PHY2403F, Fall 2018, Quantum Field Theory 1, Homework # 1:

`Due on Wednesday, October 10 in class (if material delayed, we'll delay deadline).`

I. *Back to classics: relativistic electrodynamics and variational principle*

In terms of the four-vector potential A_μ, the lagrangian density of the electromagnetic field, interacting with a charged particle of mass m can be written as follows:

$$S = \int\limits_{\textit{all spacetime}} d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu \right] - m \int\limits_{\textit{worldline}} ds. \tag{1}$$

Here, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor. The current j^μ is the current corresponding to the particle which can be written as:

$$j^\mu(x) = e \int\limits_{\textit{worldline}} dX^\mu(\tau)\, \delta^{(4)}(x - X(\tau)) \ , \tag{2}$$

where $\delta^{(4)}$ is a four-dimensional delta function. All indices are raised and lowered by means of the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

The last term in (1) is the relativistic kinetic energy of the particle and the integral is over the particle's worldline, $X^\mu(\tau)$. Note that τ is a parameter used to describe the particle's location along the worldline. One can take this parameter be equal to x^0, so that $X^\mu(\tau)$ means ($X^0 = x^0$, $X^i = X^i(x^0)$), where $\vec{X}(x^0)$ is simply the trajectory of the particle (such a choice of parametrization can be useful, but is not required). Notice also that the term involving the current in (1), after substitution of (2) simply becomes

$$-e \int\limits_{\textit{worldline}} dX^\mu(\tau) A_\mu(X(\tau)) \ ,$$

which is the usual coupling of a charged particle to the electromagnetic field (choose the $\tau = x^0$ parameterization of the worldline to see this). Whether you use this form of the one of Eq. (1) depends on the problem you're solving (this is a hint).

The dynamical degrees of freedom in the action (1) are the four-vector potential A_μ and the particle position $X^\mu(\tau)$.

1. Use the identification $A^0 = \phi$, the scalar potential, and $(A^1, A^2, A^3) = \vec{A}$, the vector potential, to convince yourself that $F_{01} = (\vec{E})_x$, $F_{02} = (\vec{E})_y$, $F_{03} = (\vec{E})_z$, and that $F_{12} = -(\vec{B})_z$, $F_{13} = (\vec{B})_y$, $F_{23} = -(\vec{B})_x$.

2. Write the Euler-Lagrange equations obtained when varying (1) with respect to A_μ. Show that they can be cast in terms of the field strength tensor F and j. Note that when varying with respect to A_μ, the current is kept fixed. Using the E and B fields as the appropriate components of F, show that the Euler-Lagrange equations for A_μ from (1) reduce to the Maxwell equations familiar to you from electrodynamics.

3. Finally, write the Euler-Lagrange equation varying with respect to the worldline of the particle. Show that they give $m\frac{dU^\mu}{ds} = e\ F^{\mu\nu}U_\nu$, where $U^\mu = \frac{dX^\mu}{ds}$ is the four velocity of the particle and F is, of course, taken at the particle's position. Convince yourself that this is the relativistic Lorentz force equation.

The point of this problem is to make sure you remember/learn how the action principle works in electrodynamics. The two coupled equations, obtained by varying w.r.t. A_μ and X^μ complete the equations of classical electrodynamics. Feel free to use Landau and Lifshitz, vol. II, or my PHY450 notes online while solving this problem.

II. *Part of Problem 2.2 from Peskin and Schroeder (reproduced below)*

Consider a complex scalar field with action $S = \int d^4x(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi)$. When doing the variational principle consider ϕ and ϕ^* as independent, rather than their real and imaginary parts (this is equivalent, but more convenient).

1. Show that $H = \int d^3x(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi)$ and that the Klein-Gordon equation is obeyed by ϕ and ϕ^*.

2. Introduce complex amplitudes, diagonalize the Hamiltonian, and quantize the theory. Show that the theory has now *two* sets of particles.

3. Write the charge conserved due to the global $U(1)$ symmetry, $Q = \int d^3x\frac{i}{2}(\phi^*\pi^* - \pi\phi)$, in terms of creation and annihilation operators and find the charge of the particles of each type.

III. *Zero point energy, an exercise in unit conversion, and scales related to the "cosmological constant problem"*

In class, we showed that the zero-point energy of the quantized massless scalar field (we are taking this case, because in the physically relevant case of electrodynamics, the number of degrees of freedom and the associated vacuum energy is the same as that of two massless scalar fields) can be written as:

$$E_{vac} = V_3 \int \frac{d^3k}{(2\pi)^3}\ \frac{\omega_k}{2}\ , \tag{3}$$

where V_3 is the (large, i.e., almost infinite) volume of space. This expression diverges, because we assume that electromagnetic fields and photons of arbitrarily large momenta exist. There's no justification to this, as particle physicists have only probed the Standard Model up to energies of order a few TeV. Assume, then, that the integral above is cut off at some maximum value of the momentum Λ (called the "UV cutoff"), say of order 10 TeV.

1. What is the value of the vacuum energy density ρ_{vac}, in units of g/cm^3.

2. What value should Λ have in order that ρ_{vac} matches the observed value of the "dark energy", of order $\rho_{dark} \sim 10^{-29}$ g/cm^3. Express Λ both as a high-energy scale cutoff and as a short-distance cutoff.

3. What is the ratio of ρ_{vac} for $\Lambda \sim M_{Planck}$ to ρ_{dark}?

4. Note that the zero-point energies of phonons—the zero point energies of the quantized collective sound oscillations of nuclei in a crystal—are given, up to simple numerical factors counting the numbers of polarizations (which we won't worry about here) by an expression similar to the above. This is because phonons are massless scalar fields propagating with the speed of sound instead of speed of light. Notice that this difference is irrelevant as c appears in E_{vac} very simply: k is a wavevector and $\omega_k = ck$—a frequency (secretly multiplied by $\hbar$, of course). In the case of phonons, however, we are well aware that a cutoff scale exists and we understand well its nature: it is given by the interatomic separation, as the notion of phonons does not make sense for shorter wavelengths. Now take $k_{max} = \Lambda \sim 1/a_0$, with a_0 of order the Bohr radius and estimate the energy density of the zero point fluctuations in a crystal. Compare your result to the typical rest energy (i.e. mass) density of crystals.

The results from the first three items above lead to a puzzle commonly referred to as the "cosmological constant problem". There are various proposals for its solution, ranging from cancellations between the contributions of high and low momentum oscillators, anthropic principle (multiverse) considerations, modifications of gravity at long distances, to name a few. The issue awaits your input!

IV. *Scale invariance and conserved charge*

Consider classical electrodynamics with the lagrangian

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] . \tag{4}$$

Consider the following "dilatation" (or "scale") transformation:

$$\begin{aligned} x_\mu &\rightarrow x'_\mu = e^d \; x_\mu \\ A_\mu(x) &\rightarrow A'_\mu(x') = e^{-d} A_\mu(x) \; , \end{aligned} \tag{5}$$

where d is a constant, called the dilatation parameter.

1. Show that the action is invariant under dilatations.

2. Find the corresponding Noether current.

3. Show that—perhaps, after a redefinition of j_μ; notice that any conserved current j_μ can be redefined by adding to it $\partial^\nu C_{\mu\nu}$, where $C_{\mu\nu}$ is antisymmetric, without spoiling its conservation (in this case C can depend on x^μ, ∂^μ and A^μ, of course) the dilatation current is simply related to the energy-momentum tensor: $j_\mu^{conf} = x_\nu T^{\nu\; conf}_{\;\mu}$, where the symbol $conf$ indicates that these are the conformal energy-momentum tensor and dilatation current. Notice that this problem, secretly, requires you to also derive $T^{\mu\nu}$ for the electromagnetic field.

4. Show, then, that conservation of j_μ^{conf} implies that the energy-momentum tensor of classical electrodynamics is traceless (the trace of the tensor is defined as usual to be $g_{\mu\nu}T^{\mu\nu}$).

5. Finally, open your classical electrodynamics books and recall the interpretation of the T^{00}, T^{xx}, T^{yy}, etc., components of the energy momentum tensor as energy density and pressure. Show that the tracelessness of $T^{\mu\nu}$ is equivalent to the familiar relation $p = \rho/3$ between the energy density and pressure of isotropic radiation—the equation of state of blackbody radiation.[1]

Dilatation invariance in QED (and QCD) is perhaps the simplest example of a symmetry, where the classical action is invariant, but the quantum theory is not (as you will learn later, in the spring class). Broken scale invariance arises because one has to introduce a short-distance cutoff (a UV "regulator") to define the quantum theory. (We already saw an indication of the need for a regulator when we considered the divergent zero point energy of the free quantum scalar field.)

V. *Observability of the zero point energy:*[2] *the Casimir force.*

In class, when discussing the quantization of the real scalar field, we found the sum of zero point energies of the harmonic oscillators (one per each $\vec{k}$) into which we decomposed the field:

$$E_{zero\ point} = V_3 \int \frac{d^3k}{(2\pi)^3} \, \frac{\omega_{\vec{k}}}{2} \ , \tag{6}$$

Expression (6) gives the zero point energy of the field in a spatial volume V_3. This energy is, of course, infinite and is usually discarded (as we learned, by applying a "normal ordering" procedure) as unobservable. Nevertheless, there are circumstances under which *changes* in the zero point energy lead to measurable effects. The most celebrated example is the *Casimir effect*, predicted by Casimir in 1948 ["On the attraction between two perfectly conducting plates," H.B.G. Casimir, Kon. Ned. Akad. Wetensch.Proc. 51:793-795, 1948] and discovered experimentally in 1958 (see Lamoreaux's more recent article linked to in the "Summary of Sept. 25th class"). Another instance where this has been "observed" (in numerical simulations) is the Lüscher term in the confining string in QCD. Casimir energies generally also appear whenever the topology of space(time) is changed and people have speculated that dark energy may have something to do with that...

The Casimir effect can be described very simply (!): the zero point energy of the electromagnetic field between two infinite conducting plates is smaller than it would be in the absence of the plates. This is because the boundary conditions on the plates eliminate some of the modes of the field that would be otherwise present. The vacuum energy in the space between the plates should be proportional to the area A of the plates, as well as to $\hbar$ (as zero point energies are proportional to $\hbar$). It can also depend on a, the distance between the plates, and the speed of light c. By dimensional analysis, the excess energy (negative) in the volume aA between the plates should be

$$\Delta E_{vac}(a) \sim -aA \, \frac{\hbar c}{a^4} = -A \, \frac{\hbar c}{a^3} \ , \tag{7}$$

[1] In class, I promised you some finite-temperature problem, but this homework got long. For now, this will remain the only connection. I'll try to keep my promise... may be in the final?

[2] Notice that just like for the Planck derivation of blackbody radiation formula, where some people would say that it does not imply that the electromagnetic radiation is quantized, but only its sources (as radiation is emitted by the atoms of the cavity), there are similar claims for the Casimir force (my take is to ignore these, as we know that the radiation is quantized). See article by Lamoreaux that I put a link to online.

where the aA factor is the volume, $\hbar$ has dimensions of energy $\times$ time, c/a has dimensions of inverse time, and the extra factor of $1/a^3$ is there to make the dimension of energy right. Thus, to minimize E_{vac} the plates "want to" get closer. In other words, there should be an attractive force per unit area of the plates, called "Casimir pressure"

$$p_{\text{Casimir}} \sim \frac{\hbar c}{a^4} \,, \tag{8}$$

proportional to the inverse fourth power of the distance between the plates. In what follows we shall calculate this force.

We will use our real scalar *massless* field theory as a model for the real thing (the electromagnetic field, that we have not formally learned how to quantize yet). Casimir considered two infinite, conducting plates stretching in the y, z plane and located at $x = 0$ and $x = a$, respectively; furthermore, he used perfect conductor boundary conditions on the plates. These require that the tangential component of the vector potential, $\vec{A}_{tang.}$, vanishes at the plates (in Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, $A^0 = 0$). Our two toy "conducting plates" will be made of a "material" that requires that the scalar field ϕ vanish at the plates.

1. Show that the boundary conditions on the plates impose a quantization condition on the allowed values of field momentum perpendicular to the plates, i.e. $k_x = n\pi/a$, $n = 0, \pm1, \pm2, \ldots$ [e.g., recall your waveguide physics].

2. Consider now the contribution to the energy of the vacuum fluctuations of the field in the space between the plates and find the zero point energy per unit area of the plates. To do this, replace the integral over k_x in (6) by a sum over n, $\int dk_x = (\pi/a)\sum_n$ [Hint: to save work, use the fact that the correct expression should have the property that as the plates are removed, $a \to \infty$, the energy (per unit volume) should give back (6)]. Does the resulting expression for the zero point energy still diverge?

3. Show now, starting from (6), with integral replaced by sum, that the difference between the zero point energies *per unit area, in the space between the plates* in the presence of the plates and without the plates is:

$$\Delta E_{vac}(a) = \int_0^\infty \frac{dk\, k}{2\pi} \left(\frac{k}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \sqrt{k^2 + \frac{n^2\pi^2}{a^2}} - \frac{1}{2}\int_0^\infty dn \sqrt{k^2 + \frac{n^2\pi^2}{a^2}} \right) \,, \tag{9}$$

where, obviously, k is radial wave vector in y, z-directions.

4. The expression (9) is still ill-defined, as every single term is infinite.

   Now, to make progress, we note that the idealization of perfect conducting plates and the corresponding macroscopic boundary conditions do not make sense for wavelengths smaller than the atomic size. In particular, for frequencies above $1/a_0$ (a_0 is of the order of the Bohr radius) the conducting plates are totally invisible for the electromagnetic field. To incorporate this in our calculation, introduce a function $f(k)$ into the integrand in (9) such that $f(k) = 1$

for $k < 1/a_0$ and $f(k) = 0$ for $k > 1/a_0$, somehow smoothly interpolating between these two values.

The integrals in (9) thus become absolutely convergent—all momenta larger than the inverse Bohr size are cut off. Show that expression (9) can be written as:

$$\Delta E_{vac.}(a) = \frac{\pi^2}{8a^3}\left(\frac{1}{2}F(0) + F(1) + F(2) + \ldots - \int\limits_{n=0}^{\infty} dnF(n)\right) \ , \tag{10}$$

where $F(n) = \int_0^\infty du\sqrt{u+n^2}f(\frac{\pi}{a}\sqrt{u+n^2})$.

5. To calculate (10), use the Euler-Maclaurin formula:[3]

$$\frac{1}{2}F(0) + F(1) + F(2) + \ldots - \int\limits_{n=0}^{\infty} dnF(n) = -\frac{1}{2!}B_2F'(0) - \frac{1}{4!}B_4F'''(0) + \ldots \ , \tag{11}$$

where $B_2 = 1/6$, $B_4 = -1/30$, etc. are Bernoulli numbers, and primes denote derivatives. Now, $f(0) = 1$ as stated above; furthermore, assume that all derivatives of our smearing function $f(k)$ vanish at zero (it is not difficult to construct examples of such functions). Show that $F'(0) = 0$, $F'''(0) = -4$, and that all higher derivatives of F vanish.

Thus the "cutoff" function f does not enter the final result—or the fact that we assumed a cutoff at scales of order the inverse Bohr radius; it only mattered that $a_0 \ll L$.

6. Show, now, that the final result for the Casimir energy per unit area of the plates is:

$$\Delta E_{vac.}(a) = \frac{\pi^2}{2a^3}\,\frac{B_4}{4!} = -\frac{\pi^2}{2\times 720}\,\frac{1}{a^3} \ , \tag{12}$$

giving rise to an attractive force between the plates. This force—for the electromagnetic field, where there is an additional factor of two—was measured in 1958, and not only the sign, but also the $\sim a^{-4}$ distance dependence was observed! In fact, measuring the distance dependence is crucial for verifying the nature of this force—at atomic distances the Casimir force competes with Van-der-Vaals forces, which however have a different, $\sim a^{-7}$, dependence on the distance.

7. To get some idea of what experimentalists have to go through, estimate the force acting on plates of area 1cm^2 a micron apart. Compare with the magnitude of forces whose measurements you are familiar with. Note that the 1990's Lamoreaux measurements are accurate within 5%.

[3]This formula is used to approximate sums with integrals. See, e.g., Wikipedia article for a derivation by induction. Other, fun ways to proceed exist, my favorite is ζ-function regularization, see S. Hawking, Commun.Math.Phys. 55 (1977) 133.

Most importantly, the result is independent of the method of regularization. "By definition", this is what we call a physical result in QFT (=cutoff independent). Notice the striking difference with the E_{vac} of Eqn. (3), which inherently depends on the cutoff and can not be made physical sense within QFT ... as you see, many lessons lurk in this "simple" problem!

8. A final bonus question: what if the scalar field had a mass, m? Would you expect an effect if $m \gg 1/a$? What if $m \ll 1/a$?

You just saw the first example of extracting a finite and physically meaningful result from seemingly infinite expressions. Infinities result from assuming that quantum field theory makes sense at arbitrarily short distances, or large momenta k in (9). The possibility of extracting finite results (e.g., the Casimir force) from quantum field theory simply means that in many cases (most cases, in fact: the so-called "renormalizable" ones—and even in "non-renormalizable" if one is happy with finite precision—see QFT2) the long-distance physics is independent of the details of the short-distance, most often not understood, physics, when expressed only through quantities observed at long distances.[4]

In this example, this was seen by the independence of the final answer on the cutoff function $f(k)$. This independence really means that field modes with wave vectors $\gg 1/L$ do not contribute to the Casimir effect, i.e., it is an IR (infrared) effect.

[4]This is already familiar from classical electrodynamics although may not be always stressed. The electrostatic energy of a point charge diverges, as is well known, hence it gives an infinite contribution to the charge's rest energy. However, in the non relativistic limit (to order v^2/c^2, in fact) the equations describing the motion of charged particles do not depend at all on whatever structure one might ascribe to the electron (it could be a ball, a hollow sphere, or a tiny string). The relative motion of particles in this limit (and, of course, at relative distances larger than the "classical radius of the electron") is determined by two "relevant" parameters: their mass m and charge e. These are quantities determined by experiment, not calculated from first principles. These experiments are made at the long distance/time scales, where classical electromagnetic theory applies. There is no way to calculate m and e from first principles.

The situation in QFT is not that different—its calculational tools are a way to relate measurable quantities to measurable quantities. It usefulness is in that there are more measurable quantities than the number of measurements required to fix the relevant parameters in the Lagrangian (e.g., the same m and e for QED), so it has predictive power. When QFT is used to relate observables to observables, no infinities appear.

There we go. QFT in a nutshell.