B52 Nov 10 Lec 1 Notes

Expected Values for Continuous RVs

For continuous RV X with PDF $f_X(x)$, expected value is given by

$$E(x) = \int_{-\infty}^{+\infty} x f_X(x)\, dx$$

For function of multiple RVs X, Y with joint PDF $f_{X,Y}(x,y)$

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx dy$$

All properties of expectations hold in continuous case, in particular,

$$E(aX + bY) = aE(X) + bE(y)$$

$$X \perp Y \Rightarrow E(XY) = E(X)E(Y)$$

Ex 1:

Find the expected value of the exponential (λ) distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\, dx \\ &= \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^{\infty} x \lambda e^{-\lambda x}\, dx \\ &= 0 + [-xe^{-\lambda x}]_0^{\infty} - \int_0^{\infty} (1)\cdot(-e^{-\lambda x})\, dx \\ &= \lim_{x\to\infty} [-xe^{-\lambda x}] - (0e^{-\lambda 0}) + \int_0^{\infty} e^{-\lambda x}\, dx \\ &= \lim_{x\to\infty} \frac{-x}{e^{\lambda x}} + \frac{1}{\lambda}\int_0^{\infty} \lambda e^{-\lambda x}\, dx \\ &= \lim_{x\to\infty} \frac{-1}{\lambda e^{\lambda x}} + \frac{1}{\lambda}(1) \\ &= 0 + \frac{1}{\lambda} \\ &= \frac{1}{\lambda} \end{aligned}$$

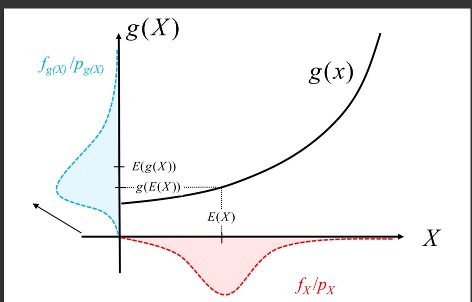
By parts: Let $u = x$ $\quad dv = \lambda e^{-\lambda x}$
$du = 1$ $\quad v = -e^{-\lambda x}$

Inequalities

Derive probabilistic statements about RV using only its expectations, without exact knowledge of underlying distribution.

Jensen Inequality

For any RV X & convex functions g we have $E(g(x)) \geq g(E(x))$. (opposite holds for concave functions)



Ex 2:

For positive RV X, find relationship of $E(x)$ to $E(x^2)$

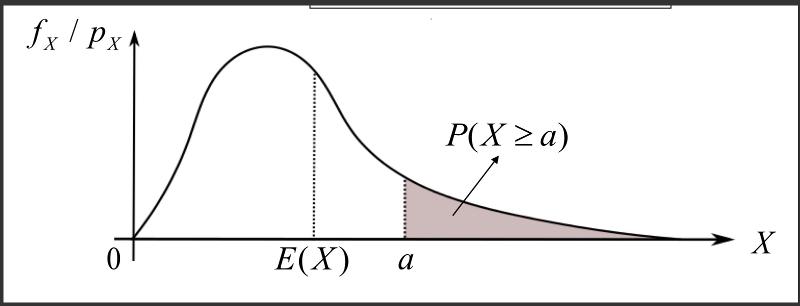
$E(g(x))$, where $g(x) = x^2$

Jensen's inequality: $E[g(x)] \geq g(E(x)) \iff E[X^2] \geq (E(x))^2$
$\Rightarrow E[x^2] - (E(x))^2 \geq 0$
$= Var[X]$
$= E[(X - E(x))^2]$

Markov's Inequality

For positive RV $X > 0$, right tail probability is bounded by mean

$$P(X \geq a) \leq \frac{E(x)}{a}$$



Proof: For continuous case

$$E(x) = \int_0^\infty x \cdot f_x(x)\,dx$$

$$= \underbrace{\int_0^a \underbrace{x}_{>0}\,\underbrace{f_x(x)}_{\geq 0}\,dx}_{\geq 0} + \underbrace{\int_a^\infty x \cdot f_x(x)\,dx}_{\geq \int_a^\infty a f_x(x)\,dx}$$

$$\geq 0 + a\int_a^\infty f_x(x)\,dx$$

$$= P(x \geq a) \Rightarrow P(X \geq a) \leq \frac{E(x)}{a}$$

$\boxtimes$

Ex 3: For Markov

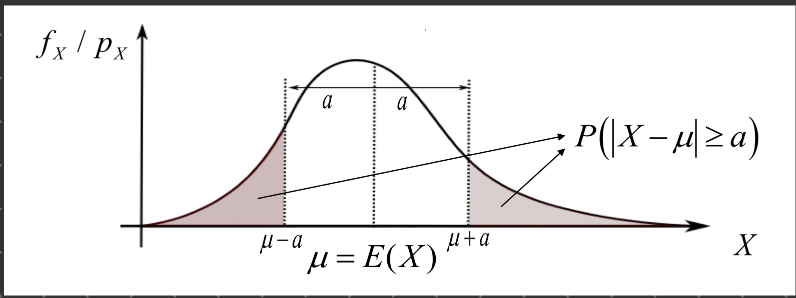
Your commute time X has a mean of 30 min and a SD of 4 min. Find the bound for the probability that your commute takes more than 1hr.

$$P(X \geq 1) \leq \frac{E(x)}{1\text{hr}} = \frac{30\text{min}}{60\text{min}} = \frac{1}{2}$$

Chebyshev Inequality

For any RV X, probability of both tails is bounded by variance.

$$P(|X-\mu| \geq a) \leq \frac{V(X)}{a^2}$$



Proof:

Apply Markov's inequality to $g(x) = (X-\mu)^2 \geq 0$

$$P(g(x) \geq a) \leq \frac{E[g(x)]}{a^2}$$

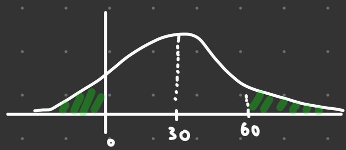
$$\Rightarrow P((x-\mu)^2 \geq a^2) \leq \frac{E[(x-\mu)^2]}{a^2}$$

$$\Rightarrow P(|x-\mu| \geq a) \leq \frac{Var[x]}{a^2}$$

Ex 4: For Chebyshev

Your commute time X has a mean of 30 min and a SD of 4 min. Find the bound for the probability that your commute takes more than 1hr.

$$Var(x) = (SD(x))^2 = 4^2 = 16$$



$$P(X \geq 60) \leq P(|X-30| \geq 30)$$
$$\leq \frac{V[x]}{30^2} = \frac{16}{900} = 1.77\%$$

Limit Results

Many probability problems involve sequence of RVs $Y_1, Y_2, \cdots$ where we are interested in limiting behaviour of Y_n as $n \to \infty$.

Typical scenario involves average of independent RVs $X_1, \ldots, X_n$ with common mean μ & variance σ^2: $\bar{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$

Find mean & variance of $\bar{X}_n$

$$E[\bar{X}_n] = E[\tfrac{1}{n}(X_1 + \ldots + X_n)] = \tfrac{1}{n}(E(X_1) + \ldots + E(X_n))$$
$$= \tfrac{1}{n}(\mu + \ldots + \mu)$$
$$= \frac{n\mu}{n}$$
$$= \mu$$

$$V(\bar{X}_n) = V(\tfrac{1}{n}(X_1 + \ldots + X_n))$$
$$= \tfrac{1}{n^2}[V(X_1) + \ldots + V(X_n)] \quad \text{B/c } X_i \perp X_j$$
$$= \tfrac{1}{n^2}(n\sigma^2)$$
$$= \frac{\sigma^2}{n}$$

Weak Law of Large Numbers

Average of independent RVs with finite variance "converges" to their common mean μ.

$$P(|\bar{X}_n - \mu| \geq \varepsilon) \to 0 \quad \text{as} \quad n \to \infty, \ \forall \varepsilon > 0$$

Distribution of $\bar{X}_n$ becomes increasingly concentrated around μ.

Proof: $P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{V(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$, $\forall \varepsilon > 0$