Q3 Review Seminar

Supremum and Infimum

Theorem:

If M is the least upper bound of the set S and ε is a positive number, then there is at least one number $x \in S$ such that $M - \varepsilon < x < M$.

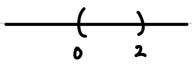
Theorem:

If M is the greatest lower bound of the set S and ε is a positive number, then there is at least one number $x \in S$ such that $m < x < m + \varepsilon$.

Examples:

1. Find LUB and GLB for the following sets.

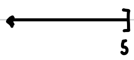
a) $(0, 2)$



LUB = 2

GLB = 0

b) $(-\infty, 5]$



LUB = 5

GLB = DNE

c) $\{-4, 4, -4.1, 4.1, -4.11, 4.11, -4.111\ldots, 4.111\ldots\}$

$4.1 = 4 + 0.1 = 4 + \frac{1}{10}$

$4.11 = 4 + 0.11 = 4 + 0.1 + 0.01 = 4 + \frac{1}{10} + \frac{1}{10^2}$

$\therefore\ 0.1111\ldots = \frac{1}{10} + \frac{1}{10^2} + \ldots$

Infinite geometric series, with $r = \frac{1}{10}$

$= \frac{1}{10}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right]$

$= \frac{1}{10}\left[\frac{1}{1 - 1/10}\right]$ $\qquad S = a\left[\frac{1}{1-r}\right]$

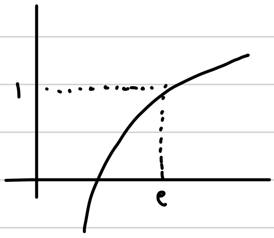
$= \frac{1}{10 - 1}$

$= \frac{1}{9}$

LUB $= 4\,1/9$

GLB $= -4\,1/9$

d) $\{x : \ln x < 1\}$



$x \in (0, e)$

LUB = e

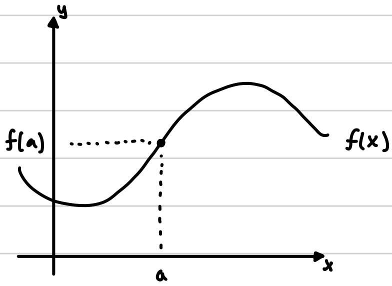
GLB = 0

e) $\{x : x^2 + x + 2 \geq 0\}$

$$x^2+x+2 = x^2 + x + \tfrac{1}{4} - \tfrac{1}{4} + 2$$
$$= (x+\tfrac{1}{2})^2 + \tfrac{7}{4} \geq 0 \quad \forall x \in \mathbb{R} = (-\infty, \infty)$$

$\therefore$ No GLB nor LUB

Continuity



Definition

A function is continuous at a if $\lim_{x \to a} f(x) = f(a)$

1) $f(a)$ exists (defined)
2) $\lim_{x \to a} f(x)$ exists $\Rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
3) $f(a) = \lim_{x \to a} f(x)$

For f to be continuous at a, all the conditions must be true.

Examples:

2. Is $f(x) = \begin{cases} \sin(x)\, e^{2x}, & x \geq 0 \\ \frac{1}{2} x^2, & x < 0 \end{cases}$

continuous at a) $x=0$, b) $x = \frac{\pi}{2}$

a) For continuity at $x=a$

we must have $\lim_{x \to 0} f(x) = f(0)$

$f(0) = \sin(0) \cdot e^{2(0)} = 0$

a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{2}x^2 = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin(x)\cdot e^{2x} = 0$

$\because$ $f(c) = \lim_{x\to 0} f(x)$

$\therefore$ f is continuous at $x=0$

b) $f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) e^{2(\frac{\pi}{2})} = e^{\pi}$

$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \sin x\, e^{2x} = \sin(\frac{\pi}{2})\cdot e^{2(\pi/2)} = f(\frac{\pi}{2})$

$\therefore$ f is continuous at $x=\frac{\pi}{2}$

3. Find $a\in\mathbb{R}$ s.t. $f(x) = \begin{cases} \frac{2}{x-1}, & x\le a \\ x, & x>a \end{cases}$

is continuous on $\mathbb{R} = (-\infty\ \infty)$

Let $a\in(-\infty,\infty)$

We need $f(a) = \lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \frac{2}{x-1} = \frac{2}{a-1}$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} x = a$

$\frac{2}{a-1} = a$

$2 = a(a-1)$

$2 = a^2-a$

$a^2-a-2=0$

$(a+1)(a-2)=0$

$\therefore$ $a=-1$ or $a=2$

Note $a\neq 2$ $\because$ $\frac{2}{x-1}$ is not continuous for $x\le 2$ since it will be undefined at $x=1$.

$\therefore$ $a=-1$

4. Show that $\sin(2x)$ is continuous at $x = \frac{\pi}{4}$.

Let $f(x) = \sin(2x)$

For continuity at $x = \pi/4$, we need $f(\pi/4) = \lim_{x \to \frac{\pi}{4}} f(x)$

$\therefore \lim_{x \to \frac{\pi}{4}} \sin(2x) = \sin(\frac{\pi}{2}) = f(\frac{\pi}{2}) = 1$

Let $x = h + \frac{\pi}{4} \Rightarrow x \to \frac{\pi}{4},\ h \to 0$

$x - \frac{\pi}{4} = h$ ← makes sure that limit is 1 from either sides of $\frac{\pi}{4}$

$$
\begin{aligned}
\lim_{x \to \frac{\pi}{4}} \sin(2x) &= \lim_{h \to 0} \sin(2(h + \tfrac{\pi}{4})) \\
&= \lim_{h \to 0} 2 \cdot \sin(h + \tfrac{\pi}{4}) \cos(h + \tfrac{\pi}{4}) \\
&= \lim_{h \to 0} 2\sin(h + \tfrac{\pi}{4}) \cdot \lim_{h \to 0} \cos(h + \tfrac{\pi}{4}) \\
&= \lim_{h \to 0} 2[\sin h \cdot \cos \tfrac{\pi}{4} + \sin(\tfrac{\pi}{4}) \cos h] \cdot \lim_{h \to 0} \cos(h + \tfrac{\pi}{4}) \\
&= 2[0 + \tfrac{1}{\sqrt{2}}(1)] \cdot \tfrac{1}{\sqrt{2}} = 2(\tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) = 1
\end{aligned}
$$

Formal ε-δ Definition of Continuity at $x=c$

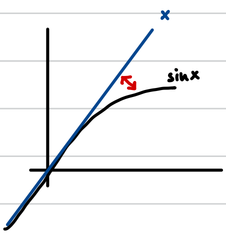
$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

5. Prove that $\sin(2x)$ is continuous at $x = 0$ using ε-δ definition.

Proof:

$\forall \varepsilon > 0$, choose $\delta =$. Then if $|x| < \delta$, we have

$$
\begin{aligned}
|\sin(2x) - \sin(0)| &= |\sin 2x| \\
&= 2\underbrace{|\sin x|}_{<|x|}\underbrace{|\cos x|}_{<1} < 2|x| < 2\delta = \cancel{2}\frac{\varepsilon}{\cancel{2}} = \varepsilon
\end{aligned}
$$



$\because |\sin x| \le |x|$

QED

IVT

If f is continuous on $[a,b]$ and K is any number strictly between $f(a)$ and $f(b)$, then there is at least one number $c \in (a,b)$ such that $f(c) = K$.

Examples:

6. Show that $10 \sin x \cos \frac{x}{2} = \frac{2}{\sqrt{2}}$ has a solution on $[0, \frac{\pi}{2}]$.

Let $f(x) = 10 \sin x \cos \frac{x}{2}$, $K = \frac{2}{\sqrt{2}}$

f is continuous on $\mathbb{R}$ $\Rightarrow$ continuous on $[0, \frac{\pi}{2}]$

$f(0) = 10 \sin(0) \cos\left(\frac{0}{2}\right) = 0$

$f(\frac{\pi}{2}) = 10 \sin(\pi) \cos\left(\frac{\pi}{2}\right) = \frac{10}{\sqrt{2}}$

$\because$ f is continuous on $[0, \frac{\pi}{2}]$ and $f(0) < K < f(\frac{\pi}{2})$

$\Leftrightarrow$ $0 < \frac{2}{\sqrt{2}} < \frac{10}{\sqrt{2}}$

$\therefore$ By IVT $\exists\, c \in (0, \frac{\pi}{2})$ s.t. $f(c) = \frac{2}{\sqrt{2}}$

7. Show that $x^2 + \frac{1}{x} - 4 = 0$ has a solution.

Let $f(x) = x^2 + \frac{1}{x} - 4$ on $[1,4]$

Clearly f is continuous on $[1,4]$

$f(1) = 1^2 + \frac{1}{1} - 4 = -2 < 0$

$f(4) = 16 + \frac{1}{4} - 4 = \frac{49}{4} > 0$

$f(1) < K < f(4)$ $\Leftrightarrow$ $-2 < 0 < \frac{49}{4}$

$\therefore$ By IVT $\exists\, c (1,4)$ s.t. $f(c) = K = 0$

EVT

If f is continuous on a bounded closed interval $[a,b]$, then on that interval f takes both a max and min value.

8. Show that $\frac{\pi^3}{32} \le x^2 \sin^2 x \le \frac{\pi^2}{4}$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$.

Apply EVT on $f(x) = x^2 \sin^2 x$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$

$\because$ $x^2 \sin^2 x$ continuous on $[\frac{\pi}{4}, \frac{\pi}{2}]$

$$-1 \le \sin x \le 1, \quad \forall x \in \mathbb{R}$$
$$\frac{1}{\sqrt{2}} \le \sin x \le 1, \quad x \in [\tfrac{\pi}{4}, \tfrac{\pi}{2}]$$
$$\frac{1}{2} \le \sin^2 x \le 1$$
$$\left(\frac{\pi}{4}\right)^2 \frac{1}{2} \le x^2 \sin^2 x \le 1 \left(\frac{\pi}{2}\right)^2$$
$$\frac{\pi^2}{32} \le x^2 \sin^2 x \le \frac{\pi^2}{4}$$

9. Show that $\frac{x \cos x}{x^2 - 8x + 25} \le \frac{5}{9}$ $\forall x \in [0, 5]$

Apply EVT on $[0,5]$ for $f(x) = \frac{x \cos x}{x^2 - 8x + 25}$ $\quad \downdownarrows \ {\downarrow \atop \uparrow} \ \frac{0}{0} \ {\uparrow \atop \downarrow} \ \upuparrows$

Consider $x^2 - 8x + 25$

Complete square so we will have only one x. $\quad = x^2 - 8x + 16 - 16 + 25$

$$= (x-4)^2 + 9$$

For $x \in [0,5]$

$$9 \le (x-4)^2 + 9 \le 25$$
$$9 \le x^2 - 8x + 25 \le 25$$
$$\frac{1}{25} \le \frac{1}{x^2 - 8x + 25} \le \frac{1}{9}$$
$$\frac{-1}{25} \le \frac{\cos x}{x^2 - 8x + 25} \le \frac{1}{9}$$
$$0\left(\frac{-1}{25}\right) \le \frac{x \cos x}{x^2 - 8x + 25} \le \frac{1}{9}(1)(5)$$
$$0 \le \frac{x \cos x}{x^2 - 8x + 25} \le \frac{5}{9}$$

Evaluating Limits

1) limit at $x=a$ $\quad \lim_{x\to a} f(x)$
2) limit at infinity $\quad \lim_{x\to\pm\infty} f(x)$

Some basic tools / techniques to evaluate limits

Rule: When evaluating limit, first substitute $x=a$ in $f(x)$ and observe value of $f(x)$.

$f(a)$

defined → Done, $L=f(a)$

Undefined or indeterminate → Try:

- ↳ Factoring
- ↳ Rationalization
- ↳ Some algebra
- ↳ Squeeze Theorem
- ↳ Facts / definitions
- ↳ Analyze
- ↳ L'Hopitals Rule

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

$$e = \lim_{h\to 0}(1+h)^{1/h}$$

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

When evaluating $\lim_{x\to\pm\infty}$ divide the expression by the highest power of x in the expression.

Examples:

10. $$\lim_{x\to\frac{\pi}{4}}\frac{\tan(x)+x^2}{\sec^2 x+1} = \frac{\tan\frac{\pi}{4}+\left(\frac{\pi}{4}\right)^2}{\sec^2\left(\frac{\pi}{4}\right)+1} = \frac{1+\frac{\pi^2}{16}}{2+1} = 16+\frac{\pi^2}{48}$$

11. $$\lim_{x\to 3}\frac{\sqrt{x}}{\tan^{-1}(\sqrt{x})} = \frac{\sqrt{3}}{\tan^{-1}(\sqrt{3})} = \frac{\sqrt{3}}{\frac{\pi}{3}} = \frac{3\sqrt{3}}{\pi}$$

12. $$\lim_{x\to 1}\frac{x^3-1}{x^2-1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \frac{3}{2}$$

13. $$\lim_{x\to -4}\frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4}\frac{(x+1)(x+4)}{(x-1)(x+4)} = \frac{-3}{-5} = \frac{3}{5}$$

14. $$\lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$
$$= \lim_{x\to 1}\sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$
$$= \sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\right)$$
$$= \sin^{-1}\left(\lim_{x\to 1}\left(\frac{1-\sqrt{x}}{1-x}\right)\left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)\right)$$
$$= \sin^{-1}\left(\lim_{x\to 1}\frac{\cancel{(1-x)}}{\cancel{(1-x)}(1+\sqrt{x})}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Squeeze Theorem

If near $x=a$ $g(x) \le f(x) \le h(x)$

If $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$
then $\lim_{x\to a} f(x) = L$

Examples:

15. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

$-1 \le \sin x \le 1 \quad \forall x \in \mathbb{R}$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \quad , x \ne 0$

g(x) $\rightarrow$ $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ $\leftarrow$ h(x)

$\lim_{x\to 0} g(x) = 0 = \lim_{x\to 0} h(x)$ $\therefore$ By Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$

16. $\lim_{x\to -\infty} e^x \cos\left(x+\frac{1}{x}\right)$

Let $\theta = \frac{1}{x} \Rightarrow$ as $x \to -\infty$, $\theta = \frac{1}{x} \to 0^-$

$$\lim_{x\to -\infty} e^x \cos\left(x+\frac{1}{x}\right) = \lim_{\theta\to 0^-} e^{1/\theta} \cdot \cos\left(\frac{1}{\theta}+\theta\right)$$

$-1 \le \cos\theta \le 1 \quad , \forall \theta \in \mathbb{R}$

$-1 \le \cos\left(\frac{1}{\theta}+\theta\right) \le 1 \quad , \theta \ne 0$

$-e^{1/\theta} \le e^{1/\theta} \cdot \cos\left(\frac{1}{\theta}+\theta\right) \le e^{1/\theta}$

$\lim_{\theta\to 0^-} -e^{1/\theta} = 0 = \lim_{\theta\to 0^-} e^{1/\theta}$

17. $\lim_{\theta\to 0} \csc(2\theta)\cdot\tan(3\theta) = \lim_{\theta\to 0} \frac{1}{\sin 2\theta} \cdot \frac{\sin 3\theta}{\cos 3\theta}$

$= \lim_{\theta\to 0} \frac{1}{\sin 2\theta} \cdot \frac{2\theta}{2\theta} \cdot \sin 3\theta \cdot 3\theta \cdot \frac{1}{\cos 3\theta}$

as $\theta \to 0$, 2θ, $3\theta \to 0$

$= \left(\lim_{2\theta\to 0} \frac{2\theta}{\sin 2\theta}\right)\left(\lim_{3\theta\to 0} \frac{\sin 3\theta}{3\theta}\right)\left(\lim_{\theta\to 0} \frac{3\theta}{2\theta}\right)\left(\lim_{\theta\to 0} \frac{1}{\cos 3\theta}\right)$

$= (1)(1)(3/2)(1)$

$= 3/2$

18. $\lim_{x\to 0} (2x+1)^{1/x}$

Let $h = 2x$

$\frac{1}{h} = \frac{1}{2x} \Rightarrow \frac{2}{h} = \frac{1}{x}$

$$\lim_{x\to 0} (2x+1)^{1/x} = \lim_{h\to 0} (1+h)^{2/h}$$

$$= \left(\lim_{h\to 0} (1+h)^{1/h}\right)^2$$

$$= e^2$$

19. $\lim_{x\to\infty} \left(1+\frac{1}{3x}\right)^{2x}$

Let $t = 3x \Rightarrow \frac{1}{3x} = \frac{1}{t}$

$$= \lim_{t\to\infty} \left(1+\frac{1}{t}\right)^{\frac{2t}{3}}$$

$$= \left(\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^{t}\right)^{2/3}$$

$$= (e)^{2/3}$$

20. $\lim_{x\to\infty} \frac{2x+4}{\sqrt{4x^2+x}}$

$$= \lim_{x\to\infty} \frac{2x+4}{\sqrt{x^2(4+\frac{1}{x})}} = \lim_{x\to\infty} \frac{2x+4}{\sqrt{x^2}\sqrt{4+\frac{1}{x}}}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}(2+\cancel{4/x})^{0}}{\cancel{x}(\sqrt{4+\cancel{1/x}}^{0}} = \frac{2}{\sqrt{4}} = 1$$

21. $\lim_{x\to-\infty} \frac{x+3}{\sqrt{\pi x^2-\pi x+1}}$

$$= \lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2(\pi-\frac{\pi}{x}+\frac{1}{x^2})}}$$

$\sqrt{x^2} \Rightarrow |x| \Rightarrow -x \Rightarrow x<0$

$$= \lim_{x\to-\infty} \frac{x+3}{\sqrt{x^2}\sqrt{\pi-\frac{\pi}{x}+\frac{1}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{\cancel{x}(1+\cancel{\frac{3}{x}})^{0}}{-\cancel{x}\sqrt{\pi-\cancel{\frac{\pi}{x}}^{0}+\cancel{\frac{1}{x^2}}^{0}}}$$

$$= -\frac{1}{\sqrt{\pi}}$$

Prove Limit Laws

1) Using ε-δ definition prove the following

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

then $\lim_{x\to a} 2f(x) + 3g(x) = 2L + 3M$

① $\lim_{x\to a} f(x) = L$ means $\forall \varepsilon_1 > 0\ \exists \delta_1 > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon_1$
② $\lim_{x\to a} g(x) = M$ means $\forall \varepsilon_2 > 0\ \exists \delta_2 > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |g(x) - M| < \varepsilon_2$

We need to prove $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |2f + 3g - (2L+3M)| < \varepsilon$

<u>Proof:</u>

Suppose statements 1 and 2 are true.
Given any $\varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$, $\varepsilon_1 = \frac{\varepsilon}{4}$, $\varepsilon_2 = \frac{\varepsilon}{6}$.

For $0 < |x-a| < \delta$,

using $\triangle$

$$|2f + 3g - (2L+3M)| = |2(f-L) + 3(g-M)| \le 2|f-L| + 3|g-M|$$
$$< 2\varepsilon_1 + 3\varepsilon_2 = 2\frac{\varepsilon}{4} + 3\frac{\varepsilon}{6} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

QED

2) Prove that

if $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to\infty} g(x) = \infty$
then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

① $\lim_{x\to\infty} f(x) = 2$ means $\forall \varepsilon_1 > 0\ \exists N_1 > 0$ s.t. $x > N_1 \Rightarrow |f(x) - 2| < \varepsilon_1$
② $\lim_{x\to\infty} g(x) = \infty$ means $\forall M > 0\ \exists N_2 > 0$ s.t. $x > N_2 \Rightarrow g(x) > M$

We need to prove that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, this means

$$\forall \varepsilon > 0\ \exists N > 0 \text{ s.t. } x > N \Rightarrow \left|\frac{f(x)}{g(x)} - 0\right| < \varepsilon$$

Proof:

Suppose statements ① and ② are true.
Let $N = \max\{N_1, N_2\}$, $\varepsilon_1 = M\varepsilon - 2$

If $x > N$ then

$$\left|\frac{f}{g} - 0\right| < \frac{\varepsilon_1 + 2}{M} = \frac{\cancel{M}\varepsilon \cancel{-2+2}}{\cancel{M}} = \varepsilon$$

QED