A37 Apr 8 Lec 2 Notes

Ex 1:

Does $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(2n)!}$ conv. or div ?

Proof:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{3^{n+2}}{(2(n+1))!} \Big/ \frac{3^{n+1}}{(2n)!} \right|$$

$$= \lim_{n\to\infty} \frac{3^{n+2}}{(2n+2)!} \Big/ \frac{3^{n+1}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{3^{n+2}}{3^{n+1}} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} 3 \cdot \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$$

$$= \lim_{n\to\infty} \frac{3}{(2n+2)(2n+1)}$$

$$= 0$$

$\therefore$ By RT, $\sum a_n$ AC $\Rightarrow \sum a_n$ conv. ▨

Power Series

Def (pg 633):

Let $a \in \mathbb{R}$. Given a sequence $\{C_n\}_{n=0}^{\infty}$ of real numbers.

The formal sum

$$C_0 + C_1(x-a) + C_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} C_n(x-a)^n$$

is called a power series.

- C_n is called the n^{th} term coefficient
- a is called the center of the PS
- $C_n(x-a)^n$ is called the general term of the PS.

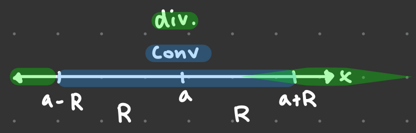
Def: Taylor Series

Let f be a function that has derivatives of all orders at pt $a \in \mathbb{R}$.
Then the PS:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{is called a taylor series.}$$

If we have a TS with $a=0$, we call this a Maclaurin Series

Question: For what values of x does $\sum c_n (x-a)^n$ conv. or div.?



Def. Radius of Convergence

The radius of convergence, denoted R, for the power series $\sum c_n (x-a)^n$ is the largest value $R \in [0,\infty) \cup \{\infty\}$ s.t. the PS:

↳ AC for x satisfying $|x-a| < R$, and
↳ diverges for x satisfying $|x-a| > R$

Def: Interval of convergence

The interval of convergence, denoted I, for $\sum c_n (x-a)^n$ is the set

$$I = \{ x \in \mathbb{R} \mid \sum c_n (x-a)^n \text{ converges} \}$$

Ex 2.

Find I for $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{\sqrt{n}\, 4^n}$ $\quad a = 2 \,;\; c_n = \frac{(-1)^n}{\sqrt{n}\, 4^n}$

Step 1: Find the radius of convergence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{\sqrt{n+1}\; 4^{n+1}} \Big/ \frac{(-1)^n (x-2)^n}{\sqrt{n}\, 4^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-2)^{n+1}}{\sqrt{n+1}\; 4^{n+1}} \cdot \frac{\sqrt{n}\, 4^n}{(-1)^n (x-2)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{|(x-2)^{n+1}|}{\sqrt{n+1}\; 4^{n+1}} \cdot \frac{\sqrt{n}\, 4^n}{|(x-2)^n|} \right)$$

$$= \lim_{n\to\infty} \left(\frac{|(x-2)^{n+1}|}{|(x-2)^n|} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{4^n}{4^{n+1}} \right)$$

$$= \lim_{n\to\infty} \left(|x-2| \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{4}|x-2| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \frac{|x-2|}{4}$$

$\therefore$ By RT,

AC when $\frac{|x-2|}{4} < 1 \iff |x-2| < 4$

div. when $\frac{|x-2|}{4} > 1 \iff |x-2| > 4$

$\therefore R = 4$

Ex 2 continued ...:

Step 2: Check end points $x = a \pm R$

$x = a - R$
$= 2 - 4$
$= -2$

; $\sum_{n=1}^{\infty} \frac{(-1)^n (-2-2)^n}{\sqrt{n}\, 4^n} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

= diverges by p-series test

$x = a + R$
$= 2 + 4$; converges
$= 6$

$\therefore\ I = (-2, 6]$

$\blacksquare$