Chapter 1

Solutions of Differential Equation

Chapter 2

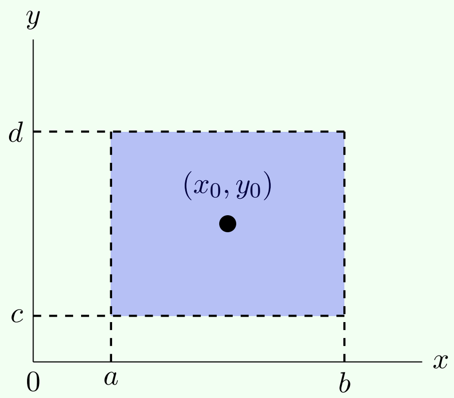
Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.1}$$

Theorem 2.1 Peano's Existence theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0). If the function $f(x, y)$ is continuous in R.

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 2.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy-plane except on the line $x = 0$ (which is the y-axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◄

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

Example 2.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Theorem 2.2 Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0).

Example 2.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀