PHY2403F, Fall 2018, Quantum Field Theory 1, Homework # 2:

`Due [tentatively] on Wednesday, October 31 in class.`

I. *Spacetime behaviour of various Green's functions*

Here, you'll study some properties of $D(x) \equiv [\hat{\phi}_-(x), \hat{\phi}_+(0)] = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-i\omega_p t + i\vec{p}\cdot\vec{x}}$.

1. For $m = 0$ ("photon"), show that:

$$D(x) = -\frac{1}{4\pi^2}\mathcal{P}\frac{1}{t^2 - r^2} - \frac{i}{8\pi}\left(\frac{\delta(t-r)}{r} - \frac{\delta(t+r)}{r}\right) ,$$

   where $r = |\vec{x}|$. Notice that $D(x)$ is singular on the light cone $t = r$. Does it vanish for spacelike separations?

   Hint: Please recall that (and why!) $\frac{1}{a \pm i\epsilon} = \mathcal{P}\frac{1}{a} \mp i\pi\delta(a)$ (here $\mathcal{P}$ denotes "principal value integration", as this relation is to be understood in terms of generalized functions, i.e. in the back of your mind it always needs to be integrated over a with suitable smooth and integrable "test functions"). Note also that what looks like a "half-delta-function integral" $\int\limits_0^\infty dy e^{ixy}$ should really be understood as $\lim\limits_{\epsilon \to 0} \int\limits_0^\infty dy e^{-\epsilon y + ixy}$.

2. For $m^2 > 0$, study the behavior of $D(x)$ for spacelike x and find the asymptotic behavior for $-x^2 \gg 1/m^2$ (i.e., at spacelike separations larger than the particle's Compton wavelength).

II. *A model with $SU(2)_L \times SU(2)_R$ internal global symmetry: chiral symmetry and the Higgs*

This problem introduces a model to describe the symmetry realization of the nonabelian chiral symmetry in QCD (quantum chromodynamics). The word "chiral" should become clear later in this class, but the "nonabelian" part will be clear below. $SU(2)_L \times SU(2)_R$ is an exact symmetry of QCD in the limit when the "current masses" of the u and d quark, m_u and m_d, are taken to vanish. In the real world, it is an approximate symmetry, in the sense that m_u and m_d are small compared to the intrinsic scale of QCD, given, say, by the proton mass ($m_{u,d} \sim$ MeV $\ll 1$ GeV). This is, thus, an example of an "approximate symmetry".

Closer to the theory you will study below, the scalar model with $SU(2)_L \times SU(2)_R$ symmetry, is really the same as the Higgs sector in the Standard Model, in the limit when the electromagnetic and weak interactions are turned off. $SU(2)_L \times SU(2)_R$ becomes a symmetry in this limit. It is only an approximate symmetry, as the electromagnetic and weak couplings (which explicitly break it) are dimensionless numbers smaller then unity.

Finally, to end the preaching preamble, the notion of approximate symmetries is not new and you have, for sure, been exposed to its usefulness when studying the hydrogen atom spectrum in quantum mechanics.

1. The Lagrangian you will study is that of two complex scalar fields, assembled into a column $\Phi = (\phi^1, \phi^2)^T$ (the ^{T} is here so I do not have to go through the trouble to write a column instead of a row). It is given by:

$$L = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \ . \tag{1}$$

Show that (1) is invariant under an $SU(2)_L$ global symmetry transformation $\Phi \to U_L \Phi$, where $U_L^\dagger U_L = 1$ is a 2×2 unitary matrix of unit determinant. In addition, the Lagrangian has a $U(1)$ symmetry, not part of $SU(2)_L$, acting as $\Phi \to e^{i\alpha}\Phi$. Find the currents and conserved charges under these symmetries.

Hint: recall that an infinitesimal $SU(2)_L$ transformation can be written as $U_L \simeq \sigma^0 + i\omega_a \frac{\sigma^a}{2}$, where σ^0 is the unit 2×2 matrix, $\sigma^a, a = 1, 2, 3$ are the Pauli matrices, and ω_a are the three parameters of infinitesimal $SU(2)_L$ transformations.

2. Show that the charge operators, $\hat{Q}_a^L$, $a = 1, 2, 3$, conserved due to $SU(2)_L$ invariance, obey the angular momentum algebra, i.e., $[\hat{Q}_1^L, \hat{Q}_2^L] = i\hat{Q}_3^L$ (plus cyclic permutations).

3. The Lagrangian (1) has, however, a larger symmetry than simply the above $SU(2)_L$. To begin seeing this, instead of using $\Phi = (\phi^1, \phi^2)^T$ introduce the real and imaginary parts of $\phi^{1,2}$. Use $\phi^1 = \psi^1 + i\psi^2, \phi^2 = \psi^3 + i\psi^4$, and introducing $\Psi = (\psi^1, \psi^2, \psi^3, \psi^4)^T$, show that (1) can be written as:

$$L = a \partial_\mu \Psi^T \partial^\mu \Psi - b m^2 \Psi^T \Psi - c\lambda (\Psi^T \Psi)^2 \ , \tag{2}$$

on the way determining the (pure numbers) a, b, c. The Lagrangian (2) has, clearly, an $O(4)$ symmetry, i.e., is invariant under $\Psi \to O\Psi$, where O is a 4×4 orthogonal matrix, $O^T O = 1$. Is there a continuous $U(1)$ allowed in this case?

Comment: I will spare you finding the currents for $SO(4)$ ($SO(4)$ matrices are the restriction of $O(4)$ matrices to the ones with unit determinant). What you will do next, instead, is to use the equivalence of Lie algebras $SO(4) \simeq SU(2) \times SU(2)$, which will come about by another change of variables (see below). Notice also that, as it comes, $SO(4)$ happens to be the Euclidean version of $SO(1, 3)$.

4. To expose the $SU(2)_L \times SU(2)_R$ symmetry of (1), now use the following change of variables. Consider, instead of Φ in (1) the 2×2 matrix H made up by components of Φ as follows:

$$H \equiv \frac{1}{\sqrt{2}} (i\sigma_2 \Phi^*, \Phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix} \ . \tag{3}$$

Show that under $SU(2)_L$ transformations, $H \to \frac{1}{\sqrt{2}}(i\sigma_2(U_L\Phi)^*, U_L\Phi) = \frac{1}{\sqrt{2}}(U_L i\sigma_2 \Phi^*, U_L \Phi) = U_L H$.

Hint: the tricky part is to show that $i\sigma_2 (U_L \Phi)^* = i\sigma_2 U_L^* \Phi^* = U_L i\sigma_2 \Phi^*$. What you need to show, then, is that $\sigma_2 U_L \sigma_2 = U_L^*$ (this fact will be very useful in our future studies of spinors, so make sure you understand it).

5. Using the change of variables (3), show that

$$H^\dagger H = \frac{1}{2}\begin{pmatrix} |\phi_1|^2 + |\phi_2|^2 & 0 \\ 0 & |\phi_1|^2 + |\phi_2|^2 \end{pmatrix} , \tag{4}$$

and, hence, that (1) can be written as:

$$L = \mathrm{Tr}\left(\partial_\mu H^\dagger \partial^\mu H\right) - m^2 \mathrm{Tr} H^\dagger H - \lambda \left(\mathrm{Tr} H^\dagger H\right)^2 , \tag{5}$$

where Tr denotes the matrix trace. Show that now (5) has $SU(2)_L \times SU(2)_R$ symmetry, acting on H as

$$H \to U_L H U_R^\dagger , \tag{6}$$

where the action of $U_R^\dagger$ on the right is pure convention (we could have taken U_R instead). U_L and U_R are two sets of independent $SU(2)$ transformations. The L and R (left and right) names are self-evident in the way (6) is written. Show that under $SU(2)_L \times SU(2)_R$, we have $\delta H = i\omega_a^L \frac{\sigma^a}{2} H - i\omega_b^R H \frac{\sigma^b}{2}$.

Hint: clearly, the only thing you need to show is $SU(2)_R$ invariance, as $SU(2)_L$ was already shown.

6. Show that the left and right $SU(2)$ conserved currents can be written as

$$j_L^{\mu,a} = \frac{i}{2}\,\mathrm{tr}\left(\partial_\mu H^\dagger \sigma^a H - H^\dagger \sigma^a \partial_\mu H\right) \text{ and } j_R^{\mu,b} = \frac{i}{2}\,\mathrm{tr}\left(\partial_\mu H \sigma^b H^\dagger - H \sigma^b \partial_\mu H^\dagger\right) , \tag{7}$$

and that the corresponding generators $\hat{Q}_a^{L,R}$ obey the commutation relations of two commuting angular momentum algebras.

Hint: notice that both currents are Hermitean and that the left is obtained from the right by interchanging H with $H^\dagger$.

III. *$SU(2)_L \times SU(2)_R$, realized in the Wigner and Nambu-Goldstone modes.*

Consider now our Lagrangian (5) and imagine that $m^2 < 0$, for whatever reason (nobody knows, really), while λ is still positive. This now becomes the Higgs Lagrangian of the Standard Model.

1. Show that the classical potential in (5) now becomes:

$$V = -|m^2| \mathrm{Tr} H^\dagger H + \lambda \left(\mathrm{Tr} H^\dagger H\right)^2 = \lambda \left(|\phi_1|^2 + |\phi_2|^2 - \frac{|m|^2}{2\lambda} \right)^2 + const. . \tag{8}$$

2. Clearly, there are extrema of the potential when $|\phi_1|^2 + |\phi_2|^2 = 0$ and when $|\phi_1|^2 + |\phi_2|^2 = \frac{|m|^2}{2\lambda}$. The second one has, clearly, smaller energy density. To quantize the theory, we now have to choose which classical minimum to expand around. Show that, if we expand around $|\phi_1|^2 + |\phi_2|^2 = 0$, we will find that the $\phi_{1,2}$ excitations are tachyons, even classically. This signals an instability, rather than a faster-than-light propagation and shows that we have chosen the wrong value of ϕ to build our quantum theory.

3. Thus, consider the $|\phi_1|^2 + |\phi_2|^2 = \frac{|m|^2}{2\lambda}$ minimum of V. This is really a set of minima. In fact the set parameterized by $|\phi_1|^2+|\phi_2|^2 = const.$ is also known as a three sphere (S^3, embedded in a four-dimensional space parameterized by $\psi^{1...4}$—not the spacetime!). To build the quantum theory, we will choose a point on this three sphere (a.k.a. the "vacuum manifold" — the set of field values that minimize the potential). We will now study the small fluctuations around the chosen point and the spectrum of the theory in this vacuum. There is an infinite number of parameterizations that can be used to do this, but I will suggest one that makes the symmetries the clearest.

   Thus, use the H-representation and take

$$H(x) = \frac{|m|}{2\sqrt{\lambda}}(1 + h(x))e^{i\varphi^a(x)\sigma^a}. \tag{9}$$

   The logic here is as follows. When $h(x)$ and $\varphi^a(x)$ vanish (i.e. there are no excitations), the parameterization (9) is equivalent, by (4), to taking a specific point on the vacuum manifold, i.e. the one where $\phi_1 = 0$ and $\phi_2 = \frac{|m|}{\sqrt{2\lambda}}$. The fields $h(x)$ and $\varphi^a(x)$ parameterize the fluctuations around this ground state (for sure, they can be mapped - the map is nonlinear - to the fluctuations of the fields $\phi_{1,2}$ around the chosen vacuum value for ϕ_2).[1]

   What you will do now is take the form (9), plug it into the Lagrangian (5) with $m^2 = -|m|^2$, and expand what you find *to second order* in the fields $h(x)$ and $\varphi^a(x)$. Show that the field $h(x)$ has a mass and find an expression for it. Show that the fields $\varphi^a(x)$ remain massless and that their Lagrangian (not just to quadratic order) only contains derivatives.

   The latter point can be seen pretty simply by noting that $H(x)$ from (9) can be written as $H(x) = \frac{|m|}{2\sqrt{\lambda}}\Omega(x)(1+h(x))$, with $\Omega^\dagger(x)\Omega(x) = 1$ and $\det\Omega(x) = 1$. In this parameterization $\Omega(x)$ fluctuations correspond to going around the vacuum manifold S^3, while the $h(x)$ fluctuations are along the "radial" directions away from the minimum. The latter cost energy, hence h is massive (the Higgs field!), while the $\Omega(x)$ only cost energy if the x-dependence is nontrivial. The $\varphi^a(x)$ (or $\Omega(x)$) are equivalent parameterizations of the Goldstone fields. What you found here is an example of a general story: if a theory has a continuous symmetry, which is not a symmetry of the ground state, there is a number of massless Goldstone (or Nambu-Goldstone) modes. For internal symmetries like the ones we are considering here, their number is equal to the number of broken generators.

   In the Standard Model, $h(x)$ is indeed the Higgs field. The fields $\varphi^a(x)$ actually become the longitudinal components of the W and Z-bosons (one usually says that they are "eaten", a manifestation of the Landau-Anderson-Higgs-Brout-Englert-Guralnik-Hagen-... mechanism).

4. One question that was not discussed and remained a bit obscure is that of the unbroken part of the symmetry. The original Lagrangian has $SU(2)_L \times SU(2)_R$ symmetry. The value of

[1] As in classical mechanics, which variables one uses to describe physics is a matter of choice and convenience. The Euler-Lagrange equations have the property that they are invariant under changes of variables, so long as no singularity occurs in the process. In fact, one of the main motivations of using Lagrangians in classical mechanics is that the change of variables is much easier to do. In other words, it is much easier to first transform the Lagrangian to spherical coordinates and then find the Euler-Lagrange equations then to transform the equations found in Cartesian coordinates to spherical coordinates (in the latter case you need to differentiate twice...). Invariance of physics under nonsingular changes of variables in the Lagrangian is, of course, inherited in field theory.

$H(x)$ in the vacuum, denoted by $\langle H\rangle$, is given by Eq. (9) with $h = \varphi^a = 0$ and is $\langle H\rangle \sim$unit matrix.

Show that, while $\langle H\rangle$ is not invariant under $SU(2)_L \times SU(2)_R$ for arbitrary $SU(2)_L$ and $SU(2)_R$ transformations, it is invariant under (6) with $U_L = U_R$. Such $SU(2)_L \times SU(2)_R$ transformations with $U_L = U_R$ are called "diagonal" or "vector" $SU(2)_V$ transformations. These remain unbroken in the vacuum. In the electroweak theory, the third component of $SU(2)_V$ is identified with electromagnetic $U(1)$. Show that the current associated with $SU(2)_V$ transformations has the form:

$$j_\mu^{V,a} = \frac{i}{2}\,\mathrm{tr}\left(\partial_\mu H^\dagger[\sigma^a, H] + \partial_\mu H[\sigma^a, H^\dagger]\right)\ . \tag{10}$$

Show also that the other "linear" combination of $SU(2)_L$ and $SU(2)_R$, Eq. (6) with $U_R = U_L^\dagger$ corresponds to the current (not conserved!) usually called the "axial current"

$$j_\mu^{A,a} = \frac{i}{2}\,\mathrm{tr}\left(\partial_\mu H^\dagger\{\sigma^a, H\} - \partial_\mu H\{\sigma^a, H^\dagger\}\right)\ , \tag{11}$$

where $\{A, B\} = AB + BA$ denotes the anticommutator.

5. Show that to linear order in the fields $h(x), \varphi^a(x)$, the a-th axial current is simply

$$j_\mu^{A,a} \sim \langle H\rangle \partial_\mu \varphi^a\ , \tag{12}$$

and find the constant in front. Thus, when the quantum operator corresponding to (12) acts on the vacuum, it creates a quantum of the Goldstone boson (times the momentum and the "Goldstone boson decay constant" which is really equal to $\langle H\rangle$).

Show also that, to leading nontrivial order in the fields, the conserved vector current $j_\mu^{V,a}$ is quadratic in the fields φ^a.

In QCD, the relation (12) and the algebra of the currents $j_\mu^{V,A}$ constitute the basis of an approach to soft-pion physics (soft means low energy) known as "current algebra".

Here, we studied the Nambu-Goldstone mode. In the Wigner mode, when $m^2 > 0$, there are no massless particles, as is easy to convince yourselves.

IV. *Playing with the non-relativistic limit*

Consider a real scalar relativistic field theory of mass m with $\lambda\phi^4$ interaction. Let there be N particles of momenta labeled by $p_1, ...p_N$, whose energies are such that they are insufficient to create any new particles. Nevertheless, the particles can scatter and exchange momenta. In what follows you will study this *N-particle nonrelativistic limit* in some detail.

1. Write down the Hamiltonian of the field theory, including the interaction term, *restricted to the N-particle sector* of Hilbert space. (Use the creation and annihilation operator representation, i.e. write the result as sums of products of creation and annihilation operators of particles of various momenta.)

2. Does the resulting Hamiltonian preserve particle number? Is there an associated symmetry? What is the operator that generates it?

   In anticipation (or a hint) of the answer, notice that even though the interacting real scalar field theory has no continuous global symmetry that can give rise to a conserved "particle number", a particle number symmetry appears as an "accidental" symmetry in the nonrelativistic limit. (Of course, this never happens for massless fields, such as the electromagnetic field, as there is no such thing as a nonrelativistic limit there.)

3. Consider now the interaction term in your reduced (to the N-particle sector of Hilbert space) Hamiltonian. How does a typical interaction term (for given configurations of momenta) act on an N-particle state? What kinds of scattering processes does it describe?

4. What do you think is the potential, in x-space, that allows the various particles to scatter and exchange momentum? How would you describe the resulting nonrelativistic quantum system to friends who never took QFT but are well-versed in quantum mechanics?

`Hint`: For part 4, consider $N = 2$ first. Start with a two particle nonrelativistic quantum mechanics with Hamiltonian:

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1 - x_2) ,$$

where p_i, x_i are the operators of momentum and position of the $i-th$ particle (three vectors, arrows ommitted for brevity). Use as a basis the eigenstates of the free Hamiltonian, i.e. plane waves, $|\vec{p}_1, \vec{p}_2\rangle$, symmetric with respect to interchange of the momenta (even better, use the corresponding wavefunctions $\psi_{p_1,p_2}(x_1, x_2) = \langle x_1, x_2 | p_1, p_2\rangle$). Compute the matrix elements

$$\langle q_1, q_2 | H | p_1, p_2\rangle$$

in this basis. To compare to the nonrelativistic limit of the scalar field theory, compute the same matrix elements of the Hamiltonian you found in (1.) above, in the basis of states of the restricted ($N = 2$) Hilbert space $|p_1, p_2\rangle$. Are they similar to the matrix elements you found in the quantum mechanics problem for some choice of $V(x_1 - x_2)$? Explain the difference (if any). Then go on to answer (4.) for any N.