Fast high-order solvers on simplices for the de Rham complex

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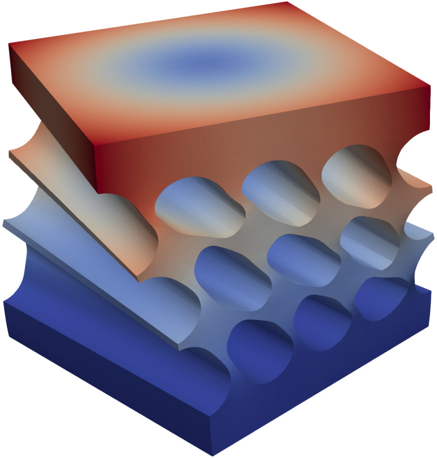

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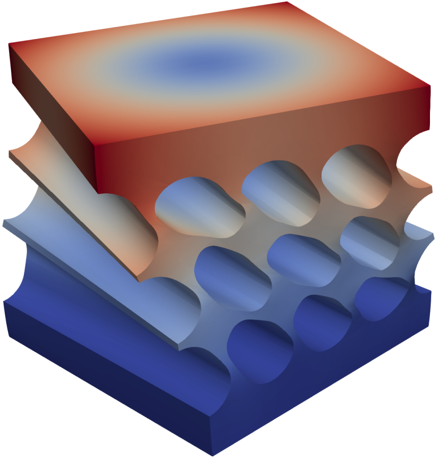
- ✓ rapid convergence,
- ✓ high arithmetic intensity.



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- ✗ require high-order meshes,
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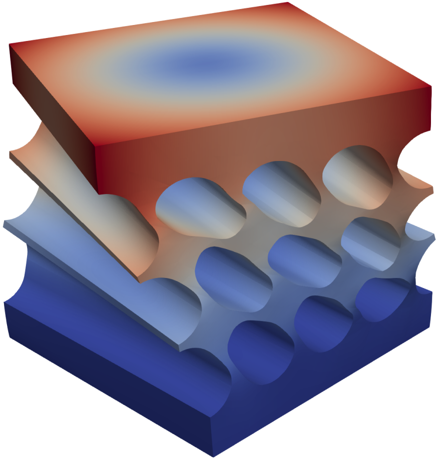

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This talk . . .

. . . is about fast solvers for high-order discretisations of some canonical PDEs on simplices.

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$$\begin{aligned}
&\text{find } u \in H(\mathrm{grad}) : (\beta u, v) + (\alpha \,\mathrm{grad}\, u, \mathrm{grad}\, v) = (f, v) \quad \forall\, v \in H(\mathrm{grad}),\\
&\text{find } \mathbf{u} \in H(\mathrm{curl}) : (\beta \mathbf{u}, \mathbf{v}) + (\alpha \,\mathrm{curl}\, \mathbf{u}, \mathrm{curl}\, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall\, \mathbf{v} \in H(\mathrm{curl}),\\
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R. Hiptmair. “Operator Preconditioning”. In: *Computers & Mathematics with Applications* 52.5 (2006), pp. 699–706.

K.-A. Mardal and R. Winther. “Preconditioning discretizations of systems of partial differential equations”. In: *Numerical Linear Algebra with Applications* 18.1 (2011), pp. 1–40.

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Co-designed basis + solver with $\mathcal{O}(p^6)$ flops and $\mathcal{O}(p^4)$ storage (vs $\mathcal{O}(p^9)$ and $\mathcal{O}(p^6)$).

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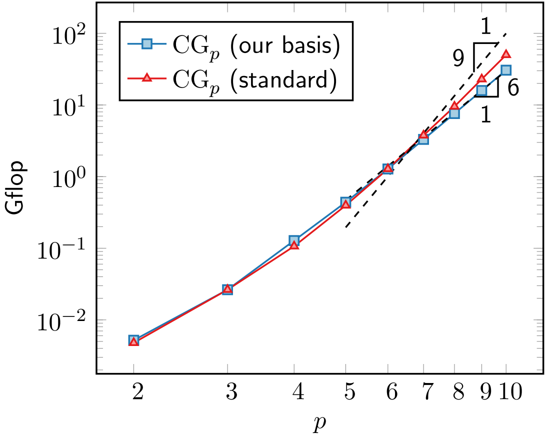
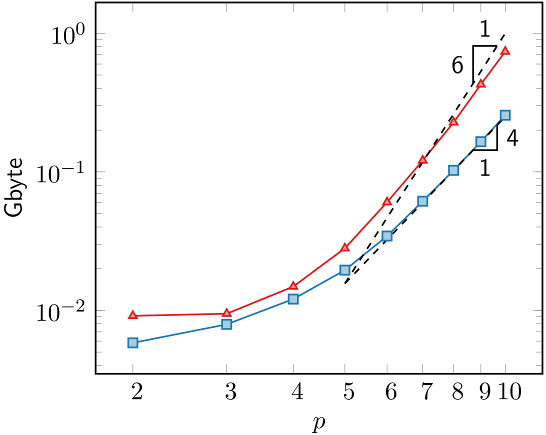


Flop count, $H(\mathrm{grad})$



Peak memory, $H(\mathrm{grad})$

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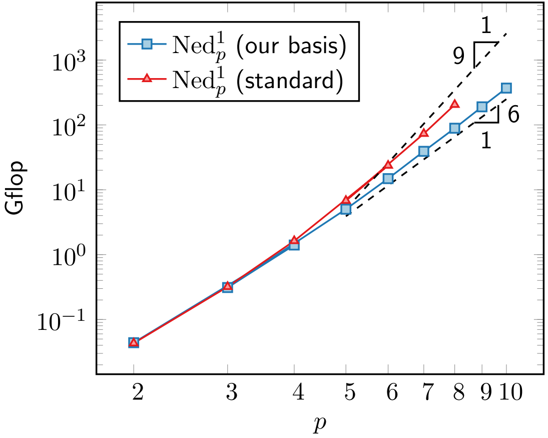

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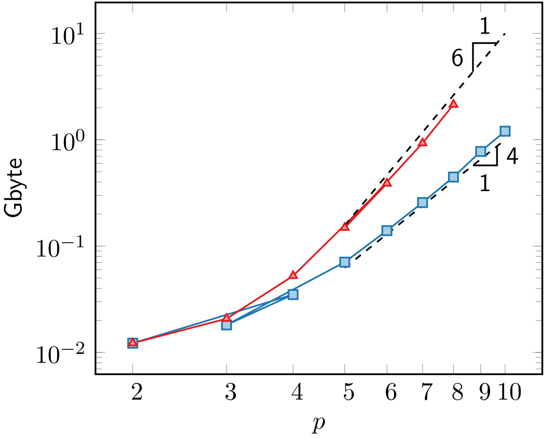

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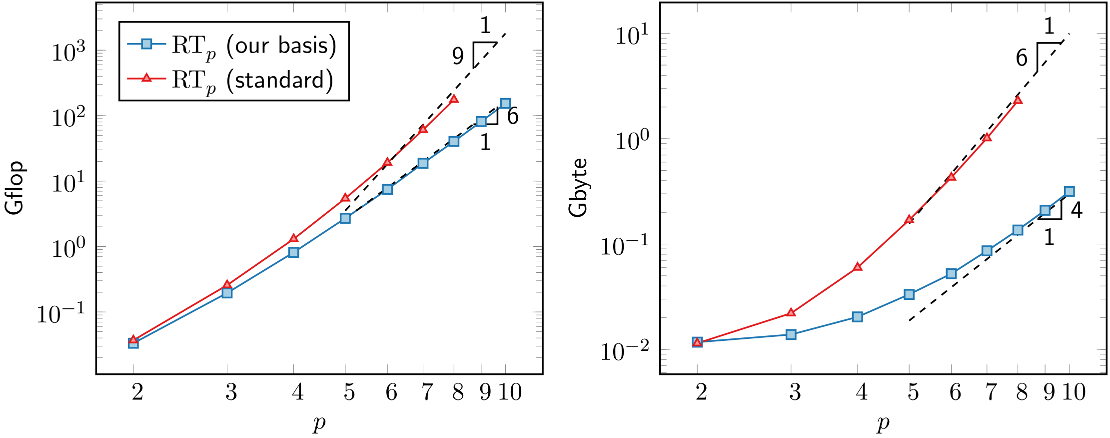

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Offline setup costs

involve solving $\mathcal{O}(1)$ small eigenproblems, with complexities $\mathcal{O}(p^9)$ time and $\mathcal{O}(p^6)$ storage.

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Open question

Can we achieve $\mathcal{O}(p^4)$ flops and $\mathcal{O}(p^3)$ storage for solver + matvecs on simplices?

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Section 2

The key idea

All multigrid/domain decomposition solvers must do some relaxation on fine grids.

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Star patches give effective and p-robust relaxation methods.

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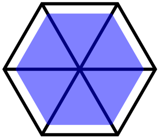

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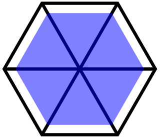

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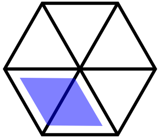

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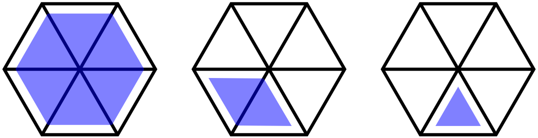

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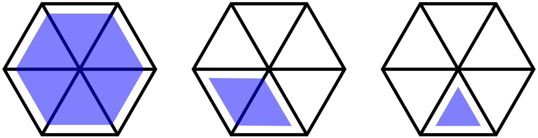

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Let's see some examples of how star patches are useful in solvers.

For the Poisson problem in $H(\mathrm{grad})$, the space decomposition

$$X_p^0 = X_1^0 + \sum_{v \in \Delta_0(\mathcal{T}_h)} X_p^0\big|_{\star v}$$

is p-robust: only $\mathcal{O}(1)$ conjugate gradient iterations.

Here $\Delta_k(\mathcal{T}_h)$ is the set of k-entities in the mesh $\mathcal{T}_h$.

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For the $H(\mathrm{curl})$ ($k = 1$) and $H(\mathrm{div})$ ($k = 2$) Riesz maps, the space decomposition

$$X_p^k = X_1^k + \sum_{J \in \Delta_{k-1}(\mathcal{T}_h)} \mathrm{d}^{k-1} X_p^{k-1}\Big|_{\star J} + \sum_{L \in \Delta_k(\mathcal{T}_h)} X_p^k\Big|_{\star L}$$

is p-robust (but the proof is an open problem).

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Our response

Solve lower-dimensional patch problems: $\mathcal{O}(p^{3(d-1)})$ flops and $\mathcal{O}(p^{2(d-1)})$ storage.

Complexities for some p-robust high-order solvers in three dimensions

	Year	Cell	Problem	Time	Storage
▶ Pavarino & Widlund	1993		$H(\mathrm{grad})$	p^9	p^6
Arnold, Falk & Winther	2000		all	p^9	p^6
Beuchler & Pillwein	2007		$H(\mathrm{grad})$	-	-
Schöberl et al.	2008		$H(\mathrm{grad})$	p^9	p^6
Farrell & Brubeck	2024		all	p^4	p^3
This work	2025		all	p^6	p^4

Pavarino & Widlund

proved that two-level p-multigrid with *star patches* on quads and hexes is p-robust.

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Arnold, Falk & Winther

proved that *star patches* are good h-multigrid relaxation for $H(\mathrm{curl})$ and $H(\mathrm{div})$.

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Beuchler & Pillwein

proposed a basis in which cell interior problems can be solved in $\mathcal{O}(p^6)$ time.

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Schöberl, Melenk, Pechstein, & Zaglmayr

proved the Pavarino–Widlund result (star patches are p-robust) on simplices.

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Farrell & Brubeck

propose tensor product bases with the same complexities as matrix-vector products.

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This work

much harder than hexes: suboptimal complexity, but much better than a naïve approach!

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Interior decoupling on a cell

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where C is p-robust.

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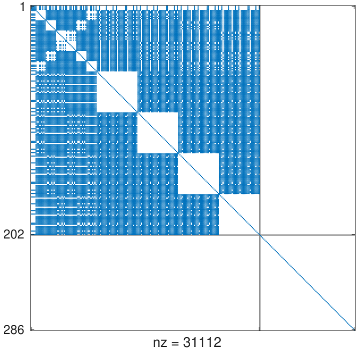
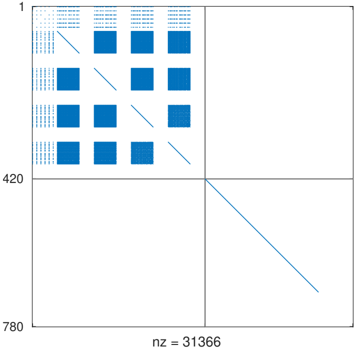
So we pay the dense linear algebra costs for a problem *one dimension lower*, on Γ.

Sparsity patterns for $(\mathrm{d}u, \mathrm{d}v)_K$ on reference cell, $p = 10$

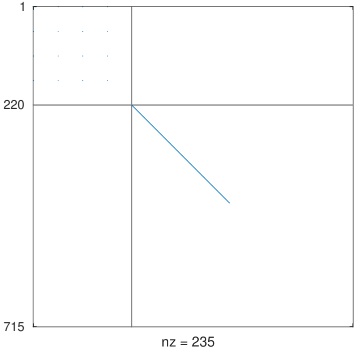
Lagrange, $\mathrm{d} = \mathrm{grad}$



Nédélec, $\mathrm{d} = \mathrm{curl}$



Raviart–Thomas, $\mathrm{d} = \mathrm{div}$



Section 3

First numerical example

A p-robust solver for the Riesz maps:

- Conjugate gradients
  - Two-level vertex-star additive Schwarz
    - p-fine: Interior-interface additive fieldsplit
      - Interior: point-Jacobi
      - Interface: dense linear algebra
    - p-coarse ($p = 1$): Cholesky factorisation

CG iteration counts (rel. tol. $= 10^{-8}$).

d	p	$H(\mathrm{grad})$	$H(\mathrm{curl})$	$H(\mathrm{div})$
2	4	20		24
	7	20		23
	10	20		23
	14	20		23
3	4	23	39	19
	7	23	42	19
	10	23	42	19

Additive space decompositions on the interface Schur complement

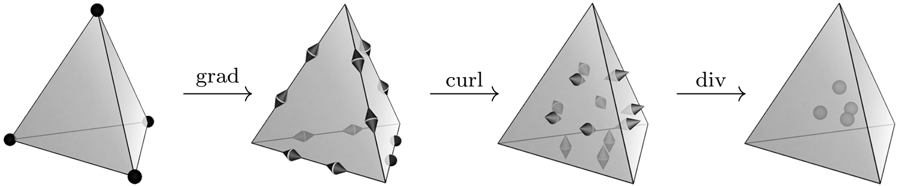
- Vertex-star + Lowest-order (à la Schöberl et al.)
- Edge-star + Vertex-star on $\mathrm{grad}\, H(\mathrm{grad})$ + Lowest-order (à la Hiptmair)
- Edge-star + Lowest-order (à la Arnold–Falk–Winther)

Section 4

Construction of the basis

We're going to build *different bases* for the *usual spaces*

$$\begin{tikzcd}
H(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d] & L^2 \arrow[d] \\
\mathrm{CG}_p \arrow[r, "\mathrm{grad}"] \arrow[d] & \mathrm{Ned}_p^2 \arrow[r, "\mathrm{curl}"] \arrow[d] & \mathrm{BDM}_p \arrow[r, "\mathrm{div}"] \arrow[d] & \mathrm{DG}_p \arrow[d] \\
{} & {} & {} & {}
\end{tikzcd}$$



and their first-kind brothers.

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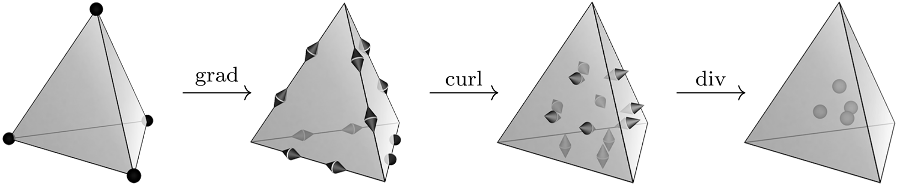
$$\begin{tikzcd}
H(\mathrm{grad}) \arrow[r, "\mathrm{grad}"] \arrow[d] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d] & L^2 \arrow[d] \\
\mathrm{CG}_p \arrow[r, "\mathrm{grad}"] & \mathrm{Ned}_p^2 \arrow[r, "\mathrm{curl}"] & \mathrm{BDM}_p \arrow[r, "\mathrm{div}"] & \mathrm{DG}_p
\end{tikzcd}$$



and their first-kind brothers.

We do this by *choosing different degrees of freedom*.

Definition (Finite element)

A finite element is a triple $(K, \mathcal{V}, \mathcal{L})$ where

- The cell K is a bounded, closed subset of $\mathbb{R}^d$ with nonempty connected interior and piecewise smooth boundary;

Philippe Ciarlet

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Philippe Ciarlet

Intuition

The degrees of freedom are what you store to uniquely specify a function on the cell.

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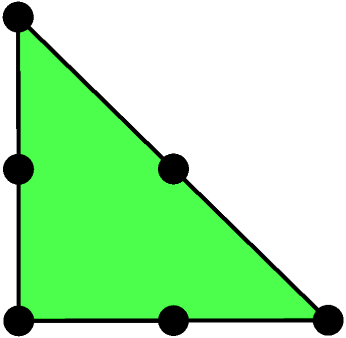
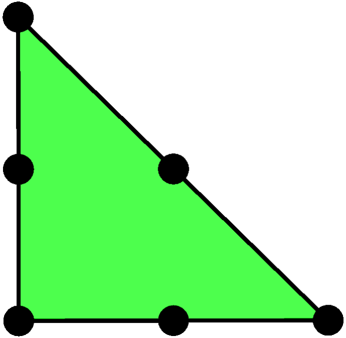
Basis functions are biorthogonal

The basis functions $\{\phi_j\}$ are constructed to satisfy $\ell_i(\phi_j) = \delta_{ij}$.



The quadratic Lagrange finite element CG_2 in two dimensions.



The quadratic Lagrange finite element CG_2 in two dimensions.

Example (CG_2 in 2D)

$K = \triangle$, $\mathcal{V} = \mathrm{span}(1, x, y, x^2, y^2, xy)$, $\mathcal{L} = \{\ell_1, \ldots \ell_6\}$.

Each ℓ_i evaluates the function at a vertex or edge midpoint.

Subsection 1

Degrees of freedom for $H(\mathrm{grad})$

Usually CG_p just has point evaluations as degrees of freedom.

Leszek Demkowicz

Peter Monk

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Demkowicz et al. proposed different degrees of freedom for the whole de Rham complex that induce discrete commuting de Rham complexes on each entity. Each entity can have a different polynomial degree.

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The degrees of freedom are pointwise evaluation at vertices

$$\ell_V : v \mapsto v(V) \qquad \forall\, V \in \Delta_0(\triangle),$$

and moments on each higher-dimensional entity

$$\ell_S : v \mapsto (\mathrm{grad}_S\, q, \mathrm{grad}_S\, v)_S \quad \forall\, S \in \bigcup_{k=1}^{d} \Delta_k(\triangle)\, \forall\, q \in \mathbb{P}_{p,0}(S),$$

where $\mathbb{P}_{p,0}(S)$ is the *bubble space* on S

$$\mathbb{P}_{p,0}(S) := \{v \in \mathbb{P}_p(S) : v = 0 \text{ on } \partial S\}.$$

Peter Monk

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We choose the basis for $\mathbb{P}_{p,0}(S)$ such that

$$(\mathrm{grad}_S\,\phi_j^S, \mathrm{grad}_S\,\phi_i^S)_S = \delta_{ij}, \quad (\phi_j^S, \phi_i^S)_S = \lambda_j \delta_{ij}.$$

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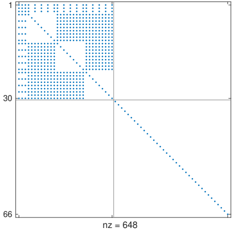
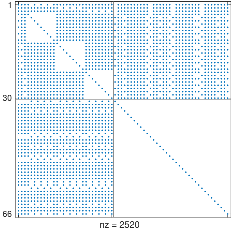
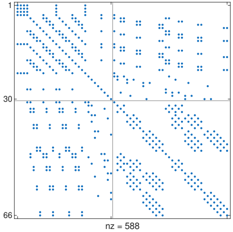
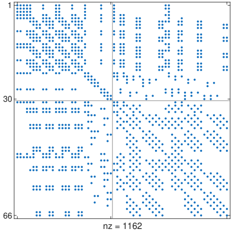
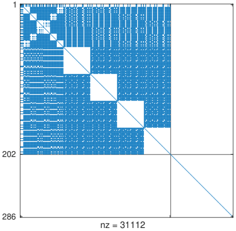
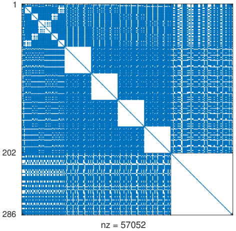
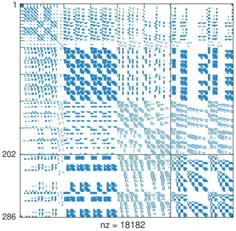
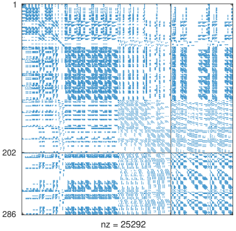
This requires one small dense offline eigensolve for each type of entity S ($\mathcal{O}(p^9)$ offline cost).

	stiffness	mass
our basis		
hierarchical		

$$H(\mathrm{grad}),\ d = 2,\ p = 10$$

	stiffness	mass
our basis		
hierarchical		

$$H(\mathrm{grad}),\ d = 3,\ p = 10$$

Subsection 2

Degrees of freedom for $H(\mathrm{curl})$ and $H(\mathrm{div})$

For $H(\mathrm{curl})$, our eigenidea with Demkowicz et al. is to employ tangential moments on edges

$$\ell_E : v \mapsto (q, v \cdot t)_E \qquad \forall\, E \in \Delta_1(\triangle)\, \forall\, q \in \mathbb{P}_p(E),$$

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and two kinds of moments on entities of higher dimension

$$\ell_{S,1} : v \mapsto (\mathrm{grad}_S\, \phi_j^S, v)_S \qquad \forall\, S \in \bigcup_{k=2}^{d} \Delta_k(\triangle)\, \forall\, \phi_j^S,$$

where $\{\phi_j^S\}$ are the eigenfunctions from $H(\mathrm{grad})$, and

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where $\{\phi_j^S\}$ are the eigenfunctions from $H(\mathrm{grad})$, and

$$\ell_{S,2} : v \mapsto (\mathrm{curl}_S\, \Phi_j^S, \mathrm{curl}\, v)_S \quad \forall\, S \in \bigcup_{k=2}^{d} \Delta_k(\triangle)\, \forall\, \Phi_j^S,$$

where $\{\mathrm{curl}_S\, \Phi_j^S\}$ is a basis for $\mathrm{curl}_S\, \mathbb{X}$, $\mathbb{X} = [\mathbb{P}_p(S)]^d \cap H_0(\mathrm{curl}, S)$, such that

$$(\mathrm{curl}_S\, \Phi_j^S, \mathrm{curl}_S\, \Phi_i^S)_S = \delta_{ij}, \quad (\Phi_j^S, \Phi_i^S)_S = \lambda_j \delta_{ij}, \quad \Phi_j^S \times \mathbf{n} = 0 \text{ on } \partial S.$$

For $H(\mathrm{div})$, our eigenidea with Demkowicz et al. is to employ normal moments on faces

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and two kinds of moments on the only entity of higher dimension, the interior K

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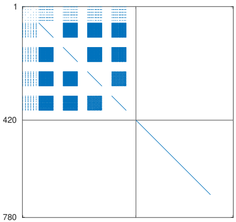
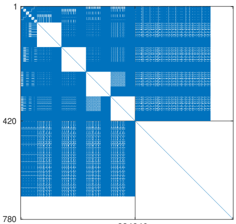
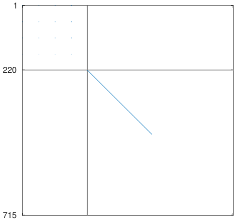
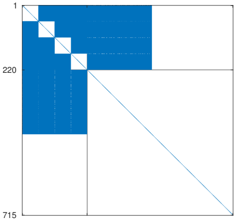
space	stiffness	mass
$H(\mathrm{curl})$	 nz = 31366	 nz = 334940
$H(\mathrm{div})$	 nz = 235	 nz = 132599

$$d = 3,\ p = 10$$

Section 5

Analysis

Lemma (Unisolvence)

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Theorem (Kernel and range)

The kernel and range of d have a low-order part from the Whitney forms, and a high-order part we characterise precisely (as certain basis functions).

Interior decoupling on a physical cell

$$\operatorname{cond}\left(\begin{bmatrix}\operatorname{diag}\left(\mathrm{A}_{\mathcal{I}\mathcal{I}}\right) & 0 \\ 0 & \mathrm{A}_{\Gamma\Gamma}\end{bmatrix}^{-1}\begin{bmatrix}\mathrm{A}_{\mathcal{I}\mathcal{I}} & \mathrm{A}_{\mathcal{I}\Gamma} \\ \mathrm{A}_{\Gamma\mathcal{I}} & \mathrm{A}_{\Gamma\Gamma}\end{bmatrix}\right) \leq C \max\left\{1, \frac{\beta}{\alpha}h^2\right\},$$

where C depends on shape regularity, d, k, and $\operatorname{diam}(\Omega)$.

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How do we use this to efficiently implement patch solves?

Theorem (interior-interface splitting)

Given an energy-stable space decomposition

$$X^k = \sum_{m=1}^{M} X_m^k$$

the *interior-interface split decomposition*

$$X^k = \sum_{m=1}^{M} X_m^k \cap X_\Gamma^k + X_\mathcal{I}^k$$

is also energy-stable provided $\beta h^2/\alpha \leq 1$.

The ratio of stability constants is independent of h, p, α, and β.

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We can take a robust space decomposition and robustly split it for fast patch solves!

Section 6

Application: the Hodge Laplacian

We employ our Riesz map solvers in a new application of operator preconditioning.

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Operator preconditioning

Let $\mathbb{Z}$ be a Hilbert space. Consider the problem: find $z \in \mathbb{Z}$ such that

$$a(z, w) = L(w) \quad \forall\, w \in \mathbb{Z},$$

where a is a symmetric bilinear form and $L \in \mathbb{Z}^*$.

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Using Z, the Riesz map for $\mathbb{Z}$, as a preconditioner for the operator A associated with a yields

$$\mathrm{cond}(Z^{-1}A) \leq C\beta^{-1}.$$

We consider the *Hodge Laplacian*, a fundamental PDE associated with a complex: find $(\sigma, u) \in X^{k-1} \times X^k$ such that

$$\begin{aligned} -(\sigma, \tau) + (u, \mathrm{d}^{k-1}\,\tau) &= 0 && \forall \tau \in X^{k-1}, \\ (\mathrm{d}^{k-1}\,\sigma, v) + (\mathrm{d}^k\,u, \mathrm{d}^k\,v) &= F(v) && \forall v \in X^k. \end{aligned}$$

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This is a mixed formulation for the problems

$$-\operatorname{div} \operatorname{grad} u = f, \quad \operatorname{curl} \operatorname{curl} \mathbf{u} - \operatorname{grad} \operatorname{div} \mathbf{u} = \mathbf{f}.$$

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$$-\operatorname{div}\operatorname{grad} u = f, \quad \operatorname{curl}\operatorname{curl}\mathbf{u} - \operatorname{grad}\operatorname{div}\mathbf{u} = \mathbf{f}.$$

These are well-posed with

$$\mathbb{Z} = \begin{cases} H(\mathrm{grad}) \times H(\mathrm{curl}) & k = 1, \\ H(\mathrm{curl}) \times H(\mathrm{div}) & k = 2, \\ H(\mathrm{div}) \times L^2 & k = 3. \end{cases}$$

A standard application of the operator preconditioning framework yields that the Riesz map

$$\langle(\sigma, u), (\tau, v)\rangle_{\mathbb{Z}} = (\sigma, \tau) + (\mathrm{d}^{k-1}\sigma, \mathrm{d}^{k-1}\tau) + (u, v) + (\mathrm{d}^{k} u, \mathrm{d}^{k} v)$$

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For $\gamma > 0$, let $\mathbb{Z}_\gamma$ be the Hilbert space with weighted inner product

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Then all eigenvalues of $Z_\gamma^{-1}A$ converge to ± 1 as $\gamma \to \infty$ like $\mathcal{O}(\gamma^{-1})$.

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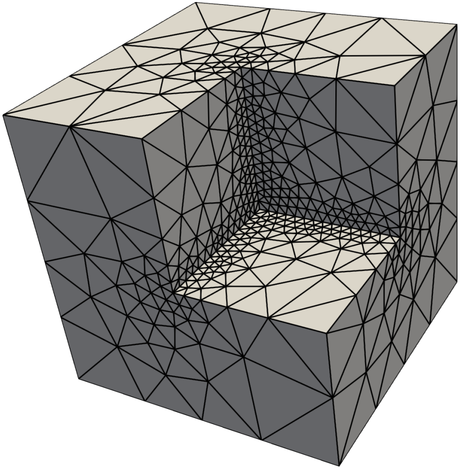
For both Riesz maps, $\gamma \gg 1$ takes us in to the regime our solvers work best.

We consider the Fichera cube with right hand side $f = 1$ or $\mathbf{f} = (1, 1, 1)^\top$.



The reentrant corner causes singularities and low regularity in the solution.

MINRES iteration counts: interior-interface split (not split)

p	$\mathrm{CG}_p \times \mathrm{Ned}_p^1$		$\mathrm{RT}_p \times \mathrm{DG}_{p-1}$	
	$\gamma = 1$	$\gamma = 10^3$	$\gamma = 1$	$\gamma = 10^3$
1	6 (6)	3 (3)	5 (5)	3 (3)
2	41 (41)	37 (37)	3 (25)	14 (16)
3	46 (48)	40 (41)	26 (28)	17 (17)
4	52 (50)	44 (44)	29 (28)	19 (18)
5	59 (51)	51 (44)	31 (29)	19 (19)
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Observations:

- $\gamma = 10^3$ gives perfect control of the Schur complement for $p = 1$.

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Observations:

- $\gamma = 10^3$ gives perfect control of the Schur complement for $p = 1$.
- Moving from $\gamma = 1$ to $\gamma = 10^3$ saves $\sim$ 10 iterations, for free.
- The interior-interface split is *much* cheaper to compute than the unsplit solver.

Section 7

Conclusion

Riesz maps

We have a solver with $\mathcal{O}(p^6)$ flops and $\mathcal{O}(p^4)$ storage for the Riesz maps on tetrahedra.

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Take-home message

For efficiency, design formulation + basis + solver together!