T_A is invertible $\Rightarrow$ A is $n \times n$

T_A is 1-1 $\Leftrightarrow$ $A\vec{x} = \vec{y}$ $\forall \vec{y} \in \mathbb{R}^n$ has at most one solution

T_A is 1-1 $\Leftrightarrow$ rref (A) has pivot in every column.

T_A is onto $\Leftrightarrow$ $A\vec{x} = \vec{y}$ $\forall \vec{y} \in \mathbb{R}^n$ is consistent

T_A is onto $\Leftrightarrow$ rref (A) has pivot in every row.

$\therefore$ $T_A : \mathbb{R}^n \to \mathbb{R}^n$, $\vec{x} \mapsto A\vec{x}$ is invertible iff rref (A) $= I_n$.

Theorem:

A is invertible iff $\exists B$ s.t. $AB = BA = I_n$.

Such a B, if exists, is called inverse of A.

$$\left[\begin{array}{c:c} A_{n \times m} & I_n \end{array}\right]$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{c} \quad A_1 \qquad\qquad E_1 \\ \left[\begin{array}{ccc:ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \qquad\qquad\qquad E_2 E_1 \\ \left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \quad I_n \qquad\qquad E_3 E_2 E_1 = A^{-1} \\ \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -2 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

product of elementary matrices is A^{-1}

Theorem:

If A is an invertible matrix then

$$[A \mid I_n] \sim [I_n \mid B], \text{ where } B = A^{-1}$$

Proof:

If A is invertible, A row reduces to I_n. i.e.

$$A \overset{①}{\sim} A_2 \overset{②}{\sim} A_3 \sim \cdots \overset{Ⓚ}{\sim} A_K = I_n$$

where ⓘ are row reducing steps.

By PS1, $\exists$ elementary matrices $E_1, E_2, \ldots, E_K$ corresponding to ①, ②, ..., Ⓚ respectively. That is

$$E_K \cdots E_3 E_2 E_1 A = I_n$$

By theorem 2.4.3,

$$A(E_K \cdots E_3 E_2 E_1) = I_n$$

therefore $A^{-1} = E_K \ldots E_3 E_2 E_1$

By PS1

$$I_n \overset{①}{\sim} A_{K-1} \overset{②}{\sim} \cdots \overset{③}{\sim} E_K \ldots E_3 E_2 E_1$$

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