Chapter 1

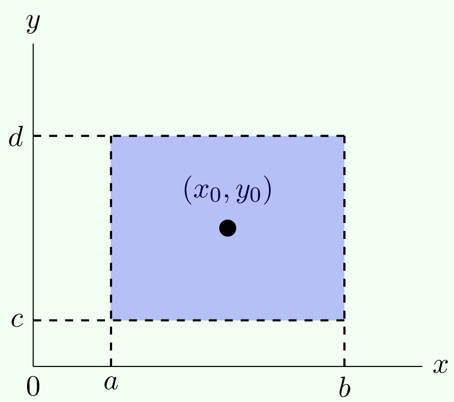
Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.1}$$

Theorem 1.1 Peano's Existence theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0). If the function $f(x, y)$ is continuous in R.

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy-plane except on the line $x = 0$ (which is the y-axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

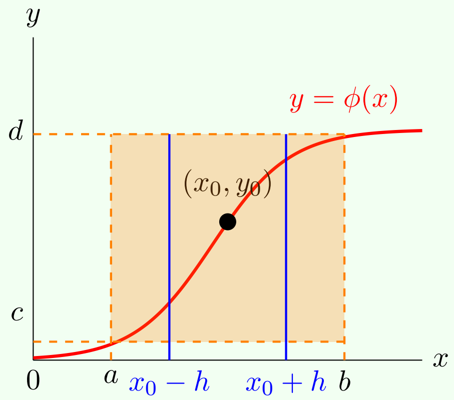
Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0).



If the functions $f(x, y)$ and $\partial f/\partial y$ are continuous on region R, then the first-order IVP

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.2}$$

has a unique solution $y = \phi(x)$ in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Remark. From the diagram above,

1. Without knowledge of existence theory, we might for example, use a computer software to find a numerical approximation to a "solution" that does not exist.
2. The theorem guarantees us a unique solution that is defined on some interval of width $2h$, but it say nothing about the size of h.

Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x, y) = \frac{x(y^2-9)}{x^2+1} & (1) \\ \quad y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x, y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0\,(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.

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Chapter 2

Solving First-order Differential Equation

Theorem 2.1
If function $f(x)$ and function $g(x)$ are continuous, then the DE is solvable by performing integration on both sides, said

$$\int f(x)dx = \int g(y)\,dy + C$$

Example 2.0.1. Solve $e^{x+y}\,dy - 1\,dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y}\,dy = 1\,dx \quad &\Rightarrow e^x * e^y\,dy = 1\,dx \\ &\Rightarrow e^y\,dy = e^{-x}\,dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^y\,dy = \int e^{-x}\,dx \quad &\Rightarrow e^y = -e^{-x} + C && e^y > 0 \text{ so that RHS } > 0 \\ &\Rightarrow y = \ln|-e^{-x} + C| && \text{general solution in implicit form} \end{aligned}$$

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Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x\,dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} = -\frac{1}{2}x^2 + c_1 \Rightarrow \frac{1}{y} &= x^2 - 2c_1 \\ \Rightarrow y &= \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).

Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$. ◀

2.1 Exact Equation

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1}$$

is exact if $\exists F(x,y)$ such that $M\,dx + N\,dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x,y) = C$, C is a constant.

Definition 2.1 Total differential
Let $F(x,y)$ be a function that has continuous first derivative in a domain D.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \forall (x,y) \in D \tag{2.2}$$

Theorem 2.2 Test for Exactness
Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectanger $R : a < x < b,\ c < y < d$. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.3}$$

Proof. ($\Rightarrow$) If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R, according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \tag{2.4}$$

at each point of R.

($\Leftarrow$) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.5}$$

to prove $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.6}$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\phi x \phi y} \tag{2.7}$$

Integrating both sides with respect to x, we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \tag{2.8}$$

$\square$

Example 2.1.1. Solve $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$.

Solution The DE is in the form of $M\,dx + N\,dy = 0$, with the test of exactness

$$\begin{cases} M = 3x(xy-2) & \Rightarrow M_y = 3x^2 - 0 = 3x^2 \\ N = x^3 + 2y & \Rightarrow N_x = 3x^2 \end{cases}$$

To find the general solution of Eq(1): $F(x,y) = C$.

Find the function F by solving the system

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = 3x(xy-2) = 3x^2y - 6x \\ \dfrac{\partial F}{\partial y} = N = x^3 + 2y \end{cases}$$

Now integrate (a) with respect to x, regarding y as a constant.

$$(a) \Rightarrow F = \int_y (3x^2y - 6x)\,dx = x^3y - 3x^2 + g(y)$$

where g is a function of y alone. Again, we partial derivative on y to obtain $g'(y)$.

$$\frac{\partial F}{\partial y} = x^3 - 0 + g'(y) = x^3 + g'(y)$$

comparing to Eq(b), we have $g'(y) = 2y$. Which means $g(y) = y^2$. (ignored constant). In result, $F = x^3y - 3x^2 + y^2$ is the general solution of the DE.

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Example 2.1.2. Solve the initial-value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$\begin{aligned} M &= 2x\cos y + 3x^2y, \quad N = x^3 - x^2\sin y - y \\ M_y &= -2x\sin y + 3x^2, \quad N_x = 3x^2 - 2x\sin y \end{aligned}$$

Since $M_y = N_x = 3x^2 - 2x\sin y \quad \forall (x,y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D. Next, we must find F such that

$$\begin{cases} F_x = M = 2x\cos y + 3x^2y & (a) \\ M_y = x^3 - x^2\sin y - y & (b) \end{cases}$$

From $(a) \Rightarrow F = \int(2x\cos y + 3x^2y)\,dx = x^2\cos y + x^3y + g(y)$. (c)
where g is a function of y.

Again, $(c) \Rightarrow F_y = -x^2 \sin y + x^3 + g'(y)$ (d)

Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2 \cos y + x^3 y - \frac{1}{2}y^2 + C$$

Hence a 1-parameter family of solutions is $F(x, y) = 0$ or $x^2 \cos y + x^3 y - \frac{1}{2}y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C: Subtituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0, 2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2 \cos y + x^3 y - \frac{1}{2}y^2 + 2$. ◀

Example 2.1.3. Determine the constant a so that the equation

$$\frac{1}{t^2} + \frac{1}{y^2} + \left(\frac{at + 1}{3}\right)\frac{dy}{dt} = 0$$

is exact, and then solve the resulting equation.

Theorem 2.3
The general solution to an exact equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is defined implicitly by

$$F(x, y) = C \tag{2.11}$$

where F is a potential function of the DE and C is an arbitrary constant.

Remark. We can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N \end{cases} \tag{2.12}$$

Proof. If $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact, then $\exists$ a potential function F such that $M(x, y)\, dx + N(x, y)\, dy = dF$.

This gives us $dF = 0$ so that $F(x, y) = C$, where C is an arbitrary constant. □

Example 2.1.4. Solve the DE

$$3x(xy - 2)\, dx + (x^3 + 2y)\, dy = 0$$

2.2 Making an Equation Exact: Integrating Factors

Sometimes it is impossible to transform a nonexact DE that into an exact equation by multiplying it by a function. The resulting DE can be resolved using the technique of the previous section. However, it is impossible for a solution to be lost or gained as a result of the multiplication.

Definition 2.2
If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact but $I(x,y)M(x,y)\,dx + N(x,y)I(x,y)\,dy = 0$ is exact, then $I(x,y)$ is called an integrating factor of the DE.

Remark. We may be able to determine $I(x,y)$ from the equation

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial y}(IN) \tag{2.13}$$

Example 2.2.1. Verify that $I(x,y) = x^{-1}$ is an integrating factor of $(x+y)\,dx + x\ln x\,dy = 0$ on the interval $(0,\infty)$. Hence, find the solution for this DE.

Solution

$$(x+y)\,dx + x\ln x\,dy = 0 \qquad (1)$$

First, show that Eq(1) is not exact. Suppose $M = x+y$, and $N = x\ln x$. Then

$$M_y = 1, \quad N_x = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

Because $M_y \neq N_x$, so Eq(1) is not exact.

Next, multiplying Eq(1) by $I(x,y) = \dfrac{1}{x}$, we obtain

$$\frac{(x+y)}{x}\,dx + \ln x\,dy = 0 \qquad (2)$$

Now we show that Eq(2) is exact. Let $\tilde{M} = 1 + \dfrac{1}{y}$, and $\tilde{N} = \ln x$, then

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} = \frac{1}{x}$$

The general solution will be $F(x,y) = C$, we can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = 1 + \dfrac{y}{x} & (3) \\ \dfrac{\partial F}{\partial y} = \ln x & (4) \end{cases}$$

Integrating Eq(3) with respect to x we have

$$\int (3)dx \quad \Rightarrow F = \int \left(1 + \frac{y}{x}\right) dx = x + y\ln x + g(y) \qquad (5)$$

where $g(y)$ is a function of y alone.

Differentiate Eq(5) with respect to y,

$$\frac{\partial F}{\partial y} = \ln|x| + g'(y) \qquad (6)$$

Now comparing Eq(4) and Eq(6), we obtain $g'(y) = g(y) = 0$.

Thus, the general solution is $F = x + y\ln|x| + 0$.

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Theorem 2.4 Existence and Uniqueness Theorem for a 1st-order DE
If the functions p and q are continuous on an open interval $I : a < x < b$ containing the point x_0, then the IVP

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0 \tag{2.16}$$

has a unique solution in the largest open interval containing x_0, in which both p and q are continuous.

Proof. The DE can be written as $y' = f(x,y)$, where $f(x,y) = q(x) - p(x)y$.

Moreover, since p and q are continuous on I, the solution

$$y(x) = e^{-\int_{x_0}^{x} p(t)\,dt}\left[\int_{x_0}^{x} q(t)e^{-\int_{x_0}^{x} p(t)\,dt}\,dt + y_0\right] \tag{2.17}$$

is well defined $\forall x \in I$. □

Example 2.2.2. Find the largest interval I on which the initial value problem

$$xy' + 2y = 4x^2, \quad y(-1) = 2$$

has a unique problem.

Example 2.2.3 (Radioactive Decay)**.** A certain radioactive isotope is known to decay at a rate of proportional to the amount present. Initially, 100 grams of the isotope are present, but after 75 years its mass decays to 75 grams.

1. Setup and solve an initial-value problem for $N(t)$, the mass of the isotope at time t.
2. What is the half-life of the substance?

   Note: Half-life of a radioactive substance is the time required for half of it to decay.

Solution Let $N(t)$ be the amount of material at time t,

Given $\frac{dN}{dt} \propto N$. So $\frac{dN}{dt} = kN$ where k is the constant of proportionality.

Given the initial amount $N(0) = 100$, and $N(50) = 75$.

1. The IVP is
$$\frac{dN}{dt} = kN, \quad N(0) = 100, \quad N(50) = 75$$

   Identify the DE: Separable, 1st-order, linear DE

   To solve the DE, we can use either integrating factor or by writting the DE in the form of $\frac{dN}{dt} + p(t)N = q(t)$.

$$IF = e^{\int p(t)dt} = e^{\int -k\,dt} = e^{-kt}$$

   and we obtained

$$N(t) = 100e^{\frac{1}{50}\ln(3/4)t} \quad \forall t \in [0, +\infty)$$

2. Let the half-life of isotope be T, and
$$N(T) = \frac{1}{2}N(0) = 50$$

   using the formula that we found on (a),

$$\begin{aligned} N(T) = 100e^{\frac{1}{50}\ln(3/4)T} = 50 \Rightarrow e^{\frac{1}{50}\ln(3/4)T} &= \frac{1}{2} \\ \Rightarrow T = \frac{\ln(1/2)}{\frac{1}{50}\ln(3/4)} &= 120.471042 \text{ years} \end{aligned}$$

   which means this radioactive substance required roughly 120 years to decay to half of its mass.

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2.3 Substitution Method

In this section, we will use an appropriate substitution to transform a given ODE into one that could be solved by one of the standard methods.

Theorem 2.5 Substitution method
The substitution
$$u = ax + by + c, b \neq 0 \tag{2.18}$$
transforms the equation
$$\frac{dy}{dx} = f(ax + by + c) \tag{2.19}$$
into a separable equation.

Proof. Consider a differential equation of the form (2.18)

Let
$$u = ax + by + c \tag{2.20}$$

Taking the drivative with respect to x we obtained

$$\frac{du}{dx} = a + b\frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(a - \frac{du}{dx}\right) \tag{2.21}$$

Substituting this result back to Eq(2.18)

$$\frac{1}{b}\left(a - \frac{du}{dx}\right) = f(u) \tag{2.22}$$

which is clearly a separable equation:

$$\frac{1}{a + bf(u)}\, du = dx \tag{2.23}$$

□

Example 2.3.1. Use an appropriate substitution to solve

$$\frac{dy}{dx} = \sin^2(3x - 3y + 1)$$

Solution Substituting $u = 3x - 3y + 1$, $\frac{du}{dx} = 3 - 3\frac{dy}{dx}$ or $\frac{dy}{dx} = 1 - \frac{1}{3}\frac{du}{dx}$ into the given DE. We obtain

$$\begin{aligned}
1 - \frac{1}{3}\frac{du}{dx} = \sin^2 u \Rightarrow \frac{du}{dx} &= 3\cos^2 u \\
\Rightarrow \sec^2 u\, du &= 3\, dx \\
\Rightarrow \tan u &= 3x + C && (C \text{ is a constant}) \\
\Rightarrow u &= \arctan(3x + C) \\
\Rightarrow 3x - 3y + 1 &= \arctan(3x + C)
\end{aligned}$$

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Thus, $3x - 3y + 1 = \arctan(3x + C)$ is the solution of the original DE.

Example 2.3.2. Use an appropriate substitution to solve

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$$

Solution Substituting $u = y - 2x + 3$, $\frac{du}{dx} = \frac{dy}{dx} - 2$ or $\frac{dy}{dx} = 2 + \frac{du}{dx}$ into the given DE. We obtain

$$\begin{aligned}
2 + \frac{du}{dx} = 2 + \sqrt{u} \Rightarrow \frac{du}{dx} &= \sqrt{u} \\
\Rightarrow \frac{1}{\sqrt{u}}\, du &= dx \\
\Rightarrow \int \frac{1}{\sqrt{u}}\, du &= \int dx \\
\Rightarrow 2\sqrt{u} &= x + C && (C \text{ is a constant}) \\
\Rightarrow 4u &= (x + C)^2 \\
\Rightarrow 4(y - 2x + 3) &= (x + C)^2
\end{aligned}$$

and thus the solution is

$$y = \frac{(x+C)^2}{4} + 2x - 3$$

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2.4 Homogeneous Equation

A first-order DE $y' = f(x, y)$ is homogeneous if $f(x, y)$ can be expressed as a function of the ratio y/x alone. In other words, the DE

$$y' = f(x, y) \tag{2.24}$$

is homogeneous if it can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{2.25}$$

Theorem 2.6 Substitution method for homogeneous equation
The substitution $v = y/x$ (or $v = x/y$) will reduce the homogeneous DE

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{2.26}$$

to a separable DE.

Example 2.4.1. Solve $(x - y)y' = x + y$

Solution Certainly, $(x - y)y' = x + y \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$ is homogeneous.

By using the substitution

$$\begin{cases} v = \frac{y}{x} \Rightarrow y = vx \\ \frac{dy}{dx} = v + x\frac{dv}{dx} \end{cases}$$

substitute into Eq(1), we have

$$\begin{aligned} (1) \Rightarrow \frac{dy}{dx} &= \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}} \\ \Rightarrow v + x\frac{dv}{dx} &= \frac{1+v}{1-v} \\ \Rightarrow x\frac{dv}{dx} &= \frac{1+v}{1-v} - v \\ \Rightarrow \frac{1-v}{1+v^2}\, dv &= \frac{1}{x}\, dx \end{aligned}$$

integrating both sides

$$\begin{aligned} \int \frac{1-v}{1+v^2}\, dv = \int \frac{1}{x}\, dx \quad &\Rightarrow \int \left[\frac{1}{1+v^2} + \frac{-v}{1+v^2}\right] dv = \ln|x| + C \\ &\Rightarrow \arctan(v) - \frac{1}{2}\ln|1+v^2| = \ln|x| + C \end{aligned}$$

again, we substitute $v = \frac{y}{x}$ back to the result

$$\begin{aligned}
&\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left|1 + \frac{y^2}{x^2}\right| = \ln|x| + C \\
&\Rightarrow 2\arctan\left(\frac{y}{x}\right) = \ln\left|1 + \frac{y^2}{x^2}\right| + 2\ln|x| + 2C \\
&\Rightarrow 2\arctan\left(\frac{y}{x}\right) = \ln(x^2 + y^2) + 2C
\end{aligned}$$

is the general solution of the original DE. ◀

Example 2.4.2 (Bernoulli differential equation). A first-order differential equation of the form

$$y' + P(x)y = Q(x)y^n$$

where n is any real number, is called a **Bernoulli differential equation**.

For $n = 0, 1$, the equation is linear. Otherwise it is nonlinear.

Solution 1. For $n \geq 2$, consider the equation

$$y' + P(x)y = Q(x)y^n \tag{2.27}$$

Multiply (2.27) by $(1-n)y^{-n}$, we have

$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

using the following substitution

$$\begin{cases} w = y^{1-n} \\ \dfrac{dw}{dx} = (1-n)y^{(1-n)-1}\dfrac{dy}{dx} = (1-n)y^{-n}\dfrac{dy}{dx} \end{cases}$$

The original DE is now a linear first-order separable DE.

$$\begin{aligned}
\frac{dw}{dx} + (1-n)wP(x) = (1-n)Q(x) \\
\frac{dw}{dx} = (1-n)[Q(x) - wP(x)]
\end{aligned}$$

2. A Bernoulli equation with $n = 2$ is

$$xy' + y = -xy^2 \tag{2.28}$$

which can be rewrite into

$$y' + \frac{1}{x}y = -y^2 \tag{$\diamond$}$$

Now use the substitution $w = y^{1-n} = y^{1-2} = y^{-1}$. We have

$$\frac{dw}{dx} = -y^{-2}\frac{dy}{dx} \tag{2.29}$$

Now multiply ($\diamond$) by $-y^{-2}$,

$$\begin{aligned} -y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} &= 1 \\ \frac{dw}{dx} - \frac{1}{x}w &= 1 \end{aligned} \tag{$\blacksquare$}$$

We can solve this using the integrating factor I, applying the formula and compute the integrating factor

$$I = e^{\int 1/x\,dx} = e^{-\ln|x|} = \frac{1}{x} \tag{2.30}$$

Now multiplying ($\blacksquare$) with integrating factor I

$$\begin{aligned} \frac{1}{x}\left(\frac{dw}{dx} - \frac{w}{x}\right) &= \frac{1}{x} \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \frac{1}{x} \\ \frac{w}{x} &= \int \frac{1}{x}\,dx = \ln|x| + C \qquad C \text{ is a constant} \end{aligned}$$

This arrive that the solution is $w = x\ln|x| + Cx$, where C is a constant. Recall that $w = 1/y$, hence the last step is solve for y.

$$\begin{aligned} \frac{1}{y} &= x\ln|x| + Cx \\ y &= \frac{1}{x\ln|x| + Cx}, \quad x \neq 0 \end{aligned}$$

◀

Summary: Solving 1st order DE

1. Separable equation: $f(x)\,dx = g(y)\,dy$. (Method of solving: integrating both sides)
2. Exact DE: Use **Exactness Test**, whether $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

2.5 Tutorials

Exercise 2.5.1 Solve $\dfrac{dx}{dt} = 4(x^2+1), \quad x\left(\dfrac{\pi}{4}\right) = 1.$

Exercise 2.5.2 Solve $\dfrac{dy}{dx} = e^{x^2}, \quad y(3) = 5$. The functions defined by integrals are listed as below:

Error Function	$\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$
Complementary error function	$\operatorname{erfc}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^\infty e^{-t^2}\, dt$

Exercise 2.5.3 Solve the initial-value problem

$$(e^2y - y)\cos x \frac{dy}{dx} = e^y \sin(2x), \quad y(0) = 0$$

Exercise 2.5.4 Solve $(x^2+y^2)\,dx + (x^2 - xy)\,dy = 0.$

Chapter 3

Second-order differential equations

We now consider what constitutes the so-called general solution of a homogeneous linear DE. To understand this, we first introduce the concepts of linear dependence and linear independent.

3.1 Linear dependence and linear independence

Definition 3.1 Linear independence
The functions $f_1, f_2, \ldots, f_n$ are said to be linearly independent on an interval I if there exist constants $c_1, c_2, \ldots, c_n$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{3.1}$$

Consider

$$S = \{y : y'' + p(x)y' + q(x)y = 0\} \tag{3.2}$$

is a vector space with dimension 2. If y_1 and y_2 are two linearly independent solution to the HLDE, then its general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3.3}$$

where c_1 and c_2 are constants. The set of solution $\{y_1, y_2\}$ is called the fundamental set of solutions (a basis of S) to the HLDE.

To show that $\{y_1, y_2\}$ is a fundamental set of solutions of HLDE. We can perform the Wronskian test, by Wronskian test we must show they are both **solutions** and **linearly independent**.

Definition 3.2 Wronskian
The **Wronskian** of two differential functions, said $f(x)$ and $g(x)$ is the determinant

$$W(f(x), g(x)) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)\,g'(x) - g(x)\,f'(x) \tag{3.4}$$

Remark. If Wronskian is nonzero, then the solutions y_1 and y_2 are linearly independent. Otherwise, they are linearly dependent.

Example 3.1.1. Given the equation

$$y'' - 4y = 0 \tag{3.5}$$

1. Show that

$$y_1 = e^{2x}, \quad y_2 = e^{-2x}$$

form a fundamental set of solutions of (3.5) on $\mathbb{R}$.

3.2 Second-order HLDE with constant coefficients

Definition 3.3
A homogeneous 2nd order linear DE with constant coefficients has the form

$$ay'' + by' + cy = 0 \tag{3.6}$$

where $a \neq 0$, b and c are constants.

$ar^2 + br + c = 0$ or $P(r) = 0$ is called the characteristic equation (or auxiliary equation) The characteristic equation is always quadratic, and its two roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{3.7}$$

Similar to what we had learned in quadratic equation, there are **three** possible forms for the general solution of (3.6) depending on the nature of the characteristic roots r_1 and r_2.

Example 3.2.1. Find a general solution to

1. $2y'' - 2y' - 5y = 0$
2. $y'' + 8y' + 16y = 0$
3. $y'' + 2y' + 4y = 0$

Solution The DEs above can be solve using the method of characteristic equation.

1. The roots of $2r^2 - 2r - 5 = 0$ are

$$r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-15)}}{2(2)} = \boxed{\frac{1 \pm \sqrt{11}}{2}}$$

The two roots are real and distinct. Thus the general solution is

$$y = c_1 \exp\left[\frac{(1 + \sqrt{11})}{2}\, x\right] + c_2 \exp\left[\frac{(1 - \sqrt{11})}{2}\, x\right]$$

where c_1 and c_2 are arbitrary constants.

2. The characteristic equation is

$$r^2 + 8r + 16 = (r + 4)^2 = 0$$

On solving, it has root $r = -4$ with multiplicity 2. Thus the general solution is

$$y = c_1 e^{-4x} + c_2\, x e^{-4x}$$

where c_1 and c_2 are arbitrary constants.

3. The characteristic equation $r^2 + 2r + 4 = 0$ has complex roots

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} = \boxed{-1 \pm \sqrt{3}i}, \quad \text{with } i = \sqrt{-1}$$

Thus, the general solution is

$$y = e^{-x}[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)]$$

where c_1 and c_2 are arbitrary constants.

◀

3.2.1 Non-homogeneous DE with constant coefficients

The general solution of non-homogeneous DE

$$ay'' + by' + cy = g(x) \tag{3.8}$$

is

$$y = y_h + y_p \tag{3.9}$$

where y_p is the general solution of the associated homogeneous equation

$$ay'' + by' + cy = 0 \tag{3.10}$$

3.3 Method of undetermined coefficients

Example 3.3.1. Solve the initial value problem

$$y'' + 4y = 12\cos 2x, \quad y(0) = 3, \quad y'(0) = 4$$

Solution Consider the DE

$$y'' + 4y = 12\cos 2x \tag{$\bigstar$}$$

The original DE is a second-order non-homogeneous DE. The general solution should be

$$y = y_p + y_h$$

We first finding the complementary solution y_h using the method of characteristic equation

$$r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

the general solution for y_h is

$$y_h = c_1 \cos 2x + c_2 \sin 2x$$

where c_1 and c_2 are arbitrary coefficients.

Find y_p, we consider

$$y_p = x[A \cos 2x + B \sin 2x]$$

when differentiating we have

$$y_p' = A\cos 2x + B\sin 2x + x(-2A\cos 2x - 2B\sin 2x])$$
$$y_p'' = (-2A\sin 2x + 2B\cos 2x) + (-2A\sin 2x + 2B\cos 2x) + x(-4A\cos 2x - 4B\sin 2x)$$

which A and B are the constants that we need to solve. Now substituting y_p, y_p', and y_p'' back into ($\bigstar$), the equation is

$$[-4A\sin 2x + 4B\cos 2x + x(-4A\sin 2x - 4B\sin 2x)] + 4x[A\cos 2x + B\sin 2x] = 12\cos 2x(-4A - 4Ax)$$

Equating the coefficients we have

$$\begin{cases} -4A = 0 & \Rightarrow A = 0 \\ 4B = 12 & \Rightarrow B = 3 \end{cases}$$

this arrived that $\boxed{y_p = 3x\sin 2x}$. The general solution of ($\bigstar$) is

$$y = c_1\cos 2x + c_2 \sin 2x + \boxed{3x\sin 2x}$$

From the given IVP conditions, we can determine the constants c_1 and c_2 ◀

3.4 Variation of Parameters

This method is more powerful than that of undetermined coefficients. It can be used to find y_p even for the LDE with variable coefficients:

$$y'' + p(x)y' + q(x)y = \varphi(x) \tag{3.11}$$

Theorem 3.1 Variation of parameters
Consider the differential equation.

$$y'' + p(x)y' + q(x)y = \varphi(x) \tag{3.12}$$

Assume that $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions for

$$y'' + p(x)y' + q(x)y = 0 \tag{3.13}$$

Then a particular solution to the non-homogeneous DE is

$$y_p(t) = u_1' y_1(t) + u_2' y_2(t) \tag{3.14}$$

where

$$u_1' = -\frac{y_2(x)\varphi(x)}{W(y_1, y_2)}, \quad u_2' = \frac{y_1(x)\varphi(x)}{W(y_1, y_2)} \tag{3.15}$$

$W(y_1, y_2)$ is the Wronskian in which defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) \tag{3.16}$$

Example 3.4.1. Find the general solution of

$$y'' + y = \sec x, \quad 0 < x < \pi/2$$

Example 3.4.2. Find the general solution of

$$y'' - 2y' + 2y = e^x \sin x$$

Solution This equation is a second-order non-homogeneous DE. The general solution should take the form of

$$y = y_p + y_h$$

We find y_h with characteristic equation

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4(2)}}{2} = \boxed{1 \pm i}$$

has two complex roots, so the general solution of y_h is

$$y_h = e^x[c_1 \cos x + c_2 \sin x]$$

Next, we are going to use the method of variation of parameters to determine the particular solution y_p. According to the theorem, y_p to the non-homogeneous DE is

$$y_p(t) = u_1' y_1(t) + u_2' y_2(t)$$

Compute the Wronskian,

$$\begin{aligned} W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + \cos x & e^x \sin x + e^x \cos x \end{vmatrix} \\ &= e^x \cos x(e^x \sin x + e^x \cos x) - e^x \sin x(e^x \cos x - e^x \sin x) \\ &= e^{2x} \end{aligned}$$

and we compute u_1' and u_2',

$$u_1' = -\frac{y_2\,\varphi(x)}{W} = -\frac{e^x \sin x(e^x \sin x)}{e^{2x}} = -\sin^2 x$$

$$u_2' = \frac{y_1\,\varphi(x)}{W} = \frac{e^x \cos x(e^x \sin x)}{e^{2x}} = \sin x \cos x$$

Itegrating u_1' and u_2' with respect to x,

$$\begin{aligned} u_1 = \int -\sin^2 x\, dx &= \int -\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx \\ &= -\frac{x}{2} + \frac{\sin(2x)}{4} && \text{Ignore the constant} \end{aligned}$$

$$\begin{aligned} u_2 = \int \sin x \cos x\, dx &= \frac{1}{2}\int (2\sin x \cos x)\, dx \\ &= -\frac{1}{4}\cos 2x + C \\ &= -\frac{1}{4} + \frac{1}{2}\sin^2 x + C \\ &= \frac{1}{2}\sin^2 x && \text{Ignore the constant} \end{aligned}$$

this yield that the particular solution is

$$y_p = -\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) e^x \cos x + \frac{1}{2}\sin^2 x(e^x \sin x)$$

thus the general solution of the non-homogeneous DE is

$$y = e^x[c_1 \cos x + c_2 \sin x] - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) e^x \cos x + \frac{1}{2} e^x \sin^3 x$$

where c_1 and c_2 are arbitrary constants. ◀

Chapter 4

Power Series Solutions

Example 4.0.1. Solve the IVP

$$\begin{cases} y' + 2xy = x^3 & (1) \\ y(1) = 1 & (2) \end{cases}$$

Find the first 3 nonzero-terms of the Taylor series of $y(x)$ about $x = 1$.

Solution The Taylor series of $y(x)$ about $x = 1$ is

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y^{(3)}(1)}{3!}(x-1)^2 + \cdots$$

by definition.

It remains to find $y^{(n)}(1)$ for $n \geq 1$ (until we get the first 3 nonzero-term)

From Eq(1),

$$y'(x) = x^3 - 2xy(x) \Rightarrow y'(1) = 1^3 - 2(1)y(1) = 1 - 2 = -1$$

From Eq(1), differentiate again,

$$y''(x) = 3x^2 - \left(2x\frac{dy}{dx} + y(2)\right) = 3x^2 - 2xy' - 2y \qquad (3)$$

Substitute $y(1) = 1, y'(1) = -1$ into Eq(3),

$$y''(1) = 3(1)^2 - 2(1)(-1) - 2 = 3$$

Thus the Taylor series is

$$\begin{aligned} y(x) &= 1 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 1 + (1-x) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 2 - x + \frac{3}{2!}(x-1)^2 + \cdots \end{aligned}$$

◀

Q: Here is the question, under what condition does a DE has a solution of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \tag{4.3}$$

This bring us to another section: which is about analytic at a point for a series.

4.1 Analytic at a point

Definition 4.1
If the Taylor series of f, where

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{4.4}$$

exists and converges to $f(x) \quad \forall x \in I$, an open interval containing $x = a$, then function f is analytic at $x = a$.

Example of Analytic functions

All polynomials $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ are analytic $\forall x \in \mathbb{R}$.

Example 4.1.1. Legendre Equation

$$(1-x^2)y'' + 2xy' + \lambda y = 0 \qquad (1)$$

Find a power series solution for this DE.

Solution In the standard form $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = \frac{2}{1-x^2}, \quad q(x) = \frac{\lambda}{1-x^2}$$

Both p and q are analytic at $x = 0$. As $x = 0$ is an ordinary point , Eq(1) will have two linearly independent solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

Substitute $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} a_n x^{n-1}, \ y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \qquad (2)$

and,

$$-x^2 y'' = \sum_{n=2}^{\infty} -(n-1)na_n x^n \Rightarrow 2xy' = \sum_{n=1}^{\infty} 2na_n x^n = 2a_1 x + \sum_{n=2}^{\infty} 2na_n x^n$$
$$\Rightarrow \lambda y = \sum_{n=0}^{\infty} \lambda a_n x^n = \lambda a_0 + \lambda a_1 x + \sum_{n=2}^{\infty} \lambda a_n x^n$$

into Eq(1).

Replacing n to $n+2$ in Eq(2), we obtain

$$\begin{aligned}
(2) &= \sum_{n+2=2}^{\infty} (n+2)(n+1)a_{n+2}\, x^{n+2-2} \\
&= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}\, x^{n} \\
&= (0+2)(0+1)a_2 + (1+2)(1+1)a_3 + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n
\end{aligned}$$

So that

$$(2a_2 + \lambda a_0) + (6a_3 + 2a_1 + \lambda a_1)x + \sum_{n=2}^{\infty}\Big\{(n+1)(n+2)a_{n+2} - n(n-1)a_n + 2na_n + \lambda a_n\Big\}x^n = 0$$

Equating coefficients both sides to zero,

$$\begin{cases} 2a_2 + \lambda q = 0 & (3) \\ 6a_3 + (2+\lambda)a_1 = 0 & (4) \end{cases}$$

[Recurrence relation]

$$(n+1)(n+2)a_{n+2} + [2n + \lambda - n(n-1)]a_n = 0 \text{ for } n \geq 2$$

Write all the a_n's in terms of a_0 and a_1,

$$(3) \Rightarrow a_2 = -\frac{\lambda}{2}a_0$$

$$(4) \Rightarrow a_3 = -\frac{(2+\lambda)}{6}a_1$$

$$\begin{aligned}
(5) \Rightarrow a_4 &= \frac{2(2-3) - \lambda}{4(3)}\, a_0 \\
&= \frac{-2-\lambda}{4(3)}\left(\frac{\lambda}{2}\right) a_0 \\
&= \frac{\lambda(\lambda+2)a_0}{4!}
\end{aligned}$$

The solution is

$$\begin{aligned}
y &= (a_0 + a_2x^2 + a_4x^4 + \cdots) + (a_1 + a_3x^3 + a_5x^5 + \cdots) \\
&= a_0\left(1 - \frac{\lambda}{2}x^2 + \frac{\lambda(\lambda+2)}{4!}x^4 + \cdots\right) + a_1\left(x - \frac{\lambda+2}{3!}x^3 + \frac{\lambda(\lambda+2)}{5!}x^5 + \cdots\right)
\end{aligned}$$

◀

4.2 Regular Singular Point

Theorem 4.1 Frobenius Theorem
Given an equation $P(x)y'' + Q(x)y' + R(x)y = 0$, if x is a regular singular point at $\mathbb{R}$, then there exists a solution of the form

$$y = (x - x_0)^n \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{2n}$$

Example 4.2.1. Solve the DE

$$2xy'' + y' - y = 0$$

and check whether $x = 0$ is a regular singular point of of this equation.

Solution The DE is

$$2xy'' + y' - y = 0 \qquad (1)$$

Firstly, we need to check whether $x = 0$ is a RSP of Eq(1).

By Theorem 3.1, (1) has a Frobenius solution of the form $\sum_{n=0}^{\infty} a_n x^{n+r}$.

To find the solutions, substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

into Eq(1),

$$2xy'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}$$
$$\Rightarrow 2xy'' = 2r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} 2(n+r)(n+r-1) x^{n+r-1} a_n$$

Assumption: Assume that $a_0 \neq 0$,

$$\begin{aligned}
(2) &\Rightarrow 2r(r-1) + r = 0 && \\
&\Rightarrow 2r^2 - r = 0 && \text{Indicial equation} \\
&\Rightarrow r = 0 \quad \text{or} \quad r = \frac{1}{2} && \text{Indicial roots}
\end{aligned}$$

Case 1: When $r = 0$, from Eq(3) we have

$$(3) \Rightarrow [2(n+1)n + (n+1)]a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \; \forall n \geq 0$$

Iterate through $n = 0, 1, 2, \ldots$ and find a_n in terms of a_0,

$$\begin{aligned}
n = 0: &\qquad & a_1 &= \frac{a_0}{1 \times 1} = a_0 \\
n = 1: &\qquad & a_2 &= \frac{a_1}{2 \times 3} = \frac{a_0}{(1 \times 2)(1 \times 3)} \\
n = 2: &\qquad & a_3 &= \frac{a_2}{3 \times 5} = \frac{a_0}{(1 \times 2 \times 3)(1 \times 3 \times 5)} \\
n = 3: &\qquad & a_4 &= \frac{a_3}{4 \times 7} = \frac{a_0(2 \times 4 \times 6 \times 8)}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)(2 \times 4 \times 6 \times 8)} \\
& & &= \frac{a_0}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)}
\end{aligned}$$

by mathematical induction, y_1 can be express as

$$y_1 = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{i=1}^{n}(2n-1)} \right]$$

Case 2: When $r = 1/2$, from Eq(3) we have

$$\begin{aligned}
n = 0: &\qquad & a_1 &= \frac{a_0}{1 \times 3} = \frac{1}{3} a_0 \\
n = 1: &\qquad & a_2 &= \frac{a_1}{5 \times 2} = \frac{a_0}{(1 \times 3 \times 5)(1 \times 2)} \\
n = 2: &\qquad & a_3 &= \frac{a_2}{7 \times 3} = \frac{a_0}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} \\
n = 3: &\qquad & a_4 &= \frac{a_0}{(1 \times 3 \times 5 \times 7 \times 9)(1 \times 2 \times 3 \times 4)} \\
\vdots & & & \vdots
\end{aligned}$$

The second solution is

$$\begin{aligned}
y_2 = \sum_{n=0}^{\infty} a_n x^{n+1/2} &= x^{1/2} \sum_{n=0}^{\infty} a_n x^n \\
&= x^{1/2}[a_0 + a_1 + a_2 + \cdots] \\
&= x^{1/2} \left[1 + \frac{x}{1 \times 3} + \frac{x^2}{(1 \times 3 \times 5)(1 \times 2)} + \frac{x^3}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} + \cdots \right]
\end{aligned}$$

which can be written as

$$y_2 = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{i=1}^{n}(2n+1)} \right]$$

By inspection, y_1 and y_2 are not scalar multiples, implies that they are linearly independent. Therefore the general solution of Eq(1) is $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants. ◄

Remark. In general, we may not get two linearly independent solutions. We are guaranteed by Frobenius theorem that there is at least one solution in the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4.7}$$

then we can use reduction of order to find for the 2nd linearly independent solution.

Example 4.2.2. For the given DE:

$$xy'' + 3y' - y = 0$$

Given that one of its solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{2}{n!\,(n+2)!} x^n = 1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots$$

Find the 2nd LI solution using reduction of order.

Solution We write the original DE in standard form: $y" + p(x)y' + q(x)y = 0$, with $p(x) = \dfrac{3}{x}$.

$$\begin{aligned}\exp\big[-\int p(x)\,dx\big] &= \exp\big[-\int -\frac{3}{x}\,dx\big] \\ &= e^{-3|x|} \\ &= \frac{1}{x^3}\end{aligned}$$

$$\frac{e^{-\int p(x)\,dx}}{y_1^2} = \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots\right)^2} \tag{a}$$

To expand the squared series, we can apply the rule of multiplication for power series:

How to multiply two power series?

If there are two power series such that $f(x) = \sum_{n=0}^{N} a_n (x - x_0)^n$, $g(x) = \sum_{n=0}^{N} b_n (x - x_0)^n$, then the multiplication of these two series are

$$f(x)g(x) = \sum_{n=0}^{N} c_n (x - x_0)^n$$

where $c_n = a_0 b_N + a_1 b_{N-1} + a_2 b_{N-2} + \cdots + a_N b_0$.

In this case, $f(x) = g(x)$ with $a_0 = b + 0 = 1$, $a_1 = b_1 = \frac{1}{3}$, $a_2 = b_2 = \frac{1}{24}$.

$$
\begin{aligned}
c_0 &= a_0 b_0 = 1 \\
c_1 &= a_0 b_1 + a_1 b_0 = 1 \times \frac{1}{3} + \frac{1}{3} \times 1 = \frac{2}{3} \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 = \frac{1}{24} + \frac{1}{9} + \frac{1}{24} = \frac{7}{36} \\
c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = \frac{1}{360} + \frac{1}{72} + \frac{1}{72} = \frac{1}{360} = \frac{1}{30} \\
&\vdots
\end{aligned}
$$

now we can continue to work on Eq(a).

$$
\begin{aligned}
\frac{e^{-\int p(x)\,dx}}{y_1^2} &= \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots\right)^2} \\
&= \frac{1}{x^3(c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3 + \cdots)} \\
&= \frac{1}{x^3\left(1 + \frac{2}{3}x + \frac{7}{36}x^2 + \frac{1}{30}x^3 + \cdots\right)}
\end{aligned} \tag{b}
$$

Using long division to expand Eq(b):

$$
\begin{array}{rl}
 & 1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \cdots \\
\hline
1 + \frac{2}{3}x + \frac{7}{36}x^3 + \cdots &)1 \\
 & \underline{1 + \frac{2}{3}x + \frac{7}{36}x^3 + \cdots} \\
 & -\frac{2}{3}x - \frac{7}{36}x^2 - \frac{1}{30}x^3 + \cdots \\
 & \underline{-\frac{2}{3}x - \frac{4}{9}x^2 - \frac{7}{54}x^3 + \cdots} \\
 & \frac{1}{4}x^2 + \frac{13}{136}x^3 + \cdots \\
 & \underline{\frac{1}{4}x^2 + \frac{1}{6}x^3 + \cdots} \\
 & -\frac{19}{270}x^3 + \cdots
\end{array}
$$

we obtained

$$
\begin{aligned}
\frac{e^{-\int p(x)\,dx}}{y_1^2} &= \frac{1}{x^3}\left(1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \cdots\right) \\
&= x^{-3} - \frac{2}{3}x^{-2} + \frac{1}{4}x^{-1} - \frac{19}{270} + \cdots
\end{aligned}
$$

Now continue and integrate the result,

$$
\begin{aligned}
\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \int \left(x^{-3} - \frac{2}{3}x^{-2} + \frac{1}{4}x^{-1} - \frac{19}{270} + \cdots\right) dx \\
&= -\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots
\end{aligned}
$$

Finally, we can now apply the rule of reduction of order and find y_2.

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \\
&= \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^2 + \cdots\right)\left(-\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots\right) \\
&= -\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots - \frac{1}{6}x^{-1} + \frac{1}{12}x\ln|x| - \frac{19}{810}x^2 + \cdots \\
&\quad - \frac{1}{48} + \frac{1}{36}x + \frac{1}{96}x^2\ln|x| + \frac{1}{540}x^2 + \cdots
\end{aligned}
$$

◀

Example 4.2.3. 1. Show that

$$\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty}(a_{k-1} + a_{k+1})x^k$$

Solution From LHS,

$$\text{LHS} = \sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = \left(a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k\right) + \sum_{k=0}^{\infty} a_k x^{k+1}$$

◀

Chapter 5

Laplace Transform

5.1 Partial Fraction

Example 5.1.1. Decompose the following fractions as a sum of partial fractions

1. $\dfrac{2x^2 - x + 4}{x^3 + 4x}$

2. $\dfrac{3x^2 - 4x + 5}{(x+1)^2(x-2)}$

5.2 Laplace Transform

Definition 5.1 Laplace Transform
Let $f(t)$ be a function defined for all $t \geq 0$. The Laplace Transform (LT) of $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\, dt = \lim_{N \to +\infty} \int_0^N e^{-st} f(t)\, dt \tag{5.1}$$

Example 5.2.1. Evaluate $\mathcal{L}\{\sin \omega t\}$ and $\mathcal{L}\{\cos \omega t\}$, where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2}{3}\pi i}$ is a cubic root of unity.

Solution From complex analysis, the De Moivre's theorem state that

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$$

this implies that

$$\begin{cases} e^{i\theta} + e^{-i\theta} = 2\cos\theta \\ e^{i\theta} - e^{-i\theta} = 2i\sin\theta \end{cases}$$

We let $\theta = \omega t$,

$$
\begin{aligned}
\mathcal{L}\{\cos \omega t\} &= \frac{1}{2}[\mathcal{L}\{e^{i\omega t}\} + \mathcal{L}\{e^{-i\omega t}\}] \\
&= \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s - (-i\omega)}\right] \\
&= \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right] \\
&= \frac{1}{2}\frac{(s + i\omega) + (s - i\omega)}{(s - i\omega)(s + i\omega)} \\
&= \frac{s}{s^2 - (i\omega^2)} \\
&= \frac{s}{s^2 + \omega^2}
\end{aligned}
$$

◀

[Behavior of $F(s)$]

5.2.1 Inverse Laplace Transforms

Consider $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $F(s) = \mathcal{L}\{f(t)\}$. we then said that $f(t)$ is the inverse Laplace transform of $F(s)$ and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{5.2}$$

that is, the original function $f(t)$ itself is the differentiation of the Laplace transform.

Theorem 5.1 Differentiation of Transform
Given the transform $F(s)$ to be

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{+\infty} e^{-st}\, f(t)\, dt \tag{5.3}$$

then the differentiation of transform is

$$\frac{dF}{ds} = \int_0^{+\infty} (-t\, f(t))\, e^{-st}\, dt = \mathcal{L}\{-t\, f(t)\} \tag{5.4}$$

Corollary 5.1.1. The inverse transform of $F(s)$ can be write as

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} \tag{5.5}$$

As with Laplace transforms, we've got the following fact to help us take the inverse transform.

Theorem 5.2 Linearity of Inverse Laplace Transforms
Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\,\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\} \tag{5.6}$$

for any constants a and b.

Example 5.2.2. Solve the following IVP problem:

$$\begin{cases} y'' + 2ty' - 4y = 1 \\ y(0) = y'(0) = 0 \end{cases}$$

Solution Applying Laplace transform to the DE, we have

$$\begin{aligned}
&[s^2Y - sy(0) - y'(0)] - 2\left(s\frac{dY}{ds} + Y\right) - 4Y = \frac{1}{s} \\
&s^2Y - 2s\frac{dY}{ds} - 6Y = \frac{1}{s} \\
&-2s\frac{dY}{ds} + (s^2 - 6)Y = \frac{1}{s} \\
&\frac{dY}{ds} + \frac{6 - s^2}{2s}Y = -\frac{1}{2s^2} \qquad (\triangle)
\end{aligned}$$

upon transform, it now become an 1st-order DE. We further use the integrating factor technique to solve this problem.

$$IF = e^{\int p(s)\,ds} = \exp\left\{\int \frac{6 - s^2}{2s}\,ds\right\} = s^3\, e^{-s^2/4}$$

multiply IF with $(\triangle)$,

$$\begin{aligned}
\frac{d}{ds}\left(Y s^2 e^{-s^2/4}\right) &= -\frac{s}{2}\,e^{-s^2/4} \\
Y s^2 e^{-s^2/4} &= \int -\frac{s}{2}\,e^{-s^2/4}\,ds = e^{-s^2/4} + C \qquad \text{where } C \text{ is a constant} \\
Y &= \frac{1}{s^3} + \frac{Ce^{s^2/4}}{s^3} \qquad (\bullet)
\end{aligned}$$

As $s \to \infty$, we calculate the limit of $(\bullet)$ (use L'Hospital Rule to verify)

$$\lim_{s\to+\infty} Y = \lim_{s\to+\infty}\left(\frac{1}{s^3} + \frac{Ce^{s^2/4}}{s^3}\right) = 0$$

if and only if $C = 0$. Thus

$$Y = \frac{1}{s^3} \quad \Rightarrow y = \frac{t^2}{2!} = \frac{1}{2}t^2$$

◀

5.3 Convolution

Example 5.3.1. Solve the equation

$$y = t + \int_0^t y(\tau)\sin(t - \tau)\,d\tau$$

Solution The original equation can be rewrite as

$$y = t + y \star \sin t$$

Applying Laplace Transform on the original equation, we have

$$Y = \frac{1}{s^2} + Y \cdot \frac{1}{s^2+1}$$

Solving for Y,

$$\begin{aligned}
Y\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2} &\Rightarrow Y\left(\frac{s^2}{s^2+1}\right) = \frac{1}{s^2} \\
&\Rightarrow Y = \frac{s^2+1}{s^4} \\
&\Rightarrow Y = \frac{1}{s^2} + \frac{1}{s^4}
\end{aligned}$$

Applying inverse Laplace transform on Y, we obtain

$$\begin{aligned}
\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^4}\right\} &\Rightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} \\
&\Rightarrow y = t + \frac{t^3}{3!} = t + \frac{1}{6}t^3
\end{aligned}$$

◀

Theorem 5.3 First Shifting Theorem
If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(x)\} = F(s-a)$.

Proof.

$$\mathcal{L}\{e^{at} f(x)\} = \int_0^{+\infty} e^{-st} e^{at} f(t)\, dt = \int_0^{+\infty} e^{-(s-a)t} f(t)\, dt = F(s-a)$$

which has been shown. □

The Laplace Second Shifting Theorem, on the other hand, states that the Laplace transform of the delayed function equals the product of the Laplace transform of the original function and the shifted function.

5.4 More Properties of Laplace Transform

5.4.1 Step function

Definition 5.2 Unit Step Function
The step function $H(t-a)$ is defined to be

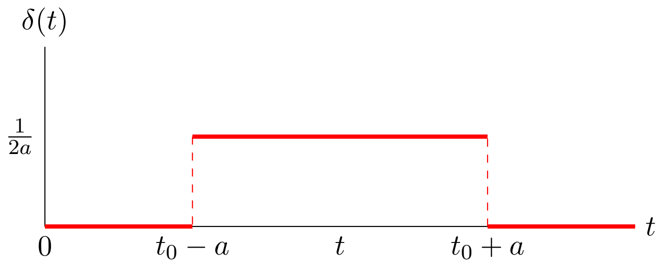
$$H(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases} \tag{5.7}$$

the unit step function is also known as the Heaviside function.

Theorem 5.4 Second Shifting Theorem
Suppose

5.4.2 Dirac Delta function

A unit impulse function is a function that is "on" for a short period and 0 ("off") otherwise



mathematically,

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \le t_0 < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t_0 < t_0 + a \\ 0, & t_0 \ge t_0 + a \end{cases} \tag{5.8}$$

The unit impulse function has the property

$$\int_0^{+\infty} \delta_a(t - t_0)\, dt = 1 \tag{5.9}$$

the limit of this function as $a \to 0$ is the Dirac Delta function. Note that $\delta(t - a)$ is not a proper function.

Definition 5.3 Dirac Delta function
The Dirac Delta function is defined as

$$\delta(t - a) = \lim_{k \to 0^+} f_k(t) = \begin{cases} +\infty, & \text{if } t = a \\ 0, & \text{if } t \ne a \end{cases} \tag{5.10}$$

where

$$f_k(t) = \begin{cases} \dfrac{1}{k}, & \text{if } a \le t \le a + k \\ 0, & \text{otherwise} \end{cases} \tag{5.11}$$

Example 5.4.1. Solve the IVP problem:

$$y'' + 2y' + 5y = 25t - \delta(t - \pi), \quad y(0) = -2, \quad y'(0) = 5$$

where δ is known as Dirac Delta function.

Solution Applying Laplace transform on the original DE, we have

$$
\begin{aligned}
[s^2Y - sy(0) - y'(0)] + 2[sY - y(0)] + 5Y &= \frac{25}{s^2} - e^{-\pi s} \\
s^2 + 2s - 5 + 2[sY + 2] + 5Y &= \frac{25}{s^2} - e^{-\pi s}
\end{aligned}
$$

Solving for Y,

$$
\begin{aligned}
(s^2 + 2s + 5)Y &= \frac{25}{s^2} - e^{-\pi s} - 2s + 1 \\
Y &= \frac{25}{s^2(s^2 + 2s + 5)} - \frac{e^{-\pi s}}{s^2 + 2s + 5} + \frac{1 - 2s}{s^2 + 2s + 5} \\
Y &= \frac{5}{s^2} - \frac{2}{s} - \frac{e^{-\pi s}}{s^2 + 2s + 5} \\
Y &= \frac{5}{s^2} - \frac{2}{s} - e^{-\pi s} F(s) \qquad (\diamond)
\end{aligned}
$$

Notice that $F(s)$ can be simplify for ease of inverse transformation,

$$
\begin{aligned}
F(s) = \frac{1}{s^2 + 2s + 5} &= \frac{1}{(s^2 + 2s + 2) + 2} \\
&= \frac{1}{(s + 1)^2 + 1} \\
&= \frac{1}{2}\left[\frac{2}{(s + 1)^2 + 2^2}\right]
\end{aligned}
$$

inverting $F(s)$ will obtain $f(t) = \frac{1}{2}e^{-t}\sin(2t)$.

Furthermore, we again invert $(\diamond)$,

$$
\begin{aligned}
y &= 5t - 2 - f(t - \pi)\,H(t - \pi) \\
&= 5t - 2 - \frac{1}{2}e^{-(t-\pi)}\sin 2(t - \pi)\,H(t - \pi) \\
&= 5t - 2 - \frac{1}{2}e^{-(t-\pi)}[\sin 2t \cos \pi - \cos 2t \sin 2\pi]\,H(t - \pi) \\
\Rightarrow y &= 5t - 2 - \frac{1}{2}e^{-(t-\pi)}\sin(2t)\,H(t - \pi)
\end{aligned}
$$

which this is the solution for the DE. ◀

5.5 Periodic Function

Definition 5.4 Periodic function
The function f is said to be periodic with period $p > 0$, if $f(t+p) = f(t)$ for all $t \in$ Domain f.

For example, $\sin(t + 2\pi) = \sin(t)$, so the function $\sin(t)$ is a periodic function with period of 2π.

Theorem 5.5 Laplace Transform of Periodic Function
Suppose $\mathcal{L}\{f(t)\}$ exists and $\exists \hat{\tau} > 0$ s.t. $f(t+\hat{\tau}) = f(t)$ for all $t \geq 0$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-st}} \int_0^{\hat{\tau}} e^{-st} f(t)\, dt \tag{5.12}$$

5.6 Tutorials

Exercise 5.6.1 Express the fraction $\dfrac{x^2+7x-3}{(x-2)(x^2+1)}$ as the sum of partial fractions.

Exercise 5.6.2

1. Show that $\sin 3x = 3\sin x - 4\sin^3 x$.
2. Hence, using the identity to evaluate $\mathcal{L}\{\sin^3 x\}$.

Exercise 5.6.3 Find $\mathcal{L}\left\{ \right\}$

Exercise 5.6.4 Solve the initial-value problem

$$x'' + 16x = \cos(4t), \quad x(0) = 0, \quad x'(0) = 1$$

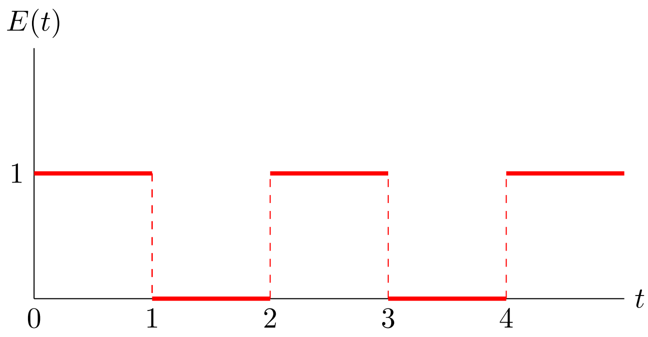
Exercise 5.6.5 Solve the equation

$$y' + 4y + 5\int_0^t y\, dt = e^{-t}, \quad y(0) = 0$$

Exercise 5.6.6 Solve the integral equation

$$y(t) = e^{-t} - 2\int_0^t y(u)\cos(t-u)\, du$$

Exercise 5.6.7 Find the Laplace transform of the periodic function shown in figure below.



Exercise 5.6.8 In the two-mesh network shown below, the switch is closed at $t = 0$ and the voltage source is given by $V(t) = 150\sin(1000t)$. Find the mesh currents i_1 and i_2.