Answers to Homework 2

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CSCI-480, Randomised Algorithms

due February 7th, 2024

Problem 1

Suppose we have a deck of 52 cards with assigned card values from 1 to 13; we draw 8 cards from the deck. We want to find what the probability is that the sum of the values of the drawn 8 cards gives a remainder of 1 when divided by 13.
Since each draw is independent and the cards are replaced to the deck after each draw, the sum of the values of the seven cards can be any value with equal probability modulo 13 due to the uniform distribution of card values.
Let S_7 be the sum of the card values after seven draws from the deck. The value of $S_7 \mod 13$ can be any integer $x \in [0, 12]$ with equal probability. For the sum of the eight draws S_8 to have a remainder of 1 when divided by 13, the eighth card we draw must satisfy $(S_7 + \text{value of the 8th card}) \mod 13 = 1$.
Given the uniform distribution of card values modulo 13, the probability that the eighth card will result in a total sum S_8 that satisfies $S_8 \mod 13 = 1$ is $\frac{1}{13}$, since each remainder from 0 to 12 is equally likely for S_7, and there is exactly one value for the 8th card that achieves the desired remainder for all values of S_7.
Therefore, the probability that the sum of the card values when divided by 13 has a remainder of 1 after eight draws is $\frac{1}{13}$.

Problem 2

We have two decks of cards: one fully red and the other half red and half black.
a) By the law of total probability, the probability that the chosen card is red is:

$$P(\text{Red}) = P(\text{Red}|\text{Standard Deck})P(\text{Standard Deck}) + P(\text{Red}|\text{All-Red Deck})P(\text{All-Red Deck})$$

$$= \frac{26}{52} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

b) If the drawn card is red, the probability that we chose the deck that is fully red is:

$$P(\text{All-Red Deck}|\text{Red}) = \frac{P(\text{Red}|\text{All-Red Deck})P(\text{All-Red Deck})}{P(\text{Red})}$$

$$= \frac{1 \cdot \frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

Problem 3

We have a test for leukemia that has a 99% accuracy. It also gives false positives 10% of the time. We want to find what the probability is that the probability of a patient having leukemia given the fact that 0.1% of the population has leukimia is $< 1\%$. Using Bayes' theorem, the probability that a patient has leukemia given a positive test result is:

$$P(\text{Leukemia}|\text{Positive}) = \frac{P(\text{Positive}|\text{Leukemia})P(\text{Leukemia})}{P(\text{Positive})}$$

Where:

$$P(\text{Positive}) = P(\text{Positive}|\text{Leukemia})P(\text{Leukemia}) + P(\text{Positive}|\text{No Leukemia})P(\text{No Leukemia})$$

$$= 0.99 \cdot 0.001 + 0.10 \cdot (1 - 0.001) \approx 0.1009$$

Therefore:

$$P(\text{Leukemia}|\text{Positive}) = \frac{0.99 \cdot 0.001}{0.1009} \approx 0.0098$$

Problem 4

a) We have the QuickFind algorithm where we want to find the probability of comparison between two elements e_i and e_j for $i < j$. This comparison happens if one of them is chosen as a pivot before the other is eliminated. We have three cases to consider based on the positions of i, j, and k.

1. $i < k < j$: either e_i or e_j must be chosen as the pivot for them to be compared; the probability of this happening is $\frac{2}{j-i+1}$ since there are $j - i + 1$ elements between i and j, and either e_i or e_j being chosen as the pivot will lead to their comparison.

2. $i < j \leq k$: For e_i and e_j to be compared, one of them must be chosen as the pivot before any element larger than e_j is chosen. The probability is $\frac{2}{j-i+1}$.

3. $k \leq i < j$: for a comparison to occur, one of them must be chosen as the pivot before any element that is smaller than e_i is chosen. The probability is $\frac{2}{j-i+1}$.

b) We assume $n = 2m + 1$ and $k = m + 1$. This assumption makes the second and third cases symmetric.
The expected number of comparisons $\sum_{i<j} E_{ij}$ can be analyzed by considering the contribution of each pair i, j such that e_i and e_j are compared:

1. For $i < k < j$, we have $l = j - i + 1$ and sum over all of the possible values of l. We know that $n = 2m + 1$; this mean thatthe range of l starts from 3 to $m + 2$. The expected number of comparisons is:

$$\sum_{l=3}^{m+2} \frac{2}{l}$$

2. For $i < j \leq k$ and $k \leq i < j$, which are symmetric, we sum over all possible values of j for $i < j \leq m + 1$, which gives us:

$$\sum_{i=1}^{m} \sum_{j=i+1}^{2m+1} \frac{2}{j - i + 1}$$