A22 Mar 26 Lec 2 Notes

Def:

Given the entry a_{ij} in A, the cofactor of a_{ij} is

$$a_{ij}' = (-1)^{i+j} |A_{ij}|$$

Def:

$$\det A := a_{11} a_{11}' + a_{12} a_{12}' + \dots + a_{1n} a_{1n}'$$

Theorem: Properties of Determinant

(i) $\det A = \det(A^T)$

Proof (i):

$A = (a_{ij})$, $A^T = (b_{ij})$

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{1j}|$$
$$= \sum (-1)^{1+i} b_{i1} |A_{i1}|$$
$$= \det(A^T)$$

(ii) Switch rows $A \sim B$

$\det A = -\det B$

Proof (ii):

Base case: $n = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\det A = ad - bc = -(cb - ad) = -\det B$$

I.H: Assume statement holds for any $k \times k$ matrix, $k \lneq n$ basically for minors

I.S: Suppose $A_{n\times n} = (a_{ij})$, $A \overset{R_i \leftrightarrow R_j}{\sim} B$. Pick row ℓ $(\ell \neq i, \ell \neq j)$

$$\det A = \sum_{j=1}^{n} a_{\ell j} (-1)^{\ell + j} |A_{\ell j}| \qquad \det B = \sum_{j=1}^{n} b_{\ell j} (-1)^{\ell + j} |B_{\ell j}|$$

expanded along ℓ^{th} row

Proof (ii) continued...:

$$A = \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_i & \text{---} \\ a_{\ell j} & \vec{r}_\ell & \text{---} \\ \text{---} & \vec{r}_j & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix} \qquad B = \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_j & \text{---} \\ b_{\ell j} & \vec{r}_\ell & \text{---} \\ \text{---} & \vec{r}_i & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix}$$

$a_{\ell j} = b_{\ell j} \quad \forall\, 1 \le j \le n$

Note: $A_{\ell j} \overset{R_j \Leftrightarrow R_i}{\sim} B_{\ell j}$, by hypothesis $|A_{\ell j}| = -|B_{\ell j}|$

$$\det B = \sum b_{\ell j} (-1)^{\ell+j} |B_{\ell j}| = \sum a_{\ell j} (-1)^{\ell+j} (-1) |A_{\ell j}|$$

$$= -\sum a_{\ell j} (-1)^{\ell+j} |A_{\ell j}|$$

$$= -\det A$$

$\blacksquare$

(iii)

$$A = \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ a_{ij} & \vec{r}_i & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix} \qquad B = \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ b_{ij} & k\vec{r}_i & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix}$$

$\det B = k \det A$

Proof (iii):

$$\det B = \sum_{j=1}^{n} b_{ij} (-1)^{i+j} |B_{ij}| \qquad \text{expand } B \text{ along } i^{th} \text{ row}$$

$$= \sum_{j=1}^{n} k\, a_{ij} (-1)^{i+j} |A_{ij}| \qquad \text{Note: } b_{ij} = k a_{ij}$$

$$= k \sum_{j=1}^{n} a_{ij} (-1)^{i+j} |A_{ij}|$$

$$= k \det A$$

$\blacksquare$

(iv) $A = \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ \text{---} & \vec{r}_i & \text{---} \\ \text{---} & \vec{r}_j & \text{---} \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix}$, $\vec{r}_i = \vec{r}_j$

Proof (iv):

$$\det \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ \text{---} & \vec{r}_i & \text{---} \\ \text{---} & \vec{r}_j & \text{---} \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix} = -\det \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ \text{---} & \vec{r}_j & \text{---} \\ \text{---} & \vec{r}_i & \text{---} \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix} = -\det \begin{bmatrix} \text{---} & \vec{r}_1 & \text{---} \\ \text{---} & \vec{r}_i & \text{---} \\ \text{---} & \vec{r}_j & \text{---} \\ \text{---} & \vec{r}_n & \text{---} \end{bmatrix}$$

$\det A = -\det A \Rightarrow \det A = 0$

$$(v)\quad A = \begin{bmatrix} \text{———} & \vec{r}_1 & \text{———} \\ a_{k1}\ a_{k2} & \vec{r}_k & a_{kn} \\ a_{i1}\ a_{i2} & \vec{r}_i & a_{in} \\ \text{———} & \vec{r}_n & \text{———} \end{bmatrix} , \quad B = \begin{bmatrix} \text{———} & \vec{r}_1 & \text{———} \\ \text{—} & \vec{r}_k + \ell \vec{r}_i & \text{—} \\ \text{———} & \vec{r}_i & \text{———} \\ \text{———} & \vec{r}_n & \text{———} \end{bmatrix} \to a_{k1} + \ell a_{i1} \quad a_{k2} + \ell a_{i2} \ \cdots \ a_{kj} + \ell a_{ij} \ \cdots$$

Proof (v): $A \overset{R_k \leftrightarrow R_k + \ell R_i}{\sim} B \qquad \det A = \det B$

Let's expand along k^{th} row.

$\vec{r}_k = (a_{k1} \ a_{k2} \ \cdots \ a_{kn})$, $\vec{r}_i = (a_{i1} \ a_{i2} \ \cdots \ a_{in})$

$$\det B = \sum_{j=1}^{n} (-1)^{k+j} (a_{kj} + \ell a_{ij}) |B_{kj}|$$

$$= \sum_{j=1}^{n} (-1)^{k+j} a_{kj} |B_{kj}| + \ell \sum_{j=1}^{n} (-1)^{k+j} a_{ij} |B_{kj}|$$

$$= \sum_{j=1}^{n} (-1)^{k+j} a_{kj} |A_{kj}| + \ell \sum_{j=1}^{n} (-1)^{k+j} a_{ij} |A_{kj}| \qquad \text{Note: } |B_{kj}| = |A_{kj}|$$

$$= \det A \quad + \ell\, 0$$

$$= \det A$$