Chapter 1

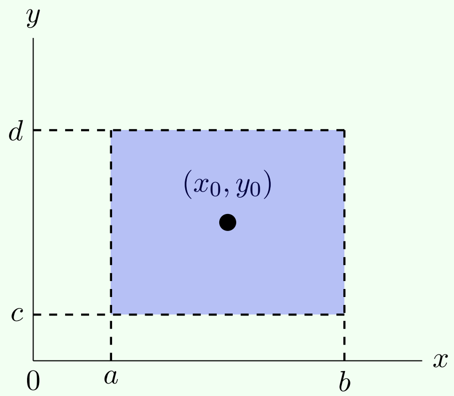
Existence and Uniqueness of Solutions

In this topic, we would like to address the existence and uniqueness to the general first-order IVP:

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.1}$$

Theorem 1.1 Peano's Existence theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be a open rectangular region containing the point (x_0, y_0). If the function $f(x, y)$ is continuous in R.

$$y' = f(x, y), \quad y(x_0) = y_0$$



in some interval $x_0 - h < x < x_0 + h$ contained in $a < x < b$.

Example 1.0.1. Determine whether Peano's Existence theorem does or does not guarantee existence of a solution of the initial value problem:

$$xy' = y, \quad y(1) = 0$$

Solution The DE can be written as $y' = f(x, y)$ where $f(x, y) = \frac{y}{x}$. Observe that f is continuous everywhere in the xy-plane except on the line $x = 0$ (which is the y-axis). Since the initial point $(1, 0)$. Hence, the theorem guarantees the existence of a solution of the IVP. ◀

The next example tells us that there are first-order initial value problems that have more than one solutions.

An IVP with more than one solution

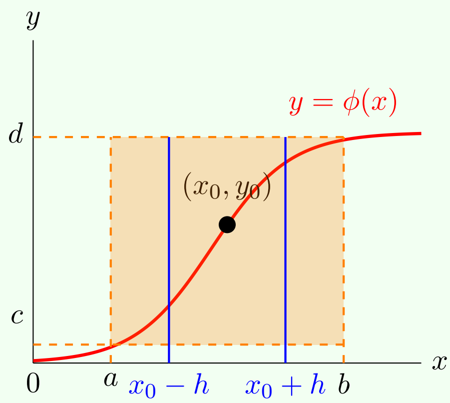
Example 1.0.2. Verify that the function $y_1 = 0$ and $y_2 = x$ are solutions of the initial value problem

$$xy' = y, \quad y(0) = 0$$

Remark. The function $f(x, y) = \frac{y}{x}$ is continuous everywhere in the plane except at the points (x, y) where $x = 0$. Thus, Peano Existence theorem does not guarantee the existence of a solution in some neighbourhood of the initial point $(0, 0)$.

Obviously, the next thing we would like to find out is that if an IVP does have a solution, what conditions could we impose on (5) to

Theorem 1.2 Picard's Existence and Uniqueness Theorem
Let $R = \{(x, y) \mid a < x < b, c < y < d\}$ be an open rectangular region containing the point (x_0, y_0).



Example 1.0.3. Determine whether Picard's theorem guarantees that the first-order IVP

$$y' = y^2 + x^3, \quad y(2) = 5$$

has a unique solution.

Solution Consider the following IVP

$$\begin{cases} y' = f(x, y) = y^2 + x^3 \\ y(2) = 5 \end{cases}$$

Observe that f is continuous $\forall (x, y) \in \mathbb{R}$. And since

$$f_y(x, y) = \frac{\partial f}{\partial y} = 2y \text{ is continuous } \forall (x, y) \in \mathbb{R}$$

Thus, f and $\frac{\partial f}{\partial y}$ are continuous near the initial point (2, 5). By Picard's theorem, this IVP has a unique solution. ◀

Example 1.0.4. Use Picard's theorem or Peano Existence theorem to discuss the existence and uniqueness of the solutions of the following IVP

$$y' = 3y^{2/3}, \quad y(x_0) = y_0$$

Example 1.0.5. Use the Picard's existence and uniqueness theorem to prove that $y(x) = 3$ is the only solution to the IVP

$$y' = \frac{x(y^2 - 9)}{x^2 + 1}, \quad y(0) = 3$$

Solution

$$\begin{cases} y' = f(x, y) = \frac{x(y^2-9)}{x^2+1} & (1) \\ y(x_0) = y_0 & (2) \end{cases}$$

Being rational function, f is continuous $\forall (x, y) \in \mathbb{R}^2$ except at the points where $x^2 + 1 = 0$. Since $x^2 + 1 \neq 0 \quad \forall x \in \mathbb{R}$, so f is continuous, the same condition can be apply on $\partial f / \partial y$.

First off, we need to show that $y(x) = 3$ is a solution to the IVP. By direct substitution, we substitute $y(x) = 3, y'(x) = 0$ into Eq (1).

$$\text{LHS of Eq (1)} = 0 = \frac{0\,(3^2 - 9)}{0^2 + 1} = \text{RHS of Eq (1)}$$

Also, $y(x) = 3 \Rightarrow y(0) = 3 \Rightarrow y(x) = 3$ satisfies condition (2).

By Picard's theorem, $y(x) = 3$ is the only solution to the IVP.

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Chapter 2

Solving First-order Differential Equation

Theorem 2.1
If function $f(x)$ and function $g(x)$ are continuous, then the DE is solvable by performing integration on both sides, said

$$\int f(x)dx = \int g(y)\,dy + C$$

Example 2.0.1. Solve $e^{x+y}\,dy - 1\,dx = 0$.

Solution The DE is separable and can be formulate as

$$\begin{aligned} e^{x+y}\,dy = 1\,dx \quad &\Rightarrow e^x * e^y\,dy = 1\,dx \\ &\Rightarrow e^y\,dy = e^{-x}\,dx \end{aligned}$$

Integrating both sides we have

$$\begin{aligned} \int e^y\,dy = \int e^{-x}\,dx \quad &\Rightarrow e^y = -e^{-x} + C && e^y > 0 \text{ so that RHS } > 0 \\ &\Rightarrow y = \ln|-e^{-x} + C| && \text{general solution in implicit form} \end{aligned}$$

◀

Example 2.0.2. Find all solutions to $y' = -2y^2x$. Be sure to describe any singular solutions if there is one.

Solution Is this DE separable? Yes, since it can be written as

$$-\frac{dy}{y^2} = 2x\,dx$$

Integrating both sides of the equation, we have

$$\begin{aligned} -\frac{1}{2y} = -\frac{1}{2}x^2 + c_1 &\Rightarrow \frac{1}{y} = x^2 - 2c_1 \\ &\Rightarrow y = \frac{1}{x^2 - 2c_1} \end{aligned}$$

By inspection, $y = 0$ is another solution (obvious solution).
Therefore, the solutions are $y = 0$ and $y = (x^2 - 2c)^{-1} \quad \forall x \in \mathbb{R}$. ◀

2.1 Exact Equation

The equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1}$$

is exact if $\exists F(x,y)$ such that $M\,dx + N\,dy = dF$. In this case, the solution to the DE is given by $dF = 0$ or $F(x,y) = C$, C is a constant.

Definition 2.1 Total differential
Let $F(x,y)$ be a function that has continuous first derivative in a domain D.

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \forall (x,y) \in D \tag{2.2}$$

Theorem 2.2 Test for Exactness
Suppose $M, N, \frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous in the open rectanger $R : a < x < b,\ c < y < d$. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ if and only if } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.3}$$

Proof. ($\Rightarrow$) If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, then we can find a potential function F such that $F_x = M$ and $F_y = N$. As the first-order partial derivatives of M and N are continuous in R, according to the commutative law of partial derivative operator,

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x} \tag{2.4}$$

at each point of R.

($\Leftarrow$) On the other hand, consider

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.5}$$

to prove $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, we must show that we can construct a function F such that $F_x = M$ and $F_y = N$.

Let ϕ to be a function such that $\frac{\partial \phi}{\partial x} = M$. Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2.6}$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\phi x \phi y} \tag{2.7}$$

Integrating both sides with respect to x, we get

$$N = \frac{\partial \phi}{\partial y} + B'(y) \tag{2.8}$$

□

Example 2.1.1. Solve $3x(xy-2)\,dx + (x^3+2y)\,dy = 0$.

Solution The DE is in the form of $M\,dx + N\,dy = 0$, with the test of exactness

$$\begin{cases} M = 3x(xy-2) & \Rightarrow M_y = 3x^2 - 0 = 3x^2 \\ N = x^3 + 2y & \Rightarrow N_x = 3x^2 \end{cases}$$

To find the general solution of Eq(1): $F(x,y) = C$.
Find the function F by solving the system

$$\begin{cases} \dfrac{\partial F}{\partial x} = M = 3x(xy-2) = 3x^2y - 6x \\ \dfrac{\partial F}{\partial y} = N = x^3 + 2y \end{cases}$$

Now integrate (a) with respect to x, regarding y as a constant.

$$(a) \Rightarrow F = \int_y (3x^2y - 6x)\,dx = x^3y - 3x^2 + g(y)$$

where g is a function of y alone. Again, we partial derivative on y to obtain $g'(y)$.

$$\frac{\partial F}{\partial y} = x^3 - 0 + g'(y) = x^3 + g'(y)$$

comparing to Eq(b), we have $g'(y) = 2y$. Which means $g(y) = y^2$. (ignored constant). In result, $F = x^3y - 3x^2 + y^2$ is the general solution of the DE.

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Example 2.1.2. Solve the initial-value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0, \quad y(0) = 2$$

Solution First, we have to determine whether or not the equation is exact. Here

$$M = 2x\cos y + 3x^2y, \quad N = x^3 - x^2\sin y - y$$
$$M_y = -2x\sin y + 3x^2, \quad N_x = 3x^2 - 2x\sin y$$

Since $M_y = N_x = 3x^2 - 2x\sin y \quad \forall (x,y) \in \mathbb{R}^2$, the DE is exact in every rectangular domain D. Next, we must find F such that

$$\begin{cases} F_x = M = 2x\cos y + 3x^2y & (a) \\ M_y = x^3 - x^2\sin y - y & (b) \end{cases}$$

From $(a) \Rightarrow F = \int (2x\cos y + 3x^2y)\,dx = x^2\cos y + x^3y + g(y)$. (c)
where g is a function of y.
Again, $(c) \Rightarrow F_y = -x^2\sin y + x^3 + g'(y)$ (d)
Now comparing (b) and (d),

$$g'(y) = -y \xrightarrow{\text{Integrate with respect to } y} g(y) = -\frac{y^2}{2} + C \quad \text{where } C \text{ is an arbitrary constant}$$

Thus, we have potential function

$$F = x^2\cos y + x^3y - \frac{1}{2}y^2 + C$$

Hence a 1-parameter family of solutions is $F(x,y) = 0$ or $x^2\cos y + x^3y - \dfrac{1}{2}y^2 + C$.

Finally, we can now use the initial condition $y(0) = 2$ to find C: Subtituting $x = 0, y = 2$ into the above solution, we obtain

$$F(0, 2) = 0 + 0 - 2 + C = 0 \Rightarrow C = 2$$

Therefore, the solution to the IVP is $x^2 \cos y + x^3 y - \frac{1}{2}y^2 + 2$. ◀

Example 2.1.3. Determine the constant a so that the equation

$$\frac{1}{t^2} + \frac{1}{y^2} + \left(\frac{at+1}{3}\right)\frac{dy}{dt} = 0$$

is exact, and then solve the resulting equation.

Theorem 2.3
The general solution to an exact equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is defined implicitly by

$$F(x, y) = C \tag{2.11}$$

where F is a potential function of the DE and C is an arbitrary constant.

Remark. We can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = M \\ \dfrac{\partial F}{\partial y} = N \end{cases} \tag{2.12}$$

Proof. If $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact, then $\exists$ a potential function F such that $M(x, y)\,dx + N(x, y)\,dy = dF$.

This gives us $dF = 0$ so that $F(x, y) = C$, where C is an arbitrary constant. □

2.2 Making an Equation Exact: Integrating Factors

Sometimes it is impossible to transform a nonexact DE that into an exact equation by multiplying it by a function. The resulting DE can be resolved using the technique of the previous section. However, it is impossible for a solution to be lost or gained as a result of the multiplication.

Definition 2.2
If $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact but $I(x, y)M(x, y)\,dx + N(x, y)I(x, y)\,dy = 0$ is exact, then $I(x, y)$ is called an integrating factor of the DE.

Remark. We may be able to determine $I(x, y)$ from the equation

$$\frac{\partial}{\partial y}(IM) = \frac{\partial}{\partial y}(IN) \tag{2.13}$$

Example 2.2.1. Verify that $I(x, y) = x^{-1}$ is an integrating factor of $(x + y)\,dx + x \ln x\,dy = 0$ on the interval $(0, \infty)$. Hence, find the solution for this DE.

Solution

$$(x + y)\,dx + x \ln x\,dy = 0 \qquad (1)$$

First, show that Eq(1) is not exact. Suppose $M = x + y$, and $N = x \ln x$. Then

$$M_y = 1, \quad N_x = x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

Because $M_y \neq N_x$, so Eq(1) is not exact.
Next, multiplying Eq(1) by $I(x, y) = \dfrac{1}{x}$, we obtain

$$\frac{(x+y)}{x}\, dx + \ln x\, dy = 0 \qquad (2)$$

Now we show that Eq(2) is exact. Let $\tilde{M} = 1 + \dfrac{1}{y}$, and $\tilde{N} = \ln x$, then

$$\frac{\partial \tilde{M}}{\partial y} = \frac{\partial \tilde{N}}{\partial x} = \frac{1}{x}$$

The general solution will be $F(x, y) = C$, we can find F by solving the system of equations

$$\begin{cases} \dfrac{\partial F}{\partial x} = 1 + \dfrac{y}{x} & (3) \\ \dfrac{\partial F}{\partial y} = \ln x & (4) \end{cases}$$

Integrating Eq(3) with respect to x we have

$$\int (3) dx \quad \Rightarrow F = \int \left(1 + \frac{y}{x}\right) dx = x + y \ln x + g(y) \qquad (5)$$

where $g(y)$ is a function of y alone.
Differentiate Eq(5) with respect to y,

$$\frac{\partial F}{\partial y} = \ln|x| + g'(y) \qquad (6)$$

Now comparing Eq(4) and Eq(6), we obtain $g'(y) = g(y) = 0$.
Thus, the general solution is $F = x + y \ln|x| + 0$.

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Theorem 2.4 Existence and Uniqueness Theorem for a 1st-order DE
If the functions p and q are continuous on an open interval $I : a < x < b$ containing the point x_0, then the IVP

$$\frac{dy}{dx} + p(x)y = q(x), \quad y(x_0) = y_0 \qquad (2.16)$$

has a unique solution in the largest open interval containing x_0, in which both p and q are continuous.

Proof. The DE can be written as $y' = f(x, y)$, where $f(x, y) = q(x) - p(x)y$.
Moreover, since p and q are continuous on I, the solution

$$y(x) = e^{-\int_{x_0}^{x} p(t)\, dt} \left[\int_{x_0}^{x} q(t) e^{-\int_{x_0}^{x} p(t)\, dt}\, dt + y_0\right] \qquad (2.17)$$

is well defined $\forall x \in I$. □

Example 2.2.2. Find the largest interval I on which the initial value problem

$$xy' + 2y = 4x^2, \quad y(-1) = 2$$

has a unique problem.

Example 2.2.3 (Radioactive Decay)**.** A certain radioactive isotope is known to decay at a rate of proportional to the amount present. Initially, 100 grams of the isotope are present, but after 75 years its mass decays to 75 grams.

1. Setup and solve an initial-value problem for $N(t)$, the mass of the isotope at time t.
2. What is the half-life of the substance?

Note: Half-life of a radioactive substance is the time required for half of it to decay.

Solution Let $N(t)$ be the amount of material at time t,
Given $\dfrac{dN}{dt} \propto N$. So $\dfrac{dN}{dt} = kN$ where k is the constant of proportionality.
Given the initial amount $N(0) = 100$, and $N(50) = 75$. ◀

2.3 Substitution Method

Example 2.3.1. Use an appropriate substitution to solve

$$\frac{dy}{dx} = \sin^2(3x - 3y + 1)$$

Solution Substituting $u = 3x - 3y + 1$, $\frac{du}{dx} = 3 - 3\frac{dy}{dx}$ or $\frac{dy}{dx} = 1 - \frac{1}{3}\frac{du}{dx}$ into the given DE. We obtain

$$\begin{aligned}
1 - \frac{1}{3}\frac{du}{dx} = \sin^2 u \Rightarrow \frac{du}{dx} &= 3\cos^2 u \\
&\Rightarrow \sec^2 u \, du = 3\, dx \\
&\Rightarrow \tan u = 3x + C && (C \text{ is a constant}) \\
&\Rightarrow u = \arctan(3x + C) \\
&\Rightarrow 3x - 3y + 1 = \arctan(3x + C)
\end{aligned}$$

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Thus, $3x - 3y + 1 = \arctan(3x + C)$ is the solution of the original DE.

Example 2.3.2. Use an appropriate substitution to solve

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$$

Solution Substituting $u = y - 2x + 3$, $\frac{du}{dx} = \frac{dy}{dx} - 2$ or $\frac{dy}{dx} = 2 + \frac{du}{dx}$ into the given DE. We obtain

$$\begin{aligned}
2 + \frac{du}{dx} = 2 + \sqrt{u} \Rightarrow \frac{du}{dx} &= \sqrt{u} \\
&\Rightarrow \frac{1}{\sqrt{u}} \, du = dx \\
&\Rightarrow \int \frac{1}{\sqrt{u}} \, du = \int dx \\
&\Rightarrow 2\sqrt{u} = x + C && (C \text{ is a constant}) \\
&\Rightarrow 4u = (x + C)^2 \\
&\Rightarrow 4(y - 2x + 3) = (x + C)^2
\end{aligned}$$

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2.4 Homogeneous Equation

A first-order DE $y' = f(x,y)$ is homogeneous if $f(x,y)$ can be expressed as a function of the ratio y/x alone.

Theorem 2.5
The substitution $v = y/x$ (or $v = x/y$) will reduce the homogeneous DE

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{2.18}$$

to a separable DE.

Example 2.4.1. Solve $(x-y)y' = x+y$

Solution Certainly, $(x-y)y' = x+y \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$ is homogeneous.
By using the substitution

$$\begin{cases} v = \frac{y}{x} \Rightarrow y = vx \\ \frac{dy}{dx} = v + x\frac{dv}{dx} \end{cases}$$

substitute into Eq(1), we have

$$\begin{aligned} (1) \Rightarrow \frac{dy}{dx} &= \frac{1+\frac{y}{x}}{1-\frac{y}{x}} \\ \Rightarrow v + x\frac{dv}{dx} &= \frac{1+v}{1-v} \\ \Rightarrow x\frac{dv}{dx} &= \frac{1+v}{1-v} - v \\ \Rightarrow \frac{1-v}{1+v^2}\,dv &= \frac{1}{x}\,dx \end{aligned}$$

integrating both sides

$$\begin{aligned} \int \frac{1-v}{1+v^2}\,dv = \int \frac{1}{x}\,dx \quad &\Rightarrow \int \left[\frac{1}{1+v^2} + \frac{-v}{1+v^2}\right] dv = \ln|x| + C \\ &\Rightarrow \arctan(v) - \frac{1}{2}\ln|1+v^2| = \ln|x| + C \end{aligned}$$

again, we substitute $v = \frac{y}{x}$ back to the result

$$\begin{aligned} &\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left|1+\frac{y^2}{x^2}\right| = \ln|x| + C \\ &\Rightarrow 2\arctan\left(\frac{y}{x}\right) = \ln\left|1+\frac{y^2}{x^2}\right| + 2\ln|x| + 2C \\ &\Rightarrow 2\arctan\left(\frac{y}{x}\right) = \ln(x^2+y^2) + 2C \end{aligned}$$

is the general solution of the original DE. ◀

Summary: Solving 1st order DE

1. Separable equation: $f(x)\,dx = g(y)\,dy$. (Method of solving: integrating both sides)
2. Exact DE: Use **Exactness Test**, whether $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Chapter 3

Power Series Solutions

Example 3.0.1. Solve the IVP

$$\begin{cases} y' + 2xy = x^3 & (1) \\ y(1) = 1 & (2) \end{cases}$$

Find the first 3 nonzero-terms of the Taylor series of $y(x)$ about $x = 1$.

Solution The Taylor series of $y(x)$ about $x = 1$ is

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y^{(3)}(1)}{3!}(x-1)^2 + \cdots$$

by definition.

It remains to find $y^{(n)}(1)$ for $n \geq 1$ (until we get the first 3 nonzero-term)

From Eq(1),

$$y'(x) = x^3 - 2xy(x) \Rightarrow y'(1) = 1^3 - 2(1)y(1) = 1 - 2 = -1$$

From Eq(1), differentiate again,

$$y''(x) = 3x^2 - \left(2x\frac{dy}{dx} + y(2)\right) = 3x^2 - 2xy' - 2y \qquad (3)$$

Substitute $y(1) = 1, y'(1) = -1$ into Eq(3),

$$y''(1) = 3(1)^2 - 2(1)(-1) - 2 = 3$$

Thus the Taylor series is

$$\begin{aligned} y(x) &= 1 + (-1)(x-1) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 1 + (1-x) + \frac{3}{2!}(x-1)^2 + \cdots \\ &= 2 - x + \frac{3}{2!}(x-1)^2 + \cdots \end{aligned}$$

◀

Q: Here is the question, under what condition does a DE has a solution of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{3.3}$$

This bring us to another section: which is about analytic at a point for a series.

3.1 Analytic at a point

Definition 3.1
If the Taylor series of f, where

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{3.4}$$

exists and converges to $f(x)$ $\quad \forall x \in I$, an open interval containing $x = a$, then function f is analytic at $x = a$.

Example of Analytic functions

All polynomials $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ are analytic $\forall x \in \mathbb{R}$.

Example 3.1.1. Legendre Equation

$$(1-x^2)y'' + 2xy' + \lambda y = 0 \qquad (1)$$

Find a power series solution for this DE.

Solution In the standard form $y'' + p(x)y' + q(x)y = 0$, we have

$$p(x) = \frac{2}{1-x^2}, \quad q(x) = \frac{\lambda}{1-x^2}$$

Both p and q are analytic at $x = 0$. As $x = 0$ is an ordinary point , Eq(1) will have two linearly independent solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

Substitute $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \qquad (2)$
and,

$$-x^2 y'' = \sum_{n=2}^{\infty} -(n-1)na_n x^n \Rightarrow 2xy' = \sum_{n=1}^{\infty} 2na_n x^n = 2a_1 x + \sum_{n=2}^{\infty} 2na_n x^n$$
$$\Rightarrow \lambda y = \sum_{n=0}^{\infty} \lambda a_n x^n = \lambda a_0 + \lambda a_1 x + \sum_{n=2}^{\infty} \lambda a_n x^n$$

into Eq(1).
Replacing n to $n+2$ in Eq(2), we obtain

$$\begin{aligned}(2) = &\sum_{n+2=2}^{\infty} (n+2)(n+1)a_{n+2}\, x^{n+2-2} \\ &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\, x^n \\ &= (0+2)(0+1)a_2 + (1+2)(1+1)a_3 + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}x^n\end{aligned}$$

So that

$$(2a_2 + \lambda a_0) + (6a_3 + 2a_1 + \lambda a_1)x + \sum_{n=2}^{\infty}\Big\{(n+1)(n+2)a_{n+2} - n(n-1)a_n + 2na_n + \lambda a_n\Big\}x^n = 0$$

Equating coefficients both sides to zero,

$$\begin{cases} 2a_2 + \lambda q = 0 & (3) \\ 6a_3 + (2+\lambda)a_1 = 0 & (4) \end{cases}$$

[Recurrence relation]

$$(n+1)(n+2)a_{n+2} + [2n + \lambda - n(n-1)]a_n = 0 \text{ for } n \geq 2$$

Write all the a_n's in terms of a_0 and a_1,

$$(3) \Rightarrow a_2 = -\frac{\lambda}{2}a_0$$

$$(4) \Rightarrow a_3 = -\frac{(2+\lambda)}{6}a_1$$

$$\begin{aligned}(5) \Rightarrow a_4 &= \frac{2(2-3)-\lambda}{4(3)}\,a_0 \\ &= \frac{-2-\lambda}{4(3)}\left(\frac{\lambda}{2}\right)\,a_0 \\ &= \frac{\lambda(\lambda+2)a_0}{4!}\end{aligned}$$

The solution is

$$\begin{aligned} y &= (a_0 + a_2x^2 + a_4x^4 + \cdots) + (a_1 + a_3x^3 + a_5x^5 + \cdots) \\ &= a_0\left(1 - \frac{\lambda}{2}x^2 + \frac{\lambda(\lambda+2)}{4!}x^4 + \cdots\right) + a_1\left(x - \frac{\lambda+2}{3!}x^3 + \frac{\lambda(\lambda+2)}{5!}x^5 + \cdots\right)\end{aligned}$$

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3.2 Regular Singular Point

Theorem 3.1 Frobenius Theorem
Given an equation $P(x)y'' + Q(x)y' + R(x)y = 0$, if x is a regular singular point at $\mathbb{R}$, then there exists a solution of the form

$$y = (x-x_0)^n \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n(x-x_0)^{2n}$$

Example 3.2.1. Solve the DE

$$2xy'' + y' - y = 0$$

and check whether $x = 0$ is a regular singular point of of this equation.

Solution The DE is

$$2xy'' + y' - y = 0 \qquad (1)$$

Firstly, we need to check whether $x = 0$ is a RSP of Eq(1).

By Theorem 3.1, (1) has a Frobenius solution of the form $\displaystyle\sum_{n=0}^{\infty} a_n x^{n+r}$.

To find the solutions, substitute

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = r a_0 x^{r-1} + \sum_{n=1}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

into Eq(1),

$$2xy'' = \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1}$$

$$\Rightarrow 2xy'' = 2r(r-1) a_0 x^{r-1} + \sum_{n=1}^{\infty} 2(n+r)(n+r-1) x^{n+r-1} a_n$$

Assumption: Assume that $a_0 \neq 0$,

$$\begin{aligned} (2) &\Rightarrow 2r(r-1) + r = 0 \\ &\Rightarrow 2r^2 - r = 0 && \text{Indicial equation} \\ &\Rightarrow r = 0 \quad \text{or} \quad r = \frac{1}{2} && \text{Indicial roots} \end{aligned}$$

Case 1: When $r = 0$, from Eq(3) we have

$$(3) \Rightarrow [2(n+1)n + (n+1)] a_{n+1} = a_n \quad \Rightarrow \quad a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \; \forall n \geq 0$$

Iterate through $n = 0, 1, 2, \ldots$ and find a_n in terms of a_0,

$$\begin{aligned} n = 0: &\qquad a_1 = \frac{a_0}{1 \times 1} = a_0 \\ n = 1: &\qquad a_2 = \frac{a_1}{2 \times 3} = \frac{a_0}{(1 \times 2)(1 \times 3)} \\ n = 2: &\qquad a_3 = \frac{a_2}{3 \times 5} = \frac{a_0}{(1 \times 2 \times 3)(1 \times 3 \times 5)} \\ n = 3: &\qquad a_4 = \frac{a_3}{4 \times 7} = \frac{a_0(2 \times 4 \times 6 \times 8)}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)(2 \times 4 \times 6 \times 8)} \\ &\qquad\quad = \frac{a_0}{(1 \times 2 \times 3 \times 4)(1 \times 3 \times 5 \times 7)} \end{aligned}$$

by mathematical induction, y_1 can be express as

$$y_1 = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \, \prod_{i=1}^{n} (2n-1)} \right], \quad n \in \mathbb{Z}^+$$

Case 2: When $r = 1/2$, from Eq(3) we have

$$\begin{aligned}
n = 0 : \quad & a_1 = \frac{a_0}{1 \times 3} = \frac{1}{3} a_0 \\
n = 1 : \quad & a_2 = \frac{a_1}{5 \times 2} = \frac{a_0}{(1 \times 3 \times 5)(1 \times 2)} \\
n = 2 : \quad & a_3 = \frac{a_2}{7 \times 3} = \frac{a_0}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} \\
n = 3 : \quad & a_4 = \frac{a_0}{(1 \times 3 \times 5 \times 7 \times 9)(1 \times 2 \times 3 \times 4)} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

The second solution is

$$\begin{aligned}
y_2 = \sum_{n=0}^{\infty} a_n x^{n+1/2} &= x^{1/2} \sum_{n=0}^{\infty} a_n x^n \\
&= x^{1/2} [a_0 + a_1 + a_2 + \cdots] \\
&= x^{1/2} \left[1 + \frac{x}{1 \times 3} + \frac{x^2}{(1 \times 3 \times 5)(1 \times 2)} + \frac{x^3}{(1 \times 3 \times 5 \times 7)(1 \times 2 \times 3)} + \cdots \right]
\end{aligned}$$

which can be written as

$$y_2 = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{i=1}^{n} (2n+1)} \right]$$

By inspection, y_1 and y_2 are not scalar multiples, implies that they are linearly independent. Therefore the general solution of Eq(1) is $y = c_1 y_1 + c_2 y_2$, where c_1 and c_2 are arbitrary constants. ◀

Remark. In general, we may not get two linearly independent solutions. We are guaranteed by Frobenius theorem that thre is at least one solution in the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{3.7}$$

then we can use reduction of order to find for the 2nd linearly independent solution.

Example 3.2.2. For the given DE:

$$xy'' + 3y' - y = 0$$

Given that one of its solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{2}{n!\,(n+2)!} x^n = 1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots$$

Find the 2nd LI solution using reduction of order.

Solution We write the original DE in standard form: $y" + p(x)y' + q(x)y = 0$, with $p(x) = \dfrac{3}{x}$.

$$\begin{aligned}
\exp\left[-\int p(x)\,dx\right] &= \exp\left[-\int -\frac{3}{x}\,dx\right] \\
&= e^{-3|x|} \\
&= \frac{1}{x^3}
\end{aligned}$$

$$\frac{e^{-\int p(x)\,dx}}{y_1^2} = \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots\right)^2} \tag{a}$$

To expand the squared series, we can apply the rule of multiplication for power series:

How to multiply two power series?

If there are two power series such that $f(x) = \sum_{n=0}^{N} a_n(x - x_0)^n$, $g(x) = \sum_{n=0}^{N} b_n(x - x_0)^n$, then the multiplication of these two series are

$$f(x)g(x) = \sum_{n=0}^{N} c_n(x - x_0)^n$$

where $c_n = a_0 b_N + a_1 b_{N-1} + a_2 b_{N-2} + \cdots + a_N b_0$.

In this case, $f(x) = g(x)$ with $a_0 = b + 0 = 1$, $a_1 = b_1 = \frac{1}{3}$, $a_2 = b_2 = \frac{1}{24}$.

$$\begin{aligned}
c_0 &= a_0 b_0 = 1 \\
c_1 &= a_0 b_1 + a_1 b_0 = 1 \times \frac{1}{3} + \frac{1}{3} \times 1 = \frac{2}{3} \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 = \frac{1}{24} + \frac{1}{9} + \frac{1}{24} = \frac{7}{36} \\
c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = \frac{1}{360} + \frac{1}{72} + \frac{1}{72} = \frac{1}{360} = \frac{1}{30} \\
&\vdots
\end{aligned}$$

now we can continue to work on Eq(a).

$$\begin{aligned}
\frac{e^{-\int p(x)\,dx}}{y_1^2} &= \frac{1}{x^3} \cdot \frac{1}{\left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^3 + \cdots\right)^2} \\
&= \frac{1}{x^3(c_0 + c_1\,x + c_2\,x^2 + c_3\,x^3 + \cdots)} \\
&= \frac{1}{x^3\left(1 + \frac{2}{3}x + \frac{7}{36}x^2 + \frac{1}{30}x^3 + \cdots\right)}
\end{aligned} \tag{b}$$

Using long division to expand Eq(b):

$$\begin{array}{r l}
 & 1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \cdots \\ \cline{2-2}
1 + \frac{2}{3}x + \frac{7}{36}x^3 + \cdots & \big)1 \\
 & 1 + \frac{2}{3}x + \frac{7}{36}x^3 + \cdots \\ \cline{2-2}
 & \quad - \frac{2}{3}x - \frac{7}{36}x^2 - \frac{1}{30}x^3 + \cdots \\
 & \quad - \frac{2}{3}x - \frac{4}{9}x^2 - \frac{7}{54}x^3 + \cdots \\ \cline{2-2}
 & \qquad \frac{1}{4}x^2 + \frac{13}{136}x^3 + \cdots \\
 & \qquad \frac{1}{4}x^2 + \frac{1}{6}x^3 + \cdots \\ \cline{2-2}
 & \qquad\quad - \frac{19}{270}x^3 + \cdots
\end{array}$$

we obtained

$$\begin{aligned}\frac{e^{-\int p(x)\,dx}}{y_1^2} &= \frac{1}{x^3}\left(1 - \frac{2}{3}x + \frac{1}{4}x^2 - \frac{19}{270}x^3 + \cdots\right)\\ &= x^{-3} - \frac{2}{3}x^{-2} + \frac{1}{4}x^{-1} - \frac{19}{270} + \cdots\end{aligned}$$

Now continue and integrate the result,

$$\begin{aligned}\int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx &= \int\left(x^{-3} - \frac{2}{3}x^{-2} + \frac{1}{4}x^{-1} - \frac{19}{270} + \cdots\right)dx\\ &= -\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots\end{aligned}$$

Finally, we can now apply the rule of reduction of order and find y_2.

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx\\ &= \left(1 + \frac{1}{3}x + \frac{1}{24}x^2 + \frac{1}{360}x^2 + \cdots\right)\left(-\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots\right)\\ &= -\frac{1}{2}x^{-2} + \frac{2}{3}x^{-1} + \frac{1}{4}\ln|x| - \frac{19}{270}x + \cdots - \frac{1}{6}x^{-1} + \frac{1}{12}x\ln|x| - \frac{19}{810}x^2 + \cdots\\ &\quad - \frac{1}{48} + \frac{1}{36}x + \frac{1}{96}x^2\ln|x| + \frac{1}{540}x^2 + \cdots\end{aligned}$$

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Example 3.2.3. 1. Show that

$$\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = a_1 + \sum_{k=1}^{\infty}(a_{k-1} + a_{k+1})x^k$$

Solution From LHS,

$$\text{LHS} = \sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} = \left(a_1 + \sum_{k=1}^{\infty} a_{k+1}x^k\right) + \sum_{k=0}^{\infty} a_k x^{k+1}$$

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