W10 Lecture 19 Notes

Examples:

1. $f(x) = \dfrac{x^2}{1-x^2}$

1) Domain $f(x) = (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

2) X-ints $\Rightarrow f(x)=0,\ x=0$

   y-ints $\Rightarrow x=0,\ f(x)=0$

3) Symmetry

   $f(-x) = \dfrac{(-x)^2}{1-(-x)^2} = \dfrac{x^2}{1-x^2} \Rightarrow f(x) = f(-x)$

   $f(x)$ is even and we can consider $[0,1) \cup (1,\infty)$ only.

4) Asymptotes

   a) VA $\Rightarrow x=1$

   $\lim\limits_{x\to1^-} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^+} = \infty$; $\lim\limits_{x\to1^+} \dfrac{x^2}{1-x^2} = \dfrac{1}{0^-} = -\infty$

   b) $f(x) = \dfrac{P_2(x)}{Q_2(x)}$ – No slant asymptote.

   c) $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{1-x^2} = -1$

   HA $\Rightarrow y=-1$

Aside: $-1 = \dfrac{x^2}{1-x^2} \Rightarrow -1+x^2 = x^2$

$-1 \neq 0$

No intersection between HA and the graph.

5) Derivative

   $f'(x) = \dfrac{2x}{(1-x)^2(1+x)^2}$

6) Critical Points

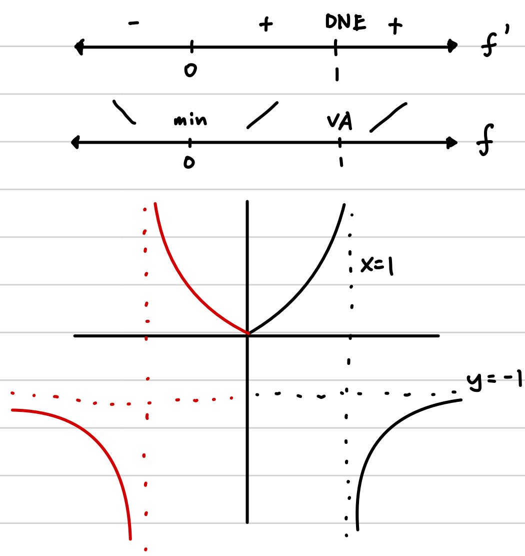
   $f'(x) = 0 \Rightarrow x=0 \in \text{Dom } f(x)$

   $x_c = 0$

   $f'(x) = \text{DNE} \Rightarrow x = \pm1 \notin \text{Dom } f(x)$

7) Concavity

8) Graph



2. $f(x) = \dfrac{(x-1)^2}{x^2+x-6} = \dfrac{(x-1)^2}{(x-2)(x+3)}$

1) Domain $f(x) = (-\infty,-3) \cup (-3,2) \cup (2,\infty)$

2) Intercepts

$x=0 \Rightarrow f(x) = \frac{(-1)^2}{-6} = -\frac{1}{6}$ Y-int $= (0,-\frac{1}{6})$

$f(x)=0 \Rightarrow x-1=0 \Rightarrow x=1$ x-int $= (1,0)$

3) Symmetry

$f(-x) \neq -f(x) \neq f(x)$

No symmetry

4) Asymptotes

a) VA $\Rightarrow x=2$, $x=-3$

$\lim_{x\to2^-} f(x) = \frac{1}{0^-} = -\infty$ $\lim_{x\to2^+} f(x) = \frac{1}{0^+} = \infty$

$\lim_{x\to-3^-} f(x) = \frac{16}{0^+} = \infty$ $\lim_{x\to-3^+} f(x) = \frac{16}{0^-} = -\infty$

b) SA

$f(x) = \dfrac{P_2(x)}{Q_2(x)} \Rightarrow$ No SA

c) HA

$$\lim_{x\to\pm\infty} \frac{x^2-2x+1}{x^2+x-6} = \lim_{x\to\pm\infty} \frac{x^2/x^2 - 2x/x^2 + 1/x^2}{x^2/x^2 + x/x^2 - 6/x^2} = 1$$

$\lim_{x\to+\infty} f(x)$

$\lim_{x\to-\infty} f(x)$

d) Check if HA intersects graph

$1 = \dfrac{(x-1)^2}{(x-2)(x+3)} \Rightarrow 3x=7 \Rightarrow x = 7/3$

Point of intersection $= (7/3, 1)$

5) Derivative

$f'(x) = \dfrac{(x-1)(3x-11)}{(x-2)^2(x+3)^2}$

6) Critical Points

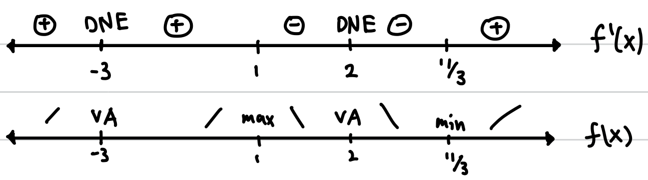
$f'(x)=0 \Rightarrow x=1,\ x=11/3 \in \text{Dom} f(x)$

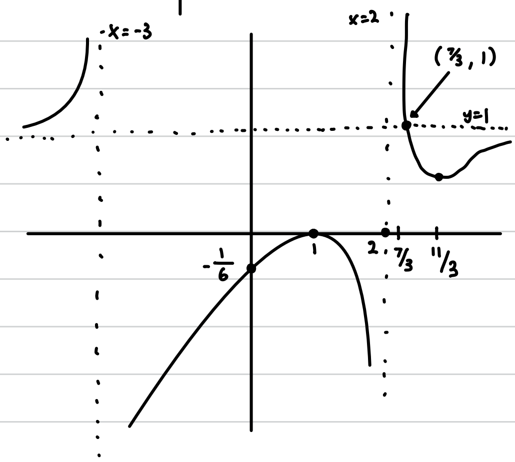
$x_{c_1}=1$; $x_{c_2}=11/3$

$f'(x) = \text{DNE} \Rightarrow x=2,\ x=-3 \notin \text{Dom} f(x)$

7) Concavity

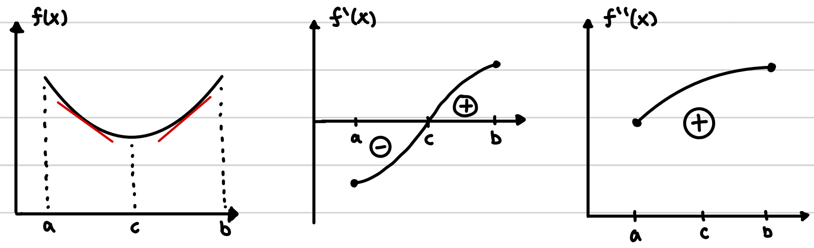
8) Graph





Definition 1

a) If graph of f **bends upward** on an interval I then it is called **concave upward** on I. $f(x)$ is concave up on (a,b) if $f'(x)$ increases from negative to positive on (a,b).



b) If graph of f **bends downward** on an interval I then it is called **concave downward** on I. $f(x)$ is concave down on (a,b) if $f'(x)$ decreases from positive to negative on (a,b).

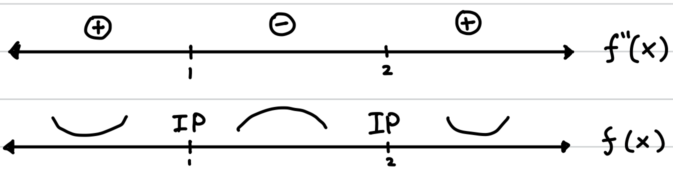
Concavity Test

a) If $f''(x) > 0$ for all x in I, then the graph of f is **concave upward**.
b) If $f''(x) < 0$ for all x in I, then the graph of f is **concave downward**.

Example:

3. Find the intervals on which the graph of $y = \frac{1}{6}x^4 - x^3 + 2x^2$ is concave up and down.

1) $\text{Dom} f(x) = (-\infty, \infty)$
2) Derivatives
   $f'(x) = \frac{2}{3}x^3 - 3x^2 + 4x$; $f''(x) = 2x^2 - 6x + 4$
3) Inflection Points
   $f''(x) = 0 \Rightarrow 2x^2 - 6x + 4 = 0 \Rightarrow x_1 = 1, x_2 = 2$



Definition 5

A point $x=c$ on the graph $f(x)$ is called an inflection point if $f(x)$ is continuous at $x=c$ and the concavity of curve is changes from concave up to down or vice versa.

Second Derivative Test

a) If $f'(c)=0$ and $f''(c)<0$, then $f(c)$ has a local max at c.
b) If $f'(c)=0$ and $f''(c)>0$, then $f(c)$ has a local min at c.
c) If $f''(x)=0$, then $f(c)$ has a point of inflection at c iff $f''(x)$ changes its sign at $x=c$.

Examples:

4. Find and classify critical points of $f(x)=2x^2-\frac{4}{x}$.

1) Domain $f(x) = (-\infty,0)\cup(0,\infty)$
2) Derivatives
$f'(x) = 4x+\frac{4}{x^2}$; $f''(x) = 4-2\cdot\frac{4}{x^3} = \frac{4x^3-8}{x^3}$
3) Critical Points
$f'(x)=0 \Rightarrow 4x+\frac{4}{x^2}=0 \Rightarrow x_c=-1 \in \text{Dom } f(x)$
$f'(x)=\text{DNE} \Rightarrow x=0 \notin \text{Dom } f(x)$
4) Classification :

$f''(-1) = \frac{-4-8}{-1} > 0 \Rightarrow$ f(x) has local min at $x=-1$.