Free Surface Flows Generated by a Moving Pressure Distribution

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Outline

- **Formulation**
  - Assumptions
  - The Non-Linear Problem
  - Deriving the Linear Problem
- **Solution**
  - Rayleigh Viscosity
  - Integral Expression for Free-Surface
- **Discussion**
  - Model Predictions
  - Critical Point

Assumptions

Our Assumptions

- Fluid is incompressible
- Fluid is irrotational
- Fluid is inviscid
- Surface tension is constant
- Non-negligible gravity
- Infinite depth

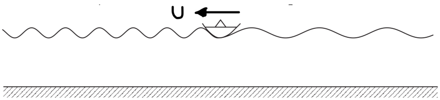

Figure: Moving Frame of disturbance

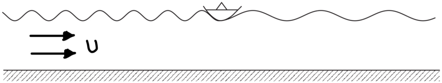

Figure: Rest Frame of disturbance

Formulation of the Non-Linear Problem

- Laplace's Equation

$$\phi_{xx} + \phi_{yy} = 0 \,, \ -\infty < y < \eta(x)$$

- Kinematic Boundary Condition

$$\phi_y = \phi_x \eta_x \quad \text{on} \quad y = \eta(x)$$

- Dynamic Boundary Condition

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + \epsilon\frac{P(x)}{\rho} = \frac{1}{2}U^2 \quad \text{on} \quad y = \eta(x)$$

- Boundary Condition at infinity

$$\phi_x \to U, \ \ \phi_y \to 0 \quad \text{as} \quad y \to -\infty$$

Deriving the Linear Problem

- Linearisation

$$\phi(x,y) = Ux + \epsilon\phi_1(x,y) + O(\epsilon^2)$$
$$\eta(x) = \epsilon\eta_1(x) + O(\epsilon^2)$$

Deriving the Linear Problem

- Linearisation

$$\phi(x,y) = Ux + \epsilon\phi_1(x,y) + O(\epsilon^2)$$

$$\eta(x) = \epsilon\eta_1(x) + O(\epsilon^2)$$

- Substituting into above

$$\epsilon\phi_{1y} = (U + \epsilon\phi_{1x})(\epsilon\eta_{1x})$$

$$\frac{1}{2}((U+\epsilon\phi_{1x})^2 + (\epsilon\phi_{1y})^2) + g(\epsilon\eta_1) - \frac{T}{\rho}\frac{(\epsilon\eta_{1xx})}{(1+(\epsilon\eta_{1x})^2)^{\frac{3}{2}}} + \epsilon\frac{P(x)}{\rho} = \frac{1}{2}U^2$$

$$(U+\epsilon\phi_{1x}) \to U, \quad (\epsilon\phi_{1y}) \to 0$$

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$$(U+\epsilon\phi_{1x}) \to U, \quad (\epsilon\phi_{1y}) \to 0$$

- By expanding, ignoring ϵ^2 terms and collecting the ϵ terms we get our **linear problem**

The Linear Problem

$$\phi_{1xx} + \phi_{1yy} = 0, \quad y < 0$$

$$\phi_{1y} = U\eta_{1x} \quad \text{on} \quad y = 0$$

$$U\phi_{1x} + g\eta_1 - \frac{T}{\rho}\eta_{1xx} + \frac{P}{\rho} = 0 \quad \text{on} \quad y = 0$$

$$\phi_{1x} \to 0, \ \phi_{1y} \to 0 \quad \text{as} \quad y \to -\infty$$

Remark: Viscosity

- To get a unique solution, we enforce the **Radiation Condition**
- We introduce **Rayleigh viscosity** μ, and later we take the limit as μ tends to 0.
- Then our Dynamic Boundary Condition becomes

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + \epsilon\frac{P(x)}{\rho} + \mu\phi_1 = \frac{1}{2}U^2$$

Solution in Infinite Depth

$$\phi_1 = \int_{-\infty}^{\infty} F(a, y) e^{iax} \, dx$$

- Then Laplace's Equation becomes

$$\frac{\partial^2 F}{\partial y^2} - a^2 F = 0$$

$$\frac{\partial F}{\partial y} \to 0 \text{ as } y \to -\infty$$

- This gives the solution

$$F = A(a) e^{|a|y}$$

- Thus we have

$$\phi_1 = \int_{-\infty}^{\infty} A(a) e^{|a|y} e^{iax} \, dx$$

Solution in Infinite Depth

$$\phi_1 = \int_{-\infty}^{\infty} A(a) e^{|a|y} e^{iax}\ dx \qquad P(x) = \int_{-\infty}^{\infty} B(a) e^{iax}\ da$$

- Modified Dynamic Boundary Condition

$$U\phi_{1xx} + \frac{g}{U}\phi_{1y} - \frac{T}{\rho U}\phi_{1yxx} + \frac{P_x}{\rho} + \mu\phi_{1x} = 0 \quad \text{on} \quad y = 0.$$

- Substituting the Integral Expressions to

$$A(a) = \frac{iB(a)}{\rho U D(a)}$$

$$\text{where} \quad D(a) = a - \frac{g|a|}{U^2 a} - \frac{Ta^2}{\rho U^2} - i\mu_1 \quad \text{and} \quad \mu_1 = \frac{\mu}{U}$$

Integral Expression for the Free-Surface

Using the kinematic boundary condition and the integral expression for ϕ_1 we have:

$$\eta_1(x) = \frac{1}{\rho U^2} \int_{-\infty}^{\infty} \frac{|a|\, B(a)\, e^{iax}}{a\, D(a)}\, da$$

By symmetry arguments, we simplify the expression for η :

$$\eta_1(x) = \frac{2}{\rho U^2} \Re \int_{0}^{\infty} \frac{e^{iax} B(a)}{D(a)}\, da$$

Evaluating the Integral

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We want to take the limit as $\mu_1 \to 0$ so we consider two cases:

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1. **Case 1:** $\Re D(a) \neq 0 \quad \forall a > 0$

$$\eta_1(x) = \frac{2}{\rho U^2} \Re \int_0^\infty \frac{e^{iax} B(a)}{a - (g/U^2) - (Ta^2/\rho U^2)} \, da$$

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2. **Case 2:** $\exists\, a^* > 0 \quad \text{such that} \quad \Re D(a*) = 0$

Dispersion Relation

If $\exists\, a^* > 0 \quad \text{such that} \quad \Re D(a*) = 0$ then we have:

$$a^* - \frac{g}{U^2} - \frac{T(a^*)^2}{\rho U^2} = 0$$

This can be rearranged as follows:

$$U^2 = \frac{g}{a^*} + \frac{Ta^*}{\rho}$$

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Thus a* is the wave number satisfying the dispersion relation for gravity capillary waves in infinite depth with c = U.

$$C(k)^2 = \frac{g}{k} + \frac{Tk}{\rho} \implies (C(a^*))^2 = U^2$$

Dispersion Relation

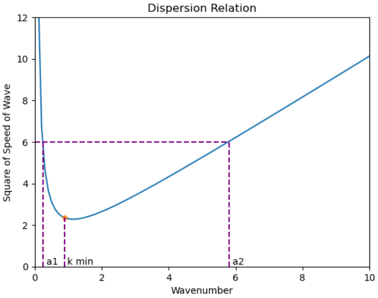


Figure: How speed varies with depth

- For $U > c_{min}$ we have two **positive** roots for $\Re D(a)$, which are $0 < a_1^* < k_{min} < a_2^*$
- Since $\Re D(a)$ is odd in a, the roots of $\Re D(a)$ are $\pm a_1^*$ and $\pm a_2^*$

Finding the roots of D(a)

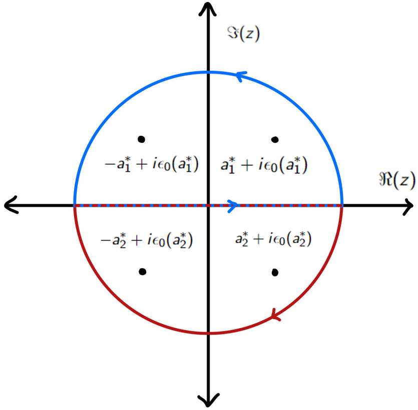
- Approximate roots of D(a) (call it a) using the roots of $\Re D(a)$ (call it a^*). Then:

$$a = a^* + \epsilon(a^*)$$

- By using a Taylor expansion and keeping only leading order terms we have:

$$a = a^* + i\epsilon_0(a^*) \quad \epsilon_0(a^*) = \frac{-\mu_1 U}{2a^* C'(a^*)}$$

Application of Residue Theorem



- If $x > 0$, close the contour in the upper half-plane

$$\eta_1(x) = \frac{2\pi}{\rho U} \frac{B(a_1^*)}{a_1^* C'(a_1^*)} sin(a_1^* x)$$

- If $x < 0$, close the contour in the lower half-plane

$$\eta_1(x) = -\frac{2\pi}{\rho U} \frac{B(a_2^*)}{a_2^* C'(a_2^*)} sin(a_2^* x)$$

Discussing the Solution

- Recall $c_{min} = \left(\frac{4Tg}{\rho}\right)^{\frac{1}{4}}$ when $k = k_{min} = \left(\frac{\rho g}{T}\right)^{\frac{1}{2}}$

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Thus with $\alpha = \frac{Tg}{\rho U^4}$:

$$\boxed{U > c_{min} \quad \text{when} \quad \alpha < 0.25}$$

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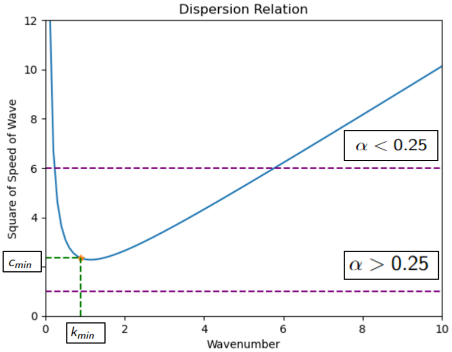

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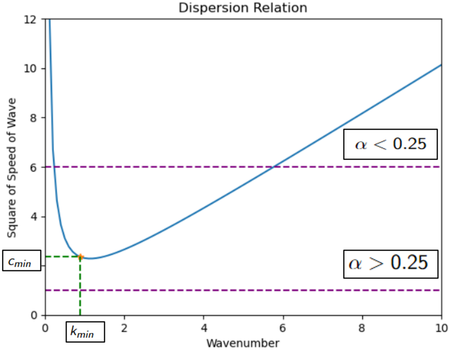

$\Re D(a)$ has real roots when $\alpha < 0.25$

$\Re D(a)$ has complex roots when $\alpha > 0.25$

Model Prediction

- If $\alpha < 0.25$ then:

$$\eta_1(x) = \frac{2\pi}{\rho U}\frac{B(a_1^*)}{a_1^* C'(a_1^*)} sin(a_1^* x) \quad \text{as } x \to +\infty$$

$$\eta_1(x) = -\frac{2\pi}{\rho U}\frac{B(a_2^*)}{a_2^* C'(a_2^*)} sin(a_2^* x) \quad \text{as } x \to -\infty$$

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- If $\alpha > 0.25$ then:

$$\eta_1(x) = \frac{2}{\rho U^2}\Re \int_0^\infty \frac{e^{iax}B(a)}{a - (g/U^2) - (Ta^2/\rho U^2)}\, da$$

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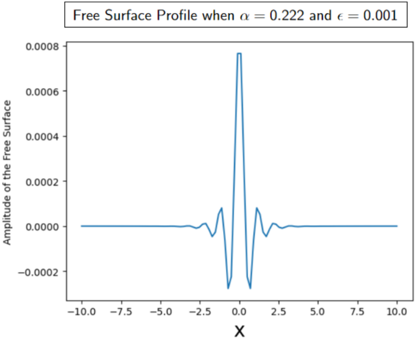
Riemann - Lebesgue

If $f \in L^1(\mathbb{R})$, then $\hat{f}(k) \to 0$ as $k \to \pm\infty$

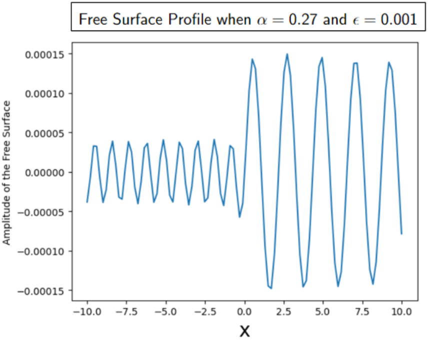
Model Prediction

$$P(x) = \frac{\rho U^2}{2} \exp \frac{-5gx^2}{u^4}$$

$\alpha < 0.25$



$\alpha > 0.25$



A Limitation of the Linear Theory

Using $\eta_1(x) = \frac{2}{\rho U^2} \Re \int_0^\infty \frac{e^{iax} B(a)}{a - (g/U^2) - (Ta^2/\rho U^2)}\, da$, we let $\alpha \to 0.25^+$

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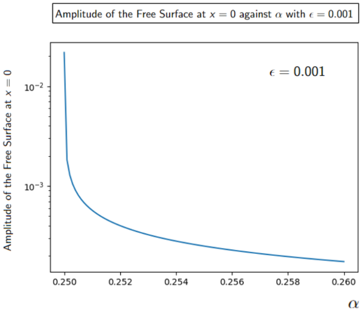

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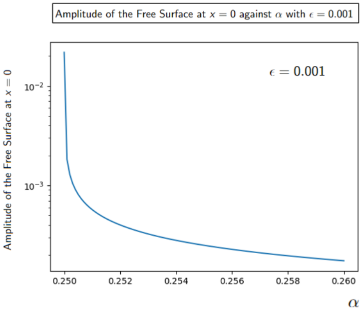

- Our linearisation breaks down as $\alpha \to 0.25^+$

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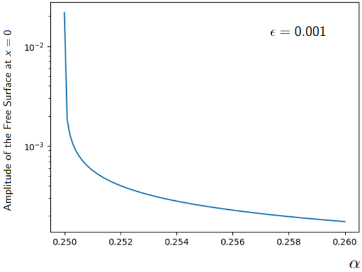
Amplitude of the Free Surface at $x = 0$ against α with $\epsilon = 0.001$



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- Need non-linear theory to describe this better

- Thank You -