

Chapter 1

Existence and Uniqueness of Solutions

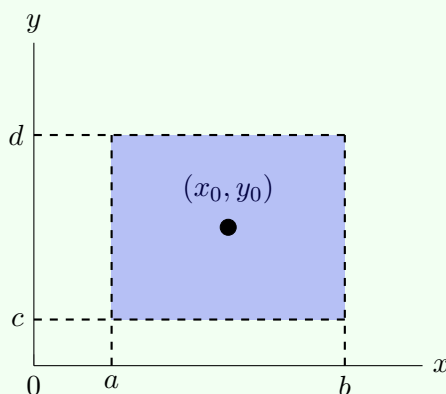
In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (1.1)$$

Theorem 1.1 Peano's Existence theorem

Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0) . If the function $f(x, y)$ is continuous in R .

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy -plane except on the line $x = 0$ (which is the y -axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

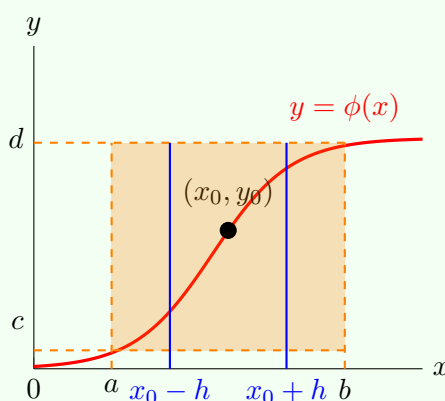
$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem

Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0) .



Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point $(2, 5)$. By Picard's theorem, this IVP has a unique solution. \blacktriangleleft

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x, y) = \frac{x(y^2 - 9)}{x^2 + 1} & (1) \\ y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x, y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.



Chapter 2

Solving First-order Differential Equation

Theorem 2.1

If function $f(x)$ and function $g(y)$ are continuous, then equation

$$\int f(x)dx = \int g(y)dy + C$$

Example 2.0.1. Solve $e^{x+y} dy - 1 dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y} dy &= 1 dx &\Rightarrow e^x * e^y dy &= 1 dx \\ &&\Rightarrow e^y dy &= e^{-x} dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^y dy &= \int e^{-x} dx &\Rightarrow e^y &= -e^{-x} + C && e^y > 0 \text{ so that RHS } > 0 \\ &&\Rightarrow y &= \ln | -e^{-x} + C | && \text{general solution in implicit form} \end{aligned}$$

◀

Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} &= -\frac{1}{2}x^2 + c_1 &\Rightarrow \frac{1}{y} &= x^2 - 2c_1 \\ &&\Rightarrow y &= \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).

Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$.

◀

2.1 Exact Equation

The equation

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1)$$

is exact if $\exists F(x, y)$ such that $M dx + N dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x, y) = C$, C is a constant.

Definition 2.1 Total differential

Let $F(x, y)$ be a function that has continuous first derivative in a domain D .

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \quad \forall (x, y) \in D \quad (2.2)$$

Theorem 2.2 Test for Exactness

Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectangle $R : a < x < b, c < y < d$. Then

$$M(x, y) dx + N(x, y) dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.3)$$

Proof. (\Rightarrow) If $M(x, y) dx + N(x, y) dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R , according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \quad (2.4)$$

at each point of R .

(\Leftarrow) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.5)$$

to prove $M(x, y) dx + N(x, y) dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2.6)$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} \quad (2.7)$$

Integrating both sides with respect to x , we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \quad (2.8)$$

□

Example 2.1.1. Solve $3x(xy - 2) dx + (x^3 + 2y) dy = 0$.

Solution The DE is in the form of $M dx + N dy = 0$, where

$$\begin{cases} M = \frac{1}{t^2} + \frac{1}{y^2} \\ N = \frac{at+1}{y^3} \end{cases}$$

In order to make DE to be exact, we must have $M_y = N_t \Rightarrow a = \dots$ ◀

Example 2.1.2. Solve the initial-value problem

$$(2x \cos y + 3x^2 y) dx + (x^3 - x^2 \sin y - y) dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$M = 2x \cos y + 3x^2 y, \quad N = x^3 - x^2 \sin y - y$$

$$M_y = -2x \sin y + 3x^2, \quad N_x = 3x^2 - 2x \sin y$$

Since $M_y = N_x = 3x^2 - 2x \sin y \quad \forall (x, y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D . Next, we must find F such that

$$\begin{cases} F_x = M = 2x \cos y + 3x^2 y & (a) \\ M_y = x^3 - x^2 \sin y - y & (b) \end{cases}$$

$$\text{From (a)} \Rightarrow F = \int (2x \cos y + 3x^2 y) dx = x^2 \cos y + x^3 y + g(y). \quad (c)$$

where g is a function of y .

$$\text{Again, (c)} \Rightarrow F_y = -x^2 \sin y + x^3 + g'(y) \quad (d)$$

Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2 \cos y + x^3 y - \frac{1}{2} y^2 + C$$

Hence a 1-parameter family of solutions is $F(x, y) = 0$ or $x^2 \cos y + x^3 y - \frac{1}{2} y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C : Substituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0, 2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2 \cos y + x^3 y - \frac{1}{2} y^2 + 2$. ◀

Example 2.1.3. Determine the constant a so that the equation

$$\frac{1}{t^2} + \frac{1}{y^2} + \left(\frac{at+1}{3} \right) \frac{dy}{dt} = 0$$

is exact, and then solve the resulting equation.

Theorem 2.3

The general solution to an exact equation $M(x, y) dx + N(x, y) dy = 0$ is defined implicitly

by

$$F(x, y) = C \quad (2.11)$$

where F is a potential function of the DE and C is an arbitrary constant.

Remark. We can find F by solving the system of equations

$$\begin{cases} \frac{\partial F}{\partial x} = M \\ \frac{\partial F}{\partial y} = N \end{cases} \quad (2.12)$$

Proof. If $M(x, y) dx + N(x, y) dy = 0$ is exact, then \exists a potential function F such that $M(x, y) dx + N(x, y) dy = dF$.

This gives us $dF = 0$ so that $F(x, y) = C$, where C is an arbitrary constant. \square

2.2 Making an Equation Exact: Integrating Factors

Sometimes it is impossible to transform a nonexact DE that into an exact equation by multiplying it by a function. The resulting DE can be resolved using the technique of the previous section. However, it is impossible for a solution to be lost or gained as a result of the multiplication.

Definition 2.2

If $M(x, y) dx + N(x, y) dy = 0$ is not exact but $I(x, y)M(x, y) dx + N(x, y)I(x, y) dy = 0$ is exact, then $I(x, y)$ is called an integrating factor of the DE.

Remark. We may be able to determine $I(x, y)$ from the equation

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial x}(IN) \quad (2.13)$$

Example 2.2.1. Verify that $I(x, y) = x^{-1}$ is an integrating factor of $(x + y) dx + x \ln x dy = 0$ on the interval $(0, \infty)$. Hence, find the solution for this DE.

Solution

$$(x + y) dx + x \ln x dy = 0 \quad (1)$$

First, show that Eq(1) is not exact. Suppose $M = x + y$, and $N = x \ln x$. Then

$$M_y = 1, \quad N_x = x \left(\frac{1}{x} \right) + \ln x = 1 + \ln x$$

Because $M_y \neq N_x$, so Eq(1) is not exact.

Next, multiplying Eq(1) by $I(x, y) = \frac{1}{x}$, we obtain

$$\frac{(x + y)}{x} dx + \ln x dy = 0 \quad (2)$$

Now we show that Eq(2) is exact. Let $\tilde{M} = 1 + \frac{1}{y}$, and $\tilde{N} = \ln x$, then

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} = \frac{1}{x}$$

The general solution will be $F(x, y) = C$, we can find F by solving the system of equations

$$\left\{ \begin{array}{l} \frac{\partial F}{\partial x} = 1 + \frac{y}{x} \\ \frac{\partial F}{\partial y} = \ln x \end{array} \right. \quad (3)$$

Integrating Eq(3) with respect to x we have

$$\int (3) dx \Rightarrow F = \int \left(1 + \frac{y}{x} \right) dx = x + y \ln x + g(y) \quad (5)$$

where $g(y)$ is a function of y alone.

Differentiate Eq(5) with respect to y ,

$$\frac{\partial F}{\partial y} = \ln |x| + g'(y) \quad (6)$$

Now comparing Eq(4) and Eq(6), we obtain $g'(y) = g(y) = 0$.

Thus, the general solution is $F = x + y \ln |x| + 0$.



Summary: Solving 1st order DE

1. Separable equation: $f(x) dx = g(y) dy$. (Method of solving: integrating both sides)
2. Exact DE: Use **Exactness Test**, whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Chapter 3

Power Series Solutions