W7 Lecture 12 Notes

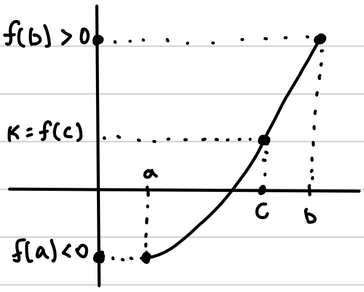
$S \subset \mathbb{R},\ x \in S$

$\inf S = m,\ x \geq m$ $\overset{m}{|}$ - - - - - - - - $\overset{M}{|}$ $x \leq M,\ \sup S = M$

1. LUB Axiom – if set S is bound from above, then it has a Supremum
2. GLB Axiom – if set S is bound from below, then it has an Infinum.

The Intermediate Value Theorem

If $f(x)$ is continuous on $[a,b]$, then for any $K \in \mathbb{R}$, $f(a) < K < f(b)$ there exists at least one $c \in (a,b)$ such that $f(c) = K$.

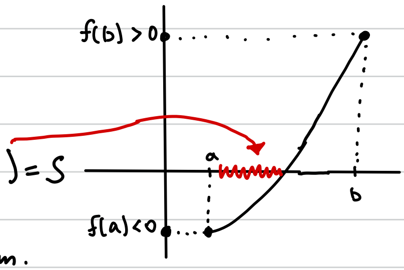


Lemma

If $f(x)$ is continuous on $[a,b]$ and $f(a) < 0 < f(b)$ then there exists at least one number $c \in (a,b)$ such $f(c) = 0$. (when $K = 0$)

Lemma Proof:

$\left.\begin{array}{l} f(x) \text{ is continuous on } [a,b] \\ f(a) < 0,\ f(b) > 0 \end{array}\right\} \Rightarrow$ then there exists such number ξ (x_i) that $f(x)$ is negative on $[a, \xi) = S$

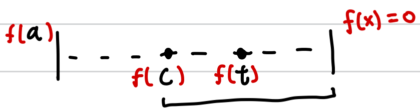


Set S is bounded from above by b, so it has a supremum. (by LUB Axiom)

Assume $\sup S = c$ ($c \neq b$, because $f(b) > 0$ and $f(x) < 0$ on set S, so $c < b$)

$c < b$:

- $f(c) > 0$ (not possible because $f(x) < 0$ on set S)
- $f(c) < 0$ (not possible because if $f(c) < 0$ then there exists such number t that $f(x)$ is negative on $[a, t)$



But if $f(x) < 0$ on $[a, t)$ then $t = \sup S$, which contradicts our definition of $\sup S$.)

- $f(c) = 0$

Therefore $f(c) = 0$

QED

IVT Proof:

Introduce $g(x) = f(x) - K$, $\forall K \in (f(a), f(b))$

$$\left.\begin{aligned} g(b) &= f(b) - K > 0 \\ g(a) &= f(a) - K < 0 \end{aligned}\right\} \Rightarrow \text{by lemma, there exists such number } c \in (a,b) \text{ that } g(c) = 0$$

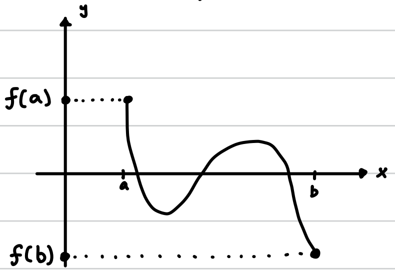
$g(c) = 0$, $g(c) = f(c) - K = 0$

$f(c) = K$

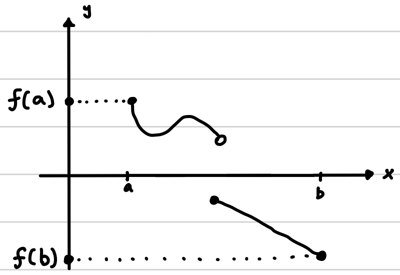
QED

Corollary for IVT:

$f(x)$ can change sign at roots



$f(x)$ can change sign at a discontinuity.



Example:

1. Show that $f(x) = x^3 - x - 1$ has a root on interval $[1, 2]$.

$f(x)$ has a root on $[1,2]$ means that there exists such number $c \in (1,2)$ that $f(c) = 0$. To prove this, we can use IVT.

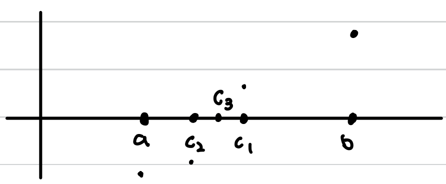
$f(x) = x^3 - x - 1$ must satisfy all the conditions of IVT.

① $f(x)$ is continuous on $[1,2]$ because we proved that cubic and linear functions are continuous on their domains.

② $$\left.\begin{aligned} f(1) &= 1 - 1 - 1 = -1 < 0 \\ f(2) &= 8 - 2 - 1 = 5 > 0 \end{aligned}\right\} \Rightarrow -1 < f(x) < 5$$

Therefore by IVT there exists such number $c \in (1, 2)$ that $f(c) = 0$, so $x = c$ is the root of $f(x)$ on the interval $[1, 2]$.

Method of bisections

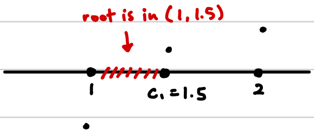


1st Approximation

$C_1 = 1 + \frac{2-1}{2} = 1.5$, $f(1) < 0$ and $f(1.5) > 0$, so root is between 1 and 1.5 $(1, 1.5)$

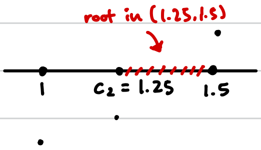
$f(1.5) = 0.875 > 0$

$f(1) = -1 < 0$



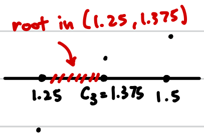
2nd Approximation

$C_2 = 1 + \frac{1.5-1}{2} = 1.25$, $f(1) = -1 < 0$ and $f(1.25) = -0.296 < 0$, so root is in $(1.25, 1.5)$



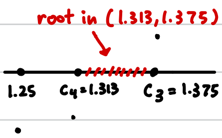
3rd Approximation

$C_3 = 1.25 + \frac{1.5-1.25}{2} = 1.375$, $f(1.375) = 0.224 > 0$ and $f(1.25) = -0.296 < 0$



4th Approximation

$C_4 = 1.25 + \frac{1.375-1.25}{2} = 1.3125$, $f(1.3125) = -0.05151 < 0$ and $f(1.375) = 0.224 > 0$.



We can choose

$C_5 = 1.313 + \frac{1.375 - 1.313}{2}$

$= 1.344$ or we can continue the algorithm as long as we need to obtain a higher accuracy.

Continued...

2. Prove $f(x) = \frac{1}{\sqrt{x}}$ is continuous on its domain.

This means that $\lim_{x \to a} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}$; $\mathrm{Dom}\, f(x) = (0, \infty)$

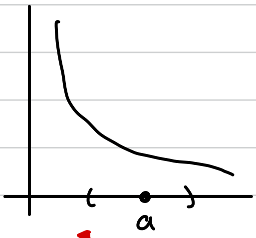
$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow \left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| < \varepsilon$

Aside:

$$\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| = \left| \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}} \cdot \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}+\sqrt{x}} \right| = \left| \frac{a-x}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} \right| = \frac{|a-x|}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})}$$

$$< \frac{\delta}{\sqrt{a}} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{a}+\sqrt{x}}$$

$$< \frac{\delta}{\sqrt{a}} \cdot \frac{\sqrt{2}}{\sqrt{a}} \cdot \frac{1}{\sqrt{a}} = \frac{\sqrt{2}\,\delta}{\sqrt{a^3}}$$



we do not want to choose a delta that crosses over to asymptote

Proof:

Given $\varepsilon > 0$, choose $\delta = \min\left\{ \frac{|a|}{2}, \frac{\sqrt{a^3}}{\sqrt{2}}\varepsilon \right\}$. Then if $|x-a| < \delta$ we have:

$$\left| \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} \right| \ldots\ldots = \frac{|x-a|}{\sqrt{x}\sqrt{a}(\sqrt{x}+\sqrt{a})} < \frac{\delta \cdot \sqrt{2}}{\sqrt{a^3}} =$$

$$= \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{a^3}}} \cdot \frac{\cancel{\sqrt{a^3}}\,\varepsilon}{\cancel{\sqrt{2}}}$$

$$= \varepsilon$$

QED

Guess $\delta = \frac{\sqrt{a^3}\,\varepsilon}{\sqrt{2}}$

To estimate $\frac{1}{\sqrt{x}}$, let $\delta \leq \frac{|a|}{2}$

$$|a-x| = |x-a| < \frac{|a|}{2}$$

$$-\frac{|a|}{2} < x-a < \frac{|a|}{2}$$

$$a - \frac{a}{2} < x < \frac{a}{2} + a$$

$|a| = a$ because $a > 0$

$$\frac{a}{2} < x < \frac{3}{2}a$$

$$\sqrt{\frac{a}{2}} < \sqrt{x} < \sqrt{\frac{3}{2}a}$$

$$\sqrt{\frac{2}{a}} > \frac{1}{\sqrt{x}} > \sqrt{\frac{2}{3a}}$$

$$\frac{1}{\sqrt{x}} < \sqrt{\frac{2}{a}}$$

To estimate $\frac{1}{\sqrt{a}+\sqrt{x}}$

$$\frac{1}{\sqrt{a}+\sqrt{x}} < \frac{1}{\sqrt{a}}$$

WRONG $\delta \leq |a|$

$$-|a| < x-a < |a|$$

$$a-a < x < a+a$$

$$0 < x < 2a$$

$$0 < \sqrt{x} < \sqrt{2a}$$

$$\frac{1}{0} < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{2a}}$$