi/3} \int_0^{2\pi} \int_0^a \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= a^3 \frac{\pi}{3}(\sqrt{3}-1)$$

Ex 2:

Compute the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Let $x = au$, $y = bv$ and $z = cw$ to get $u^2+v^2+w^2 = 1$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Polar coordinates method:

$$\iiint_V dv = \iiint_B \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$= abc \iiint_B du\,dv\,dw$$ ← Volume of a unit ball

$$= abc \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} dw\,dv\,du$$

$$= 8abc \int_0^1 \int_0^{\sqrt{1-u^2}} \sqrt{1-u^2-v^2} \, dv\,du$$ By symmetry

$$= 8abc \int_0^{\pi/2} \int_0^1 \sqrt{1-r^2}\, r \, dr\,d\theta$$

$$= \frac{4}{3} abc\pi$$

Spherical coordinates method:

$$\iiint_V dv = \iiint_B \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$= abc \iiint_B du\,dv\,dw$$

$$= abc \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \, d\rho\,d\theta\,d\phi$$

$$= \frac{4}{3} abc\pi$$

Ex 3:

Integrate $(x^2+y^2)z^2$ over the part of the cylinder $x^2+y^2 \le 1$ inside the sphere $x^2+y^2+z^2=4$.

$D = \{(x,y) \mid x^2+y^2 \le 1\}$

$W = \{(x,y,z) \mid (x,y) \in D, -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2}\}$

Cylindrical coordinates

$W^* = \{(r,\theta,z) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi, -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\}$

$$\iiint_W (x^2+y^2)z^2 \, dv = \iiint_{W^*} r^2 z^2 |r| \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3 z^2 \, dz \, dr \, d\theta$$

$$\approx 6.385$$