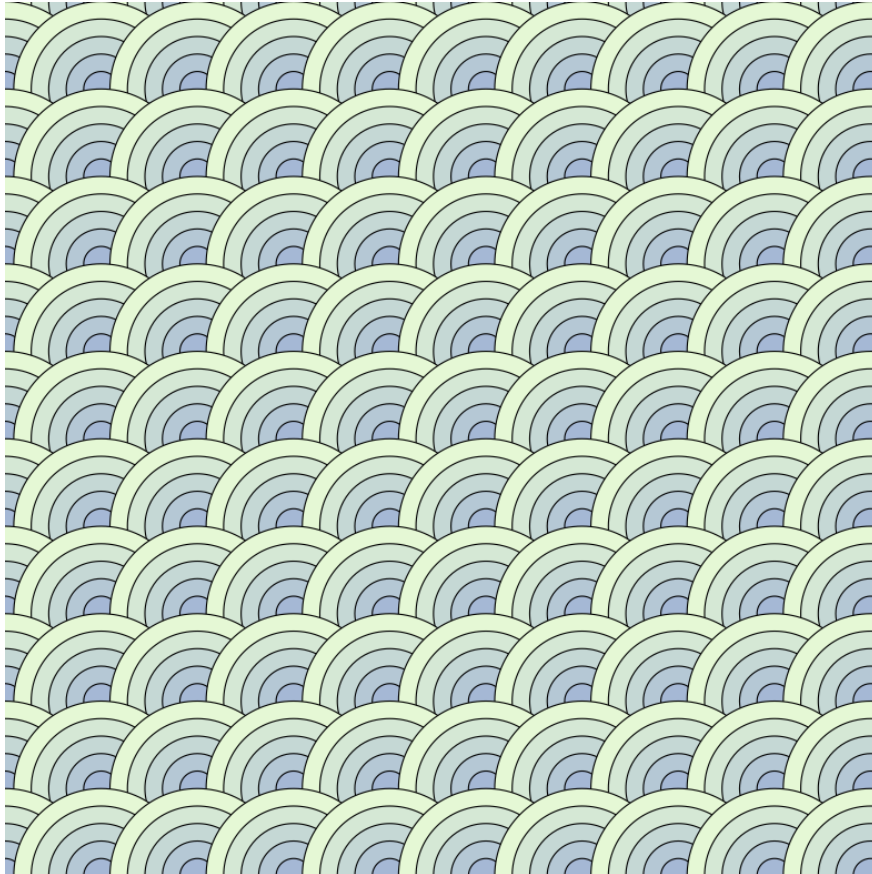


# Abstract Algebra

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Based on lectures UECM 3363 Abstract Algebra in Universiti Tunku Abdul Rahman in 2020  
Notes taken by MurphyShark.

# Groups

## Definition 1.1 Closure

Let  $G$  be a set. A binary operation on  $G$  is a function that assigns each ordered pair of elements of  $G$  an element of  $G$ . This condition is called **closure**.

The most familiar binary operations are ordinary addition, subtraction and multiplication of integers. However, the division of integers is not a binary operation on the integers.

## Definition 1.2 Binary operation

Let  $G$  be a set. A **binary operation** is a map of sets:

$$* : G \times G \rightarrow G.$$

For ease of notation we write  $*(a, b) = a * b \quad \forall a, b \in G$ . Any binary operation on  $G$  gives a way of combining elements. As we have seen, if  $G = \mathbb{Z}$  then  $+$  and  $\times$  are natural example of binary operations.

Additive Group	Multiplicative Group
Let $G$ be a set, and $+$ be an operation, then $(G, +)$ is an additive group provided	Let $G$ be a set, and $\circ$ be an operation, then $(G, \circ)$ is an multiplicative group provided
1. $\forall a, b \in G, a + b \in G$	6. $\forall a, b \in G, a \circ b \in G$
2. $\forall a, b, c \in G, a + (b + c) = (a + b) + c$	7. $\forall a, b, c \in G, a \circ (b \circ c) = (a \circ b) \circ c$
3. $\forall a \in G, \exists 0 \in G$ (identity) s.t. $a + 0 = a = 0 + a$	8. $\forall a \in G, \exists 1 \in G$ (unity) s.t. $a \circ 1 = a = 1 \circ a$
4. $\forall a \in G, \exists -a \in G$ (additive inverse) s.t. $a + (-a) = 0 = (-a) + a$	9. $\forall a \in G, \exists a^{-1} \in G$ (unity) s.t. $a \circ a^{-1} = 1 = a^{-1} \circ a$
5. (Commutative) $\forall a, b \in G, a + b = b + a$	10. (Commutative) $\forall a, b \in G, a \circ b = b \circ a$

Joining additive and multiplicative groups together, we form a ring with **distributive laws**

$$11. \quad \forall a, b, c \in G, (a + b) \circ c = (a \circ c) + (b \circ c)$$

$$12. \quad \forall a, b, c \in G, c \circ (a + b) = (c \circ a) + (c \circ b)$$

- Abelian group: (1-5) or (6-10)
- Associative Ring: 1-6, with 11 and 12
- Semigroup: 1, 2 only
- Monoid (Semigroup with identity): 1, 2, 3 only
- Commutative ring: 1-5, 6, 10, 11, and 12
- Ring: 1-5, with 11 and 12
- Ring with unity: 1-6, with 8, 11, and 12
- Field: 1-12

### Axiom 1.1 Groups

Let  $G$  be a set together with a *binary operation* that assigns to each ordered pair  $(a, b)$  of elements of  $G$  an element in  $G$  denoted by  $a * b$ . We say that  $(G, *)$  is a group under this operation if the following properties are satisfied.

1. **(Closure)**  $\forall a, b \in G, \quad a * b \in G$ .
2. **(Associativity)**  $\forall a, b, c \in G, \quad a * (b * c) = (a * b) * c \in G$ .
3. **(Existence of Identity)**  $\forall a \in G, \exists e \in G \text{ s.t. } a * e = a = e * a \in G$ .
4. **(Existence of Inverse)**  $\forall a \in G, \exists a^{-1} \in G \text{ s.t. } a * a^{-1} = e = a^{-1} * a \in G$ .

**Example 1.0.1.** The set of integers  $\mathbb{Z}$ , the set of rational numbers  $\mathbb{Q}$  and the set of real numbers  $\mathbb{R}$  are all groups under normal addition.

**Example 1.0.2.** The set

$$GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

is a non-abelian group under matrix multiplication.

**Solution** Check if  $GL(2, \mathbb{R})$  is closure, associative, has identity and has inverse.

1. (Closure) For all  $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$  in  $GL(2, \mathbb{R})$ , with  $a_1d_1 - b_1c_1 \neq 0$  and  $a_2d_2 - b_2c_2 \neq 0$ .

$$\begin{aligned} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix} \in GL(2, \mathbb{R}) \\ &= (a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)(c_1a_2 + d_1c_1) \\ &= a_1a_2c_1b_2 + a_1a_2d_1d_2 + b_1b_2c_1c_2 + b_1c_2d_1d_2 \\ &\quad - a_1a_2b_2c_1 - a_2b_2c_2d_1 - a_2b_1c_1d_2 - b_1c_2d_1d_2 \\ &= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0 \in GL(2, \mathbb{R}). \end{aligned}$$

Matrix multiplication is closed under  $GL(2, \mathbb{R})$ .

2. (Associativity) For all  $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$  in  $GL(2, \mathbb{R})$ , we have

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \left( \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \right) = \left( \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}.$$

Matrix multiplication in  $GL(2, \mathbb{R})$  is associative.

3. (Existence of identity)  $\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}), \exists \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in GL(2, \mathbb{R})$  s.t.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4. (Existence of inverse)  $\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}), \exists \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in GL(2, \mathbb{R})$  s.t.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ab + ba \\ cd - cd & -bc + ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Similarly, we can verify that

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R})$$

since

$$\frac{1}{ad - bc} \det \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} (da - bc) = 1 \neq 0.$$

The inverse does exist whenever  $a, b, c, d$  in  $\mathbb{R}$ .

◀

**Example 1.0.3 (Non-example).** The set  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  is not a group under multiplication modulo 4.

**Solution** Because  $\gcd(2, 4) = 2 \neq 1$ , which means  $2^{-1}$  does not exist in  $\mathbb{Z}_4$ . Each element in the group should have its unique inverse. Thus  $(\mathbb{Z}_4, \cdot)$  is not a group. ◀

**Example 1.0.4** (Non-example). The set of integers under subtraction is not a group.

**Solution** For all  $a, b, c \in \mathbb{Z}$ ,

$$a - (b - c) = a - b + c \neq (a - b) - c.$$

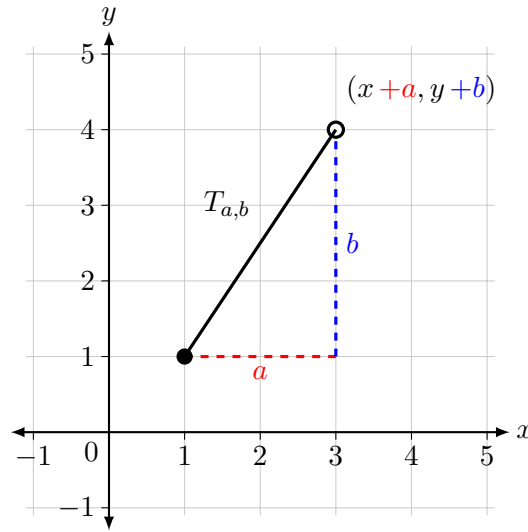
Which violates the associative property. So the subtraction in the set of integers is not a group. ◀

**Example 1.0.5.** The set  $\mathbb{Q}^+$  of positive rationals is a group under ordinary multiplication.

**Example 1.0.6.** For a fixed point  $(x, y)$  in 2-dimensional cartesian plane  $\mathbb{R}^2$ , we define the geometrical translation  $T_{a,b} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T_{a,b}(x, y) = (x + a, y + b).$$

The set  $G = \{T_{a,b} \mid a, b \in \mathbb{R}\}$  is a group under function composition.



**Solution** 1. (Closure) We want to show:

$$\forall T_{a,b}, T_{c,d} \in G, \quad T_{a,b} \circ T_{c,d} \in G$$

We compute the composition

$$\begin{aligned} (T_{a,b} \circ T_{c,d})(x, y) &= T_{a,b}(T_{c,d}(x, y)) \\ &= T_{a,b}(x + c, y + d) \\ &= (x + a + c, y + b + d) \\ &= (x + (a + c), y + (b + d)) \quad \text{associativity of ordinary addition} \\ &= T_{a+c, b+d}(x, y) \end{aligned}$$

which is closed under  $G$ .

2. (Associativity) For all  $T_{a,b}, T_{c,d}, T_{g,h} \in G$ , we have

$$\begin{aligned} T_{a,b} \circ (T_{c,d} \circ T_{g,h}) &= T_{a,b} \circ T_{c+g,d+h} \\ &= T_{a+(c+g),b+(d+h)} \\ &= T_{(a+c)+g,(b+d)+h} \\ &= T_{a+c,b+d} \circ T_{g,h} \\ &= (T_{a,b} \circ T_{c,d}) \circ T_{g,h} \end{aligned}$$

so the translation closed under the function composition.

3. (Existence of identity)  $\forall T_{a,b} \in G, \exists T_{e_1,e_2} \in G$  such that

$$T_{a,b} \circ T_{e_1,e_2} = T_{a,b} = T_{e_1,e_2} \circ T_{a,b}.$$

We need to find the value of  $e_1$  and  $e_2$ .

$$\begin{aligned} T_{a,b} \circ T_{e_1,e_2} = T_{a,b} &\Rightarrow T_{a+e_1,b+e_2} = T_{a,b} \\ &\Rightarrow a + e_1 = a \quad \text{and} \quad b + e_2 = b \end{aligned}$$

On solving, we have  $e_1 = e_2 = 0$ . Thus  $T_{0,0} \in G$  is the identity.

4. (Existence of inverse)  $\forall T_{a,b} \in G, \exists T_{\alpha,\beta} \in G$  such that

$$T_{a,b} \circ T_{\alpha,\beta} = T_{0,0} = T_{\alpha,\beta} \circ T_{a,b}.$$

Compute

$$\begin{aligned} T_{a,b} \circ T_{\alpha,\beta} = T_{0,0} &\Rightarrow T_{a+\alpha,b+\beta} = T_{0,0} \\ &\Rightarrow a + \alpha = 0 \quad \text{and} \quad b + \beta = 0 \end{aligned}$$

solving equations give us  $\alpha = -a$  and  $\beta = -b$ . The inverse of  $T_{a,b}$  in  $G$  is  $T_{-a,-b}$ .



### Definition 1.3 Multiplicative group modulo $n$

The multiplicative group of integers modulo  $n$ , denoted  $\mathbb{Z}_n^*$  or  $U(n)$ , is the group

$$U(n) := \{k \in \mathbb{Z}_n \mid \gcd(n, k) = 1\}$$

where the binary operation is multiplication, modulo  $n$ .

**Example 1.0.7.** The set

$$U(n) = \{x \in \mathbb{Z}^+ \mid x < n, \gcd(x, n) = 1\}$$

is a group under multiplication modulo  $n$ .

**Solution** 1. (Closure) For all  $x, y \in U(n)$ , where  $x, y < n$  and  $\gcd(x, n) = \gcd(y, n) = 1$ , then  $xy \in U(n)$  since  $\gcd(xy, n) = \gcd(x, n) \gcd(y, n) = 1$ .

2. (Associativity) Associative holds since  $x(yz) = (xy)z$  whenever  $x, y, z$  in  $U(n)$ .

3. (Existence of identity)  $\forall x \in U(n), \exists 1 \in U(n)$  s.t.  $x \cdot 1 = x = 1 \cdot x$ .

4. (Existence of inverse) For all  $x \in U(n)$ , since  $\gcd(x, n) = 1$ . Then by *extended Euclidean algorithm* we have

$$ax + bn = 1 \quad \text{for some } a, b \in U(n) \quad (\heartsuit)$$

taking modulo  $n$  on  $(\heartsuit)$  yields  $ax = 1 \implies x^{-1} = a$ . Thus the inverse of  $x$  does exist.



**Example 1.0.8.** Draw a cayley table for  $U(10)$ .

**Solution**  $U(10)$  contains all the integers that are coprime to 10. That is,

$$U(10) = \{1, 3, 7, 9\}.$$

$U(10)$  is a group under multiplication modulo 10.

$\cdot_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1



### Lemma 1.1 Uniqueness of group identity

In a group  $G$ , there is one and only one identity element  $e$ .

*Proof.* For the sake of contradiction. Suppose not, Suppose that  $e$  and  $e'$  are both identity elements of group  $G$ . Since  $e$  is an identity element of  $G$ , then  $e \in G$  and

$$ea = a = ae \quad \forall a \in G. \quad (\heartsuit)$$

Since  $e'$  is also an identity element of  $G$ . we said that  $e' \in G$  and

$$e'a = a = ae' \quad \forall a \in G. \quad (\clubsuit)$$

From  $(\heartsuit)$ , if we take  $a = e'$ , then  $e \cdot e' = e'$ .

From  $(\clubsuit)$ , if we take  $a = e$ , then  $e = e \cdot e'$ .

Combining the results we have  $e = e \cdot e' = e'$ , and so  $e = e'$ . There is only one identity element in  $G$ . We proved the uniqueness of identity.  $\square$

### Lemma 1.2 Cancellation rule

In a group  $G$ ,  $ba = ca$  implies  $b = c$ ; and  $ab = ac$  implies  $b = c$ .

*Proof.* Consider  $G$  is a group, then

$$\forall a \in G, \exists a' \in G \quad \text{s.t.} \quad aa' = e = a'a.$$

To show the right cancellation works, we further consider  $ba = ca$ . Multiplying  $a'$  on both sides of the previous equation on right, we obtained

$$(ba)a' = (ca)a'$$

Then,  $b(aa') = c(aa')$  and so  $be = ce \Rightarrow \boxed{b = c}$ . The proof is now complete.  $\square$

### Theorem 1.1 Socks-shoes property

$$(a \circ b)^{-1} = b^{-1} \circ a^{-1} \quad (1.1)$$

*Proof.* Since we know that  $G$  is a group, then  $ab \in G$  for all  $a, b \in G$  since  $G$  is closure. Next, we consider the following equation

$$\begin{aligned} (ab)(b^{-1}a^{-1}) &= a(bb^{-1})a^{-1} && G \text{ is associative} \\ &= aea^{-1} \\ &= aa^{-1} \\ &= \boxed{e} && \text{cancellation rule returns identity} \end{aligned}$$

this equation states that

$$\boxed{(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}} = e$$

now we cancel off  $ab$  from both sides of the equations, we now arrive at

$$(ab)^{-1} = b^{-1}a^{-1}$$

and we have done the proof.  $\square$

**Remark.** In abstract algebra, the position of inputs in binary operator is very important! The commutative property no necessary hold.  $a \circ b \neq b \circ a$ . E.g. matrix multiplication  $AB \neq BA$ .

**Example 1.0.9** (Tutorial). Show that every group with identity  $e$  and  $x * x = x$  for all  $x \in G$  is abelian.

**Solution** Given  $(G, *)$  is a group, there is an identity  $e \in G$  and for all  $x \in G$ ,  $x * x = x$ . We want to show  $G$  is abelian.

$\forall a, b \in G, a * a = a$  and  $b * b = b$ . Since  $G$  is a group, so  $a * b \in G$ . Observe that

$$\begin{aligned} (a * b) * (a * b) &= a * b \Rightarrow a * (b * a) * b = (a * a) * (b * b) \\ &\Rightarrow a * (b * a) * b = a * a * b * b \\ &\Rightarrow a * (b * a) * b = a * (a * b) * b \\ &\Rightarrow \cancel{a} * (b * a) * \cancel{b} = \cancel{a} * (a * b) * \cancel{b} \\ &\Rightarrow \boxed{b * a = a * b}. \end{aligned}$$

Thus  $*$  commute and  $G$  is an abelian group.  $\blacktriangleleft$



## 1.1 Finite groups and Subgroups

### Definition 1.4 Order of group

The number of elements of a group (finite or infinite) is called its order. We will use  $|G|$  to denote the order of  $G$ .

The order of an element  $g$  in  $G$  is the **smallest positive integer**  $n$  such that  $g^n = e$  (In additive notation, this would be  $ng = 0$ ). If no such integer exists, we said that  $g$  has infinite order. The order of an element  $g \in G$  is denoted by  $\text{ord}(g)$ .

**Example 1.1.1.**  $(\mathbb{Z}_5, +) = \{0, 1, 2, 3, 4\}$  is a group. The order of  $G$  is 5. We can write  $|G| = 5$ .

Take an element, said 2 in  $\mathbb{Z}_5$ . The order of 2 is  $\text{ord}(2) = 5$  since

$$2 + 2 + 2 + 2 + 2 = 10 \equiv 0 \pmod{5}.$$

It takes five iterations on the same operation to return back to the identity 0.

The order of 0 is infinity,  $\text{ord}(0) = \infty$ . No matter how many times you add on the zero, it is still zero and it is identity in  $\mathbb{Z}_5$ .

**Example 1.1.2.** Take an element in  $U(6)$ , said 5.  $\text{ord}(5) = 2$  because

$$5 \cdot 5 = 5^2 = 1 \pmod{6}.$$

### Definition 1.5 Subgroups

If a subset  $H$  of a group  $G$  is itself a group under the same operation of  $G$ , we say that  $H$  is a subgroup of  $G$ .

**Remark.** We use the notation  $H \leq G$  to mean that  $H$  is a subgroup of  $G$ . We use the notation  $H < G$  to denote that  $H$  is a proper subgroup of  $G$ .

The subgroup  $\{e\}$  is called the trivial subgroup of  $G$ ; a subgroup that is not  $\{e\}$  is called a nontrivial subgroup of  $G$ .

### 1.1.1 Subgroup tests

#### Theorem 1.2 One step subgroup test

Suppose  $G$  is a multiplicative group and  $H \subseteq G$ . If

1.  $H \neq \emptyset$ ,
2.  $\forall a, b \in H, ab^{-1} \in H$

then  $H$  is a subgroup of  $G$ .

*Proof.* Given that  $G$  is a group and  $\emptyset \neq H \subseteq G$  such that for any  $a, b$  in subgroup  $H$ , we have

$$ab^{-1} \in H \quad (\heartsuit)$$

Then, what we need to do is to show that  $H \leq G$ , which is equivalent to show that  $H$  itself is a group, and  $H$  definitely inherits the operation of  $G$ . So  $H$  is closed under the same operation of  $G$ .

(Closure) Take  $a = x$  and  $b = y^{-1}$  into ( $\heartsuit$ ), which for all  $x, y \in H$ . We have

$$x(y^{-1})^{-1} = xy \in H$$

which is closed under  $H$ .

(Associativity) Since associative law holds in  $G$ , so as  $H$ , since both  $G$  and  $H$  are sharing the same operation.

(Existence of identity) Since  $H$  is nonempty, then we can randomly pick an element  $x \in H$ . If we replace  $a$  and  $b$  in the hypothesis ( $\heartsuit$ ) with  $a = b = x$ , then we have

$$\forall x \in H, \quad xx^{-1} = e \in H$$

(Existence of inverse) Replacing  $a = e$  and  $b = x$  in ( $\heartsuit$ ), we have

$$ex^{-1} = x^{-1} \in H \quad \forall x \in H$$

□

**Example 1.1.3.** Let  $G$  be an abelian group with identity  $e$ . Then

$$H = \{x \in G \mid x^2 = e\}$$

is a subgroup of  $G$ .

**Solution** We can use one-step subgroup test to check:

1. Since  $G$  is a group, then  $e \in G$  and so  $e = e^2 \in H \neq \emptyset$ .
2. For all  $a, b \in H$ , we see that  $a^2 = e$  and  $b^2 = e$ . Next, we compute

$$\begin{aligned} (ab^{-1})^2 &= (ab^{-1})(ab^{-1}) \\ &= a^2(b^{-1})^2 \\ &= a^2(b^2)^{-1} \\ &= ee^{-1} \\ &= e \end{aligned}$$

implies  $ab^{-1}$  is in  $H$ . Thus  $H \leq G$ .

◀

### Theorem 1.3 Two-step subgroup test

Suppose  $G$  is a multiplicative group and  $H \subseteq G$ .  $H$  is a subgroup of  $G$  provided

1.  $H \neq \emptyset$ ,
2. For any  $a, b \in H$ ,  $ab \in H$ ,
3. For all  $a \in H$ ,  $a^{-1} \in H$

**Theorem 1.4 Finite subgroup test**

Suppose  $G$  is a multiplicative group and  $H \subseteq G$ .  $H$  is a subgroup of  $G$  provided

1.  $|H| < \infty$
2. For all  $a, b \in H$ ,  $ab \in H$ . (which means  $H$  closed under the same operation of  $G$ )

**Example 1.1.4.** Let  $G$  be an abelian group under multiplication with identity  $e$ . Then

$$H = \{x^2 \mid x \in G\}$$

is a subgroup of  $G$ .

**Solution**

Since  $G$  is a group, then  $e^2 = e \in G$  and  $G$  is nonempty.

For all  $a, b \in H$ , we see that  $a = x^2$  and  $b = y^2 \quad \forall x, y \in G$ . Next, we compute

$$\begin{aligned} ab^{-1} &= (x^2)(y^2)^{-1} \\ &= xxy^{-1}y^{-1} \\ &= xy^{-1}xy^{-1} \\ &= (xy^{-1})^2 \end{aligned}$$

implies  $ab^{-1} = (xy^{-1})^2$  is in  $H$ . Thus  $H \leq G$ . ◀

**Example 1.1.5.** Let  $G = \mathbb{R}^* \setminus \{0\}$ ,  $(G, \cdot)$  is a multiplicative group. And

$$H = \{x \in G \mid x = 1 \text{ or } x \text{ is irrational}\}$$

is not a subgroup of  $G$ .

**Solution** With two-step subgroup test, step 2 is wrong.  
(Counterexample)  $\sqrt{2} \in H$ , but  $\sqrt{2}\sqrt{2} = 2 \notin H$ . ◀

**Example 1.1.6.** Let  $G = \mathbb{R}^* \setminus \{0\}$ ,  $(G, \cdot)$  is a multiplicative group. And

$$K = \{x \in G \mid x \geq 1\}$$

is not a subgroup of  $G$ .

**Solution** With two-step subgroup test, step 3 is wrong.

(Counterexample) Take  $2 \in K$ , but  $2^{-1} = \frac{1}{2} \notin K$ . ◀

## 1.2 Cyclic groups

Cyclic groups are groups in which every element is a power of some fixed element. In additive group, then every element is a multiple of some fixed element. For instance,

$$\underbrace{a + a + \cdots + a}_{n \text{ times}} = na, \quad n \text{ is integer}$$

**Definition 1.6 Generating subgroup**

If  $G$  is a multiplicative group and  $g \in G$ , then the subgroup generated by element  $g$  is

$$\langle g \rangle = \{\underbrace{a \cdot a \cdots a}_{n \text{ times}} \mid n \in \mathbb{Z}\} = \{g^n \mid n \in \mathbb{Z}\} \quad (1.2)$$

If the group is abelian and is additive, then

$$\langle g \rangle = \{\underbrace{a + a + \cdots + a}_{n \text{ times}} \mid n \in \mathbb{Z}\} = \{ng \mid n \in \mathbb{Z}\} \quad (1.3)$$

**Remark.**  $\langle g \rangle$  is called a **cyclic subgroup** generated by  $g$  in group  $G$ . When  $G = \langle g \rangle$ , then  $G$  is called a **cyclic group**.

**Definition 1.7 Cyclic group**

A group  $G$  is **cyclic** if  $G = \langle g \rangle$  for some  $g \in G$ .  $g$  is a **generator** of  $\langle g \rangle$ .

**Lemma 1.3**

$\langle g \rangle$  is a subgroup of  $G$ .

*Proof.* We can use 2-step subgroup test to verify  $\langle g \rangle \leq G$ :

1. Since  $g \in \langle g \rangle \neq \emptyset$ .
2. For all  $g_1, g_2 \in \langle g \rangle$ , we have

$$g_1 = g^{n_1}, \quad g_2 = g^{n_2}$$

where  $n_1$  and  $n_2$  are integers. And since

$$g_1 g_2 = g^{n_1} g^{n_2} = g^{n_1+n_2}$$

and  $n_1+n_2 \in \mathbb{Z}$  implies that  $g_1 g_2 \in \langle g \rangle$ .

3. For all  $g_1 \in \langle g \rangle$ , we have  $g_1 = g^k$ , where  $k$  is integer. We compute the inverse

$$g_1^{-1} = (g^k)^{-1} = g^{-k}, \quad -k \in \mathbb{Z}$$

which tells us that  $g_1^{-1} \in \langle g \rangle$ .

Therefore, by 2-step subgroup test,  $\langle g \rangle$  is a subgroup of  $G$ . □

**Lemma 1.4**

If  $G$  is a cyclic group, then  $G$  is abelian.

*Proof.* Consider a cyclic group  $G$ . We want to show  $G$  is also an abelian group.

Since  $G$  is a group, we say

$$\forall g_1, g_2 \in G, \quad g_1 = g^{n_1}, \quad g_2 = g^{n_2}$$

where  $n_1$  and  $n_2$  are integers. In order to show that  $G$  is abelian, we need to show that the commutative law applied in group  $G$ .

now compute

$$\begin{aligned} g_1 g_2 &= a^{n_1} a^{n_2} \\ &= g^{n_1+n_2} \\ &= g^{n_2+n_1} && \text{commutative in normal addition} \\ &= g^{n_2} g^{n_1} = \boxed{g_2 g_1} \end{aligned}$$

thus  $G$  is an abelian group. □

### Definition 1.8 Center of group

The **center**,  $Z(G)$ , of a group  $G$  is a subset of elements in  $G$  that commute with every element of  $G$ , that is,

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}. \quad (1.4)$$

### Lemma 1.5

The center of a group  $G$  is also a subgroup of  $G$ .

*Proof.* We use one-step subgroup test to verify:

1. Since we know that  $G$  is a group, certainly the identity  $e \in G$  and

$$ex = x = xe \quad \forall x \in G.$$

implies that  $e \in Z(G)$  and  $Z(G)$  is nonempty.

2. For any  $a_1, a_2$  in  $Z(G)$ , we need to show

$$a_1 a_2^{-1} \in Z(G).$$

Since  $Z(G)$  is the center, we have  $a_1 x = xa_1$  and  $a_2 x = xa_2$  for all  $x \in G$ . Proving  $a_1 a_2^{-1} \in Z(G)$  is equivalent to show

$$a_1 a_2^{-1} x = x a_1 a_2^{-1} \quad \forall x \in G$$

compute

$$\begin{aligned} a_1 a_2^{-1} x &= a_1 (a_2^{-1} x) && \text{Associativity of } Z(G) \\ &= a_1 (x a_2^{-1}) && \text{Since } a_2^{-1} x = x a_2^{-1} \\ &= (a_1 x) a_2^{-1} && \text{Associativity of } Z(G) \\ &= (x a_1) a_2^{-1} && \text{Since } a_1 x = x a_1 \\ &= \boxed{x a_1 a_2^{-1}} \end{aligned}$$

which is what we desired.

Therefore the center  $Z(G)$  is a subgroup of  $G$  by one-step subgroup test. □

**Definition 1.9 Group centralizer**

Let  $a$  be a **fixed** element of a group  $G$ . The centralizer of  $a$  in  $G$  is

$$C(a) = \{g \in G \mid ga = ag\}. \quad (1.5)$$

**Theorem 1.5**

Let  $a$  be a **fixed** element in group  $G$ . If  $a$  has infinite order, then  $a^i = a^j$  if and only if  $i = j$ . However, if  $a$  has finite order, said,  $n$ , then

$$\langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\} \quad (1.6)$$

and  $a^i = a^j$  if and only if  $n \mid i - j$ .

*Proof.* Consider a group  $G$ , and take an  $a$  from  $G$ . If  $a$  has infinite order, say,  $\text{ord}(a) = \infty$ , then there is no nonzero integer  $n$  such that  $a^n = e$ . We assume an equation  $a^i = a^j$  for some  $i, j \in \mathbb{Z}$ , we have

$$a^{i-j} = e \Rightarrow i - j = 0 \Rightarrow \boxed{i = j}.$$

and we are done.

On the other hand, if  $a$  has finite order, just say  $\text{ord}(a) = n$ . We want to show

$$\langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}.$$

Apparently,  $e, a, a^2, \dots, a^{n-1}$  are all belongs to  $\langle a \rangle$ , so as the list  $\{e, a, a^2, \dots, a^{n-1}\} \subseteq \langle a \rangle$ . Now we continue to check if  $\{e, a, a^2, \dots, a^{n-1}\} \supseteq \langle a \rangle$ .

By *division algorithm*, there exists some integers  $q$  and  $r$  such that

$$k = nq + r, \quad 0 \leq r < n$$

compute

$$a^k = a^{qn+r} = (a^n)^q a^r = e^q a^r = a^r$$

this implies  $a^k = a^r \in \{e, a, a^2, \dots, a^{n-1}\}$ . Thus we have

$$\{e, a, a^2, \dots, a^{n-1}\} \supseteq \langle a \rangle.$$

Now the final part is to show  $a^i = a^j$  iff  $n \mid i - j$ , we are going to proof on two directions.

( $\Rightarrow$ ) If  $a^i = a^j$ , we need to show that  $n$  is divisible by  $i - j$ . Again we applying the *division algorithm*,

$$i - j = nq + r, \quad 0 \leq r < n$$

which  $q$  is quotient and  $r$  is remainder.

compute

$$\begin{aligned}
 a^{i-j} = e &\Rightarrow a^{nq+r} = e \\
 &\Rightarrow a^{nq} a^r = e \\
 &\Rightarrow (a^n)^q a^r = e \\
 &\Rightarrow e^q a^r = e \\
 &\Rightarrow e a^r = e \\
 &\Rightarrow a^r = e
 \end{aligned}$$

division algorithm

since  $a^n = e$

but  $n$  is the least integer such that  $a^n = e$  and so the condition  $0 \leq r < n$  implies  $r = 0$ . Now we continue on the opposite side of the statement.

( $\Leftarrow$ ) This part is more straightforward. Conversely, if  $n|i - j$ , then

$$\begin{aligned}
 a^{i-j} &= a^{nq+r} \\
 &= a^{nq} \\
 &= (a^n)^q \\
 &= e^q \\
 &= e
 \end{aligned}$$

division algorithm

remainder  $r$  is zero

since  $a^n = e$

and we are done. □

### Corollary 1.1

For any group element  $a$ ,  $\text{ord}(a) = |\langle a \rangle|$ .

*Proof.* By previous theorem,  $\langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}$  has  $n$  elements. Thus  $|\langle a \rangle| = \text{ord}(a) = n$ . □

### Theorem 1.6

Let  $a$  be an element of order  $n$  in a group and let  $k$  be a positive integer. Then

$$\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$$

and

$$\text{ord}(a) = \frac{n}{\gcd(n, k)}.$$

*Proof.* 1. To show that  $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$  is equivalent of showing  $\langle a^k \rangle \subseteq \langle a^{\gcd(n,k)} \rangle$ .

Consider  $a^k \in \langle a^k \rangle$ , and let  $d = \gcd(n, k)$ . This implies that  $d$  divide  $k$  and  $k = dr$  for some integer  $r$ . Thus

$$a^k = a^{dr} = (a^d)^r \in \langle a^d \rangle = \langle a^{\gcd(n,k)} \rangle.$$

On the other hand, we want to show  $\langle a^k \rangle \supseteq \langle a^{\gcd(n,k)} \rangle$ , which is equivalent to show  $\langle a^d \rangle \subseteq \langle a^k \rangle$

$\langle \mathfrak{a}^k \rangle$ . Consider  $\mathfrak{a}^d \in \langle \mathfrak{a}^d \rangle$ . By extended Euclidean algorithm,

$$\begin{aligned}
 \mathfrak{a}^d &= \mathfrak{a}^{\gcd(n,k)} \\
 &= \mathfrak{a}^{nt+ks} && \text{for some integers } k, s \\
 &= (\mathfrak{a}^n)^t (\mathfrak{a}^k)^s \\
 &= e^t (\mathfrak{a}^k)^s \\
 &= (\mathfrak{a}^k)^s \in \langle \mathfrak{a}^k \rangle.
 \end{aligned}$$

2. Certainly,

$$\begin{aligned}
 \text{ord}(\mathfrak{a}^k) &= |\langle \mathfrak{a}^k \rangle| \\
 &= |\langle \mathfrak{a}^d \rangle| \\
 &= \text{ord}(\mathfrak{a}^d) \\
 &= \frac{n}{d} \\
 &= \frac{n}{\gcd(n, k)}.
 \end{aligned}$$

□

### Theorem 1.7 Fundamental theorem of cyclic groups

Suppose  $G = \langle g \rangle$  is cyclic.

1. Every subgroup of  $G$  is cyclic.
2. If  $|G| = n$ , then the order of any subgroup of  $G$  divides  $n$ .
3. If  $|G| = n$ , then for any positive integer  $k$ ,  $n$  the subgroup  $\langle g^{n/k} \rangle$  is the unique subgroup of order  $k$ .

*Proof.* 1. Let  $H$  is a subgroup of  $G$ , if  $H = \{e\}$  then we are done.

Assume that  $H \neq \{e\}$ , choose  $g^m \in H$  with minimal  $m \in \mathbb{Z}^+$  by well-ordering. Clearly  $\langle g^m \rangle \subseteq H$ . If some  $g^k \in H$  then by *division algorithm* we have

$$k = qm + r \implies r = k - qm \quad 0 \leq r < m$$

and then  $g^r = g^k (g^m)^{-q} \in H$  and so  $r = 0$  by minimality of  $m$  and so  $g^k = (g^m)^q$  and hence  $g^k \in \langle g^m \rangle$ .

2. Take a subgroup  $H \leq G$ . From (1) we know  $H$  is cyclic and  $H = \langle g^m \rangle$  with minimal positive integer  $m$ . Again we apply *division algorithm* and write

$$n = qm + r \implies r = n - qm \quad 0 \leq r < m$$

and  $g^r = g^n (g^m)^{-q} \in H$  and so  $r = 0$ , and then

$$|H| = |\langle g^m \rangle| = \text{ord}(g^m) = \frac{n}{\gcd(n, m)} = \frac{n}{m}$$

and thus  $m|H| = n$  and  $|H|$  divide  $n$ .



3. Observe first that  $k|n$  we have

$$|\langle g^{n/k} \rangle| = |g^{n/k}| = \frac{n}{\gcd(n, n/k)} = \frac{n}{n/k} = k.$$

Thus certainly  $\langle g^{n/k} \rangle$  is a subgroup of order  $k$ . We must show that it is unique. Let  $H$  be a subgroup of  $G$  such that  $|H| = k|n$ . Since  $H \leq G$  by (1) and (2) we have  $H = \langle g^m \rangle$  with  $m|n$ . Then we have

$$k = |H| = |\langle g^m \rangle| = \text{ord}(g^m) = \frac{n}{\gcd(n, m)} = \frac{n}{m}$$

Thus  $m = \frac{n}{k}$  and so  $H = \langle g^m \rangle = \boxed{\langle g^{n/k} \rangle}$ .

□

**Example 1.2.1.** In  $\mathbb{Z}_{12} = \{0, 1, 2, \dots, 11\}$  the complete list of generators is  $U(12) = \{1, 5, 7, 11\}$ . So for example

$$\begin{aligned} \langle 5 \rangle &= \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55\} \pmod{12} \\ &= \{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7\} \end{aligned}$$

**Example 1.2.2.** Consider  $U(50)$ : its order is  $\phi(50) = 20$ , and its elements are

$$\{1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49\}.$$

Given that  $U(50) = \langle 3 \rangle$ . Find all generators of  $U(50)$ .

**Solution** Since  $\langle 3^k \rangle = \langle 3 \rangle \Leftrightarrow \gcd(20, k) = 1 \Leftrightarrow k \in U(20)$ . Since

$$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\},$$

the generators of  $U(50)$  are

$$\{3, 3^3, 3^7, 3^9, 3^{11}, 3^{13}, 3^{17}, 3^{19}\} \text{ or } \{3, 27, 37, 33, 47, 23, 13, 17\}.$$

◀

**Example 1.2.3.** Find all the subgroups of  $\mathbb{Z}_{42}$ .

**Solution** Listed out all the possible divisors of 42 we have

$k$	$42/k$	subgroup order $k$ , $\langle (42/k) \rangle$ with $k 42$ .
1	42	$\langle 42 \rangle = \{0\}$
2	21	$\langle 21 \rangle = \{0, 21\}$
3	14	$\langle 14 \rangle = \{0, 14, 18\}$
6	7	$\langle 7 \rangle = \{0, 7, 14, 21, 28, 35\}$
7	6	$\langle 6 \rangle = \{0, 6, 12, 18, 24, 30, 36\}$
14	3	$\langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39\}$
21	2	$\langle 2 \rangle$ is set of all even numbers in $\mathbb{Z}_{42}$
42	1	$\langle 1 \rangle = \mathbb{Z}_{42}$

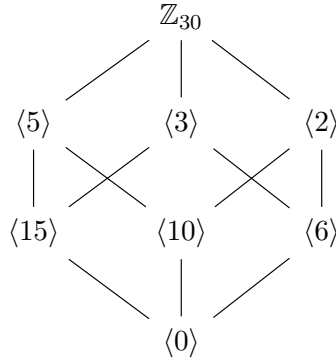
◀

**Example 1.2.4.** Draw a subgroup lattice of  $\mathbb{Z}_{30}$ .

**Solution** By prime factorizing  $30 = 2 \cdot 3 \cdot 5$ . The factors of 30 are

$$1, 2, 3, 5, 6, 10, 15, 30$$

The lattice diagram is



## 1.3 Permutation

### Definition 1.10

A permutation of a set  $\mathcal{A}$  is a function from  $\mathcal{A}$  to  $\mathcal{A}$  that is both one-to one and onto. A permutation group of a set  $\mathcal{A}$  is the set of permutations of  $\mathcal{A}$  that forms a group under function composition

**Example 1.3.1.** Let  $S_3$  denote the set of all one-to-one functions from  $\{1, 2, 3\}$  to itself. The  $S_3$  under function composition, is a group with six elements.

$$\begin{aligned} S_3 &= \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \\ &= \{1, (1\ 2), (1\ 3), (2\ 3), (1\ 3)(1\ 2), (1\ 2)(1\ 3)\} \end{aligned}$$

**Example 1.3.2.** Let  $A = \{1, 2, 3, 4\}$ , and

$$S_4 = \{\sigma \mid \sigma : A \rightarrow A \text{ is a permutation from } A \text{ onto } A\}.$$

We defined  $\sigma$  as follow:

$$\sigma(1) \mapsto 2, \quad \sigma(2) \mapsto 3, \quad \sigma(3) \mapsto 4, \quad \sigma(4) \mapsto 1.$$

Now we can express  $\sigma$  as a permutation such that

$$\sigma = (1\ \sigma(1)\ \sigma^2(1)\ \sigma^3(1)) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

where

$$\begin{aligned}\sigma(1) &= 2 \\ \sigma^2(1) &= \sigma(\sigma(1)) = \sigma(2) = 3 \\ \sigma^3(1) &= \sigma(\sigma(\sigma(1))) = \sigma(\sigma(2)) = \sigma(3) = 4 \\ \sigma^4(1) &= \sigma(\sigma(\sigma(\sigma(1)))) = \sigma(\sigma(\sigma(2))) = \sigma(\sigma(3)) = \sigma(4) = 1\end{aligned}$$

**Example 1.3.3.** Given  $\alpha = (2\ 3\ 1\ 5\ 4)$  and  $\beta = (2\ 1\ 3\ 4)$  in  $S_5$ . Compute  $\alpha\beta$ , and  $\beta^{-1}$ .

**Solution** Compute

$$\begin{aligned}\alpha\beta &= (2\ 3\ 1\ 5\ 4)(2\ 1\ 3\ 4) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 3 & 4 \end{pmatrix} \\ &= (1)(2\ 5\ 4\ 3) \\ &= (2\ 5\ 4\ 3)\end{aligned}$$

and

$$\begin{aligned}\beta^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \\ &= (1\ 2\ 4\ 3)(5) \\ &= (1\ 2\ 4\ 3).\end{aligned}$$

◀

**Lemma 1.6**

Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

**Theorem 1.8**

If the pair of cycles  $\alpha = (a_1, a_2, \dots, a_m)$  and  $\beta = (b_1, b_2, \dots, b_m)$  have no entries in common, then  $\alpha\beta = \beta\alpha$ . In other words, any two disjoint cycles commute.

*Proof.* Let  $\alpha = (a_1, a_2, \dots, a_m)$  and  $\beta = (b_1, b_2, \dots, b_m)$  be two disjoint cycles. These cycles are defined on the set

$$\mathcal{A} = \{a_1, a_2, \dots, a_t, b_1, b_2, \dots, b_s, \underbrace{c_1, c_2, \dots, c_r}_{\text{Fixed points}}\}.$$

and so  $\alpha, \beta \in S_{t+s+r}$ . We want to show that  $\alpha \circ \beta = \beta \circ \alpha$  is equivalent to show

$$(\alpha \circ \beta)(x) = (\beta \circ \alpha)(x) \quad \forall x \in \mathcal{A}.$$

We are considering three possible cases:

Case 1: Suppose that  $x = a_i$ , where  $1 \leq i \leq t$ . On LHS

$$\begin{aligned} (\alpha \circ \beta)(a_i) &= \alpha(\beta(a_i)) \\ &= \alpha(a_i) \\ &= a_{i+1}. \end{aligned}$$

and on RHS

$$\begin{aligned} (\beta \circ \alpha)(a_i) &= \beta(\alpha(a_i)) \\ &= \beta(a_{i+1}) \\ &= a_{i+1}. \end{aligned}$$

Thus, LHS = RHS for first case.

Case 2: Suppose that  $x = b_j$ , where  $1 \leq j \leq s$ . On LHS

$$\begin{aligned} (\alpha \circ \beta)(b_j) &= \alpha(\beta(b_j)) \\ &= \alpha(b_{j+1}) \\ &= b_{j+1}. \end{aligned}$$

and on RHS

$$\begin{aligned} (\beta \circ \alpha)(b_j) &= \beta(\alpha(b_j)) \\ &= \beta(b_j) \\ &= b_{j+1}. \end{aligned}$$

Thus, LHS = RHS for second case.

Case 3: At last, suppose that  $x = c_k$ , where  $1 \leq k \leq r$ . Each  $c_k$  are fixed points and they always stay in the same value whenever any cycles.

$$(\alpha \circ \beta)(c_k) = c_k = (\beta \circ \alpha)(c_k).$$

Hence, we conclude that any disjoint cycles are commute. □

**Definition 1.11**

A permutation that can be expressed as a product of an even (or odd) number of 2-cycles is called an even (or odd) permutation.

**Example 1.3.4.** Write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 4 & 1 & 2 & 5 & 8 & 6 \end{pmatrix} \in S_8$$

into the product of 2-cycles.

**Solution** We can express  $\sigma$  as

$$\begin{aligned}\sigma &= (1\ 3\ 4)(2\ 7\ 8\ 6\ 5) \\ &= (1\ 4)(1\ 3)(2\ 5)(2\ 6)(2\ 8)(2\ 7).\end{aligned}$$

◀

**Example 1.3.5.** Find the order of  $\beta = (1\ 2\ 3\ 4)(5\ 6\ 7)$  in  $S_7$ .

**Solution** The order of a disjoint cycle is the LCM of the order of each disjoint cycles. Namely,

$$\begin{aligned}\text{ord}(\beta) &= LCM(\text{ord}(1\ 2\ 3\ 4), \text{ord}(5\ 6\ 7)) \\ &= LCM(4, 3) \\ &= \frac{4 \times 3}{\gcd(4, 3)} \\ &= 12.\end{aligned}$$

◀

**Definition 1.12 Alternating group**

The group of even permutations of  $n$  symbols is denoted by  $A_n$  and is called the **alternating group** of degree  $n$ .

**Theorem 1.9**

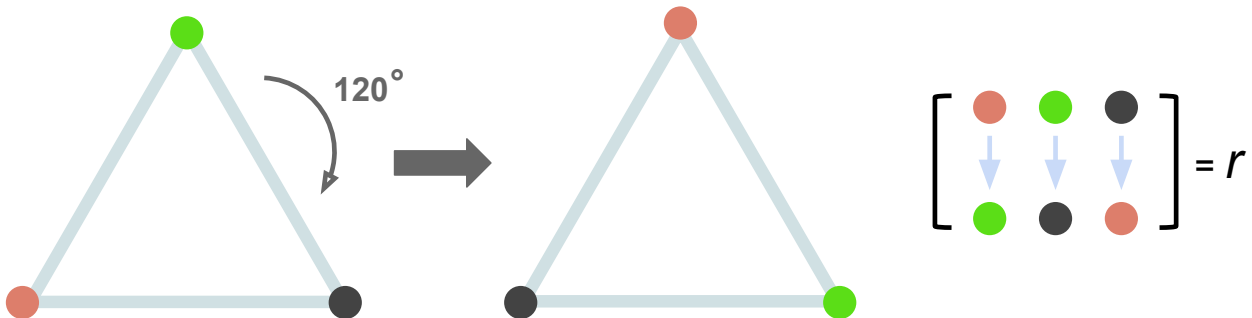
For  $n > 1$ ,  $A_n$  has order  $n!/2$ .

## 1.4 Dihedral Group

Dihedral groups are an essential class in group theory that arise naturally in geometry and other areas of mathematics.

For  $n \geq 3$ , the dihedral group  $D_n$  is described as the rigid motions taking a regular  $n$ -gon back to itself, with the operations

We could said that the rotational symmetry group of an equilateral triangle,  $C_3$ , is isomorphic to  $\mathbb{Z}_3$ . We can combine the horizontal reflection and rotations and form another reflection lines, which these reflection lines runs from one of the vertices to the center of the opposing side.



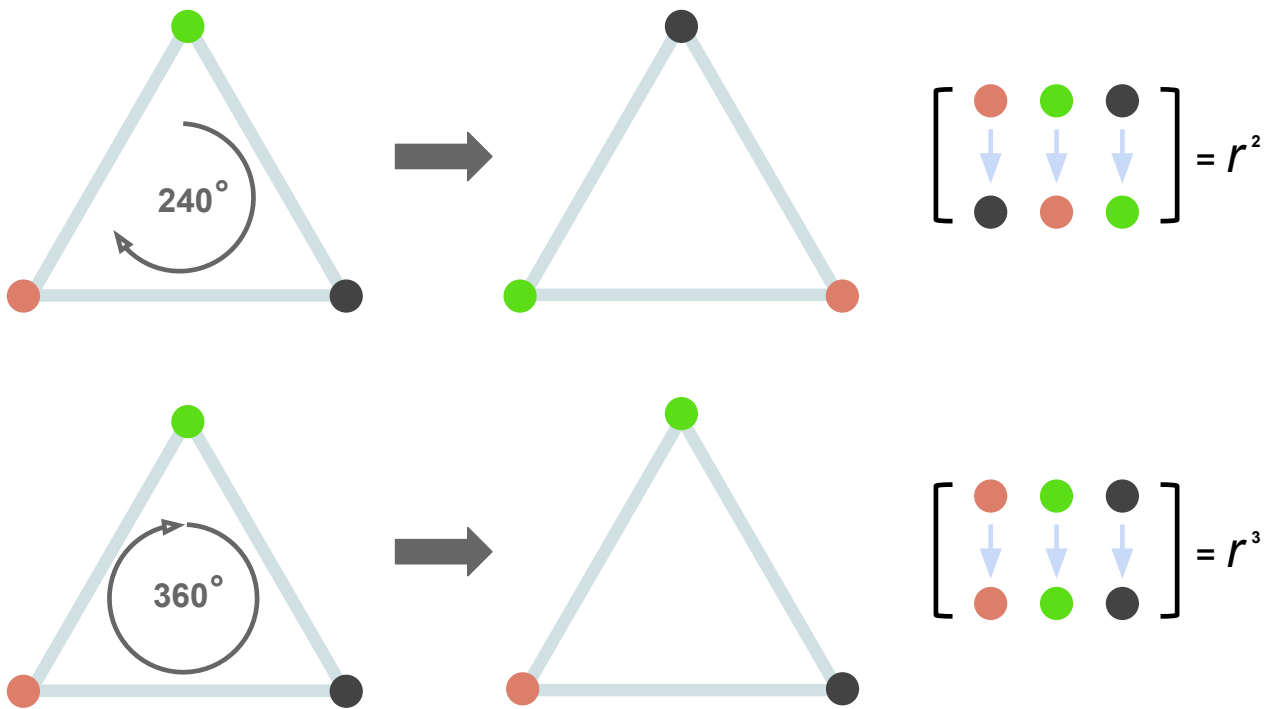


Figure 1.1: The rotation  $r$  on dihedral group  $D_3$  with order 6.  $r$  in  $D_3$  is described as rotating equilateral triangle 120 degree.

### Theorem 1.10

Let the  $n$ -degree dihedral group

$$D_n = \langle r, s \mid r^n = e, s^2 = e, srs = r^{-1} \rangle.$$

Then

1.  $r^k s = sr^{-k}$ .
2. The order of  $r^k$  is  $\frac{n}{\gcd(n, k)}$ .

*Proof.* 1. Compute

$$\begin{aligned} r^k s &= er^k s \\ &= s^2 r^k s \\ &= ssr^k s \\ &= \boxed{sr^{-k}}. \end{aligned}$$

and we are done.

2. We will first show that  $r^k = e$  if and only if  $n|k$ .

( $\Rightarrow$ ) Consider  $e = r^k$ , then by *division algorithm* we have

$$k = na + b \quad \text{where } 0 \leq b < n.$$

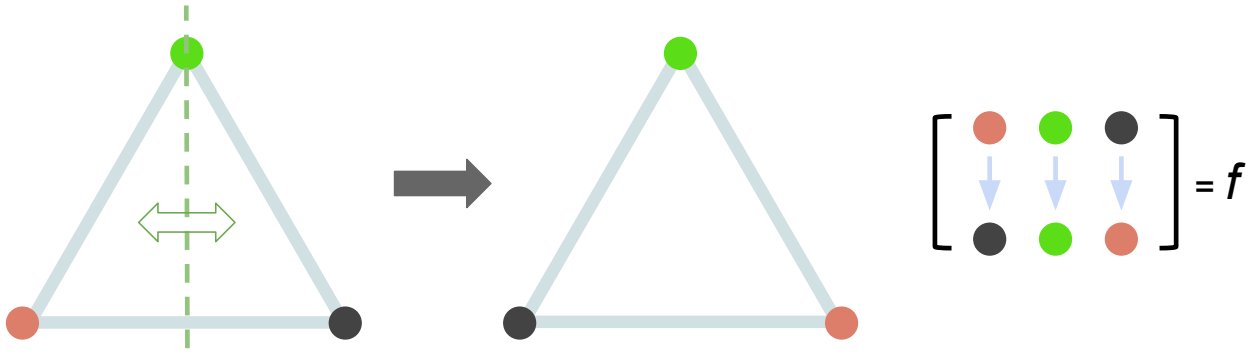


Figure 1.2: The group action  $f$  on dihedral group  $D_3$  with order 6.  $f$  is a horizontal flip.

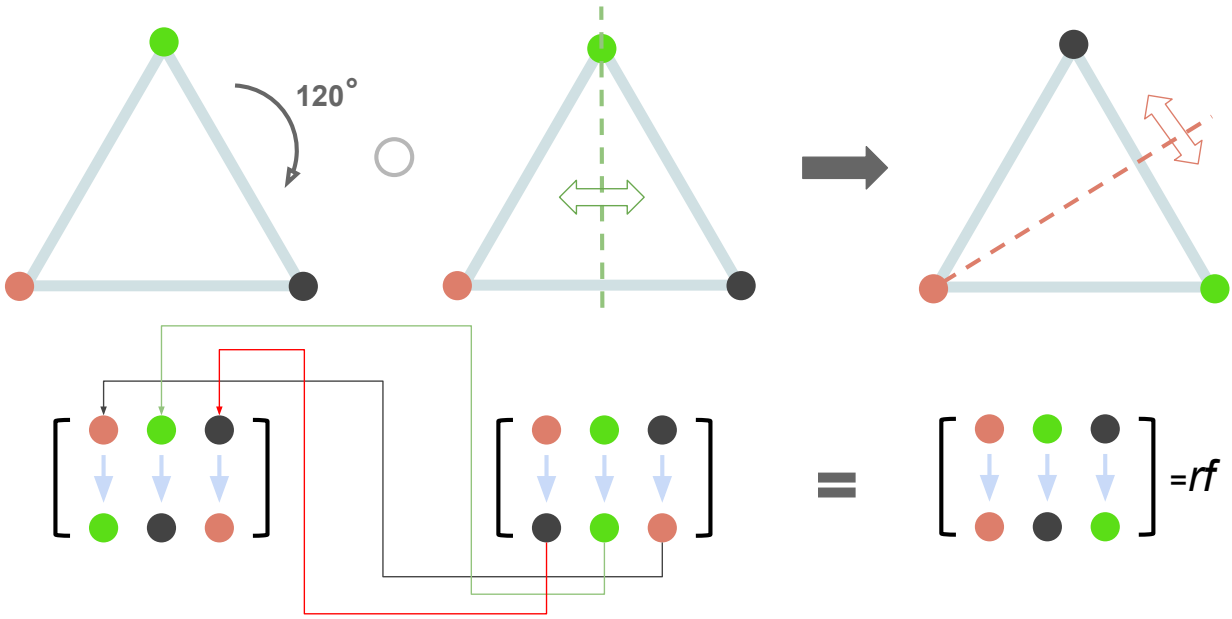


Figure 1.3: The composition of 120 deg rotation with horizontal reflection form another reflection line at vertex

Thus

$$e = r^k = r^{na+b} = (r^n)^a r^b = e^a r^b = r^b.$$

Since the smallest possible integer  $m$  by well-ordering, such that  $r^m = e$ , is  $n$ ,  $b = 0$ .

( $\Leftarrow$ ) Conversely, if  $n$  divide  $k$ , then  $k = ns$  for some integer  $s$ . Hence

$$r^k = r^{ns} = (r^n)^s = e^s = e.$$

Thus  $r^k = e \iff n|k$ .

Now let  $b = r^k \in D_n$ , since  $r$  is a generator of  $D_n$ . We shall show that the smallest integer  $m$  such that  $r^k = e$  is  $n/k$ . Let  $d = \gcd(n, k)$ . Consider

$$e = b^m = r^{km}.$$

Since this is the smallest integer  $m$  such that  $n|km$ . Thus  $\frac{n}{d}$  divide  $\frac{mk}{d}$ . Because  $d$  is the greatest

common divisor of  $n$  and  $k$ , implies  $\frac{n}{d}$  and  $\frac{k}{d}$  are relatively prime. Hence

$$\frac{n}{d} \mid \frac{mk}{d} \implies \frac{n}{d} \mid m$$

The smallest such  $m$  is  $\frac{n}{d}$ . Thus

$$\text{ord}(r^k) = \frac{n}{\gcd(n, k)}.$$

□

**Example 1.4.1.** Let

$$G = SL_2(\mathbb{Z}_3) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = 1; a, b, c, d \in \mathbb{Z}_3 \right\}.$$

Show that  $|G| = 48$ .

**Solution** From the first row, for all  $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus \{(0, 0)\}$ . There are  $({}^3P_1 \times {}^3P_1) - 1 = 8$  possibilities.

For the second row, for all  $(c, d) \in \mathbb{Z}_3 \times \mathbb{Z}_3 \setminus (a\mathbb{Z}_3, b\mathbb{Z}_3)$ . There are  $({}^3P_1 \times {}^3P_1) - 3 = 6$  possibilities.

Thus the order of group  $G$  is the product of the number of possibilities of these two rows.  
 $|G| = 8 \times 6 = 48$ . ◀

## 1.5 Normal subgroups, Quotient groups

### 1.5.1 Cosets

#### Definition 1.13 Cosets

Let  $G$  be a group and let  $H$  be a subset of  $G$ . For any  $a \in G$ , the set

$$aH := \{ah \mid h \in H\}$$

is called the **left coset** of  $H$  in  $G$  containing element  $a$ .

Analogously, the set

$$Ha := \{ha \mid h \in H\}$$

is called the **right coset** of  $H$  in  $G$  containing element  $a$ . In this case, the element  $a$  is called the **coset representative** of  $aH$  (or  $Ha$ ).

We use  $|aH|$  (or  $|Ha|$ ) to be the number of elements in the left (or right) coset.

**Example 1.5.1.** Consider  $G = \mathbb{Z}_9 = \{0, 1, 2, \dots, 8\}(\text{mod } 9)$ . We take a cyclic subgroup

$$H = \langle 3 \rangle = \{0, 3, 6\}$$



which came from  $(G, +_9)$ . All **left cosets** of  $G$  with respect to  $H$  are  $\{H, 1 +_9 H, 2 +_9 H\}$  where

$$\begin{aligned} 0 + H &= \{0 + 0, 0 + 3, 0 + 6\} \pmod{9} = \{0, 3, 6\} = H \\ 1H &= 1 + H = \{1 + 0, 1 + 3, 1 + 6\} \pmod{9} = \{1, 4, 7\} \\ 2H &= 2 + H = \{2 + 0, 2 + 3, 2 + 6\} \pmod{9} = \{2, 5, 8\} \\ 3H &= 3 + H = \{3 + 0, 3 + 3, 3 + 6\} \pmod{9} = \{3, 6, 0\} = H \end{aligned}$$

As for the right cosets of  $G$  with respect to  $H$  are  $\{H, H +_9 1, H +_9 2\}$ . Pay attention that now the element of coset are being added to right-hand side instead of from left side.

$$\begin{aligned} H + 0 &= \{0 + 0, 0 + 3, 0 + 6\} \pmod{9} = \{0, 3, 6\} = H \\ H1 &= H + 1 = \{0 + 1, 3 + 1, 6 + 1\} \pmod{9} = \{1, 4, 7\} \\ H2 &= H + 2 = \{0 + 2, 3 + 2, 6 + 2\} \pmod{9} = \{2, 5, 8\} \\ H3 &= H + 3 = \{0 + 3, 3 + 3, 6 + 3\} \pmod{9} = \{3, 6, 0\} = H \end{aligned}$$

Also, we can draw multiplication table of the cosets.

	$H$	$1 + H$	$2 + H$
$H$	$H$	$1 + H$	$2 + H$
$1 + H$	$1 + H$	$2 + H$	$H$
$2 + H$	$2 + H$	$H$	$1 + H$

Two cosets can also be multiply together. For instance,

$$(1 + H) \cdot (2 + H) = (1 + 2) + H = 0 + H = H$$

and

$$(2 + H) \cdot (2 + H) = (2 + 2) + H = 4 + H = 1 + H.$$

**Example 1.5.2.** The reals under addition  $(\mathbb{R}, +)$ , the subgroup  $(\mathbb{Z}, +)$  of integers. Now

$$\mathbb{R}/\mathbb{Z} = \{r + \mathbb{Z} \mid r \in \mathbb{R}\}.$$

The cosets are  $r + \mathbb{Z}$  with  $r \in [0, 1)$ .  $\mathbb{R}/\mathbb{Z}$  is isomorphic to the circle group  $\mathbb{S}^1$  of complex numbers of absolute value 1. The isomorphism is

$$\phi[(r + \mathbb{Z})] = e^{i2\pi r}.$$

## 1.5.2 Normal subgroups

### Definition 1.14 Normal subgroups

A subgroup  $H$  of  $(G, \cdot)$  is called a normal subgroup if for all  $g \in G$  we have

$$gH = Hg. \tag{1.7}$$

We shall denote that  $H$  is a subgroup of  $G$  by  $H < G$ , and that  $H$  is a normal subgroup of  $G$

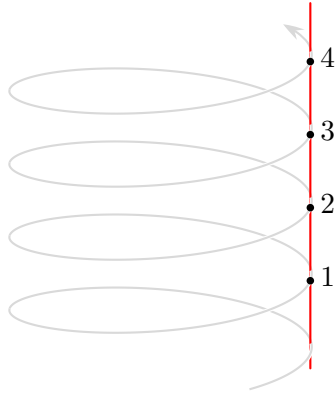


Figure 1.4: The abelian group  $\mathbb{R}/\mathbb{Z}$  can be generated by  $e^{i2\pi r}$ .

by  $H \triangleleft G$ .

If  $H$  is a normal subgroup of  $G$ , and the order of  $H$  is equal to the order of  $G$ , we called  $H$  the proper normal subgroup, write as  $H \trianglelefteq G$ .

You should be very careful here. The equality  $gH = Hg$  is a set equality. They are not constants or numbers! It says that a right coset is equal to left a coset, it is not an equality elementwise.

**Example 1.5.3.** Let  $\mathbb{R}[x]$  denote the group of all polynomial with real coefficients under normal addition.

For any  $f$  in  $\mathbb{R}[x]$ , let  $f'$  denote the derivative of  $f$ . Then the mapping  $f \rightarrow f'$  is a homomorphism from  $\mathbb{R}[x]$  to itself. The kernel of the derivative mapping is the set of all constant polynomials  $f(x) = c$ .

Now suppose we have a group  $(G, \cdot)$ , and  $H$  is a normal subgroup of  $G$ , just said  $H \triangleleft G$ . The set  $G/H$  is defined by

$$G/H = \{gH \mid g \in G\}.$$

$G/H$  is known as a **quotient group**.

**Example 1.5.4.** Show that if  $H$  and  $K$  are normal subgroups of a group  $G$  such that  $H \cap K = \{e\}$ , then  $hk = kh$  for all  $h \in H$  and  $k \in K$ .

**Solution** We knew that  $H \trianglelefteq G$  and  $K \trianglelefteq G$ , these conditions imply

$$gHg^{-1} \subseteq H, \quad gKg^{-1} \subseteq K \quad \forall g \in G.$$

Since  $khk^{-1} \in kHk^{-1} \subseteq kHk^{-1} \subseteq H$ . We want to show  $hk = kh$  for all  $h \in H$  and  $k \in K$ . Compute

$$\begin{aligned} h(kh^{-1}k^{-1}) &= e \Rightarrow hkh^{-1}k^{-1} = e \\ &\Rightarrow hkh^{-1} = ke \\ &\Rightarrow hkh^{-1} = khh^{-1} \\ &\Rightarrow hk = kh. \end{aligned}$$



### 1.5.3 Quotient group

#### Definition 1.15 Quotient group

Let  $G$  be a group, with  $H$  a subgroup such that  $gH = Hg$  for any  $g \in G$ . The set

$$G/H = \{gH \mid g \in G\}$$

of cosets of  $H$  in  $G$  is called a **quotient group**.

We can check that  $G/H$  is indeed a group:

- Because  $gH = Hg \implies gHg' = gg'H$  and  $G/H$  is closed under the same binary operator.
- Binary operation  $\circ : G/H \times G/H, (gH, g'H) \mapsto gHg'H$  is associative.
- The identity element is  $H$  since  $H(gH) = gH$  for any  $g$  in  $G$ .
- The inverse of  $gH$  is  $g^{-1}H$ , since

$$\begin{aligned} (gH)(g^{-1}H) &= (g^{-1}H)(gH) \\ &= (gg^{-1})H \\ &= eH \\ &= H. \end{aligned}$$

## 1.6 Group homomorphisms

#### Definition 1.16 Group homomorphisms

A group homomorphism is a map  $f : (G, \diamond_G) \rightarrow (H, \bullet_H)$  that respects binary operations:

$$f(a) \bullet_H f(b) = f(a \diamond_G b) \quad \forall a, b \in G \quad (1.8)$$

#### Theorem 1.11 Properties of homomorphism

Let  $\phi$  be a homomorphism from a group  $G$  to a group  $G'$  and let  $g$  be an element of  $G$ . Then

1.  $\phi$  carries the identity of  $G$  to the identity of  $G'$ .
2.  $\phi(g^n) = (\phi(g))^n$  for all  $n \in \mathbb{Z}$ .
3. If  $\text{ord}(g)$  is finite, then  $\text{ord}(\phi(g))$  divides  $\text{ord}(g)$ .
4.  $\ker(\phi)$  is a subgroup of  $G$ .
5.  $\phi(a) = \phi(b)$  if and only if  $a \ker(\phi) = b \ker(\phi)$ .
6. If  $\phi(g) = g'$ , then  $\phi^{-1}(g') = \{x \in G \mid \phi(x) = g'\} = g \ker(\phi)$ .

**Example 1.6.1.** Let  $G$  be any group and let  $a$  be any element of  $G$ . Let  $\phi : \mathbb{Z} \rightarrow G$  be defined by

$$\phi(n) = a^n$$

Show that  $\phi$  is a homomorphism. Find the kernel of  $\phi$ .

**Solution** For all  $n_1, n_2$  in  $\mathbb{Z}$ , we want to show

$$\phi(n_1 + n_2) = \phi(n_1) \phi(n_2).$$

In fact,

$$\phi(n_1 + n_2) = a^{n_1+n_2} = a^{n_1} a^{n_2} = \phi(n_1) \phi(n_2).$$

which means  $\phi$  preserves the group operation of  $G$ . Again we want to show  $\phi$  is both *one-to-one* and *onto*.

1. (One-to-one) For all  $x, y \in \mathbb{Z}$ ,

$$\begin{aligned} \phi(x) = y &\Rightarrow a^x = y \\ &\Rightarrow x = \frac{\log y}{\log a}, \quad a > 0 \end{aligned}$$

2. (Onto) For all  $x, y \in \mathbb{Z}$ ,

$$\phi(x) = \phi(y) \Rightarrow a^x = a^y \Rightarrow x = y.$$

Hence, we compute the kernel of  $\phi$ ,

$$\begin{aligned} \text{Ker}(\phi) &= \{n \in \mathbb{Z} \mid \phi(n) = 1\} \\ &= \{n \in \mathbb{Z} \mid a^n = 1\} \\ &= \{n \in \mathbb{Z} \mid a^n = a^0\} \\ &= \{n \in \mathbb{Z} \mid n = 0\} \\ &= \{0\}. \end{aligned}$$



## 1.7 Isomorphism

### Definition 1.17 Group isomorphisms

An isomorphism  $\phi$  from a group  $G$  to a group  $G'$  is a one-to-one mapping from  $G$  onto  $G'$  that preserves the group operation. That is,

$$\phi(a) \bullet_G \phi(b) = \phi(a \diamond_{G'} b)$$

for all  $a, b \in G$ . If there is an isomorphism from  $G \rightarrow G'$ , we say that  $G$  and  $G'$  are isomorphic and write as  $G \cong G'$ .

There are four separate steps involved in proving that a group  $G$  is isomorphic to another group  $G'$ .

1. Define a candidate for the isomorphism; that is, assume that  $\phi(a) = \phi(b)$  and hence prove that  $a = b$ .
2. Prove that  $\phi$  is one-to-one; that is, assume that  $\phi(a) = \phi(b)$  and hence prove that  $a = b$ .
3. Prove that  $\phi$  is onto; that is, for any element  $g' \in G'$ , find an element  $g \in G$  such that  $\phi(g) = g'$ .

4. Prove that  $\phi$  is operation-preserving; that is, show that

$$\phi(a) \bullet_G \phi(b) = \phi(a \diamond_{G'} b)$$

for all  $a, b \in G$ .

**Example 1.7.1.** Let  $G$  be the real numbers under addition and  $G'$  be the positive real numbers under multiplication.  $G$  and  $G'$  are isomorphic under the mapping

$$\phi(x) = 2^x.$$

**Solution** Define the mapping  $\phi : (G = \mathbb{R}, +) \rightarrow (G' = \mathbb{R}^+, \cdot)$ . For all  $x, y$  in  $G$ , we want to show

$$\phi(x + y) = \phi(x) \phi(y).$$

In fact,

$$\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x) \phi(y).$$

which means  $\phi$  is homomorphism. Again we want to show  $\phi$  is both *one-to-one* and *onto*.

1. (One-to-one) For all  $x, y \in G$ ,

$$\begin{aligned} \phi(x) = y &\Rightarrow 2^x = y \\ &\Rightarrow x = \log_2 y, \quad y > 0 \end{aligned}$$

2. (Onto) For all  $x, y \in G$ ,

$$\phi(x) = \phi(y) \Rightarrow 2^x = 2^y \Rightarrow x = y.$$

Hence, we compute the kernel of  $\phi$ ,

$$\begin{aligned} \text{Ker}(\phi) &= \{n \in \mathbb{R} \mid \phi(n) = 1\} \\ &= \{n \in \mathbb{R} \mid 2^n = 1\} \\ &= \{n \in \mathbb{R} \mid n = 0\} \\ &= \{0\}. \end{aligned}$$

◀

**Example 1.7.2.** Let  $G = SL(2, \mathbb{R})$ , the group of  $2 \times 2$  real matrices with determinant 1. Let  $M$  be any  $2 \times 2$  matrix with determinant 1. The mapping  $\phi_M$  from  $G \rightarrow G$  defined by

$$\phi_M(A) = MAM^{-1}$$

is an isomorphism.

**Solution** 1. First we are going to show that  $\phi_M$  is one-to-one; that is, for any  $A_1, A_2 \in G$ , if  $\phi_M(A_1) = \phi_M(A_2)$ , then  $A_1 = A_2$ .

$$\begin{aligned} \phi_M(A_1) = \phi_M(A_2) &\Rightarrow MA_1M^{-1} = MA_2M^{-1} \\ &\Rightarrow M^{-1}MA_1M^{-1}M = M^{-1}MA_2M^{-1}M \\ &\Rightarrow IA_1I = IA_2I \\ &\Rightarrow A_1 = A_2. \end{aligned}$$

2. Show  $\phi_M$  is onto. For all  $A_2 \in G$ , we need to find  $A_1 \in G$  such that

$$\phi_M(A_1) = A_2.$$

We find an equation for  $A_1$ ,

$$\begin{aligned}\phi_M(A_1) = A_2 &\Rightarrow MA_1M^{-1} = A_2 \\ &\Rightarrow A_1 = M^{-1}A_2M\end{aligned}$$

Now we verify that  $\phi_M(A_1) = A_2$ . Compute

$$\begin{aligned}\phi_M(A_1) &= \phi_M(M^{-1}A_2M) \\ &= M(M^{-1}A_2M)M^{-1} \\ &= IA_2I \\ &= A_2.\end{aligned}$$

3. At last we are going to show  $\phi_M$  is a homomorphism. That is,

$$\forall A_1, A_2 \in G, \quad \phi_M(A_1 A_2) = \phi_M(A_1)\phi_M(A_2).$$

We start from LHS,

$$\begin{aligned}\phi_M(A_1 A_2) &\Rightarrow MA_1 A_2 M^{-1} \\ &\Rightarrow MA_1 I A_2 M^{-1} \\ &\Rightarrow MA_1 (M^{-1}M) A_2 M^{-1} \\ &\Rightarrow (MA_1 M^{-1})(MA_2 M^{-1}) \\ &\Rightarrow \phi_M(A_1)\phi_M(A_2).\end{aligned}$$

Therefore  $\phi_M$  is an isomorphism. ◀

**Example 1.7.3.** The group  $U(10)$  is not isomorphic to  $U(12)$ .

**Theorem 1.12 Cayley's theorem**

Every group is isomorphism to a group of permutation.

*Proof.* Let  $G$  be a multiplication group. From group  $G$ , we need to construct a permutation group  $\bar{G}$  that is isomorphic to  $G$ .

**Step 1: construct permutation group  $\bar{G}$**

Given  $G$ , for all  $g \in G$ . We define a map  $T_g : G \rightarrow G$  such that

$$T_g(x) = gx.$$

Let  $\bar{G} = \{T_g \mid g \in G\}$ . We need to show that  $\bar{G}$  under function composition is a group.

1. (Closure) For all  $T_g, T_h \in \bar{G}$ , we want to show  $T_g \circ T_h \in \bar{G}$ .

$$\begin{aligned} T_g \circ T_h &= T_g(T_h(x)) \\ &= T_g(hx) \\ &= g(hx) \\ &= (gh)x \\ &= T_{gh}(x) \in \bar{G}. \end{aligned}$$

2. (Associativity) For all  $T_g, T_h, T_k \in \bar{G}$ , we have

$$T_g \circ (T_h \circ T_k) = T_{g(hk)} = T_{(gh)k} = (T_g \circ T_h) \circ T_k.$$

The associativity holds in  $\bar{G}$ .

3. (Existence of identity) For all  $T_g \in \bar{G}$ , there exists an  $T_{g'} \in \bar{G}$  such that

$$\begin{aligned} T_g \circ T_{g'} &= T_g \Rightarrow T_{gg'} = T_g \\ &\Rightarrow gg' = g \\ &\Rightarrow g' = 1 \end{aligned}$$

so that  $g'(x) = 1 \cdot x = x$ .

4. (Existence of inverse) For all  $T_g \in \bar{G}$ , the inverse of  $T_g$  is

$$T_{g^{-1}}(x) = g^{-1}x \quad \forall g \in G.$$

Therefore  $\bar{G}$  is a group under function composition. In the next step we are going to prove that the mapping from  $G$  to  $\bar{G}$  is an isomorphism.

### Step 2: show that $\phi : G \rightarrow \bar{G}$ is isomorphism

We now define a mapping  $\phi : G \rightarrow \bar{G}$ , where

$$\phi(g) = T_g(x) = gx \quad \forall g \in G.$$

We perform the 3 steps to check  $\phi(g)$  is an isomorphism.

1. ( $\phi$  is one-to-one) For all  $g, h \in G$ ,

$$\begin{aligned} \phi(g) = \phi(h) &\Rightarrow T_g = T_h \\ &\Rightarrow T_g(x) = T_h(x) \quad \forall x \in G \\ &\Rightarrow gx = hx \quad \forall x \in G \\ &\Rightarrow g = h. \end{aligned}$$

2. ( $\phi$  is onto) For all  $T_{g'} \in \bar{G}$ , we need to find an  $g \in G$  such that  $\phi(g) = T_{g'}$ .

$$\begin{aligned} \phi(g) = T_{g'} &\Rightarrow T_g(x) = T_{g'}(x) \\ &\Rightarrow gx = g'x \\ &\Rightarrow g = g'. \end{aligned}$$

3. ( $\phi$  is homomorphism) To show that  $\phi$  is homomorphism, it is equivalent to show that

$$\phi(g \circ h) = \phi(g) \phi(h).$$

From LHS,

$$\begin{aligned} \phi(g \circ h) &= T_{gh} = T_{gh}(x) \\ &= (g \circ h)x \\ &= gx \cdot hx \\ &= \phi(g) \phi(h). \end{aligned}$$

□

**Definition 1.18 Group stabilizer**

If  $G$  is a group of permutations on the set  $S$  and  $s \in S$  then we define the **stabilizer** of  $s$  to be the set

$$\text{Stab}_G(s) = \{\phi \in G \mid \phi(s) = s\}.$$

**Lemma 1.7**

If  $G$  is a group of permutations of the set  $S$  and  $s \in S$ . Then  $\text{Stab}_G(s)$  is a subgroup of  $G$ .

*Proof.* Using two-step subgroup test to verify:

1.  $\exists \phi : S \rightarrow S \in \text{Stab}_G(s)$  s.t.  $\phi(x) = x$ . Thus  $\text{Stab}_G(s) \neq \emptyset$ .
2. For all  $\phi_1, \phi_2 \in \text{Stab}_G(s)$ , we have

$$(\phi_1 \circ \phi_2)(s) = \phi_1(\phi_2(s)) = \phi_1(s) = s \in S$$

Therefore  $\phi_1 \circ \phi_2 \in \text{Stab}_G(s)$ .

3. For all  $\phi \in \text{Stab}_G(s)$ ,

$$\begin{aligned} \phi(s) = s &\Rightarrow \phi^{-1}(\phi(s)) = \phi^{-1}(s) \\ &\Rightarrow \phi^{-1}(s) = s \in S \end{aligned}$$

So  $\phi^{-1}$  is also in  $\text{Stab}_G(s)$ .

Therefore  $\text{Stab}_G(s)$  is a subgroup of  $G$ .

□

**Definition 1.19 Group orbit**

If  $G$  is a group of permutations of the set  $S$  and  $s \in S$ . We define the **orbit** to be the set

$$\text{Orbit}_G(s) = \{\phi(s) \mid \phi \in G\}.$$

**Theorem 1.13 Orbit-Stabilizer theorem**

For any group action  $\phi : G \rightarrow \text{Permutation}(S)$ , and for any  $s \in S$ ,

$$|\text{Orbit}_G(s)| \cdot |\text{Stab}_G(s)| = |G|. \quad (1.9)$$



*Proof.* We define a mapping  $f : G/\text{Stab}_G(s) \rightarrow \text{Orbit}_G(s)$  such that

$$f(\phi \text{Stab}_G(s)) = \phi(s).$$

$f$  is a homomorphism, and we want to show  $f$  is one-to-one and onto.

1. (One-to-one) For all  $\phi_1, \phi_2 \in \text{Stab}_G(s)$ , we have

$$\begin{aligned} \phi_1 \text{Stab}_G(s) = \phi_2 \text{Stab}_G(s) &\Rightarrow (\phi_1^{-1} \circ \phi_2) \in \text{Stab}_G(s) \\ &\Rightarrow (\phi_1^{-1} \circ \phi_2)(s) = s \\ &\Rightarrow \phi_1^{-1}(\phi_2(s)) = s \\ &\Rightarrow \phi_2(s) = \phi_1(s) \end{aligned}$$

2. (Onto) We again want to show  $f$  is onto. For all  $\phi \in \text{Orbit}_G(s)$ , we have

$$f(\phi \text{Stab}_G(s)) = \phi(s).$$

So  $f$  is both one-to-one and onto. Which means

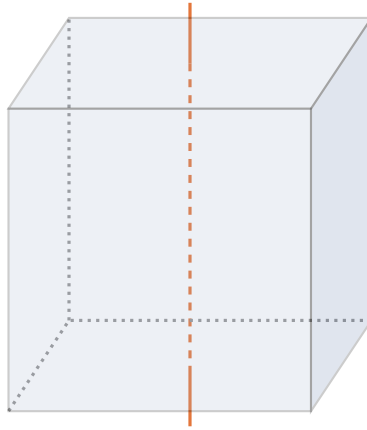
$$\begin{aligned} |G/\text{Stab}_G(s)| = |\text{Orbit}_G(s)| &\implies \frac{|G|}{|\text{Stab}_G(s)|} = |\text{Orbit}_G(s)| \\ &\implies |\text{Orbit}_G(s)| \cdot |\text{Stab}_G(s)| = |G|. \end{aligned}$$

□

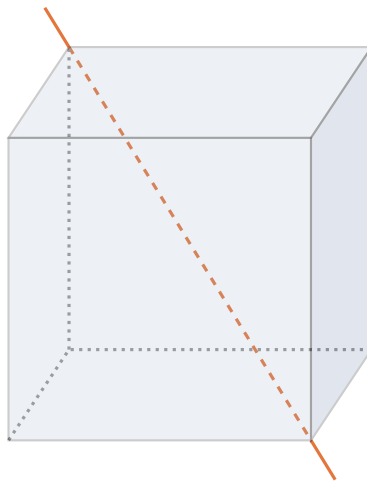
**Theorem 1.14**

The group of rotations of a cube is isomorphic to  $S_4$ .

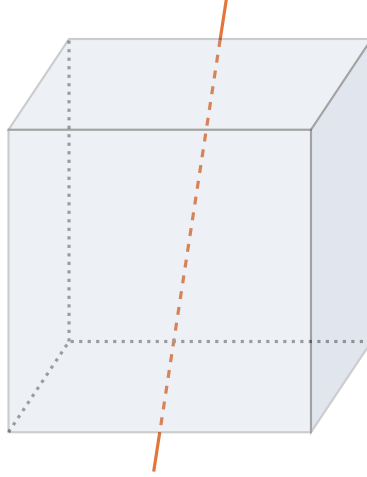
*Proof.* We can prove it by visualizing the rigid motions of a cube rotated along the possible axes.



One of three possible axes of rotation through the centers of opposite faces. Each rotation could be  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$ , or  $270^\circ$ , for a total of  $3 \times 4 = 12$  rotations of this type. But three in this count are the trivial identity rotation, which we only count once, so there are really 10 unique rotations along these axes.



One of four possible axes of rotation through opposite vertices. Each could be either  $120^\circ$  or  $240^\circ$ , so there are  $4 \times 2 = 8$  rotations of this type.



One of six possible axes of rotation through the centers of opposite edges. Only a  $180^\circ$  rotation around these axes would preserve the shape, so we have only 6 rotations possible.

There are  $10 + 8 + 6 = 24$  possible ways to rotate a cube, this is equal to the order of  $S_4$ . The order of  $S_4$  is

$$|S_4| = 4! = 4 \times 3 \times 2 \times 1 = 24.$$

□

### Theorem 1.15 Properties of isomorphisms

Suppose that  $\phi$  is an isomorphism from a group  $G$  onto a group  $G'$ . Then

1.  $\phi$  carries the identity of  $G$  to the identity of  $G'$ .
2. For every integer  $n$  and for every group element  $a$  in  $G$ ,
$$\phi(a^n) = \phi(a)^n.$$
3. For any elements  $a$  and  $b$  of  $G$ ,  $a$  and  $b$  commute if and only if  $\phi(a)$  and  $\phi(b)$  commute.
4.  $G = \langle a \rangle$  if and only if  $G' = \langle \phi(a) \rangle$ .
5.  $\text{ord}(a) = \text{ord}(\phi(a))$  for all  $a \in G$ .
6. For a fixed integer  $k$  and fixed group element  $b \in G$ , the equation  $x^k = b$  has the same number of solutions in  $G$  as does the equation  $x^k = \phi(b) \in G'$ .
7. If  $G$  is finite, then  $G$  and  $G'$  have exactly the same number of elements of every order.

*Proof.* 1. Work with  $G$ , we know  $e = g^n g^{-n}$ , so

$$\begin{aligned} \phi(e) &= \phi(g^n g^{-n}) \Rightarrow e' = \phi(g^n) \phi(g^{-n}) \\ &\Rightarrow e' = \underbrace{\phi(g * g * \dots * g)}_{n \text{ times}} \underbrace{\phi(g^{-1} * g^{-1} * \dots * g^{-1})}_{n \text{ times}} \\ &\Rightarrow e' = \underbrace{\phi(g) * \phi(g) * \dots * \phi(g)}_{n \text{ times}} \underbrace{\phi(g^{-1}) * \phi(g^{-1}) * \dots * \phi(g^{-1})}_{n \text{ times}} \end{aligned}$$

2. Using the similar technique, we have

$$\phi(g^n) = \phi(\underbrace{g * g * \cdots * g}_{n \text{ times}}) = \underbrace{\phi(g) * \phi(g) * \cdots * \phi(g)}_{n \text{ times}} = \phi(g)^n.$$

3. For all  $a, b \in G$ ,  $a$  and  $b$  commute if and only if

$$\begin{aligned} ab = ba &\iff \phi(ab) = \phi(ba) \\ &\iff \phi(a)\phi(b) = \phi(b)\phi(a) \\ &\iff \phi(a) \text{ and } \phi(b) \text{ commute} \\ &\iff G' \text{ is abelian.} \end{aligned}$$

4. For all  $a \in G$ ,

$$\begin{aligned} G = \langle a \rangle &\iff \phi(a^n) = \phi(e) \\ &\iff \phi(a^n) = e' \\ &\iff \phi(a)^n = e' \\ &\iff G' = \langle \phi(a) \rangle. \end{aligned}$$

(5, 6, 7) leave as tutorial. □

### Theorem 1.16

Suppose that  $\phi$  is an isomorphism from a group  $G$  onto a group  $G'$ . Then

1.  $\phi^{-1}$  is an isomorphism from  $G'$  onto  $G$ .
2.  $G$  is abelian if and only if  $G'$  is abelian.
3.  $G$  is cyclic if and only if  $G'$  is cyclic.
4. If  $K$  is a subgroup of  $G$ , then

$$\phi(K) = \{\phi(k) \mid k \in K\}$$

is a subgroup of  $G'$ .

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \left| \begin{array}{c} \leq \\ \leq \end{array} \right. & & \\ K & \xrightarrow{\phi} & \phi(K) \end{array}$$

*Proof.* 1. Given that  $\phi$  is a homomorphism, then for all  $a, b \in G$  we have

$$\phi(ab) = \phi(a)\phi(b).$$

We need to prove that  $\phi^{-1}$  is also a homomorphism, that is,

$$\forall a', b' \in G' \quad \phi^{-1}(a'b') = \phi^{-1}(a')\phi^{-1}(b').$$

Certainly

$$\begin{cases} \phi(a) = a' \\ \phi(b) = b' \end{cases} \implies \begin{cases} \phi^{-1}(a') = a \\ \phi^{-1}(b') = b \end{cases}$$

Since  $\phi$  is homomorphism,

$$\begin{aligned} \phi(ab) &= \phi(a)\phi(b) \Rightarrow \phi(ab) = a'b' \\ &\Rightarrow ab = \phi^{-1}(a'b') \\ &\Rightarrow \phi^{-1}(a')\phi^{-1}(b') = \phi^{-1}(a'b') \end{aligned}$$

2. Refer to property (3) of previous theorem.
3. Refer to property (4) of previous theorem.
4. Given  $K$  is a subgroup of  $G$ , we want to show that  $\phi(K) \leq G'$ . The main idea is to use one-step subgroup test:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \left| \leq \right. & & \left| \leq \right. \\ \phi^{-1}(N) & \xleftarrow{\phi^{-1}} & N \end{array}$$

- Clearly,  $e' = \phi(e) \in \phi(K)$ . Thus  $\phi(K)$  is nonempty. (Since  $e \in K \neq \emptyset$  as  $K \leq G$ ).
- For all  $x, y \in \phi(K)$ ,

$$x \in \phi(K) \Rightarrow x = \phi(\alpha_1)$$

and

$$y \in \phi(K) \Rightarrow y = \phi(\alpha_2)$$

where  $\alpha_1, \alpha_2 \in K$ . Compute

$$\begin{aligned} xy^{-1} &= \phi(\alpha_1)\phi(\alpha_2)^{-1} \\ &= \phi(\alpha_1)\phi(\alpha_2^{-1}) \\ &= \phi(\alpha_1\alpha_2^{-1}) \end{aligned}$$

Since  $\alpha_1\alpha_2^{-1} \in K$ , thus  $\phi(\alpha_1\alpha_2^{-1}) \in \phi(K) \implies xy^{-1} \in \phi(K)$ .

□

### Theorem 1.17 Sylow's theorem

Let  $G$  be a finite group such that  $p^n$  divides  $|G|$ , where  $p$  is prime. Then there exists a subgroup of order  $p^n$ .

*Proof.* Assume that  $|G| = p^n m$ , where  $m = p^r k$  with  $\gcd(p, k) = 1$ . Our central strategy is to consider a cleverly chosen group action of  $G$  and prove one of the stabilizer subgroups has order  $p^n$ . We will need to heavily exploit the orbit-stabilizer theorem.

Let  $S$  be the set of all subsets of  $G$  of order  $p^n$ . An element of  $S$  is an unordered  $n$ -tuple of distinct elements in  $G$ . There is a natural action of  $G$  on  $S$  by term-by-term composition on the left.

Consider  $\sigma \in S$ . If we fix an ordering  $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_{p^n}\} \in S$ , then

$$g(\sigma) := \{g * \sigma_1, g * \sigma_2, \dots, g * \sigma_{p^n}\}.$$

We first claim that  $|\text{Stab}(\sigma)| \leq p^n$ . To see this we define the function

$$f : \text{Stab}(\sigma) \rightarrow \sigma$$

$$g \rightarrow g * \sigma_1$$

By the cancellation property for groups this is an injective map. Hence

$$|\text{Stab}(\sigma)| \leq |\sigma| = p^n.$$

On the other hand, observe that

$$|S| = \binom{p^n m}{p^n} = \frac{p^n m!}{(p^n)!(p^n m - p^n)!} = \prod_{j=0}^{p^n-1} \frac{p^n m - j}{p^n - j} = m \prod_{j=1}^{p^n-1} \frac{p^n m - j}{p^n - j}.$$

If  $1 \leq j \leq p^n - 1$  then  $j$  is divisible by  $p$  at most  $n - 1$  times. This means that  $p^n m - j$  and  $p^n - j$  have the same number of  $p$  factors, namely the number of  $p$  factor of  $j$ . This means that

$$\prod_{j=1}^{p^n-1} \frac{p^n m - j}{p^n - j}$$

has no  $p$  factors. Hence  $p^r u$ , where  $\gcd(p, u) = 1$ .

Now recall that  $S$  is the disjoint union of the orbits of our action of  $G$  on  $S$ . Hence there must be an  $\sigma \in S$  such that

$$|\text{Orbit}(\sigma)| = p^s t$$

where  $s \leq r$  and  $\gcd(p, t) = 1$ . By the *orbit-stabilizer theorem* we know that

$$|\text{Stab}(\sigma)| = \frac{p^{n+r-s} u}{t}$$

Because  $|\text{Stab}(\sigma)| \in \mathbb{N}$  and  $u, t$  are coprime to  $p$ , we deduce that  $u/t$  is natural number. Hence  $|\text{Stab}(\sigma)| \geq p^n$ .

For this choice of  $\sigma \in S$ ,  $\text{Stab}(\sigma)$  is thus a subgroup of size  $p^n$ . □

Historically this is a slight extension of what is called Sylow's First Theorem. There are two more which describe the properties of such subgroups in greater depth.

## 1.8 Automorphisms

### Definition 1.20 Automorphisms

An isomorphism from a group  $G$  onto itself is called an automorphism.

**Example 1.8.1.** Let the 2-dimensional cartesian plane

$$\mathbb{R}^2 = \{(a, b) | a, b \in \mathbb{R}\}.$$

Then

$$\phi(a, b) = (b, a)$$

is an automorphism of the group  $\mathbb{R}^2$  under componentwise addition.

**Example 1.8.2.** Compute  $\text{Aut}(\mathbb{Z}_{10})$ .

**Solution** For any  $\alpha \in \text{Aut}(\mathbb{Z}_{10})$  and for any  $k \in \mathbb{Z}_{10}$ . We define  $k \mapsto k\alpha(1)$  such that

$$1 \mapsto \alpha_1 : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}, \quad \alpha_1(x) = x$$

$$3 \mapsto \alpha_3 : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}, \quad \alpha_3(x) = 3x$$

$$7 \mapsto \alpha_7 : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}, \quad \alpha_7(x) = 7x$$

$$9 \mapsto \alpha_9 : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10}, \quad \alpha_9(x) = 9x$$

In fact,  $\text{Aut}(\mathbb{Z}_{10})$  is isomorphic to  $U(10) = \{1, 3, 7, 9\}$ .  $\text{Aut}(\mathbb{Z}_{10})$  is defined as

$$\text{Aut}(\mathbb{Z}_{10}) = \{\alpha : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{10} \mid \alpha \text{ is an automorphism}\}.$$

$\cdot_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

$\circ$	$\alpha_1$	$\alpha_3$	$\alpha_7$	$\alpha_9$
$\alpha_1$	$\alpha_1$	$\alpha_3$	$\alpha_7$	$\alpha_9$
$\alpha_3$	$\alpha_3$	$\alpha_9$	$\alpha_1$	$\alpha_7$
$\alpha_7$	$\alpha_7$	$\alpha_1$	$\alpha_9$	$\alpha_3$
$\alpha_9$	$\alpha_9$	$\alpha_7$	$\alpha_3$	$\alpha_1$



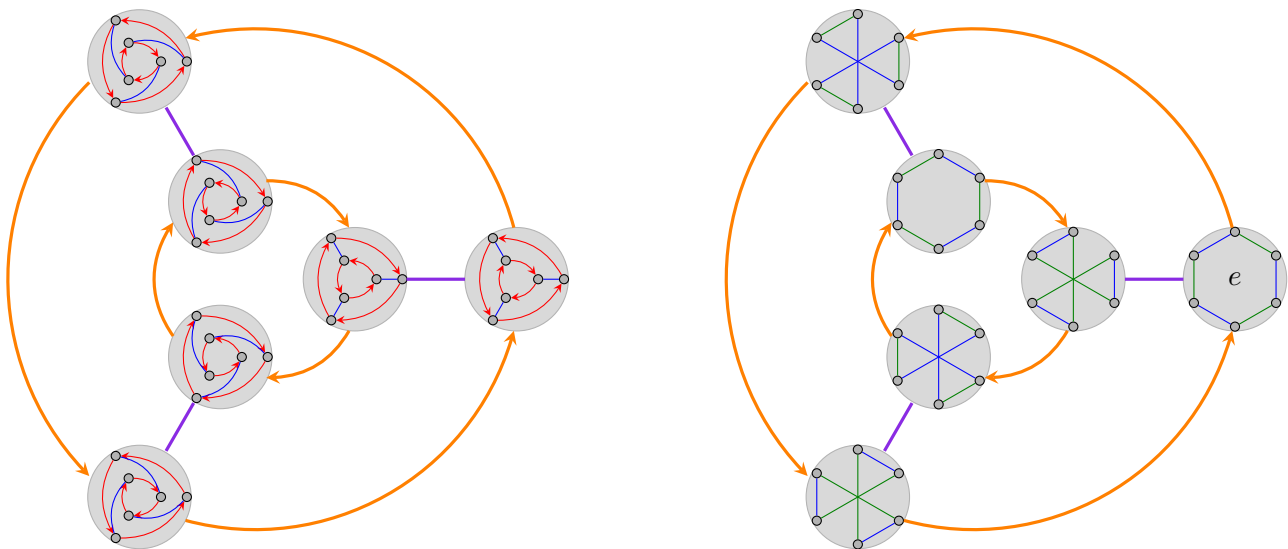
**Theorem 1.18**

For every positive integer  $n$ ,  $\text{Aut}(\mathbb{Z}_n)$  is isomorphic to  $U(n)$ .

**Example 1.8.3.** The automorphisms of  $D_3$  is  $\text{Aut}(D_3) = \langle \alpha, \beta \rangle \cong D_3$ , where

$$\begin{cases} \alpha(r) = r \\ \alpha(f) = rf \end{cases} \quad \begin{cases} \beta(r) = r^2 \\ \beta(f) = f \end{cases}$$

All of these automorphisms are *inner* (of the form  $f_x : g \mapsto x^{-1}gx$ ). Two Cayley diagrams for  $\text{Aut}(D_3)$  are shown below.



### Definition 1.21 Inner automorphisms

Let  $G$  be a group, and let  $a \in G$ . The function  $\phi_a$  defined by

$$\phi_a(x) = axa^{-1} \quad \text{for all } x \in G$$

is called the inner automorphism of  $G$  included by  $a$ .

When  $G$  is a group, we use  $\text{Aut}(G)$  to denote the set of all automorphisms of  $G$  and  $\text{Inn}(G)$  to denote the set of all inner automorphisms of  $G$ .

### Theorem 1.19

The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under the operation of function composition.

*Proof.* The set of inner automorphisms of  $G$  included by  $a$  is

$$\text{Inn}(G) = \{\phi_a \mid \phi_a \text{ is an inner automorphism}\}.$$

Then satisfied the group properties:

1. We want to show  $\forall \phi_a, \phi_b \in \text{Inn}(G), \quad \phi_a \circ \phi_b \in \text{Inn}(G)$ .

Compute  $(\phi_a \circ \phi_b)(g)$  for all  $g$  in  $G$ ,

$$\begin{aligned} (\phi_a \circ \phi_b)(g) &= \phi_a(\phi_b(g)) \\ &= \phi_a(bgb^{-1}) && \text{(Defn. of inner automorphism)} \\ &= a(bgb^{-1})a^{-1} \\ &= (ab)g(b^{-1}a^{-1}) \\ &= (ab)g(ab)^{-1} && \text{(Socks-shoes property)} \\ &= \phi_{ab}(g) \in \text{Inn}(G) \end{aligned}$$

Thus  $\text{Inn}(G)$  is closed under function composition.



2. Next we want to show the associativity in  $\text{Inn}(G)$ , that is,

$$\forall \phi_a, \phi_b, \phi_c \in \text{Inn}(G), \quad \phi_a \circ (\phi_b \circ \phi_c) = (\phi_a \circ \phi_b) \circ \phi_c$$

we compute  $\phi_a \circ (\phi_b \circ \phi_c)$ .

$$\begin{aligned} [\phi_a \circ (\phi_b \circ \phi_c)](g) &= a(bc)g(bc)^{-1}a^{-1} \\ &= (ab)cg c^{-1} b^{-1} a^{-1} \\ &= (ab)\underline{cg c^{-1}}(ab)^{-1} \\ &= [(\phi_a \circ \phi_b) \circ \phi_c](g) \end{aligned}$$

3. Suppose  $e$  is the identity element of  $G$ , then  $\phi_e(g) = ege^{-1} = g \in \text{Inn}(G)$ .  $\phi_e$  is the identity of  $\text{Inn}(G)$ .

4. For all  $\phi_a \in \text{Inn}(G)$ , there exists  $\phi_{a^{-1}} \in \text{Inn}(G)$  such that

$$\begin{aligned} \phi_a \circ \phi_{a^{-1}} &= a(ag^{-1}a^{-1})a^{-1} \\ &= (aa^{-1})g^{-1}(aa^{-1}) \\ &= g^{-1} \end{aligned}$$

We have shown that the inner automorphisms are group. Is  $\text{Inn}(G)$  a subgroup of  $\text{Aut}(G)$ ? Of course it is. We are going to use one-step subgroup test to find out.

**One-step subgroup test:**

1. First of all, we want to show

$$\forall \phi_a, \phi_b \in \text{Inn}(G), \phi_a \circ \phi_{b^{-1}} \in \text{Inn}(G).$$

we compute

$$\begin{aligned} (\phi_a \circ \phi_{b^{-1}})(g) &= \phi_a(bg^{-1}b^{-1}) \\ &= a(bg^{-1}b^{-1})a^{-1} \\ &= (ab)g^{-1}(ab)^{-1} \\ &= \phi_{(ab)^{-1}} \in \text{Inn}(G) \end{aligned}$$

□

### 1.8.1 Isomorphism theorems

#### **Theorem 1.20 First Isomorphism theorem**

Let  $f : G \rightarrow H$  be a homomorphism of groups with kernel  $K$ . Then the quotient group  $G/K$  is isomorphic to  $\text{Im } f$ .

*Proof.* **Step 1:** Check if  $\text{Ker}(\phi)$  is a subgroup of  $G$ .

The main idea is to use one-step subgroup test to check if  $\text{Ker}(\phi) \leq G$ :

1. Since  $\phi$  is a homomorphism, then  $\phi(e) = e$ ; and so

$$e \in \text{Ker}(\phi) \neq \emptyset.$$

So  $\text{Ker}(\phi)$  is nonempty.

2. For all  $g, h \in \text{Ker}(\phi)$ , we need to show that

$$gh^{-1} \in \text{Ker}(\phi).$$

Given that  $g \in \text{Ker}(\phi)$ , then  $\phi(g) = e$ . Also,  $h \in \text{Ker}(\phi)$ , then  $\phi(h) = e$ . Now compute

$$\begin{aligned} \phi(gh^{-1}) &= \phi(g) \phi(h^{-1}) \\ &= \phi(g) \phi(h)^{-1} \\ &= ee^{-1} \\ &= e \in \text{Ker}(\phi). \end{aligned}$$

Since  $gh^{-1} \in \text{Ker}(\phi)$ , thus  $\text{Ker}(\phi) \leq G$ .

**Step 2:** Check if  $\text{Ker}(\phi) \trianglelefteq G$

We want to show  $a\text{Ker}(\phi)a^{-1} \subseteq \text{Ker}(\phi)$  for all  $a \in G$ . This is equivalent to show if  $y \in a\text{Ker}(\phi)a^{-1}$ , then  $y$  is in  $\text{Ker}(\phi)$ . Using normal subgroup test and check:

$$\begin{aligned} y \in a\text{Ker}(\phi)a^{-1} &\Rightarrow a^{-1}ya \in \text{Ker}(\phi) \\ &\Rightarrow \phi(a^{-1}ya) = e \\ &\Rightarrow \phi(a^{-1}) \phi(y) \phi(a) = e \\ &\Rightarrow \phi(y) = \phi(a) \phi(e) \phi(a)^{-1} \\ &\Rightarrow \phi(y) = \phi(a) \phi(e) \phi(a^{-1}) \\ &\Rightarrow \phi(y) = \phi(aea^{-1}) = \phi(aa^{-1}) = \phi(e) = e \end{aligned}$$

Thus  $y$  is in  $\text{Ker}(\phi)$ . So  $\text{Ker}(\phi) \trianglelefteq G$ .

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ \downarrow \trianglelefteq & & \downarrow \leq \\ \text{Ker}(\phi) & & \phi(G) \\ \uparrow \cong & & \\ G/\text{Ker}(\phi) & & \end{array}$$

Moreover, we still need to show  $\phi(G) \leq G'$ .  $\phi(G)$  is the image of  $G$  that is define as

$$\phi(G) = \{\phi(g) \mid g \in G\}.$$

Again we use one-step subgroup test to check. For all  $x, y \in \phi(G)$ , we want to show  $xy^{-1}$  is in  $\phi(G)$ .

Let  $x = \phi(g_1)$ , and  $y = \phi(g_2)$  where  $g_1, g_2 \in G$ . Compute

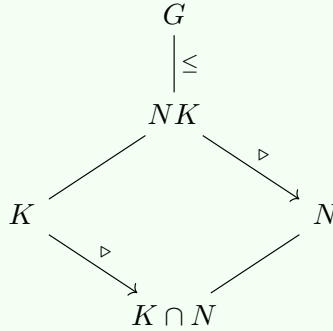
$$\begin{aligned} xy^{-1} &= \phi(g_1) \phi(g_2^{-1}) \\ &= \phi(g_1 g_2^{-1}) \in \phi(G) \end{aligned}$$

The map  $T : G/\text{Ker}(\phi) \rightarrow \phi(G)$  (where  $T(g\text{Ker}(\phi)) = \phi(g)$ ) is one-to-one and onto. So  $T$  is an isomorphism.  $\square$

### Theorem 1.21 Second Isomorphism Theorem

Let  $K$  be a proper subgroup of  $G$ , and  $N \triangleleft G$ . Then

1.  $NK = \{nk \mid n \in N, k \in K\}$  is a proper subgroup of  $G$ , write  $NK < G$ .
2.  $N \triangleleft NK$  and  $K \cap N \triangleleft K$ .
3.  $K/(K \cap N)$  is isomorphism to  $NK/N$ .



*Proof.* 1. Clearly  $NK \subset G$ . By using two-step subgroup test. For all  $nk, n'k' \in NK$

$$nkn'k' = (nn')(kk') \in NK.$$

and

$$nk = e \implies nkk^{-1}n^{-1} = e \implies (nk)^{-1} = k^{-1}n^{-1} \in NK.$$

Thus  $NK < G$ .

2. Clearly  $N \triangleleft NK < G$ , since  $aN = Na$  whenever  $a$  in  $NK$ .  $N \triangleleft G$  implies that the mapping  $\pi : G \rightarrow G/N$  that maps  $a \rightarrow Na$  is a surjective homomorphism. We define  $f : K \rightarrow G/N$  which  $k \mapsto Nk$  whenever  $k$  in  $K$ . Apparently  $f$  is also homomorphism.

The kernel of  $f$  is

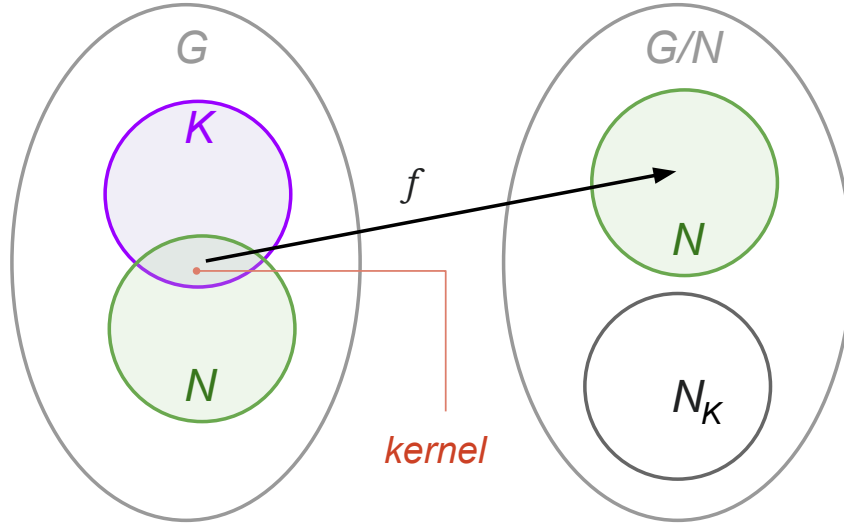
$$\begin{aligned} \ker f &= \{k \in K \mid f(k) = Ne\} \\ &= \{k \in K \mid Nk = Ne\} \\ &= \{k \in K \mid k \in N\} \\ &= K \cap N. \end{aligned}$$

Thus  $K \cap N \triangleleft K$ .

3. Find the image of  $f$ ,

$$\begin{aligned}\text{Im } f &= \{Nk \in G/N \mid k \in K\} \\ &= \{Nnk \in G/N \mid n \in N, k \in K\} \\ &= \{Nnk \in G/N \mid nk \in NK\} \\ &= NK/N.\end{aligned}$$

By First Isomorphism Theorem, we now have  $K/(K \cap N)$  is isomorphism to  $NK/N$ .



□

**Theorem 1.22 Subgroup of the quotient group  $G/N$**

Let  $N \triangleleft G$ , and let  $N \subset K < G$ . Then  $K/N$  is a proper subset of  $G/N$ .

*Proof.* Clearly if  $N \triangleleft K$ , then

$$K/N = \{N_k \mid k \in K\} \subset G/N = \{N_g \mid g \in G\}.$$

Thus  $K/N < G/N$ .

□

## 1.9 Lagrange Theorem

**Theorem 1.23 Lagrange theorem**

If  $G$  is a finite group and  $H$  is a subgroup of  $G$ , then  $|H|$  divides  $|G|$ .

*Proof.* Let  $a_1H, a_2H, \dots, a_rH$  denote the distinct left cosets and right cosets of  $H$  in  $G$ . For all  $a \in G$ ,  $\exists i$  s.t.  $aH = a_iH$  and  $a \subseteq a_iH$ . Thus we can say that each members in  $G$  belongs to the one of the cosets of  $a_iH$ . Since  $|a_iH| = |H|$ , so

$$G = \bigcup_{k=1}^r a_kH \implies |G| = \sum_{k=1}^r |a_kH| = \sum_{k=1}^r |H| = r(|H|).$$

Thus  $|H|$  divides  $|G|$  □

**Example 1.9.1.** Given  $G$  is a group of order 6. Find all possible subgroups of  $G$ .

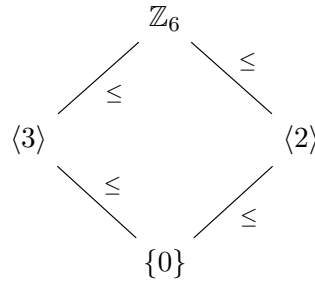
**Solution** Suppose  $H$  is a subgroup of  $G$ ,  $H \leq G$ , and so by Lagrange's theorem we have

$$|H| \mid |G| = 6$$

which implies  $|H| \in \{1, 2, 3, 6\}$ .

If the order of  $H$  is one, then  $H = \langle e \rangle = \{e\}$ .

On the other hand, when  $|H| = 6$  which means  $|H| = |G| = 6$  and  $H = G$ ; that is,  $H$  is trivial subgroup.



#### Theorem 1.24

If  $G$  is a group of order  $p$ , where  $p$  is a prime. Then  $G$  is cyclic.

*Proof.* Suppose  $H$  is a subgroup of  $G$  and we apply Lagrange's theorem, which we have

$$|H| \mid |G| = p.$$

Assume  $H \neq \{e\}$  and the order of  $H$  is greater than one, implies  $|H| = p$  and so  $|G| = |H| = p$ .

Hence, since  $H \neq \{e\}$ , then there exists  $a \neq e$  and  $a \in H$ . So using  $a$  to generate we obtained

$$H = G = \langle a \rangle$$

implies that  $H$  and  $G$  are both cyclic. □

**Example 1.9.2 (Tutorial).** Draw a subgroup lattice of  $\mathbb{Z}_{60}$ .

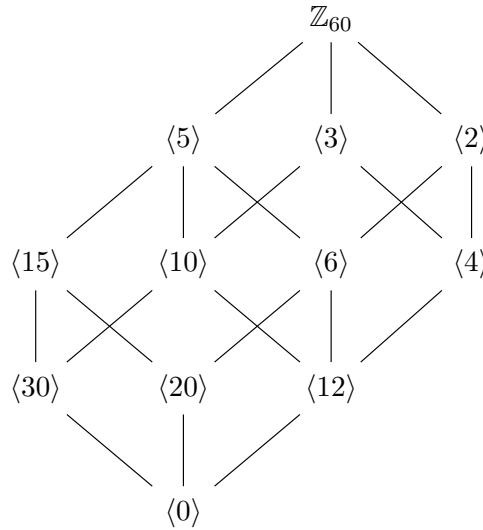
**Solution** Suppose  $H$  is a subgroup of  $\mathbb{Z}_{60}$ . By Lagrange theorem,

$$|H| \mid 60 = 2^2 \cdot 3 \cdot 5.$$

Thus,

$$|H| \in \{1, 2, 4, 3, 6, 12, 5, 10, 20, 15, 30, 60\}.$$

The lattice diagram is as follow:



## 1.10 External Direct Product

### Definition 1.22 External direct product

Given groups  $(G, *)$  and  $(H, +)$ , the external direct product of  $G, H$  written as  $G \oplus H$ , is the set of all ordered pairs  $(g, h)$  for which the binary operation on  $G \oplus H$  is defined component-wise:

$$(g_1, h_1) \oplus (g_2, h_2) = (g_1 * g_2, h_1 + h_2).$$

The resulting algebraic object satisfies the axioms for a group. Specifically:

- **Associativity**, the binary operation on  $G \times H$  is indeed associative.
- **Existence of identity**, the direct product has an identity element, namely  $(e_1, e_2)$ , where  $e_1$  is the identity element of  $G$  and  $e_2$  is the identity element of  $H$ .
- **Existence of inverses**, the inverse of an element  $(g, h)$  of  $G \times H$  is the pair  $(g^{-1}, h^{-1})$ , where  $g^{-1}$  is the inverse of  $g \in G$ , and  $h^{-1}$  is the inverse of  $h \in H$ .

**Example 1.10.1.** Find  $U(8) \oplus U(10)$ .

**Solution** We have  $U(8) = \{1, 3, 5, 7\}$  and  $U(10) = \{1, 3, 7, 9\}$ . Then  $U(8) \oplus U(10)$  is the cartesian product of each element.

	1	3	7	9
1	(1, 1)	(1, 3)	(1, 7)	(1, 9)
3	(3, 1)	(3, 3)	(3, 7)	(3, 9)
7	(7, 1)	(7, 3)	(7, 7)	(7, 9)
9	(9, 1)	(9, 3)	(9, 7)	(9, 9)

**Theorem 1.25**

The order of an element in a direct product of a finite number of finite groups is the *least common multiple* of the orders of the components of the element. In symbols,

$$|(g_1, g_2, \dots, g_n)| = LCM(\text{ord}(g_1), \text{ord}(g_2), \dots, \text{ord}(g_n)).$$

*Proof.* First we consider the case where the direct product has two components.

Consider  $(g_1, g_2) \in G_1 \oplus G_2$ . Let  $s = LCM(\text{ord}(g_1), \text{ord}(g_2))$  and  $t = |(g_1, g_2)|$ . Then

$$(g_1, g_2)^s = (g_1^s, g_2^s) = (e, e) \implies t|s.$$

Thus  $t \leq s$ . But

$$(g_1^t, g_2^t) = (g_1, g_2)^t = (e, e) \implies g_1|t \text{ and } g_2|t.$$

Thus  $t$  is a common multiple of  $\text{ord}(g_1)$  and  $\text{ord}(g_2)$ , which means  $s \leq t$  since

$$s = LCM(\text{ord}(g_1), \text{ord}(g_2)).$$

Hence  $s = t$  and  $|(g_1, g_2)| = LCM(\text{ord}(g_1), \text{ord}(g_2))$ .

For the general case, suppose the result holds for

$$G_1 \oplus G_2 \oplus \dots \oplus G_{n-1}.$$

But  $G_1 \oplus G_2 \oplus \dots \oplus G_n = (G_1 \oplus G_2 \oplus \dots \oplus G_{n-1}) \oplus G_n$ . So applying the previous argument, the result holds for

$$G_1 \oplus G_2 \oplus \dots \oplus G_n$$

by induction. □

**Theorem 1.26**

Let  $G_1$  and  $G_2$  be two groups. Let  $H_1$  and  $H_2$  be normal subgroups of  $G_1$  and  $G_2$  respectively then

$$H_1 \oplus H_2 \trianglelefteq G_1 \oplus G_2.$$

*Proof.* Since  $H_1$  is a subgroup of  $G_1$  and  $H_2$  is a subgroup of  $G_2$ , therefore  $H_1 \oplus H_2$  is subgroup of  $G_1 \oplus G_2$ .

Now let  $(h_1, h_2) \in H_1 \oplus H_2$  and  $(g_1, g_2) \in G_1 \oplus G_2$ , then

$$\begin{aligned} (g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} &= (g_1 h_1, g_2 h_2)(g_1^{-1}, g_2^{-1}) \\ &= (g_1 h_1 g_1^{-1}, g_2 h_2 g_2^{-1}) \in H_1 \oplus H_2. \end{aligned}$$

$g_1 h_1 g_1^{-1} \in H_1$  and  $g_2 h_2 g_2^{-1} \in H_2$  because  $H_1 \trianglelefteq G_1$  and  $H_2 \trianglelefteq G_2$ . Thus  $H_1 \oplus H_2 \trianglelefteq G_1 \oplus G_2$ . □

## 1.11 Internal Direct Product

### Definition 1.23

Let  $(G, *)$  be a group and let  $(H, *)$  and  $(K, *)$  be two subgroups of  $G$ . Then  $G$  is said to be the internal direct product of  $H$  and  $K$  (write  $G = H \times K$ ) if:

1.  $G = \{h * k \mid h \in H, k \in K\}$ .
2.  $H \cap K = \{e\}$  where  $e$  is the identity in  $G$ .
3.  $h * k = k * h$  for all  $h \in H$  and for all  $k \in K$ .

**Remark.** If  $H, K \triangleleft G$ , then

$$H \times K \cong H \oplus K.$$

**Example 1.11.1.** Consider  $H = \{0, 2, 4\}$  and  $K = \{0, 3\}$ , show that  $G = \mathbb{Z}_6$  is the internal direct product of  $H$  and  $K$ .

**Solution** 1. We check  $G = H \times K$ , compute

$$\begin{aligned} G &= H \times K \\ &= \{0, 2, 4\} \times \{0, 3\} \\ &= \{0 +_6 0, 0 +_6 3, 2 +_6 0, 2 +_6 3, 4 +_6 0, 4 +_6 3\} \\ &= \{0, 3, 2, 5, 4, 1\} = \mathbb{Z}_6. \end{aligned}$$

2.  $H \cap K = \{0\}$ , 0 is identity of  $\mathbb{Z}_6$ .

3.  $\forall a \in H, b \in K, \quad a +_6 b = b +_6 a$ .

$H$  and  $K$  are subgroups of  $G = \mathbb{Z}_6$ . Thus  $G$  is the internal direct product of  $H$  and  $K$ . ◀

**Example 1.11.2.**  $\mathbb{Z}_6$  is internal direct product of  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$ .

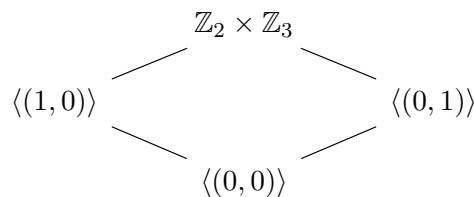
**Solution** The direct product of  $\mathbb{Z}_2$  and  $\mathbb{Z}_3$  form a tuple,

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_3 &= \{(x, y) \mid x \in \mathbb{Z}_2 \text{ and } y \in \mathbb{Z}_3\} \\ &= \{(0, 0), (0, 1), (0, 2) \\ &\quad (1, 0), (1, 1), (1, 2)\} \end{aligned}$$

For all  $(a_1, b_1), (a_2, b_2)$ , we define the product as

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 +_2 a_2, b_1 +_3 b_2).$$

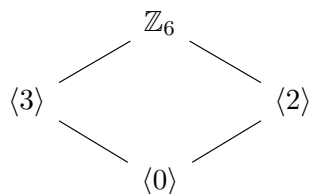
For example,  $(1, 0) \cdot (1, 2) = (1 + 0 \bmod 2, 1 + 2 \bmod 3) = (1, 0)$ . Compute the Cayley table for all elements in  $\mathbb{Z}_2 \times \mathbb{Z}_3$  (it is actually a  $6 \times 6$  **Latin square**)





·	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(0, 0)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(0, 1)	(0, 1)	(0, 2)	(0, 0)	(1, 1)	(1, 2)	(1, 0)
(0, 2)	(0, 2)	(0, 0)	(0, 1)	(1, 2)	(1, 0)	(1, 1)
(1, 0)	(1, 0)	(1, 1)	(1, 2)	(0, 0)	(0, 1)	(0, 2)
(1, 1)	(1, 1)	(1, 2)	(1, 0)	(0, 1)	(0, 2)	(0, 0)
(1, 2)	(1, 2)	(1, 0)	(1, 1)	(0, 2)	(0, 0)	(0, 1)

Which is quite matches with the Cayley table of  $\mathbb{Z}_6$ .



·	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

In fact,  $\mathbb{Z}_2 \times \mathbb{Z}_3$  is isomorphic to  $\mathbb{Z}_6$ . ◀

## 1.12 Finite Abelian groups

### Theorem 1.27 Fundamental theorem of Finite Abelian groups

Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.

Since a cyclic group of order  $n$  is isomorphic to  $\mathbb{Z}_n$ . Then every finite Abelian group  $G$  is isomorphic to a group of the form

$$\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{n_k}}$$

where  $i$ 's are not necessarily distinct primes and the prime powers  $p_1^{n_1}, p_2^{n_2}, \dots, p_k^{n_k}$  are uniquely determined by  $G$ .

Look at groups whose orders have the form  $p^k$ , where  $p$  is prime and  $j \leq 4$ . There is one group of order  $p^k$  for each set of positive integers whose sum is  $k$  (such a set is called a partition of  $k$ ); that is, if  $k$  can be written as  $k = n_1 + n_2 + \cdots + n_t$ , where each  $n_i$  is a positive integer, then  $\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{n_t}}$  is an Abelian group of order  $p^k$ .

Order of $G$	Partitions of $k$	Possible direct products of $G$
$p$	1	$\mathbb{Z}_p$
$p^2$	2 1 + 1	$\mathbb{Z}_{p^2}$ $\mathbb{Z}_p \oplus \mathbb{Z}_p$
$p^3$	3 1 + 2 1 + 1 + 1	$\mathbb{Z}_{p^3}$ $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$
$p^4$	4 1 + 3 2 + 2 1 + 1 + 2 1 + 1 + 1 + 1	$\mathbb{Z}_{p^4}$ $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$

### 1.12.1 Greedy algorithm for an Abelian group of order $p^n$

Here are the procedure to find all the possible abelian group of order  $p^n$ . Note that  $p$  is prime.

1. Compute the orders of the elements of the group  $G$ .
2. Select an  $a_1$  of maximum order and define  $G_1 = \langle a_1 \rangle$ . Set  $i = 1$ .
3. If  $|G| = |G_i|$ , stop. Otherwise, replace  $i$  by  $i + 1$ .
4. Select an element  $a_i$  of maximum order  $p^k$  such that

$$p^k \leq |G|/|G_{i-1}|$$

and none of  $a_i, a_i^p, a_i^{p^2}, \dots, a_i^{p^{k-1}}$  is in  $G_{i-1}$ , and define  $G_i = G_{i-1} \times \langle a_i \rangle$ .

5. Return to step 3.

In the general case where  $|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ , we simply use the algorithm to build up a direct product of order  $p_1^{n_1}$ , then another of order  $p_2^{n_2}$ , and so on. The direct product of all these pieces is the desired factorization of  $G$ .

**Example 1.12.1.** Let  $G = \{1, 8, 12, 14, 18, 21, 31, 34, 38, 44, 47, 51, 52, 57, 64\}$  under multiplication modulo 65. Show that  $G$  is isomorphic to  $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ .

**Solution** Since  $G$  has order 16, we have

$$\begin{aligned}
G &\cong \mathbb{Z}_{16} \\
&\cong \mathbb{Z}_{2^4} && \text{take } p = 2, k = 4 \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^3} && \text{Partitions of } k = 1 + 3 \\
&\cong \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2} && \text{Partitions of } k = 2 + 2 \\
&\cong \mathbb{Z}_4 \oplus \mathbb{Z}_4
\end{aligned}$$



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**Pseudocode 1:** Greedy algorithm for an Abelian group of order  $p^n$ 


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// Main body of greedy algorithm for finding Abelian group of order  $p^k$ .
1 function Main_greedy_algorithm( $G$ : FiniteGroup)  $\rightarrow$  Product of CyclicGroups
2    $\bar{a} \leftarrow \text{find\_element\_with\_max\_order}(G)$ 
   // Generated subgroup with first element  $\bar{a}_1$  from  $\bar{a}$ .
3    $H \leftarrow \langle \bar{a}_1 \rangle$ 
4    $i \leftarrow 1$ 
5   while  $|H| \neq |G|$  do ▷ Greedy search for all cyclic groups
6      $\text{max\_cyclic\_order} = |G|/|H|$ 
7      $\langle a_i \rangle = []$ 
8     while  $p^k \leq \text{max\_cyclic\_order}$  do
9       if  $\bar{a}_i^{p^{k-1}} \notin H$  then ▷ Check if  $\bar{a}_i^{p^{k-1}}$  already in previous  $H$ 
10         $\langle a_i \rangle.\text{push}(\bar{a}_i^{p^{k-1}})$ 
11      else
12        continue
13       $k \leftarrow k + 1$ 
14     $H \leftarrow H \times \langle a_i \rangle$ 
15     $i \leftarrow i + 1$ 
16  return  $H$ 

// Compute the orders for all elements in group  $G$ .
17 function compute_orders( $G$ : FiniteGroup)  $\rightarrow$  orders
18   $\text{orders} = []$ 
19  for  $a_i \in G$  do ▷ elements in  $G$ 
20    if  $a_i == \text{identity}(G)$  then
21      return 1 ▷ The order of identity element is one.
22    else
23       $j \leftarrow 1$ 
24      while  $a_i^j \neq e$  do
25         $j \leftarrow j + 1$  ▷ Find the order of each element.
26       $\text{orders}.\text{push}(j)$ 
27  return  $\text{orders}$ 

// Filter out  $a_i$  with maximum order, return a list of tuple  $(a_i, |\langle a_i \rangle|)$ .
28 function find_element_with_max_order( $G$ : FiniteGroup)  $\rightarrow$  Array<tuple(element, order)>
29   $\text{orders} \leftarrow \text{max\_cyclic\_order}(G)$ 
30  return  $\text{zip}(G, \text{orders}).\text{filter}((_, o) \rightarrow o == \text{Max}(\text{orders}))$ 

```

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**Example 1.12.2.** Let  $G = \{1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 109, 116, 118, 127, 134\}$  under multiplication modulo 135. Show that  $G$  is isomorphic to  $\mathbb{Z}_{12} \oplus \mathbb{Z}_2$ .

**Solution** Since  $G$  has order 24, and the prime factorization of  $24 = 3 \cdot 2^3$ . We have

$$\begin{aligned}
 G &\cong \mathbb{Z}_{24} \\
 &\cong \mathbb{Z}_3 \oplus \mathbb{Z}_{2^3} \\
 &\cong \mathbb{Z}_3 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \\
 &\cong \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
 &\cong \mathbb{Z}_{3 \times 4} \oplus \mathbb{Z}_2 && \text{Since } \mathbb{Z}_p \oplus \mathbb{Z}_q \cong \mathbb{Z}_{pq} \\
 &\cong \mathbb{Z}_{12} \oplus \mathbb{Z}_2
 \end{aligned}$$



## Tutorials

**Exercise 1.12.1** Prove whether the following group  $G$  together with operation  $*$  is a group.

1. Let  $*$  defined on  $G = \mathbb{R}$  by letting  $a * b = ab \quad \forall a, b \in \mathbb{R}$ .
2. Let  $*$  defined on  $G = 2\mathbb{Z}$  by letting  $a * b = a + b \quad \forall a, b \in 2\mathbb{Z}$ .
3. Let  $*$  defined on  $G = \mathbb{R}^*$  by letting  $a * b = \sqrt{ab} \quad \forall a, b \in \mathbb{R}^*$ .
4. Let  $*$  defined on  $G = \mathbb{Z}$  by letting  $a * b = \max\{a, b\} \quad \forall a, b \in \mathbb{Z}$ .

**Exercise 1.12.2** Determine whether the given set of matrices under the specified operation, matrix addition or multiplication, is a group.

1. All  $2 \times 2$  diagonal matrices under matrix addition.
2. All  $2 \times 2$  diagonal matrices under matrix multiplication.
3. All  $2 \times 2$  diagonal matrices with no zero diagonal entry under matrix multiplication.
4. All  $2 \times 2$  diagonal matrices with all diagonal entries either 1 or  $-1$  under matrix multiplication.
5. All  $2 \times 2$  upper-triangular matrices under matrix multiplication.
6. All  $2 \times 2$  upper-triangular matrices under matrix addition.
7. All  $2 \times 2$  upper-triangular matrices with determinant 1 under matrix multiplication.
8. All  $2 \times 2$  upper-triangular matrices with determinant either 1 or  $-1$  under matrix multiplication.

**Exercise 1.12.3** Prove whether

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc \neq 0, a, b, c, d \in \mathbb{Z} \right\}$$

is a group under matrix multiplication.

**Exercise 1.12.4** Prove whether

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid ad \neq 0, a, b, d \in \mathbb{Z} \right\}$$

is a non-abelian group under matrix multiplication.

**Exercise 1.12.5** Prove whether

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \neq 0, a, b \in \mathbb{Z} \right\}$$

is an abelian group under matrix multiplication.

**Exercise 1.12.6** Let  $(G, *)$  be a group and suppose that

$$a * b * c = e \quad \forall a, b, c \in G.$$

Show that  $b * c * a = e$ .

**Exercise 1.12.7** Show that if every element of the group  $G$  is its own inverse, then  $G$  is abelian.

**Exercise 1.12.8** Show that every group with identity  $e$  and  $x \cdot x = x$  for all  $x \in G$  is abelian.

**Exercise 1.12.9** Show that if  $G$  is a finite group with identity  $e$  and with even number of elements, then there is an  $a \neq e$  in  $G$  such that  $a * a = e$ .

**Exercise 1.12.10** Suppose  $G$  is a group such that

$$(ab)^2 = a^2 b^2 \quad \forall a, b \in G.$$

Show that  $G$  is abelian.

**Exercise 1.12.11** Find the order of the following cyclic groups.

1. The subgroup of  $U(6)$  generated by  $\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$ .
2. The subgroup of  $U(5)$  generated by  $\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)$ .
3. The subgroup of  $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$  generated by  $(1, 5)$ .

**Exercise 1.12.12** Let  $a$  and  $b$  be elements of a group  $G$ . Show that if  $ab$  has finite order  $n$ , then  $ba$  also has order  $n$ .

**Exercise 1.12.13** Show that a group with no proper nontrivial subgroup is cyclic.

**Exercise 1.12.14** Let  $G$  be a nonabelian group with center  $Z(G)$ . Show that there exists an abelian subgroup  $H$  of  $G$  such that  $Z(G) \subset H$  but  $Z(G) \neq H$ .

**Exercise 1.12.15** Find all subgroups of the following groups and draw the subgroups diagram for the subgroups. Hence, list all orders of the subgroups of the given groups.

1.  $\mathbb{Z}_{36}$
2.  $\mathbb{Z}_{60}$

**Exercise 1.12.16**

1. Find all the proper nontrivial subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .
2. Find all the subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_4$  of order 4.

**Exercise 1.12.17**

1. Are the groups  $\mathbb{Z}_2 \times \mathbb{Z}_{12}$  and  $\mathbb{Z}_4 \times \mathbb{Z}_6$  isomorphic?
2. Are the groups  $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$  and  $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$  isomorphic?

**Exercise 1.12.18** Find the conjugacy classes of dihedral group  $D_8$ .

**Exercise 1.12.19** Show that a group that has only finite number of subgroups must be a finite group.

**Exercise 1.12.20** Find all cosets of the subgroup  $4\mathbb{Z}$  of  $\mathbb{Z}$ .

**Exercise 1.12.21** Compute the quotient group  $\mathbb{Z}_{12}/\langle 2 \rangle$ .

**Exercise 1.12.22** Show that if  $H$  is a subgroup of index 2 in a finite group  $G$ , then every left coset of  $H$  is also a right coset of  $H$ .

**Exercise 1.12.23** Let  $\phi : G \rightarrow G$  be a mapping defined by

$$\phi(x) = x^3 \quad \forall x \in G$$

where  $G = \mathbb{R} \setminus \{0\}$  is a group defined under usual multiplication. Show that  $\phi$  is a homomorphism, and hence find  $\ker(\phi)$ .

**Exercise 1.12.24** Let  $\phi : G \rightarrow G$  be a mapping defined by

$$\phi(x) = 5^x \quad \forall x \in G$$

where  $G = \mathbb{R} \setminus \{0\}$  is a group defined under usual multiplication. Show that  $\phi$  is a homomorphism, and hence find  $\ker(\phi)$ .

**Exercise 1.12.25** Let  $\phi : G \rightarrow G$  be a mapping defined by

$$\phi(x) = 7x \quad \forall x \in G$$

where  $G = \mathbb{Z}$  is a group defined under usual addition. Show that  $\phi$  is a homomorphism, and hence find  $\ker(\phi)$ .

**Exercise 1.12.26** Let  $G$  be a group and  $g$  an element in  $G$ . Consider the mapping  $\phi : G \rightarrow G$  defined as  $\phi(x) = gxg^{-1}$ . Show that  $\phi$  is an isomorphism.

**Exercise 1.12.27** Find  $\ker(\phi)$  for map  $\phi : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{20}$  such that  $\phi(1) = 8$ .

**Exercise 1.12.28** Find  $\ker(\phi)$  for map  $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  such that  $\phi(1, 0) = (2, -3)$  and  $\phi(0, 1) = (-1, 5)$ .

**Exercise 1.12.29** Let  $\phi : G \rightarrow H$  be a group homomorphism. Show that  $\phi(G)$  is abelian if and only if

$$xyx^{-1}y^{-1} \in \ker(\phi) \quad \forall x, y \in G.$$

**Exercise 1.12.30** Consider  $A$  the set of affine maps of  $\mathbb{R}$ , that is

$$A = \{f : x \mapsto ax + b, a \in \mathbb{R}^*, b \in \mathbb{R}\}$$

1. Show that  $A$  is a group with respect to the composition of map.

2. Let

$$N = \{g : x \mapsto x + b, b \in \mathbb{R}\}$$

Show that  $N \triangleleft A$ .

3. Show that the quotient group  $A/N$  is isomorphic to  $\mathbb{R}^*$ .

**Exercise 1.12.31** Let  $G = S_4$  and let

$$H = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$$

1. Show that  $H$  is a normal subgroup of  $G$ .

2. Let  $\overline{H} = \{\sigma \in S_4 \mid \sigma(4) = 4\}$ . Define  $\sigma : \overline{H} \rightarrow \text{Aut}(H)$  by  $\sigma(\tau) = \sigma\tau\sigma^{-1}$  for  $\sigma \in \overline{H}$ . Prove that

$$\overline{H} \ltimes_{\sigma} H \cong S_4.$$

**Exercise 1.12.32** Find (up to isomorphism) all abelian groups of order 45.

**Exercise 1.12.33** Show that any group of order  $p^2$  is abelian.

**Exercise 1.12.34** Let  $G$  be a group of order  $pq$ , where  $p$  and  $q$  are prime numbers. Show that every proper subgroup of  $G$  is cyclic.

**Exercise 1.12.35** If  $H, K \leq G$ , show that  $H \cap K \leq G$ .

**Exercise 1.12.36** If  $N \triangleleft G$  and  $H \leq G$ , show that  $NH \leq G$ .

**Exercise 1.12.37** If  $N_1, N_2 \triangleleft G$ , show that  $N_1 \cap N_2 \triangleleft G$ .

**Exercise 1.12.38** If  $N \triangleleft G$  and  $H \leq G$ , show that  $H \cap N \triangleleft G$ .



# Rings

Ring is an algebraic structure which is a set of elements with two operations: addition and multiplication.

## Axiom 2.1 Rings

A ring  $R$  is a set with two binary operation, addition (usually denoted by  $+$ ) and multiplication (usually denoted by  $ab$ ), such that for all  $a, b, c \in R$ .

1. Addition is commutative,  $a + b = b + a$ .
2. Associativity holds in addition,  $(a + b) + c = a + (b + c)$ .
3. There is an additive identity  $0_R$ . That is, there is an element  $0_R \in R$  such that

$$a + 0_R = a = 0_R + a$$

for all  $a \in R$ .

4. There is an additive inverse  $-a \in R$  such that

$$a + (-a) = 0_R = -a + a.$$

5. Associativity holds in multiplication,  $a(bc) = (ab)c$ .

6. Distributive law holds in  $R$ ,

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca.$$

Here are a few things you should take notes:

1. A ring is an Abelian group under addition, also having an associative multiplication that is left and right distributive over addition.
2. Note that multiplication need not be commutative. When it is, we say that the ring is commutative.
3. A ring need not have an identity under multiplication. A unity (or identity) in a ring is a nonzero element that is an identity under multiplication.
4. A nonzero element of a commutative ring with unity need not have multiplicative inverse. When it does, we say that it is a unit of the ring. Thus,  $x$  is a unit if  $x^{-1}$  exists.
5. We follow the following terminology and notation. If  $x$  and  $y$  belong to a commutative ring  $R$  and  $x$  is nonzero, we say that  $x$  divides  $y$  and write  $x|y$ , if there exists an element  $c$  in  $R$  such that  $y = xc$ .

6. If  $x$  is an element from a group under the operation of addition and  $n$  is a positive integer,  $nx$  means  $\underbrace{x + x + \cdots + x}_{n \text{ times}}$ , where there are  $n$  summands.

**Example 2.0.1.** The set  $\mathbb{Z}$  of integers under ordinary addition and multiplication is a commutative ring with unity 1. The units of  $\mathbb{Z}$  are 1 and  $-1$ .

**Example 2.0.2.** The set  $M_2(\mathbb{Z})$  of  $2 \times 2$  matrices with integer entries is a noncommutative ring with unity  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ .

**Example 2.0.3.** The set of all continuous real-valued functions of a real variable whose graphs pass through the point  $(1, 0)$  is a commutative ring without unity under the operations of pointwise addition and multiplication, that is,

$$(f + g)(x) = f(x) + g(x)$$

and

$$(fg)(x) = f(x)g(x).$$

### Theorem 2.1

Let  $a, b$  and  $c$  belong to a ring  $R$ . Then

1.  $a0 = 0a = 0$ .
2.  $a(-b) = (-a)b = -(ab)$ .
3.  $(-a)(-b) = ab$ .
4.  $a(b - c) = ab - ac$  and  $(b - c)a = ba - ca$ .
5. Furthermore, if  $R$  has a unity element 1, then  $(-1)a = -a$ .
6.  $(-1)(-1) = (-1)^2 = 1$ .

*Proof.* We are going to prove each of them:

1.

$$\begin{aligned} a(0 + 0) &= a0 \Rightarrow a0 + a0 = a0 \\ &\Rightarrow (-a0) + a0 + a0 = (-a0) + a0 \\ &\Rightarrow 0 + a0 = 0 \\ &\Rightarrow a0 = 0. \end{aligned}$$

2.

$$\begin{aligned} a0 = 0 &\Rightarrow a(b + (-b)) = 0 \\ &\Rightarrow ab + a(-b) = 0 \\ &\Rightarrow - (ab) + ab + a(-b) = - (ab) + 0 \\ &\Rightarrow a(-b) = -(ab). \end{aligned}$$

3.

$$\begin{aligned} (-a)(-b) &= -(a(-b)) \\ &= -(a(-b)) \\ &= -(-(ab)) \\ &= ab. \end{aligned}$$

4.

$$a(b - c) = a(b + (-c)) = ab + a(-c) = ab - ac.$$

5.

$$\begin{aligned} 0a = 0 &\Rightarrow (1 + (-1))a = 0 \\ &\Rightarrow 1a + (-1)a = 0 \\ &\Rightarrow a + (-1)a = 0 \\ &\Rightarrow (-a) + a + (-1)a = (-a) + 0 \\ &\Rightarrow 0 + (-1)a = (-a) \\ &\Rightarrow (-1)a = -a. \end{aligned}$$

6.

$$(-1)(-1) = -(1(-1)) = -(-(1 \cdot 1)) = 1.$$

□

### Lemma 2.1

If a ring has a unity, it is unique.

*Proof.* Suppose  $1_R$  and  $1'_R$  are both unity of ring  $R$ . Then For all  $a \in R$ , we must have

$$a \cdot 1_R = a = 1_R \cdot a$$

and

$$a \cdot 1'_R = a = 1'_R \cdot a.$$

Thus, we say that  $a \cdot 1_R = a = a \cdot 1'_R$ . By cancellation law this yields

$$1_R = 1'_R.$$

So ring  $R$  has unique unity.

□

## 2.1 subrings

### Theorem 2.2 Subring test

A nonempty subset  $S$  of the ring  $R$  is a subring if  $S$  is closed under subtraction and multiplication, that is,

1.  $S \neq \emptyset$ .
2.  $a - b \in S \quad \forall a, b \in S$ .

$$3. ab \in S \quad \forall a, b \in S.$$

*Proof.* Since addition in  $R$  is commutative and  $S$  closed under subtraction, we know that by the *one-step subgroup test* that  $S$  is an Abelian group under addition.

Also, since multiplication in  $R$  is associative as well as distributive over addition, the same is true for multiplication in  $S$ .

Therefore, the only condition remaining to be checked is that multiplication is a binary operation on  $S$ . But this is exactly what closure means.  $\square$

**Example 2.1.1.**  $S = \{0, 2, 4\}$  is a subring of  $\mathbb{Z}_6$ .

**Solution** Using subring test, the subtraction and multiplication form a group in  $S$ .

−	0	2	4
0	0	4	2
2	2	0	4
4	4	2	0

·	0	2	4
0	0	0	0
2	0	4	2
4	0	2	4

Thus  $S \leq \mathbb{Z}_6$ . ◀

## 2.2 Quotient rings, Ideals

### 2.2.1 Ideals

#### Definition 2.1 Ideals

A subring  $A$  of a ring  $R$  is called an ideal if for every  $r \in R$  and every  $a \in A$  both  $ra$  and  $ar$  are in  $A$ .

So, a subring  $A$  of a ring  $R$  is an ideal of  $R$  if

$$rA = \{ra \mid a \in A\} \subseteq A$$

and

$$Ar = \{ar \mid a \in A\} \subseteq A$$

for all  $r \in R$ . An ideal  $A$  of  $R$  is called a **proper ideal** of  $R$  if  $A$  is a proper subset of  $R$ .

#### Lemma 2.2

Let  $I$  be a subring of a ring  $R$ . Then  $I$  is an ideal in  $R$  if and only if multiplication

$$(a + I)(b + I) = (ab + I)$$

is a well-defined operation on the cosets of  $I$  in  $R$ .

*Proof.* ( $\Rightarrow$ ) Assume that  $I$  is an ideal in  $R$ , and suppose that  $a_1 + I = a_2 + I$  and  $b_1 + I = b_2 + I$ .

This implies that  $a_1 = a_2 + k$  and  $b_1 = b_2 + j$  for some  $i, j \in I$ . Then we have

$$a_1b_1 = a_2b_2 + a_2j + kb_2 + kj.$$

Since  $I$  is a subring of  $R$ , and therefore it closed under multiplication, as well as addition, and  $kj \in I$ . Since  $I$  is an ideal,  $a_2j \in I$  and  $kb_2 \in I$ , and so  $a_2j + kb_2 + kj \in I$ .

Therefore,  $a_1b_1 \in a_2b_2 + I$  and  $a_1b_1 + I = a_2b_2 + I$ . Thus, the multiplication on the set of cosets of  $I$  is well-defined.

( $\Leftarrow$ ) Assume that the indicated operation is well-defined. We need to show that for all  $r \in R$ , and for all  $x \in I$ , we have  $rx \in I$  and  $xr \in I$ . So we have  $x + I = 0 + I = I$ . Hence

$$rx + I = (r + I)(x + I) = (r + I)(0 + I) = 0 + I = I.$$

Again we have  $xr \in I$ . Thus  $I$  is an ideal in  $R$ . □

### Theorem 2.3 Ideal test

A nonempty subset  $A$  of a ring  $R$  is an ideal of  $R$  provided

1.  $A \neq \emptyset$ .
2.  $a - b \in A$  whenever  $a, b \in A$ .
3.  $ra$  and  $ar$  are in  $A$  for all  $a \in A$  and  $r \in R$ .

**Example 2.2.1.** For any ring  $R$ ,  $\{0\}$  and  $R$  are ideals of  $R$ . The ideal  $\{0\}$  is called the trivial ideal.

**Example 2.2.2.** For any positive integer  $n$ , the set

$$n\mathbb{Z} = \{0, \pm n, \pm 2n, \pm 3n, \dots\}$$

is an ideal of  $\mathbb{Z}$ .

**Solution** We can show  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$  using ideal test.

1. Since  $0 \in \mathbb{Z}$ , then  $n \cdot 0 = 0 \in n\mathbb{Z} \neq \emptyset$ .
2. For all  $a, b \in n\mathbb{Z}$ , we let  $a = nt_1$  and  $b = nt_2$  for  $t_1, t_2 \in \mathbb{Z}$ . We have

$$a - b = nt_1 - nt_2 = n(t_1 - t_2) \in n\mathbb{Z}$$

since  $t_1 - t_2$  is also an integer.

3. Whenever  $a \in A$  and  $r \in R$ , let  $a = nt'$ ,  $t' \in \mathbb{Z}$ . We have

$$ar = (nt')r = n(t'r) \in n\mathbb{Z}, \quad t'r \in \mathbb{Z}.$$

and

$$ra = r(nt') = nrt' = n(rt') = n(t'r) = ar \in n\mathbb{Z}.$$

Therefore  $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ .



**Example 2.2.3.** Let  $R$  be a commutative ring with unity and let  $a \in R$ . The set

$$\langle a \rangle = \{ra \mid r \in R\}$$

is an ideal of  $R$  called the **principal ideal** generated by  $a$ .

**Solution** Using ideal test to check

1. Since  $R$  is a ring, then  $0_R \in R$  and so  $0_R = 0_R \cdot a \in \langle a \rangle \neq \emptyset$ .
2. For all  $b, c \in \langle a \rangle$ , let  $b = r_1a, c = r_2a$ , where  $r_1, r_2 \in R$ . Then

$$b - c = r_1a - r_2a = (r_1 - r_2)a \in \langle a \rangle.$$

and  $r_1 - r_2$  is in  $R$ .

3. For all  $\bar{a} \in \langle a \rangle, r \in R$ , we let  $\bar{a} = r'a$  and

$$\bar{a}r = (r'a)r = a(r'r) \in \langle a \rangle.$$

where  $r'r \in R$ .

On the other hand,

$$r\bar{a} = r(r'a) = a(rr') = a(r'r) = \bar{a}r \in \langle a \rangle.$$

Therefore  $\langle a \rangle$  is a subring of  $R$ .

◀

## 2.2.2 Quotient Rings

### Theorem 2.4

Let  $R$  be a ring and let  $A$  be a subring of  $R$ . The set of cosets

$$R/A = \{r + A \mid r \in R\}$$

is a ring under the operations

- $(s + A) + (t + A) = s + t + A$
- $(s + A)(t + A) = st + A$

if and only if  $A$  is an ideal of  $R$ .

*Proof.* Let  $R$  to be a ring, and let  $A \trianglelefteq R$ . For all  $s + A, t + A$  in  $R/A$  we define addition as

$$\boxplus(s + A, t + A) := (s + A) \boxplus (t + A) = s + t + A$$

and multiplication as

$$\odot(s + A, t + A) := (s + A) \odot (t + A) = st + A.$$

We want to show  $(R/A, \boxplus, \odot)$  is a ring.

1. (Closureness) Suppose that  $s + A = s' + A$  and  $t + A = t' + A$  for all  $s, s', t, t' \in \mathbb{R}$ . First we need to show

$$(s + t) + A = (s' + t') + A.$$

We are going to express  $s, t$  in term of  $s', t'$  respectively.

$$\begin{aligned} s + A = s' + A &\Rightarrow s - s' \in A \\ &\Rightarrow s - s' = a_1 \in A \\ &\Rightarrow \boxed{s = a_1 + s'}, \quad a_1 \in A. \end{aligned} \quad (\heartsuit)$$

and

$$\begin{aligned} t + A = t' + A &\Rightarrow t - t' \in A \\ &\Rightarrow t - t' = a_2 \in A \\ &\Rightarrow \boxed{t = a_2 + t'}, \quad a_2 \in A. \end{aligned} \quad (\clubsuit)$$

Summing up  $(\heartsuit)$  together with  $(\clubsuit)$  we have

$$s + t = (a_1 + s') + (a_2 + t') = a_1 + a_2 + s' + t', \quad a_1 + a_2 \in A.$$

Subtracting  $s' + t'$  on both side of the equation yields

$$s + t - (s' + t') = a_1 + a_2, \implies s + t - (s' + t') \in A.$$

We have shown  $R/A$  closed under addition  $\boxplus$ , we continue to proof

$$st + A = s't' + A.$$

Which is equivalent to show  $R/A$  closed under the multiplication  $\odot$ . Applying the results that we found from  $(\heartsuit), (\clubsuit)$  we have

$$st - s't' = a_1a_2 + a_1t' + s'a_2$$

Is  $a_1t'$  in  $A$ ? Of course, since  $a_1 \in A, t' \in R \implies a_1t' \in A \triangleleft R$ . So as  $s'a_2 \in A \triangleleft R$ .

2. (Existence of additive identity) For all  $s + A \in R/A$ , there exists  $e + A \in R/A$  such that

$$\begin{aligned} (s + A) \boxplus (e + A) &= s + A \Rightarrow (s + e) + A = 0 + s + A \\ &\Rightarrow s + e = s \\ &\Rightarrow e = 0_A. \end{aligned}$$

Thus the additive identity is  $0_A + A$ .

3. (Existence of additive inverse) For all  $s + A \in R/A$ , there exists  $r + A \in R/A$  such that

$$\begin{aligned} (s + A) \boxplus (r + A) &= 0 + A \Rightarrow (s + r) + A = 0 + A \\ &\Rightarrow s + r = 0 \\ &\Rightarrow r = -s. \end{aligned}$$

Thus the additive inverse of  $s + A$  is  $-s + A$  in  $R/A$ .

4. (Associativity of multiplication) For all  $s + A, t + A, u + A$  in  $R/A$ , compute

$$\begin{aligned}
 (s + A) \odot [(t + A) \odot (u + A)] &= (s + A) \odot (tu + A) \\
 &= s(tu) + A \\
 &= (st)u + A \\
 &= [(st) + A] \odot (u + A) \\
 &= [(s + A) \odot (t + A)] \odot (u + A).
 \end{aligned}$$

Associativity in  $\odot$  holds.

5. (Existence of unity) For all  $s + A \in R/A$ , there exists  $e' + A \in R/A$  such that

$$\begin{aligned}
 (s + A) \odot (e' + A) &= s + A \Rightarrow se' + A = s + A \\
 &\Rightarrow se' = s \\
 &\Rightarrow e' = 1_A \in R.
 \end{aligned}$$

The multiplicative identity is  $1_A + A$  in  $R/A$ .

6. (Existence of multiplicative inverse) For all  $s + A \in R/A$ , there exists  $r + A \in R/A$  such that

$$\begin{aligned}
 (s + A) \odot (r + A) &= 1_A + A \Rightarrow sr + A = 1_A + A \\
 &\Rightarrow sr = 1_A \\
 &\Rightarrow r = s^{-1}.
 \end{aligned}$$

The multiplicative inverse of  $s + A$  is  $s^{-1} + A$  in  $R/A$ , provided  $s^{-1}$  exists in  $R$ .

7. (Distributive Law) For all  $s + A, t + A, u + A$  in  $R/A$ , compute

$$\begin{aligned}
 (s + A) \odot [(t + A) \boxplus (u + A)] &= (s + A) \odot [(t + u) + A] \\
 &= s(t + u) + A \\
 &= st + su + A \\
 &= (st + A) + (su + A) \\
 &= (st + A) \boxplus (su + A) \\
 &= (s + A) \odot (t + A) \boxplus (s + A) \odot (u + A)
 \end{aligned}$$

Distributive law holds in  $(R/A, \boxplus, \odot)$ .

Therefore  $(R/A, \boxplus, \odot)$  is a ring. □

**Example 2.2.4.**  $\mathbb{Z}/4\mathbb{Z} = \{4\mathbb{Z}, 1 + 4\mathbb{Z}, 2 + 4\mathbb{Z}, 3 + 4\mathbb{Z}\}$

**Solution** The integers with multiple of 4 is

$$4\mathbb{Z} = \{\dots, -8, -4, 0, 4, 8, 12, \dots\}.$$

The left ideals are

$$\begin{aligned}
 1 +_4 4\mathbb{Z} &= \{\dots, -7, -3, 1, 5, 9, 13, \dots\} \\
 2 +_4 4\mathbb{Z} &= \{\dots, -6, -2, 2, 6, 10, 14, \dots\} \\
 3 +_4 4\mathbb{Z} &= \{\dots, -5, -1, 3, 7, 11, 15, \dots\} \\
 4 +_4 4\mathbb{Z} &= \{\dots, -4, 0, 4, 8, 12, 16, \dots\} = 4\mathbb{Z}
 \end{aligned}$$



**Definition 2.2 characteristic of ring**

The characteristic of a ring  $R$  is the least positive integer  $n$  such that  $nx = 0$  for all  $x \in R$ . If no such integer exists, we say that  $R$  has characteristic 0. The characteristic of  $R$  is denoted by  $\text{char}(R)$ .

**Example 2.2.5.** The characteristic of ring  $\mathbb{Z}_7$  is 7.

**Example 2.2.6.** Find the characteristic of  $R = \mathbb{Z}[i]/\langle 2 - i \rangle$ .

**Solution** Given that

$$R = \mathbb{Z}[i]/\langle 2 - i \rangle = \{a + bi + \langle 2 - i \rangle\}.$$

Observe that

$$2 - i + \langle 2 - i \rangle = 0 + \langle 2 - i \rangle$$

and so we treat  $2 - i$  as zero, and  $i = 2$  and so

$$-1 = i^2 = 4 \implies 5 = 0 \in R.$$

Thus  $R = \{0 + \langle 2 - i \rangle, 1 + \langle 2 - i \rangle, 2 + \langle 2 - i \rangle, 3 + \langle 2 - i \rangle, 4 + \langle 2 - i \rangle\}$ . So  $\text{Char}(R) = 5$ . ◀

**Definition 2.3 Prime ideal**

An ideal  $I$  in a commutative ring  $R$  is said to be prime if  $I \neq R$  and whenever  $ab \in I$ , then either  $a \in I$  or  $b \in I$ .

**Lemma 2.3**

Let  $R$  be a commutative ring with unity, and  $I$  be an ideal in  $R$ . Then  $I$  is a prime ideal in  $R$  if and only if  $R/I$  is an integral domain.

*Proof.*  $R/I$  will therefore be an integral domain and only if it has no zero divisors. This condition is equivalent to the condition that

$$(a + I)(b + I) = I \iff a + I = I \text{ or } b + I = I.$$

Thus  $R/I$  is an integral domain if and only if  $ab + I = I$  implies that  $a + I = I$  or  $b + I = I$  or, in other words, if and only if  $ab \in I$  implies that  $a \in I$  or  $b \in I$ , which is to say that  $I$  is a prime ideal in  $R$ . □

**Definition 2.4 Maximal ideal**

An ideal  $I$  in a ring  $R$  is said to be maximal if  $I \neq R$  and whenever  $J$  is an ideal such that

$$I \subset J \subset R$$

then  $I = J$  or  $J = R$ .

**Lemma 2.4**

Consider  $R$  is a ring with nonzero unity, and  $M$  is an ideal such that  $M \neq R$ . If  $R/M$  is a division ring, then  $M$  is a maximal ideal.

*Proof.* Suppose  $I$  is an ideal such that  $M \subsetneq I \subseteq R$ . Then  $\exists a \in I$  s.t.  $a \notin M$ . Then  $a + M \neq 0 + M$  and there exists  $b + M \in R/M$  such that

$$(a + M)(b + M) = 1_R + M \implies (1_R - ab) \in M \implies ab + m = 1_R$$

for some  $m \in M$ . Since  $ab \in I$  and  $m \in M \subset I$ . Also  $1_R \in I \implies I = R$ . Thus  $M$  is a maximal ideal.  $\square$

**Theorem 2.5**

Let  $M$  be an ideal in a commutative ring  $R$  with identity. Then  $M$  is a maximal ideal if and only if the quotient ring  $R/M$  is a field.

*Proof.* ( $\Leftarrow$ ) If  $R/M$  is a field, then  $M$  is a maximal ideal by previous lemma.

( $\Rightarrow$ ) Since  $M \neq R$ ,  $R/I$  is a commutative ring with  $1_R + R \neq 0_R + M$ . Take any nonzero  $a + M \in R/M$ ,  $a \notin M$  and put

$$N := Ra + M = \{ra + m \mid r \in R, m \in M\}.$$

Note that  $Ra$  is an ideal and  $M$  is also an ideal ( $Ra = \langle a \rangle$ ). Thus  $Ra + M$  is ideal that include  $M$ .

Since  $M$  is maximal, this implies that  $N = R \implies 1_R \in N$ .  $ra + m = 1_R$  for some  $r \in R, m \in M$ . Compute

$$\begin{aligned} ra + m = 1_R &\implies ra + M = 1_R + M && \text{Since } (ra - 1_R) \in M \\ &\implies (a + M)(r + M) = 1_R + M. \end{aligned}$$

We can now see that  $a + M$  is actually a unit in  $R/M$ . Hence  $R/M$  is a field.  $\square$

**Corollary 2.1**

In a commutative ring  $R$  with unity, every maximal ideal is a prime ideal.

*Proof.* If  $I$  is a maximal ideal in  $R$ , then  $R/I$  is a field. Every field is an integral domain, so  $R/I$  is also an integral domain, and  $I$  is a prime ideal.  $\square$

**Example 2.2.7.** The ideal  $\langle x^2 + 1 \rangle$  is maximal in  $\mathbb{R}[x]$ .

**Solution** Suppose that  $A$  is an ideal of  $\mathbb{R}[x]$  that properly contains  $\langle x^2 + 1 \rangle$ . We will prove that  $A = \mathbb{R}[x]$  by showing that  $A$  contains some nonzero real number  $c$ . Then

$$1 = \frac{1}{c}c \in A$$

and therefore  $A = \mathbb{R}[x]$ .

Let a polynomial  $f(x) \in A$ , but  $f(x) \notin \langle x^2 + 1 \rangle$ . Then

$$f(x) = q(x)(x^2 + 1) + r(x),$$

where  $r(x) = ax + b$ , where  $a, b$  both are nonzero, and hence

$$a^2x^2 - b^2 = (ax + b)(ax - b) \in A \text{ and } a^2(x^2 + 1) \in A.$$

So that

$$0 \neq a^2 + b^2 = (a^2x^2 + a^2) - (a^2x^2 - b^2) \in A.$$

◀

**Example 2.2.8.** The ideal  $\langle x^2 + 1 \rangle$  is not prime in  $\mathbb{Z}_2[x]$ .

**Solution** (Counterexample) Take

$$(x + 1)^2 = x^2 + 2x + 1 = x^2 + 1 \in \langle x^2 + 1 \rangle$$

but  $1 + x \notin \langle x^2 + 1 \rangle$ .

◀

## 2.3 Ring homomorphism

### Definition 2.5 Ring homomorphism

A ring homomorphism  $f$  from a ring  $(R, \oplus, \odot)$  to a ring  $(S, \boxplus, \boxdot)$  is a mapping from  $R$  to  $S$  that preserves the ring additions  $(\oplus, \boxplus)$  and multiplications  $(\odot, \boxdot)$ ; that is,

$$f(a \oplus b) = f(a) \boxplus f(b)$$

and

$$f(a \odot b) = f(a) \boxdot f(b)$$

A ring homomorphism that is one-to-one and onto is called the **ring isomorphism**.

**Example 2.3.1.** The map  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_3$  defined by

$$\phi(x) = x \pmod{3} \quad \forall x \in \mathbb{Z}$$

is a ring homomorphism.

**Solution** Clearly, for all  $x, y \in \mathbb{Z}$

$$\begin{aligned} \phi(x + y) &= (x + y) \pmod{3} = (x \pmod{3}) + (y \pmod{3}) \\ &= \phi(x) +_3 \phi(y) \end{aligned}$$

and

$$\begin{aligned} \phi(xy) &= (xy) \pmod{3} = (x \pmod{3}) \cdot (y \pmod{3}) \\ &= \phi(x) \cdot_3 \phi(y) \end{aligned}$$

This is an example of a map that respects both operations.

◀

**Example 2.3.2.** Consider the map  $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_6, \phi(x) = 3x$  for all  $x$  in  $\mathbb{Z}_4$ .  $\phi$  is a ring homomorphism.

**Solution** For all  $x, y \in \mathbb{Z}$ , we check that

$$\begin{aligned}\phi(x + y) &= 3(x + y) \pmod{6} = (3x \pmod{6}) + (3y \pmod{6}) \\ &= \phi(x) +_6 \phi(y)\end{aligned}$$

and

$$\begin{aligned}\phi(xy) &= 3(xy) \pmod{6} = 9(xy) \pmod{6} = (3x \pmod{6}) \cdot (3y \pmod{6}) \\ &= \phi(x) \cdot_6 \phi(y)\end{aligned}$$

this map preserves both operations. So  $\phi$  is a ring homomorphism.

In our calculation, we can have used the fact that  $3 = 9 \pmod{6}$ . The jump from 3 to 9 modulo 6 can be better seen in

$$3 \pmod{6} = \phi(1) = \phi(1 \cdot 1) = \phi(1) \phi(1) = 3 \cdot 3 = 9 \pmod{6}.$$

◀

**Example 2.3.3.** For  $a, b \in \mathbb{R}$ , let  $A(a, b) = M_2(\mathbb{R})$  be defined by

$$A(a, b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Let  $R = \{A(a, b) \mid a, b \in \mathbb{R}\} \subseteq M_2(\mathbb{R})$ . Then  $R \cong \mathbb{C}$ .

**Solution** Let  $\phi : R \rightarrow \mathbb{C}$  be defined by

$$\phi(A(a, b)) = a + bi \in \mathbb{C}.$$

We show firstly that  $\phi$  is a ring homomorphism.

For addition, we have

$$\begin{aligned}\phi(A(a, b) + A(c, d)) &= \phi(A(a + c, b + d)) \\ &= (a + c) + (b + d)i \\ &= (a + bi) + (c + di) \\ &= \phi(A(a, b)) + \phi(A(c, d)).\end{aligned}$$

For multiplication, we have

$$\begin{aligned}\phi(A(a, b) + A(c, d)) &= \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) \\ &= \phi\left(\begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}\right) \\ &= \phi(A(ac - bd, ad + bc)) \\ &= (a + bi)(c + di) \\ &= \phi(A(a, b)) \phi(A(c, d)).\end{aligned}$$

Now,  $\phi$  is one-to-one and onto since  $\phi(A(a, b)) = a + bi = 0$  if and only if  $a = b = 0$ , and  $\text{Ker } \phi = \{A(0, 0)\}$  is trivial. ◀

**Example 2.3.4.** Show that the equation  $2x^3 - 5x^2 + 7x - 8 = 0$  has no solutions in  $\mathbb{Z}$ .

**Solution** Let  $\phi : \mathbb{Z} \rightarrow \mathbb{Z}_3$  be the natural homomorphism  $\phi(x) = x \bmod 3$ . Suppose that there is an integer  $a \in \mathbb{Z}$  such that

$$2a^3 - 5a^2 + 7a - 8 = 0.$$

Then

$$0 = \phi(0) = \phi(2a^3 - 5a^2 + 7a - 8) = 2\phi(a)^3 - 5\phi(a)^2 + 7\phi(a) - 8.$$

Since  $-5 = 7 = -8 = 1 \pmod{3}$  in  $\mathbb{Z}_3$ , we have

$$2\phi(a)^3 - 5\phi(a)^2 + 7\phi(a) - 8 = 2\phi(a)^3 + \phi(a)^2 + \phi(a) + 1$$

and thus  $2b^3 + b^2 + b + 1 = 0$ , where  $b = \phi(a)$  in  $\mathbb{Z}_3$ .

However, one can easily check that no element  $b \in \{0, 1, 2\}$  in  $\mathbb{Z}_3$  is a solution to this equation. Therefore there is no such integer  $a \in \mathbb{Z}$  to the original equation. ◀

**Example 2.3.5 (Tutorial).** Show that the rings  $2\mathbb{Z}$  and  $3\mathbb{Z}$  are not isomorphic.

**Solution** Assume the contrary and let  $\phi : 2\mathbb{Z} \rightarrow 3\mathbb{Z}$  to be an isomorphism. Let us examine  $\phi(2)$ . Note that for some  $k \in \mathbb{Z}$ ,  $\phi(2) = 3k$ . Since  $\phi$  is a homomorphism,

$$\phi(k) = \phi(2 + 2) = \phi(2) + \phi(2) = 3k + 3k = 6k.$$

But  $\phi$  is a ring homomorphism and

$$\phi(k) = \phi(2 \cdot 2) = \phi(2) \phi(2) = (3k)(3k) = 9k^2.$$

This implies that  $6k = 9k^2 \implies k = 0$  or  $k = \frac{2}{3}$ .

For  $k = 0 \implies \phi(x) = 0$  is not one-to-one and not onto. Also,  $k = \frac{2}{3} \notin \mathbb{Z}$ , and thus  $\phi$  cannot be an isomorphism. ◀

**Example 2.3.6.** Determine all ring homomorphism from  $\mathbb{Z}$  to  $\mathbb{Z}_6$ .

**Solution** Since  $\mathbb{Z}$  is generated from 1 by addition and subtraction, if a ring homomorphism  $f : \mathbb{Z} \rightarrow \mathbb{Z}_6$ , then for any  $a \in \mathbb{Z}$ , we have

$$f(a) = am$$

where  $m = f(1)$ . Then  $f$  is linear, so

$$f(a) + f(b) = am + bm = (a + b)m = f(a + b) \quad \forall a, b \in \mathbb{Z}.$$

So  $f$  is a ring homomorphism if and only if

$$0 = f(ab) - f(a)f(b) = abm - (am)(bm) = ab(m - m^2)$$

for any  $a, b \in \mathbb{Z}$ .

In particular, taking  $a = b = 1$ , we need to find  $m$  such that  $0 = m - m^2 \pmod{6}$ . Working

modulo 6 one by one

$$\begin{array}{ll} 0 - 0^2 = 0 - 0 = 0, & 1 - 1^2 = 1 - 1 = 0, \\ 3 - 3^2 = 3 - 9 = -6 = 0, & 4 - 4^2 = 4 - 16 = -12 = 0, \\ 2 - 2^2 = 2 - 4 = -2 \neq 0, & 5 - 5^2 = 5 - 25 = -20 \neq 0 \end{array}$$

The possible values of  $m$  are 0, 1, 3 and 4. So the homomorphisms are as follow

- $f(a) = 0 \pmod{6}, \quad \forall a \in \mathbb{Z}.$
- $f(a) = a \pmod{6}, \quad \forall a \in \mathbb{Z}.$
- $f(a) = 3a \pmod{6}, \quad \forall a \in \mathbb{Z}.$
- $f(a) = 4a \pmod{6}, \quad \forall a \in \mathbb{Z}.$



### Theorem 2.6 The first isomorphism for rings

Let  $f$  be a ring homomorphism from ring  $R$  to  $S$ . Then the mapping from  $R/\ker(f)$  to  $f(R)$ , given by

$$r + \ker(f) \rightarrow f(r)$$

is an isomorphism. In symbols,  $R/\ker(f) \cong f(R)$ .

*Proof.* Define a map  $\bar{f} : R/K \rightarrow \text{Im } f$  by

$$\bar{f}(a + K) = f(a) \quad \forall a \in R, a + K \in R/K.$$

1. Since  $f$  is well defined, so

$$\begin{aligned} a + K = b + K &\Rightarrow a - b \in K \\ &\Rightarrow \bar{f}(a - b) = 0_S \\ &\Rightarrow \bar{f}(a) - \bar{f}(b) = 0_S \\ &\Rightarrow \bar{f}(a) = \bar{f}(b) \end{aligned}$$

2.  $\bar{f}$  is injective since

$$\begin{aligned} \bar{f}(a + K) = \bar{f}(b + K) &\Rightarrow \bar{f}(a) = \bar{f}(b) \\ &\Rightarrow \bar{f}(a - b) = 0_S \\ &\Rightarrow a - b \in K \\ &\Rightarrow a + K = b + K. \end{aligned}$$

3.  $\bar{f}$  is surjective. For all  $f(a) \in \text{Im } f, \exists a + K \in R/K$  such that  $\bar{f}(a + K) = f(a)$ .

4.  $\bar{f}$  is homomorphism,

$$\begin{aligned} \bar{f}(a + K + b + K) &\Rightarrow \bar{f}((a + b) + K) \\ &\Rightarrow \bar{f}(a + b) \\ &\Rightarrow \bar{f}(a) + \bar{f}(b) \\ &\Rightarrow f(a + K) + f(b + K). \end{aligned}$$

$$\begin{aligned}
f(a + K) \cdot f(b + K) &\Rightarrow f(ab + K) \\
&\Rightarrow \bar{f}(ab) \\
&\Rightarrow \bar{f}(a) \cdot \bar{f}(b) \\
&\Rightarrow f(a + K) \cdot f(b + K).
\end{aligned}$$

Thus  $f : R/K \cong \text{Im } f$  as rings. □

**Example 2.3.7.** Let  $\phi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_3$  be the ring homomorphism defined by

$$\phi((a, b)) = b \bmod 3.$$

Then  $\ker(\phi) = \mathbb{Z} \times 3\mathbb{Z}$  and  $(\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times 3\mathbb{Z})$  is isomorphic to  $\mathbb{Z}_3$ , which is a field. Thus  $\mathbb{Z} \times 3\mathbb{Z}$  is a maximal ideal of  $\mathbb{Z} \times \mathbb{Z}$ .

## 2.4 Polynomial rings

### Definition 2.6

Let  $R$  be a commutative ring. We define

$$R[x] = \{r_n x^n + r_{n-1} x^{n-1} + \cdots + r_1 x + r_0 \mid r_i \in R\}. \quad (2.1)$$

The letter  $x$  here can be thought of a variable or just a placeholder. Either way the familiar structure allows us to add, subtract and multiply these as we do traditional polynomials even if the ring were some strange abstract entity.

## 2.5 Factorization of polynomials

### Theorem 2.7 Division algorithm

Let  $R$  be a ring with identity and  $f(x), g(x) \in R[x]$  with  $g(x) \neq 0$ . Then there exists unique polynomials  $q(x)$  and  $r(x)$  in  $R[x]$  such that

$$f(x) = q(x)g(x) + r(x) \quad (2.2)$$

and  $\deg(r) < \deg(g)$ .  $r(x) = 0$  if there is no remainder.

*Proof.* The basic idea is to formalize the process of long division in an inductive sense. We omit the details here. They're boring here. □

**Example 2.5.1.** In  $\mathbb{Z}_3$  we can divide  $2x^2 + 1$  into  $x^4 + 2x^3 + 2x + 1$ . Then we have

$$x^4 + 2x^3 + 2x + 1 = (2x^2 + 1)(2x^2 + x + 2)$$

**Theorem 2.8 Factor theorem**

Let  $F$  be a field,  $a \in F$  and  $f(x) \in F[x]$ . Then  $a$  is a **root** (or **zero**) of  $f(x)$  if and only if  $x - a$  is a factor of  $f(x)$ .

*Proof.* ( $\Rightarrow$ ) Assume that  $a \in F$  is a zero of  $f(x) \in F[x]$ . We wish to show that  $x - a$  is a factor of  $f(x)$ . To do so, apply the division algorithm. By division algorithm,  $\exists$  unique polynomials  $q(x)$  and  $r(x)$  such that

$$f(x) = (x - a)q(x) + r(x)$$

and the  $\deg(r) < \deg(x - a) = 1$ , so  $r(x) = c \in F$ , where  $c$  is a constant. Also, the fact that  $a$  is a zero of  $f(x)$  implies  $f(a) = 0$ . So

$$f(x) = (x - a)q(x) + c \implies 0 = f(a) = (a - a)q(a) + c.$$

Thus  $c = 0$ , and  $x - a$  is a factor of  $f(x)$ .

( $\Leftarrow$ ) On the other way, we want to show □

**Definition 2.7 Algebraically closed**

Given  $F$  a field, we call  $F$  **algebraically closed** if every  $f \in F[x]$  such that  $\deg(f) > 0$  has a root in  $F$ .

**Example 2.5.2.** Show that  $x^2 + 3x - 4 \in \mathbb{Z}_{12}[x]$  has 4 roots.

**Solution** We list down all the values of  $f(x) = x^2 + 3x - 4$  for  $x = 0, 1, \dots, 11$ .

$x$	0	1	2	3	4	5	6	7	8	9	10	11
$x^2 + 3x - 4 \pmod{12}$	8	0	6	2	0	0	2	6	0	8	6	6

which now we can see:  $x^2 + 3x - 4$  has 4 zeros in  $\mathbb{Z}_{12}[x]$ . Thus, a polynomial of degree  $n$  can have more than  $n$  roots in a ring. The problem is that  $\mathbb{Z}_{12}$  is not a domain:  $(x + 4)(x - 1) = 0$  does not imply one of the factors must be zero. ◀

**Example 2.5.3.** Show that the polynomial  $2x^3 + 3x^2 - 7x - 5$  can be factored into linear factors in  $\mathbb{Z}_{11}[x]$ .

**Solution** We can use synthetic division,

$$\begin{array}{r|rrrr}
 2 & 3 = -8 & -7 = 4 & 6 & \\
 & -2 & -10 & -6 & -1 \\
 \hline
 2 & -10 = 1 & -6 & & \\
 & -4 & 6 & & -2 \\
 \hline
 2 & -3 & & & 
 \end{array}$$

Thus,  $2x^3 + 3x^2 - 7x - 5 = (x + 1)(x + 2)(2x - 3)$  in  $\mathbb{Z}_{11}[x]$ . ◀

**2.5.1 Irreducibility tests**

There are various methods to check if a polynomial in  $\mathbb{Z}[x]$  is irreducible in  $\mathbb{Q}[x]$ .



**Theorem 2.9 Rational root test**

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (\star)$$

be a polynomial with integers coefficients. If  $r \neq 0$  and the rational number  $r/s$  (in lowest terms) is a root of  $f(x)$ , then  $r|a_0$  and  $s|a_n$ .

*Proof.* Plug  $x = r/s$  into  $(\star)$  and equating with zero. The equation is now

$$a_n \left(\frac{r}{s}\right)^n + a_{n-1} \left(\frac{r}{s}\right)^{n-1} + \cdots + a_1 \left(\frac{r}{s}\right) + a_0 = 0.$$

Again multiplying  $s^n$  on both sides

$$a_n r^n + a_{n-1} r^{n-1} s + \cdots + a_1 r s^{n-1} + a_0 s^n = 0.$$

Factoring  $r$  out and moving  $a_0 s^n$  to the right-hand side. We obtained

$$r(a_n r^{n-1} + a_{n-1} r^{n-2} s + \cdots + a_1 s^{n-1}) = -a_0 s^n.$$

Since  $\gcd(r, s) = 1$ , thus  $r|a_0$  and similarly  $s|a_n$ . □

**Example 2.5.4.** The polynomial  $f(x) = 2x^4 + x^3 - 21x^2 - 14x + 12$  is reducible in  $\mathbb{Q}[x]$ .

**Solution** If  $r/s$  is a root of  $f(x)$ , where  $r|12$  and  $s|2$ . Thus the possible roots are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}.$$

In fact,  $f(x) = (x + 3)(x - \frac{1}{2})(2x^2 - 4x - 8) \in \mathbb{Q}[x]$ . ◀

**Example 2.5.5.** The polynomial  $g(x) = x^3 + 4x^2 + x - 1$  is irreducible in  $\mathbb{Q}[x]$ .

**Solution** The possible roots are  $\{-1, 1\}$ . However

$$g(1) = 1 + 4 + 1 - 1 = 5 \quad \text{and} \quad g(-1) = -1 + 4 - 1 - 1 = 1$$

So  $g(x)$  has no root and  $\deg g(x) = 3$ . Thus  $g(x)$  is irreducible over  $\mathbb{Q}[x]$ . ◀

**Theorem 2.10 Mod  $p$  Irreducibility test**

Let  $p$  be a prime and let  $f(x) \in \mathbb{Z}[x]$  with degree 1 or greater. Let  $\bar{f} \in \mathbb{Z}_p[x]$  obtained by reducing all of  $f(x)$ 's coefficients mod  $p$ . Then if

$$\deg(\bar{f}) = \deg(f) \quad (2.3)$$

and  $\bar{f}$  is irreducible over  $\mathbb{Z}_p$  then  $f(x)$  is irreducible over  $\mathbb{Q}$ .

*Proof.* Assume that  $f(x) = p(x)q(x)$  in  $\mathbb{Z}[x]$ . Since  $\phi : \mathbb{Z}[x] \rightarrow \mathbb{Z}_p[x]$  defined by  $\phi f(x) = \bar{f}(x)$  is a ring homomorphism. So

$$\bar{f}(x) = \overline{p(x)q(x)} = \bar{p}(x)\bar{q}(x).$$

If  $p \nmid a_k$ , then  $p$  does not divide the leading coefficients of  $p(x)$  and  $q(x)$ . Thus  $\deg \bar{p}(x) = \deg p(x)$  and  $\deg \bar{q}(x) = \deg q(x)$ .  $\square$

**Example 2.5.6.** The polynomial  $f(x) = x^5 + 8x^4 + 3x^2 + 4x + 7$  is irreducible in  $\mathbb{Q}$ .

**Solution** We define

$$\bar{f}(x) = x^5 + x^2 + 1 \in \mathbb{Z}_2[x].$$

By rational root test, the only possible root is 0.1 from  $\mathbb{R}$  but it is not an inetger. There are several quadratic polynomials in  $\mathbb{Z}_2[x]$  such as

$$x^2 + x + 1, \quad x^2 + 1, \quad x^2 + x, \quad x^2.$$

Since  $x^2 + 1, x^2 + x, x^2$  both have roots, they cannot be factor of  $\bar{f}$ . The only possible factor of  $\bar{f}$  is  $x^2 + x + 1$ . Thus

$$x^5 + x^2 + 1 = (x^2 + x + 1)(x^3 + ax^2 + bx + c).$$

Equating coefficients of both sides, we have

$$\begin{cases} 1 + a = 0 \\ 1 + a + b = 0 \\ a + b + c = 0 \\ b + c = 0 \\ c = 1 \end{cases}.$$

On solving yields  $a = -1 = 1(\text{mod } 2)$ ,  $b = 0$  and  $c = 1$  but  $b + c \neq 0$  and is contradiction. So  $f(x)$  does not has a quadratic factor. It means that  $f(x)$  is irreducible in both  $\mathbb{Z}_2[x]$  and  $\mathbb{Z}$ . So  $f(x)$  is also irreducible in  $\mathbb{Q}[x]$ .  $\blacktriangleleft$

**Theorem 2.11 Eisenstein's criterion**

Let  $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x] \setminus \{0\}$ . If there is a prime number  $p$  such that  $p \nmid a_n$ , but  $p|a_{n-1}, \dots, p|a_2$  and  $p^2|a_0$ . Then  $f(x)$  is irreducible over  $\mathbb{Q}$ .

*Proof.* Suppose that  $f(x)$  is reducible over  $\mathbb{Q}$  then

$$f(x) = g(x)h(x)$$

and  $g(x), h(x)$  are nonconstant polynomials.

Let

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

$$g(x) = b_rx^r + b_{r-1}x^{r-1} + \cdots + b_1x + b_0,$$

$$h(x) = c_sx^s + c_{s-1}x^{s-1} + \cdots + c_1x + c_0.$$

Since  $p|a_0 = b_0c_0 \implies p|b_0$  or  $p|c_0$ , and  $p^2 \nmid a_0$ . This implies that  $p$  divides only one of them. Assume that  $p|b_0$  and  $p \nmid c_0$ , then

$$p|a_0 = b_0c_1 + b_1c_0.$$

Since  $p|b_0c_1$  and  $p \nmid c_0 \implies p|b_1$ . Assume that  $p|b_i \forall 0 \leq i < m$  for some  $m \leq r$ . Then

$$p|a_m = \sum_{\substack{i+j=m \\ j \leq s}} b_i c_j \implies p|b_m c_0 \implies p|b_m.$$

By mathematical induction,  $p|b_r$ . Thus  $p|a_n = b_r c_s$ . This contradicting the fact that  $f(x)$  is reducible.  $\square$

**Example 2.5.7.**  $x^9 + 5$  is irreducible in  $\mathbb{Q}[x]$  with  $p = 5$ .

**Example 2.5.8.**  $x^{17} + 6x^{13} - 15x^4 + 3x^2 - 9x + 12$  is irreducible in  $\mathbb{Q}[x]$  with  $p = 3$ .

**Example 2.5.9.**  $x^n + 5$  is irreducible in  $\mathbb{Q}[x]$  for all  $n \geq 1$ . There are irreducible polynomials of every degree in  $\mathbb{Q}[x]$ .

### Corollary 2.2

For any prime  $p$ , the  $p$ -th cyclotomic polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over  $\mathbb{Q}$ .

*Proof.* Let  $\zeta = e^{2\pi i/n}$ . Then  $\zeta, \zeta^2, \dots, \zeta^n$  are the  $n$ -th roots of unity. They form the vertices of a regular  $n$ -gon in the complex plane. If  $\gcd(a, n) > 1$  then  $\zeta^a$  is a root of unity of order  $n/\gcd(a, n) < n$ , but if  $\gcd(a, n) = 1$  then  $\zeta$  is not a root of lower order, and in this case we call  $\zeta^a$  a primitive  $n$ -th root of unity. We define the  $n$ -th cyclotomic polynomial  $\Phi_n(x)$  to be the monic polynomial of degree  $\phi(n)$  whose roots are the primitive  $n$ -th root of unity:

$$\Phi_n(x) = \prod_{\substack{a=1 \\ \gcd(a,n)=1}}^n (x - \zeta^a). \quad (2.4)$$

The first few cyclotomic polynomials are as follows:

$n =$	
1	$\Phi_1(x) = x - 1$
2	$\Phi_2(x) = x + 1$
3	$\Phi_3(x) = x^2 + x + 1$
4	$\Phi_4(x) = x^2 + 1$
5	$\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$
6	$\Phi_6(x) = x^2 - x + 1$
7	$\Phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
8	$\Phi_8(x) = x^4 + 1$
9	$\Phi_9(x) = x^6 + x^3 + 1$
10	$\Phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$
11	$\Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
12	$\Phi_{12}(x) = x^4 - x^2 + 1$
13	$\Phi_{13}(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
14	$\Phi_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
15	$\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$
16	$\Phi_{16}(x) = x^8 + 1$

Let  $p$  denote a given prime number. For any polynomial  $f(x)$  with integral coefficients let  $\bar{f}(x)$  be the polynomial whose coefficients are the residue classes (mod  $p$ ) determined by the coefficients of  $f(x)$ . Thus the assertion  $\bar{f} = \bar{g}$  means that there is a polynomial  $h(x)$  with integral coefficients such that  $f(x) = ph(x)$ .

**Lemma 2.5**

(Schönemann, 1846) Let  $A(x)$  be a monic polynomial with integral coefficients, for instance

$$A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 = \prod_{i=1}^n (x - \alpha_i).$$

Let  $p$  be a prime, and put

$$C(x) = \prod_{i=1}^n (x - \alpha_i^p).$$

Then  $\bar{C} = \bar{A}$ .

*Proof.* Let  $\sigma_k(\alpha)$  denote the  $k$ -th symmetric function of  $\alpha_i$ . When  $\sigma_k(\alpha)^p$  is expanded by the multinomial theorem, all coefficients except the extreme ones are divisible by  $p$ . That is,

$$\frac{\sigma_k(\alpha_1, \alpha_2, \dots, \alpha_n)^p - \sigma_k(\alpha_1^p, \alpha_2^p, \dots, \alpha_n^p)}{p}$$

is a symmetric polynomial in the  $\alpha$ , with integral coefficients, and hence by the symmetric function theorem the quantity must be a rational integer.  $\square$

**Lemma 2.6**

Put  $f(x) = x^n - 1$ . Then  $\bar{f}$  is a squarefree if and only if  $p \nmid n$ .

*Proof.* By previous lemma we can see that if  $p \nmid n$ . Then  $\gcd(\bar{f}, \bar{f}') = 1$ , and hence that  $\bar{f}$  is square-free. On the other hand, if  $p|n$ , say  $n = mp$  for some integer  $m$ , then

$$\bar{f} = \overline{(x^m - 1)^p}$$

and hence  $\bar{f}$  is not squarefree.

Let  $\Phi_n(x)$  denote the  $n$ -th cyclotomic polynomial. Since  $\Phi_n|f$ , it follows from the above that if  $p \nmid n$ , then  $\Phi_n$  is also squarefree.  $\square$

### Theorem 2.12

(Kronecker, 1854) The polynomial  $\Phi_n(x)$  is irreducible over  $\mathbb{Q}$ .

*Proof.* Suppose that  $A$  and  $B$  are monic polynomials with rational coefficients such that  $\Phi_n = AB$ , and suppose also that  $\deg A > 0$ . We know that  $A$  and  $B$  have integral coefficients. Let  $Z$  denote the roots of  $A$ . Let  $C$  be the monic polynomial whose roots are the numbers  $\zeta^p$  for  $\zeta \in Z$ . Here  $p$  is an arbitrary prime not dividing  $n$ . Our first step is to show that  $A = C$ .

Since the map  $\zeta \mapsto \zeta^p$  merely permutes the roots of  $\Phi_n$ , we know that  $C|\Phi_n$ . Let  $G = \gcd(B, C)$ . Then  $\bar{G}|B$  and  $\bar{G}|C$ . But  $\bar{A} = \bar{C}$  by previous lemma, and hence  $\bar{G}^2|\bar{A}\bar{B}$ . But  $\Phi_n$  is squarefree, by previous lemma. Hence  $G = \bar{1}$ , so  $G = 1$ , and consequently  $C|A$ . But  $C$  and  $A$  have the same degree, so in fact  $A = C$ .

Now let  $\zeta$  be a root of  $A$ , and  $\zeta'$  a root of  $\Phi_n$ . Then there exists a positive integer  $a$ ,  $\gcd(a, n) = 1$ , such that  $\zeta' = \zeta^a$ . We factor  $a$ ,  $a = p_1 p_2 \dots p_k$ . Since  $\zeta$  is a root of  $A$ , it follows from the argument above that  $\zeta^{p_1}$  is also a root of  $A$ . Then by a second application of the above argument, we see that  $\zeta^{p_1 p_2}$  is also a root of  $A$ . Continuing in this manner, we deduce that  $\zeta'$  is a root of  $A$ . Since this is valid for every root  $\zeta'$  of  $\Phi_n$ , we conclude that  $A = \Phi_n$ . Hence  $\Phi_n$  is irreducible.  $\square$

### Theorem 2.13

Let  $F$  be a field and let  $p(x) \in F[x]$ . Then  $\langle p(x) \rangle$  is a maximal ideal in  $F[x]$  if and only if  $p(x)$  is irreducible over  $F$ .

*Proof.* Suppose  $\langle p(x) \rangle$  is a maximal ideal in  $F[x]$ . We know that  $p(x) \neq 0$  and  $p(x)$  is not a unit since neither  $\{0\}$  nor  $\langle 1_F \rangle = F[x]$  is a maximal ideal in  $F[x]$ . Let

$$p(x) = g(x)h(x)$$

be a factorization. Then  $\langle p(x) \rangle \subseteq \langle g(x) \rangle \subseteq \langle F[x] \rangle$  and since  $\langle p(x) \rangle$  is maximal we either have  $\langle g(x) \rangle = \langle p(x) \rangle$  or  $\langle g(x) \rangle = F[x]$ . In the first case we get  $\square$

### Theorem 2.14 Fundamental Theorem of Algebra

Every nonconstant polynomial in  $\mathbb{C}[x]$  has a root in  $\mathbb{C}$ .

**Remark.** The field  $\mathbb{C}$  is algebraically closed.

**Corollary 2.3**

A polynomial is irreducible in  $\mathbb{C}[x]$  if and only if it has a degree 1.

*Proof.* All linear equation with degree 1 only have one root in  $\mathbb{R}$ . □

**Corollary 2.4**

Every nonconstant polynomial  $f(x)$  of degree  $n$  can be written in the form

$$c(x - a_1)(x - a_2) \dots (x - a_n)$$

for some  $c, a_1, a_2, \dots, a_n \in \mathbb{C}$ . This factorization is unique except for the order of the factors.

*Proof.* By the fundamental theorem of algebra,

$$\begin{aligned} f(x) &= (r_1x + s_1)(r_2x + s_2) \dots (r_nx + s_n) \\ &= r_1 r_2 \dots r_n (x + s_1 r_1^{-1})(x + s_2 r_2^{-1}) \dots (x + s_n r_n^{-1}). \end{aligned}$$

Since  $f(x)$  has  $n$  unique roots, factorization is also unique. □

**Lemma 2.7**

If  $f(x)$  is a polynomial in  $\mathbb{R}[x]$  and  $a + bi$  is a root of  $f(x)$  in  $\mathbb{C}$ , then  $a - bi$  is also a root of  $f(x)$ .

*Proof.* Let  $z = a + bi$  and the conjugate  $\bar{z} = a - bi$ . Define a map  $\varphi : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  by  $\varphi f(x) = \bar{f}$ . Bijection is trivial in  $\varphi$ , e.g.  $\varphi(f + g) = \bar{f} + \bar{g}$  and  $\varphi(fg) = \bar{f}\bar{g}$  since

$$\overline{(a + bi)(c + di)} = \overline{(a + bi)} \overline{(c + di)}.$$

If  $f(x)$  has a root  $z$  then  $\bar{f}(x)$  will have a root  $\bar{z}$ . If coefficients of  $f(x)$  are all real numbers, then  $f(x) = \bar{f}(x)$ . Thus  $f(x)$  has a root  $\bar{z}$ . □

**Theorem 2.15**

A polynomial  $f(x)$  is irreducible in  $\mathbb{R}[x]$  if and only if  $f(x)$  is a first-degree polynomial or  $f(x) = ax^2 + bx + c$  with  $b^2 - 4ac < 0$ .

*Proof.* In  $\mathbb{C}[x]$ ,

$$f(x) = c(x - a_1)(x - a_2) \dots (x - a_n).$$

If  $a_i = c + di, a_j = c - di$  for some  $1 \leq j \leq n$ . The product of the conjugates are

$$(x - a_i)(x - a_j) = (x - c - di)(x - c + di) = x^2 - 2cx + c^2 + d^2 \in \mathbb{R}[x].$$

Thus we can pair them and so  $f(x)$  can be split by irreducible polynomials whose degree is either 1 or 2.

Now we know every irreducible polynomial has a degree 1 or 2. When its degree is 2, then

$$f(x) = ax^2 + bx + c \quad \forall a, b, c \in \mathbb{R} \quad (\clubsuit)$$

We now continue to work on the “formula” to solve  $x$ . Completing the square on (♣)

$$\begin{aligned}
 ax^2 + bx + c = 0 &\Rightarrow a \left[ x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 \right] - \left( \frac{b}{2a} \right)^2 = 0 \\
 &\Rightarrow \left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \\
 &\Rightarrow x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
 &\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a \neq 0
 \end{aligned}$$

Now we can take a look on determinant  $\Delta = b^2 - 4ac$ . If  $\Delta < 0$ , the two roots will be in  $\mathbb{C} \setminus \mathbb{R}$ , else the two roots are in  $\mathbb{R}$ . (Either  $\Delta > 0$  or  $\Delta = 0$ ). Hence the first-degree polynomial or quadratic polynomial is irreducible in  $\mathbb{R}[x]$ .  $\square$

### Corollary 2.5

Every polynomial  $f(x)$  of odd degree in  $\mathbb{R}[x]$  has a root in  $\mathbb{C}$ .

*Proof.* Consequently, we can tell if a polynomial in  $\mathbb{R}[x]$  or  $\mathbb{C}[x]$  is irreducible without any elaborate tests.  $\square$

## 2.6 Integral Domains

Let  $R$  be a commutative ring. A **zero divisor** is a nonzero element  $a \in R$  such that

$$ab = 0 \tag{2.5}$$

for some nonzero  $b \in R$ . The most familiar integral domain is  $\mathbb{Z}$ . It is a commutative ring with unity one. If  $a, b \in \mathbb{Z}$  and  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .

### Definition 2.8

A ring with unity 1 having no zero divisors is an integral domain.

### Lemma 2.8

Fields are integral domain

*Proof.* Let  $F$  be a field. We want to show that  $F$  has no zero divisors. Suppose  $ab = 0$  and  $a \neq 0$ . Then  $a$  must have an inverse  $a^{-1}$  such that  $a^{-1}ab = a^{-1} \cdot 0 \implies b = 0$ . Therefore,  $F$  has no zero divisors, and so  $F$  is an integral domain.  $\square$

### Corollary 2.6

$\mathbb{Z}_p$  is a field.

*Proof.* According to previous theorem, we need only prove that  $\mathbb{Z}_p$  has no zero divisors.

Suppose that  $a, b \in \mathbb{Z}_p$  and  $ab = 0$ . Then  $ab = pk$  for some integer  $k$ . But then we have either  $p \mid a$  or  $p \mid b$  and in  $\mathbb{Z}_p$  is equivalent to either  $a = 0$  or  $b = 0$ .  $\square$

### Lemma 2.9

If  $R$  is an integral domain, then the characteristic of  $R$  is either 0 or a positive prime.

**Solution** Suppose not, suppose  $R$  has characteristic  $n = ab$  with  $1 < a < b < n$ . Then

$$\underbrace{(1_R + \cdots + 1_R)}_{a \text{ times}} \underbrace{(1_R + \cdots + 1_R)}_{b \text{ times}} = ab1_R = n1_R = 0.$$

Since  $R$  is an integral domain, it is either  $a \cdot 1_R = 0$  or  $b \cdot 1_R = 0$ . This contradicts with our assumption that  $n$  is not prime.  $\blacktriangleleft$

### Definition 2.9

If  $F$  is a field, then the only ideals are  $\{0\}$  and  $F$  itself.

*Proof.* Let  $F$  be a field, and let  $I \subset F$  be an ideal. Assume  $I \neq \{0\}$ , and find  $x \neq 0 \in I$ . Since  $F$  is a field,  $x$  is invertible; Since  $I$  is an ideal,  $1 = x^{-1} \cdot x \in I$ . Therefore  $I = F$ .  $\square$

**Example 2.6.1.** The extended ring

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$

is a field and that every nonzero element has a multiplicative inverse.

**Solution** This is clearly a ring. To show that every nonzero element has a multiplicative inverse. Consider  $a + b\sqrt{2} \neq 0 \in \mathbb{Q}[\sqrt{2}]$ . The multiplicative inverse is

$$\frac{1}{a + b\sqrt{2}}$$

Then multiplying top and bottom by conjugate, we have

$$\frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

Now we want to show  $a^2 - 2b^2 \neq 0$ .

If  $a = 0$  and  $b \neq 0$  or if  $a \neq 0$  and  $b = 0$ , then  $a^2 - 2b^2 \neq 0$ . Since  $a^2 - 2b^2 \neq 0$ , the only other possibility is  $a, b \neq 0$ .

Thus,  $a^2 = 2b^2$  with  $a, b \neq 0$ . We may assume that  $a$  and  $b$  are integers – in fact, now we can see 2 divides  $2b^2$ , so  $2 \mid a^2 \implies 2 \mid a$ . So  $a = 2c$  for some integer  $c$ . Plugging in gives  $4c^2 = 2b^2 \implies 2c^2 = b^2$ .

It follows that every nonzero element of  $\mathbb{Q}[\sqrt{2}]$  is invertible, so  $\mathbb{Q}[\sqrt{2}]$  is a field.  $\blacktriangleleft$

**Example 2.6.2** (Non-example).  $M_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$  is not an integral domain.



**Solution** Choose

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

from  $M_2(\mathbb{Z})$ , and compute the matrix product

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

$A, B$  are zero divisors but none of them are zero. Thus  $M_2(\mathbb{Z})$  is not an integral domain. ◀

### Theorem 2.16

A finite integral domain is a field

*Proof.* Let  $D$  be a finite integral domain. Since  $D$  is an integral domain, then  $D$  is a commutative ring with unity, and hence we need to show that  $D$  is a field. In order to do this, we want to show  $\forall a \neq 0 \in D, \exists a^{-1} \in D$  such that

$$a \cdot a^{-1} = 1_D = a^{-1} \cdot a.$$

Without loss of generality, we let

$$D = \{a, a^2, a^3, \dots, a^t\}$$

where  $a \neq 0$  for some  $t \in \mathbb{N}$ . Consider two elements  $a^i, a^j$  from  $D$ , we have

$$\begin{aligned} a^i = a^j &\Rightarrow a^{i-j} = 1_D \\ &\Rightarrow a a^{i-j-1} = 1_D \\ &\Rightarrow a^{-1} = a^{i-j-1} \\ &\Rightarrow a \cdot a^{-1} = a^{i-j} = 1_D. \end{aligned}$$

and the multiplication is commutative, therefore  $D$  is a field. ◻

**Remark.**

$$\begin{aligned} \text{if } \mathbb{Z}_p \text{ is a field} &\implies \mathbb{Z}_p \text{ has no zero divisors.} \\ &\implies \mathbb{Z}_p \text{ is an integral domain.} \\ &\implies |\mathbb{Z}_p| = p \text{ is prime and is finite.} \end{aligned}$$

### Theorem 2.17

Let  $R$  be a commutative ring with unity and let  $A$  be an ideal of  $R$ . Then  $R/A$  is an integral domain if and only if  $A$  is prime.

*Proof.* We are going to prove in two directions:

( $\Rightarrow$ ) Suppose  $R/A$  is an integral domain and  $ab \in A$ . Then

$$(a + A)(b + A) = ab + A = A$$

is the zero element of the ring  $R/A$ . So it is either  $a + A = A$  or  $b + A = A$ ; that is, either  $a \in A$  or  $b \in A$ . Thus  $A$  is prime.

( $\Leftarrow$ ) On the other hand, observe that  $R/A$  is a commutative ring with unity for any proper ideal  $A$ . Thus, we only need to show that when  $A$  is prime,  $R/A$  has no zero divisors.

Now suppose that  $A$  is prime and

$$(a + A)(b + A) = 0 + A = A.$$

Then  $ab \in A$ , and therefore, it is either  $a \in A$  or  $b \in A$ . Thus, one of  $a + A$  or  $b + A$  is the zero coset in  $R/A$ . Now we are done.  $\square$

## 2.7 Principal Ideal Domain

### Definition 2.10

An integral domain  $R$  is called a **principal ideal domain** (or **PID**) if every ideal in  $R$  is principal.

**Example 2.7.1.** The integers  $\mathbb{Z}$  and polynomial rings over fields are principal ideal domains.

### Theorem 2.18

If  $F$  is a field then  $F[x]$  is a PID.

*Proof.* We know  $F[x]$  is integral domain since  $F$  is an integral domain. Let  $I$  be an ideal of  $F[x]$ .

**Case 1:** If  $I = \{0\}$  then  $I = \langle 0 \rangle$  and we are done.

**Case 2:** If  $I \neq \{0\}$  let  $g(x)$  be a nonzero polynomial of minimal degree in  $I$  (which exists by well-ordering). If  $g(x)$  is constant then  $g(x) = \alpha \in F$  and then  $I = F = \langle \alpha \rangle$  because for any  $r \in F$  we have

$$r = r\alpha^{-1}\alpha \in \langle \alpha \rangle.$$

Suppose then that  $g(x)$  is not constant, we claim  $I = \langle g(x) \rangle$ . Since  $g(x) \in I$  we have  $\langle g(x) \rangle \subseteq I$ . We claim  $I \subseteq \langle g(x) \rangle$ . Let  $f(x) \in I$ . By the *division algorithm*, we can write

$$f(x) = q(x)g(x) + r(x)$$

with  $0 \leq \deg(r(x)) < \deg(g(x))$ . Since

$$r(x) = f(x) - q(x)g(x)$$

we have  $r(x) \in I$  and the fact that  $g(x)$  is a nonzero polynomial of minimal degree implies that  $r(x) = 0$  and so  $f(x) = q(x)g(x) \implies f(x) \in \langle g(x) \rangle$ .  $\square$

## 2.8 Unique Factorization Domain

### Definition 2.11

An integral domain  $D$  is a **unique factorization domain** (UFD in short) if

1. Every nonzero element of  $D$  that is not a unit can be written as a product of irreducibles of  $D$ , and
2. The factorization into irreducibles is unique up to associates and the order in which the factors appear.

### Theorem 2.19

Every PID is a UFD.

*Proof.* Let  $R$  be a PID and suppose that a nonzero element  $a$  of  $R$  can be express in two different ways as a product of irreducibles. Suppose

$$a = p_1 p_2 \cdots p_r \quad \text{and} \quad a = q_1 q_2 \cdots q_s$$

where each  $p_i$  and  $q_j$  is irreducible in  $R$ , and  $s \geq r$ . Then  $p_1$  divides the product  $q_1, q_2, \dots, q_s$  and so  $p_1 | q_j$  for some  $j$ , as  $p_1$  is prime. After reordering the  $q_j$  we can consider  $p_1 | q_1$ . Then  $q_1 = u_1 p_1$  for some unit  $u_1$  of  $R$ , since  $q_1$  and  $p_1$  are both irreducible. Thus

$$p_1 p_2 \cdots p_r = u_1 p_1 q_2 \cdots q_s$$

and cancelling  $p_1$  on both side

$$p_2 \cdots p_r = u_1 q_2 \cdots q_s.$$

Continuing this process we reach

$$1 = u_1 u_2 \cdots u_r q_{r+1} \cdots q_s.$$

Since none of the  $q_j$  is a unit, this means that  $r = s$  and  $p_1 p_2 \cdots p_r$  are associates of  $q_1 q_2 \cdots q_r$  in some order. Thus  $R$  is a unique factorization domain.  $\square$

### Theorem 2.20

Every field is a UFD.

*Proof.* Every field  $F$  is a UFD because it is PID  $\Rightarrow$  it is an integral domain and every nonzero is a unit, and it contains no prime.  $\square$

## 2.9 Euclidean Domains

### Definition 2.12

An integral domain  $D$  is called a **Euclidean Domain** if there is a function  $d$  (called the measure) from the nonzero elements of  $D$  to the positive integers such that

1.  $d(a) \leq d(ab)$  for all nonzero  $a, b \in D$ ; and
2. If  $a, b \in D, b \neq 0$ , then there exists elements  $q, r \in D$  such that

$$a = bq + r$$

where  $r = 0$  or  $d(r) < d(b)$ .

**Example 2.9.1.** The ring  $\mathbb{Z}$  is a Euclidean domain with  $d(a) = |a|$ .

**Example 2.9.2.** Let  $F$  be a field. Then  $F[x]$  is a Euclidean domain with  $d(f(x)) = \deg f(x)$ .

**Example 2.9.3.** Gaussian integers  $\mathbb{Z}[i]$  is a Euclidean domain with

$$d(a + bi) = a^2 + b^2.$$

### Theorem 2.21

Every Euclidean domain is a PID.

*Proof.* Let  $E$  be a Euclidean domain. Consider an ideal  $I$  of  $E$ . If  $I = \{0\}$ , then  $I = \langle 0 \rangle$ .

Let  $I \neq \{0\}$ . Then  $N = \{d(x) \mid x \in I, x \neq 0\}$  is a nonempty set of nonnegative integers; and so, by the well-ordering principle it has the least element.

Let  $a \in I, a \neq 0$  such that  $d(a)$  is the least element of  $N$ . i.e.  $d(a) \leq d(x)$  for all nonzero  $x$  in  $I$ . We want to show  $I = Ea$ . Since  $a \in I$ , it follows that

$$Ea \subseteq I.$$

Let  $b \in I$ . Since  $E$  is a Euclidean domain, there exist  $q, r \in E$  such that

$$b = aq + r, \quad \text{where } r = 0 \text{ or } d(r) < d(a).$$

If  $r \neq 0$ , then  $r = b - aq \in I$  shows that  $d(r) \in N$ ; and since  $d(r) < d(a)$ , this contradicts the minimality of  $d(a)$  in  $N$ . Therefore,  $r = 0$  and so  $b = aq \in Ea$ . Thus  $I \subseteq Ea$  and hence  $I = Ea$ .  $\square$

**Remark.** If  $D$  is Euclidean domain, then it is both PID and UFD.

### Theorem 2.22

If  $D$  is UFD, then  $D[x]$  is UFD.

*Proof.* Let  $D$  be an Euclidean domain and  $\emptyset \neq I \triangleleft D$ .

Among all the nonzero elements of  $I$ , let  $a \in I$  be such that  $d(a)$  is a minimum. Then

$$I = \langle a \rangle.$$

For if  $b \in I$ , there exist some  $q, r$  such that

$$b = aq + r,$$

where  $r = 0$  or  $d(r) < d(a)$ . But  $r = b - aq \in I$ , so  $d(r)$  cannot be less than  $d(a)$ . Thus,  $r = 0$  and  $b = aq \in \langle a \rangle$   $\square$

## Tutorials

**Exercise 2.9.1** For each of the following, decide whether the indicated operations on the set will form a ring. If a ring is not formed, state the reason why this is the case. If a ring is formed state whether the ring is commutative, whether it has unity, and whether it is a field.

1.  $n\mathbb{Z}$ , under the usual addition and multiplication.
2.  $n\mathbb{R}^+$ , under the usual addition and multiplication.
3.  $n\mathbb{Z} \times \mathbb{Z}$  with addition and multiplication by components.
4.  $n\mathbb{Z} \times 2\mathbb{Z}$  with addition and multiplication by components.
5.  $\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$  with the usual addition and multiplication.
6.  $\{ri \mid r \in \mathbb{R}\}$  with the usual addition and multiplication where  $i^2 = -1$ .

**Exercise 2.9.2** Let  $\alpha = \sqrt[3]{5}$  and  $\mathbb{Z}[\alpha] = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathbb{Z}\}$ . Prove whether  $\mathbb{Z}[\alpha]$  is a subring of  $\mathbb{R}$ .

**Exercise 2.9.3** Let  $X$  be some arbitrary set, and  $P(X)$  be the set of all subsets of  $X$ . Define operators on  $P(X)$  as follows, where  $a, b$  in  $P(X)$ :

$$a + b = (a \cup b) \setminus (a \cap b)$$

and

$$ab = a \cap b.$$

Show that  $P(X)$  is a commutative ring.

**Exercise 2.9.4** Let  $\mathbb{A}$  be the set  $\mathbb{A} = \{a + bi \mid a, b \in \mathbb{Q}\}$  where  $i^2 = -1$ . Here,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi)(c + di) = (ac - bd) + (ad - bc)i.$$

Show that  $\mathbb{A}$  is a field.

**Exercise 2.9.5** Show that the rings  $2\mathbb{Z}$  and  $3\mathbb{Z}$  are not isomorphic.

**Exercise 2.9.6** Show that a ring  $R$  has no nonzero nilpotent element if and only if 0 is the only solution of  $x^2 = 0$  in  $R$ .

**Exercise 2.9.7** Show that if  $R$  is a ring with unity and  $N$  is an ideal of  $R$  such that  $N \neq R$ , then  $R/N$  is a ring with unity.

**Exercise 2.9.8** If  $F$  is a field, show that  $(F \setminus \{0\}, \cdot)$  is a group.

**Exercise 2.9.9** Show that in a field  $F$ , the only ideals are  $F$  and  $\{0\}$ .

**Exercise 2.9.10** Show that each homomorphism from a field to a ring is either one to one or maps everything onto 0.

**Exercise 2.9.11** Find the characteristic of the following rings:

1.  $2\mathbb{Z}$ .
2.  $\mathbb{Z}_3 \times 3\mathbb{Z}$ .
3.  $\mathbb{Z}_5 \times \mathbb{Z}_5$ .

**Exercise 2.9.12** Show that the matrix  $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  is a zero divisor in  $M_2(\mathbb{Z})$ .

**Exercise 2.9.13** An element  $a$  of a ring  $R$  is idempotent if  $a^2 = a$ . Show that a division ring contains exactly two idempotent elements.

**Exercise 2.9.14** If  $A$  and  $B$  are ideals of a ring  $R$ , then the sum  $A + B$  of  $A$  and  $B$  is defined by

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

1. Show that  $A + B$  is an ideal of  $R$ .
2. Show that  $A \subseteq A + B$ .

**Exercise 2.9.15** If  $A$  and  $B$  are ideals of a ring  $R$ , then the product  $AB$  of  $A$  and  $B$  is defined by

$$AB = \left\{ \sum_{i=1}^n a_i b_i \mid a_i \in A, b_i \in B, n \in \mathbb{Z}^+ \right\}.$$

1. Show that  $AB$  is an ideal of  $R$ .
2. Show that  $AB \subseteq (A \cap B)$ .

**Exercise 2.9.16** Find  $q(x)$  and remainder  $r(x)$  as described by the division algorithm so that

$$f(x) = g(x)q(x) + r(x)$$

with  $r(x) = 0$  or of degree less than the degree of  $g(x)$ .

1.  $f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2$  and  $g(x) = x^2 + 2x - 3$  in  $\mathbb{Z}_7[x]$ .
2.  $f(x) = x^5 - 2x^4 + 3x - 5$  and  $g(x) = 2x + 1$  in  $\mathbb{Z}_{11}[x]$ .

# Fields

## 3.1 Extension Fields

### 3.1.1 Simple Extension

#### Definition 3.1

Let  $E$  be an extension field of a field  $F$  and let  $\alpha \in E$ . We call an element  $\alpha$  **algebraic** over  $F$  if  $\alpha$  is the zero of some nonzero polynomial in  $F[x]$ . If  $\alpha$  is not algebraic over  $F$ , it is called **transcendental** over  $F$ .

**Example 3.1.1.**  $\mathbb{C}$  is an extension field of  $\mathbb{R}$ .

$$\begin{array}{c} \mathbb{C} \\ | \\ \mathbb{R} \end{array}$$

The imaginary number  $i = \sqrt{-1}$  is said to be algebraic since  $x^2 + 1 = 0 \in \mathbb{R}[x]$ . While  $\pi$  is transcendental since it is not a zero in  $\mathbb{R}[x]$ .

#### Theorem 3.1

Let  $K$  be an extension field of  $F$ , and  $u \in K$  is an algebraic element over  $F$ . Then there exists a unique monic irreducible polynomial  $p(x)$  in  $F[x]$  that has  $u$  as a root. Furthermore if  $u$  is a root of  $g(x) \in F[x]$ , then  $p(x)$  divide  $g(x)$ .

*Proof.* Notice that  $u \in K$  is algebraic over  $F$  if and only if there is a nonzero polynomial  $f(x) \in F[x]$  such that  $f(u) = 0_K$ .

Let  $S$  be the set of all nonzero polynomials in  $F[x]$  that have  $u$  as a root, then  $S$  is nonempty set. By well-ordering principle,  $\exists p(x) \in S$  such that  $p(x)$  has the smallest degree in  $S$ .

Suppose that  $f(x) \in F[x]$  with  $f(u) = 0_K$ . By division algorithm,

$$f(x) = p(x)q(x) + r(x)$$

with  $\deg p(x) > \deg r(x)$  or  $r(x) = 0$ .

If  $r(x) \neq 0$ ,

$$f(x) - p(x)q(x) = r(x) \implies f(u) - p(u)q(u) = r(u) = 0.$$

This contradicting the fact that  $p(x)$  is the smallest polynomial. Thus  $r(x) = 0$  and  $p(x)$  divide  $f(x)$ . And now if we let  $p(x)$  and  $q(x)$  be the smallest polynomial. Then,  $p(x)|q(x)$  and  $q(x)|p(x)$  implies that  $p(x) = q(x)$ .  $\square$



**Remark.** The  $p(x)$  is called the "minimal polynomial of  $u$  over  $F$ ".

**Example 3.1.2.**  $\mathbb{R}$  is an extension field of  $\mathbb{Q}$  and  $\sqrt{3} \in \mathbb{R}$  is algebraic, then

$$p(x) = x^2 - 3 \in \mathbb{Q}[x]$$

**Example 3.1.3.**  $\mathbb{R}$  is an extension field of  $\mathbb{Q}$  and  $\sqrt{3} + \sqrt{5} \in \mathbb{R}$  is algebraic, Then

$$p(x) = x^4 - 16x^2 + 4 \in \mathbb{Q}[x].$$

**Solution** Let  $x = \sqrt{3} + \sqrt{5} \in \mathbb{R}$ , then

$$\begin{aligned} x^2 &= 3 + 2\sqrt{15} + 5 \Rightarrow x^2 - 8 = 2\sqrt{15} \\ &\Rightarrow (x^2 - 8)^2 = 4 \cdot 15 \\ &\Rightarrow x^4 - 16x^2 + 4 = 0 \end{aligned}$$

Thus  $p(x) = x^4 - 16x^2 + 4 \in \mathbb{Q}[x]$ . ◀

### Theorem 3.2

Let  $K$  be an extension field of  $F$  and  $u \in K$  is an algebraic element over  $F$  with minimal polynomial  $p(x)$  of degree  $n$ , then

1.  $F[u]$  is a field isomorphism of  $F[x]/p(x)$ .
2.  $\{1, u, u^2, \dots, u^{n-1}\}$  is a basis of the vector space  $F(u)$  over  $F$ .
3.  $|F(u) : F| = n$ .

*Proof.* 1. Since  $F(u)$  contains  $F$  and  $u$ , so  $F(u)$  contains every element of the form

$$b_0 + b_1u + b_2u^2 + \dots + b_tu^t \quad \forall b_i \in F.$$

We again define a function  $\varphi : F[x] \rightarrow F(u)$  by

$$\varphi(f(x)) = f(u).$$

Then  $\varphi$  is ring homomorphism.

Note that  $\ker \varphi = \langle p(x) \rangle$  where  $p(x)$  is the minimal polynomial of  $u$  over  $F$ . By the first isomorphism theorem,  $F[x]/p(x) \cong \text{Im } \varphi$ . Since  $p(x)$  is irreducible, the quotient ring  $F[x]/\langle p(x) \rangle$  is a field and  $\text{Im } \varphi$  is also a field.

Note that  $\varphi(c) = c \quad \forall c \in F$  and  $\varphi(x) = u$ . Thus  $F \subset \text{Im } \varphi$  and  $u \in \text{Im } \varphi$ . By definition of simple extension,  $F(u) = \text{Im } \varphi$ .

2. Since  $F(u) = \text{Im } \varphi$ ,  $\forall w \in F(u)$ ,  $\exists f(x) \in F[x]$  s.t.  $f(u) = w$ . If  $\deg f(x) > n$ . By division algorithm, we have

$$f(x) = p(x)q(x) + r(x)$$

If  $r(x) = 0$ ,  $f(u) = p(u)q(u) = 0 = w$ .

Otherwise, if  $r(x) \neq 0$ , then

$$r(x) = f(x) - p(x)q(x) \implies r(u) = f(u) - p(u)q(u) = w$$

with  $\deg r(x) < n$ . Thus

$$F(u) = \text{Span}\{1, u, u^2, \dots, u^{n-1}\}.$$

3. Let  $c_0 + c_1u + c_2u^2 + \dots + c_{n-1}u^{n-1} = 0$ . If  $\exists c_k \in F$  s.t.  $c_k \neq 0$ . Then  $p(x)$  is not the minimal polynomial of  $u$  over  $F$ . Thus  $c_i = 0 \forall c_i \in F$ . Hence, we can say that  $\{1, u, u^2, \dots, u^{n-1}\}$  is linearly independent and it is also a basis of  $F(u)$  over  $F \implies |F(u) : F| = n$ .

□

**Example 3.1.4.**  $\mathbb{Q}[\sqrt{3}]$  is isomorphic to  $\mathbb{Q}[\sqrt{3}]/\langle x^2 - 3 \rangle$

**Example 3.1.5.** If  $u$  and  $v$  have the same minimal polynomial  $p(x)$  in  $F[x]$ , then  $F(u)$  is isomorphic to  $F(v)$ . For instance,

$$\mathbb{Q}[\sqrt{3}] \cong \mathbb{Q}[-\sqrt{3}].$$

Let  $\sigma : F \rightarrow E$  be an isomorphism then we again define  $\bar{\sigma} : F[x] \rightarrow E[x]$  by for  $a_0 + a_1x + \dots + a_nx^n \in F[x]$ . We can write

$$\sigma(a_0 + a_1x + \dots + a_nx^n) = \sigma(a_0) + \sigma(a_1x) + \dots + \sigma(a_nx^n) \quad (3.1)$$

and  $\bar{\sigma}$  is also isomorphism.

### Corollary 3.1

Let  $\sigma : F \rightarrow E$  be an isomorphism of fields. Let  $u$  be an algebraic element in "some" extension field of  $F$  with minimal polynomial  $p(x) \in F[x]$ . Again we let  $v$  be an algebraic element in some extension field of  $E$  with minimal polynomial  $\sigma p(x) \in E[x]$ . Then  $\sigma$  extends to an isomorphism of fields  $\bar{\sigma} : F(u) \rightarrow E(v)$  such that

$$\bar{\sigma}(u) = v \text{ and } \bar{\sigma}(c) = \sigma(c) \quad \forall c \in F.$$

*Proof.* By previous theorem,  $\varphi : F[x]/(p(x)) \rightarrow F(u)$  and  $\bar{\varphi} : E[x]/(\sigma p(x)) \rightarrow E(v)$  are isomorphism where  $\varphi([f(x)]) = f(u)$  and  $\bar{\varphi}([g(x)]) = g(v)$ .

Furthermore, we let  $\xi$  be the surjective isomorphism

$$\bar{\xi} : E[x] \rightarrow E[x]/(\sigma p(x))$$

defined by  $\bar{\xi}(g(x)) = [g(x)]$ .

Note that

$$\begin{array}{ccccccc} F[x] & \xrightarrow{\sigma} & E[x] & \xrightarrow{\bar{\xi}} & E[x]/(\sigma p(x)) & \xrightarrow{\bar{\varphi}} & E[v] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ f(x) & \longrightarrow & \sigma f(x) & \longrightarrow & [\sigma f(x)] & \longrightarrow & \sigma f(v) \end{array}$$

Since  $\sigma$ ,  $\bar{\varphi}$  and  $\bar{\xi}$  are surjective, so is the composite function.

$$\begin{aligned} \ker \bar{\phi}(\bar{\xi}(\sigma)) &= \{f(x) \in F[x] \mid \sigma f(v) = 0\} \\ &= \{f(x) \in F[x] \mid \sigma f(x) \in \langle \sigma f(x) \rangle\} \\ &= \langle p(x) \rangle \end{aligned}$$

By First isomorphism theorem,

$$F(u) \cong F[x]/\langle p(x) \rangle \cong^\theta E(v)$$

Since  $\theta([f(x)]) = \sigma f(v)$ . Note that

$$\theta([x]) = \sigma \cdot 1_F \cdot v = 1_V \cdot v = v$$

so we have the following situation

$$\begin{array}{ccccc} F(u) & \xleftarrow{\varphi} & F[x]/\langle p(x) \rangle & \xrightarrow{\theta} & E(v) \\ f(u) & \xleftarrow{\varphi} & [f(x)] & \xrightarrow{\theta} & \sigma f(v) \\ c & \xleftarrow{\varphi} & [c] & \xrightarrow{\theta} & \sigma(c) \end{array}$$

The composite function  $\theta \circ \varphi^{-1} : F(u) \rightarrow E(v)$  is an isomorphism that extends  $\sigma$  and maps  $u$  to  $v$ .

$$\begin{array}{ccccc} F & \xrightarrow{\sigma} & E \\ \downarrow \subseteq & & \downarrow \subseteq \\ F[x] & \xrightarrow{\sigma} & E[x] \\ \downarrow \xi & & \downarrow \bar{\xi} \\ F(u) \xrightarrow{\varphi} F[x]/\langle p(x) \rangle & \xrightarrow{\sigma} & E[x]/\langle \sigma p(x) \rangle & \xrightarrow{\bar{\varphi}} & E(v) \end{array}$$

By First isomorphism thm.

□

**Example 3.1.6.**  $x^3 - 2$  is irreducible in  $\mathbb{Q}[x]$  by Eisenstein's criterion.  $\sqrt[3]{2} \in \mathbb{R}$  is a root of it. Verify that  $\sqrt[3]{2}\omega$  is also a root of  $x^3 - 2$  in  $\mathbb{C}$  where

$$\omega = \frac{-1 + \sqrt{3}i}{2}$$

is a complex cube root of 1.

**Solution** Let  $\sigma$  be the identity function from  $\mathbb{Q}$  to  $\mathbb{Q}$ . By applying the previous corollary, we have

$$\mathbb{Q}(\sqrt[3]{2}) \cong^\theta \mathbb{Q}(\sqrt[3]{2}\omega)$$

such that  $\bar{\sigma}(\sqrt[3]{2}) = \sqrt[3]{2}\omega$ . And now  $(\sqrt[3]{2}\omega)^3 = 2\omega^3 = 2$ . ◀

### 3.1.2 Algebraic Extension

#### Definition 3.2 Algebraic extension

An extension field  $K$  of a field  $F$  is said to be an algebraic extension of  $F$  if every element of  $K$  is algebraic over  $F$ .

**Example 3.1.7.**  $\mathbb{C}$  is an algebraic extension of  $\mathbb{R}$ .  $\forall a + bi \in \mathbb{C}$ , where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ . We have

$$(x + a + bi)(x + a - bi) = x^2 + 2ax + a^2 + b^2.$$

Thus  $a + bi$  is a root of  $x^2 + 2ax + a^2 + b^2 = 0$ .

#### Theorem 3.3

If  $K$  is a finite-dimensional extension field of  $F$ , then  $K$  is an algebraic extension of  $F$ .

*Proof.* Let  $\{V_1, V_2, \dots, V_n\}$  be the basis of  $K$  over  $F$ . For all  $u \in K$ ,  $\{1, u, u^2, \dots, u^n\}$  is linearly dependent. That is,

$$\exists u^k \in K \text{ s.t. } u^k = \text{Span}\{1, u, u^2, \dots, u^n\} = c_0 + c_1u + c_2u^2 + \dots + c_{k-1}u^{k-1} (k \geq 1).$$

Thus  $u$  is a root of  $f(x) = x^k - c_{k-1}u^{k-1} - \dots - c_0$ , this implies  $K$  is an algebraic extension.  $\square$

In fact, a simple extension is an algebraic extension if  $u$  is algebraic. If extension field  $K$  contains a transcendental element  $u$ , then  $K$  must be infinite dimensional over  $F$ .

Non algebraic  $\implies$  Infinite dimension

Note that  $F(u)$  denote the intersection of all subfields of  $K$  that contains both  $F$  and  $u$ . It said to be a simple extension of  $F$ . If  $u_1, u_2, \dots, u_n$  are elements of an extension field  $K$  of  $F$ . Let  $F(u_1, \dots, u_n)$  denote the intersection of all the subfields of  $K$  that contain  $F$  and every  $u$  (known as generalized simple extension);  $F(u, u_1, \dots, u_n)$  is said to be a finitely generated extension of  $F$ .

#### Theorem 3.4

If  $K = F(u_1, u_2, \dots, u_n)$  is a finitely generated extension field of  $F$  and each  $u_i$  is algebraic over  $F$ , then  $K$  is a finite-dimensional algebraic extension of  $F$ .

*Proof.* Note that if  $u, v$  is algebraic over  $F$ , then  $v$  is algebraic over  $F(u)$ . Thus

$$|F(u, v) : F(u)| \cdot |F(u) : F| < \infty \implies |F(u, v) : F| = |F(u, v) : F(u)| \cdot |F(u) : F| < \infty.$$

By mathematical induction, we have

$$|F(u_1, u_2, \dots, u_n) : F(u_1, u_2, \dots, u_{n-1})| \dots |F(u_1) : F| < \infty$$

which is also finite.  $\square$

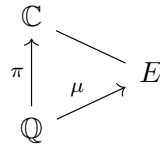
**Corollary 3.2**

If  $L$  is algebraic extension of  $K$  and  $K$  is an algebraic extension of  $F$ . Then  $L$  is an algebraic extension field of  $F$ .

*Proof.*  $\forall \omega \in L, \exists f(x) \in K[x] \text{ s.t. } f(\omega) = a_0 + a_1\omega + \cdots + a_n\omega^n.$

Note that  $F(a_0, a_1, \dots, a_n)$  is finitely generated extension of  $F$  and all  $a_i$ 's are algebraic. Thus it is finite dimensional algebraic extension of  $F$ . Since  $\omega$  is algebraic over  $F(a_0, a_1, \dots, a_n)$ . So  $F(a_0, a_1, \dots, a_n)$  is finite dimensional extension of  $F \implies \omega$  is algebraic over  $F$ . Thus  $L$  is an algebraic extension of  $F$ .  $\square$

**Remark.** Algebraic subfield  $E$  of  $\mathbb{C}$  over  $\mathbb{Q}$  is called the **field of algebraic numbers**. Where  $E$  is a finite-dimensional algebraic extension over  $\mathbb{Q}$ .



- $\mu$  denote algebraic extension over  $\mathbb{Q}$ , e.g.:  $\sqrt{2}, \sqrt{3}, i, \dots$
- $\pi$  denote non-algebraic extension.

**Corollary 3.3**

Let  $K$  be an extension field of  $F$  and let  $E$  be the set of all elements of  $K$  that are algebraic over  $F$ . Then  $E$  is a subfield of  $K$  and an algebraic extension field of  $F$ .

*Proof.* We only need to show that  $E$  is a field. Let  $u, v \in E$ , note that  $F(u, v)$  is finitely generated extension of  $F$ , so  $E$  is algebraic extension.  $E$  is closed under subtraction and multiplication. Moreover  $u^{-1}$  is algebraic over  $F$ . Thus  $E$  is a subfield of  $K$ .  $\square$

**Example 3.1.8.**

$$\mathbb{Q}(i, -i) = \mathbb{Q}(i)$$

**Example 3.1.9.**

$$\mathbb{Q}(\sqrt{3}, i) = \mathbb{Q}(\sqrt{3})(i)$$

**Solution**

$$\begin{aligned} |\mathbb{Q}(\sqrt{3}, i)| &= |\mathbb{Q}(\sqrt{3})(i) : \mathbb{Q}| \\ &= |\mathbb{Q}(\sqrt{3})(i) : \mathbb{Q}(\sqrt{3})| \cdot |\mathbb{Q}(\sqrt{3}) : \mathbb{Q}| \\ &= 2 \cdot 2 \\ &= 4 \end{aligned}$$



**Example 3.1.10.** Every finite-dimensional extension is also finitely generated. If  $\{u_1, u_2, \dots, u_n\}$  is a basis of  $K$  over  $F$ . This implies  $F(u_1, u_2, \dots, u_n) \subseteq K$  and  $K \subseteq F(u_1, u_2, \dots, u_n)$ . Thus,

$$K = F(u_1, u_2, \dots, u_n) = \text{Span}\{u_1, u_2, \dots, u_n\}.$$

**Example 3.1.11** (Non-example).

$$\mathbb{Q}(\sqrt{3}, \sqrt{5}) \neq \mathbb{Q}(\sqrt{3})$$

**Solution** For the sake of contradiction, consider  $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(\sqrt{3})$ , then

$$\sqrt{5} = a + b\sqrt{3}, \quad \forall a, b \in \mathbb{Q}$$

Altering this equation by moving  $a$  to left-hand side, then squaring both sides. We obtain

$$\begin{aligned} (\sqrt{5} - a)^2 &= (b\sqrt{3})^2 \Rightarrow 5 - 2\sqrt{5}a + a^2 = 3b^2 \\ &\Rightarrow \frac{5 + a^2 - 3b^2}{2a} = \sqrt{5} \quad (a \neq 0) \end{aligned}$$

However, when  $a = 0$ , we have  $5 = 3b^2$ . Which is a contradiction. ◀

## 3.2 Splitting Field

In last chapter we had discussed about the integral domain. Suppose polynomial  $f(x)$  has degree  $n$ . Then  $f(x)$  has at most  $n$  roots in *any* field. Suppose that  $K$  contains fewer than  $n$  roots of  $f(x)$ . It might be possible to find an extension field of  $K$  that contains additional roots of  $f(x)$ .

### Definition 3.3 Splitting field

If  $F$  is a field and  $f(x) \in F[x]$ , then an extension field  $K$  of  $F$  is said to be a **splitting field** (or **root field**) of  $f(x)$  over  $F$  provided that

- $f(x)$  splits over  $K$ , say

$$f(x) = c(x - u_1) \dots (x - u_n) \tag{3.2}$$

- and

$$K = \underbrace{F(u_1, u_2, \dots, u_n)}_{\text{smallest field}}. \tag{3.3}$$

**Example 3.2.1.** If  $f(x) = x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1)$  in  $\mathbb{Q}[x]$ . Then

$$\mathbb{Q}(\sqrt{2}, -\sqrt{2}, i, -i) = \mathbb{Q}(\sqrt{2}, i)$$

is a splitting field of  $f(x)$  over  $\mathbb{Q}$ .

### 3.3 Finite Fields

#### Theorem 3.5

Let  $R$  be a ring with identity. Then

1. The set

$$\mathfrak{P} = \{k \cdot 1_R \mid k \in \mathbb{Z}\}$$

is a subring of  $R$ .

2. If  $R$  has characteristic 0, then  $\mathfrak{P}$  is isomorphic to  $\mathbb{Z}$ .
3. If  $R$  has characteristic  $n > 0$ , then  $\mathfrak{P}$  is isomorphic to  $\mathbb{Z}_n$ .

*Proof.* We prove each of the statements listed above.

1. First of all, we use subring test to check if  $\mathfrak{P}$  is a subring of  $R$ .

$$\begin{cases} a \cdot 1_R - b \cdot 1_R = (a - b) \cdot 1_R \in \mathfrak{P} \\ a \cdot 1_R \cdot b \cdot 1_R = ab \cdot 1_R \in \mathfrak{P} \end{cases}$$

so  $\mathfrak{P}$  is a subring of  $R$ .

We now prove (2), (3) at once. We consider a map  $f : \mathbb{Z} \rightarrow R$  defined by

$$f(n) = n \cdot 1_R \quad \forall n \in \mathbb{Z}.$$

Then  $f$  is homomorphism because

$$f(n + m) = (n + m) \cdot 1_R = f(n) + f(m)$$

and the kernel is

$$\ker f = \{n \in \mathbb{Z} \mid n \cdot 1_R = 0_R\}.$$

By the first isomorphism theorem,  $\mathbb{Z}/\ker f$  is isomorphic to  $\mathbb{R}$ .

- If  $R$  has a characteristic 0, then  $\ker f = \langle 0 \rangle \implies \mathbb{Z} \cong \mathbb{R}$ .
- If  $R$  has a characteristic  $n$ , then  $\ker f = \langle n \rangle \implies \mathbb{Z}/\langle n \rangle \cong \mathbb{R}$ .

□

#### Corollary 3.4

Every finite field has characteristic  $p$  for some prime  $p$ .

*Proof.* Since finite field is integral domain, it must has characteristic prime  $p$ .

□

**Remark.** 1. The converse of it is false. There are infinite fields of characteristic  $p$  (i.e.  $\mathbb{Z}_p[x]$ ).

$$\mathfrak{P} = \{k \cdot 1_R \mid k \in \mathbb{Z}\}$$

is called the prime subfield of  $K$  and is contained in every subfield of  $K$ . Which means every field of characteristic  $p$  must contains  $\mathbb{Z}_p$ .

2. The number of elements in a finite field  $K$  is called the order of  $K$ .

### 3.3.1 Order of finite field

#### Theorem 3.6

A finite field  $K$  has order  $p^n$ , where  $p$  is the characteristic of  $K$  and  $n = |K : \mathbb{Z}_p|$ .

*Proof.* Let  $K$  be a finite dimensional extension of  $\mathbb{Z}_p$ . Let  $n = |K : \mathbb{Z}_p|$ , then  $\{u_1, u_2, \dots, u_n\}$  is a basis of  $K$ .

$\forall k \in K$ ,  $k$  is represented uniquely be

$$k = c_1 u_1 + c_2 u_2 + \dots + c_n u_n.$$

There are precisely  $p^n$  distinct linear combinations of the form. Thus  $|K| = p^n$ .  $\square$

#### Lemma 3.1 The Freshman's dream

Let  $R$  be a commutative ring with identity of characteristic  $p$ , where  $p$  is a prime. Then for every  $a, b \in R$  and for all positive integer  $n$  we have

$$(a + b)^{p^n} = a^{p^n} + b^{p^n}. \quad (3.4)$$

*Proof.* We will use the induction on  $n$ .

Assume  $n = 1$ , we expand  $(a + b)^p$  with binomial theorem.

$$(a + b)^p = \sum_{k=0}^p \binom{p}{k} a^{p-k} b^k = a^p + \binom{p}{1} a^{p-1} b + \dots + \binom{p}{p-1} a b^{p-1} + b^p.$$

Note that

$$\binom{p}{k} = \frac{p!}{(p-k)! k!}, \quad k, p-k < p \text{ for } 1 \leq k < p.$$

This implies that  $p$  divide  $\binom{p}{k} \implies \binom{p}{k} a^{p-k} b^k = 0 \pmod{p}$ . Thus  $(a + b)^p = a^p + b^p$ . We are done for base case.

Assume that it holds for all less than  $n$ .

$$\begin{aligned} (a + b)^{p^n} &= ((a + b)^p)^{p^{n-1}} \\ &= (a^p + b^p)^{p^{n-1}} \\ &= (a^p)^{p^{n-1}} + (b^p)^{p^{n-1}} \\ &= a^{p^n} + b^{p^n}. \end{aligned}$$

Therefore the theorem is true for every positive integer  $n$ . Now we are done.  $\square$

#### Theorem 3.7 Existence of finite field

Let  $K$  be an extension field  $\mathbb{Z}_p$ . For all positive integer  $n$ ,  $K$  has order  $p^n$  if and only if  $K$  is a splitting field of  $x^{p^n} - x$  over  $\mathbb{Z}_p$ .



*Proof.* ( $\Rightarrow$ ) Let  $f(x) = x^{p^n} - x$  over  $\mathbb{Z}_p$ . Then

$$f(x) = p^n x^{p^n-1} - 1 = -1$$

and so  $f(x)$  and  $f'(x)$  are relatively prime  $\Rightarrow f(x)$  is separable. Then  $f(x)$  has  $p^n$  distinct roots in splitting field. Let  $c$  be any nonzero element of  $K$ . Let  $\{c_1, c_2, \dots, c_{p^n-1}\}$  be all the nonzero elements of  $K$  and let

$$u = c_1 c_2 \cdots c_k.$$

Note that  $\{cc_1, cc_2, \dots, cc_k\}$  are all distinct and nonzero elements of  $K$ . Thus

$$\{cc_1, cc_2, \dots, cc_k\} = \{c_1, c_2, \dots, c_{p^n-1}\}$$

and

$$cc_1 \cdot cc_2 \cdots cc_{p^n-1} = c_1 \cdot c_2 \cdots c_{p^n-1} \Rightarrow c^{p^n-1} = 1_K \Rightarrow c^{p^n} - c = 0.$$

So all nonzero elements of  $K$  consisting of the  $p^n$  distinct roots of  $x^{p^n} - x$  over  $\mathbb{Z}_p$ .

( $\Leftarrow$ ) On the other hand, let  $E$  be the subset of  $K$  consisting of the  $p^n$  distinct roots of  $x^{p^n} - x$ . We want to show  $E$  is a subfield of  $K$ .

For all  $a, b$  in  $E$ ,  $a^{p^n} = a$  and  $b^{p^n} = b$ . So

$$(a+b)^{p^n} = a^{p^n} + b^{p^n} = a + b \Rightarrow a + b \in E$$

$$(ab)^{p^n} = a^{p^n} b^{p^n} = ab \Rightarrow ab \in E$$

$$(-a)^{p^n} = (-1)^{p^n} a^{p^n} = -a \Rightarrow -a \in E$$

and

$$(a^{-1})^{p^n} = (a^{p^n})^{-1} = a^{-1}.$$

Clearly  $0, 1 \in E$ . Since  $K$  is a field, the rest of axioms are true. Since  $K$  is splitting field of  $x^{p^n} - x$ ,  $K \subseteq E$ . Hence  $K = E$  and  $|K| = |E| = p^n$ .  $\square$

### 3.4 Galois Theory

The classical question of algebra : Whether or not there were formulas for the solution of higher degree polynomial equations. There are no formula for the solution of all polynomial equations of degree  $n$  when  $n \geq 5$ .

Galois theory had a profound influence on the development of later mathematics for beyond the scope of the original solvability problem. Why? Because his theory connecting the field extensions with groups.

#### Definition 3.4 Galois group

Let  $K$  be an extension field of  $F$ . An  $F$ -automorphism of  $K$  is an isomorphism  $\sigma : K \rightarrow K$  that fixes  $F$  elementwise. That is,  $\sigma(c) \rightarrow c, \forall c \in F$ .

The set of all  $F$ -automorphisms of  $K$  is denoted  $\text{Gal}_F K$  and is called the Galois group of  $K$  over  $F$ .

#### Theorem 3.8

If  $K$  is an extension field of  $F$ , then  $\text{Gal}_F K$  is a group under the operation of composition of functions.

*Proof.* We need to show that the function composition from a group:

[Closedness]:  $\forall \sigma, \tau \in \text{Gal}_F K, \sigma \circ \tau(c) = c \forall c \in F$ .

[Associativity]: Associativity holds because of function composition's property.

[Existence of identity]:  $\forall \sigma \in \text{Gal}_F K, \sigma \circ \tau = \tau \circ \sigma = \sigma$ .

[Existence of inverse]:  $\forall \sigma \in \text{Gal}_F K, \sigma^{-1}(c) = c \forall c \in F$ . So  $\sigma^{-1}$  is the inverse in  $\text{Gal}_F K$ .  $\square$

### Theorem 3.9

Let  $K$  be an extension field of  $F$  and  $f(x) \in F[x]$ . If  $u \in K$  is a root of  $f(x)$  and  $\sigma \in \text{Gal}_F K$ , then  $\sigma(u)$  is also a root of  $f(x)$ .

*Proof.* Take a  $u \in K, f(u) = 0 \implies \sigma(f(u)) = \sigma(0) = 0$ .

Since  $\sigma \in \text{Gal}_F K, \sigma(f(u)) = f(\sigma(u)) = 0$ .  $\square$

**Remark.** Every root of  $p(x)$  in  $K$  is the image of  $u$  under some automorphism of  $\text{Gal}_F K$ .

### Theorem 3.10

Let  $K$  be the splitting field of some polynomial over  $F$  and  $u, v \in K$ . Then there exists an  $\sigma \in \text{Gal}_F K$  such that  $\sigma(u) = v$  if and only if  $u$  and  $v$  have the same minimal polynomial.

*Proof.* ( $\implies$ ) Note that if  $K$  is splitting field of  $f(x)$  with  $\deg f = n$ . Then

$$|K : F| \leq n! \implies K \text{ is algebraic over } F.$$

So  $u$  has a minimal polynomial  $q(x)$  over  $F$ .

By previous theorem,  $p(v) = 0$  and  $q(u) = 0$ . These imply  $p(x)|q(x)$  and  $q(x)|p(x)$  and hence  $p(x) = q(x)$ . We are done for this direction.

( $\impliedby$ ) On the other hand, in simple extension, we can extend an isomorphism  $\sigma : F \rightarrow F$  to  $\bar{\sigma} : \bar{F} \rightarrow \bar{F}$  such that  $\bar{\sigma}(u) = v$  and  $\bar{\sigma}(c) = c$  for all constant  $c \in F$ .

Since  $K$  is splitting field of some polynomial over  $F$ , which is also a splitting field of  $F(u)$  and  $F(v)$ . We can extend an isomorphism  $\bar{\sigma}$  to an isomorphism  $\tilde{\sigma} : K \rightarrow K$ . If  $\sigma = 1$ , then  $\tilde{\sigma} \in \text{Gal}_F K$ .  $\square$

**Example 3.4.1.** Given that  $\sigma \in \text{Gal}_{\mathbb{R}} \mathbb{C}$ , find  $\sigma(i)$ .

**Solution** By previous theorem,

$$\sigma(i) = \{i, -i\}.$$

Thus  $\text{Gal}_F K = \{1, \sigma\}$  is a group of order 2 and hence  $\text{Gal}_{\mathbb{R}} \mathbb{C}$  is isomorphic to  $\mathbb{Z}_2$ .  $\blacktriangleleft$

### Theorem 3.11 Galois group of finitely generated extension

Let  $K = F(u_1, u_2, \dots, u_n)$  be an algebraic extension field of  $F$ . If  $\sigma \circ \tau \in \text{Gal}_F K$  and  $\sigma(u_i) = \tau(u_i)$  for each  $i = 1, 2, \dots, n$ . Then  $\sigma = \tau$ .

In other words, an automorphism in  $\text{Gal}_F K$  is completely determined by its action on  $u_1, u_2, \dots, u_n$ .

*Proof.* Let  $\beta = \sigma^{-1} \circ \tau$ , then

$$\sigma^{-1} \circ \tau(u_i) = \sigma^{-1} \circ \sigma(u_i) = u_i.$$

Let  $v \in F(u_i)$ , then there exists a constant  $c_i \in F$  such that

$$c_0 + c_1 u_1 + \cdots + c_{n-1} u_1^{n-1} = w \implies \beta(v) = v.$$

Again we let  $w \in F(u_1, u_2)$ , then there exists a constant  $c_i \in F$  such that

$$c_0 + c_1 u_2 + \cdots + c_{n-1} u_2^{n-1} = w \implies \beta(w) = w.$$

Repeating this process, we conclude that for all  $v \in K$ ,  $\beta(v) = v \implies \beta = 1$ . Thus  $\tau = \sigma$ .  $\square$

**Example 3.4.2.** Consider  $\sigma \in \text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt{3}, \sqrt{5}]$ . We let two actions

$$\tau(\sqrt{3}) = -\sqrt{3}, \quad \tau(\sqrt{5}) = \sqrt{5}$$

and

$$\alpha(\sqrt{3}) = \sqrt{3}, \quad \alpha(\sqrt{5}) = -\sqrt{5}$$

and defined  $\beta = \alpha \circ \tau$ .

Then  $\text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt{3}, \sqrt{5}] = \{1, \tau, \alpha, \beta\}$  such that

	1	$\tau$	$\alpha$	$\beta$
$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$
$\sqrt{5}$	$\sqrt{5}$	$\sqrt{5}$	$-\sqrt{5}$	$-\sqrt{5}$

Note that  $\tau, \alpha, \beta$  both have an order 2. Thus  $\text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt{3}, \sqrt{5}] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

### Corollary 3.5

If  $K$  is splitting field of a separable polynomial  $f(x)$  of degree  $n$  in  $F[x]$ , then  $\text{Gal}_F K$  is isomorphic to a subgroup of  $S_n$ .

*Proof.* Let  $u_1, u_2, \dots, u_n$  be distinct roots of  $f(x)$ . Then  $K = F(u_1, u_2, \dots, u_n)$  and let

$$\mathcal{U} = \{u_1, u_2, \dots, u_n\}.$$

For all permutation  $\sigma \in \text{Gal}_F K$ ,  $\forall 1 \leq i \leq n$ ,  $\sigma(u_i) = u_j$  for some  $j$ .

Now define a function  $\theta : \text{Gal}_F K \rightarrow S_n$  defined by  $\theta : \sigma \mapsto \sigma \circ 1_{\mathcal{U}}$ .  $\theta$  is well defined. Note that

$$\sigma \circ 1_{\mathcal{U}} = \tau \circ 1_{\mathcal{U}} \implies \sigma = \tau$$

by previous theorem. Thus  $\theta$  is injective, and  $\theta$  is homomorphism.

By the first isomorphism theorem, we say that

$$\text{Gal}_F K \cong \text{Im } \theta = \text{a subgroup of } S_n.$$

$\square$

**Definition 3.5 Intermediate field**

Let  $K$  be an extension field of  $F$ . A field  $E$  such that

$$F \subseteq E \subseteq K$$

is called an **intermediate field** of the extension. Clearly that  $\text{Gal}_E K \cong \text{Gal}_F K$ .

**Theorem 3.12**

Let  $K$  be an extension field of  $F$ . If  $H$  is a subgroup of  $\text{Gal}_F K$ , let

$$E_H = \{k \in K \mid \sigma(k) = k \text{ for every } \sigma \in H\}.$$

Then  $E_H$  is an intermediate field of the extension  $K$ . The field  $E_H$  is called the "fixed field" of subgroup  $H$ .

*Proof.* It is clear that  $F \subseteq E_H \subseteq K$ . So we only want to show that  $E_H$  is a subfield.

Let  $a, b \in E_H$ . For all  $\sigma \in H$ , we have

$$\sigma(a + b) = \sigma(a) + \sigma(b) = a + b$$

$$\sigma(ab) = \sigma(a)\sigma(b) = ab$$

$$\sigma(0_K) = 0_K, \sigma(1_K) = 1_K$$

$$\sigma(-a) = -\sigma(a) = -a$$

$$\sigma(a^{-1}) = (\sigma(a))^{-1} = a^{-1}$$

Thus  $E_H$  is a subfield of  $K$ . □

**Example 3.4.3.** From previous example,

$$\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{3}] \subseteq \mathbb{Q}[\sqrt{3}, \sqrt{5}].$$

And  $\text{Gal}_{\mathbb{Q}[\sqrt{3}]} \mathbb{Q}[\sqrt{3}, \sqrt{5}] = \{1, \alpha\}$ , and  $\text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt{3}, \sqrt{5}] = \{1, \tau, \alpha, \beta\}$ .

**Example 3.4.4.**  $\text{Gal}_{\mathbb{Q}[\sqrt{3}]} \mathbb{Q}[\sqrt{3}, \sqrt{5}] = \{1, \alpha\}$  is the fixed field of  $\mathbb{Q}[\sqrt{3}] = \{1, \alpha\}$ , where

$$\alpha(\sqrt{3}) = \sqrt{3}, \alpha(\sqrt{5}) = -\sqrt{5}.$$

**Solution** For all  $a_i \in \mathbb{Q}$ . Compute

$$\begin{aligned} \alpha(a_0 + a_1\sqrt{3} + a_2\sqrt{5} + a_3\sqrt{15}) &= a_0 + a_1\sqrt{3} + a_2\sqrt{5} + a_3\sqrt{15} \\ \iff a_0 + a_1\sqrt{3} - a_2\sqrt{5} - a_3\sqrt{15} &= a_0 + a_1\sqrt{3} + a_2\sqrt{5} + a_3\sqrt{15} \\ \iff a_2\sqrt{5} + a_3\sqrt{15} &= 0 \\ \iff a_2 + a_3\sqrt{3} &= 0 \end{aligned}$$

Since  $a_2a_3^{-1} \in \mathbb{Q} \implies -\sqrt{3} \in \mathbb{Q}$ , which is a contradiction. ◀

**Example 3.4.5.**  $\text{Gal}_{\mathbb{R}} \mathbb{C} = \{1, \alpha\}$ , and  $\alpha(a + bi) \implies a - bi = a + bi \implies b = 0$ . Thus fixed field of  $\text{Gal}_{\mathbb{R}} \mathbb{C}$  is the field  $\mathbb{R}$ .

**Remark.** The ground field  $F$  need not always be the fixed field of the group  $\text{Gal}_F K$ .

**Example 3.4.6.**  $\sqrt[3]{2}$  is the root of  $x^3 - 2 = 0$ . So  $\sqrt[3]{2} \mapsto \{\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2\}$ , where  $\omega$  is the cube root of unity.

However,  $\sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2 \notin \mathbb{Q}[\sqrt[3]{2}]$ .  $\forall \sigma \in \text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$ ,  $\sigma(\sqrt[3]{2}) = \sqrt[3]{2}$ . Thus  $\sigma = 1$ . The fixed field of  $\text{Gal}_{\mathbb{Q}} \mathbb{Q}[\sqrt[3]{2}]$  is  $\mathbb{Q}[\sqrt[3]{2}]$ .

### 3.4.1 Fundamental Theorem of Galois theory

#### Definition 3.6 Galois correspondence

Let  $K$  be a finite-dimensional extension field of  $F$ , and let  $S$  be the set of all intermediate fields. Again we let  $T$  be the set of all subgroups of the Galois group  $\text{Gal}_F K$ .

Define a map  $\phi : T \rightarrow S$  by this rule. For each intermediate field  $E$ ,

$$\phi(E) = \text{Gal}_E K. \quad (3.5)$$

This function  $\phi$  is called the Galois correspondence.

$$\begin{array}{ccc} \text{Gal}_K K & \longrightarrow & K \\ \uparrow T & & \downarrow S \\ \text{Gal}_F K & \longrightarrow & F \end{array}$$

**Example 3.4.7.**

$$\mathbb{Q} \rightarrow \text{Gal}_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}, \sqrt{5}) = \{1, \tau, \alpha, \beta\}$$

#### Lemma 3.2

Let  $K$  be a finite-dimensional extension field of  $F$ . If  $H$  is a subgroup of the Galois group  $\text{Gal}_F K$  and  $E$  is the **fixed field** of  $H$ , then  $K$  is *simple, normal, separable extension* of  $E$ .

$$\begin{array}{ccc} \text{Gal}_F K & & K \\ \downarrow & & \downarrow \\ H & \longrightarrow & E_H \end{array}$$

*Proof.* Since  $K$  is finite-dimensional extension field, so  $K$  is algebraic over  $F$ . Let  $\mathfrak{U} \in K$  and  $p(x) \in E[x]$  be minimal polynomial of  $\mathfrak{U}$ , and  $\forall \sigma \in H$ ,  $\sigma(\mathfrak{U})$  is some root of  $p(x)$ .

Therefore,  $\mathfrak{U}$  has a finite number of distinct images under automorphisms in  $H$ , said

$$\mathfrak{U} = u_1, u_2, \dots, u_t \in K, \quad \text{where } t \leq \deg p(x)$$

If  $\sigma \in H$  and  $u_i = \tau(\mathfrak{U})$  with  $\tau \in H$ , then  $\sigma(u_i) = \sigma \circ \tau(\mathfrak{U})$ .

Since  $\sigma$  is injective, so

$$\mathfrak{U} \xrightarrow{H} \{u_1, u_2, \dots, u_t\}$$

which  $\{u_1, u_2, \dots, u_t\}$  is image of  $\mathfrak{U}$ . And  $u_i = \tau(\mathfrak{U})$  for some  $\tau \in H$ . is injective

$$\{u_1, u_2, \dots, u_t\} \xrightarrow{\text{permutation } \sigma} \{u_1, u_2, \dots, u_t\}$$

Every automorphism in  $H$  permutes  $u_1, u_2, \dots, u_t$ . Let

$$f(x) = (x - u_1)(x - u_2) \dots (x - u_t)$$

Since all  $u_i$ 's are distinct,  $f(x)$  is separable.

Now we claim that  $f(x) \in E[x]$ . Note that  $\sigma f(x) = f(x)$  for all  $\sigma \in H$ . All coefficients of  $f(x)$  is fixed by  $\sigma \in H$ . Thus  $f(x) \in E[x]$ . Since  $u = u_1$  is a root of  $f(x) \in E[x]$ ,  $u$  is separable over  $E$ .  $\implies K$  is separable extension of  $E$ .

From the previous theorem,  $K = E(V)$  for some  $V \in K$ . Let  $g(x) = (x - v)(x - v_2) \dots (x - v_s)$  where  $\{v_1 = v, v_2, \dots, v_s\}$  are images of  $\sigma \in H$ . Similarly,  $g(x) \in E[x]$  and  $K = E(v)$  is splitting field of  $g(x)$ . Therefore  $K$  is normal extension of  $E$ .  $\square$

### Definition 3.7 Galois extension

If  $K$  is a finite dimensional, normal, separable extension field of the field  $F$ , we say that  $K$  is a Galois extension of  $F$  or that  $K$  is Galois over  $F$ .

### Corollary 3.6

Let  $K$  be a finite dimensional extension field of  $F$ . Then  $K$  is Galois over  $F$  if and only if  $F$  is fixed field of the Galois group over  $F$ .

*Proof.*  $(\implies)$  By previous theorem,  $E = F$ . Thus  $F$  is fixed field of the Galois group over  $F$ .

$(\impliedby)$  By lemma,  $K$  is simple, normal, separable extension of  $F$ .  $\square$

### Theorem 3.13 Fundamental theorem of Galois Theory

If  $K$  is a Galois extension field of  $F$ ,

1. There is a bijection between the set  $S$  of all the intermediate fields of the extension and the set  $T$  of all subgroups of the Galois group  $\text{Gal}_F K$ , given by assigning each intermediate field  $E$  to the subgroup  $\text{Gal}_E K$ . Furthermore

$$|K : E| = |\text{Gal}_E K|$$

and

$$|E : F| = |\text{Gal}_F K : \text{Gal}_E K|.$$

2. An intermediate field  $E$  is normal extension of  $F$  if and only if the corresponding group  $\text{Gal}_E K$  is a normal subgroup of  $\text{Gal}_F K$ .

*Proof.* 1. By theorem, fixed field of  $\text{Gal}_E K$  is  $E$ . By theorem we have

$$|\text{Gal}_E K| = |K : E|.$$

Similarly,

$$|\text{Gal}_F K| = |K : F|.$$

In fact  $|K : F| = |K : E| |E : F|$ . Thus

$$|\text{Gal}_F K| = |\text{Gal}_E K| \cdot |E : F| \implies |E : F| = |\text{Gal}_F K : \text{Gal}_E K|.$$

2. ( $\Leftarrow$ ) Assume first that  $\text{Gal}_E K$  is a normal subgroup of  $\text{Gal}_F K$ . If  $p(x)$  is an irreducible polynomial in  $F[x]$  with a root  $u \in E$ , we must show that  $p(x)$  splits in  $E[x]$ .

Since  $K$  is normal over  $F$ . We know that  $p(x)$  splits in  $K[x]$ . So we know that  $p(x)$  also splits in  $E[x]$ . So we need to show that each root  $v$  of  $p(x)$  in  $K$  is actually in  $E$ . There is an automorphism  $\sigma \in \text{Gal}_F K$  such that  $\sigma(u) = v$  by theorem. So  $\text{Gal}_F K = |K : F| = E$ .

$$\begin{array}{ccc} K = F(u) & \xrightarrow{\sigma} & F(u) = K \\ \downarrow & & \downarrow \\ F & \xrightarrow{1} & F \end{array}$$

Now if  $\tau$  is any element of  $\text{Gal}_E K$ , then normality implies

$$\tau \circ \sigma = \sigma \circ \tau' \quad \text{for some } \tau' \in \text{Gal}_E K.$$

Since  $u \in E$ , we have

$$\begin{aligned} \tau(v) &= \tau(\sigma(u)) \\ &= \sigma(\tau'(u)) \\ &= \sigma(u) \\ &= v. \end{aligned}$$

So  $\text{Gal}_E K$  fixes other roots for all  $v \in E$ . Thus  $E$  is normal extension of  $F$ .

( $\Rightarrow$ ) Assume that  $E$  is normal subgroup of  $F$ . Then there exists a surjective homomorphism of groups  $\theta : \text{Gal}_F K \rightarrow \text{Gal}_F E$  whose kernel is  $\text{Gal}_E K$ . Then  $\text{Gal}_E K$  is normal subgroup of  $\text{Gal}_F K$ .

Therefore by First isomorphism theorem,  $\text{Gal}_F E \cong \text{Gal}_F K / \text{Gal}_E K$ .

□

### Lemma 3.3

Let  $K$  be a finite-dimensional normal extension field of  $F$  and  $E$  an intermediate field which is normal over  $F$ . Then there is a surjective homomorphism of groups

$$\theta : \text{Gal}_F K \rightarrow \text{Gal}_F E$$

where kernel of  $\theta$  is  $\text{Gal}_E K$ .

*Proof.* Let  $\sigma \in \text{Gal}_F K$  and an  $u \in E$ . Then  $u$  is algebraic over  $F$  with minimal polynomial  $p(x)$ .

Since  $E$  is normal, all roots of  $p(x)$  are in  $E$ . Since  $\sigma(u)$  is also root of  $p(x)$ ,  $\sigma(u) \in E$ . Therefore

$$\sigma(E) \subseteq E \quad \forall \sigma \in \text{Gal}_F K.$$

Thus we can restrict  $\sigma$  to  $E$  and  $\sigma \circ 1_E$  is an  $F$ -isomorphism that is  $E \cong \sigma(E)$ . Hence,

$$|E : F| = |\sigma(E) : F|.$$

Since  $K$  is splitting field over  $F$ ,  $K$  is also splitting field over  $E$ .  $\forall \sigma \in \text{Gal}_F E$ .  $\sigma$  can be extended to an  $F$ -automorphism in  $\text{Gal}_F K$ . Its kernel consists of the automorphisms of  $K$  whose restriction to  $E$  is the identity map as  $\text{Gal}_E K$ .  $\square$

**Example 3.4.8.** Let  $K$  be the splitting field of  $x^3 - 2$ . Note that

$$\mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{2}] \subseteq K$$

and  $|\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}| = 3$ .  $K$  is Galois extension of  $\mathbb{Q}$  such that

$$|\text{Gal}_{\mathbb{Q}} K| = |K : \mathbb{Q}|$$

and  $\text{Gal}_{\mathbb{Q}} K$  is isomorphic to a subgroup of  $S_3$ .

$$\begin{aligned} 3 < |K : \mathbb{Q}| \leq 6 &\implies 3 < |K : \mathbb{Q}[\sqrt[3]{2}]| \cdot |\mathbb{Q}[\sqrt[3]{2}] : \mathbb{Q}| \leq 6 \\ &\implies 3 < 3 |K : \mathbb{Q}[\sqrt[3]{2}]| \leq 6 \\ &\implies 1 < |K : \mathbb{Q}[\sqrt[3]{2}]| \leq 2. \end{aligned}$$

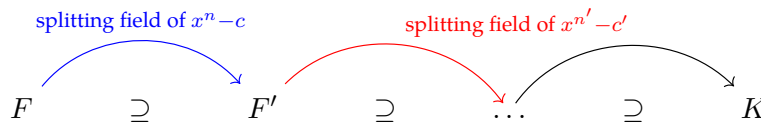
We must have  $|K : \mathbb{Q}| = 6$  and  $\text{Gal}_{\mathbb{Q}} K \cong S_3$ .

$\mathbb{Q}[\omega]$  is the splitting field of  $x^2 + x + 1$  and hence  $\mathbb{Q}[\omega]$  is normal and separable. Thus  $K$  is Galois over  $\mathbb{Q}$ .

## 3.5 Solvability by Radicals

We shall assume that all fields have characteristic 0. A "formula" is a specific procedure that starts with coefficients of the polynomial  $f(x) \in F[x]$  and arrives at the solutions of equation  $f(x) = 0_F$  by using only the field operations  $(+_F, -_F, \times_F, \div_F)$  and the extraction of roots (such as  $\sqrt[n]{\cdot}$ ).

In this context, an  $n$ -th root of an element  $c$  in field  $F$  is any root of the polynomial  $x^n - c$  in some extension field of  $F$ . If that "formula" really exists, then there exists an extension field  $K$  of  $F$  such that



**Example 3.5.1.** The solutions of  $x^3 + 3x + 2 = 0 \in \mathbb{Q}[x]$  are

$$\boxed{\sqrt[3]{-1 + \sqrt{2}} + \sqrt[3]{-1 - \sqrt{2}}} \quad \boxed{\omega \sqrt[3]{-1 + \sqrt{2}} + (\omega^2) \sqrt[3]{-1 - \sqrt{2}}} \quad \boxed{(\omega^2) \sqrt[3]{-1 + \sqrt{2}} + \omega \sqrt[3]{-1 - \sqrt{2}}}$$

**Definition 3.8** Radical extensions



A field  $K$  is said to be a radical extension of a field  $F$  if there is a chain of fields

$$F = F_0 \subseteq F_1 \subseteq \dots \subseteq F_t = K$$

such that for each  $i = 1, 2, \dots, t$ ,  $F_i = F_{i-1}(u_i)$  and some power of  $u_i$  is in  $F_{i-1}$  ( $u_i^{K_i} \in F_{i-1}$ ). The equation  $f(x) = 0_F$  is said to be solvable by radicals if there is a radical extension of  $F$  that contains a splitting field of  $f(x)$ . If not, then  $f(x)$  is not solvable by radical.

### Definition 3.9 Solvable groups

A group  $G$  is said to be solvable if it has a "chain" of subgroups

$$G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_{n-1} \triangleright G_n = \langle e \rangle$$

and  $G_{i-1}/G_i$  is abelian.

**Example 3.5.2.** Every abelian group  $G$  is solvable.  $G \supseteq \langle e \rangle$  and  $G/\langle e \rangle \cong G$  is abelian.

**Solution** Since  $S_3 \triangleright \langle (123) \rangle$ , and

$$(12) \langle (123) \rangle = \{(12), (13), (23)\}$$

$$\langle (123) \rangle (12) = \{(12), (23), (13)\}$$

◀

### Theorem 3.14

For  $n \geq 5$  the group  $S_n$  is not solvable.

*Proof.* For the sake of contradiction, suppose that  $S_n$  is solvable and that

$$S_n = G_0 \supseteq G_1 \supseteq \dots \supseteq G_t = \langle 1 \rangle.$$

Let  $(rst)$  be any 3-cycle in  $S_n$  and let  $\alpha, \beta$  be any element of  $\{1, 2, \dots, n\}$  other than  $r, s, t$  (they always exist since  $n \geq 5$ ). Since  $S_n/G_1$  is abelian, by theorem of dihedral group,

$$\begin{aligned} (t\alpha s)(sr\beta)(t\alpha s)^{-1}(sr\beta)^{-1} &= (t\alpha s)(sr\beta)(ts\alpha)(s\beta r) \\ &= (rst) \in G_1 \end{aligned}$$

Note that the cycle  $\langle (rst) \rangle \subseteq G_1$ , and  $G_1$  definitely contains all the 3-cycles since  $G_1/G_2$  is abelian, repeating upper process,  $G_2$  also contains  $\langle (rst) \rangle$ .

In conclusion,  $\forall i \in \{1, 2, \dots, n\}$ ,  $G_i$  contains all the 3-cycles. This contradicting the fact that  $S_n$  is solvable.  $\square$

### Theorem 3.15

Every homomorphic image of a solvable group  $G$  is solvable.

*Proof.* Let  $G$  be a solvable group. Then there exists a chain of groups

$$G = G_0 \supseteq G_1 \dots \supseteq G_n = \langle e \rangle$$

such that for all  $i$ ,  $G_{i-1} \triangleright G_i$  and  $G_{i-1}/G_i$  is abelian.

Consider  $f$  is the homomorphism of  $G$ . Then  $f(G_i)$  is also a group, and the chain of group is now

$$f(G) = f(G_0) \supseteq f(G_1) \dots \supseteq f(G_n) = \langle e \rangle$$

and  $aba^{-1}b^{-1} \in G_i$  whenever  $a, b$  in  $G_{i-1}$ . This implies

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in f(G_i) \quad \forall a, b \in G_{i-1}$$

and

$$\forall c, d \in f(G_{i-1}), \exists a, b \in G_{i-1} \quad \text{s.t. } f(a) = c \text{ and } f(b) = d.$$

So we have

$$f(a)f(b)f(a)^{-1}f(b)^{-1} = cdc^{-1}d^{-1} \in f(G_i).$$

Therefore  $f(G_{i-1})/f(G_i)$  is abelian. Hence  $f(G)$  is solvable group.  $\square$

### Definition 3.10

A generator of this cyclic group of  $n$ -th roots of unity in  $K$  is called a primitive  $n$ -th root of unity.

This definition states that  $\zeta$  is a primitive  $n$ -th roots of unity iff  $\zeta, \zeta^2, \dots, \zeta^n$  are the  $n$  distinct  $n$ -th roots of unity.

**Example 3.5.3.** Consider  $x^4 - 1 \in \mathbb{Q}[x]$ . The 4-th roots of unity in  $\mathbb{C} = \{1, -1, i, -i\} = \langle i \rangle$ .  $i$  and  $-i$  are primitive 4-th root of unity in  $\mathbb{C}$ .

**Example 3.5.4.** According to De Moivre's theorem,

$$\cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$$

is a primitive  $n$ -th root of unity in  $\mathbb{C}$ .

### Theorem 3.16

Every homomorphic image of a solvable group  $G$  is solvable.

*Proof.* Let  $G$  be a solvable group. Then there exists a chain of groups

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = \langle e \rangle$$

such that  $G_{i-1} \triangleright G_i$  and  $G_{i-1}/G_i$  is abelian  $\forall i$ .

Let  $f$  be the homomorphism of  $G$ , then  $f(G_i)$  is also a group. The chain of group is now

$$f(G) = f(G_0) \supseteq f(G_1) \supseteq \dots \supseteq f(G_n) = \langle e \rangle$$

and  $\forall a, b \in G_{i-1}$ , we have  $aba^{-1}b^{-1} \in G_i$ . And

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in f(G_i).$$

And for all  $c, d$  in  $f(G_i)$ , there exists some  $a, b \in G_i$  such that  $f(a) = c$  and  $f(b) = d$ . So

$$f(a)f(b)f(a)^{-1}f(b)^{-1} = cdc^{-1}d^{-1} \in f(G_i).$$

Thus  $f(G_{i-1})/f(G_i)$  is abelian. Thus  $f(G)$  is solvable group.  $\square$

**Lemma 3.4**

Let  $F$  be a field and  $\zeta$  a primitive  $n$ -th root of unity of  $F$ . Then  $F$  contains a primitive  $d$ -th root of unity for every positive divisor  $d$  of  $n$ .

*Proof.* Because  $\zeta$  is a primitive  $n$ -th root of unity of  $F$ , that is,  $\zeta^n = 1_F$ . If  $\zeta$  has order  $n$  and  $n = dk$ ,  $\zeta^k$  has order  $d$ .

Note that  $\{\zeta^k, \zeta^{2k}, \dots, \zeta^{nk}\}$  are all distinct and roots of  $x^d - 1_F$ . Thus  $\zeta^k$  is a primitive  $d$ -th root of unity.  $\square$

**Example 3.5.5.** There is no formula (involving only field operations and extraction of roots) for the solution of all 5th-degree polynomial equations.

**Solution** Consider  $f(x) = 2x^5 - 10x + 5 \in \mathbb{Q}[x]$ . We check the zeros of the derivatives.

$$f'(x) = \frac{d}{dx}2x^5 - 10x + 5 = 10x^4 - 10 \implies \text{roots are } \pm 1, \pm i$$

$$f''(x) = \frac{d^2}{dx^2}2x^5 - 10x + 5 = 40x^3 \implies \text{the only root is } 0.$$

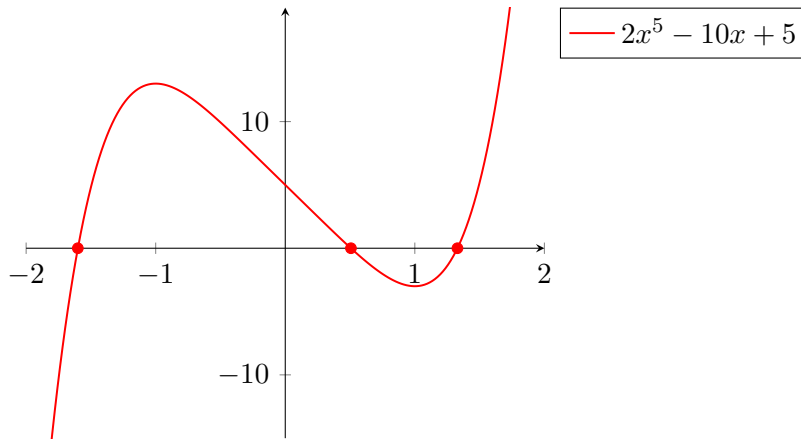


Figure 3.1: By graph,  $f(x) = 2x^5 - 10x + 5$  has 3 real roots.

Note that by Eisenstein's criterion,  $5|10$ ,  $5|4$ , but  $5 \nmid 2$  and  $5^2 \nmid 5$ . So  $f(x)$  is irreducible in  $\mathbb{Q}[x]$ .

If  $K$  is splitting field of  $f(x)$  in  $\mathbb{C}$ . Then

$$|\text{Gal}_{\mathbb{Q}} K| = [K : \mathbb{Q}]$$

Since  $K$  is Galois field of  $\mathbb{Q}$ .

If  $r$  is any root of  $f(x)$ , then

$$|K : \mathbb{Q}| = |K : \mathbb{Q}(r)| \cdot |\mathbb{Q}(r) : \mathbb{Q}| = 5 |K : \mathbb{Q}(r)|.$$

Thus  $|K : \mathbb{Q}|$  is divisible by 5. By Cauchy's theorem, there exists  $\sigma \in \text{Gal}_{\mathbb{Q}} K$  with  $\text{ord}(\sigma) = 5$ . Note that  $\text{Gal}_{\mathbb{Q}} K \subseteq S_5$  and the only elements of order 5 are (5-cycles), the 5-cycle is in  $\text{Gal}_{\mathbb{Q}} K$ . The complex conjugation  $a + bi \mapsto a - bi$  induces an automorphism on  $K$  since  $\sigma \in \text{Gal}_{\mathbb{Q}} \mathbb{C}$ ,  $\sigma|_K \in \text{Gal}_{\mathbb{Q}} K$ .

Since  $\sigma|_K$  interchanges the two nonreal roots of  $f(x)$ , implies that  $\text{Gal}_{\mathbb{Q}} K$  contains a transposition (2-cycle). Note that the only subgroup of  $S_5$  that contains both a 5-cycle and a transposition is  $S_5$  itself. So  $\text{Gal}_{\mathbb{Q}} K = S_5 \implies f(x)$  is not solvable by radicals. ◀

## Tutorials

**Exercise 3.5.1** Let  $E$  be an extension field of a field  $F$  and let  $\alpha \in E$  be transcendental over  $F$ . Show that every element of  $F(\alpha)$  is not in  $F$  is also transcendental over  $F$ .

**Exercise 3.5.2** Let  $F$  be a finite field of characteristic  $p$ . Show that every element of  $F$  is algebraic over the prime field  $\mathbb{Z}_p \leq F$ .