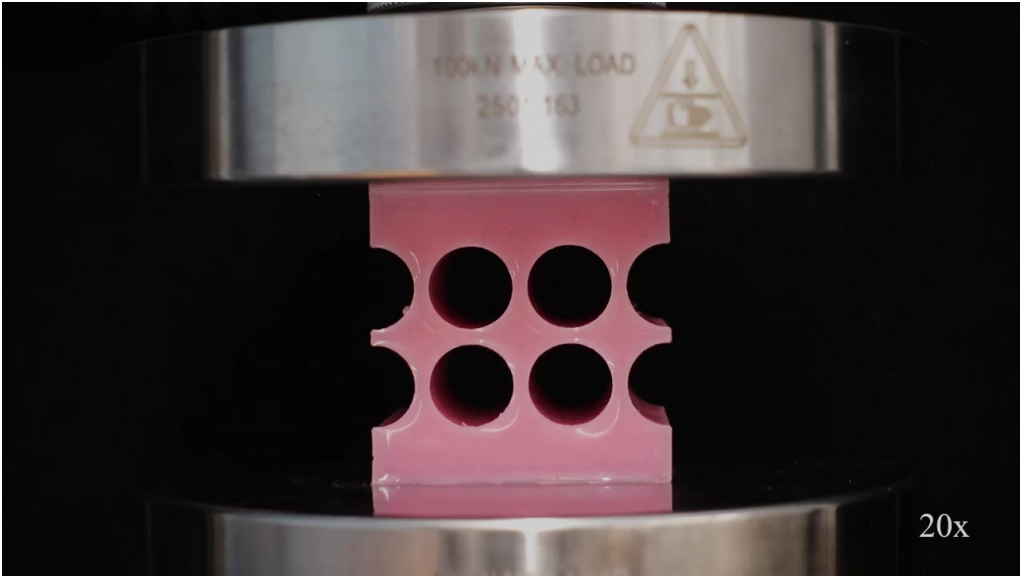
Controlling the energy jump of multistable structures using shape optimisation

Patrick E. Farrell[1,2] Arselane Hadj-Slimane[3] Àlex Ferrer[4,5] Alberto Paganini[6]

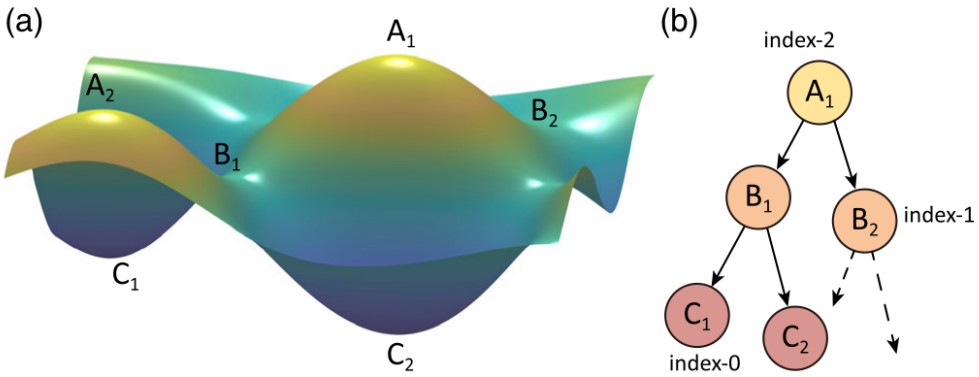



[1]University of Oxford [2]Charles University [3]ÉNS Paris-Saclay [4]Universitat Politècnica de Catatlunya
[5]CIMNE [6]University of Leicester



From Medina, Farrell, Bertoldi, Rycroft (2000).

Many PDEs have multiple solutions for the same data.



Multiple solutions for the stationary points of an energy. Credit: Yin et al. (2020)

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In the mechanical context, this is used in impact padding: energy is dissipated on impact to switch the structure from one state to another.

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How do we formulate such problems? How do we analyse them? How do we solve them?

Section 2

Formulation

Let $\mathcal{U}_{\text{ad}}$ denote a collection of admissible shapes. We want to solve

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where the PDE solutions satisfy the PDE

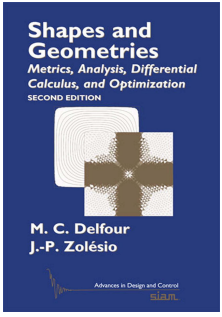
$$u_k(\Omega) \in V(\Omega) \text{ and } F(\Omega, u_k(\Omega)) = 0 \text{ for all } k = 1, \ldots, n,$$

and are all distinct on all domains:

$$u_k(\Omega) \neq u_l(\Omega) \text{ for all } \Omega \in \mathcal{U}_{\text{ad}}, k = 1, \ldots, n, l = k+1, \ldots, n.$$

For our admissible set, we make the standard choice of mapping from a reference domain

$$\mathcal{U}_{\mathsf{ad}} = \{T(\Omega_0) \mid T \in \mathcal{T}_{\mathsf{ad}}\},$$



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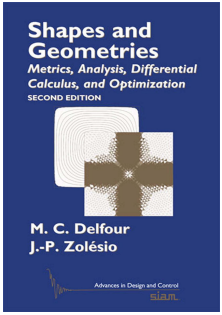
for $\Theta = C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$.



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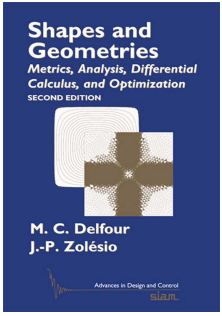

Good news

This preserves the property that the mapped domain is Lipschitz if the initial domain is.

For the PDE, we take a standard primal neo-Hookean hyperelasticity.

The function space is

$$V(\Omega) \subset H^1(\Omega; \mathbb{R}^N) \text{ with boundary conditions,}$$

and the PDE is the first-order optimality condition for an energy minimisation problem

$$\langle F(\Omega, u_k(\Omega)), v \rangle = \mathcal{E}'(\Omega, u_k; v).$$

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The hyperelastic energy is

$$\mathcal{E}(\Omega, u) = \int_\Omega \psi(I + \nabla u) \, \mathrm{d}x - \int_\Omega b \cdot u \, \mathrm{d}x,$$

for a (neo-Hookean) strain energy density ψ

$$\psi(\mathbf{F}) = \mu \left(\mathrm{tr}(\mathbf{F}^\top \mathbf{F}) - N \right) - \mu \ln \det \mathbf{F} + \frac{\lambda}{2} (\ln \det \mathbf{F})^2.$$

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and construct functionals like

$$J(\Omega) = \left(\Delta\mathcal{E}(\Omega, u_1, u_2) - \frac{1}{5}\Delta\mathcal{E}(\Omega_0, u_1, u_2)\right)^2 + \left(\Delta\mathcal{E}(\Omega, u_2, u_3) - \frac{1}{5}\Delta\mathcal{E}(\Omega_0, u_2, u_3)\right)^2$$

where solutions 1 and 3 are local minimisers and solution 2 is the saddle point between them.

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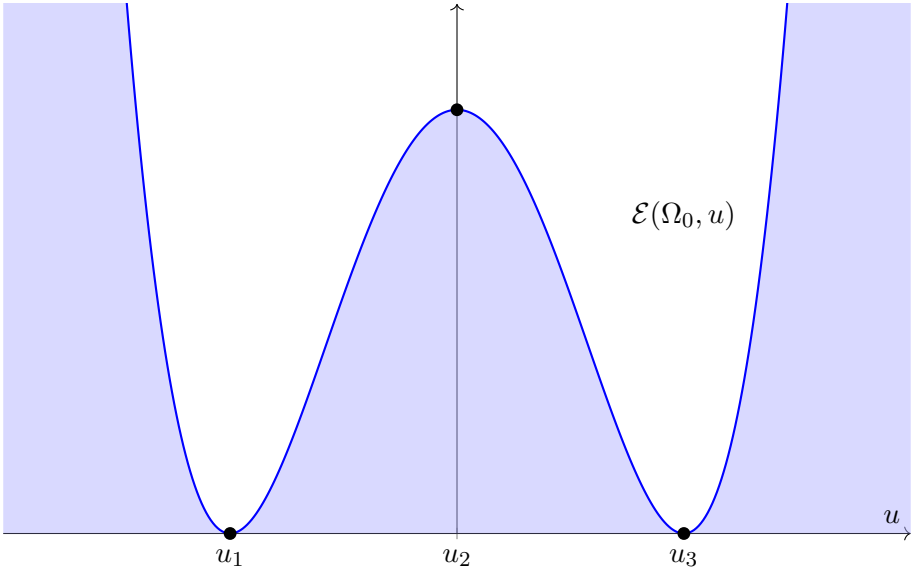
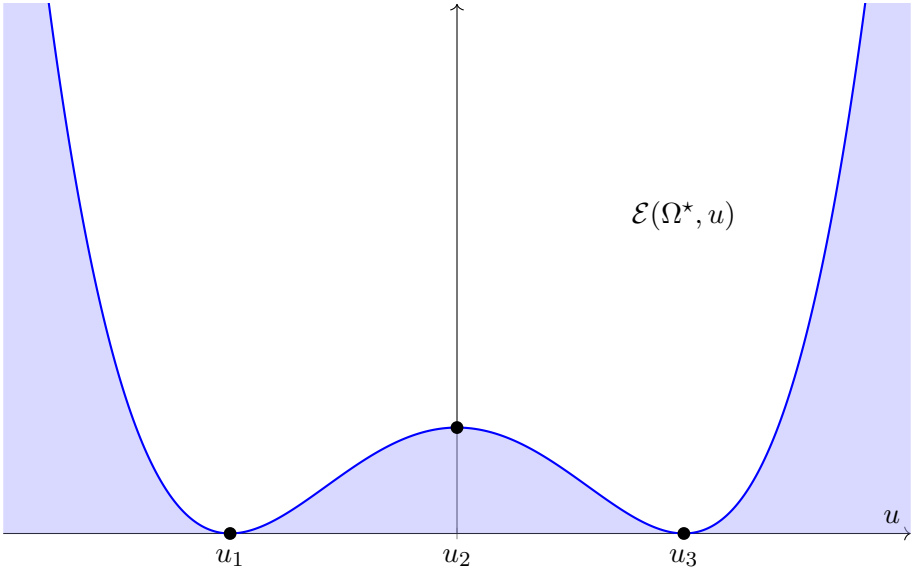
The number of solutions $n(\Omega)$ might vary, but we need $n(\Omega) \geq 3$.



Before optimisation.



After optimisation.

Section 3

Preliminary analysis

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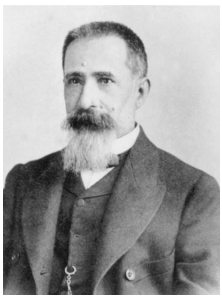

Ulisse Dini

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We can apply the implicit function theorem by rewriting the PDE via pullback onto Ω_0, and by parameterising the transformation $T_t \in \mathcal{T}_{\text{ad}}$ as

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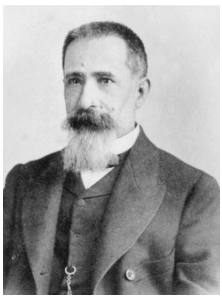

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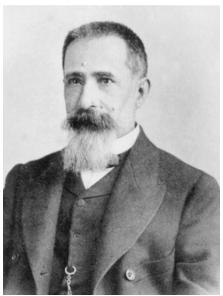
This gives sufficient conditions that guarantee that the solutions in the statement of the problem continue to exist.



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Section 4

Computational results

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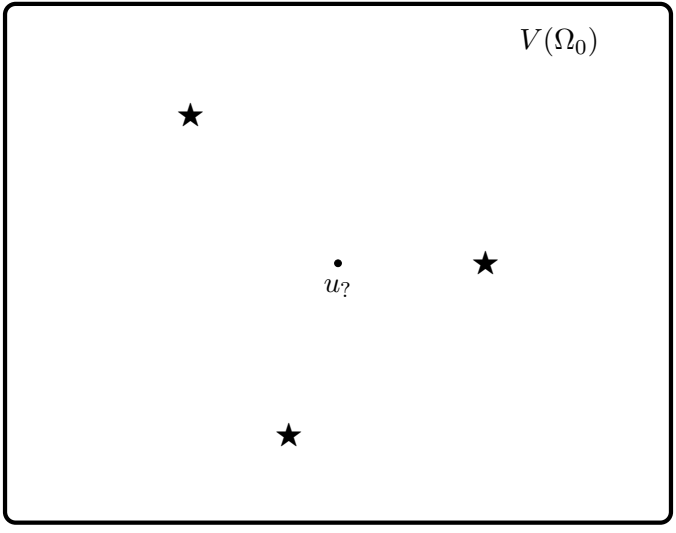
This gives mesh-independent convergence of the outer optimisation iteration.

We use deflation + continuation to compute the multiple solutions on the initial domain.



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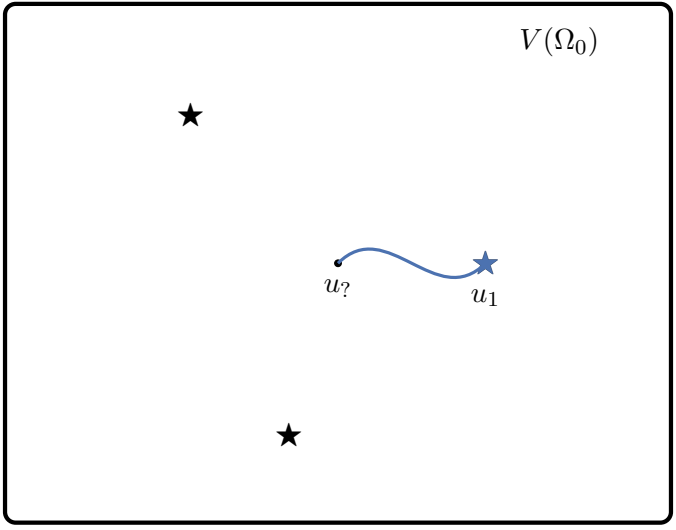

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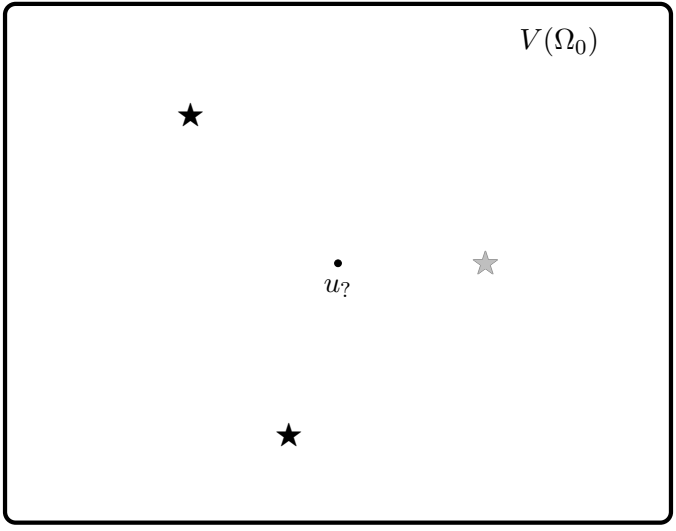

Deflate solution found.

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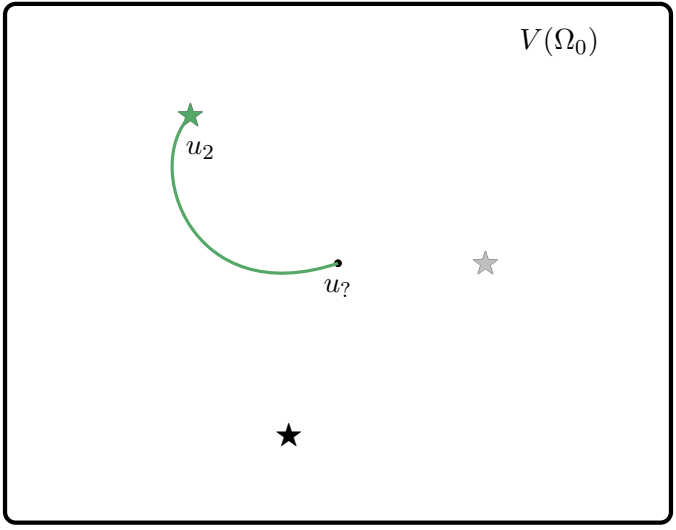

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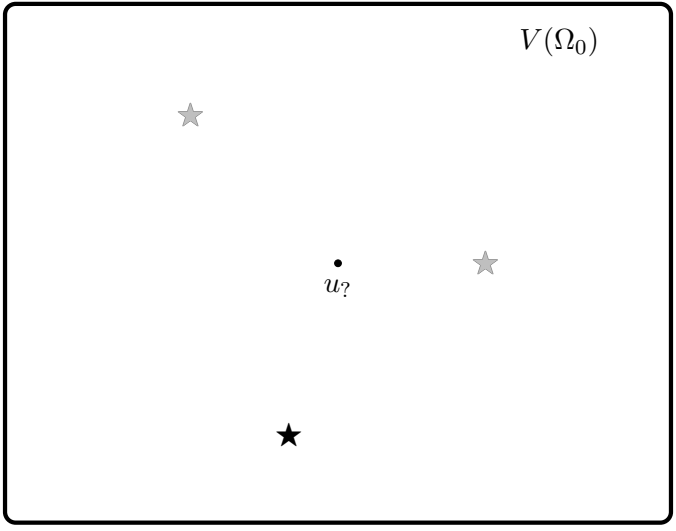

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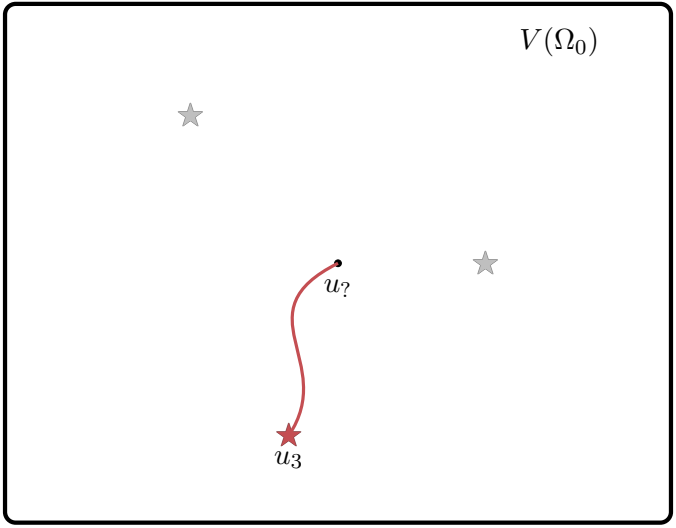

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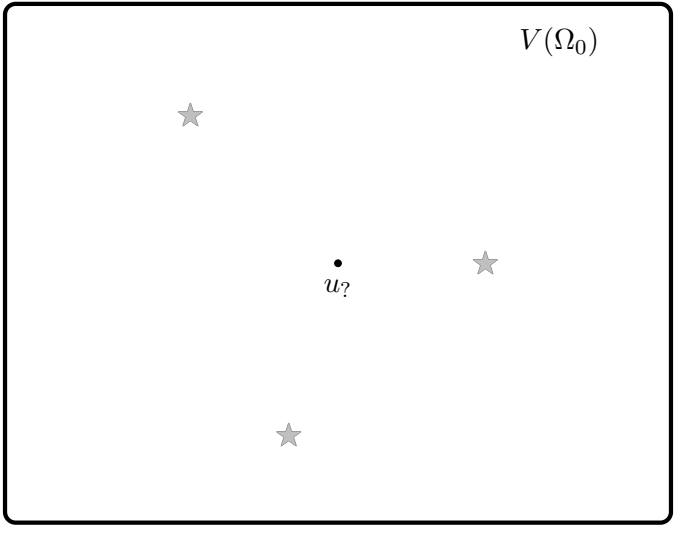

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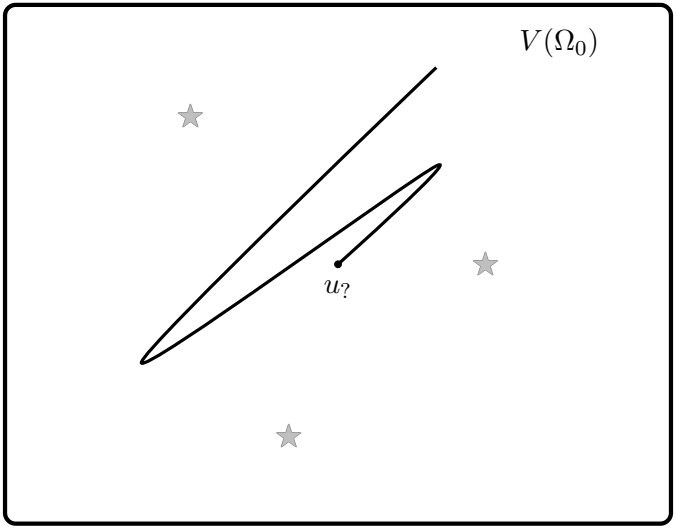

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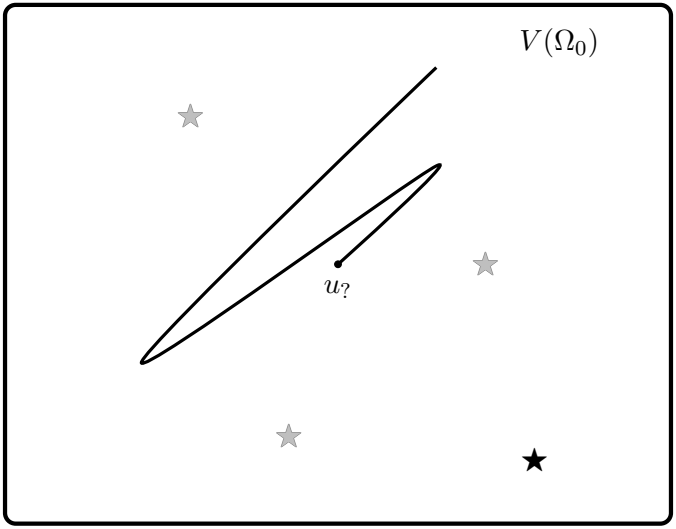

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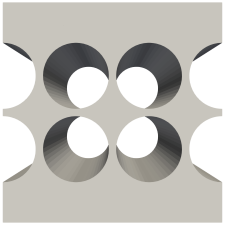
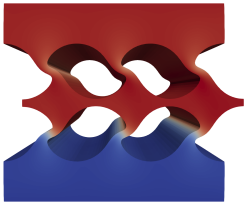
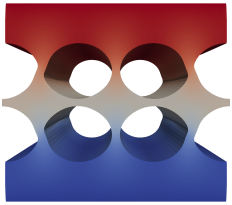
With these the optimisation algorithm works robustly in practice.



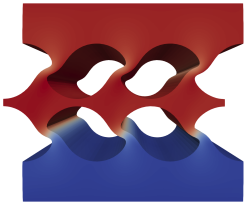
Ω_0



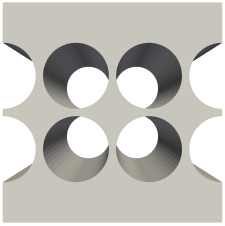
$u_1(\Omega_0)$



$u_2(\Omega_0)$



$u_3(\Omega_0)$



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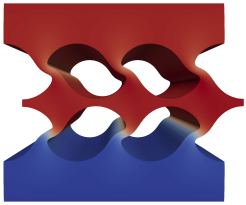

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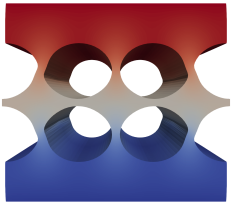

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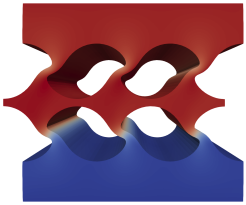

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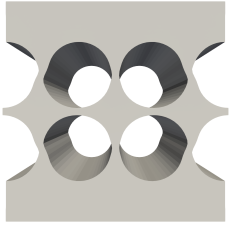
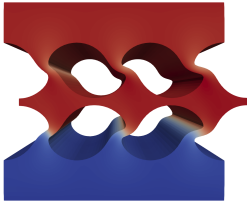


$\Omega^\star$



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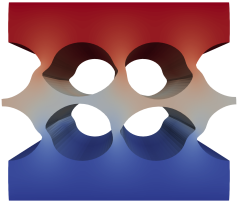

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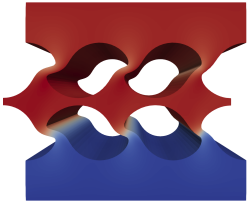
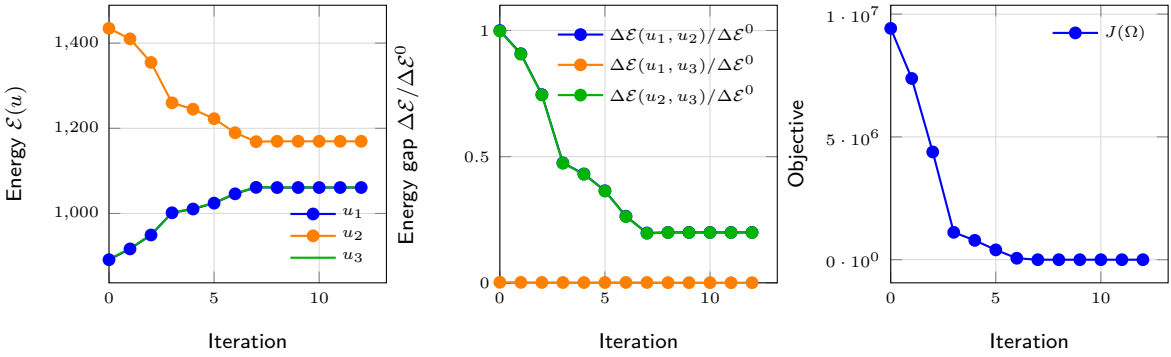
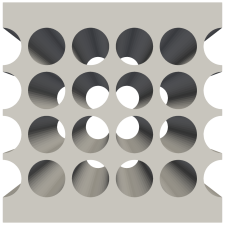


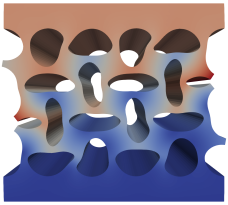
$u_3(\Omega^\star)$



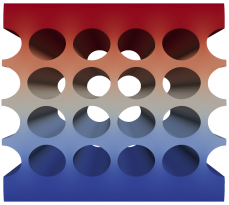
The energy gap converges to the target value in 7 iterations.



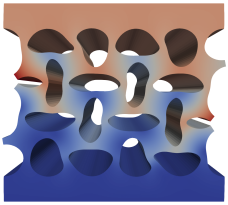
Ω_0



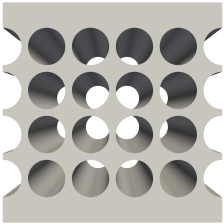
$u_1(\Omega_0)$



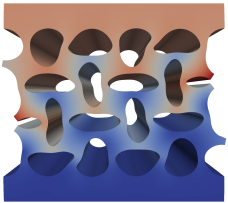
$u_2(\Omega_0)$



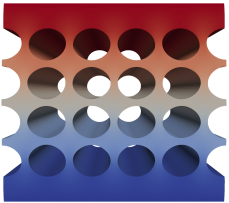
$u_3(\Omega_0)$



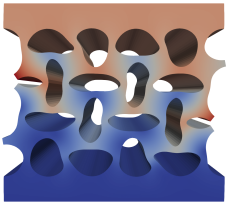
Ω_0



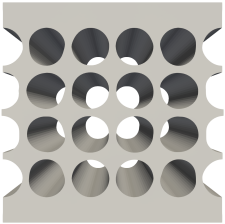
$u_1(\Omega_0)$



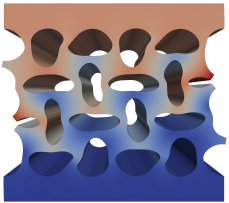
$u_2(\Omega_0)$



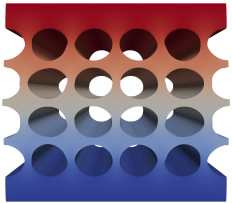
$u_3(\Omega_0)$



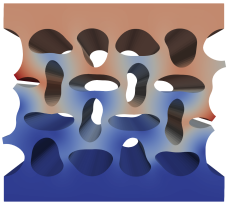
$\Omega^\star$



$u_1(\Omega^\star)$



$u_2(\Omega^\star)$



$u_3(\Omega^\star)$

Section 5

Conclusions

Good news

We can formulate and solve PDE-constrained optimisation problems with multivalued parameter-to-solution maps.

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Many open questions remain:

1. Analysis of the problem and the algorithm.

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Many open questions remain:

1. Analysis of the problem and the algorithm.
2. What should we do if the number of solutions varies dynamically?
3. Self-contact can be important here. How would this work for a QVI?