Chapter 1

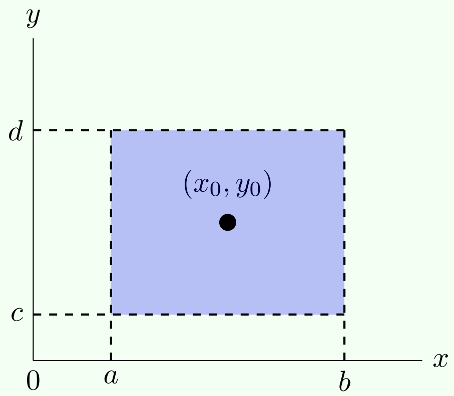
Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{1.1}$$

Theorem 1.1 Peano's Existence theorem
Let $R = \{(x,y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0). If the function $f(x,y)$ is continuous in R.

$$y' = f(x,y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x,y)$ where $f(x,y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy-plane except on the line $x = 0$ (which is the y-axis). Since the initial point $(1,0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

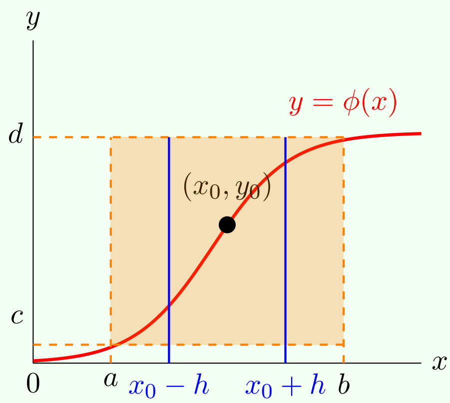
Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0).



Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x, y) = \frac{x(y^2-9)}{x^2+1} & (1) \\ y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x, y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0\,(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.

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Chapter 2

Solving First-order Differential Equation

Theorem 2.1
If function $f(x)$ and function $g(x)$ are continuous, then equation

$$\int f(x)dx = \int g(y)dy + C$$

Example 2.0.1. Solve $e^{x+y}\,dy - 1\,dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y}\,dy = 1\,dx \quad &\Rightarrow e^{x} * e^{y}\,dy = 1\,dx \\ &\Rightarrow e^{y}\,dy = e^{-x}\,dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^{y}\,dy = \int e^{-x}\,dx \quad &\Rightarrow e^{y} = -e^{-x} + C && e^{y} > 0 \text{ so that RHS } > 0 \\ &\Rightarrow y = \ln|-e^{-x} + C| && \text{general solution in implicit form} \end{aligned}$$

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Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x\,dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} = -\frac{1}{2}x^2 + c_1 &\Rightarrow \frac{1}{y} = x^2 - 2c_1 \\ &\Rightarrow y = \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).
Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$. ◀

2.1 Exact Equation

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1}$$

is exact if $\exists F(x,y)$ such that $M\,dx + N\,dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x,y) = C$, C is a constant.

Definition 2.1 Total differential
Let $F(x,y)$ be a function that has continuous first derivative in a domain D.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \forall (x,y) \in D \tag{2.2}$$

Theorem 2.2 Test for Exactness
Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectanger $R : a < x < b,\ c < y < d$. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.3}$$

Proof. ($\Rightarrow$) If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R, according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \tag{2.4}$$

at each point of R.

($\Leftarrow$) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.5}$$

to prove $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.6}$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\phi x \phi y} \tag{2.7}$$

Integrating both sides with respect to x, we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \tag{2.8}$$

□

Example 2.1.1. Solve $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$.

Solution The DE is in the form of $M\,dx + N\,dy = 0$, where

$$\begin{cases} M = \dfrac{1}{t^2} + \dfrac{1}{y^2} \\ N = \dfrac{at+1}{y^3} \end{cases}$$

In order to make DE to be exact, we must have $M_y = N_t \Rightarrow a = \ldots$. ◀

Example 2.1.2. Solve the initial-value problem

$$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$M = 2x\cos y + 3x^2 y, \quad N = x^3 - x^2\sin y - y$$

$$M_y = -2x\sin y + 3x^2, \quad N_x = 3x^2 - 2x\sin y$$

Since $M_y = N_x = 3x^2 - 2x\sin y \quad \forall (x,y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D. Next, we must find F such that

$$\begin{cases} F_x = M = 2x\cos y + 3x^2 y & (a) \\ M_y = x^3 - x^2\sin y - y & (b) \end{cases}$$

From $(a) \Rightarrow F = \int (2x\cos y + 3x^2 y)\,dx = x^2\cos y + x^3 y + g(y)$. (c)
where g is a function of y.

Again, $(c) \Rightarrow F_y = -x^2\sin y + x^3 + g'(y)$ (d)

Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2\cos y + x^3 y - \frac{1}{2}y^2 + C$$

Hence a 1-parameter family of solutions is $F(x,y) = 0$ or $x^2\cos y + x^3 y - \frac{1}{2}y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C: Subtituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0,2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2\cos y + x^3 y - \frac{1}{2}y^2 + 2$. ◀

Example 2.1.3. Determine the constant a so that the equation

$$\frac{1}{t^2} + \frac{1}{y^2} + \left(\frac{at+1}{3}\right)\frac{dy}{dt} = 0$$

is exact, and then solve the resulting equation.

Theorem 2.3
The general solution to an exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is defined implicitly

by

$$F(x, y) = C \tag{2.11}$$

where F is a potential function of the DE and C is an arbitrary constant.

Remark. We can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N \end{cases} \tag{2.12}$$

Proof. If $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact, then $\exists$ a potential function F such that $M(x, y)\,dx + N(x, y)\,dy = dF$.

This gives us $dF = 0$ so that $F(x, y) = C$, where C is an arbitrary constant. $\square$

2.2 Making an Equation Exact: Integrating Factors

Sometimes it is impossible to transform a nonexact DE that into an exact equation by multiplying it by a function. The resulting DE can be resolved using the technique of the previous section. However, it is impossible for a solution to be lost or gained as a result of the multiplication.

Definition 2.2
If $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact but $I(x, y)M(x, y)\,dx + N(x, y)I(x, y)\,dy = 0$ is exact, then $I(x, y)$ is called an integrating factor of the DE.

Remark. We may be able to determine $I(x, y)$ from the equation

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial y}(IN) \tag{2.13}$$

Example 2.2.1. Verify that $I(x, y) = x^{-1}$ is an integrating factor of $(x + y)\,dx + x \ln x\,dy = 0$ on the interval $(0, \infty)$. Hence, find the solution for this DE.

Solution

$$(x + y)\,dx + x \ln x\,dy = 0 \qquad (1)$$

First, show that Eq(1) is not exact. Suppose $M = x + y$, and $N = x \ln x$. Then

$$M_y = 1, \quad N_x = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

Because $M_y \neq N_x$, so Eq(1) is not exact.

Next, multiplying Eq(1) by $I(x, y) = \dfrac{1}{x}$, we obtain

$$\frac{(x + y)}{x}\,dx + \ln x\,dy = 0 \qquad (2)$$

Now we show that Eq(2) is exact. Let $\tilde{M} = 1 + \dfrac{1}{y}$, and $\tilde{N} = \ln x$, then

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} = \frac{1}{x}$$

The general solution will be $F(x, y) = C$, we can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = 1 + \dfrac{y}{x} & (3) \\ \dfrac{\partial F}{\partial y} = \ln x & (4) \end{cases}$$

Integrating Eq(3) with respect to x we have

$$\int (3)dx \quad \Rightarrow F = \int \left(1 + \frac{y}{x}\right) dx = x + y \ln x + g(y) \qquad (5)$$

where $g(y)$ is a function of y alone.

Differentiate Eq(5) with respect to y,

$$\frac{\partial F}{\partial y} = \ln|x| + g'(y) \qquad (6)$$

Now comparing Eq(4) and Eq(6), we obtain $g'(y) = g(y) = 0$.

Thus, the general solution is $F = x + y \ln|x| + 0$.

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Summary: Solving 1st order DE

1. Separable equation: $f(x)\,dx = g(y)\,dy$. (Method of solving: integrating both sides)
2. Exact DE: Use **Exactness Test**, whether $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Chapter 3

Power Series Solutions

Example 3.0.1. Solve the IVP

$$\begin{cases} y' + 2xy = x^3 & (1) \\ y(1) = 1 & (2) \end{cases}$$

Find the first 3 nonzero-terms of the Taylor series of $y(x)$ about $x = 1$.

Solution The Taylor series of $y(x)$ about $x = 1$ is

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y^{(3)}(1)}{3!}(x-1)^2 + \cdots$$

by definition.

It remains to find $y^{(n)}(1)$ for $n \geq 1$ (until we get the first 3 nonzero-term)

From Eq(1),

$$y'(x) = x^3 - 2xy(x) \Rightarrow y'(1) = 1^3 - 2(1)y(1) = 1 - 2 = -1$$

From Eq(1), differentiate again,

$$y''(x) = 3x^2 - \left(2x\frac{dy}{dx} + y(2)\right) = 3x^2 - 2xy' - 2y \qquad (3)$$

Substitute $y(1) = 1, y'(1) = -1$ into Eq(3),

$$y''(1) = 3(1)^2 - 2(1)(-1) - 2 = 3$$

Thus the Taylor series is

$$\begin{aligned} y(x) &= 1 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 1 + (1-x) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 2 - x + \frac{3}{2!}(x-1)^2 + \cdots \end{aligned}$$

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Q: Here is the question, under what condition does a DE has a solution of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \qquad (3.3)$$

This bring us to another section: which is about analytic at a point for a series.

3.1 Analytic at a point

Definition 3.1
If the Taylor series of f, where

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3.4}$$

exists and converges to $f(x) \quad \forall x \in I$, an open interval containing $x = a$, then function f is analytic at $x = a$.

Example of Analytic functions

All polynomials $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ are analytic $\forall x \in \mathbb{R}$.

Example 3.1.1. Legendre Equation

$$(1-x^2)y'' + 2xy' + \lambda y = 0 \qquad (1)$$

Find a power series solution for this DE.

Solution In the standard form $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = \frac{2}{1-x^2}, \quad q(x) = \frac{\lambda}{1-x^2}$$

Both p and q are analytic at $x = 0$. As $x = 0$ is an ordinary point , Eq(1) will have two linearly independent solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

Substitute $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \qquad (2)$

and,

$$-x^2 y'' = \sum_{n=2}^{\infty} -(n-1)n a_n x^n \Rightarrow 2xy' = \sum_{n=1}^{\infty} 2n a_n x^n = 2a_1 x + \sum_{n=2}^{\infty} 2n a_n x^n$$

$$\Rightarrow \lambda y = \sum_{n=0}^{\infty} \lambda a_n x^n = \lambda a_0 + \lambda a_1 x + \sum_{n=2}^{\infty} \lambda a_n x^n$$

into Eq(1).
Replacing n to $n+2$ in Eq(2), we obtain

$$\begin{aligned}
(2) &= \sum_{n+2=2}^{\infty} (n+2)(n+1)a_{n+2}\, x^{n+2-2} \\
&= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}\, x^n \\
&= (0+2)(0+1)a_2 + (1+2)(1+1)a_3 + \sum_{n=2}^{\infty} (n+1)(n+2)a_{n+2} x^n
\end{aligned}$$

So that

$$(2a_2 + \lambda a_0) + (6a_3 + 2a_1 + \lambda a_1)x + \sum_{n=2}^{\infty}\Big\{(n+1)(n+2)a_{n+2} - n(n-1)a_n + 2na_n + \lambda a_n\Big\}x^n = 0$$

Equating coefficients both sides to zero,

$$\begin{cases} 2a_2 + \lambda q = 0 & (3) \\ 6a_3 + (2+\lambda)a_1 = 0 & (4) \end{cases}$$

[Recurrence relation]

$$(n+1)(n+2)a_{n+2} + [2n + \lambda - n(n-1)]a_n = 0 \text{ for } n \geq 2$$

Write all the a_n's in terms of a_0 and a_1,

$$(3) \Rightarrow a_2 = -\frac{\lambda}{2}a_0$$

$$(4) \Rightarrow a_3 = -\frac{(2+\lambda)}{6}a_1$$

$$\begin{aligned} (5) \Rightarrow a_4 &= \frac{2(2-3)-\lambda}{4(3)}\,a_0 \\ &= \frac{-2-\lambda}{4(3)}\left(\frac{\lambda}{2}\right)a_0 \\ &= \frac{\lambda(\lambda+2)a_0}{4!} \end{aligned}$$

The solution is

$$\begin{aligned} y &= (a_0 + a_2x^2 + a_4x^4 + \cdots) + (a_1 + a_3x^3 + a_5x^5 + \cdots) \\ &= a_0\left(1 - \frac{\lambda}{2}x^2 + \frac{\lambda(\lambda+2)}{4!}x^4 + \cdots\right) + a_1\left(x - \frac{\lambda+2}{3!}x^3 + \frac{\lambda(\lambda+2)}{5!}x^5 + \cdots\right) \end{aligned}$$

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3.2 Regular Singular Point

Theorem 3.1 Frobenius Theorem
Given an equation $P(x)y'' + Q(x)y' + R(x)y = 0$, if x is a regular singular point at $\mathbb{R}$, then there exists a solution of the form

$$y = (x - x_0)^n \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n(x-x_0)^{2n}$$

Example 3.2.1. Solve the DE

$$2xy'' + y' - y = 0$$

and check whether $x = 0$ is a regular singular point of of this equation.

Solution The DE is

$$2xy'' + y' - y = 0 \qquad (1)$$

Firstly, we need to check whether $x = 0$ is a RSP of Eq(1).

By Theorem 3.1, (1) has a Frobenius solution of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^{n+r}$.

To find the solutions, substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

into Eq(1),

$$2xy'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}$$

$$\Rightarrow 2xy'' = 2r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} 2(n+r)(n+r-1) x^{n+r-1} a_n$$

Assumption: Assume that $a_0 \neq 0$,

$$\begin{aligned} (2) &\Rightarrow 2r(r-1) + r = 0 && \\ &\Rightarrow 2r^2 - r = 0 && \text{Indicial equation} \\ &\Rightarrow r = 0 \quad \text{or} \quad r = \frac{1}{2} && \text{Indicial roots} \end{aligned}$$

Case 1: When $r = 0$, from Eq(3) we have

$$(3) \Rightarrow [2(n+1)n + (n+1)] a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \ \forall n \geq 0$$

Iterate through $n = 0, 1, 2, \ldots$ and find a_n in terms of a_0,

$$\begin{aligned} n = 0: &\quad a_1 = \frac{a_0}{1 \times 1} = a_0 \\ n = 1: &\quad a_2 = \frac{a_1}{2 \times 3} = \frac{a_0}{(1 \times 2)(1 \times 3)} \\ n = 2: &\quad a_3 = \frac{a_2}{3 \times 5} = \frac{a_0}{(1 \times 2 \times 3)(1 \times 3 \times 5)} \\ n = 3: &\quad a_4 = \frac{a_3}{4 \times 7} = \frac{a_0 (2 \times 4 \times 6 \times 8)}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)(2 \times 4 \times 6 \times 8)} \\ &\quad = \frac{a_0}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} \end{aligned}$$

by mathematical induction, y_1 can be express as

$$y_1 = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \, \prod_{i=1}^{n} (2n-1)} \right], \quad n \in \mathbb{Z}^+$$

Case 2: When $r = 1/2$, from Eq(3) we have

$$\begin{aligned}
n = 0: &\quad a_1 = \frac{a_0}{1 \times 3} = \frac{1}{3} a_0 \\
n = 1: &\quad a_2 = \frac{a_1}{5 \times 2} = \frac{a_0}{(1 \times 3 \times 5)(1 \times 2)} \\
n = 2: &\quad a_3 = \frac{a_2}{7 \times 3} = \frac{a_0}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} \\
n = 3: &\quad a_4 = \frac{a_0}{(1 \times 3 \times 5 \times 7 \times 9)(1 \times 2 \times 3 \times 4)}
\end{aligned}$$

The second solution is

$$\begin{aligned}
y_2 = \sum_{n=0}^{\infty} a_n x^{n+1/2} &= x^{1/2} \sum_{n=0}^{\infty} a_n x^n \\
&= x^{1/2}[a_0 + a_1 + a_2 + \cdots] \\
&= x^{1/2}\left[1 + \frac{x}{1 \times 3} + \frac{x^2}{(1 \times 3 \times 5)(1 \times 2)} + \frac{x^3}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} + \cdots\right]
\end{aligned}$$

which can be written as

$$y_2 = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, \prod_{i=1}^{n}(2n+1)}\right]$$

By inspection, y_1 and y_2 are not scalar multiples, implies that they are linearly independent. Therefore the general solution of Eq(1) is $y = c_1 y_1 + c_2 y_2$. ◀

Example 3.2.2. 1. Show that

$$\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty} (a_{k-1} + a_{k+1}) x^k$$

Solution ◀