

Chapter 1

Existence and Uniqueness of Solutions

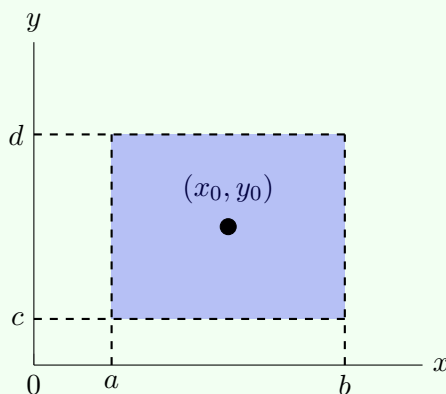
In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (1.1)$$

Theorem 1.1 Peano's Existence theorem

Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0) . If the function $f(x, y)$ is continuous in R .

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy -plane except on the line $x = 0$ (which is the y -axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

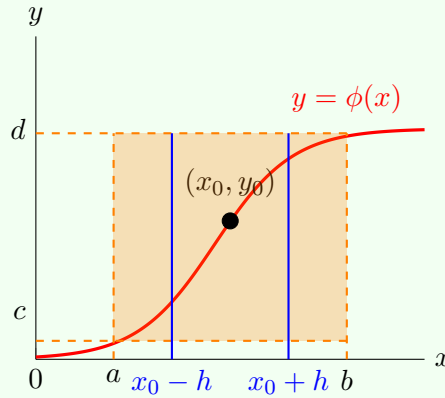
$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem

Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0) .



Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point $(2, 5)$. By Picard's theorem, this IVP has a unique solution. ◀

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x, y) = \frac{x(y^2 - 9)}{x^2 + 1} & (1) \\ y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x, y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.

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Chapter 2

Solving First-order Differential Equation

Theorem 2.1

If function $f(x)$ and function $g(y)$ are continuous, then equation

$$\int f(x)dx = \int g(y)dy + C$$

Example 2.0.1. Solve $e^{x+y} dy - 1 dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y} dy &= 1 dx &\Rightarrow e^x * e^y dy &= 1 dx \\ &&\Rightarrow e^y dy &= e^{-x} dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^y dy &= \int e^{-x} dx &\Rightarrow e^y &= -e^{-x} + C && e^y > 0 \text{ so that RHS } > 0 \\ &&\Rightarrow y &= \ln | -e^{-x} + C | && \text{general solution in implicit form} \end{aligned}$$

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Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} &= -\frac{1}{2}x^2 + c_1 \Rightarrow \frac{1}{y} = x^2 - 2c_1 \\ &\Rightarrow y = \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).

Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$.

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2.1 Exact Equation

The equation

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1)$$

is exact if $\exists F(x, y)$ such that $M dx + N dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x, y) = C$, C is a constant.

Definition 2.1 Total differential

Let $F(x, y)$ be a function that has continuous first derivative in a domain D .

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \quad \forall (x, y) \in D \quad (2.2)$$

Theorem 2.2 Test for Exactness

Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectangle $R : a < x < b, c < y < d$. Then

$$M(x, y) dx + N(x, y) dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.3)$$

Proof. (\Rightarrow) If $M(x, y) dx + N(x, y) dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R , according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \quad (2.4)$$

at each point of R .

(\Leftarrow) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.5)$$

to prove $M(x, y) dx + N(x, y) dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.6)$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} \quad (2.7)$$

Integrating both sides with respect to x , we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \quad (2.8)$$

□

Example 2.1.1. Solve $3x(xy - 2) dx + (x^3 + 2y) dy = 0$.

Solution The DE is in the form of $M dx + N dy = 0$, where

$$\begin{cases} M = \frac{1}{t^2} + \frac{1}{y^2} \\ N = \frac{at+1}{y^3} \end{cases}$$

In order to make DE to be exact, we must have $M_y = N_t \Rightarrow a = \dots$ ◀

Example 2.1.2. Solve the initial-value problem

$$(2x \cos y + 3x^2 y) dx + (x^3 - x^2 \sin y - y) dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$M = 2x \cos y + 3x^2 y, \quad N = x^3 - x^2 \sin y - y$$

$$M_y = -2x \sin y + 3x^2, \quad N_x = 3x^2 - 2x \sin y$$

Since $M_y = N_x = 3x^2 - 2x \sin y \quad \forall (x, y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D . Next, we must find F such that

$$\begin{cases} F_x = M = 2x \cos y + 3x^2 y & (a) \\ M_y = x^3 - x^2 \sin y - y & (b) \end{cases}$$

From (a) $\Rightarrow F = \int (2x \cos y + 3x^2 y) dx = x^2 \cos y + x^3 y + g(y)$. (c)
where g is a function of y .

$$\text{Again, (c) } \Rightarrow F_y = -x^2 \sin y + x^3 + g'(y) \quad (d)$$

Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2 \cos y + x^3 y - \frac{1}{2} y^2 + C$$

Hence a 1-parameter family of solutions is $F(x, y) = 0$ or $x^2 \cos y + x^3 y - \frac{1}{2} y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C : Substituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0, 2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2 \cos y + x^3 y - \frac{1}{2} y^2 + 2$. ◀

Example 2.1.3. Determine the constant a so that the equation

$$\frac{1}{t^2} + \frac{1}{y^2} + \left(\frac{at+1}{3} \right) \frac{dy}{dt} = 0$$

is exact, and then solve the resulting equation.

Theorem 2.3

The general solution to an exact equation $M(x, y) dx + N(x, y) dy = 0$ is defined implicitly

by

$$F(x, y) = C \quad (2.11)$$

where F is a potential function of the DE and C is an arbitrary constant.

Remark. We can find F by solving the system of equations

$$\begin{cases} \frac{\partial F}{\partial x} = M \\ \frac{\partial F}{\partial y} = N \end{cases} \quad (2.12)$$

Proof. If $M(x, y) dx + N(x, y) dy = 0$ is exact, then \exists a potential function F such that $M(x, y) dx + N(x, y) dy = dF$.

This gives us $dF = 0$ so that $F(x, y) = C$, where C is an arbitrary constant. \square

2.2 Making an Equation Exact: Integrating Factors

Sometimes it is impossible to transform a nonexact DE that into an exact equation by multiplying it by a function. The resulting DE can be resolved using the technique of the previous section. However, it is impossible for a solution to be lost or gained as a result of the multiplication.

Definition 2.2

If $M(x, y) dx + N(x, y) dy = 0$ is not exact but $I(x, y)M(x, y) dx + N(x, y)I(x, y) dy = 0$ is exact, then $I(x, y)$ is called an integrating factor of the DE.

Remark. We may be able to determine $I(x, y)$ from the equation

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN) \quad (2.13)$$

Example 2.2.1. Verify that $I(x, y) = x^{-1}$ is an integrating factor of $(x + y) dx + x \ln x dy = 0$ on the interval $(0, \infty)$. Hence, find the solution for this DE.

Solution

$$(x + y) dx + x \ln x dy = 0 \quad (1)$$

First, show that Eq(1) is not exact. Suppose $M = x + y$, and $N = x \ln x$. Then

$$M_y = 1, \quad N_x = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$$

Because $M_y \neq N_x$, so Eq(1) is not exact.

Next, multiplying Eq(1) by $I(x, y) = \frac{1}{x}$, we obtain

$$\frac{(x + y)}{x} dx + \ln x dy = 0 \quad (2)$$

Now we show that Eq(2) is exact. Let $\tilde{M} = 1 + \frac{1}{y}$, and $\tilde{N} = \ln x$, then

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} = \frac{1}{x}$$

The general solution will be $F(x, y) = C$, we can find F by solving the system of equations

$$\begin{cases} \frac{\partial F}{\partial x} = 1 + \frac{y}{x} & (3) \\ \frac{\partial F}{\partial y} = \ln x & (4) \end{cases}$$

Integrating Eq(3) with respect to x we have

$$\int (3) dx \Rightarrow F = \int \left(1 + \frac{y}{x}\right) dx = x + y \ln x + g(y) \quad (5)$$

where $g(y)$ is a function of y alone.

Differentiate Eq(5) with respect to y ,

$$\frac{\partial F}{\partial y} = \ln |x| + g'(y) \quad (6)$$

Now comparing Eq(4) and Eq(6), we obtain $g'(y) = g(y) = 0$.

Thus, the general solution is $F = x + y \ln |x| + 0$.

Theorem 2.4

If the functions p and q are continuous on an open interval $I : a < x < b$ containing the point x_0 , then the IVP

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0 \quad (2.16)$$

has a unique solution in the largest open interval containing x_0 , in which both p and q are continuous.

Example 2.2.2. Find the largest interval I on which the initial value problem

$$xy' + 2y = 4x^2, \quad y(-1) = 2$$

has a unique problem.

Summary: Solving 1st order DE

1. Separable equation: $f(x) dx = g(y) dy$. (Method of solving: integrating both sides)
2. Exact DE: Use **Exactness Test**, whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Chapter 3

Power Series Solutions

Example 3.0.1. Solve the IVP

$$\begin{cases} y' + 2xy = x^3 & (1) \\ y(1) = 1 & (2) \end{cases}$$

Find the first 3 nonzero-terms of the Taylor series of $y(x)$ about $x = 1$.

Solution The Taylor series of $y(x)$ about $x = 1$ is

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!} (x-1)^n = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!} (x-1)^2 + \frac{y^{(3)}(1)}{3!} (x-1)^3 + \dots$$

by definition.

It remains to find $y^{(n)}(1)$ for $n \geq 1$ (until we get the first 3 nonzero-term)

From Eq(1),

$$y'(x) = x^3 - 2xy(x) \Rightarrow y'(1) = 1^3 - 2(1)y(1) = 1 - 2 = -1$$

From Eq(1), differentiate again,

$$y''(x) = 3x^2 - \left(2x \frac{dy}{dx} + y(2) \right) = 3x^2 - 2xy' - 2y \quad (3)$$

Substitute $y(1) = 1, y'(1) = -1$ into Eq(3),

$$y''(1) = 3(1)^2 - 2(1)(-1) - 2 = 3$$

Thus the Taylor series is

$$\begin{aligned} y(x) &= 1 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \dots \\ &= 1 + (1-x) + \frac{3}{2!}(x-1)^2 + \dots \\ &= 2 - x + \frac{3}{2!}(x-1)^2 + \dots \end{aligned}$$

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Q: Here is the question, under what condition does a DE has a solution of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad (3.3)$$

This bring us to another section: which is about analytic at a point for a series.

3.1 Analytic at a point

Definition 3.1

If the Taylor series of f , where

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad (3.4)$$

exists and converges to $f(x) \quad \forall x \in I$, an open interval containing $x = a$, then function f is analytic at $x = a$.

Example of Analytic functions

All polynomials $P(x) = a_0 + a_1x + a_2x^2 + \dots$ are analytic $\forall x \in \mathbb{R}$.

Example 3.1.1. Legendre Equation

$$(1-x^2)y'' + 2xy' + \lambda y = 0 \quad (1)$$

Find a power series solution for this DE.

Solution In the standard form $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = \frac{2}{1-x^2}, \quad q(x) = \frac{\lambda}{1-x^2}$$

Both p and q are analytic at $x = 0$. As $x = 0$ is an ordinary point, Eq(1) will have two linearly independent solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

$$\text{Substitute } y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad (2)$$

and,

$$\begin{aligned} -x^2 y'' &= \sum_{n=2}^{\infty} -(n-1)n a_n x^n \Rightarrow 2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = 2a_1 x + \sum_{n=2}^{\infty} 2n a_n x^n \\ &\Rightarrow \lambda y = \sum_{n=0}^{\infty} \lambda a_n x^n = \lambda a_0 + \lambda a_1 x + \sum_{n=2}^{\infty} \lambda a_n x^n \end{aligned}$$

into Eq(1).

Replacing n to $n+2$ in Eq(2), we obtain

$$\begin{aligned} (2) &= \sum_{n+2=2}^{\infty} (n+2)(n+1) a_{n+2} x^{n+2-2} \\ &= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n \\ &= (0+2)(0+1) a_2 + (1+2)(1+1) a_3 + \sum_{n=2}^{\infty} (n+1)(n+2) a_{n+2} x^n \end{aligned}$$

So that

$$(2a_2 + \lambda a_0) + (6a_3 + 2a_1 + \lambda a_1)x + \sum_{n=2}^{\infty} \left\{ (n+1)(n+2)a_{n+2} - n(n-1)a_n + 2na_n + \lambda a_n \right\} x^n = 0$$

Equating coefficients both sides to zero,

$$\begin{cases} 2a_2 + \lambda a_0 = 0 & (3) \\ 6a_3 + (2 + \lambda)a_1 = 0 & (4) \end{cases}$$

[Recurrence relation]

$$(n+1)(n+2)a_{n+2} + [2n + \lambda - n(n-1)]a_n = 0 \text{ for } n \geq 2$$

Write all the a_n 's in terms of a_0 and a_1 ,

$$(3) \Rightarrow a_2 = -\frac{\lambda}{2}a_0$$

$$(4) \Rightarrow a_3 = -\frac{(2+\lambda)}{6}a_1$$

$$\begin{aligned} (5) \Rightarrow a_4 &= \frac{2(2-3) - \lambda}{4(3)} a_0 \\ &= \frac{-2 - \lambda}{4(3)} \left(\frac{\lambda}{2} \right) a_0 \\ &= \frac{\lambda(\lambda+2)a_0}{4!} \end{aligned}$$

The solution is

$$\begin{aligned} y &= (a_0 + a_2x^2 + a_4x^4 + \cdots) + (a_1 + a_3x^3 + a_5x^5 + \cdots) \\ &= a_0 \left(1 - \frac{\lambda}{2}x^2 + \frac{\lambda(\lambda+2)}{4!}x^4 + \cdots \right) + a_1 \left(x - \frac{\lambda+2}{3!}x^3 + \frac{\lambda(\lambda+2)}{5!}x^5 + \cdots \right) \end{aligned}$$

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3.2 Regular Singular Point

Theorem 3.1 Frobenius Theorem

Given an equation $P(x)y'' + Q(x)y' + R(x)y = 0$, if x is a regular singular point at \mathbb{R} , then there exists a solution of the form

$$y = (x - x_0)^n \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{2n}$$

Example 3.2.1. Solve the DE

$$2xy'' + y' - y = 0$$

and check whether $x = 0$ is a regular singular point of this equation.

Solution The DE is

$$2xy'' + y' - y = 0 \quad (1)$$

Firstly, we need to check whether $x = 0$ is a RSP of Eq(1).

By Theorem 3.1, (1) has a Frobenius solution of the form $\sum_{n=0}^{\infty} a_n x^{n+r}$.

To find the solutions, substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r) a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

into Eq(1),

$$\begin{aligned} 2xy'' &= \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} \\ \Rightarrow 2xy'' &= 2r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} 2(n+r)(n+r-1) x^{n+r-1} a_n \end{aligned}$$

Assumption: Assume that $a_0 \neq 0$,

$$\begin{aligned} (2) \Rightarrow 2r(r-1) + r &= 0 \\ \Rightarrow 2r^2 - r &= 0 && \text{Indicial equation} \\ \Rightarrow r = 0 \quad \text{or} \quad r = \frac{1}{2} && \text{Indicial roots} \end{aligned}$$

Case 1: When $r = 0$, from Eq(3) we have

$$(3) \Rightarrow [2(n+1)n + (n+1)] a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \quad \forall n \geq 0$$

Iterate through $n = 0, 1, 2, \dots$ and find a_n in terms of a_0 ,

$$\begin{aligned} n = 0 : & \quad a_1 = \frac{a_0}{1 \times 1} = a_0 \\ n = 1 : & \quad a_2 = \frac{a_1}{2 \times 3} = \frac{a_0}{(1 \times 2)(1 \times 3)} \\ n = 2 : & \quad a_3 = \frac{a_2}{3 \times 5} = \frac{a_0}{(1 \times 2 \times 3)(1 \times 3 \times 5)} \\ n = 3 : & \quad a_4 = \frac{a_3}{4 \times 7} = \frac{a_0(2 \times 4 \times 6 \times 8)}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)(2 \times 4 \times 6 \times 8)} \\ & \quad = \frac{a_0}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} \end{aligned}$$

by mathematical induction, y_1 can be express as

$$y_1 = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{i=1}^n (2i-1)} \right], \quad n \in \mathbb{Z}^+$$

Case 2: When $r = 1/2$, from Eq(3) we have

$$\begin{aligned}
 n = 0 : & & a_1 &= \frac{a_0}{1 \times 3} = \frac{1}{3}a_0 \\
 n = 1 : & & a_2 &= \frac{a_1}{5 \times 2} = \frac{a_0}{(1 \times 3 \times 5)(1 \times 2)} \\
 n = 2 : & & a_3 &= \frac{a_2}{7 \times 3} = \frac{a_0}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} \\
 n = 3 : & & a_4 &= \frac{a_0}{(1 \times 3 \times 5 \times 7 \times 9)(1 \times 2 \times 3 \times 4)} \\
 & \vdots & & \vdots
 \end{aligned}$$

The second solution is

$$\begin{aligned}
 y_2 &= \sum_{n=0}^{\infty} a_n x^{n+1/2} = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \\
 &= x^{1/2} [a_0 + a_1 + a_2 + \cdots] \\
 &= x^{1/2} \left[1 + \frac{x}{1 \times 3} + \frac{x^2}{(1 \times 3 \times 5)(1 \times 2)} + \frac{x^3}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} + \cdots \right]
 \end{aligned}$$

which can be written as

$$y_2 = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{i=1}^n (2i+1)} \right]$$

By inspection, y_1 and y_2 are not scalar multiples, implies that they are linearly independent. Therefore the general solution of Eq(1) is $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants. ◀

Remark. In general, we may not get two linearly independent solutions. We are guaranteed by Frobenius theorem that there is at least one solution in the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad (3.7)$$

then we can use reduction of order to find for the 2nd linearly independent solution.

Example 3.2.2. For the given DE:

$$xy'' + 3y' - y = 0$$

Given that one of its solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{2}{n! (n+2)!} x^n = 1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots$$

Find the 2nd LI solution using reduction of order.

Solution We write the original DE in standard form: $y'' + p(x)y' + q(x)y = 0$, with $p(x) = \frac{3}{x}$.

$$\begin{aligned}
 \exp \left[- \int p(x) dx \right] &= \exp \left[- \int -\frac{3}{x} dx \right] \\
 &= e^{-3|x|} \\
 &= \frac{1}{x^3}
 \end{aligned}$$

$$\frac{e^{-\int p(x) dx}}{y_1^2} = \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \dots\right)^2} \quad (\text{a})$$

To expand the squared series, we can apply the rule of multiplication for power series:

How to multiply two power series?

If there are two power series such that $f(x) = \sum_{n=0}^N a_n(x-x_0)^n$, $g(x) = \sum_{n=0}^N b_n(x-x_0)^n$, then the multiplication of these two series are

$$f(x)g(x) = \sum_{n=0}^N c_n(x-x_0)^n$$

where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \dots + a_nb_0$.

In this case, $f(x) = g(x)$ with $a_0 = b_0 = 1$, $a_1 = b_1 = \frac{1}{3}$, $a_2 = b_2 = \frac{1}{24}$.

$$c_0 = a_0b_0 = 1$$

$$c_1 = a_0b_1 + a_1b_0 = 1 \times \frac{1}{3} + \frac{1}{3} \times 1 = \frac{2}{3}$$

$$c_2 = a_0b_2 + a_1b_1 + a_2b_0 = \frac{1}{24} + \frac{1}{9} + \frac{1}{24} = \frac{7}{36}$$

$$c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = \frac{1}{360} + \frac{1}{72} + \frac{1}{72} + \frac{1}{360} = \frac{1}{30}$$

\vdots

now we can continue to work on Eq(a).

$$\begin{aligned} \frac{e^{-\int p(x) dx}}{y_1^2} &= \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \dots\right)^2} \\ &= \frac{1}{x^3(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)} \\ &= \frac{1}{x^3\left(1 + \frac{2}{3}x + \frac{7}{36}x^2 + \frac{1}{30}x^3 + \dots\right)} \end{aligned} \quad (\text{b})$$

Using long division to expand Eq(b):

$$\begin{array}{r} 1 + \frac{2}{3}x + \frac{7}{36}x^2 + \dots \quad \overline{) 1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \dots} \\ \underline{1} \phantom{+ \frac{2}{3}x + \frac{7}{36}x^2 + \dots} \\ -\frac{2}{3}x \phantom{+ \frac{7}{36}x^2 + \dots} \\ \underline{+\frac{2}{3}x} \phantom{+ \frac{7}{36}x^2 + \dots} \\ \frac{1}{4}x^2 + \frac{7}{36}x^2 - \frac{19}{270}x^3 + \dots \\ \underline{-\frac{1}{4}x^2} \phantom{+ \frac{7}{36}x^2 + \dots} \\ \frac{7}{36}x^2 - \frac{19}{270}x^3 + \dots \\ \underline{-\frac{7}{36}x^2} \\ -\frac{19}{270}x^3 + \dots \end{array}$$

we obtained

$$\begin{aligned}\frac{e^{-\int p(x) dx}}{y_1^2} &= \frac{1}{x^3} \left(1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \dots \right) \\ &= x^{-3} - \frac{2}{3}x^{-2} + \frac{1}{4}x^{-1} - \frac{19}{270} + \dots\end{aligned}$$

by using reduction of order, we can find y_2 using formula

$$y_2 = \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$$

Example 3.2.3. 1. Show that

$$\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty} (a_{k-1} + a_{k+1})x^k$$

Solution From LHS,

$$\text{LHS} = \sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = \left(a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k \right) + \sum_{k=0}^{\infty} a_k x^{k+1}$$