Free Surface Flows - Supplementary Material

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1 Introduction

These are set of footnotes for the presentation that was delivered on the free-surface slows due to a moving pressure distribution. Here we expound on certain interesting ideas that did not make it to the presentation due to time-constraints. We recommend reading these notes after the presentation, as we refer to certain slides in the subsequent sections.

2 Stationary Wave Patterns

Most commonly studied wave phenomena, such as Stokes Waves, are unsteady, where the fluid motion changes with time. However, the waves described are part of a completely uniform steady stream and as shown in the presentation these waves do not depend on time. This is called a stationary wave pattern (not to be confused with standing waves).

3 Gravity - Capillary Waves

The dispersion relation for gravity-capillary waves has a unique minimum at $c = c_{min}$ when $k = k_{min}$. By arguing graphically we showed that if $U < c_{min}$, then there are no waves generated, but if $U > c_{min}$, there are two values of k such that $c(k) = U$ and these correspond to the the two wave trains that are upstream and downstream of the obstacle. We now argue why it is the case that the upstream-travelling wave train is a capillary wave and the downstream-travelling wave is a gravity wave.

Let the two values of k such that $c(k) = U$ be k_1 and k_2, where $k_1 < k_{min} < k_2$. Given below is the dispersion relation for gravity-capillary waves:

$$\omega^2 = gk\left(1 + \frac{Tk^2}{\rho g}\right). \tag{1}$$

We define the parameter L as follows:

$$L = \frac{Tk^2}{\rho g}. \tag{2}$$

Then it follows from 1 that L determines the relative importance of surface tension compared to gravity.

We now first examine the downstream wave train where $k = k_1$. Since $k_1 < k_{min}$, we have $L < 1$. Thus this wave train is dominated by gravity. Furthermore, the energy of this gravity waves propagates at the group velocity[1] and we have the following relation between the group velocity (c_g) and phase velocity (c) for gravity waves.

$$c_g = \frac{1}{2}c. \tag{3}$$

In the rest frame of the disturbance, the energy of the wave propagates to the left at a speed of $c_g - U$. Then since we have $c_g < c$ for the gravity wave, the energy of this wave train must be travelling downstream.

Now, we proceed to the capillary waves, which correspond to $k = k_2$. Since $k > k_{min}$, we have $L > 1$, so for these waves the effects of surface tension dominate. Furthermore, for capillary waves, we have the following relation between the group velocity (c_g) and phase velocity (c) for gravity waves:

$$c_g = \frac{3}{2}c. \tag{4}$$

As a result of the fact that $c_g > c$, the energy of this wavetrain travels upstream to the left.

4 A Brief Note on the Radiation Condition

There are several different interpretations of motivation for implementing the radiation condition through the addition of Rayleigh Viscosity and in this section, we will explore some of them.

The radiation condition essentially states that there is no energy coming from infinity. This means that if we have a wave train to the right of the obstacle, then the energy of this wave must be propagating downstream. Similarly, any wave train to the left of the obstacle must have its energy propagating upstream. It is now clear why we must have gravity waves to the right of the obstacle and capillary waves to the left of the obstacle. If switched positions (i.e. capillary waves on the right and gravity waves on the left), then we would have energy propagating towards the obstacle, which violates the radiation condition.

The first interpretation is from Vanden-Broeck[2][3]. Vanden-Broek argues that the solution to the set of equations in slide 4 does not admit a unique solution unless we implement the radiation condition and using a well-known trick by Rayleigh[4] that this can be achieved for free by adding a retarding force to the velocity in the governing equation - a so-called "Rayleigh Viscosity".

Another interpretation due to Lighthill[1] suggests that the role of Rayleigh Viscosity is that is regularises the solution. If we set $\mu = 0$ in the integral on slide 11, we encounter an interesting issue when applying the residue theorem. It is that the poles of the integrand in question lie on the path of integration! This is essentially alleviated by the Rayleigh viscosity, which gives the poles of the integrand a non-zero imaginary part, which shifts them away from the real-axis.

References

[1] Lighthill J. *Waves in Fluids*. Vol. 90. 3. 1979.

[2] Jean-Marc Vanden-Broeck. *Gravity–Capillary Free-Surface Flows*. Cambridge Monographs on Mechanics. Cambridge University Press, 2010.

[3] Emilian Părău, Jean-Marc Vanden-Broeck, and M. Cooker. "Time evolution of three-dimensional nonlinear gravity–capillary free-surface flows". In: *Journal of Engineering Mathematics* 68 (Dec. 2010), pp. 291–300. DOI: `10.1007/s10665-010-9391-y`.

[4] Lord Rayleigh. "The Form of Standing Waves on the Surface of Running Water". In: *Proceedings of the London Mathematical Society* s1-15.1 (Nov. 1883), pp. 69–78. ISSN: 0024-6115. DOI: `10.1112/plms/s1-15.1.69`. eprint: `https://academic.oup.com/plms/article-pdf/s1-15/1/69/4292115/s1-15-1-69.pdf`. URL: `https://doi.org/10.1112/plms/s1-15.1.69`.