The Stability of the Three Body Problem

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Outline

1. Introduction to the problem
2. Numerical Methods
   - Runge - Kutta and Method of Finite Elements
   - Max Step in Solve IVP
   - Numerical Stability
3. Perturbations
   - Perturbing Mass
   - Perturbing Speed
   - Perturbing Angle

Aims of the Project

- We aim to answer the following questions:
  - Is the system more sensitive to perturbations in certain initial conditions (mass, angle, speed) than others ?
  - What is the smallest possible perturbation we can make such that there is no significant change in the trajectories of the masses ?

Initial Conditions

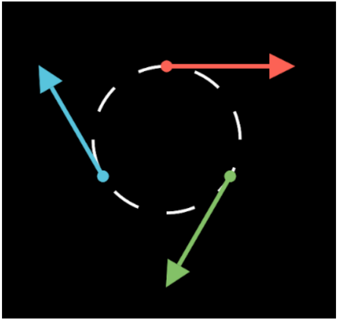


Figure: The initial Conditions

- This is a chaotic system

- **Mass** : 1 Solar Mass
- **Tangential Speed** : 1 km/s
- **Radius** : 1 pc
- **Time** : 1 Unit (14.91 Myr)

- The effective force between two masses has magnitude:

$$F = \frac{m^2}{r^2}$$

Forces acting on the system

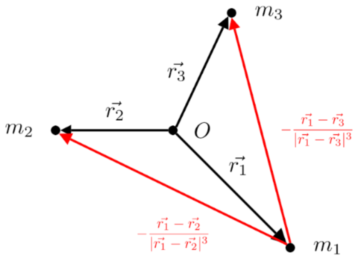


Figure: Forces on Mass 1

$$m_1 \ddot{\vec{r_1}} = -\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}-\vec{r_3}}{|\vec{r_1}-\vec{r_3}|^3} \quad (1)$$

$$m_2 \ddot{\vec{r_2}} = -\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_2}-\vec{r_1}|^3} - \frac{\vec{r_2}-\vec{r_3}}{|\vec{r_2}-\vec{r_3}|^3} \quad (2)$$

$$m_3 \ddot{\vec{r_3}} = -\frac{\vec{r_3}-\vec{r_2}}{|\vec{r_3}-\vec{r_2}|^3} - \frac{\vec{r_3}-\vec{r_1}}{|\vec{r_3}-\vec{r_1}|^3} \quad (3)$$

Runge - Kutta and Finite Element Method

The Runge - Kutta Method

- This is a way of numerically solving differential equations

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$t_{n+1} = t_n + h$$

$$k_1 = f(t_n, y_n)$$

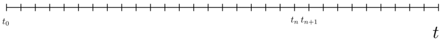
$$k_2 = f(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2})$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2})$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

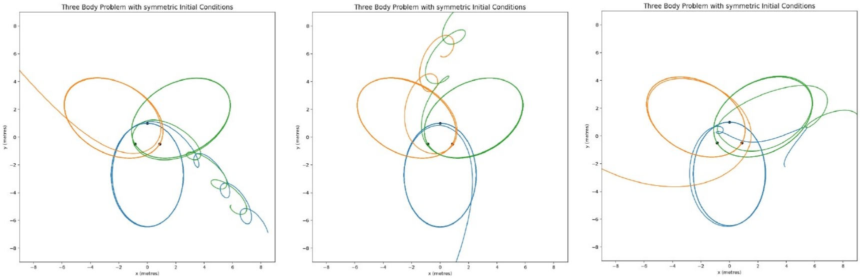
Finite - Element Method

- Splitting up time into a finite number of smaller time steps



Max Step

- The solve_ivp() function takes a parameter max_step, which alters the maximum possible value for h in the Runge-Kutta method.



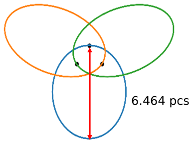
Max step = 0.1

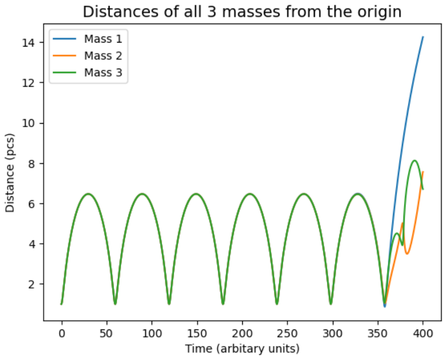
Max step = 0.01

Max step = 0.001

Numerical Instability

- Small errors in the input cause a considerably larger error in the final output
- We only have numerical stability up to **348 units** with a maxstep of 0.01





Animation

What does maximum average deviation mean?

Definition

$$Maximum\ Average\ Deviation = max\left\{\frac{1}{3}\sum_{i=1}^{3}|\boldsymbol{r_{pi}}(t) - \boldsymbol{r_i}(t)| : 0 \leq t \leq 348\right\}$$

$\boldsymbol{r_{pi}}(t)$ Position Vector of Particle i after the perturbation at time t

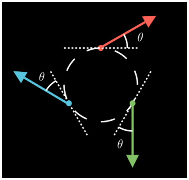
$\boldsymbol{r_i}(t)$ Position Vector of Particle i (unperturbed) at time t

348 The time for which the system is numerically stable in arbitrary units of time

Perturbations

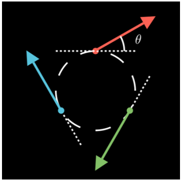
Symmetric Perturbation

This is when we apply the same (positive) perturbation to all 3 masses

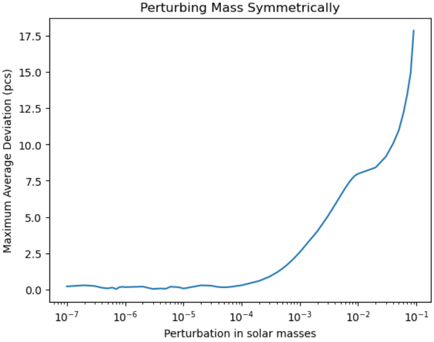


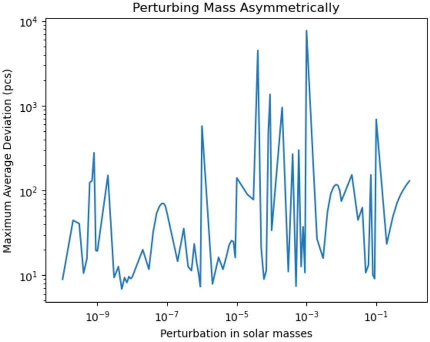
Asymmetric Perturbation

This is when we apply a (positive) perturbation to only one mass

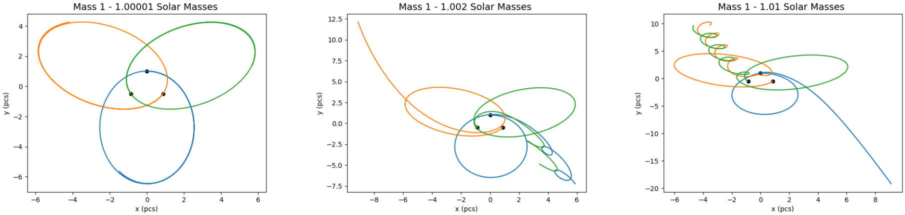


Perturbing Mass

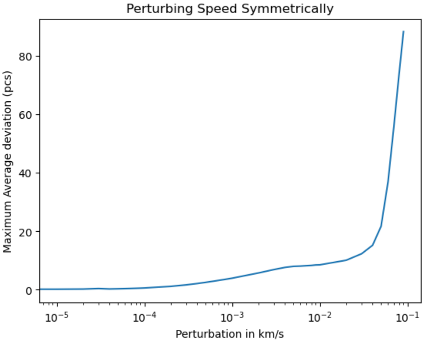


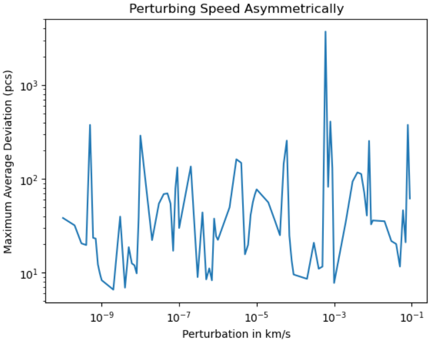


Asymmetric Perturbations in Mass



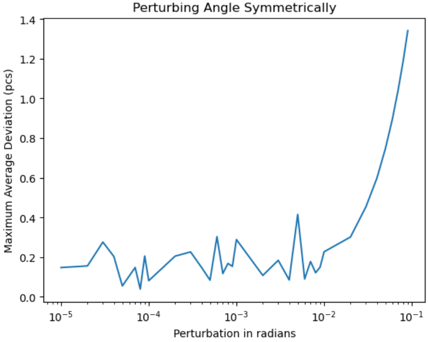
Perturbing Speed

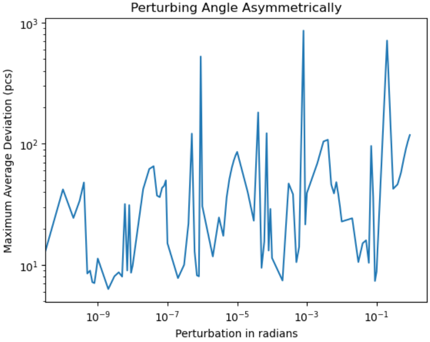




Animation

Perturbing Angle





Animation

Conclusion

- The system is more sensitive to asymmetric perturbations than symmetric perturbations
- The system is more sensitive to symmetric perturbations in mass than angle and speed
- The smallest possible symmetric perturbation we can make such that there is no significant change in the trajectories of the masses is about 10^{-2} km/s or radians for angle and speed and about 10^{-4} solar masses for mass
- No such value exists for asymmetric perturbations

- Thank You -