QUANTUM EFFECTS IN COSMOLOGY

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1. Introduction – The Present State of the Universe

The contemporary classical cosmology successfully describes the main features and evolution of the universe, but uses for this some specific initial conditions. In the framework of classical cosmology these conditions do not have their own reasonable explanation. They are just selected in such a way that the theoretical predictions be compatible with the actual observations. A more deeper understanding of why the universe has these and not the other properties can be provided by quantum cosmology. The most important unsolved issue is the nature of the cosmological singularity whose existence follows from the classical general relativity. The phenomenon of singularity is probably the most compelling reason for replacing classical cosmology with quantum cosmology.

Let us recall some properties of the present world which seem to have their origin in the very early universe (see standard textbooks [1]).

The distribution of galaxies in space as well as the distribution of their red shifts indicate that at the largest accessible scales the universe is more or less homogeneous and isotropic. The most convincing manifestation of the large scale homogeneity and isotropy of the universe is the very low level of the angular variations of the temperature of the microwave background radiation: $\Delta T/T \approx 5 \cdot 10^{-6}$ [2]. All the observational data point out to the conclusion that the overall structure and dynamics of the observable part of the universe can be described, up to small perturbations, by the Friedmann (or Friedmann-Robertson-Walker, FRW) metrics:

$$ds^2 = c^2 dt^2 - a^2(t) dl^2. \tag{1}$$

It is known that dl^2 , the spatial part of the metrics (1) , can correspond to the closed ($k = +1$), open ($k = -1$) or flat ($k = 0$) 3-dimensional spaces. The actual sign of the spatial curvature depends on the ratio $\Omega = \rho_m/\rho_c$ of the mean total matter density ρ_m to the critical density $\rho_c = \frac{3}{8\pi G} H^2$, where H is the Hubble parameter $H(t)$, $H(t) \equiv \dot{a} /a$. The majority of the available astronomical data favour the value $\Omega < 1$ which implies that $k = -1$. However the observations can not presently exclude neither $k = 0$ nor $k = +1$. In any case, the current value of the parameter Ω is close to one.

Although the overall structure of the universe is homogeneous and isotropic, it is obviously inhomogeneous and anisotropic at scales characteristic for galaxies and their clusters. It is believed that these inhomogeneities were formed as a result of growth of small initial (primordial) perturbations. In order to produce the observed inhomogeneities the initial perturbations must have had the specific amplitude and specific spectrum. There are some theoretical and observational arguments in support of the so-called "flat" Harrison-Zeldovich spectrum [3] of the initial fluctuations. In order to be compatible with the observations this picture may also require a significant amount of "dark" matter.

The dynamical characteristics of the averaged distribution of the matter, the growth and formation of the small scale inhomogeneities, the abundances of various chemical elements, as well as other features of the actual universe, are all successfully brought together by the "standard" classical cosmological theory. The trouble is, however, that the "standard" theory postulates certain properties of the universe rather than derives them from more fundamental principles. For instance, the observational fact of the angular uniformity of the temperature of the microwave background radiation over the sky does not have any rational explanation except of being a consequence of the postulated, everlasting homogeneity and isotropy (plus small perturbations). A more natural explanation to a set of observational facts can be provided by the inflationary hypothesis [4]. Of course, this hypothesis has its own limits of applicability and conditions of realization, and after all it may prove to be wrong, but the phenomenon of inflation seems to be quite general and stable. This is why it is worthwhile to investigate its consequences and compare with observations.

According to the inflationary hypothesis the spatial volume of the universe confined to the current Hubble distance $l_H = c/H \approx 2 \cdot 10^{28}$ cm or, possibly, even much larger volume, has developed from a small region which was causally connected in the very distant past. If the inflationary stage in the evolution of the very early universe did really take place, the large scale homogeneity and isotropy, as well as the closeness of Ω to one, can be explained as the consequences of the inflationary expansion.

The simplest model for the inflationary stage of expansion is provided by the De-Sitter solution. Originally it was derived as a solution to the vacuum Einstein equations with a constant cosmological Λ-term. However, it can also be treated as a solution to the Einstein equations with matter satisfying the effective equation of state p = -ϵ. The De-Sitter solution describes a space-time with a constant 4-curvature. This space-time is as symmetric as Minkowski space-time, in the sense that it also admits the 10-parameter group of motions. The line element of the De-Sitter space-time has the form (see, for example, [5]):

$$ds^2 = c^2 dt^2 - a^2(t)\Big[dr^2 + sin^2 r\Big(d\theta^2 + sin^2\theta d\psi^2\Big)\Big], \tag{2}$$

where $a(t) = r_0 ch\frac{ct}{r_0}$, and $r_0 =$ const. It is known that in the De-Sitter space-time one can also introduce the frames of reference with flat or open (hyperbolic) space sections, though these coordinate systems do not cover the whole of the De-Sitter space-time. The often used is the flat 3-space representation:

$$ds^2 = c^2 dt^2 - a^2(t)\Big(dx^2 + dy^2 + dz^2\Big), \tag{3}$$

where $a(t) = e^{H_0 t}$, and $H_0 =$ const. The constant H_0 is the Hubble parameter at the De-Sitter stage of expansion. The scale factor $a(t)$ of the line element (2) approaches the behaviour $a(t) = e^{H_0 t}$ very quickly during several characteristic time intervals $t = \frac{1}{c}\, r_0$.

In order to see the advantages of the inflationary hypothesis let us assume that intially, during the Planck era, the distance between two idealized physical objects was of order of a few Planckian scales, $l_{pl} = 10^{-33}$ cm. It can be shown that the present day distance between these objects can be as large as the present day Hubble distance $l_H \approx 2 \cdot 10^{28}$ cm, if the duration of inflationary stage Δt was sufficiently long, $H_0 \Delta t > 65$ In this way a small causally connected region could have expanded to the size of the presently observed universe, and the uniformity of T over the sky could have been established by causal physical processes in the very early universe. At the same time, a sufficiently long inflation makes the present value of the Ω parameter very close to $\Omega = 1$.

2. What Can We Expect From a Complete Cosmological Theory?

The hypothesis of the inflationary stage helps us to make some of the cosmological data more "natural". However, the origin of the inflationary stage still needs to be explained. The question still remains, what kind of evolution did the universe experience before the inflationary stage and how did the universe itself originate? A frequently made assumption is that prior to the De-Sitter stage there was a preceding radiation-dominated era. This assumption just postpones the answer to the above mentioned questions and returns us to the problem of cosmological singularity and quantum gravity. As a more fundamental solution to the problem, it was suggested [6] that the inflationary era was preceded by an essentially quantum-gravitational phenomenon called a spontaneous birth of the universe. A theory capable of describing the classical stages of evolution of the universe, as well as its quantum-gravitational origin, can be named a complete cosmological theory. Let us speculate on the main expected features of such a theory.

The desired evolution of the scale factor $a(t)$ is shown in Fig. 1. According to this scenario, the moment of appearance of the classical universe corresponded to $t = 0$. After that moment of time the inflationary evolution has started and has been governed by Eq. (2). It is natural to expect that all the characteristic parameters of the newly born universe were of order of the Planckian scales, i.e. the classical space-time came into being near the limit of applicability of classical general relativity. The inflationary expansion may be able to pickup such a micro-universe and to increase its size up, at least, to the present day Hubble radius. The wiggly line joining the points $a = 0$ and $a = l_{pl}$ at Fig. 1 was meant to describe an essentially quantum-gravitational process similar to the quantum tunnelling or quantum decay which could have resulted in the nucleation of the universe in the state of classical De-Sitter expansion. It is reasonable to suppose that at the beginning of classical evolution the deviations from the highly symmetric De-Sitter solution were negligibly small. Moreover, it seems to be sufficient to take these deviations with the minimally possible amplitude, i.e. at the level of quantum zero-point fluctuations. During the inflationary period these fluctuations could have been amplified and produce the density perturbations and gravitational waves. The density perturbations are needed to form the observed inhomogeneities in the universe. Gravitational waves seem to be the only source of impartial information about the inflationary epoch and the quantum birth of the universe.

These matters have been subjects of study in many research papers by various authors. We will discuss them in more detail below. We will see that some of the notions introduced above have acquired more precise formulation and some of the problems have been partially solved.

3. An Overview of Quantum Effects in Cosmology

From this brief exposition of a complete cosmological theory it is clear that quantum effects and quantum concepts should play a decisive role in different contexts and at different levels of approximation. It is useful to give a short classification of the areas of further discussion where the quantum notions will be dealt with. It is worth emphasizing that we will often use below the common and powerful technique which is the splitting up of a given problem into the "background" and "perturbational" parts.

We will start from a description of classical perturbations on a classical background space-time. The physical meaning of such effects as parametric amplification of cosmological perturbations, and first of all, amplification of gravitational waves, can be clearly seen already at this level of approximation. The next level of approximation treats the perturbations as quantized fields interacting with the variable gravitational field of the nonstationary universe, or, in geometrical language, with the classical background geometry. A particular, but not obligatory, example of the variable gravitational field is provided by the inflationary expansion. At this level of approximation, we will see how the initial vacuum state of the quantized fields evolves into a pure multiparticle state with very specific quantum properties. It will be shown that the final quantum state belongs to the class of the so-called squeezed quantum states. Squeezing is a very distinct feature potentially allowing to prove or disprove the quantum origin of the primordial cosmological perturbations.

At a still deeper level, the background geometry and matter fields are also treated quantum-mechanically — this is the realm of quantum cosmology. The main object of interest in quantum cosmology is the wave function of the universe which, in general, describes all degrees of freedom at the equal footing. This level of discussion is appropriate for tackling such issues as the beginning and the end of classical evolution as well as quantum birth of the universe. However, there is no one unique wave function of the universe, there are many of them. All possible wave functions constitute the whole space of the wave functions. Presently, we do not know any guiding principle allowing to prefer one cosmological wave function over others. This is why we are facing a painful job of analyzing all of them trying to introduce a probability measure in the space of all wave functions.

Going still further, one can introduce a notion of a Wave Function given in the space of all possible wave functions. In other words, a wave function of the universe becomes an operator acting on the Wave Function describing the many universes system. This is a subject of the now popular so-called third-quantized theory. It is aimed at describing the multiple production and annihilation of the baby-universes. This fascinating subject is still at the beginning of its development and is beyond the scope of the present papers. The reader is referred to the recent review and technical papers on the subject [7]. In some sense the different theories listed from above to the bottom are various approximation to the theories listed in the opposite direction.

Let us start from the classical theory of small perturbations superimposed on a given background space-time.

4. Parametric (Superadiabatic) Amplification of ClassicalWaves

Let us consider classical weak gravitational waves. The main purpose is to study the parametric (superadiabatic) amplification of gravitational waves [8, 9]. The same mechanism is applicable to other fluctuations if they are governed by similar equations.

We assume that the space-time metric $\mathrm{g}_{\mu\nu}$ can be presented in the form $g_{\mu\nu} \approx g^{(0)}_{\mu\nu} + h_{\mu\nu}$, where $g^{(0)}_{\mu\nu}$ is the background metric:

$$ds^2 = a^2(\eta)\Big(d\eta^2 - dx^2 - dy^2 - dz^2\Big) \tag{4}$$

and $h_{\mu\nu}$ are gravitational wave perturbations. The functions $h_{\mu\nu}(\mathbf{x}, \eta)$ can be simplified and some of them put to zero, $h_{0\mu} = 0$, by using the available gauge freedom. The remaining components can be decomposed into the mode functions, so that for a given mode one has

$$h^k_i = \frac{1}{a}\mu(\eta)\, G^k_i(x, y, z), \tag{5}$$

where $h^k_i = h_{ij} g^{(0)jk}$.

In the case under discussion the eigenfunctions G^k_i can be taken in the simplest form: $G^k_i = \exp\left[\,\pm i(n_1 x + n_2 y + n_3 z)\right], n_1^2 + n_2^2 + n_3^3 = n^2$. The main equation to be solved is

$$\mu'' + \mu\Big(n^2 - U(\eta)\Big) = 0, \tag{6}$$

where $' = d/d\eta$, $U(\eta) = a''/a$, n is the wave-number, and the wavelength is $\lambda = 2\pi a/n$.

Equation (6) describes an oscillator with the varying frequency, that is we are dealing with a parametrically excited oscillator. It is also worth noting that this equation is mathematically similar to the spatial part of the Schrödinger equation, with the prime playing the role of a spatial derivative, for a particle with energy n^2 and the potential $U(\eta)$. A typical potential barrier $U(\eta)$ is shown in Fig. 2.

In the intervals of η-time such that $n^2 >> |U(\eta)|$ the solutions to Eq. (6) have the form $\mu = e^{\pm in\eta}$, so that one has usual high-frequency waves with the adiabatically changing amplitude: $h = \frac{1}{a}\sin(n\eta + \phi)$. In the expanding world the amplitude decreases. The amplitudes of the waves with such n that $n^2 >> |U(\eta)|$ for all values of η decrease adiabatically for all η. These waves are shown symbolically by the wavy line with the decreasing amplitude above the potential barrier $U(\eta)$ in Fig. 2.

If for a given n there is an interval of time when $n^2 << |U(\eta)|$, the solutions to the second-order differential equation (6) are no longer oscillatory. In case $U(\eta) = a''/a$ they are $\mu_1 = a$ and $\mu_2 = a\int a^{-2}d\eta$. The waves satisfying $n^2 << |U(\eta)|$ for some η encounter the potential barrier and are governed by the solutions μ_1 and μ_2 in the under-barrier region. The amplitude μ_f of the function $\mu(\eta)$ right after exit of the wave from under the barrier depends on the initial phase of the wave ϕ. The exiting amplitude μ_f can be larger or smaller than the entering amplitude μ_i defined right before the wave encountered the barrier. However, averaging of $(\mu_f)^2$ over the initial phase ϕ (i.e., integrating from 0 to 2π) leads always to the dominant contribution from the solution μ_1. This means that the adiabatic factor $1/a$ is cancelled out by $\mu_1 = a$ and the amplitude h (with the factor $1/a$ taken into account) of a "typical" wave can be regarded as remaining constant in the region occupied by the barrier. It stays constant instead of diminishing adiabatically as the waves above the potential barrier do. Thus, the exiting amplitude h_f of a "typical" wave is equal to the entering amplitude h_i and is larger than it would have been, if the wave behaved adiabatically (see Fig. 2).

The amplification coefficient $R(n)$ for a given n is just the ratio $a(\eta_f)/a(\eta_i)$ where $a(\eta_i)$ is the value of the scale factor at the last oscillation of the wave before entering the under-barrier region, and $a(\eta_f)$ is the value of the scale factor at the first oscillation of the wave after leaving the under-barrier region. It is seen from Fig. 2 that different waves, that is waves with different wave numbers n, stay under the potential barrier for different intervals of time. This means that, in general, the amplification coefficient depends on n: $R(n) = 1$ for all n above the top of the potential, and $R(n) \gg 1$ for smaller n. The initial spectrum of the waves $h(n) = A(n)/a$, defined at some η well before the interaction began, transforms into the final spectrum $h(n) = B(n)/a$, defined at some η well after the interaction completed. The transformation occurs according to the rule: $B(n) = R(n)A(n)$. This is the essence of the mechanism of the superadiabatic (parametric) amplification of gravitational waves and, in fact, of any other fluctuations obeying similar equations.

The initial amplitudes and spectrum of classical waves can be arbitrary. It is only important to have a nonzero initial amplitude, otherwise the final amplitude will also be zero. Now, remaining at the same classical level, we will imitate the quantum zero-point fluctuations by assuming that they are classical waves with certain amplitudes and arbitrary phases. Waves with different frequencies have different amplitudes, so they form some initial vacuum spectrum. In order to derive the vacuum spectrum, we neglect the interaction with the gravitational field and consider, essentially, waves in Minkowski spacetime. The energy density of gravitational waves scales as $\epsilon_g = (c^4/G)(h^2/\lambda^2)$. For a given wavelength λ we want to have "a half of the quantum" in each mode, that is we want to have energy $\frac{1}{2}\hbar\omega$ in the volume $= \lambda^3$. It follows from this requirement that the vacuum amplitude of gravitational waves with the wavelength λ is equal to $h(\lambda) = l_{pl}/\lambda$. Hence, the initial vacuum spectrum of gravitational waves, defined at some early epoch η_b, is $h(n)\approx nl_{pl}/a_b$, where a_b is the scale factor at that epoch. This is the spectrum to be transformed by the interaction with the external gravitational field. The amplification process makes the number of quanta in each interacting mode much larger than 1/2. The renormalization (subtraction of $\frac{1}{2}\hbar\omega$) does not practically change the energy of the amplified waves, while it cancels the initial $\frac{1}{2}\hbar\omega$ of those (high-frequency) waves that did not interact with the field.

5. Graviton Creation in the Inflationary Universe

As an illustration, we will apply the above considerations to the inflationary (De-Sitter) model. In terms of η -time the De-Sitter solution (3) has the scale factor $a(\eta) = -c/H_0\eta$. (It is convenient to have η negative and growing from $-\infty$.) We assume that the De-Sitter stage ends at some $\eta = \eta_1<0$ and goes over into the radiation-dominated stage with $a(\eta) = (\eta - 2\eta_1)c/H_0\eta_1^2$. The relevant potential $U(\eta)$ is shown in Fig. 3 by a solid line 1. A wave with the wave number n, $(n\eta_1)^2\ll 1$, enters the potential and ceases to oscillate at some η_i, when $\lambda\approx c\dot{a}/a$, that is $2\pi a/n\approx a^2/a'$. This leads to the entering condition $n\eta_i \approx 1$. For different n 's this condition is satisfied at different $a(\eta_i)$: $a(\eta_i) \approx cn/H_0$. The waves leave the potential and start oscillating again at the radiation-dominated stage when the condition $2\pi a/n \approx a^2/a'$ is satisfied again. This leads to the exiting condition $n(\eta_f - 2\eta_1) \approx 1$ and $a(\eta_f) \approx c/nH_0\eta_1^2 \approx a(\eta_i)(n\eta_1)^{-2}$. The amplification coefficient $R(n) = (n\eta_1)^{-2}$ is much larger than 1 for $(n\eta_1)^2 \ll 1$ and scales as $R(n) = n^{-2}$.

Now we will see how the vacuum spectrum is transformed. By the time of entering the barrier, the amplitude $h(\eta_i) \approx n(l_{pl}/a_b)(a_b/a(\eta_i)) \approx l_{pl}H_0/c$ is the same for all n. The exiting amplitude is also independent of n : $h(\eta_f) = l_{pl}H_0/c$, that is all amplified waves start oscillating with the same amplitude. The fluctuations which start oscillating (enter the Hubble radius) with the same amplitude are said to have the "flat" Harrison-Zeldovich spectrum. One should keep in mind, however, that the comparatively shorter waves start oscillating earlier and their amplitudes decrease more by some fixed (present) time η_0 : $h(\eta_0) \approx h(\eta_f)(a(\eta_f)/a_o) \approx l_{pl}n/(n\eta_1)^2 a_0$. As expected, the amplification coefficient $R(n) = n^{-2}$ transforms the initial vacuum spectrum $h(n) = n$ into the final spectrum $h(n) = n^{-1}$.

To put some more detail in this discussion above, one can consider exact solutions to Eq. (6) at the inflationary (i) and radiation-dominated (e) stages. In the region $\eta \leq \eta_1$ the general solution to Eq. (6) has the form

$$\mu_i = A\left[\cos(n\eta + \phi) - \frac{1}{n\eta}\sin(n\eta + \phi)\right] \tag{7}$$

where A and ϕ are arbitrary constants. For $\eta \geq \eta_1$ the general solution has the form

$$\mu_e = B\sin(n\eta + \chi), \tag{8}$$

where B and χ are constants to be determined from the conditions that μ and μ' join continuously at $\eta = \eta_1$.

It is convenient to introduce $n\eta_1 \equiv x$. The joining conditions are:

$$\begin{aligned} A\left[\cos(x + \phi) - \frac{1}{x}\sin(x + \phi)\right] &= B\sin(x + \chi) \\ A\left[-\frac{1}{x}\cos(x + \phi) - \left(1 - \frac{1}{x^2}\right)\sin(x + \phi)\right] &= B\cos(x + \chi) \end{aligned} \tag{9}$$

Their consequence is $(B/A)^2 = 1 + x^{-2} + (x^{-4} - 2x^{-2})\sin^2(x + \phi) - x^{-3}\sin 2(x + \phi)$. One can see that $(B/A)^2$ depends on the initial phase ϕ. After averaging over the phase ϕ one obtains $\overline{(B/A)^2} = 1+1/2x^4$. This expression can be taken as the definition of the amplification coefficient $R(n)$. It is seen that $R(n) = 1$ for waves above the potential, $x^2 \gg 1$, and

$$R(n) \approx \frac{1}{\sqrt{2}x^2} \tag{10}$$

for the longer waves, $x^2 \ll 1$. This expression is in full agreement with the previous qualitative estimates.

The radiation-dominated era ends at some time η_2 and goes over into the matter-dominated era with the scale factor $a(\eta) = c(\eta + \eta_2 - 4\eta_1)^2 / 4H_0\eta_1^2(\eta_2 - 2\eta_1)$. For $\eta > \eta_2$ the potential $U(\eta)$ is again non-zero, it is shown by the solid line 2 in Fig. 3. The waves which satisfy the condition $n(\eta_2 - 2\eta_1) \gg 1$ do not interact with the potential and their evolution was fully described above. (This part of the spectrum was first discussed in Ref. 10.) However, the longer waves, satisfying the opposite condition $n(\eta_2 - 2\eta_1) \ll 1$, interact with the second barrier and transform additionally their spectrum. (This part of the spectrum was first discussed in Ref. 11, see also Ref. 12.) These waves start oscillating at the matter-domianted stage when the requirement $n(\eta_f + \eta_2 - 4\eta_1) \approx 1$ is satisfired. For these waves, the exiting value of the scale factor is $a(\eta_f) \approx c/H_0(n\eta_1)^2(\eta_2 - 2\eta_1)$ and the amplification coefficient $R(n) \approx 1/(n\eta_1)^2 n(\eta_2 - 2\eta_1) = n^{-3}$. The present-day amplitudes are $h(\eta_0) \approx l_{pl} n/(n\eta_1)^2 n(\eta_2 - 2\eta_1) a_0 = n^{-2}$.

The waves with the wave number n_H satisfying $n_H(\eta_0 + \eta_2 - 4\eta_1) \approx 1$ enter the Hubble radius at the present epoch. Their wavelengths λ_H are of the order of the present Hubble radius, $\lambda_H \approx 2 \cdot 10^{28}$ cm, and their frequencies ν_H are $\nu_H \approx 10^{-18}$ Hz. The amplitude of these waves is, roughly, $h_H(\eta_0) \approx l_{pl} H_0/c$. It can not be too large in order not to cause too large angular (quadrupole) variations in the temperature of the microwave background radiation, $\triangle T/T \approx h_H(\eta_0)$. This requirement places a limit on the value of the Hubble parameter H_0 at the De-Sitter stage [11]. Equally strong limit follows from the observational restrictions on $\triangle T/T$ in a few degrees angular scale, where the contribution of waves with frequencies $\nu = 10^{-16}\ Hz$ is dominant [9] . The inflationary spectrum of graviational waves in terms of the spectral flux density as a function of frequency ν, is shown in Fig. 4 (adopted from [9]). The upper position of the spectrum is determined by the observational data on $\triangle T/T$. At the same graph one can see theoretical predictions of some other models and the existing experimental limits as well as the expected levels of sensitivity of various observational techniques.

The predicted gravity-wave spectrum is more complex if the Hubble parameter at the inflationary stage was not constant. It is interesting to know that the time variations of the Hubble parameter are in one-to-one correspondence with the frequency variations of the present-day spectral energy density of waves [13]. This makes it possible, at least in principle, to study the details of the very early evolution of the universe by measuring the spectral properties of relic gravitational waves.

In our previous discussion we have been mainly interested in the amplitude of a "typical" wave. In other words, we have been calculating the r.m.s. value of the final amplitude, assuming that the initial phase ϕ is distributed randomly and evenly in the interval form 0 to 2π. However, the distribution of the final phase is also important. Equivalently, one can ask about the final distributions of the quadrature components of the wave, that is the components proportional to $\sin n\eta$ and $\cos n\eta$. This study can serve as an introduction to the notion of the quantum mechanical squeezing which we will be discussing later.

Let us return to the exact solutions (7) and (8). Initially, for $\eta \to -\infty$, solution (7) can be written as $\mu_i \approx A[v_1 \sin n\eta + v_2 \cos n\eta]$, where $v_1 \equiv -\sin\phi$, $v_2 \equiv \cos\phi$. The mean values of v_1, v_2 and $\overline{v_1 v_2}$ are zero. However, the mean values of v_1^2 and v_2^2 are nonzero and equal: $\overline{v_1^2} = \overline{v_2^2} = \frac{1}{2}$. To derive the quadrature components of solution (8) one can first find, from the joining conditions, the constant χ and rewrite Eq. (8) in the form:

$$\begin{gathered}\mu_e = A \sin n\eta \Big[+ \frac{1}{x^2} \cos x \sin(x+\phi) - \frac{1}{x} \cos\phi - \sin\phi \Big] + \\ A \cos n\eta \Big[- \frac{1}{x^2} \sin x \sin(x+\phi) - \frac{1}{x} \sin\phi + \cos\phi \Big] \equiv \\ A k_1 \sin n\eta + A k_2 \cos n\eta \end{gathered} \tag{11}$$

The mean values of k_1 and k_2 are again zero, but for the quadratic combinations one obtains:

$$\begin{aligned}\overline{k_1^2} &= \frac{1}{2}\Big[1 + \frac{1}{x^2} + \frac{1}{x^2}\Big(\frac{1}{x^2} - 2\Big)\cos^2 x - \frac{1}{x^3}\sin 2x\Big], \\ \overline{k_2^2} &= \frac{1}{2}\Big[1 + \frac{1}{x^2} + \frac{1}{x^2}\Big(\frac{1}{x^2} - 2\Big)\sin^2 x + \frac{1}{x^3}\sin 2x\Big], \\ \overline{k_1 k_2} &= \frac{1}{2}\Big[-\frac{1}{2x^2}\Big(\frac{1}{x^2} - 2\Big)\sin 2x - \frac{1}{x^3}\cos 2x\Big]\end{aligned} \tag{12}$$

Note that $\overline{k_1^2}\,\overline{k_2^2} - \overline{(k_1 k_2)}^2 = \frac{1}{4} = \overline{v_1^2}\,\overline{v_2^2}$. If one makes a constant shift $\eta = \tilde{\eta} + y$, the solution (11) transforms to $\mu_e = Al_1 \sin n\tilde{\eta} + Al_2 \cos n\tilde{\eta}$. The constant y can be chosen in such a way that $\overline{l_1 l_2} = 0$. Under this choice one obtains

$$\overline{l_1^2} = \frac{1}{4}\left[2 + \frac{1}{x^4} + \frac{1}{x^2}\sqrt{4 + \frac{1}{x^4}}\right], \quad \overline{l_2^2} = \frac{1}{4}\left[2 + \frac{1}{x^4} - \frac{1}{x^2}\sqrt{4 + \frac{1}{x^4}}\right] \tag{13}$$

This choice minimizes one of the quadrature variances and maximizes the other.

We are interested in the case $x^2 \ll 1$. For this case, $\overline{l_1^2} \approx 1/2x^4$, $\overline{l_2^2} \approx x^4/2$. We see that during the amplification process, one of the noise components strongly increased while the other decreased equally strongly. One can also say that the final phase χ is not evenly distributed as the function of ϕ, but is highly peaked near the values $tg\ \chi \approx -2x$. We will see below that the supression (squeezing) of variances in one of two quadrature components of the wave field is a characteristic feature of squeezed quantum states. Moreover, squeezing may reduce one of the two variances below the level of zero-point quantum fluctuations.

6. Quantum States of a Harmonic Oscillator

Let us first recall some properties of quantum states of an ordinary harmonic oscillator. We will need this information in our further discussion. Especially, we will be interested in the notion of squeezed quantum states.

Classical equations of motion for a harmonic oscillator, $\ddot{x} + \omega^2 x = 0$, can be derived from the Lagrange function $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mw^2x^2$ according to the rule: $\left(\frac{\partial L}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial L}{\partial x} = 0$. Associated with L is the Hamilton function $H = \frac{p^2}{2m} + \frac{m}{2}w^2x^2$, where $p = \left(\frac{\partial L}{\partial \dot{x}}\right)$. Quantization is achieved by introducing the operators $\hat{x}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and establishing the commutation relation: $[\hat{x}\hat{p}] = i\hbar$. From $\hat{x}$ and $\hat{p}$ one can construct the creation and annihilation operators a$^+$ and a: $a^+ = \left(\frac{mw}{2\hbar}\right)^{1/2}\left(\hat{x} - i\frac{\hat{p}}{mw}\right)$, $a = \left(\frac{mw}{2\hbar}\right)^{1/2}\left(\hat{x} + i\frac{\hat{p}}{mw}\right)$, $[a, a^+] = 1$ and the particle number operator $\hat{N}$: $\hat{N} = \hat{a}^+\hat{a} = \frac{\hat{H}}{\hbar w} - \frac{1}{2}$. The oscillator can be described by the wave function (or state function) $\psi(x,t)$ which satisfy the Schrödinger equation: $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$. The ground (vacuum) quantum state $|0>$ is defined by the requirement a$|0>=0$. The ground wave function has the form $\psi(x) = \left(\frac{mw}{\pi\hbar}\right)^{1/4} e^{-\frac{mw}{2\hbar}x^2}$. The n-quantum states are defined as the eigenstates of the $\hat{N}$ operator: $\hat{N}|n> = n|n>$, they are also eigenstates of $\hat{H}$ with eigenvalues $\hbar\omega\left(n+\frac{1}{2}\right)$, $\hat{H}\Big|n> = \hbar w\left(n+\frac{1}{2}\right)\Big|n>$. These states are produced by the action of the creation operator a$^+$ on the vacuum state: $|n> = \frac{\left(a^+\right)^n}{\sqrt{n!}}|0>$.

An important class of quantum states are coherent states which are regarded as the "most classical". The coherent states are generated from the vacuum state $|0>$ by the action of the displacement operator: $D(a,\alpha) \equiv \exp\left[\alpha a^+ - \alpha^* a\right]$, where α is an arbitrary complex number. We are mostly interested in squeezed states. Squeezed states involve operators quadratic in a, a^+. A one-mode squeezed state (for a review see, for example, [13]) is generated by the action of the squeeze operator:
$S_1(r,\phi) \equiv exp\Big[\frac{1}{2}r\Big(e^{-2i\phi}a^2 - e^{2i\phi}a^{+2}\Big)\Big]$, where the real numbers r and ϕ are known as the squeeze factor and squeeze angle: $0 \leq r < \infty$, $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. A squeezed state can be generated by the action of the squeeze operator on any coherent state and, in particular, on the vacuum state, in which case it is called squeezed vacuum state. If a harmonic oscillator is exposed to the time-dependent interaction, a one-mode squeezed state is produced, as a result of the Schrödinger evolution, by the interaction Hamiltonian

$$H(t) = \sigma(t)a^{+2} + \sigma^*(t)a^2 \tag{14}$$

where σ is arbitrary function of time.

The meaning of the word "squeezing " is related to the properties of these states with respect to variances (or noise moments) ΔA of different operators A: $\Delta A \equiv A - <A>$. The squeezed wave functions are always Gaussian:

$$\psi(x) \sim e^{-\frac{1}{2}\gamma x^2} \tag{15}$$

but, the variances of the variables $\hat{x}$ and $\hat{p}$ are substantially different. They can be presented in terms of the complex parameter $\gamma, \gamma \equiv \gamma_1 + i\gamma_2$, or real parameters r, ϕ:

$$\begin{aligned} &<(\Delta\hat{x})^2> = \frac{1}{2\gamma_1}, \quad <(\Delta\hat{p})^2> = \frac{|\gamma|^2}{2\gamma_1}, \\ &<\left(\Delta\hat{x}^2\right)> = \frac{1}{2}(ch2r - sh2r\cos 2\phi), \\ &<(\Delta\hat{p})^2> = \frac{1}{2}(ch2r + sh2r\cos 2\phi) \end{aligned} \tag{16}$$

These variances should be compared with those for a coherent state, in which case they are always equal to each other and are minimally possible: $<(\Delta\hat{x})^2> = <(\Delta\hat{p})^2> = \frac{1}{2}$. So, in a squeezed state, one component of the noise is always large but another is "squeezed" and can be smaller than $\frac{1}{2}$. In (x,p) plane the line of a total noise $K = \frac{1}{2}\Big[<(\Delta\hat{x})^2> + <(\Delta\hat{p})^2>\Big]$ for the coherent states can be described by a circle, while this line is an ellipse for the squeezed states (see Fig. 5). In a squeezed vacuum state the mean values of $\hat{x}$ and $\hat{p}$ are zero, so in this case, the center of the ellipse is at the origin.

The mean value of the particle number operator N is not zero in a squeezed vacuum state, it can be expressed in terms of the squeeze parameter $r: \quad <N> = sh^2 r$. For strongly squezed states, $r \gg 1$, the $<N>$ is very large $<N> \approx \frac{1}{4}e^{2r}$. The variance of N is $<(\triangle N)^2> = \frac{1}{2}sh^2 2r$. The $<(\triangle N)^2>$ is also very large for $r \gg 1: \quad <(\triangle N)^2> \approx \frac{1}{8}e^{4r}$ and $<(\triangle N)^2>^{\frac{1}{2}} \approx \frac{1}{2\sqrt{2}}e^{2r}$, that is $<(\triangle N)^2>^{\frac{1}{2}} \approx <N>$. In contrast, $<(\triangle N)^2>^{\frac{1}{2}} = <N>^{\frac{1}{2}}$ for coherent states.

A notion which is very useful for our problem of the pair particle creation is the two-mode squeezed states. The two modes under our consideration will be two particles (waves) travelling in the opposite direction. A two-mode squeezed vacuum state is generated from the vacuum $|0,0>$ by the action of the two-mode squeeze operator

$$S(r,\,\phi) = \exp\Big[r\Big(e^{-2i\phi}a_+a_- - e^{2i\phi}a_+^+a_-^+\Big)\Big] \tag{17}$$

where a_+, a_- and a_+^+, a_-^+ are annihilation and creation operators for the two modes, r is the squeeze parameter and ϕ is the squeeze angle.

7. Squeezed quantum states of relic gravitons and primordial density pertrubations

Our preceding analyses of perturbations interacting with the variable gravitational field was essentially classical. Quantum mechanics entered our calculations only as a motivation for choosing the particular initial amplitudes and for making the averaging over initial phases. We interpreted the final results as quantum-mechanical generation of gravitational waves and, possibly, other perturbations, but a rigorous treatment is still needed. We will see now that a consistent quantum-mechanical theory confirms our main results and gives a much more detailed and informative picture of the entire phenomenon.

We will start again from gravitational waves. The gravitational wave field $h_{ij}(\eta, \mathbf{x})$ becomes an operator and can be written in the general form

$$h_{ij}(\eta, \mathbf{x}) = C \int_{-\infty}^{\infty} d^3\mathbf{n} \sum_{s=1}^{2} p_{ij}^s(\mathbf{n}) \Big[a_{\mathbf{n}}^s(\eta) e^{i\mathbf{n}\mathbf{x}} + a_{\mathbf{n}}^{s+}(\eta) e^{-i\mathbf{n}\mathbf{x}} \Big] \tag{18}$$

This expression requires some explanation. In Eq. (18) we do not write the scale factor $a(\eta)$ in front of the expression (compare with Eq. (5)) which can be taken care of later. It is precisely h_{ij} 's given by Eq. (18) that appear automatically in the "field-theoretical" treatment of the problem, see Refs. [15, 16]. The normalization constant C includes all the numerical coefficients but we do not need them now and will not write C below. We will also use units $c = 1$, $\hbar = 1$. Two tensors $p_{ij}^s(\mathbf{n})$, $s = 1, 2$, represent two independent polarization states of each mode (wave). The tensors p_{ij}^s satisfy the "transverse-traceless" conditions: $p_{ij}^s n^j = 0$, $p_{ij}^s \delta^{ij} = 0$. For a wave travelling in the firection $\frac{\mathbf{n}}{n} = (\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta)$ the two polarization tensors are $p_{ij}^1(\mathbf{n}) = l_i l_j - m_i m_j$, $p_{ij}^2(\mathbf{n}) = l_i m_j + l_j m_i$ where l_j, m_j are two unit vectors ortogonal to $\frac{\mathbf{n}}{n}$ and to each other: $l_j = (\sin\varphi,\, -\cos\varphi,\, 0)$, $m_j = (\cos\theta\cos\varphi,\, \cos\theta\sin\varphi,\, -\sin\theta)$. The operators $a_{\mathbf{n}}^s(\eta)$, $a_{\mathbf{n}}^{s+}(\eta)$ are annihilation and creation operators for waves (particles) travelling in the direction of $\mathbf{n}$. These are Heisenberg operators depending on time η.

The evolution of the operators $a_{\mathbf{n}}^s(\eta)$, $a_{\mathbf{n}}^{s+}(\eta)$ is governed by the Heisenberg equations of motion for each mode $\mathbf{n}$ and for each polarization s:

$$\frac{da_{\mathbf{n}}}{d\eta} = -i[a_{\mathbf{n}},\, H], \quad \frac{da_{\mathbf{n}}^+}{d\eta} = -i\Big[a_{\mathbf{n}}^+,\, H\Big] \tag{19}$$

where H is the Hamiltonian of the problem. To derive the Hamiltonian we will proceed as follows.

The classical equations (6) are the Euler-Lagrange equations following from the Lagrangian

$$L = \frac{1}{2}\left[\mu'^2 - n^2\mu^2 - 2\frac{a'}{a}\mu\mu' + \left(\frac{a'}{a}\right)^2 \mu^2\right] \tag{20}$$

The associated Hamiltonian is

$$H = \frac{1}{2}\left[p^2 + n^2\mu^2 + 2\frac{a'}{a}\mu p\right] \tag{21}$$

where p is the canonically conjugated momentum: $p = \partial L/\partial\mu' = \mu' - (a'/a)\mu$. In the quantum treatment, μ and p are operators satisfying the commutation relation $[\mu, p] = i$. The associated annihilation and creation operators are

$$b = \sqrt{\frac{n}{2}}\Big(\mu + i\frac{p}{n}\Big), \quad b^+ = \sqrt{\frac{n}{2}}\Big(\mu - i\frac{p}{n}\Big), \quad \Big[b, b^+\Big] = 1 \tag{22}$$

In terms of b, b^+, the Hamiltonian (21) acquires the form

$$H = nb^+b + \sigma(\eta)b^{+2} + \sigma^*(\eta)b^2 \tag{23}$$

where the coupling function $\sigma(\eta)$ is $\sigma(\eta) = ia'/2a$. The Hamiltonian (23) is precisely of the form of Eq. (14). In the Schrödinger picture, the quantum state (wave-function) transforms from the vacuum state to a one-mode squeezed vacuum state as a result of the Schrödinger evolution with the Hamiltonian (23). We will use the Heisenberg picture in which the operators change with time but the quantum state of the system remains fixed. We also wish to write the Hamiltonian in the form which would explicitly demonstrate the underlying physical phenomenon: particle creation in pairs with the oppositely directed momenta. The Hamiltonian to be used in Eq. (19) has the form

$$H = na^+_{\mathbf{n}}a_{\mathbf{n}} + na^+_{-\mathbf{n}}a_{-\mathbf{n}} + 2\sigma(\eta)a^+_{\mathbf{n}}a^+_{-\mathbf{n}} + 2\sigma^*(\eta)a_{\mathbf{n}}a_{-\mathbf{n}} \tag{24}$$

This Hamiltonian can be derived from Eq. (23) if one considers the sum of two Hamiltonians (23) for two modes (with the same frequency n) b_1 and b_2 and introduces $a_{\mathbf{n}}$, $a_{-\mathbf{n}}$ according to the relations:

$$a_{\mathbf{n}} = \frac{b_1 - ib_2}{\sqrt{2}}, \quad a_{-\mathbf{n}} = \frac{b_1 + ib_2}{\sqrt{2}} \tag{25}$$

The descriptions based on the b-operators corresponds to standing waves and one-mode squeezed states while the description based on the a-operators corresponds to travelling waves and two-mode squeezed states.

By using Eq. (24) one can write the Heisenberg equations of motion (19) in the form

$$i\frac{da_{\mathbf{n}}}{d\eta} = na_{\mathbf{n}} + i\frac{a'}{a}a^+_{-\mathbf{n}}, \quad -i\frac{da^+_{\mathbf{n}}}{d\eta} = na^+_{\mathbf{n}} - i\frac{a'}{a}a_{-\mathbf{n}} \tag{26}$$

for each s and each $\mathbf{n}$. The solutions to these equations can be written as

$$\begin{aligned} a_{\mathbf{n}}(\eta) = u_n(\eta)a_{\mathbf{n}}(0) + v_n(\eta)a^+_{-\mathbf{n}}(0) \\ a^+_{\mathbf{n}}(\eta) = u^*_n(\eta)a^+_{\mathbf{n}}(0) + v^*_n(\eta)a_{-\mathbf{n}}(0) \end{aligned} \tag{27}$$

where $a_{\mathbf{n}}(0)$, $a^+_{\mathbf{n}}(0)$ are the initial values of the operators $a_{\mathbf{n}}(\eta)$, $a^+_{\mathbf{n}}(\eta)$ for some initial time and the complex functions $u_n(\eta)$, $v_n(\eta)$ satisfy the equations

$$i\frac{du_n}{d\eta} = nu_n + i\frac{a'}{a}v^*_n, \quad i\frac{dv_n}{d\eta} = nv_n + i\frac{a'}{a}u^*_n \tag{28}$$

where $|u_n|^2 - |v_n|^2 = 1$ and $u_n(0) = 1$, $v_n(0) = 0$. Note that u_n, v_n depend only on the absolute value of the vector $\mathbf{n}$, not its direction. The operators $a_{\mathbf{n}}(0)$, $a^+_{\mathbf{n}}(0)$ (Schrödinger operators) obey the commutation relations $[a_{\mathbf{n}}(0),\, a^+_{\mathbf{m}}(0)] = \delta^3(\mathbf{n}-\mathbf{m})$ and so do the evolved operators: $[a_{\mathbf{n}}(\eta),\, a_{\mathbf{m}}(\eta)] = \delta^3(\mathbf{n}-\mathbf{m})$.

It follows from Eq. (28) that the function $u_n + v^*_n$ satisfies the equation

$$(u_n + v^*_n)'' + \left(n^2 - \frac{a''}{a}\right)(u_n + v^*_n) = 0 \tag{29}$$

which is precisely Eq. (6). The two complex functions u_n, v_n restricted by one constraint $|u_n|^2 - |v_n|^2 = 1$ can be parameterized by the three real functions $r_n(\eta)$, $\phi_n(\eta)$, $\theta_n(\eta)$:

$$u = e^{i\theta} chr, \quad v = e^{-i(\theta - 2\phi)} shr \tag{30}$$

which are, corrrespondingly, squeeze parameter r, squeeze angle ϕ and rotation angle θ. For each n and s, they obey the equations

$$\begin{aligned} r' &= \frac{a'}{a}\cos 2\phi \\ \theta' &= n - \frac{a'}{a}\sin 2\phi\; thr \\ \phi' &= -n - \frac{a'}{a}\sin 2\phi\; cthr \end{aligned} \tag{31}$$

which can be used for an explicit calculation of r, ϕ, θ for a given scale factor $a(\eta)$.

The squeeze parameters have been calculated [17] (by a different method) for a model which we have considered in Sec. 5. The model includes three successive stages of expansion: De-Sitter, radiation-dominated and matter-dominated. It has been shown that the squeeze parameter r varies in the large interval from $r\approx 1$ to $r\approx 120$ over the spectrum of relic gravtational waves (see Fig. (4)). The value $r\approx 1$ applies to the shortest waves with the present day frequencies of order of $\nu\approx 10^8$ Hz, $r\approx 10^2$ is the value of r attributed to the waves with frequencies $\nu\approx 10^{-16}$ Hz, and $r\approx 120$ corresponds to the waves with the Hubble frequencies $\nu\approx 10^{-18}$ Hz.

It is important to note that Eqs. (27) can be cast in the form

$$a_{\mathbf{n}}(\eta) = RSa_{\mathbf{n}}(0)S^+R^+, \quad a^+_{\mathbf{n}}(\eta) = RSa^+_{\mathbf{n}}(0)S^+R^+ \tag{32}$$

where

$$S(r,\,\phi) = \exp\Big[r\Big(e^{-2i\phi}a_{\mathbf{n}}(0)a_{-\mathbf{n}}(0) - e^{2i\phi}a^+_{\mathbf{n}}(0)a^+_{-\mathbf{n}}(0)\Big)\Big] \tag{33}$$

is the unitary two-mode squeeze operator and

$$R(\theta) = \exp\Big[-i\theta\Big(a^+_{\mathbf{n}}(0)a_{\mathbf{n}}(0) + a^+_{-\mathbf{n}}(0)a_{-\mathbf{n}}(0)\Big)\Big] \tag{34}$$

is the unitary rotation operator. Eq. (32) demonstartes explicitly the inevitable appearance of squeezing in this kind of problem.

We will assume that the quantum state of the field is the vacuum state defined by the requirement $a_{\mathbf{n}}(0)|0\rangle = 0$ for each $\mathbf{n}$ and for both s. The values of $a_{\mathbf{n}}(\eta)$, $a^+_{\mathbf{n}}(\eta)$ determine all the statistical properties of the field in the later times. The mean values of $a_{\mathbf{n}}(\eta)$, $a^+_{\mathbf{n}}(\eta)$ are zero: $\langle 0|a_{\mathbf{n}}(\eta)|0\rangle = 0$, $\langle 0|a^+_{\mathbf{n}}(\eta)|0\rangle = 0$. The mean values of the quadratic combinations of $a_{\mathbf{n}}(\eta)$, $a^+_{\mathbf{n}}(\eta)$ are not zero:

$$\begin{array}{l}
\langle 0|a_{\mathbf{n}}(\eta)a_{\mathbf{m}}(\eta)|0\rangle = u_n(\eta)v_m(\eta)\delta^3(\mathbf{n}+\mathbf{m}) \\
\langle 0|a^+_{\mathbf{n}}(\eta)a^+_{\mathbf{m}}(\eta)|0\rangle = v^*_n(\eta)u^*_m(\eta)\delta^3(\mathbf{n}+\mathbf{m}) \\
\langle 0|a_{\mathbf{n}}(\eta)a^+_{\mathbf{m}}(\eta)|0\rangle = u_n(\eta)u^*_m(\eta)\delta^3(\mathbf{n}-\mathbf{m}) \\
\langle 0|a^+_{\mathbf{n}}(\eta)a_{\mathbf{m}}(\eta)|0\rangle = v^*_{\mathbf{n}}(\eta)v_m(\eta)\delta^3(\mathbf{n}-\mathbf{m})
\end{array} \tag{35}$$

These relationships (first two) show explicitly that the waves (modes) with the opposite momenta are not independent but, on the contrary, are strongly correlated. This means that the generated field is a combination of standing waves [17]. Let us see how this is reflected in the correlation functions of the field.

For purposes of illustration, we will first ignore the tensorial indices in Eq. (18) and will consider a scalar field

$$h(\eta,\, \mathbf{x}) = \int_{-\infty}^{\infty} d^3\mathbf{n}\Big(a_{\mathbf{n}}(\eta)e^{i\mathbf{nx}} + a_{\mathbf{n}}^{+}(\eta)e^{-i\mathbf{nx}}\Big) \tag{36}$$

It is obvious that the mean value of the field is zero, $<\!0|h(\eta,\, \mathbf{x})|0\!> = 0$, in every spatial point and for every moment of time. The mean value of the square of the field $h(\eta,\, \mathbf{x})$ is not zero and can be calculated with the help of Eq. (35):

$$<\!0|h(\eta,\, \mathbf{x}),\, h(\eta,\, \mathbf{x})|0\!> = 4\pi \int_0^{\infty} n^2 dn\Big(|u_n|^2 + |v_n|^2 + u_n v_n + u_n^* v_n^*\Big) \tag{37}$$

In term of the squeeze parameters this expression can be written as

$$<\!0|h(\eta,\, \mathbf{x}),\, h(\eta,\, \mathbf{x})|0\!> = 4\pi \int_0^{\infty} n^2 dn(ch2r_n + sh2r_n \cdot \cos 2\phi_n) \tag{38}$$

(it includes the vacuum term $4\pi \int_0^{\infty} n^2 dn$, which should be subtracted at the end). It is seen from Eq. (38) that the variance of the field does not depend on the spatial coordinate $\mathbf{x}$. The function under the integral in Eq. (38) is usually called the power spectrum of the field: $P(n) = n^2(ch2r_n + sh2r_n \cos 2\phi_n)$. The important property of squeezing is that the $P(n)$ is not a smooth function of n but is modulated and contains many zeros or, strictly speaking, very deep minima. To see this, one can return to Eq. (31). For late times, that is, well after the completion of the amplification process, the function a'/a on the right-hand side of Eq. (31) can be neglected. This is equivalent to saying that one is considering waves that are well inside the present day Hubble radius. For these late times, the squeeze parameter r_n is not growing any more and the squeeze angle ϕ_n is just $\phi_n = -n\eta - \phi_{0n}$. Since $r_n \gg 1$ for the frequencies of our interest, the $P(n)$ can be written as $P(n) \approx n^2 e^{2r_n} \cos^2(n\eta + \phi_{0n})$. The factor $\cos^2(n\eta + \phi_{0n})$ vanishes for a series of values of n. At these frequencies, the function $P(n)$ goes to zero. The position of zeros on the n axis varies with time. Similar conclusions hold for the spatial correlation function $<\!0|n(\eta,\, \mathbf{x}),\, h(\eta,\, \mathbf{x}+\mathbf{l})|0\!>$:

$$<\!0|h(\eta,\, \mathbf{x}),\, h(\eta,\, \mathbf{x}+\mathbf{l})|0\!> = 4\pi \int_0^{\infty} n^2 \frac{\sin nl}{nl}(ch2r_n + sh2r_n \cos 2\phi_n)dn \tag{39}$$

The resulting expression (39) depends on the distance between the spatial points but not on their coordinates. The power spectrum of this correlation function is also modulated by the same factor $\cos^2(n\eta + \phi_{0n})$. It is necessary to note that the power spectrum of the energy density of the field is smoooth as it includes, in addition to Eq. (39), the kinetic energy term

$$\frac{1}{n^2}<0|h'(\eta,\, \mathbf{x})h'(\eta,\, \mathbf{x})|0> = 4\pi \int_0^\infty n^2 dn(ch2r_n - sh2r_n \cos 2\phi_n) \tag{40}$$

so the contributions with the oscillating factors $\cos 2\phi_n$ cancel out.

Let us mention some new features which arise when one considers the tensor field $h_{ij}(\eta,\, \mathbf{x})$, Eq. (18), not the scalar field $h(\eta,\, \mathbf{x})$. First, one sees from Eq. (28) that the solutions $u_n^s(\eta)$, $v_n^s(\eta)$ for both polarisations $s = 1,\, 2$ are identical; they obey the same equations with the same initial conditions. This means that, for each $\mathbf{n}$, both polarisation states are necessarily generated, and with equal amplitudes. This feature can serve as a clear distinction between gravitational waves generated quantum-mechanically and by other mechanisms. The second feature is related to the properties of the correlation function analogous to Eq. (39). In case of the tensor field h_{ij}, there is one combination of the components h_{ij} which has a particular interest: $h(e) = h_{ij}e^i e^j$, where e^i is an arbitrary unit vector. The $h(e)$ enters the calculations of the fluctuations of the microwave background temperature seen in the direction e^i (Sachs-Wolfe effect):

$$\frac{\triangle T}{T}\left(e^i\right) = \frac{1}{2}\int_{\eta_E}^{\eta_R} \frac{\partial h_{ij}}{\partial \eta} e^i e^j d\eta \tag{41}$$

The relevant correlation function is $<0|h_{ij}e^i e^j(\eta,\, 0)h_{ij}e^i e^j\left(\eta,\, \tau e^k\right)|0>$, where τ is a parameter along the line of sight. It is interesting to calculate this function for the initial time $\eta = 0$ and, also, for some very late time η. Without going into the details (they will be published elsewhere), we will present some results. For $\eta = 0$ one has $v_n(\eta) = 0$, $u_n(\eta) = 1$ and

$$<0|h_{ij}e^i e^j(0,\, 0)h_{ij}e^i e^j\left(0,\, \tau e^k\right)|0> = \int_{-\infty}^{\infty} d^3\mathbf{n}\left[\left(p_{ij}^1 e^i e^j\right)^2 + \left(p_{ij}^2 e^i e^j\right)^2\right] e^{-in_j e^j \tau} \tag{42}$$

The presence of the both polarisations is very important: because of this the integration over angular variables eliminates dependence on the direction

e^j. The final result is

$$<0|h_{ij}e^ie^j(0,\,0)h_{ij}e^ie^j\left(0,\,\tau e^k\right)|0> = 16\pi\sqrt{\frac{\pi}{2}}\int_0^\infty (n\tau)^{-5/2}J_{5/2}(n\tau)n^2dn \tag{43}$$

One can see that the (vacuum) correlation function depends only on the distance between the points and does not depend on the direction e^j.

To conclude this section, we should say that the same theory of squeezed states is applicable to the density perturbations generated quantum-mechanically [18]. In a simple inflationary model governed by a massive scalar field, the progenitor of the density perturbations and, later, the density perturbations themselves, satisfy equations similar to the equations for gravitational waves: $v'' + (n^2 - V(n,\,\eta))v = 0$ where v is a gauge-invariant function which includes perturbations of the matter variables and gravitational field [19], and $V(n,\,\eta)$ is the effective potential analogous to the gravitational-wave potential $U(\eta)$, Fig. 3. Similarly to the case of gravitational waves, squeezing in density perturbations and associated (longitudinal) gravitational field, exhibits distinct observational features.

8. Quantum Cosmology, Minisuperspace Models and Inflation

Until now we have been discussing the quantum fluctuations superimposed on a given classical background spacetime. The next level of complexity is the quantization of the background geometry itself. This is the domain of full quantum gravity and quantum cosmology.

In canonical quantum gravity the role of a generalized coordinate is played by a 3–geometry $g^{(3)}$. The full set of all 3–geometries forms a superspace, where the wave function of the quantized gravitational field is defined. If some matter fields are present, the superspace includes the matter variables as well. In cosmological applications, one usually considers topologically compact geometries and calls the wave function of the entire system the wave function of the universe. The basic equation which governs the wave function of the universe is called the Wheeler – DeWitt (WD) equation. (For reviews of quantum gravity and quantum cosmology, see, for example [20].)

A simplified problem, which allows a detailed investigation, is provided by minisuperspace models. In minisuperspace models one neglects all degrees of freedom except of a few. A quantum cosmological model describing a homogeneous isotropic universe filled with a massive scalar field gives a reasonably simple, though sufficiently representative case. In this case one has only two degrees of freedom (two minisuperspace variables): the scale factor a(t) and the scalar field ϕ(t). Since the formulation of a quantum problem includes integration of some quantities, such as the Hamiltonian function, over 3 – volume, one normally considers closed 3–sphere geometries, k = +1, or torus-like geometries, k=0, in order to avoid infinities arising because of spatial integration. The total energy of a closed world is zero. This is why the analog of the Schrödinger equation takes the form $\hat{\mathrm{H}}\psi = 0$, which is the Wheeler – DeWitt equation.

For a FRW universe filled with a scalar field ϕ, $V(\phi) = \frac{1}{2}m^2\phi^2$, the Wheeler – DeWitt equation can be written as follows [21]:

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} - ka^2 + m^2\phi^2a^4\right)\psi(a,\phi) = 0 \tag{44}$$

The factor p reflects some ambiguity in the choice of operator ordering. The possible preferred choice of p for the given model is p = 1.

First, we will show how classical Einstein equations of motion follow from the quantum equation (44) in the quasi-classical approximation. For simplicity we consider the limit where the spatial curvature term ka^2 can be neglected. In this limit (and for $p = 1$) Eq. (44) reduces to

$$\left(\frac{1}{a}\frac{\partial}{\partial a}a\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} + m^2\phi^2a^4\right)\psi(a,\phi) = 0 \tag{45}$$

In the quasi-classical approximation, the wave function $\psi(\ a,\ \phi)$ has the form $\psi(a,\phi) = exp(iS(a,\phi) + i\sigma(a,\phi) + ...)$. By using this representation, the following equations can be derived from Eq (45):

$$-\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{a^2}\left(\frac{\partial S}{\partial\phi}\right)^2 + m^2\phi^2a^4 = 0 \tag{46}$$

$$i\frac{\partial^2 S}{\partial a^2} - 2\frac{\partial S}{\partial a}\frac{\partial\sigma}{\partial a} + \frac{i}{a}\frac{\partial S}{\partial a} - \frac{i}{a^2}\frac{\partial^2 S}{\partial\phi^2} + \frac{2}{a^2}\frac{\partial S}{\partial\phi}\frac{\partial\sigma}{\partial\phi} = 0 \tag{47}$$

Eq. (46) is the Hamiltonian-Jacobi equation for the action S. A real solution to Eq. (46), which describes the classical dynamics of the model, can be presented in the form

$$S(a,\phi) = -a^3f(\phi) \tag{48}$$

where the function $f(\phi)$, as a consequence of Eq. (46), satisfies the ordinary differential equation

$$9f^2 - \left(\frac{df}{d\phi}\right)^2 = m^2\phi^2 \tag{49}$$

The classical equations of motion can be obtained from Eq. (48) and the Lagrangian $L = \frac{1}{2}\Big(-a\dot{a}^2 + a^3\dot{\phi} - a^3m^2\phi^2\Big)$ of the system in the usual way. One writes $\frac{\partial L}{\partial \dot{a}} = -a\dot{a} = \frac{\partial S}{\partial a}$, $\frac{\partial L}{\partial \dot{\phi}} = a^3\dot{\phi} = \frac{\partial S}{\partial \phi}$ which leads to the relations

$$\frac{\dot{a}}{a} = 3f, \qquad \dot{\phi} = -f^{'} \tag{50}$$

where the prime denotes a derivative with respect to ϕ, and the dot denotes a derivative with respect to the time t. By taking time derivatives of Eq. (50) and using Eq. (49), one derives the equations of motion which can be cast in the usual classical form:

$$\begin{gathered}\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi = 0,\ \left(\frac{\dot{a}}{a}\right)^2 = \dot{\phi}^2 + m^2\phi^2 \\ \left(\frac{\dot{a}}{a}\right)^{\cdot} + \left(\frac{\dot{a}}{a}\right)^2 = -2\dot{\phi}^2 + m^2\phi^2\end{gathered} \tag{51}$$

Eqs. (51) are invariant under the transformation $t\rightarrow -t$. The three equations of motion (51) can be combined in one equation, in which the time parameter t does not appear at all:

$$\phi\frac{d^2\phi}{d\alpha^2} + \left(3\phi\frac{d\phi}{d\alpha} + 1\right)\left[1 - \left(\frac{d\phi}{d\alpha}\right)^2\right] = 0 \tag{52}$$

(For convenience, we use the variable $\alpha = 1\text{n}\ a$ here and below.) This equation completely describes the classical trajectories in the $(\alpha,\, \phi)$ space. The direction of motion along the trajectories is determined by the choice of the time direction.

At this point we should comment on whether or not the sign of the action S has anything to do with the expansion or contraction of a cosmological model. This issue is often discussed in the context of the so-called "tunneling" wave function [22]. Eqs. (48) and (50) may lead to the impression that $S<0\,(f>0)$ corresponds to the expansion ($\dot{a}>0$), while the opposite choice

$S>0\,(f<0)$ corresponds to the contraction ($\dot{a}<0$) of the cosmological volume. However, the choice of the parameter t in these equations is absolutely arbitrary. The functions $S>0\,(f<0)$ can perfectly well describe expansion if one changes the parameter t to $-t$ in Eq. (50). Thus, the sign of the action does not prescribe a particular meaning to the direction of evolution along the classical trajectories.

For the model defined by classical equations of motion (51), all trajectories of the model in the $\left(\phi,\, \dot{\phi}\right)$ phase plane have previously been found [23]. It has been shown that in the case of expansion (i.e., $\dot{a} > 0$), all the trajectories, except for two, start out from the ejecting nodes K_1 and K_2 (see Fig. 6). The remaining two trajectories, corresponsing to the inflationary regime, are two attracting separatrices that originate at the saddle points S_1 and S_2. The solutions to Eq. (49) have the following asymptotic behavior for trajectories that start out from the nodes: $f\approx ce^{\pm 3\phi}$, $c^2e^{\pm 3\phi}\gg m^2\phi^2$, $c = const$. And for the separatrices one has $f\approx\pm\frac{1}{3}m\phi$, $9\phi^2\gg 1$. Different values of the constant c select different trajectories leaving the nodes. Hence, a particular solution to Eq. (49) gives a difinite function S and, at the same time, a particular classical trajectory.

Now we will relate different wave functions to different classical solutions. One can distinguish different solutions to Eq. (49) by the subscript n which varies continuously and takes on two distinct values corresponding to the separatrices. By virtue of the linearity of the WD equation, we can present a full set of solutions to Eq. (45) in the form $\psi = \sum_n exp(iA_n + iS_n)$, $S_n = -exp(3\alpha)f_n$, $A_n = const$ valid in the lowest approximation. To every quasiclassical wavefunction $\psi_n = \exp(iS_n)$ one can put into correspondence a family of lines that are ortogonal to the surfaces $S_n = \text{const}$ (Fig. 7). These surfaces are constructed in the minisuperspace $(\alpha,\, \phi)$ endowed with the metric tensor

$$G^{\mu\nu} = e^{-3\alpha}diag(-1,\, +1), \qquad \mu,\nu = 1,2, \qquad x^1 = \alpha, \quad x^2 = \phi \tag{53}$$

The vector N_μ ortogonal to $S_n = \text{const}$ can be obtained by acting on $\psi_n = \exp(iS_n)$ with the momentum operators π_α and π_ϕ:

$$\hat{\pi}_\alpha\psi_n = \frac{1}{i}\frac{\partial}{\partial\alpha}\psi_n = \frac{\partial S_n}{\partial\alpha}\psi_n = N_\alpha\psi_n$$
$$\hat{\pi}_\phi\psi_n = \frac{1}{i}\frac{\partial}{\partial\phi}\psi_n = \frac{\partial S_n}{\partial\phi}\psi_n = N_\phi\psi_n$$

Taking into account Eqs. (48), (53) one obtains $N^{\alpha} = 3f$, $N^{\phi} = -f'$. The vector field $\left(N^{\alpha},\, N^{\phi}\right)$ determines the lines $x^{\mu}(\alpha,\, \phi)$ ortogonal to $S_n = \mathrm{const}$ in a parametric way: $dx^{\mu}/dt = N^{\mu}$. These lines coincide with the classical trajectories (see Eq. (50)). One can also note that by integrating the relation $d\alpha/d\phi = N^{\alpha}/N^{\phi} = -3f/f'$ along every classical path in the $(\alpha,\, \phi)$ plane one gets $z(\alpha,\, \phi) = \mathrm{const}$ where $z \equiv \alpha + 3\int (f/f')d\phi$.

In the case at hand, the family of lines ortogonal to S_n and the associated tangent vectors N^{α}, N^{ϕ} are independent of α and transform into themselves under the shift $\alpha \to \alpha +$ const, or a(t) $\to$const a(t). This symmetry is a reflection of the fact that the function a(t) alone does not appear in Eq. (51), it appears only as the Hubble factor $\dot{a}/a$. Therefore, the invariance of the vector field N^{μ} under the displacement $\alpha \to \alpha + \mathrm{const}$ means that, for a given S_n, the lines traced out by N^{μ} are all copies of one and the same physically distinct classical solution. It happened as a consequence of our assumption of a negligibly small spatial curvature, k=0; in general, it is not the case.

Thus, we see that different solutions to the Hamilton-Jacobi equation determine different wave functions in their lowest (in terms of $\hbar$) approximation. On the other hand, to a given S_n one can assign a family of classical trajectories. The next approximation to S_n defines the prefactor to the wave function $\psi_n = e^{iS_n}$. The prefactor is responsible for forming a packet from classical trajectories determined by S_n. It assigns different "weights" to different classical paths.

Let us return to Eq. (47) for σ_n. The general solution for σ_n can be expressed in terms of the function $f_n(\phi)$: $\sigma_n(\alpha, \phi) = \frac{i}{2}\Big(3\alpha + ln\, f_n^{'}\Big) + B_n(z)$ where B_n is an arbitrary function of its argument z. In the considered approximation, the general solution to WD equation can be written in the form

$$\psi = \sum_n Z_n \psi_n = \sum_n Z_n exp(iS_n + i\sigma_n) \tag{54}$$

where

$$\psi_n = \chi_n e^{i\gamma_n} \Big(a^3 f_n^{'}\Big)^{-1/2} e^{-ia^3 f_n}, \tag{55}$$

χ_n and γ_n are arbitrary real functions of z and Z_n are arbitrary complex numbers. One can see that to every path z = const in $(\alpha,\, \phi)$ plane one can assign a number $Q_n \equiv {\chi_n}^2$ (z) which is conserved along this path. A particular value of Q_n is determined by a chosen wave function (in other words, by the chosen boundary conditions for the wave function) and, specifically by the function χ_n(z). Diffrent wave functions favor the inflationary trajectories to a different degree (see, for example, [24]).

9. From the Space of Classical Solutions to the Space of Wave Functions

From the problem of distributing "weights" among different classical trajectories belonging to the same family determined by S_n we now turn to the more difficult problem of distributing "weights" among the wavefunctions themselves. As we saw above, the WKB components ψ_n, Eq. (55), participate in the general solution Eq. (54) with arbitrary complex coefficients Z_n. They determine one or other choice of possible wavefunctions. How can one classify the space of all possible wavefunctions?

To answer this question we will start from the simplest situation, when the number of the linearly independent solutions to WD equation is only two. For this aim we will first consider Eq. (44) in another limiting case, namely, when the term $-\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}$ can be neglected. In this case the variable ϕ plays the role of a parameter and the problem reduces to a one-dimensional problem. The basic Eq. (44) can be written down in the form (for k = +1):

$$\left(\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da} - a^2 + H^2a^4\right)\psi(a) = 0 \tag{56}$$

where $H \equiv m^2\phi^2$. We prefer to work with exact solutions to Eq. (56) so we choose p = − 1 or p = 3 [25]. (The case p = − 1 was first considered in Ref. [26].) We will write the exact solution for the p = − 1 case in the form:

$$\psi(a) = u^{1/2}\left[A_1 H^{(1)}_{1/3}\left(\frac{u^{3/2}}{3H^2}\right) + A_2 H^{(2)}_{1/3}\left(\frac{u^{3/2}}{3H^2}\right)\right], \quad H^2a^2 \geq 1, \tag{57}$$

$$\psi(a) = (-u)^{1/2}\left[B_1 I_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right) + B_2 K_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right)\right], \quad H^2a^2 \leq 1 \tag{58}$$

where $u = H^2a^2 - 1$ and $I_{1/3}$, $K_{1/3}$, $H^{(1)}{}_{1/3}$, $H^{(2)}{}_{1/3}$ are the Infeld, Macdonald and Hankel special functions, correspondingly.

Eq. (56) has the form of the Schrödinger equation for a 1–dimensional problem with the potential V(a) = $a^2 - H^2a^4$ (see Fig. 8). The coefficients A_1, A_2 and B_1, B_2 are two pairs of arbitrary constant coefficients in front of two pairs of linearly independent solutions. By matching the solutions (57) and (58) at the point a = 1/H one finds [25]: $B_1 = -A_1\left(1 + i\sqrt{3}\right) - A_2\left(1 - i\sqrt{3}\right), \quad B_2 = \frac{2i}{\pi}(A_2 - A_1)$.

Now let us characterize the full space of the wave functions. (Here we mainly follow Ref. [27].) In the present case, our quantun system has only two linearly independent states and therefore resembles a simple spin - 1/2 system. Let us call an arbitrarily chosen basis of states $|1>$ and $|2>$. A general state $|\psi>$ can be expanded as $|\psi> = Z_1|1> + Z_2 \mid 2>$, where Z_1 and Z_2 are arbitrary complex constants.

It is a general principle of quantum mechanics that state vectors which differ only by an overall non-zero multiple λ describe one and the same physical state. Thus, the pair of coordinates (Z_1 and Z_2) and the pair (λZ_1 and λZ_2) are equivalent. It follows that physical quantities can only depend on the ratio $\zeta = Z_1/Z_2$ which is invariant under rescaling. In our example above we may identify Z_1 with B_1 and Z_2 with B_2. It is convenient to introduce the notation $B_1 = |B_1| exp(i\beta_1), \quad B_2 = |B_2| exp(i\beta_2), \quad \beta = \beta_1 - \beta_2$ and then $\zeta = B_1/B_2 = x\, exp(i\beta)$. The ratio ζ parameterizes the points on a 2 – dimensional sphere and so we see that the set of possible wavefunctions is in 1–1 correspondence with the points on the 2 – sphere.

We now wish to place a measure on the space of quantum states. Of course there are many possible measures. However, in choosing a measure we should be guided by the priniciple that the measure should be independent of the arbitrary choice of basis states $|1>$ and $|2>$. That is if we perform a unitary change of basis, which will preserve all probability amplitudes, then the measure should remain invariant.

The invariance of the measure may be taken as the quantum analogue of the principle of general covariance in classical general relativity. In fact in the classical limit it corresponds to invariance under canonical transformations. This latter invariance was used in Ref. [28] to suggest a suitable measure on the set of classical solutions.

For a 2–state system the 2–dimensional unitary transformations will act (provided $|1>$ and $|2>$ are normalised) on the complex 2–vector (Z_1 and Z_2) by multiplication by a 2 by 2 unitary matrix. Clearly the ratio ζ is unaffected by matrices which are merely multiples of the unit matrix so we may confine attention to special unitary matrices of determinant unity, this still allows minus the identity matrix so if we want just the transformations which change the physical states we must identify to SU(2) matrices which differ by multiplication by minus one. That is, the effective physical transformations acting on the space of quantum states is the rotation group SO(3) = SU(2)/C_2 where C_2 is the group consisting of + 1 and − 1. In fact this acts on the 2–sphere in the usual way provided we identify β with the longitudinal angle and x = cotan(θ/2) where θ is the usual co-latitude.

It is now clear that we must choose for our invariant measure on the space of quantum states the usual volume element on the 2–sphere. This is clearly invariant under rotations and up to an arbitrary constant multiple it is unique. That is the measure in terms of β and θ is:

$$dV = \sin\theta d\theta d\beta, \quad 0\leq\theta\leq\pi, \quad 0\leq\beta\leq 2\pi \tag{59}$$

Of course the measure is just the Riemannian volume element with respect to the standard round metric on the 2–sphere.

It should be mentioned that the well known Hartle-Hawking wavefunction [29] is exactly the south pole ($\theta = \pi$) of the 2–sphere. This wavefunction is real. Another real wavefunction corresponds to the north pole of the 2–sphere. We call this wavefunction anti-Hartle-Hawking wavefunction. All other wavefunctions are complex.

10. On the Probability of Quantum Tunneling from "Nothing"

The measure introduced in the space of all wave-functions may allow us to formulate and solve some physically meaningful problems. We will try to pose one such a problem already in the considered simplest model. As was mentioned above, Eq. (56) looks like the Schrödinger equation for a particle of a zero energy moving in the presence of the potential V(a). The form of the potential (Fig. 8) motivates the expectation that some of the wavefunctions may be capable of describing the quantum tunneling or decay. In ordinary quantum mechanics the quantity D, where $D = \left|\frac{\psi(a_2)}{\psi(a_1)}\right|^2$, can be interpreted as the quasiclassical probability for the particle to tunnel from one classically allowed region to another (see Fig. 8). The wave function used in this expression is determined by the imposed boundary conditions, i.e. it is determined by the physical formulation of the problem. The value of D is always (much) less than unity, D<1, for wave functions describing quantum tunneling or decay. One can define a similar quantity D in our quantum cosmological model, though, the physical interpretation of D is less clear. The main difference is that in ordinary quantum mechanics one imposes suitable boundary conditions in time t and space x, while in our problem there is only one coordinate, a. (The notion of the break of classical evolution in quantum cosmology is rather delicate. We have argued in Ref. [30], that only in superspaces of more than one dimension, this notion can be clearly formulated.) Nevertheless, we will adopt the same definition of D in our problem: $D = \left|\frac{\psi\left(\frac{1}{H}\right)}{\psi(0)}\right|^2$, and will consider $p<1$. The quantity D is well defined mathematically and can be calculated for every solution of Eq. (56) regardless of its interpretation. Since in quantum–mechanical problem the "energy" ϵ of the particle is $\epsilon = 0$, we can provisionally interpret D as the probability for creation of the universe from "nothing". (This is more precise formulation of the notion, introduced at the beginning of these lectures and graphically depicted in Fig. 1.) Therefore, we are interested in dividing the wave functions into two classes which predict D<1 and D>1. It is not excluded that the wave-functions predicting D<1 can be eventually justified as describing the quantum tunneling from "nothing", or rather from the "vacuum" defined in the framework of some more deep quantum theory.

It is easy to calculate D in the approximation H << 1 [25]. One can see that the different choices of the wave-function give different values of D. For instance, the Hartle-Hawking wave-function corresponds to the choice B_1=0 and gives $D = exp\left(\frac{2}{3H^2}\right) \gg 1$. We are interested to know the value of D for a typical wave-function. In other words, we need to know how many wave-functions give D<1 or D>1? To answer this question one must consider the space of all possible wave-functions with a suitable measure. By using the measure (59) one can show that the set of wave functions predicting D>1 is very small compared with that predicting D<1. This follows from the fact that the surface area of the patch covered by the wave-functions with D>1 is very small compared with the total surface area of the 2–sphere. Indeed, the circle separating D>1 and D<1 regions on the 2–sphere corresponds to the value $\theta_0 \approx \pi - 2\, exp\left(-\frac{1}{3H^2}\right), \quad \pi - \theta_0 \ll 1$. Only a small area around the south pole $\theta = \pi$ gives the wave-functions with D>1, the rest of the surface of the 2–sphere corresponds to the wave-functions with D<1. The ratio of the surface area around the south pole to the total surface area is very small; it is equal to $exp\Big(-\frac{2}{3H^2}\Big) \ll 1$. Thus, one can say, that the probability of finding a wave function with D>1 (among them is the Hartle-Hawking wave-function) is very small. One can conclude that the overwhelming majority of the wave-functions seem to be capable of describing the quantum tunneling or decay, since they predict D<1. (It is interesting to note that the product of surface areas with D>1 and D<1 to their corresponding maximal values of D gives approximately equal numbers, both of order unity.)

The simple example presented above clarifies the notion of the measure in the space of all physically distinct wave-functions. In a similar way one can introduce the measure in the multidimensional space of the wave-functions described by Eq. (54) [27]. The use of this measure shows that the inflation is indeed a property of a typical wave function, at least, under some additional assumptions adopted in [27, 25].

11. Duration of Inflation and Possible Remnants of the Preinflationary Universe

In the framework of the inflationary hypothesis, one normally considers cosmological models whose period of inflation lasted much longer than the minimal duration necessary to increase a preinflationary scale to the size of the present-day Hubble radius. In such models, the number N of e-foldings of the scale factor during inflation is much larger than the minimal N_{min}, in which case the volume covered by inflation is much larger than the present-day Hubble volume. For this reason, one normally does not expect to find any "remnants" of the preinflationary universe (see, however, Ref. [31]) as they were enormously diluted and spread over the huge inflated volume. Nevertheless, according to the quantum cosmological considerations, the duration of inflation close to the minimally sufficient amount may happen to be the most probable prediction of some popular quantum cosmological models, as we will see below. (This section is based on Ref. [32]).

Quantum cosmology is supposed to provide initial data for classical cosmological models and resolve such issues as the likelihood of inflation and its probable duration. Obviously, we are still far away from a satisfactory answer. A part of the problem is that there are too many possible wave functions: the trouble of selecting an appropriate classical solution from the space of all possible classical solutions is replaced by an even bigger problem of selecting an appropriate wave function from the space of all possible wave functions. However, if a cosmological wave function is chosen, the derivation of the probability distribution of the permitted classical solutions seems to be more straightforward.

A wave function which has received much attention in the literature is the Hartle-Hawking wave function ψ_{HH}. As we have seen above, one cannot say that the ψ_{HH} is in any sense more probable than others. On the contrary, it looks, rather, as an exception. For simple quantum cosmological models allowing inflation, the Hartle-Hawking wave function corresponds to a single point — a pole on the two-sphere representing the space of all physically different wave functions (see Sec. 9). However, the ψ_{HH} is a real wave function while all others (except the one corresponding to the opposite pole which is also real and which we call the "anti-Hartle-Hawking" wave function) are complex. This exceptional property of the ψ_{HH} alone, if for no other reasons, justifies special attention to this wave function and makes it interesting to see what kind of predictions with regard to inflation follow from it.

For the case of homogeneous isotropic models with the scale factor $a(t)$ and a scalar field $\phi(t)$, the ψ_{HH} predicts a set of classical inflationary solutions which can be described as trajectories in the two-dimensional space $[a(t), \phi(t)]$ (see [21] and Sec. 8). These trajectories begin in the vicinity of a line which is the caustic line for the so-called Euclidean trajectories. The probability distribution P_{HH} for the classical (Lorentzian) inflationary solutions follows from the ψ_{HH} and has the form

$$P_{HH} = N \exp\left[\frac{2}{3H^2(\phi)}\right], \tag{60}$$

where N is the normalization constant and $H(\phi)$ is the Hubble factor at the beginning of inflation. The function P_{HH} varies along the caustic line and increases rapidly toward the smaller values of ϕ. This means that the probability to find a given inflationary solution is higher the lower the initial value of the scalar field $\phi(t)$ (if, of course, this interpretation of P_{HH} is correct). But smaller initial values of $\phi(t)$ correspond to the shorter periods of inflation which makes solutions with a shorter period of inflation much more probable than solutions with a longer period of inflation.

An important fact is, however, that the inflationary period cannot be too short. The reason is that the caustic line does not extend down to the very low values of ϕ; instead, it has a sharp cusp (singularity) at the point of return from which the second branch of the caustic line develops (see Fig. 9) [24]. The point of return on the caustic line divides the Euclidean trajectories into two families which touch the first or the second branch of the caustic, respectively. The Lorentzian inflationary solutions cannot begin with the initial value of the scalar field and the Hubble factor lower than the value corresponding to the point of return ϕ^* and, therefore, their periods of inflation cannot be arbitrarily short. Thus, ψ_{HH} gives more weight to inflationary solutions with lower initial values of $\phi(t)$ but does not accommodate solutions which begin with $\phi(t)$ smaller than ϕ^*. The numerical estimates for the case of the scalar field potentials $V(\phi) = m^2\phi^2/2$ and $V(\phi) = \lambda\phi^4/2$ show [24] that the number ϕ^* falls short a factor 4 or 3, respectively, to ensure the minimally sufficient inflation. The inflated scale turns out to be of order 10^{21} cm instead of the required 10^{28} cm. At the same time, the probability distribution function P_{HH} reaches its maximum value at $\phi = \phi^*$. Thus, it seems that the most probable prediction of the Hartle-Hawking wave function is a "small, underinflated universe". However, it is possible that the discrepancy between l_H and the predicted inflated scale may be weakened or even removed for other scalar field potentials. Apart from that, the deficiency of ϕ^* in being just a numerical factor 4 or 3 smaller than necessary, in the situation where the initial values of the scalar field can vary within a huge interval from ϕ^* up to about $10^5\ \phi^*$, can serve as an indication that the duration of inflation close to the minimally sufficient amount should, probably, be taken seriously, at least, as a prediction of the Hartle-Hawking wave function.

The meaning of the above discussion is that the search for the "remnants" of the preinflationary universe, in the framework of the inflationary hypothesis, may not necessarily be of a purely academic interest.

12. Relic Gravitons and the Birth of the Universe

The quantum cosmological mini-superspace models analyzed above included only two degrees of freedom and corresponded to homogeneous isotropic universes. The inclusion of all degrees of freedom at the equal footing would present a formidable problem. However, this problem can be simplified in a perturbative approximation which is a quantum-mechanical treatment of a perturbed homogeneous isotropic universe. In particular, the Schrödinger equation for gravitons, with the Hamiltonian equivalent to Eq. (21), can be derived from the fully quantum cosmological approach as an approximate equation for the linearized perturbations.

Let us consider a closed universe governed by an effective cosmological term Λ and perturbed by weak gravitational waves. The WD equation for the wave function of this system can be written in the form (see, for example, Ref. [33], [34]):

$$\left[\frac{1}{2a}\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{a}{2} + \frac{2}{3\pi}\left(\frac{l_{pl}}{l_0}\right)^2\frac{a^3}{2} + \sum_{nlm} H_{nlm}(a, h_{nlm})\right]\psi(a, \{h_{nlm}\}) = 0 \tag{61}$$

Here, the h_{nlm} denotes the amplitude of the gravity wave perturbation in a given mode (n,l,m). Since we are working in a closed 3–space, it is convenient to attribute the indices l,m to spherical harmonics. The H_{nlm} denotes the Hamiltonian of the perturbation: $H_{nlm} = \pi^2_{nlm}/2M + M\Omega^2_n h^2_{nlm}/2$, where π_{nlm} is the momentum canonically conjugated to h_{nlm}, and $M = a^3$, $\quad \Omega_n = a^{-1}(n^2-1)^{1/2}$. In what follows we will often omit the indices n,l,m for simplicity.

The total wave function $\psi(a, \{h\})$ depends on a scale factor a and a set of the gravity wave variables h_{nlm}. The ψ(a, {h}) can be presented in the form $\psi(a,\{h\}) = exp[-A(a) - A_1(a)]\Phi(a,\{h\})$, where A(a) is the "unperturbed" (background) action and A_1(a) is the prefactor of the background wave function. The Φ(a, {h}) is the part of the total wave function describing the fluctuations. We assume that ψ(a, {h}) satisfies the quasiclassical approximation with respect to the variable a. This allows us to simplify the WD-equation. We assume also that the fluctuations are weak and do not affect the background so that the term $\partial^2\Phi/\partial a^2$ can be neglected in Eq. (61).

It follows from Eq. (61) that the wave function ψ(a, h) for each mode of fluctuations obeys the Schrödinger equation

$$-\frac{1}{i}\frac{\partial\psi}{a\partial\eta} = H\psi \tag{62}$$

where $\partial/a\partial\eta = -ia^{-1}(dA/da)\partial/\partial a$, and $H = H_{nlm}$. The wave function $\Phi(a, \{h\})$ is constructed as follows: $\Phi = \prod_{nlm}\psi$. One can see that in the regime when A(a) describes classical Lorentzian evolution, that is, when the background space-time is the De-Sitter solution, Eq. (62) coincides with the Schrödinger equation for the problem considered in Sec. 7. (One has to take into account some obvious modifications related to the fact that we are considering now the case k = +1.) However, Eq. (62) has, in fact, a wider domain of applicability. The assumptions under which Eq. (62) was derived retain this equation valid in the region where A(a) describes a classically forbidden behaviour of the universe, i.e. this equation is valid in the under-barrier region $a<1/H$ (see Fig. 8) as well. In this region Eq. (62) takes the form of the Schrödinger equation written in the imaginary time. Thus, the graviton wave function ψ(a,h) extends to the classically forbidden region a<1/H and may be sensitive to the form of the background wave function in this region.

Our final goal is to show that the initial quantum state of gravitons at the beginning of the De-Sitter stage (before the parametric amplification has started) is not unrelated to the form of the background wave function of the universe in the region $a<1/H$. Everywhere in our previous discussion we were assuming that the initial state of gravitons at $\eta = \eta_b$ was the vacuum. The present-day observational predictions have also been derived under this assumption. However, this assumption, though quite usual and natural, is not obligatory. If the initial state of gravitons could have been a non-vacuum state, then it would lead to the differing predictions for the present day spectrum of relic gravitons and their squeeze parameters. In this way, by measuring the actual parameters of relic gravitons, one could learn something about the wave function of the universe in its classically forbidden regime.

One should note, however, that the possible deviations of the initial quantum state of gravitons from the vacuum state, regardless of the origin of these deviations, can not be too large. These deviations should satisfy two requirements. First, they should not violate our basic assumption that the back-action of gravitons on the background geometry is always negligibly small. Second, they should not lead to the predictions for the present day amplitudes which would exceed the existing experimental limits. By combining these requirements one can show that only for low-frequency waves and only for cosmological models with minimally sufficient duration of inflation the initial quantum state of graviton modes can possibly deviate from the vacuum [35]. In this case the deviations of the present-day spectrum can be as large as is shown by the broken line in Fig. 10 for a specific De-Sitter model with $l_0 = c/H_0 = 10^9\, l_{pl}$. In this figure the dotted line shows the spectrum produced from the initial vacuum state in the same model, and the solid line shows the highest possible inflationary spectrum compatible with the observational limits. (The solid line is just a low frequency part of the inflationary spectrum presented in Fig. 4).

Now we return to the question of which of the background wave functions are compatable with the deviations of the initial quantum state of gravitons from the vacuum. As we have already seen, the Hartle-Hawking wave function ψ_{HH} and the anti-Hartle-Hawking wave function ψ_{aHH} are, in a sense, two extremes in the description of the classically forbidden domain. Each of these extremes can be used in Eq. (62) as a background wave function. For each of them the solution to Eq. (62) can be presented in the Gaussian form (compare with Eq. (15)) $\psi(h,\eta) = C(\eta)e^{-B(\eta)h^2}$. Here $B(\eta)$ is constricted as a linear combination of two independent complex solutions to the classical wave equation (6), and the vacuum state at $\eta = \eta_b$ corresponds to the value $B(\eta) = B(\eta_b)$, where $B(\eta_b) = \frac{1}{2}\omega_b$, $\omega_b = (n^2-1)^{\frac{1}{2}}a^{-2}(\eta_b)$. It is important that the choice of ψ_{HH} or ψ_{aHH} in the classically forbidden region restricts the function $B(\eta)$ in different ways if one is willing to subject the wave function $\psi(h,\,\eta)$ to the condition of normalizability: $\int_{-\infty}^{\infty}\psi^*\psi dh{<}\infty$ [34]. If this condition is imposed, it requires Re $B(\eta){>}0$. It turns out that this requirement singles out the vacuum initial state if the background wave function is ψ_{HH} and it leaves room for non-vacuum initial states if the background wave function is ψ_{aHH}. Thus, if relic gravitational waves are detected with properties different from those following from the initial vacuum state one could conclude that the universe was described by ψ_{aHH}, and not by ψ_{HH}, in the classically forbidden regime. This would strengthen the hypothesis that the universe was created in a quantum process similar to quantum tunneling or decay. Thus, the difference between possible wave functions of the universe in the classically forbidden regime can be distinguished by exploring the properties of the gravitational wave background existing now.

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Figure Captions

1. The scale factor of a complete cosmological theory.
2. Parametric (superadiabatic) amplification of waves.
3. The potential $U(\eta)$ for the inflationary — radiation-dominated — matter-dominated cosmological model.
4. Theoretical predictions and experimental limits for stochastic gravitational waves.
5. Variancies for coherent and squeezed quantum states.
6. Classical trajectories at the compactified $\left(\phi,\, \dot{\phi}\right)$ phase diagram.
7. Classical paths in the $(\alpha,\, \phi)$ configuration space.
8. The potential $V(a)$.
9. Two branches of the caustic line and its singularity.
10. Possible spectra of relic gravitons.