Answers to Homework 1

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CSCI-480, Randomised Algorithms

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1 Question 1, Exercise 1.2

a) The probability that when you roll two dice and get the same number on both is $\frac{1}{6}$
b) To find the probability that you get a larger value on the first die than on the second, you do $1+2+3+4+5=15$. Thus, the probability is $\frac{5}{16}$

Question 2, Problem 1.3

a) To find the probability that in a shuffled deck of 52 cards one of the first two is an ace, we can use the complement.
The probability that the first card is not an ace is $\frac{48}{52}$ and for the second it is $\frac{47}{52}$. Thus the probability that one of them is an ace is:

$$1-\left(\frac{48}{52}\cdot\frac{47}{51}\right)=0.149321267$$

e) To calculate the probability of getting a full house we do:

$$\left(\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{1}\cdot\binom{4}{2}\right)\cdot\binom{52}{5}=0.0014405762304921968$$

2 Question 3

To determine the probability that a uniformly chosen number $n\in[1,1000]$ is divisible by either 4,10 or 25 we do the following:

- Count the number of multiples of 4: $\frac{1000}{4}$
- Count the number of multiples of 25: $\frac{1000}{25}$

- Count the number of mutliples of 10: $\frac{1000}{10}$

After this, we find the common multiples of pairs:

- Multiples of 4 and 10: $\frac{1000}{20}$
- Multiple of 4 and 25: $\frac{1000}{100}$
- Multiples of 10 and 25: $\frac{1000}{50}$

Then, we find the multiples of all 3 numbers which is $\frac{1000}{100}$.
Now we perform the necessary inclusion/exclusion:

$$\left(\frac{1000}{4} + \frac{1000}{25} + \frac{1000}{10}\right) - \left(\frac{1000}{20} + \frac{1000}{100} + \frac{1000}{50}\right) + \left(\frac{1000}{100}\right) = 0.32$$

Question 4, Exercise 1.6

Initially we have 1 white ball and 1 black ball; thus, the chance of picking either is 50%. When you pick a ball and add another of the same color, you add 1 more of that colour. Each sequence of picks leads, therefore, to a specific and unique number of white balls at the end.
To show that each number of white balls withing the range of 1 to $n-1$ is equally likely we can consider the decision tree. Each choice creates two branches: one in which we add a white ball and one in which we add a black ball. By induction, we can see that at any point before the last ball is added to the bin, the number of sequences which lead to each possible outcome is equal due to the fact that each choice to add a white or black ball doubles the number of sequences leading to outcomes in the next step.
Thus, when you reach n-1 balls, the number of sequences that have led to each count of white balls is the same.

Question 5

The algorithm is as follows:

- We pick 2 random integers x, y such that $x + y \equiv z \mod n$.
- We then compute $F(x)$ and $F(y)$. We know that $\frac{1}{6}$ of the entries are incorrect which means that there is a $\frac{5}{6}$ chance that $A[x] = F(x)$ and $A[y] = F(y)$
- Then we return $A[x] + A[y] \mod m$ as the estimate for $F(z)$.

THe probability that $A[x]$ and $A[y]$ are correct is $\left(\frac{5}{6}\right)^2$ which results in a $\frac{25}{36}$ chance of obtaining the correct $F(z)$.

The probability of getting the correct answer at least twice out of 3 attempts is:

$$3 \cdot \left(\frac{25}{36}\right) \cdot \left(\frac{11}{36}\right) + \left(\frac{25}{36}\right).$$

If we run the algorithm 3 times and we choose the majority value (the most common outcome) then the probability that the answer is correct is 77.7%. If there is no majority value then we can pick any of the 3 as that will give the lower bound of the correct probability since at least one run of the algorithm has given the correct answer.