B41 Oct 29 Lec 2 Notes

Theorem:

If $f \in C^\infty$ has a power series at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R$$

Then it's coefficients are given by the formula $c_n = \frac{f^n(a)}{n!}$

i.e. $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^n(a)}{n!}(x-a)^n + \dots$$

When $a=0$, we have the Maclaurin Series.

Ex 1:

Find the Maclaurin Series of $f(x) = e^x$ and its radius of convergence

Maclaurin series of e^x is :

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

By the ratio test :

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{1}{n+1}|x| \to 0 < 1 \quad \forall x$$

$\therefore$ radius $r = \infty$.

If $f(x)$ is the sum of the Taylor Series, then $f(x) = \lim_{n\to\infty} T_n(x)$

The difference $R_n(x) = f(x) - T_n(x)$ is called the n^{th} degree remainder for $f(x)$ at $x=a$.

Theorem:

If $f(x) = T_n(x) + R_n$, where T_n is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to its Taylor series on the interval $|x-a| < R$.

Theorem: Taylor's Inequality

If $|f^{n+1}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of Taylor series satisfies the inequality

$$|R_n(x)| < \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

Some Taylor Series:

$\forall x \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n}{(2n)!} x^{2n} + \dots$$

For $|x| < 1$.

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \frac{a}{1!} x + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \dots + \frac{a(a-1)\cdots(a-n+1)}{n!} x^n + \dots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

Theorem:

Let $f: U \subset \mathbb{R}^k \to \mathbb{R}$ of class C^{n+1}, the n^{th}-order Taylor Series of function f at $x = x_0$.

$$f(x_0+h) = f(x_0) + \sum_{i_1=1}^{k} h_{i_1} \frac{\partial f}{\partial x_{i_1}}(x_0) + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{k} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0) + \dots + \frac{1}{n!} \sum_{i_1, i_2, \dots, i_n = 1}^{k} h_{i_1} h_{i_2} \cdots h_{i_n} \frac{\partial^n f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}}(x_0) + R_n(x_0, h)$$

where $R_n(x_0, h) = \frac{1}{(n+1)!} \sum_{i_1, i_2, \dots, i_n, i_{n+1} = 1}^{k} h_{i_1} h_{i_2} \cdots h_{i_n} h_{i_{n+1}} \frac{\partial^{n+1} f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n} \partial x_{i_{n+1}}} (c_{i_1, i_2, \dots, i_n, i_{n+1}})$ satisfying $\frac{R_n(x_0, h)}{\|h\|^n} \to 0$. $c_{i_1, i_2, \dots, i_n, i_{n+1}}$ is a point on the line joining x_0 and $x_0 + h$.

Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ of class C^3, the second order Taylor formula of function f at $x = (0,0)$

$$T_2(x,y) = f(x_0) + \sum_{i_1=1}^{2} h_{i_1} \frac{\partial f}{\partial x_{i_1}}(x_0) + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{2} h_{i_1} h_{i_2} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x_0)$$

$$= f(0,0) + \left[x \frac{\partial f}{\partial x}(0,0) + y \frac{\partial f}{\partial y}(0,0) \right] + \frac{1}{2!} \left[x^2 \frac{\partial^2 f}{\partial x^2}(0,0) + 2xy \frac{\partial^2 f}{\partial x \partial y}(0,0) + y^2 \frac{\partial^2 f}{\partial y^2}(0,0) \right]$$