# Chapter 1

# Solutions of Differential Equation

# Chapter 2

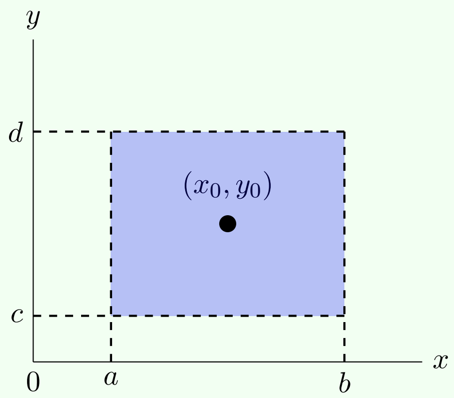
# Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.1}$$

**Theorem 2.1** Peano's Existence theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point $(x_0, y_0)$. If the function $f(x, y)$ is continuous in $R$.

$$y' = f(x, y), \quad y(x_0) = y_0$$




in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

**Example 2.0.1.** Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

**Solution** The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that $f$ is continuous everywhere in the $xy$-plane except on the line $x = 0$ (which is the $y$-axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

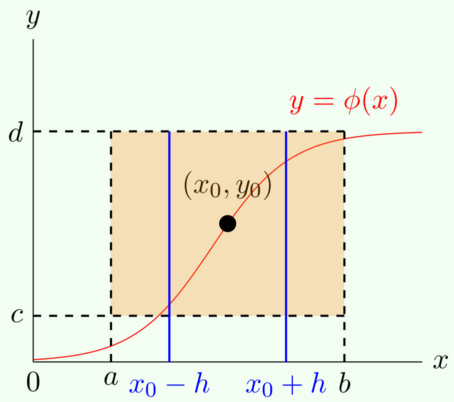
### An IVP with more than one solution

**Example 2.0.2.** Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

*Remark.* The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points $(x, y)$ where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

**Theorem 2.2** Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point $(x_0, y_0)$.




**Example 2.0.3.** Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

**Solution** Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that $f$ is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, $f$ and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀