Chapter 1

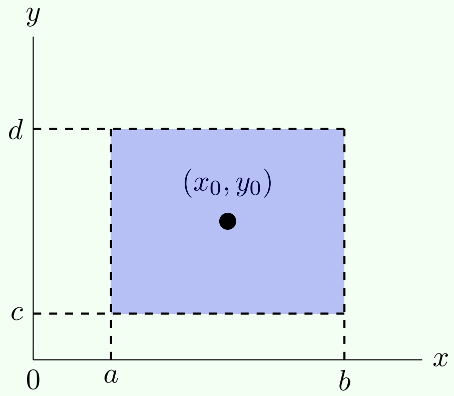
Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.1}$$

Theorem 1.1 Peano's Existence theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0). If the function $f(x, y)$ is continuous in R.

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy-plane except on the line $x = 0$ (which is the y-axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

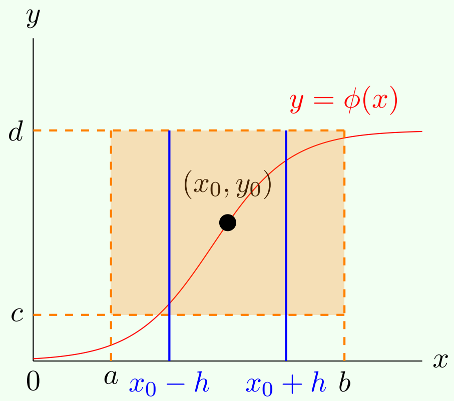
Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0).



Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x,y) = \frac{x(y^2-9)}{x^2+1} & (1) \\ y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x,y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0\,(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.

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Chapter 2

Solving First-order Differential Equation

Theorem 2.1
If function $f(x)$ and function $g(x)$ are continuous, then equation

$$\int f(x)dx = \int g(y)dy + C$$

Example 2.0.1. Solve $e^{x+y}\,dy - 1\,dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y}\,dy = 1\,dx \quad &\Rightarrow e^x * e^y\,dy = 1\,dx \\ &\Rightarrow e^y\,dy = e^{-x}\,dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^y\,dy = \int e^{-x}\,dx \quad &\Rightarrow e^y = -e^{-x} + C && e^y > 0 \text{ so that RHS } > 0 \\ &\Rightarrow y = \ln|-e^{-x} + C| && \text{general solution in implicit form} \end{aligned}$$

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Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x\,dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} = -\frac{1}{2}x^2 + c_1 &\Rightarrow \frac{1}{y} = x^2 - 2c_1 \\ &\Rightarrow y = \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).
Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$. ◀

2.1 Exact Equation

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1}$$

is exact if $\exists F(x,y)$ such that $M\,dx + N\,dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x,y) = C$, C is a constant.

Definition 2.1 Total differential
Let $F(x,y)$ be a function that has continuous first derivative in a domain D.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \forall (x,y) \in D \tag{2.2}$$

Theorem 2.2 Test for Exactness
Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectanger $R : a < x < b,\ c < y < d$. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.3}$$

Proof. ($\Rightarrow$) If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R, according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \tag{2.4}$$

at each point of R.

($\Leftarrow$) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.5}$$

to prove $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.6}$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\phi x \phi y} \tag{2.7}$$

Integrating both sides with respect to x, we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \tag{2.8}$$

$\square$

Example 2.1.1. Solve $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$.

Solution The DE is in the form of $M\,dx + N\,dy = 0$, where

$$\begin{cases} M = \dfrac{1}{t^2} + \dfrac{1}{y^2} \\ N = \dfrac{at+1}{y^3} \end{cases}$$

In order to make DE to be exact, we must have $M_y = N_t \Rightarrow a = \ldots$. ◀

Example 2.1.2. Solve the initial-value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$M = 2x\cos y + 3x^2y, \quad N = x^3 - x^2\sin y - y$$
$$M_y = -2x\sin y + 3x^2, \quad N_x = 3x^2 - 2x\sin y$$

Since $M_y = N_x = 3x^2 - 2x\sin y \quad \forall (x,y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D. Next, we must find F such that

$$\begin{cases} F_x = M = 2x\cos y + 3x^2y & (a) \\ M_y = x^3 - x^2\sin y - y & (b) \end{cases}$$

From $(a) \Rightarrow F = \int (2x\cos y + 3x^2y)\,dx = x^2\cos y + x^3y + g(y)$. (c)
where g is a function of y.

Again, $(c) \Rightarrow F_y = -x^2\sin y + x^3 + g'(y)$ (d)

Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2\cos y + x^3y - \frac{1}{2}y^2 + C$$

Hence a 1-parameter family of solutions is $F(x,y) = 0$ or $x^2\cos y + x^3y - \frac{1}{2}y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C: Subtituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0,2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2\cos y + x^3y - \frac{1}{2}y^2 + 2$. ◀

Chapter 3

Power Series Solutions