CHAPTER
FOUR

Rational B-spline Curves and Surfaces

4.1 Introduction

In this chapter we combine the concepts of Sections 1.4 and 1.5 of Chapter 1 and those of Chapter 3 to obtain **N**on**U**niform **R**ational **B-S**pline (NURBS) curves and surfaces. We present definitions and general properties and derive formulas and algorithms for the derivatives of NURBS curves and surfaces in terms of their nonrational counterparts. The earliest published works on NURBS are [Vers75; Till83]. A more recent survey can be found in [Pieg91a].

4.2 Definition and Properties of NURBS Curves

A pth-degree NURBS curve is defined by

$$\mathbf{C}(u) = \frac{\sum_{i=0}^{n} N_{i,p}(u) w_i \mathbf{P}_i}{\sum_{i=0}^{n} N_{i,p}(u) w_i} \qquad a \le u \le b \tag{4.1}$$

where the $\{\mathbf{P}_i\}$ are the *control points* (forming a *control polygon*), the $\{w_i\}$ are the *weights*, and the $\{N_{i,p}(u)\}$ are the pth-degree B-spline basis functions defined on the nonperiodic (and nonuniform) knot vector

$$U = \{\underbrace{a, \ldots, a}_{p+1}, u_{p+1}, \ldots, u_{m-p-1}, \underbrace{b, \ldots, b}_{p+1}\}$$

Unless otherwise stated, we assume that $a = 0$, $b = 1$, and $w_i > 0$ for all i. Setting

$$R_{i,p}(u) = \frac{N_{i,p}(u)w_i}{\sum_{j=0}^{n} N_{j,p}(u)w_j} \tag{4.2}$$

allows us to rewrite Eq. (4.1) in the form

$$\mathbf{C}(u) = \sum_{i=0}^{n} R_{i,p}(u)\mathbf{P}_i \tag{4.3}$$

The $\{R_{i,p}(u)\}$ are the *rational basis functions*; they are piecewise rational functions on $u \in [0,1]$.

The $R_{i,p}(u)$ have the following properties derived from Eq. (4.2) and the corresponding properties of the $N_{i,p}(u)$:

P4.1 Nonnegativity: $R_{i,p}(u) \geq 0$ for all i, p, and $u \in [0,1]$;

P4.2 Partition of unity: $\sum_{i=0}^{n} R_{i,p}(u) = 1$ for all $u \in [0,1]$;

P4.3 $R_{0,p}(0) = R_{n,p}(1) = 1$;

P4.4 For $p > 0$, all $R_{i,p}(u)$ attain exactly one maximum on the interval $u \in [0,1]$;

P4.5 Local support: $R_{i,p}(u) = 0$ for $u \notin [u_i, u_{i+p+1})$. Furthermore, in any given knot span, at most $p+1$ of the $R_{i,p}(u)$ are nonzero (in general, $R_{i-p,p}(u), \ldots, R_{i,p}(u)$ are nonzero in $[u_i, u_{i+1})$);

P4.6 All derivatives of $R_{i,p}(u)$ exist in the interior of a knot span, where it is a rational function with nonzero denominator. At a knot, $R_{i,p}(u)$ is $p-k$ times continuously differentiable, where k is the multiplicity of the knot;

P4.7 If $w_i = 1$ for all i, then $R_{i,p}(u) = N_{i,p}(u)$ for all i; i.e., the $N_{i,p}(u)$ are special cases of the $R_{i,p}(u)$. In fact, for any $a \neq 0$, if $w_i = a$ for all i then $R_{i,p}(u) = N_{i,p}(u)$ for all i.

Properties P4.1–P4.7 yield the following important geometric characteristics of NURBS curves:

P4.8 $\mathbf{C}(0) = \mathbf{P}_0$ and $\mathbf{C}(1) = \mathbf{P}_n$; this follows from P4.3;

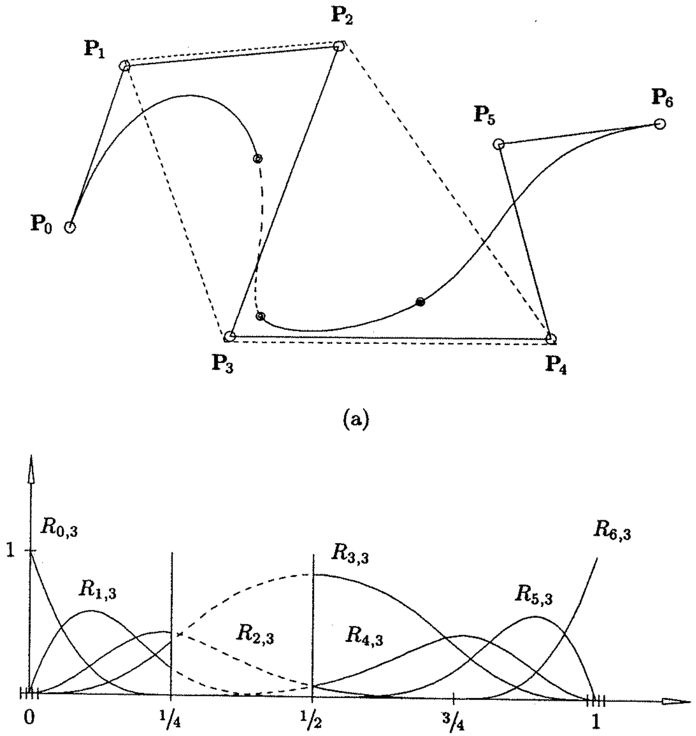
P4.9 Affine invariance: an affine transformation is applied to the curve by applying it to the control points (see P3.4, Section 3.1); NURBS curves are also invariant under perspective projections ([Lee87; Pieg91a]), a fact which is important in computer graphics;

P4.10 Strong convex hull property: if $u \in [u_i, u_{i+1})$, then $\mathbf{C}(u)$ lies within the convex hull of the control points $\mathbf{P}_{i-p}, \ldots, \mathbf{P}_i$ (see Figure 4.1, where $\mathbf{C}(u)$ for $u \in [{}^1\!/\!_4, {}^1\!/\!_2)$ (dashed segment) is contained in the convex hull of $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4\}$, the dashed area); this follows from P4.1, P4.2, and P4.5;

P4.11 $\mathbf{C}(u)$ is infinitely differentiable on the interior of knot spans and is $p-k$ times differentiable at a knot of multiplicity k;

P4.12 Variation diminishing property: no plane has more intersections with the curve than with the control polygon (replace the word 'plane', with 'line' for two-dimensional curves);

P4.13 A NURBS curve with no interior knots is a rational Bézier curve, since the $N_{i,p}(u)$ reduce to the $B_{i,n}(u)$; compare Eqs. (4.2) and (4.3) with Eq. (1.15). This, together with P4.7, implies that NURBS curves contain nonrational B-spline and rational and nonrational Bézier curves as special cases;



(b)

Figure 4.1. $U = \{0,0,0,0,{}^1\!/\!_4,{}^1\!/\!_2,{}^3\!/\!_4,1,1,1,1\}$ and $\{w_0,\ldots,w_6\} = \{1,1,1,3,1,1,1\}$. (a) A cubic NURBS curve; (b) associated basis functions.

P4.14 Local approximation: if the control point $\mathbf{P}_i$ is moved, or the weight w_i is changed, it affects only that portion of the curve on the interval $u \in [u_i, u_{i+p+1})$; this follows from P4.5.

Property P4.14 is very important for interactive shape design. Using NURBS curves, we can utilize both control point movement and weight modification to attain local shape control. Figures 4.2–4.6 show the effects of modifying a single weight. Qualitatively the effect is: Assume $u \in [u_i, u_{i+p+1})$; then if w_i increases (decreases), the point $\mathbf{C}(u)$ moves closer to (farther from) $\mathbf{P}_i$, and hence the curve is pulled toward (pushed away from) $\mathbf{P}_i$. Furthermore, the movement of $\mathbf{C}(u)$ for fixed u is along a straight line (Figure 4.6). In Figure 4.6, u is fixed and w_3 is changing. Let

$$\mathbf{B} = \mathbf{C}(u; w_3 = 0) \tag{4.4}$$

$$\mathbf{N} = \mathbf{C}(u; w_3 = 1)$$

Then the straight line defined by $\mathbf{B}$ and $\mathbf{N}$ passes through $\mathbf{P}_3$, and for arbitrary $0 < w_3 < \infty$, $\mathbf{B}_3 = \mathbf{C}(u; w_3)$ lies on this line segment between $\mathbf{B}$ and $\mathbf{P}_3$. We return to this topic in a later chapter.

As is the case for rational Bézier curves, homogeneous coordinates offer an efficient method of representing NURBS curves. Let H be the perspective map given by Eq. (1.16). For a given set of control points, $\{\mathbf{P}_i\}$, and weights, $\{w_i\}$, construct the weighted control points, $\mathbf{P}_i^w = (w_i x_i, w_i y_i, w_i z_i, w_i)$. Then define the nonrational (piecewise polynomial) B-spline curve in four-dimensional space as

$$\mathbf{C}^w(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i^w \tag{4.5}$$

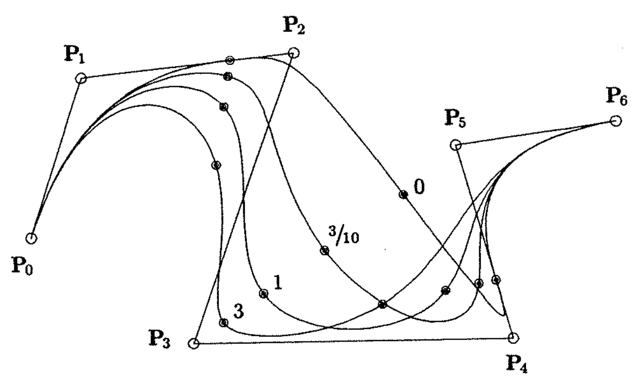


Figure 4.2. Rational cubic B-spline curves, with w_3 varying.

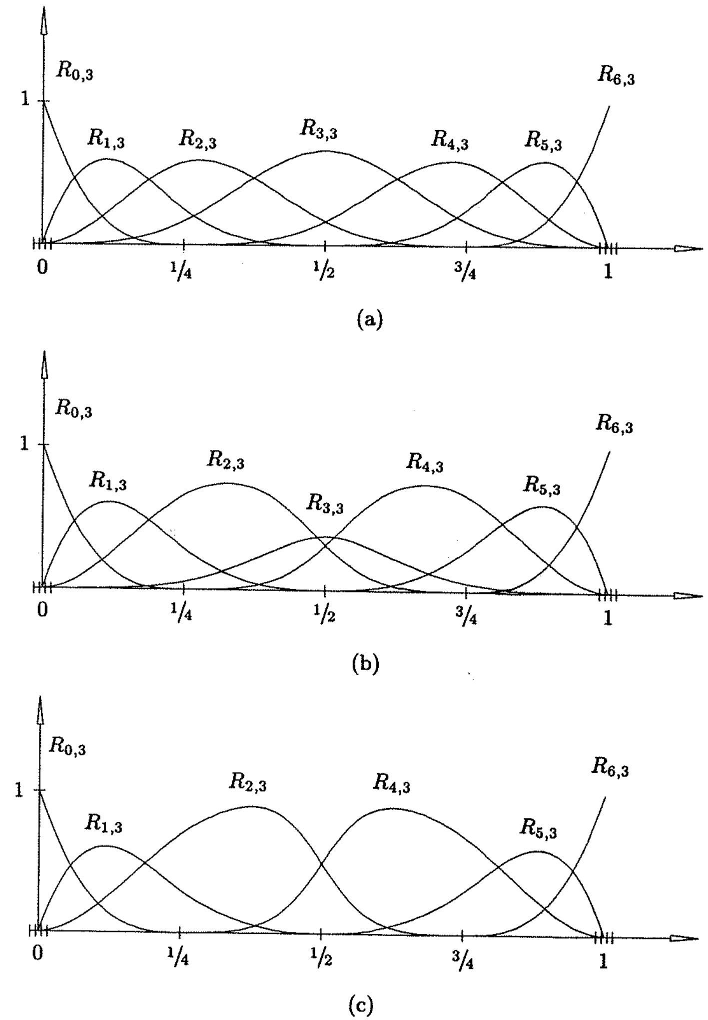


Figure 4.3. The cubic basis functions for the curves of Figure 4.2. (a) $w_3 = 1$; (b) $w_3 = 3/10$; (c) $w_3 = 0$.

Applying the perspective map, H, to $\mathbf{C}^w(u)$ yields the corresponding rational B-spline curve (piecewise rational in three-dimensional space)

$$\mathbf{C}(u) = H\{\mathbf{C}^w(u)\} = H\left\{\sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i^w\right\}$$

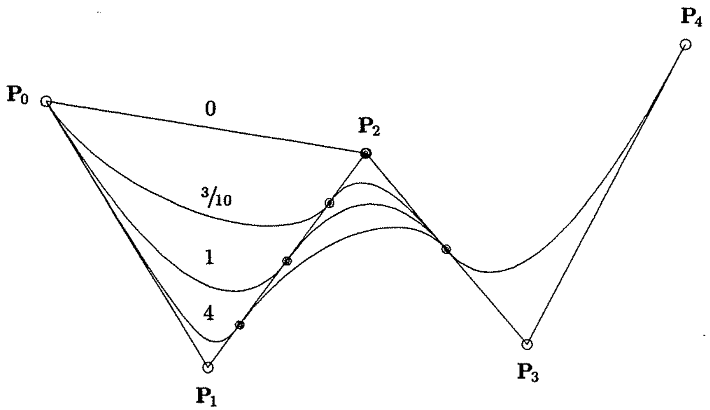


Figure 4.4. Rational quadratic curves, with w_1 varying.

$$= \frac{\sum_{i=0}^{n} N_{i,p}(u) w_i \mathbf{P}_i}{\sum_{i=0}^{n} N_{i,p}(u) w_i} = \sum_{i=0}^{n} R_{i,p}(u)\mathbf{P}_i$$

We refer interchangeably to either $\mathbf{C}^w(u)$ or $\mathbf{C}(u)$ as the NURBS curve, although strictly speaking, $\mathbf{C}^w(u)$ is not a rational curve.

Example

Ex4.1 Let $U = \{0,0,0,1,2,3,3,3\}$, $\{w_0,\ldots,w_4\} = \{1,4,1,1,1\}$, $\{\mathbf{P}_0,\ldots,\mathbf{P}_4\} = \{(0,0),(1,1),(3,2),(4,1),(5,-1)\}$. We compute the point on the rational B-spline curve at $u=1$. Now u is in the knot span $[u_3,u_4)$, and

$$N_{3,0}(1) = 1$$

$$N_{2,1}(1) = \frac{2-1}{2-1}\, N_{3,0}(1) = 1$$

$$N_{3,1}(1) = \frac{1-1}{2-1}\, N_{3,0}(1) = 0$$

$$N_{1,2}(1) = \frac{2-1}{2-0}\, N_{2,1}(1) = \frac{1}{2}$$

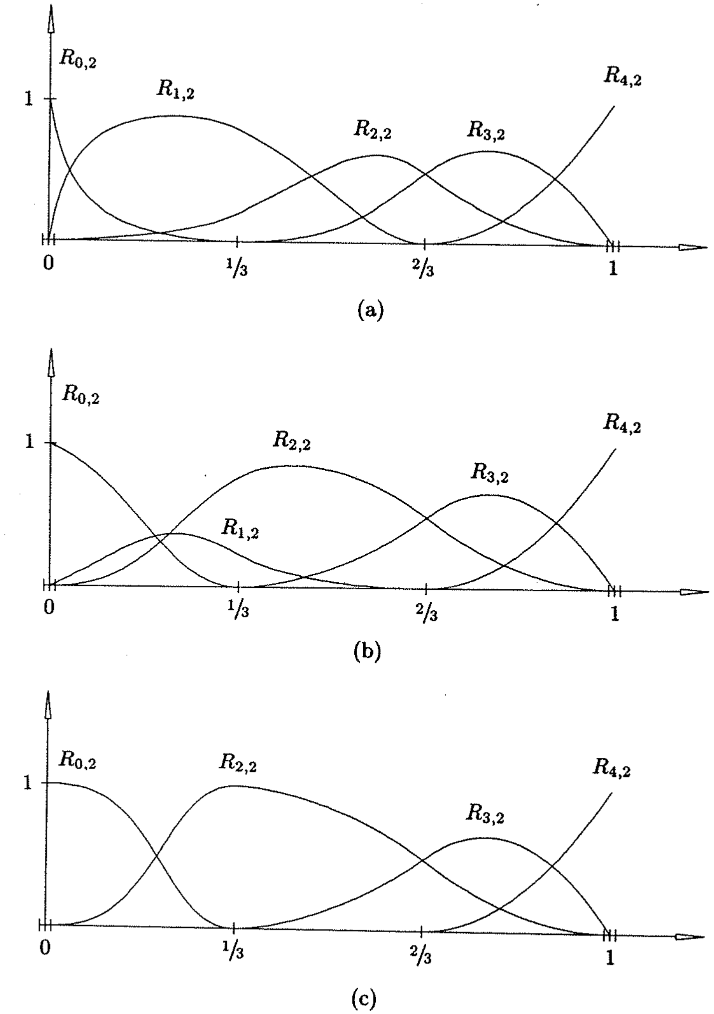


Figure 4.5. The quadratic basis functions for the curves of Figure 4.4. (a) $w_1 = 4$; (b) $w_1 = {}^3/_{10}$; (c) $w_1 = 0$.

$$N_{2,2}(1) = \frac{1-0}{2-0}\, N_{2,1}(1) = \frac{1}{2}$$

$$N_{3,2}(1) = 0$$

Hence

$$\mathbf{C}^w(1) = \frac{1}{2}\mathbf{P}_1^w + \frac{1}{2}\mathbf{P}_2^w = \frac{1}{2}(4,4,4) + \frac{1}{2}(3,2,1) = \left(\frac{7}{2}, 3, \frac{5}{2}\right)$$

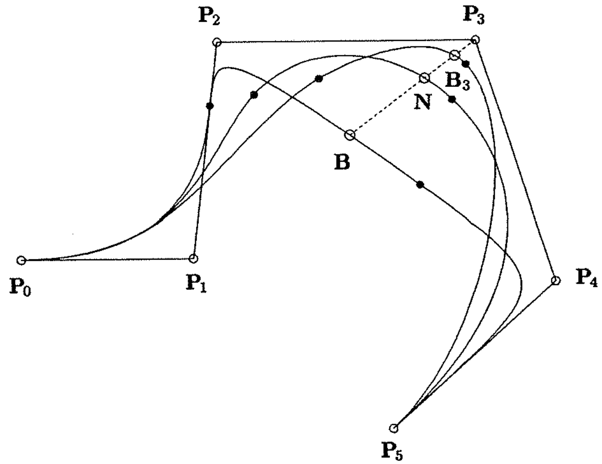


Figure 4.6. Modification of the weight w_3.

Projecting yields

$$\mathbf{C}(1) = \left(\frac{7}{5}, \frac{6}{5}\right)$$

Algorithm A4.1 computes the point on a rational B-spline curve at a fixed u value. It is based on Eq. (4.5), i.e., it assumes weighted control points in the array Pw (as do all algorithms in the remainder of this book, unless otherwise stated). Hence, Pw[i] contains $\mathbf{P}_i = (w_i x_i, w_i y_i, w_i z_i, w_i)$. Cw denotes the four-dimensional point on $\mathbf{C}^w(u)$, and C the three-dimensional point on $\mathbf{C}(u)$ (the output). For the remainder of the book we use the notation C = Cw/w to denote projection.

```
ALGORITHM A4.1
  CurvePoint(n,p,U,Pw,u,C)
    { /*  Compute point on rational B-spline curve  */
      /*  Input:  n,p,U,Pw,u  */
      /*  Output: C  */
    span = FindSpan(n,p,u,U);
    BasisFuns(span,u,p,U,N);
    Cw = 0.0;
    for (j=0; j<=p; j++)
      Cw = Cw + N[j]*Pw[span-p+j];
    C = Cw/w;    /*  Divide by weight  */
    }
```


4.3 Derivatives of a NURBS Curve

Derivatives of rational functions are complicated, involving denominators to high powers. In Section 3.3 we developed formulas and algorithms to compute the derivatives of nonrational B-spline curves. Those formulas and algorithms apply, of course, to $\mathbf{C}^w(u)$, since it is a nonrational curve in four-dimensional space. In this section we develop formulas that express the derivatives of $\mathbf{C}(u)$ in terms of the derivatives of $\mathbf{C}^w(u)$.

Let

$$\mathbf{C}(u) = \frac{w(u)\mathbf{C}(u)}{w(u)} = \frac{\mathbf{A}(u)}{w(u)}$$

where $\mathbf{A}(u)$ is the vector-valued function whose coordinates are the first three coordinates of $\mathbf{C}^w(u)$ ($\mathbf{A}(u)$ is the numerator of Eq. [4.1]). Then

$$\mathbf{C}'(u) = \frac{w(u)\mathbf{A}'(u) - w'(u)\mathbf{A}(u)}{w(u)^2}$$

$$= \frac{w(u)\mathbf{A}'(u) - w'(u)w(u)\mathbf{C}(u)}{w(u)^2} = \frac{\mathbf{A}'(u) - w'(u)\mathbf{C}(u)}{w(u)} \qquad (4.7)$$

Since $\mathbf{A}(u)$ and $w(u)$ represent the coordinates of $\mathbf{C}^w(u)$, we obtain their first derivatives using Eqs. (3.4)–(3.6). We compute higher order derivatives by differentiating $\mathbf{A}(u)$ using Leibnitz' rule

$$\mathbf{A}^{(k)}(u) = \left(w(u)\mathbf{C}(u)\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} w^{(i)}(u)\mathbf{C}^{(k-i)}(u)$$

$$= w(u)\mathbf{C}^{(k)}(u) + \sum_{i=1}^{k} \binom{k}{i} w^{(i)}(u)\mathbf{C}^{(k-i)}(u)$$

from which we obtain

$$\mathbf{C}^{(k)}(u) = \frac{\mathbf{A}^{(k)}(u) - \sum_{i=1}^{k} \binom{k}{i} w^{(i)}(u)\mathbf{C}^{(k-i)}(u)}{w(u)} \qquad (4.8)$$

Equation (4.8) gives the kth derivative of $\mathbf{C}(u)$ in terms of the kth derivative of $\mathbf{A}(u)$, and the first through $(k-1)$th derivatives of $\mathbf{C}(u)$ and $w(u)$. The derivatives $\mathbf{A}^{(k)}(u)$ and $w^{(i)}(u)$ are obtained using either Eq. (3.3) and Algorithm A3.2 or Eq. (3.8) and Algorithm A3.4.

Let us derive expressions for the first derivatives of a NURBS curve at its endpoints ($u = 0, u = 1$). From Eq. (3.7) we have

$$\mathbf{A}'(0) = \frac{p}{u_{p+1}}(w_1\mathbf{P}_1 - w_0\mathbf{P}_0) \qquad w'(0) = \frac{p}{u_{p+1}}(w_1 - w_0)$$

and from Eq. (4.7)

$$\mathbf{C}'(0) = \frac{\frac{p}{u_{p+1}}(w_1\mathbf{P}_1 - w_0\mathbf{P}_0) - \frac{p}{u_{p+1}}(w_1 - w_0)\mathbf{P}_0}{w_0}$$

from which follows

$$\mathbf{C}'(0) = \frac{p}{u_{p+1}}\frac{w_1}{w_0}(\mathbf{P}_1 - \mathbf{P}_0) \tag{4.9}$$

Analogously

$$\mathbf{C}'(1) = \frac{p}{1 - u_{m-p-1}}\frac{w_{n-1}}{w_n}(\mathbf{P}_n - \mathbf{P}_{n-1}) \tag{4.10}$$

Example

Ex4.2 Consider the quadratic rational Bézier circular arc given in Section 1.4 (Figure 1.19b). This is a NURBS curve on the knot vector $U = \{0,0,0,1,1,1\}$, with $\{\mathbf{P}_i\} = \{(1,0),(1,1),(0,1)\}$ and $\{w_i\} = \{1,1,2\}$. From Eqs. (4.9) and (4.10) we have

$$\mathbf{C}'(0) = \frac{2}{1}\frac{1}{1}(\mathbf{P}_1 - \mathbf{P}_0) = (0,2)$$

$$\mathbf{C}'(1) = \frac{2}{1-0}\frac{1}{2}(\mathbf{P}_2 - \mathbf{P}_1) = (-1,0)$$

From Eq. (4.8)

$$\mathbf{C}''(0) = \frac{\mathbf{A}''(0) - 2w'(0)\mathbf{C}'(0) - w''(0)\mathbf{C}(0)}{w_0}$$

From Eq. (3.9) or Eq. (1.10)

$$\mathbf{A}''(0) = 2(w_0\mathbf{P}_0 - 2w_1\mathbf{P}_1 + w_2\mathbf{P}_2)$$

and

$$w''(0) = 2(w_0 - 2w_1 + w_2)$$

From $w'(0) = 2(w_1 - w_0)$ it follows that

$$\mathbf{C}''(0) = \frac{2}{w_0}[w_0\mathbf{P}_0 - 2w_1\mathbf{P}_1 + w_2\mathbf{P}_2 - 4(w_1 - w_0)(\mathbf{P}_1 - \mathbf{P}_0)$$

$$-(w_0 - 2w_1 + w_2)\mathbf{P}_0]$$

$$= 2(\mathbf{P}_0 - 2\mathbf{P}_1 + 2\mathbf{P}_2 - \mathbf{P}_0) = 4(\mathbf{P}_2 - \mathbf{P}_1) = (-4,0)$$

The computation of $\mathbf{C}''(1)$ is left as an exercise.

Now assume that u is fixed, and that the zeroth through the dth derivatives of $\mathbf{A}(u)$ and $w(u)$ have been computed and loaded into the arrays `Aders` and `wders`, respectively, i.e., $\mathbf{C}^w(u)$ has been differentiated and its coordinates separated off into `Aders` and `wders`. Algorithm `A4.2` computes the point, $\mathbf{C}(u)$, and the derivatives, $\mathbf{C}^{(k)}(u)$, $1 \le k \le d$. The curve point is returned in `CK[0]` and the kth derivative is returned in `CK[k]`. The array `Bin[][]` contains the precomputed binomial coefficients (`Bin[k][i]` is $\binom{k}{i}$).

```
ALGORITHM A4.2
  RatCurveDerivs(Aders,wders,d,CK)
    { /* Compute C(u) derivatives from Cw(u) derivatives */
      /* Input:  Aders,wders,d  */
      /* Output: CK  */
    for (k=0; k<=d; k++)
      {
      v = Aders[k];
      for (i=1; i<=k; i++)
        v = v - Bin[k][i]*wders[i]*CK[k-i];
      CK[k]  = v/wders[0];
      }
    }
```

Figure 4.7 shows the first, second, and third derivatives of a cubic NURBS curve. The derivative vectors are scaled down by 0.4, 0.08, and 0.03, respectively.

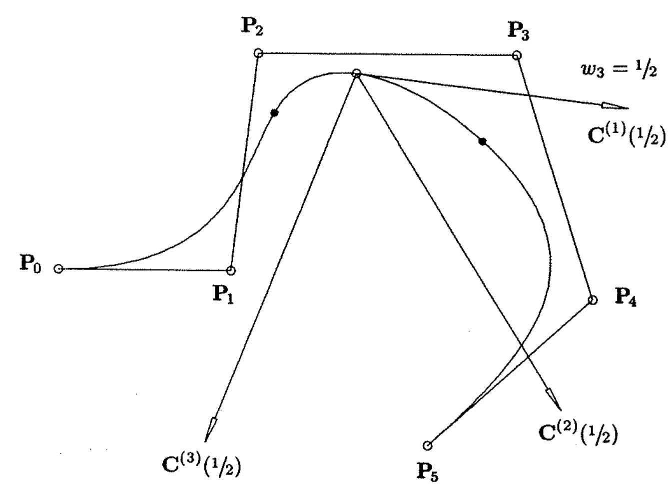


Figure 4.7. First, second, and third derivatives of a cubic NURBS curve computed at $u = 1/2$, with $w_3 = 1/2$ and $w_i = 1, i \ne 3$.

4.4 Definition and Properties of NURBS Surfaces

A NURBS surface of degree p in the u direction and degree q in the v direction is a bivariate vector-valued piecewise rational function of the form

$$\mathbf{S}(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}\,\mathbf{P}_{i,j}}{\sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u)N_{j,q}(v)w_{i,j}} \qquad 0 \le u,v \le 1 \tag{4.11}$$

The $\{\mathbf{P}_{i,j}\}$ form a bidirectional control net, the $\{w_{i,j}\}$ are the weights, and the $\{N_{i,p}(u)\}$ and $\{N_{j,q}(v)\}$ are the nonrational B-spline basis functions defined on the knot vectors

$$U = \{\underbrace{0,\ldots,0}_{p+1}, u_{p+1},\ldots,u_{r-p-1},\underbrace{1,\ldots,1}_{p+1}\}$$

$$V = \{\underbrace{0,\ldots,0}_{q+1}, v_{q+1},\ldots,v_{s-q-1},\underbrace{1,\ldots,1}_{q+1}\}$$

where $r = n+p+1$ and $s = m+q+1$.

Introducing the piecewise rational basis functions

$$R_{i,j}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)w_{i,j}}{\sum_{k=0}^{n}\sum_{l=0}^{m} N_{k,p}(u)N_{l,q}(v)w_{k,l}} \tag{4.12}$$

the surface Eq. (4.11) can be written as

$$\mathbf{S}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}(u,v)\,\mathbf{P}_{i,j} \tag{4.13}$$

Figures 4.8 and 4.9 show examples of NURBS surfaces.

The important properties of the functions $R_{i,j}(u,v)$ are roughly the same as those given in Section 3.4 for the nonrational basis functions, $N_{i,p}(u)\,N_{j,q}(v)$. We summarize them here.

P4.15 Nonnegativity: $R_{i,j}(u,v) \ge 0$ for all i, j, u, and v;

P4.16 Partition of unity: $\sum_{i=0}^{n}\sum_{j=0}^{m} R_{i,j}(u,v) = 1$ for all $(u,v) \in [0,1]\times[0,1]$;

P4.17 Local support: $R_{i,j}(u,v) = 0$ if (u,v) is outside the rectangle given by $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$;

P4.18 In any given rectangle of the form $[u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, at most $(p+1)(q+1)$ basis functions are nonzero, in particular the $R_{i,j}(u,v)$ for $i_0 - p \le i \le i_0$ and $j_0 - q \le j \le j_0$ are nonzero;

P4.19 Extrema: if $p > 0$ and $q > 0$, then $R_{i,j}(u,v)$ attains exactly one maximum value;

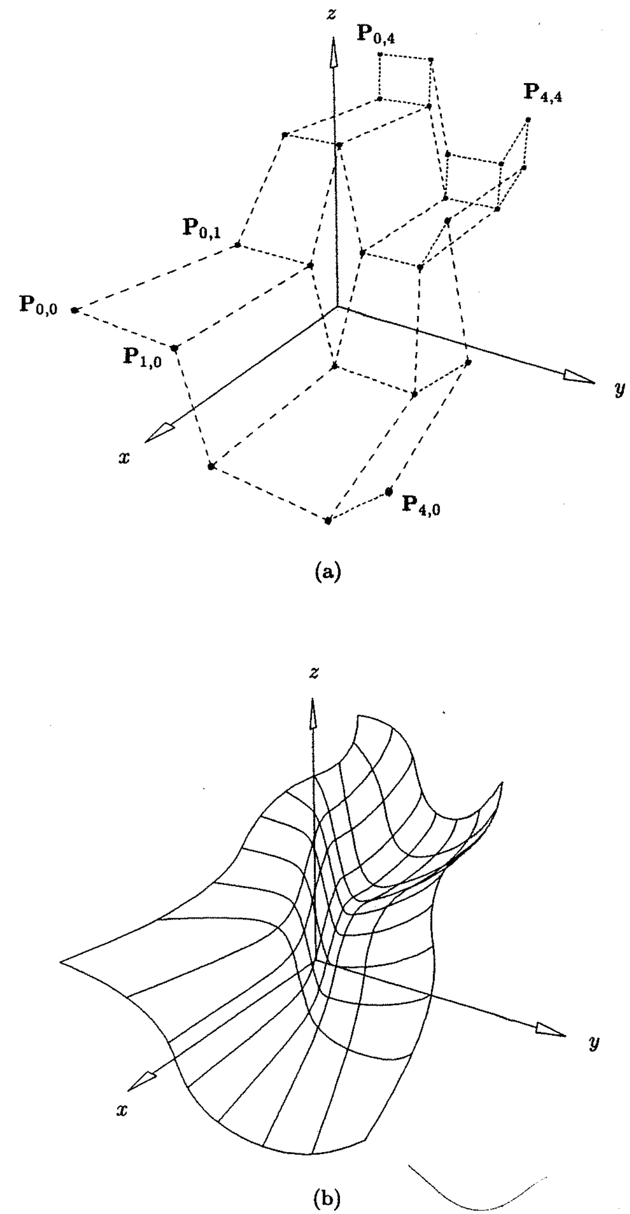


Figure 4.8. Control net and biquadratic NURBS surface, $w_{1,1} = w_{1,2} = w_{2,1} = w_{2,2} = 10$, with the rest of the weights 1. $U = V = \{0,0,0,1/3,2/3,1,1,1\}$. (a) Control net; (b) biquadratic NURBS surface.

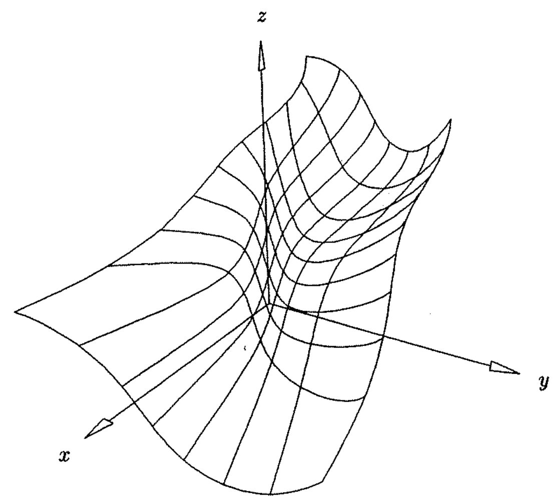


Figure 4.9. Bicubic NURBS surface defined by the control net in Figure 4.8a, with $U = V = \{0, 0, 0, 0, 1/2, 1, 1, 1, 1\}$ and with the same weights as in Figures 4.8.

P4.20 $R_{0,0}(0,0) = R_{n,0}(1,0) = R_{0,m}(0,1) = R_{n,m}(1,1) = 1$;

P4.21 Differentiability: interior to the rectangles formed by the u and v knot lines, all partial derivatives of $R_{i,j}(u,v)$ exist. At a u knot (v knot) it is $p-k$ $(q-k)$ times differentiable in the u (v) direction, where k is the multiplicity of the knot;

P4.22 If all $w_{i,j} = a$ for $0 \le i \le n$, $0 \le j \le m$, and $a \ne 0$, then $R_{i,j}(u,v) = N_{i,p}(u)N_{j,q}(v)$ for all i, j.

Properties P4.15–P4.22 yield the following important geometric properties of NURBS surfaces:

P4.23 Corner point interpolation: $\mathbf{S}(0,0) = \mathbf{P}_{0,0}$, $\mathbf{S}(1,0) = \mathbf{P}_{n,0}$, $\mathbf{S}(0,1) = \mathbf{P}_{0,m}$, and $\mathbf{S}(1,1) = \mathbf{P}_{n,m}$;

P4.24 Affine invariance: an affine transformation is applied to the surface by applying it to the control points;

P4.25 Strong convex hull property: assume $w_{i,j} \ge 0$ for all i, j. If $(u,v) \in [u_{i_0}, u_{i_0+1}) \times [v_{j_0}, v_{j_0+1})$, then $\mathbf{S}(u,v)$ is in the convex hull of the control points $\mathbf{P}_{i,j}$, $i_0 - p \le i \le i_0$ and $j_0 - q \le j \le j_0$;

P4.26 Local modification: if $\mathbf{P}_{i,j}$ is moved, or $w_{i,j}$ is changed, it affects the surface shape only in the rectangle $[u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$;

P4.27 Nonrational B-spline and Bézier and rational Bézier surfaces are special cases of NURBS surfaces;

P4.28 Differentiability: $\mathbf{S}(u,v)$ is $p-k$ $(q-k)$ times differentiable with respect to u (v) at a u knot (v knot) of multiplicity k.

We remark that there is no known variation diminishing property for NURBS surfaces (see [Prau92]).

We can use both control point movement and weight modification to locally change the shape of NURBS surfaces. Figures 4.10 and 4.11 show the effects on the basis function $R_{i,j}(u,v)$ and the surface shape when a single weight, $w_{i,j}$, is modified. Compare these figures with Figures 3.19b and 3.20a,b. Qualitatively, the effect on the surface is: Assume $(u,v) \in [u_i, u_{i+p+1}) \times [v_j, v_{j+q+1})$; then if $w_{i,j}$ increases (decreases), the point $\mathbf{S}(u,v)$ moves closer to (farther from) $\mathbf{P}_{i,j}$, and hence the surface is pulled toward (pushed away from) $\mathbf{P}_{i,j}$. As is the case for curves, the movement of $\mathbf{S}(u,v)$ is along a straight line. In Figure 4.12 (u,v) are fixed and $w_{2,2}$ is changing. Let

$$\mathbf{S} = \mathbf{S}(u, v; w_{2,2} = 0)$$
$$\mathbf{M} = \mathbf{S}(u, v; w_{2,2} = 1) \tag{4.14}$$

Then the straight line defined by $\mathbf{S}$ and $\mathbf{M}$ passes through $\mathbf{P}_{2,2}$, and for arbitrary $w_{2,2}$, $0 < w_{2,2} < \infty$, $\mathbf{S}_{2,2} = \mathbf{S}(u, v; w_{2,2})$ lies on this line segment between $\mathbf{S}$ and $\mathbf{P}_{2,2}$.

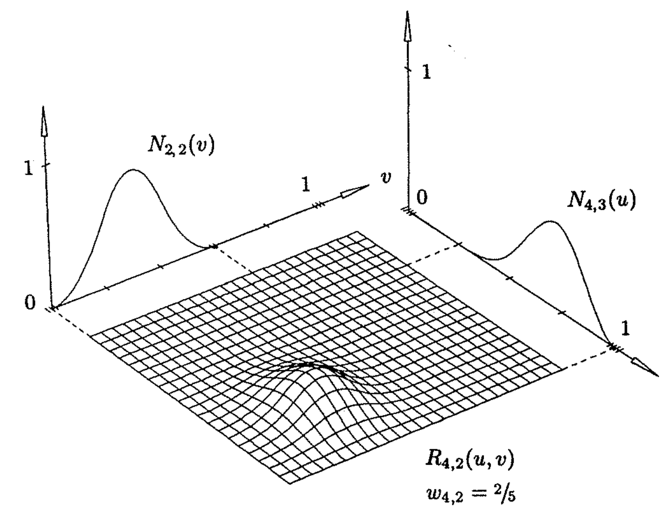
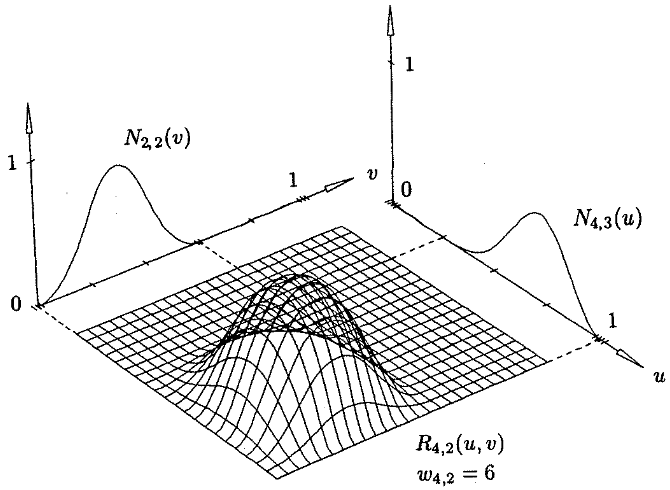


Figure 4.10. The basis function $R_{4,2}(u,v)$, with $U = \{0, 0, 0, 0, 1/4, 1/2, 3/4, 1, 1, 1, 1\}$ and $V = \{0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\}$. $w_{i,j} = 1$ for all $(i,j) \ne (4,2)$. (a) $w_{4,2} = 2/5$; (b) $w_{4,2} = 6$.



(b)

Figure 4.10. (*Continued.*)

It is convenient to represent a NURBS surface using homogeneous coordinates, that is

$$\mathbf{S}^w(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) \mathbf{P}_{i,j}^w \tag{4.15}$$

where $\mathbf{P}_{i,j}^w = (w_{i,j}x_{i,j}, w_{i,j}y_{i,j}, w_{i,j}z_{i,j}, w_{i,j})$. Then $\mathbf{S}(u,v) = H\{\mathbf{S}^w(u,v)\}$. We refer interchangeably to either $\mathbf{S}^w(u,v)$ or $\mathbf{S}(u,v)$ as the NURBS surface. Strictly speaking, $\mathbf{S}^w(u,v)$ is a tensor product, piecewise polynomial surface in four-dimensional space. $\mathbf{S}(u,v)$ is a piecewise rational surface in three-dimensional space; it is not a tensor product surface, since the $R_{i,j}(u,v)$ are not products of univariate basis functions.

Example

Ex4.3 Let $\mathbf{S}^w(u,v) = \sum_{i=0}^{7} \sum_{j=0}^{4} N_{i,2}(u) N_{j,2}(v)$, with

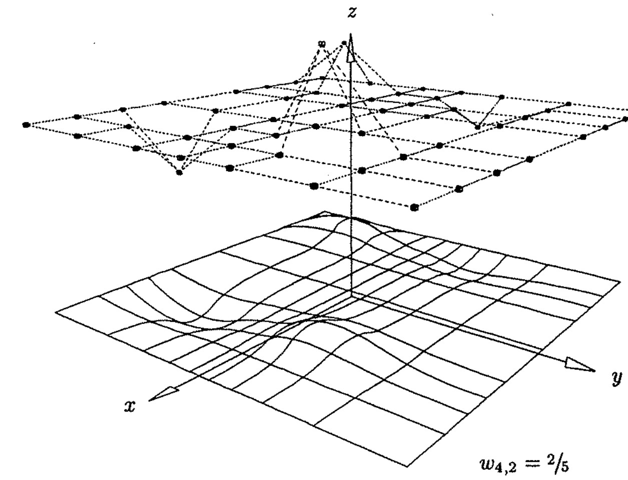
$$U = \{0,0,0,1,2,3,4,4,5,5,5\}$$

and

$$V = \{0,0,0,1,2,3,3,3\}$$

Let us evaluate the surface at $(u,v) = (5/2, 1)$. Then $u \in [u_4, u_5)$ and $v \in [v_3, v_4)$, and from Sections 3.2 and 4.2 we know that

$$N_{2,2}\left(\frac{5}{2}\right) = \frac{1}{8} \quad N_{3,2}\left(\frac{5}{2}\right) = \frac{6}{8} \quad N_{4,2}\left(\frac{5}{2}\right) = \frac{1}{8}$$



(a)

Figure 4.11. Cubic × quadratic surfaces corresponding to Figure 4.10, with the control net offset for better visualization. (a) $w_{4,2} = 2/5$; (b) $w_{4,2} = 6$.

and
$$N_{1,2}(1) = \frac{1}{2} \quad N_{2,2}(1) = \frac{1}{2} \quad N_{3,2}(1) = 0$$

Now assume that

$$[\mathbf{P}_{i,j}^w] = \begin{bmatrix} (0,2,4,1) & (0,6,4,2) & (0,2,0,1) \\ (4,6,8,2) & (12,24,12,6) & (4,6,0,2) \\ (4,2,4,1) & (8,6,4,2) & (4,2,0,1) \end{bmatrix}$$

$$i = 2,3,4; \quad j = 1,2,3$$

Then

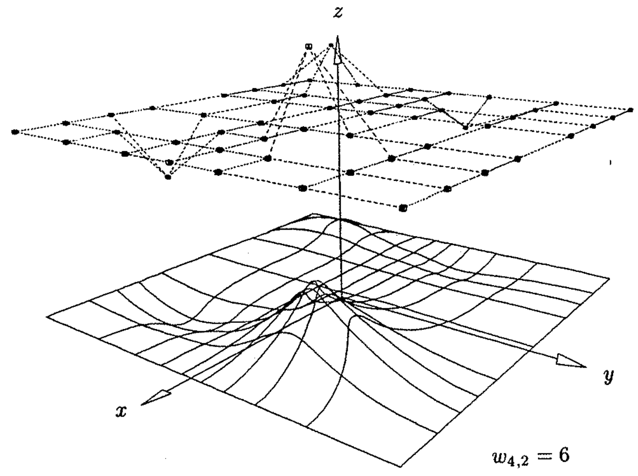
$$\mathbf{S}^w\left(\frac{5}{2}, 1\right) = \begin{bmatrix} \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \end{bmatrix} \times$$

$$\begin{bmatrix} (0,2,4,1) & (0,6,4,2) & (0,2,0,1) \\ (4,6,8,2) & (12,24,12,6) & (4,6,0,2) \\ (4,2,4,1) & (8,6,4,2) & (4,2,0,1) \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \left(\frac{54}{8}, \frac{98}{8}, \frac{68}{8}, \frac{27}{8}\right)$$

Projecting yields

$$\mathbf{S}\left(\frac{5}{2}, 1\right) = \left(2, \frac{98}{27}, \frac{68}{27}\right)$$



(b)

Figure 4.11. (*Continued.*)

Algorithm A3.5 can be adapted to compute a point on a rational B-spline surface by simply allowing the array P to contain weighted control points (use Pw), accumulating the four-dimensional surface point in Sw and inserting the line S = Sw/w at the end of the algorithm.

```
ALGORITHM A4.3
  SurfacePoint(n,p,U,m,q,V,Pw,u,v,S)
    { /*  Compute point on rational B-spline surface  */
      /*  Input:  n,p,U,m,q,V,Pw,u,v  */
      /*  Output: S  */
    uspan = FindSpan(n,p,u,U);
    BasisFuns(uspan,u,p,U,Nu);
    vspan = FindSpan(m,q,v,V);
    BasisFuns(vspan,v,q,V,Nv);
    for (l=0; l<=q; l++)
      {
      temp[l] = 0.0;
      for (k=0; k<=p; k++)
        temp[l] = temp[l] + Nu[k]*Pw[uspan-p+k][vspan-q+l];
      }
    Sw = 0.0;
    for (l=0; l<=q; l++)
      Sw = Sw + Nv[l]*temp[l];
    S = Sw/w;
    }
```

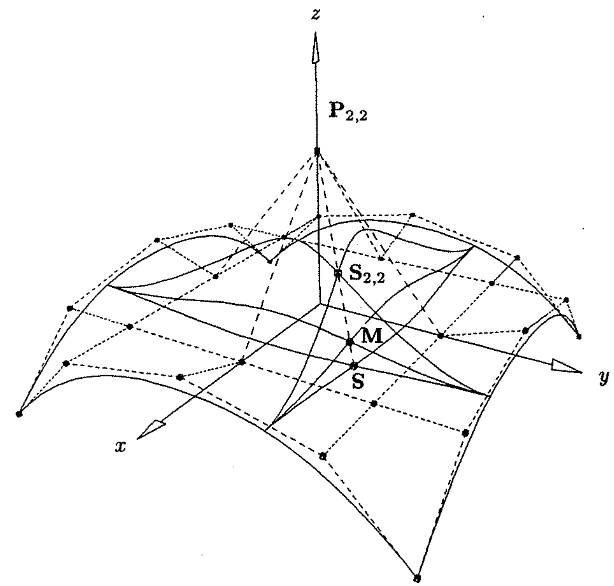



Figure 4.12. Modification of the weight $w_{2,2}$.

By applying Eqs. (3.14) and (3.15), we obtain isoparametric curves on a NURBS surface. First, fix $u = u_0$

$$\mathbf{C}^w_{u_0}(v) = \mathbf{S}^w(u_0, v) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,p}(u_0)N_{j,q}(v)\mathbf{P}^w_{i,j} \tag{4.16}$$

$$= \sum_{j=0}^{m} N_{j,q}(v)\left(\sum_{i=0}^{n} N_{i,p}(u_0)\mathbf{P}^w_{i,j}\right)$$

$$= \sum_{j=0}^{m} N_{j,q}(v)\mathbf{Q}^w_j(u_0)$$

where
$$\mathbf{Q}^w_j(u_0) = \sum_{i=0}^{n} N_{i,p}(u_0)\mathbf{P}^w_{i,j}$$

Analogously
$$\mathbf{C}^w_{v_0}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{Q}^w_i(v_0) \tag{4.17}$$

where
$$\mathbf{Q}^w_i(v_0) = \sum_{j=0}^{m} N_{j,q}(v_0)\mathbf{P}^w_{i,j}$$

$\mathbf{C}^w_{u_0}(v)$ ($\mathbf{C}^w_{v_0}(u)$) is a qth- (pth)-degree NURBS curve on the knot vector V (U). The point $\mathbf{S}^w(u_0, v_0)$ lies at the intersection of $\mathbf{C}^w_{u_0}(v)$ and $\mathbf{C}^w_{v_0}(u)$. Projecting yields

$$\mathbf{C}_{u_0}(v) = H\{\mathbf{C}^w_{u_0}(v)\} = H\{\mathbf{S}^w(u_0, v)\} = \mathbf{S}(u_0, v)$$

$$\mathbf{C}_{v_0}(u) = H\{\mathbf{C}^w_{v_0}(u)\} = H\{\mathbf{S}^w(u, v_0)\} = \mathbf{S}(u, v_0) \tag{4.18}$$

4.5 Derivatives of a NURBS Surface

The derivatives of $\mathbf{S}^w(u,v)$ are computed using Eqs. (3.17)–(3.24). We now derive formulas for the derivatives of $\mathbf{S}(u,v)$ in terms of those of $\mathbf{S}^w(u,v)$. Let

$$\mathbf{S}(u,v) = \frac{w(u,v)\mathbf{S}(u,v)}{w(u,v)} = \frac{\mathbf{A}(u,v)}{w(u,v)}$$

where $\mathbf{A}(u,v)$ is the numerator of $\mathbf{S}(u,v)$ (Eq. [4.11]). Then

$$\mathbf{S}_\alpha(u,v) = \frac{\mathbf{A}_\alpha(u,v) - w_\alpha(u,v)\mathbf{S}(u,v)}{w(u,v)} \tag{4.19}$$

where α denotes either u or v.

In general

$$\mathbf{A}^{(k,l)} = \left[(w\mathbf{S})^k\right]^l = \left(\sum_{i=0}^{k}\binom{k}{i} w^{(i,0)}\mathbf{S}^{(k-i,0)}\right)^l$$

$$= \sum_{i=0}^{k}\binom{k}{i}\sum_{j=0}^{l}\binom{l}{j} w^{(i,j)}\mathbf{S}^{(k-i,l-j)}$$

$$= w^{(0,0)}\mathbf{S}^{(k,l)} + \sum_{i=1}^{k}\binom{k}{i} w^{(i,0)}\mathbf{S}^{(k-i,l)} + \sum_{j=1}^{l}\binom{l}{j} w^{(0,j)}\mathbf{S}^{(k,l-j)}$$

$$+ \sum_{i=1}^{k}\binom{k}{i}\sum_{j=1}^{l}\binom{l}{j} w^{(i,j)}\mathbf{S}^{(k-i,l-j)}$$

and it follows that

$$\mathbf{S}^{(k,l)} = \frac{1}{w}\left(\mathbf{A}^{(k,l)} - \sum_{i=1}^{k}\binom{k}{i} w^{(i,0)}\mathbf{S}^{(k-i,l)}\right.$$

$$\left. - \sum_{j=1}^{l}\binom{l}{j} w^{(0,j)}\mathbf{S}^{(k,l-j)} - \sum_{i=1}^{k}\binom{k}{i}\sum_{j=1}^{l}\binom{l}{j} w^{(i,j)}\mathbf{S}^{(k-i,l-j)}\right) \tag{4.20}$$

From Eq. (4.20) we obtain

$$\mathbf{S}_{uv} = \frac{\mathbf{A}_{uv} - w_{uv}\mathbf{S} - w_u\mathbf{S}_v - w_v\mathbf{S}_u}{w} \tag{4.21}$$

$$\mathbf{S}_{uu} = \frac{\mathbf{A}_{uu} - 2w_u\mathbf{S}_u - w_{uu}\mathbf{S}}{w} \tag{4.22}$$

$$\mathbf{S}_{vv} = \frac{\mathbf{A}_{vv} - 2w_v\mathbf{S}_v - w_{vv}\mathbf{S}}{w} \tag{4.23}$$

From Eqs. (3.24), (4.19), and (4.20)

$$\mathbf{S}_u(0,0) = \frac{p}{u_{p+1}}\frac{w_{1,0}}{w_{0,0}}(\mathbf{P}_{1,0} - \mathbf{P}_{0,0}) \tag{4.24}$$

$$\mathbf{S}_v(0,0) = \frac{q}{v_{q+1}}\frac{w_{0,1}}{w_{0,0}}(\mathbf{P}_{0,1} - \mathbf{P}_{0,0}) \tag{4.25}$$

$$\mathbf{S}_{uv}(0,0) = \frac{pq}{w_{0,0}u_{p+1}v_{q+1}}\left(w_{1,1}\mathbf{P}_{1,1} - \frac{w_{1,0}w_{0,1}}{w_{0,0}}(\mathbf{P}_{1,0} + \mathbf{P}_{0,1})\right.$$

$$\left. + \left(\frac{2w_{1,0}w_{0,1}}{w_{0,0}} - w_{1,1}\right)\mathbf{P}_{0,0}\right) \tag{4.26}$$

Figure 4.13 shows the first- and second-order partial derivatives of a NURBS surface. The first partials are scaled down by 1/2, and the second partials are scaled down by 1/3.

Now assume that (u,v) is fixed, and that all derivatives $\mathbf{A}^{(k,l)}$, $w^{(k,l)}$ for $k,l \geq 0$ and $0 \leq k+l \leq d$, have been computed and loaded into the arrays `Aders` and `wders`, respectively. Algorithm `A4.4` computes the point $\mathbf{S}(u,v)$ and the derivatives $\mathbf{S}^{(k,l)}(u,v)$, $0 \leq k+l \leq d$. `Bin[][]` contains the precomputed binomial coefficients.

```
ALGORITHM A4.4
  RatSurfaceDerivs(Aders,wders,d,SKL)
    { /* Compute S(u,v) derivatives   */
      /* from Sw(u,v) derivatives    */
      /* Input:  Aders,wders,d  */
      /* Output: SKL  */
    for (k=0; k<=d; k++)
      for (l=0; l<=d-k; l++)
        {
        v = Aders[k][l];
        for (j=1; j<=l; j++)
          v = v - Bin[l][j]*wders[0][j]*SKL[k][l-j];
```

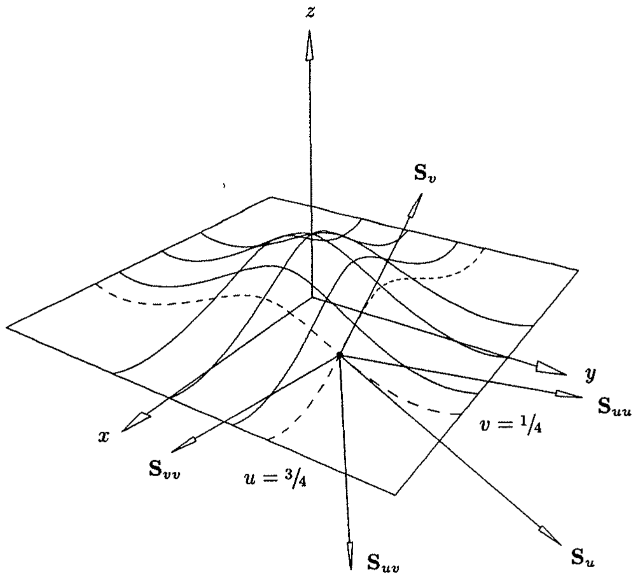


Figure 4.13. The first- and second-order partial derivatives of a bicubic NURBS surface computed at $u = 3/4$ and $v = 1/4$.

```
    for (i=1; i<=k; i++)
      {
      v = v - Bin[k][i]*wders[i][0]*SKL[k-i][l];
      v2 = 0.0;
      for (j=1; j<=l; j++)
        v2 = v2 + Bin[l][j]*wders[i][j]*SKL[k-i][l-j];
      v = v - Bin[k][i]*v2;
      }
    SKL[k][l] = v/wders[0][0];
    }
}
```

EXERCISES

4.1. Let $U = \{0, 0, 0, 1/3, 2/3, 1, 1, 1\}$ and $\{w_0, \ldots, w_4\}=\{1, 4, 1, 1, 1\}$. Using the Cox-deBoor recurrence formula (Eq. [2.5]) and Eq. (4.2), compute the five quadratic rational functions, $R_{i,2}(u)$, $0 \le i \le 4$. The graphs of these functions are shown in Figure 4.5a. Assume $\{\mathbf{P}_0, \ldots, \mathbf{P}_4\} = \{(0, 0), (1, 1), (3, 2), (4, 1), (5, -1)\}$ are control points in the xy plane. Compute the rational coordinate functions $x(u)$ and $y(u)$ representing $\mathbf{C}(u)$ in the interval $u \in [1/3, 2/3)$.

4.2. Refer to Example Ex4.2 for a quadratic rational Bézier circular arc; compute $\mathbf{C}''(1)$.

4.3. Let $\mathbf{C}^w(u) = \sum_{i=0}^{1} N_{i,1}(u)\mathbf{P}_i^w$ be a line segment in the xy plane, where $\mathbf{P}_0 = (0, 1)$, $\mathbf{P}_1 = (2, 0)$, $w_0 = 1$, $w_1 = 3$, and $U = \{0, 0, 1, 1\}$. Derive the rational functions representing the x and y coordinates of this line segment, i.e., $x(u)$ and $y(u)$, where $\mathbf{C}(u) = (x(u), y(u))$. Compute $\mathbf{C}'(0)$ using Eq. (4.9) and $\mathbf{C}''(0)$ using Eqs. (4.8) and (3.9). Then set $w_1 = 1$ and recompute $x(u)$, $y(u)$, $\mathbf{C}'(0)$, and $\mathbf{C}''(0)$.

4.4. Let $\mathbf{S}^w(u,v) = \sum_{i=0}^{1}\sum_{j=0}^{1} N_{i,1}(u)N_{j,1}(v)\mathbf{P}_{i,j}^w$, where $\{\mathbf{P}_{0,0}, \mathbf{P}_{1,0}, \mathbf{P}_{0,1}, \mathbf{P}_{1,1}\} = \{(0, 0, 1), (0, 1, 3), (2, 1, 1), (2, 0, 3)\}$, $\{w_{0,0}, w_{1,0}, w_{0,1}, w_{1,1}\} = \{2, 1, 1, 1\}$, and $U = V = \{0, 0, 1, 1\}$. Derive the four rational basis functions, $R_{i,j}(u,v)$, $0 \le i,j \le 1$, and the rational coordinate functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ of the surface $\mathbf{S}(u,v)$.

4.5. From $\mathbf{S}^w(u,v)$ in Exercise 4.4 derive the two rational isoparametric curves $\mathbf{C}_{u_0}^w(v)$ and $\mathbf{C}_{v_0}^w(u)$, for $u_0 = 1/3$ and $v_0 = 1/2$. Then evaluate the curves $\mathbf{C}_{u_0}(v)$ and $\mathbf{C}_{v_0}(u)$ at $v = 1/2$ and $u = 1/3$, respectively. Check your results by substituting $(u,v) = (1/3, 1/2)$ into the rational coordinate functions obtained in Exercise 4.4.

4.6. Let $\mathbf{S}^w(u,v) = \sum_{i=0}^{1}\sum_{j=0}^{1} N_{i,1}(u)N_{j,1}(v)\mathbf{P}_{i,j}^w$ be the surface given in Exercise 4.4. Since $N_{0,1}(1/2) = N_{1,1}(1/2) = 1/2$, it follows that

$$\mathbf{S}^w\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}\left(\mathbf{P}_{0,0}^w + \mathbf{P}_{0,1}^w + \mathbf{P}_{1,0}^w + \mathbf{P}_{1,1}^w\right) = \left(1, \frac{1}{2}, \frac{9}{4}, \frac{5}{4}\right)$$

Compute $\mathbf{S}_u(1/2, 1/2)$, $\mathbf{S}_v(1/2, 1/2)$, $\mathbf{S}_{uv}(1/2, 1/2)$, $\mathbf{S}_{uu}(1/2, 1/2)$.