W12 Reading Notes

Reading

The task is to split a chocolate bar into small squares with a minimum number of breaks.

How many breaks will it take assuming we break along the lines?

Dimensions	Breaks
1x1	0
1x2	1
2x2	3
3x2	5
4x3	11

Notice that the formula:

$\ell \times w$ chocolate bar needs $\ell \times w - 1$ breaks

Proof by Induction:

Prove that an $\ell \times w$ chocolate bar needs $\ell \times w - 1$ breaks.

Define $S(n)$: If $n \geq 1$ then a chocolate bar with n squares requires $n-1$ breaks.

Prove that $\forall n \in \mathbb{N}$ where $n \geq 1$, $S(n)$ is true.

Base Case:

$S(1) = 1 \times 1$ requires $1 - 1 = 0$ breaks.
Therefore $S(1)$ holds.

Inductive Hypothesis:

Let $n \in \mathbb{N}$. Suppose that all chocolate bars with less than n squares satisfy our claim.

Mathematically:

Let $n \in \mathbb{N}$. Suppose $\forall k \in \mathbb{N}$ such that $1 \leq k < n$, that $S(k)$ holds.

Inductive Step: Prove $S(n)$

Break the bar into smaller pieces of a and b squares, where $a, b < n$ and $a+b = n$.

Since $a<n, b<n$, the IH applies and $S(a)$ and $S(b)$ holds.

Thus total # of breaks is:

We already broke the chocolate into a and b once.

$$\underbrace{(a-1)}_{S(a)\text{ holds}} + \underbrace{(b-1)}_{S(b)\text{ holds}} + 1 = (a+b) - 1 = n - 1$$

Strong Induction

Recall the domino argument. To show that $P(n)$ is true, we can show that:

$P(0)$ and if $P(0)$ then $P(1)$. If $P(1)$ then $P(2)$. If $P(2)$ then $P(3)$ and so on until $P(n-1)$.

Thus we get:

$$(P(0) \wedge P(1) \wedge P(2) \wedge \dots P(n-1)) \rightarrow P(n)$$

So to show that $\forall n \in \mathbb{N}, P(n)$, we can show:

$$((P(0) \wedge P(1) \wedge P(2) \wedge \dots P(n-1)) \rightarrow P(n)) \rightarrow \forall n\, P(n)$$

This is how strong induction works. We assume all values smaller than n hold, and derive that $P(n)$ holds. In some situations, in order to assume that all values k smaller than n satisfy $P(k)$, we need multiple base cases.

We use multiple base cases so that when we use the induction hypothesis, we can use the fact that $P(k)$ holds for values of k such that $b \leq k < n$ where b is the smallest base case.

Examples:

1. Consider the sequence $c_1, c_2, c_3 \ldots$ defined as follows:

$$c_1 = 3, \quad c_2 = 5$$

and

$$c_k = 3c_{k-1} - 2c_{k-2} \quad \text{for every integer } k \geq 3.$$

Prove that $c_n = 2^n + 1$ for each integer $n \geq 1$.

Let $S(n)$ be $c_n = 2^n + 1$

We have to prove $\forall n \in \mathbb{N}_{\geq 1}, S(n)$ by strong induction.

Base Case: $n=1, n=2$. We can do it here or move these to IS.

IH: Assume that for all $1 \leq k < n$ that $S(k)$ holds.

IS: Prove $S(n)$

Cases:

$n=1$: $c_1 = 3$ and $2^1 + 1 = 3$ ✓

$n=2$: $c_2 = 5$ and $2^2 + 1 = 5$ ✓

$n \geq 3$: $c_n = 3c_{n-1} - 2c_{n-2}$

Since $(n \geq 3) \to (n-1 \geq 2) \land (n-2 \geq 1)$, we know that $1 \leq n-1, n-2 < n$.
Therefore the IH holds for $S(n-1)$ and $S(n-2)$. (Always have to show IH)

Applying the IH to $c_n = 3c_{n-1} - 2c_{n-2}$

$$\begin{aligned} c_n = 3(2^{n-1}+1) - 2(2^{n-2}+1) &= 3 \cdot 2^{n-1} + 3 - 2^{n-1} - 2 \\ &= 2 \cdot 2^{n-1} + 1 \\ &= 2^n + 1 \end{aligned}$$

Pre-Lecture

1. Define a series $c_0, c_1, c_2, \ldots$ as.

$c_0 = 2$, $c_1 = 2$, $c_2 = 6$ and $c_k = 3c_{k-3}$ for every integer $k \geq 3$.

Prove that c_n is even for every integer $n \geq 0$.

a) State $S(n)$:

$S(n) = C_n = 2j \ , j \in \mathbb{Z}$

b) Base cases:

$n = 0$: $C_0 = 2$ and $2 = 2j, j \in \mathbb{Z}$ is true, thus 2 is even ✓

$n = 1$. $C_1 = 2$ and $2 = 2j, j \in \mathbb{Z}$ is true, thus 2 is even ✓

$n = 2$: $C_2 = 6$ and $6 = 2j, j \in \mathbb{Z}$ is true, thus 6 is even ✓

c) IH:

Assume that for all $0 \leq k < n$ that $S(k)$ holds.

d) Prove $S(n)$:

$n \geq 3$: $C_n = 3C_{n-3}$

Since $n \geq 3 \rightarrow n-3 \geq 0$, we know that $0 \leq n-3 < n$.
Therefore IH holds for $S(n-3)$.

$C_n = 3C_{n-3} = 3(2j) \ , j \in \mathbb{Z} \Rightarrow 2(3j) \Rightarrow 2m, \ m \in \mathbb{Z}$.

Thus we have shown that C_n is even.