PHY2403F, Fall 2018, Quantum Field Theory 1, Homework # 3:

`Due [tentatively] on Wednesday, November 21 in class.`

I. *Interaction energy between static external charges*

1. Calculate the vacuum expectation value of the time ordered exponential
$$\langle 0 | T e^{\, i \int d^4x \; g \; j(x) \; \phi(x)} | 0 \rangle \tag{1}$$
for the case of a massive free real scalar field. Here, g is a coupling constant, which we shall call the "Yukawa coupling". Show, e.g. using Wick's theorem, that the answer is
$$e^{\, -\frac{g^2}{2} \int d^4x d^4y \; j(x) D_F(x-y) j(y)} \;, \tag{2}$$
which is really the exponential of the second order term and D_F is the Feynman propagator.

2. Consider the case where $j(t, \vec{x}) = \theta(T-t)\theta(T+t)\left(\delta^{(3)}(\vec{x}) - \delta^{(3)}(\vec{x} - \vec{R})\right)$. This source term represents two external opposite "charges"[1] a distance $R = |\vec{R}|$ apart, created at $t = -T$ and existing for time $2T$. Show that, in the limit $T \gg R \gg 1/m$, Eq. (2) *is proportional to*:
$$e^{-i2TV(R)}, \tag{3}$$
where $V(R)$ is the Yukawa potential.

   Hint: Recall that $\lim_{T\to\infty} \int_{-T}^{T} dx e^{ipx} = 2\pi\delta(p)$ as well as the usual relation $(2\pi\delta(p))^2 = 2\pi\delta(p) 2T$.

   The result (3) means that "*two static sources of scalar field a distance R apart interact via the Yukawa potential.*" This is because (3) is the evolution operator (it is $\sim e^{-iHt}$, for $t = 2T$) of the field theory in the presence of the static external sources (or, more appropriately, (3) is the contribution to the evolution operator that has to do with the interaction between the sources induced by the field). Thus, it is natural to call the quantity multiplying $-i2T$ and depending on R, the interaction potential $V(R)$ between the sources.

   Do opposite-sign "charges" attract or repel? How about same-sign?

   Notice that when the "charges" are also considered as part of a QFT and, therefore, $j(x)$ in (1) is replaced by an appropriate QFT expression, one finds more interesting results. Namely, the Yukawa interaction between two fermions is always attractive—whether it is between two particles, two anti-particles, or between a particle and an anti-particle. The way to establish this, as well an alternative derivation of the expression for $V(R)$ you found in (3), is to start from the scattering of (anti)fermions via scalar exchange and then take the nonrelativistic limit. A comparison with quantum-mechanical Born scattering yields then an expression for $V(R)$.

   This result quoted above is of great interest in nuclear physics, where single-pion exchange operates via $V(R)$, and turns out to be attractive between nucleons and between nucleons and anti-nucleons.

[1] In other words, classical particles linearly coupled to ϕ (if ϕ was the electrostatic potential A^0, this would really be the electromagnetic charge.) For a discussion of whether an interaction like you will study can arise from a realistic QFT, see comment in 2. below.

3. What do you think is the significance of the various limits $T \gg R \gg 1/m$? Also, what is the meaning of the factors you omitted upon going from (2) to (3)?

II. *The action of an external source perturbation and particle creation*

In class, the problem of creation of particles by an external source in quantum mechanics was discussed. Let us now study this using QFT and Feynman diagrams. Consider a massive scalar free field interacting with a classical source $j(x)$ via:

$$H = H_0 + \int d^3x(-j(x)\phi(x)) \ . \tag{4}$$

The classical source $j(x)$ is nonzero only for a finite amount of time, i.e. it is turned on and off, is assumed localized in space, and thus has a well-defined four-dimensional Fourier transform (thus the source is not itself a generalized function).

1. Argue—e.g. using our expressions for overlap of $|0\rangle$ and $|\Omega\rangle$ from class, as well as their meaning—that the probability that the source $j(x)$ creates no particles is

$$P(0) = \left| \langle 0|T \left\{ e^{i\int d^4x j(x)\phi_I(x)} \right\} |0\rangle \right|^2 \ . \tag{5}$$

2. Find the order-j^2 term in $P(0)$ and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$, where

$$\lambda = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} |\tilde{j}(p)|^2 \ , \text{ where } \ \tilde{j}(p) \equiv \int d^4y e^{ip\cdot y} j(y) \ . \tag{6}$$

3. Represent the term computed above as a Feynman diagram. Now represent the entire series for $P(0)$ in terms of Feynman diagrams. Show that the series exponentiates and, therefore, $P(0) = e^{-\lambda}$.

4. Find the probability that the source creates one particle of momentum $\vec{k}$. First, compute this to order j and then to all orders, using the trick above to sum the series.

5. Show that the probability of producing n particles is $P(n) = \frac{1}{n!}\lambda^n e^{-\lambda}$, the Poisson distribution.

6. Show that $\sum\limits_{n=0}^{\infty} P(n) = 1$ and that $\langle N \rangle = \sum\limits_{n=0}^{\infty} nP(n) = \lambda$, where λ is given in (6). Notice that the expression for the mean particle number $\langle N \rangle$ created exactly reproduces (when dimensionally reduced to $d = 1$) the one from quantum mechanics given in class. Finally, compute the mean square fluctuation $\langle (N - \langle N \rangle)^2 \rangle$.

III. *"Radiation" by accelerated source and "IR catastrophe"*

This is a baby problem having to do with radiation of scalar particles. (As we will not have too much time to study the radiation of electromagnetic fields this term, it is a good opportunity.) Consider the coupling of a classical particle to a scalar field (remember Homework 1, Problem 1, where a similar coupling to the electromagnetic field was considered):

$$S_{int} = e \int_{worldline} ds \phi(x(s)) \ , \tag{7}$$

where $x(s)$ is the worldline of the particle and e is its scalar charge (what is its dimension?). The coupling (7) corresponds to a "current" $j(x)$ coupling to ϕ as in Problem **II.** above:

$$S_{int} = e \int_{worldline} ds \phi(x(s) = \int d^4x j(x) \phi(x) \ , \quad \text{where} \ \ j(x) = e \int_{worldline} ds \delta^{(4)}(x - x(s)) \ , \tag{8}$$

is the current.

1. Consider a particle of mass M, whose worldline is given by:

$$x^\mu(s) = \frac{p_i^\mu}{M} s, \ \text{for} \ \ s < 0 \ \text{and} \ \ x^\mu(s) = \frac{p_f^\mu}{M} s, \ \text{for} \ \ s > 0 \ , \tag{9}$$

   where p_i^μ and p_f^μ are the initial and final four-momenta of the particle (both obeying $p^\mu p_\mu = M^2$, with $p^0 > 0$, of course). The physical meaning of this trajectory is that the particle undergoes acceleration at $x^0 = 0$, suddenly changing its four-momentum from p_i to p_f. Show that the Fourier transform of the current, as defined in (6) above, is given by:

$$\tilde{j}(p) = \frac{ieM}{p \cdot p_f} - \frac{ieM}{p \cdot p_i} \tag{10}$$

   To make the TA's life (and yours) easier, in getting (10), consider without loss of generality, trajectories with $p_i = (M, 0, 0, 0)$ and $p_f = (\sqrt{M^2 + q^2}, q, 0, 0)$. [2]

2. Now study the expression for the average number of particles produced, λ, or $\langle N \rangle$, of Eq. (6), as well as the average energy $\langle E \rangle$, which you can easily come up with, from (6). From now on, consider the case where the mass of the produced particles (ϕ-quanta) is zero. This has two advantages: simplifications in the various formulae as well as giving us the feeling that we are actually looking at something close to radiation of photons.

   Show that the integrals over the momenta of the emitted "photons" in $\langle N \rangle$ and $\langle E \rangle$ diverge at large p.

[2]Recall the "half-delta function" integrals from Homework 2, Problem 1 and ignore the $i\epsilon$ factors which should be present in the denominators in (10) as they will not be important for what follows.

This is because our trajectory has a sudden change of momentum at $s = 0$. We expect that the formulae for the radiated "photons" is still valid for sufficiently small momenta where the nature of the kink is not relevant (presumably for momenta less than the inverse time during which a smooth change of momenta occurs, i.e. momenta smaller than the reciprocal of the scattering time). Thus, we now suppose there is a high-momentum cutoff.

Let us then study the convergence of the small-p integrals over the momenta of the emitted particles in $\langle N \rangle$ and $\langle E \rangle$. This counts the number or energy of "soft" photons emitted. Show what while $\langle E \rangle$ is finite, the expression for $\langle N \rangle$ diverges for small $\vec{p}$.

This divergence in the number of soft photons radiated by a classical source is called the "infrared catastrophe", in the case of QED. A similar answer is obtained using a tree-level QFT calculation of the radiation of soft photons. Note one interesting fact: the divergence of the integral determining $\langle N \rangle$ is logarithmic: $\langle N \rangle \sim e^2 \log \frac{k_{max}}{k_{min}}$, where the IR cutoff k_{min} is introduced to make the integral finite. You see now that e^2 (really, the fine structure constant $\alpha \sim 1/137$, in QED) is multiplied by a large log, which can be bigger than 137. This is a first indication that perturbation theory breaks down and some resummation of diagrams may be needed. Indeed, in QED, the infrared divergence is cancelled after adding "loop" effects, see Section 6.5 of Peskin and Schroeder.

The point of this problem was to illustrate two things. First, it shows (within this classical calculation of the overlap between free and interacting vacua) how the two vacua can be orthogonal (in the case of massless ϕ, due to infrared (small momenta) problems). Second, it points toward something—the infrared divergencies in QED, and the resulting "Sudakov double logs"—that you will study later, either in QFT2 or by yourselves.

IV. *Where is the particle?*

In class, we did mention that, by analogy with non relativistic quantum mechanics, the state $\hat{\phi}(\vec{x}, t = 0)|0\rangle$ allows us to say something along the lines that *"the operator $\hat{\phi}(\vec{x})_+$ creates a particle at $\vec{x}$"*. These words are based on noticing that in QM, we have

$$|\vec{x}\rangle \sim \sum_{\vec{p}} e^{i\vec{p}\vec{x}} |\vec{p}\rangle,$$

where $|\vec{x}\rangle$ is an eigenstate of the position operator with eigenvalue $\vec{x}$ and $\vec{p}$ is, likewise, an eigenstate of momentum. On the other hand, in free massive scalar theory, the state $\hat{\phi}(\vec{x}, t = 0)|0\rangle$ can be similarly expressed as

$$\hat{\phi}(\vec{x}, t = 0)|0\rangle = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} e^{-i\vec{p}\vec{x}} \hat{a}^\dagger_{\vec{p}} |0\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}} e^{-i\vec{p}\vec{x}} |p\rangle,$$

where $|p\rangle$ is the relativistically normalized momentum eigenstate. Comparing the above two equations, reading from left to right, we are compelled to utter the words quoted in the beginning.

Accepting this interpretation literally, we are next faced with explaining the following. Consider the state $|\vec{0},0\rangle = \hat{\phi}(\vec{0},t=0)|0\rangle$, interpreted (as per the above dissussion) as a particle created at $\vec{x}=0$ at $t=0$. Similarly, the state

$$|\vec{y},t\rangle = \hat{\phi}(\vec{y},t)|0\rangle$$

is that of a particle at $\vec{y}$ at t. Notice that these are free fields so their time evolution is trivial. Then, by the usual Born rule of quantum mechanics (which we accept in QFT), the inner product

$$\langle\vec{y},t|\vec{0},0\rangle$$

would be "*the amplitude that the particle created at* $\vec{0}$ *at* $t=0$ *is found at* $\vec{y}$ *at* t". Notice that this is exactly the kind of answer that the quantum-mechanical propagator, often denoted precisely by $\langle\vec{y},t|\vec{0},0\rangle$, gives. A problem with this arises when one realizes that

$$\langle\vec{y},t|\vec{0},0\rangle = \langle 0|\hat{\phi}(\vec{y},t)\hat{\phi}(\vec{0},0)|0\rangle = D(\vec{y},t) \neq 0 \text{ for } (\vec{y},t) \sim (\vec{0},0) \ .$$

In other words, this amplitude is nonzero for spacelike separations (as you explicitly showed in Homework 2, Problem 1, Part 2). The point of the simple exercise below is to argue that the above interpretation of this amplitude should be taken with a grain of salt, i.e. not too literally, as far as relativity is concerned, of course.

The question we will ask is: to what extent is this particle at $\vec{x}=0$ localized? In quantum mechanics, we answer this question by pointing out that for an eigenstate of $\hat{x}$, whose wave function is $\delta(x-x')$, the probability to find the particle anywhere but at $x=x'$ is zero. Trying to pursue this in QFT, a conundrum that arises is that we do not have wave functions for particles. Recall that we have wave functionals, which determine the probability that *the field* has this or that value. The coordinate, on the other hand, is an argument, not an operator (hence "observable") in the theory—just like time in QM, which is also not an operator; after all we said "QM=QFT in $d=1$". The best we can do is to consider the state $|\vec{y},0\rangle$ and ask where its properties identifiable in QFT—energy or momentum—are localized.

Thus, consider the expectation value of $\hat{T}_{00}(\vec{x},t)$ (assumed normal-ordered) in this state:

$$\rho(\vec{y},\vec{x},t) \equiv \langle\vec{y},0|T_{00}(\vec{x},t)|\vec{y},0\rangle \ .$$

From the Born rule, the natural interpretation of the above quantity is the value of the energy density of the state $|\vec{y},0\rangle$ observed at $(\vec{x},t)$—spacelike or not w.r.t. $(\vec{y},0)$.

1. Show, using the translation operator, that $\rho(\vec{y},\vec{x},t) = \tilde{\rho}(0,\vec{x}-\vec{y},t) \equiv \tilde{\rho}(\vec{x}-\vec{y},t)$, where the last equality defines the new energy density $\tilde{\rho}(\vec{x},t)$.

2. Using Wick's theorem—really, a baby-version thereof—express $\tilde{\rho}(\vec{x},t)$ in terms of $D(\vec{x},t)$ and its derivatives.

3. Using the knowledge acquired from Homework 2, study how well is the particle's energy localized, already at $t=0$.

Are you surprised by the result? Are you comforted?

We didn't have time, apart from Problem 4 of Homework 2, to dwell much on the nonrelativisic limit. This limit can be achieved by forgetting the antiparticles and then defining non-relativistic fields. This is very well described in either Tong's or Luke's notes. For those of you studying cold atoms, it is definitely a must-read!

My final comment is that the most concise formulation of causality that goes beyond simply stating that the commutators vanish for spacelike separations is the one first due to Stueckelberg (1940's) and then finessed by Bogoljubov (1950's).

They consider the expectation value of an operator $\hat{O}(x)$ in a state prepared by the action of an operator $U[g]|0\rangle$. $U[g]$ is an evolution operator (see below) which is a functional of some classical fields $g(y)$ used to prepare the state of the field (e.g. external e.m. fields using to focus, accelerate, etc., the particles; $g(y)$ could also be used to turn on and off the interactions in different space time regions). Thus the object of study is:

$$\langle \hat{O}(x) \rangle = \langle 0 | U^\dagger[g] \hat{O}(x) U[g] | 0 \rangle \ .$$

The causality condition, then, is that

$$\frac{\delta \langle \hat{O}(x) \rangle}{\delta g[y]} = 0 \text{ for } x \sim y \ .$$

Now, recalling the form of the evolution operator, $U[g] = Te^{i \int dt d^3x L_I(t, \vec{x}, g(\vec{x}, t))}$, and the Baker-Campbell-Hausdorf formula, it should be clear how the vanishing of the commutators outside the light cone becomes relevant for the above condition to hold. For Bogoljubov, the vanishing commutators are a *consequence* of the causality condition given in terms of variational derivatives, as expressed above; he derives the S-matrix expansion from that requirement along with a few others (locality and Lorentz invariance, basically).

The reason to include this comment was to close the loop on something that I mentioned in class, now that we've seen what $U[g]$ may look like.