PHY2403F, Fall 2018, Quantum Field Theory 1, Homework # 4

`Due on Monday, December 3 in class.`

I. *The Wick theorem(s)*

1. **The mother of all Wick theorem(s):** Let $A_1, A_2, \ldots$ and B denote a set of either creation or annihilation operators. In other words, $A_i = a_{k_i}$ or $a^\dagger_{k_i}$ (as well as B; B is just like one of the A's, but we'll use the letter B to denote an operator which is singled out, as it is needed in the proof). Next, define a contraction $A_i A_k$ as follows:

$$O_1 \overbracket{A_i O_2 A_k} = O_1 O_2 \overbracket{A_i A_k}\ , \tag{1}$$

where O_1, O_2 are arbitrary strings of operators. The above equation signifies the fact that the "contraction" is a c-number, i.e. commutes with all operators. It is defined as follows:

$$\overbracket{A_i A_j} = \begin{cases} 0, \text{ if } \ A_i = a_{k_i}, A_j = a_{k_j} \ \text{ or } \ A_i = a^\dagger_{k_i}, A_j = a^\dagger_{k_j} \\ 0, \text{ if } \ A_i = a^\dagger_{k_i} \ \text{ and } \ A_j = a_{k_j} \\ (2\pi)^3 \delta^{(3)}(k_i - k_j), \text{ if } \ A_i = a_{k_i} \ \text{ and } \ A_j = a^\dagger_{k_j} \end{cases} \tag{2}$$

Put in words, the contraction vanishes if both A's are creation (or both are annihilation operators), as indicated in the first line in (2). The contraction is also zero if the operator to the right is an annihilation one, as per the second line in (2). Finally, the contraction is equal to the commutator of a_{k_i} with $a^\dagger_{k_j}$ in the case when the creation operator is to the left of the annihilation operator.

Finally, we use $: A \ldots B :$ to denote the expression where all annihilation operators appear to the right of all creation operators, i.e. the usual normal ordered expression. Then, Wick's theorem—as used in many body physics—is formulated as follows:

$$\begin{aligned} A_1 \ldots A_n \quad = \quad & : A_1 \ldots A_n : \\ & + : \overbracket{A_1 A_2} A_3 \ldots A_n : + \ldots + : \overbracket{A_1 \ldots A_{n-1}} A_n : + : \overbracket{A_1 \ldots A_n} : \\ & + : \overbracket{A_1 A_2} \overbracket{A_3 A_4} \ldots A_n : + \ldots \ . \end{aligned} \tag{3}$$

The first line contains the normal-ordered product of all operators without contractions, the second line—all possible terms with one contraction (not involving only A_1 of course, but all single-contraction terms, which would be painful to indicate), the third line has all possible two-contraction terms, etc.

Now, you will prove (3) in steps.

(a) Prove the following Lemma:

$$: A_1 A_2 \ldots A_n : B =: A_1 A_2 \ldots A_n B : + \sum_{1 \le k \le n} : A_1 \ldots \overbracket{A_k \ldots A_n B} : \tag{4}$$

Argue that if B is an annihilation operator, the Lemma is trivial. Thus, consider B to be a creation operator. Notice that if any of the $A_{1,\ldots,n}$ are creation operators, they can be taken to the left of the normal products in (4) (because all their contractions with B are zero). Thus, if the (4) is proven for arbitrary n for the case when all A_i's are annihilation operators, the general case is obtained by multiplying on the left with the desired number of creation operators. Thus, it suffices to prove the Lemma for the case when all A_i's are annihilation operators. Also notice Thus, after proving the Lemma for $n = 1$, use induction to show that it holds for any n. Assuming it holds for some number n, go to the case $n + 1$ by multiplying (4) by some annihilation operator A_0 on the left and show that the Lemma holds for $n + 1$ operators.

By the chain of logic described above, you have proven (4).

Notice also that the lemma (4) holds also if the product

$$: A_1 A_2 \ldots A_n :$$

is replaced by

$$: A_1 \overbracket{A_2 \ldots A_p} \ldots A_n :,$$

i.e. with the product of operators with an arbitrary number of contractions (one, as written above), with a trivial modification of the last term (since, obviously, you can not contract B with contractions).

(b) Now prove the actual Wick theorem (3). Assuming that it holds for $n = 2$. Imagine that (3) holds for n operators and prove that it holds for $n + 1$, using (4).

2. **An intermediate step:** Let now A_i and B be operators expressed as some linear combinations of creation and annihilation operators. In particular the subscripts i may now indicate spatial dependence, rather than momentum eigenvalues. Now, define the contraction as follows:

$$\overbracket{A_i A_j} = \langle 0 | A_i A_j | 0 \rangle \ , \tag{5}$$

where $|0\rangle$ is the Fock vacuum. Notice that (5) is equivalent to (2) when A_i's are either creation or annihilation operators. Argue that (3) holds verbatim.

3. **The time-ordered Wick theorem:** Use the above Wick theorem to prove the time-ordered version. Notice that, despite appearances, there is not much left to do. Now, we have space-time rather than momentum space arguments and the theorem is now formulated as follows:

$$\begin{aligned} T(A_1 \ldots A_n) \quad = \quad & : A_1 \ldots A_n : \\ & + : \overbracket{A_1 A_2} A_3 \ldots A_n : + \ldots + : \overbracket{A_1 \ldots A_{n-1}} A_n : + : \overbracket{A_1 \ldots A_n} : \\ & + : \overbracket{A_1 A_2} \overbracket{A_3 A_4} \ldots A_n : + \ldots \ , \end{aligned} \tag{6}$$

with the difference that A_i are fields (we are considering real scalar fields), $1 \ldots n$ denote space-time points, and the contraction is now the Feynman propagator, e.g. $D_F(x_1 - x_2)$, etc.

Notice that to prove (6) one can consider a particular time ordering. Then the T product becomes the normal product of operators (as they are assumed ordered). The space-time dependence can be taken out by Fourier transform which multiplies every term. Every operator is a sum of creation and annihilation operators. Their commutators are exactly the ones giving rise to the contraction in (2), on one hand, and to the function $D(x_i - x_j)$ after Fourier transform, on the other (recall that this function appears in the Feynman propagator). Convince yourselves, using (5), that this proves the theorem.

4. For extra bonus, generalize all theorems above to anti commuting fields.

II. *The "$h \to WW, ZZ$" Higgs-decay width.*

From the $SU(2)_L \times SU(2)_R$ model of Homework 2—really, the Higgs Lagrangian of the Standard Model, find the coupling of the h-particle (the Higgs boson) to the ϕ^a particles (these are now Goldstone bosons, in the electroweak theory, they become the longitudinal components of the W and Z particles). Canonically normalizing h and ϕ^a, this coupling has the form

$$const.\ h\ \partial_\mu \phi^a \partial^\mu \phi^a\ . \tag{7}$$

1. Determine the value of *const.* for canonically normalized h and ϕ^a.

2. Use this coupling to compute the width $\Gamma(h \to \phi^3 \phi^3)$ of the Higgs particle to decay to two longitudinal (say) Z-bosons (hence the index 3).

3. Plug in some numbers. Use the fact that the vacuum expectation value $|m|/\sqrt{\lambda} = 246$ GeV and the fact that $m_h = 125$GeV to get a number for the lifetime. Compare to the total width of the Higgs from `http://pdg.lbl.gov/2012/reviews/rpp2012-rev-higgs-boson.pdf`, see figure 5 there, as well to the partial width to WW given in Figure 4 there.

4. At the same time, determine the values of $|m|$ and λ separately. Is $\lambda \ll 1$ (i.e. perturbative)?

Notice that this calculation would have been physically relevant had the Higgs been heavy, $m_h \gg m_W \sim 100$ GeV. This is because the $h \to WW$ decay then is dominated (in this limit) by the decay into the longitudinal component, which is really the Goldstone boson field ϕ^a (in this limit, the result is independent of the gauge couplings $g_{1,2}$ of the Standard Model). Nonetheless, having some real numbers in this class is good.

III. *The Goldstone boson scattering cross-section, its growth with $E_{c.m.}$, and the Higgs*

This problem has:

- A great historical significance, for giving an argument in favor of the existence of a Higgs particle. The strongest argument for the Higgs particle's existence was that it was required—within the weakly coupled scenario of electroweak symmetry breaking—to tame the growth of the WW scattering amplitude and restore unitarity of the electroweak theory. Unitarity is a sacred thing and we don't want to easily give it up.

- A great future significance: measurements of WW scattering at the LHC (and future colliders) will test the Higgs model precisely, in particular the hypothesis that the Higgs particle that was found last year *completely* restores unitarity and there is no other state required. Current measurements of WW scattering at the LHC are not just not complete, they are nonexistent (and are very difficult, I am told), hence the question of whether "the Higgs is *the* Higgs" is still open.

Now, to the concrete stuff:

1. You will calculate the scattering amplitude of Goldstone boson quanta via Higgs exchange, due to the coupling you found in Eq. (1) of Problem 2. To be definite, study the amplitude $\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)$ (I am being very nice here, as I let you only look at the s-channel process!).

   For energies of the ϕ^a quanta greater than the mass of the W and Z bosons (roughly 100 GeV), this scattering amplitude via h-exchange can be shown [you got to believe me here] to be the same as the scattering of *longitudinal* W, Z-bosons.

   Show that
   $$\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)|_{h-exchange} = const. \frac{s^2}{v^2(s-m_h^2)} \ , \tag{8}$$
   where s is the appropriate Mandelstam variable (the square of the c.m. energy), m_h is the mass of h, $v = |m|/\sqrt{\lambda}$, and you will determine the constant. What you found is that the scattering amplitude (8) grows with the c.m. energy, without bound. It should intuitively clear that this may violate unitarity by leading to probabilities greater then unity at sufficiently high energies.[1]

2. Now, the interesting thing about the Higgs model is that the growth of (8) with center of mass energy is actually cancelled by the same amplitude, but now due to the direct coupling between ϕ^a quanta. To find these interactions, go to Eq. (9) of Homework 2 and study the coupling of ϕ^a: substitute $H(x)$ of Eq. (9) into Eq. (5) and find the coupling between four ϕ-quanta that gives the leading contribution to the $\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)|_{local\ \phi-interaction}$ scattering amplitude. Show that it has the form:
   $$const\ \phi^c\phi^d\partial_\mu\phi^a\partial^\mu\phi^b\ \mathrm{Tr}\left(\sigma^c\sigma^d\sigma^a\sigma^b\right) \ , \tag{9}$$
   and determine the constant.

[1] Showing this more precisely—and putting bounds on the mass on the Higgs from unitarity—requires study of partial wave decomposition (which is also widely used in quantum mechanics; while the idea is the same, it gets technically a bit more messy in QFT), which is left for future studies.

3. Finally, use (9) to calculate $\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)|_{local\ \phi-interaction}$ and show that, when added to $\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)|_{h-exchange}$, the various constants combine such that the amplitude $\mathcal{M}(\phi^1\phi^1 \to \phi^3\phi^3)|_{h-exchange+local\ \phi-interaction}$ does not grow with the center of mass energy. Hence, in the Higgs model of Homework 2 unitary (as expected) rules.

The discovery of the Higgs—expected from such theoretical arguments—is a strong evidence in favor of the recent statement:

`"Quantum field theory is how the world works." -Ed Witten` (NYT, August 12 2013)

IV. *Lorentz transforms of spinors—some useful identities*

Consider the matrix

$$\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \ .$$

Here, $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ is as defined in class, in terms of the four γ-matrices (notice that, when using the representation of the γ matrices in terms of Pauli matrices, the matrix $\Lambda_{\frac{1}{2}}$ looks like two sets of M (and M^*) matrices discussed in class, now combined into one four-by-four object).

1. Show that $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_\nu\gamma^\nu$, where Λ^μ_ν is the usual Lorentz transformation acting on vectors. (Feel free to show this for the infinitesimal form of the transformations, but then argue that the finite form holds as well.)

2. Show that $\Lambda_{\frac{1}{2}}^\dagger\gamma^0\Lambda_{\frac{1}{2}} = \gamma^0$.

3. Consider the fermion bilinear $\bar{\psi}\gamma^\mu\gamma^\nu\psi = \frac{1}{2}\bar{\psi}\{\gamma^\mu, \gamma^\nu\}\psi + \frac{1}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$, where $\{A, B\} = AB + BA$ is the anticommutator. Show that the two terms on the right transform as a scalar and a second-rank tensor, respectively, under Lorentz transformations.