Topics in Algebraic Geometry

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Contents

Chapter 1

Homological Algebra

1 Additive and abelian categories, complexes

Definition 1.1. An *additive category* is a category $\mathcal{A}$ having the following properties :

(i) For any objects L, M of $\mathcal{A}$, the set of morphisms $\mathrm{Hom}(L, M)$ is endowed with the structure of an abelian group, and for any objects L, M, N, the composition

$$\mathrm{Hom}(L, M) \times \mathrm{Hom}(M, N) \to \mathrm{Hom}(L, N)$$

is $\mathbb{Z}$-bilinear.

(ii) There exists an object which is both initial and final, which is called the *zero object* and denoted by 0 : for any object L, $\mathrm{Hom}(L, 0) = \mathrm{Hom}(0, L) = \{0\}$.

(iii) For any objects L, M of $\mathcal{A}$, the sum $L \oplus M$ and the product $L \times M$ exist.

It is easily checked that, in presence of (i) and (ii), (iii) implies that the map $L \oplus M \to L \times M$ with components $(\mathrm{Id}, 0)$ and $(0, \mathrm{Id})$ is an isomorphism. indeed, we have the following diagram:

$$\begin{tikzcd}
L \arrow[d] \arrow[dr, "{(\mathrm{Id},0)}"] & \\
L \oplus M \arrow[r, "\sim"] & L \times M \\
M \arrow[u] \arrow[ur, "{(0,\mathrm{Id})}"'] &
\end{tikzcd}$$

If $\mathcal{A}$ is an additive category, so is the dual category $\mathcal{A}^0$.

Definition 1.2. A functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories is called *additive* if $F(0) = 0$ and for any objects L, M of $\mathcal{A}$, the natural morphism $F(L) \oplus F(M) \to F(L \oplus M)$ is an isomorphism. Equivalently, F is additive if for any objects L, M of $\mathcal{A}$, the map $F : \mathrm{Hom}(L, M) \to \mathrm{Hom}(F(L), F(M))$ is $\mathbb{Z}$-linear.

The category Ab of abelian groups is in an obvious way an additive category, and if $\mathcal{A}$ is an additive category, for any object L of $\mathcal{A}$, the functor $\mathrm{Hom}(L, -)$ (resp. $\mathrm{Hom}(-, L)$) from $\mathcal{A}$ (resp. $\mathcal{A}^0$) to Ab is additive.

Definition 1.3. An *abelian category* is an additive category $\mathcal{A}$ satisfying the following axioms :

(AB 1) Any morphism $u : L \to M$ in $\mathcal{A}$ has a kernel $\mathrm{Ker}(u)$ (i.e. the equalizer of u and the 0 morphism) and a cokernel $\mathrm{Coker}(u)$ (i.e. the coequalizer of u and the 0 morphism). That is, one always has commutative diagrams

$$\begin{tikzcd} \mathrm{Ker} u \arrow[r] & L \arrow[r, "u"] & M \\ & K \arrow[u] \arrow[ur, "0"'] \arrow[ul, dotted, "\exists !"] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "u"] \arrow[dr, "0"'] & M \arrow[r] \arrow[d] & \mathrm{Coker}\, u \arrow[dl, dotted, "\exists !"] \\ & K \end{tikzcd}$$

(AB 2) For any morphism $u : L \to M$, the canonical morphism $\mathrm{CoIm}(u) \to \mathrm{Im}(u)$ is an isomorphism, where $\mathrm{CoIm}(u)$, the *coimage* of u is the cokernel of the morphism $\mathrm{Ker}(u) \to L$, and $\mathrm{Im}(u)$, the image of u, is the kernel of the morphism $M \to \mathrm{Coker}(u)$.

In presence of (AB 1), (AB 2) is equivalent to saying that a morphism which is both a monomorphism and an epimorphism is an isomorphism.

The dual category of an abelian category is abelian.

A typical example of an abelian category is the category of modules over a ring, or, more generally, of sheaves of modules over a sheaf of rings on a topological space. By a theorem of Mitchell [M, 7.1], generalizing a theorem of Freyd, any "small" abelian category $\mathcal{A}$ can be embedded as a full subcategory of a small category of modules over a ring, by a (fully faithful) functor preserving kernels and cokernels ("small" is a set-theoretic condition, meaning that the set of objects of $\mathcal{A}$ and for any two objects L, M, the set of morphisms $\mathrm{Hom}(L, M)$ belong to some "universe" (see [SGA4, I] for the definition of universes).

In an abelian category, push-outs and pull-backs exist and are defined as in the category of modules over a ring.

1.4. Exact sequence. In an abelian category $\mathcal{A}$, a sequence $L \xrightarrow{u} M \xrightarrow{v} N$ such that $v \circ u = 0$ is called *exact* if the canonical morphism $\mathrm{Ker}(v) \to \mathrm{Im}(u)$ is an isomorphism (or equivalently, $\mathrm{Ker}(v) = \mathrm{Im}(u)$ as a subobject of M). More generally, a sequence $(\cdots \to L^{i-1} \to L^i \to L^{i+1} \to \cdots)$ $(i \in \mathbb{Z})$ is called *exact* if any two consecutive morphisms $L^{i-1} \to L^i \to L^{i+1}$ form an exact sequence. A *short exact sequence* is an exact sequence of the form $0 \to L \xrightarrow{u} M \xrightarrow{v} N \to 0$ where the exactness means that u is a monomorphism, v is an epimorphism, and $\mathrm{Ker}(v) = \mathrm{Im}(u)$. The following standard result is extremely useful:

Proposition 1.5 (snake lemma). *Consider a commutative diagram in* $\mathcal{A}$

$$\begin{tikzcd} & L' \arrow[r] \arrow[d, "u'"] & L \arrow[r] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \, , \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' & \end{tikzcd}$$

in which the rows are exact. Then there exists a unique morphism

$$\delta : \mathrm{Ker} u'' \to \mathrm{Coker}\, u'$$

making the following square commutative :

$$\begin{tikzcd} L \times_{L''} \mathrm{Ker} u'' \arrow[r] \arrow[d] & \mathrm{Ker} u'' \arrow[d, "\delta"'] \\ M' \arrow[r] & \mathrm{Coker}\, u' \end{tikzcd} ,$$

in which the horizontal maps are the natural projections and the left vertical maps is induced by u. Moreover, the sequence

$$\mathrm{Ker} u' \longrightarrow \mathrm{Ker} u \longrightarrow \mathrm{Ker} u'' \xrightarrow{\delta} \mathrm{Coker}\, u' \longrightarrow \mathrm{Coker}\, u \longrightarrow \mathrm{Coker}\, u' \, ,$$

in which the maps other than δ are the natural ones, is exact.

Proof. The uniqueness of δ is clear and its existence is easy. By duality (i.e. passing to the dual category), it is enough to check exactness at $\mathrm{Ker}\, u$, which is immediate, and at $\mathrm{Ker}\, u''$, which is nontrivial. In the case of a category of modules over a ring, one can pick up elements and make a diagram chasing (cf. [B, I]). In the general case, the verification is more delicate. One can bypass it by using Mitchell's embedding theorem quoted above. $\square$

A trivial (but useful) corollary of the snake lemma is the so-called *five lemma*.

Corollary 1.6 (five lemma). *Consider a commutative diagram*

$$\begin{tikzcd}
L^1 \arrow[r] \arrow[d, "u^1"] & L^2 \arrow[r] \arrow[d, "u^2"] & L^3 \arrow[r] \arrow[d, "u^3"] & L^4 \arrow[r] \arrow[d, "u^4"] & L^5 \arrow[d, "u^5"] \\
M^1 \arrow[r] & M^2 \arrow[r] & M^3 \arrow[r] & M^4 \arrow[r] & M^5
\end{tikzcd},$$

in which the rows are exact. Then, if u^1, u^2, u^4, u^5 are isomorphisms, so is u^3.

1.7. Complex:definition and naive truncation. Let $\mathcal{A}$ be an additive category. A *complex* of $\mathcal{A}$, denoted $L^\bullet$, or just L, is a family of objects L^i of $\mathcal{A}$, $i \in \mathbb{Z}$, and morphisms $d^i : L^i \to L^{i+1}$ (sometimes just denoted d) such that $d^{i+1}d^i = 0$. One writes

$$L = (\cdots \to L^i \to L^{i+1} \to \cdots).$$

One says that d^i is the *differential* of L, and that L^i is the *component of degree* i of L. A *morphism* $u : L \to M$ of complexes of $\mathcal{A}$ is a family of morphisms $u^i : L^i \to M^i$ such that the squares

$$\begin{tikzcd}
L^i \arrow[r, "d"] \arrow[d, "u^i"] & L^{i+1} \arrow[d, "u^{i+1}"] \\
M^i \arrow[r, "d"] & M^{i+1}
\end{tikzcd}$$

commute. Complexes of $\mathcal{A}$ form an additive category $C(\mathcal{A})$.

A complex L is said to be *bounded above* (resp. *bounded below*, resp. *bounded*) if $L^i = 0$ for i sufficiently large (resp. sufficiently small, resp. outside a bounded interval of $\mathbb{Z}$). If $[a, b]$ is a bounded interval of $\mathbb{Z}$, L is said to be *concentrated* in degrees in $[a, b]$ if $L^i = 0$ for $i \notin [a, b]$. When $a = b = n$, we simply say *concentrated in degree* n. By associating to an object E of $\mathcal{A}$ the complex concentrated in degree zero and having E as component of degree zero, one identifies $\mathcal{A}$ with the full subcategory of $C(\mathcal{A})$ consisting of complexes concentrated in degree zero.

For L in $C(\mathcal{A})$ and $n \in \mathbb{Z}$, one defines the complex $L[n]$ by $L[n]^i = L^{i+n}$, the differential of $L[n]$ being given by $(-1)^n d$, where d is the differential of

L. If $u : L \to M$ is a morphism of complexes, $u[n] : L[n] \to M[n]$ is given by $u[n]^i = u^{n+i}$ (no signs involved). The functor $L \mapsto L[n]$ is called a *translation* (or *shift*) functor. The complex concentrated in degree n and having E as n-th component is $E[-n]$ (where E is identified with the corresponding complex concentrated in degree zero).

One denotes by $L^{\geq n}$ (or $\sigma_{\geq n}L$) the subcomplex of L such that $\sigma_{\geq n}L^i = L^i$ for $i \geq n$ and zero otherwise :

$$\sigma_{\geq n}L = (0 \to L^n \to L^{n+1} \to \cdots).$$

Similarly, one denotes by $L^{\leq n}$ (or $\sigma_{\leq n}L$) the quotient of L such that $\sigma_{\leq n}L^i = L^i$ for $i \leq n$ and zero otherwise :

$$\sigma_{\leq n}L = (\cdots \to L^{n-1} \to L^n \to 0).$$

The functors $\sigma_{\geq n}$ and $\sigma_{\leq n}$ are called *naive truncations*.

1.8. Cohomology of a complex. Let $\mathcal{A}$ be an abelian category and let L be a complex of $\mathcal{A}$. For $i \in \mathbb{Z}$, one defines

$$Z^iL = \operatorname{Ker} d^i : L^i \to L^{i+1}, \quad B^iL = \operatorname{Im} d^{i-1} : L^{i-1} \to L^i,$$

$$H^iL = Z^iL/B^iL.$$

One says that Z^iL (resp. B^iL, resp. H^iL) is the *cycle* object (resp. *boundary* object, resp. *cohomology* object) of L in degree i. One says that L is *acyclic* in degree i if $H^iL = 0$, and more generally, if $[a, b]$ is an interval of $\mathbb{Z}$, that L is acyclic in the interval $[a, b]$ (resp. acyclic) if L is acyclic in degree i for all i in $[a, b]$ (resp. for all i). For a fixed i, Z^i, B^i, H^i are additive functors from $C(\mathcal{A})$ to $\mathcal{A}$.

Here comes the most important notion in homological algebra.

Definition 1.9. A morphism of complexes $u : L \to M$ is called a *quasi-isomorphism* if $H^i(u) : H^iL \to H^iM$ is an isomorphism for every $i \in \mathbb{Z}$.

First notice that if $0 \to L$ is a quasi-isomorphism, then L is acyclic. If E is an object of $\mathcal{A}$, a *left resolution* of E is a quasi-isomorphism $L \to E$, where L is a complex concentrated in degree ≤ 0. It is the same as giving such a complex L, a morphism $\varepsilon : L^0 \to E$ such that the sequence

$$\cdots \longrightarrow L^i \longrightarrow \cdots \xrightarrow{d} L^0 \xrightarrow{\varepsilon} E \longrightarrow 0$$

is exact. Similarly, a *right resolution* of E is a quasi-isomorphism $E \to M$, where M is a complex concentrated in degree ≥ 0. It is the same as giving such a complex M, a morphism $\varepsilon : E \to M$ such that the sequence

$$0 \longrightarrow E \xrightarrow{\varepsilon} M^0 \xrightarrow{d} \cdots \longrightarrow M^i \longrightarrow \cdots$$

is exact.

1.10. Canonical truncation. Let $\mathcal{A}$ be an abelian category and $L \in C(\mathcal{A})$. For $n \in \mathbb{Z}$, let

$$\tau_{\leq n} L = (\cdots \xrightarrow{d} L^{n-1} \xrightarrow{d} Z^n L \longrightarrow 0)$$

be the subcomplex of L such that $(\tau_{\leq n} L)^i = L^i$ for $i < n$, Z^n for $i = n$ and 0 otherwise. Let

$$\tau_{\geq n} L = (0 \longrightarrow L^n / B^n L \xrightarrow{d} L^{n+1} \xrightarrow{d} \cdots)$$

be the quotient of L such that $(\tau_{\geq n} L)^i = L^i$ for $i > n$, $L^n / B^n L$ for $i = n$ and 0 otherwise. Finally, if $[a, b]$ is an interval of $\mathbb{Z}$, we set $\tau_{[a,b]} L = \tau_{\geq a} \tau_{\leq b} L = \tau_{\leq b} \tau_{\geq a} L$, i.e.

$$\tau_{[a,b]} L = (\, 0 \longrightarrow L^a / B^a L \xrightarrow{d} L^{a+1} \xrightarrow{d} \cdots \longrightarrow L^{b-1} \xrightarrow{d} Z^b L \longrightarrow 0\,).$$

For $a = b = n$, $\tau_{[a,b]} L = H^n(L)[-n]$. One has a natural morphism $\epsilon : \tau_{\leq n} \to L$ (resp. $\pi : \tau_{\geq n} \to L$); $H^i(\epsilon)$ (resp. $H^i(\pi)$) is an isomorphism for $i \leq n$(resp. $i \geq n$) and $H^i \tau_{\leq n} L = 0$ (resp. $H^i \tau_{\geq n} L = 0$) for $i > n$ (resp. $i < n$).

If $u : L \to M$ is a quasi-isomorphism, then $\tau_{\geq n} u : \tau_{\geq n} L \to \tau_{\geq n} M$ and $\tau_{\leq n} u : \tau_{\leq n} L \to \tau_{\leq n} M$ are quasi-isomorphisms. But in general, naive truncations $\sigma_{\leq n}$ and $\sigma_{\geq n}$ do not preserve quasi-isomorphisms.

Proposition 1.11. *Let $\mathcal{A}$ be an abelian category and*

$$0 \longrightarrow L' \xrightarrow{u} L \xrightarrow{v} L'' \longrightarrow 0$$

be a short exact sequence of complexes in $\mathcal{A}$. Then there exists a "long exact sequence of cohomology"

$$\cdots \to H^i L' \xrightarrow{H^i(u)} H^i L \xrightarrow{H^i(v)} H^i L'' \xrightarrow{\delta} H^{i+1} L' \xrightarrow{H^{i+1}(u)} H^{i+1} L \to \cdots \qquad (1.11.1)$$

Proof. By the *snake lemma* applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & L'^{i-1} \arrow[r] \arrow[d] & L^{i-1} \arrow[r] \arrow[d] & L''^{i-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'^{i} \arrow[r] & L^{i} \arrow[r] & L''^{i} \arrow[r] & 0,
\end{tikzcd}$$

we have the sequence $\tau_{[i,i+1]}L' \longrightarrow \tau_{[i,i+1]}L \longrightarrow \tau_{[i,i+1]}L''$ which has the form

$$\begin{tikzcd}
 & L'^i/B^iL' \arrow[r] \arrow[d] & L^i/B^iL \arrow[r] \arrow[d] & L'^i/B^iL' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{i+1}L' \arrow[r] & Z^{i+1}L \arrow[r] & Z^{i+1}L'' &
\end{tikzcd}$$

where the rows are exact. By the *snake lemma* again, we get the exact sequence (1.11.1). □

Corollary 1.12. *Let*

$$\begin{tikzcd}
0 \arrow[r] & L' \arrow[r] \arrow[d, "u'"] & L \arrow[r] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

be a morphism of short exact sequences in $C(\mathcal{A})$. Then if two of the morphisms u, u' , u'' are quasi-isomorphisms, so is the third one.

Proof. We only prove the case where u' and u are quasi-isomorphisms. The other two cases are analogous. Applying Proposition 1.11, we get the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^iL' \arrow[r] \arrow[d, "H^i(u')"] & H^iL \arrow[r] \arrow[d, "H^i(u)"] & H^iL'' \arrow[r] \arrow[d, "H^i(u'')"] & H^{i+1}L' \arrow[r] \arrow[d, "H^{i+1}(u')"] & H^{i+1}L \arrow[r] \arrow[d, "H^{i+1}(u)"] & \cdots \\
\cdots \arrow[r] & H^iM' \arrow[r] & H^iM \arrow[r] & H^iM'' \arrow[r] & H^{i+1}M' \arrow[r] & H^{i+1}M \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows. It follows from the five lemma(Lemma 1.6) that $H^i(u'')$ is an isomorphism for all i, i.e., u'' is a quasi-isomorphism. □

2 Bicomplexes and cones

2.1. Bicomplexes. Let $\mathcal{A}$ be an additive category. A *naive bicomplex K* of $\mathcal{A}$, is a family of objects $K^{i,j}$ in $\mathcal{A}$ indexed by $\mathbb{Z}\times\mathbb{Z}$, together with families of morphisms $d_1 = \{d_1^{i,j} : K^{i,j} \to K^{i+1,j}\}$ and $d_2 = \{d_2^{i,j} : K^{i,j} \to K^{i,j+1}\}$ in $\mathcal{A}$ such that

$$d_1^2 = 0,\ d_2^2 = 0,\ d_1 \circ d_2 = d_2 \circ d_1.$$

For convenience, the morphism $d_1^{i,j} : K^{i,j} \to K^{i+1,j}$ (resp. $d_2^{i,j} : K^{i,j} \to K^{i,j+1}$) is usually denoted by $d_1 : K^{i,j} \to K^{i+1,j}$ (resp. $d_2 : K^{i,j} \to K^{i,j+1}$).

A *bicomplex* L of $\mathcal{A}$ is a family of objects $L^{i,j}$ in $\mathcal{A}$ indexed by $\mathbb{Z}\times\mathbb{Z}$, together with a family of morphisms $d' = \{d' : L^{i,j} \to L^{i+1,j}\}$ and $d'' = \{d'' : L^{i,j} \to L^{i,j+1}\}$ in $\mathcal{A}$ such that

$$d'^2 = 0,\ d''^2 = 0,\ d' \circ d'' + d'' \circ d' = 0.$$

Starting with a naive bicomplex $K = (K^{i,j},\, d_1,\, d_2)$, we obtain a bicomplex $L =(L^{i,j},\, d',\, d'')$ by putting

$$L^{i,j} = K^{i,j},\ d' = d_1, d''^{i,j} = (-1)^i d_2^{i,j}.$$

Remark. Similarly, if we define a bicomplex L' by setting $L'^{i,j} = K^{i,j}$, $d''^{i,j} = d_2^{i,j}$ and $d'^{i,j} = (-1)^j d_1^{i,j}$, then there exists an isomorphism $L' \to L$ of bicomplexes defined by $(-1)^{ij}\,\mathrm{Id}$ in degree (i,j).

Let $M = (M^{i,j},\, d',\, d'')$ and $N = (N^{i,j},\, d',\, d'')$ be bicomplexes in $\mathcal{A}$, a morphism $f : M \to N$ is a family of morphisms $(f^{i,j} : M^{i,j} \to N^{i,j})$ such that $fd' = d'f$ and $fd'' = d''f$. Then we get an additive category $C^2(\mathcal{A})$.

Let $L = (L^{i,j},\, d',\, d'')$ be a bicomplex, and let m, n be integers. We define a new bicomplex $L[m,n] = (L[m,n]^{i,j},\, \partial',\, \partial'')$ by putting $L[m,n]^{i,j} = L^{m+i,n+j}$, $\partial'^{i,j} = (-1)^{m+n} d'^{m+i,n+j}$ and $\partial''^{i,j} = (-1)^{m+n} d''^{m+i,n+j}$.

Let M be a bicomplex. For $i \in \mathbb{Z}$, the complex $M^{i,\cdot} = (\cdots \to M^{i,j} \xrightarrow{d''} M^{i,j+1} \to \cdots)$ is called the i-th column of M. Similarly for $j \in \mathbb{Z}$, the complex $M^{\bullet,j} = (\cdots \to M^{i,j} \xrightarrow{d'} M^{i+1,j} \to \cdots)$ is called the j-th row of M.

Definition 2.2. A bicomplex K is called *biregular* if for all $n \in \mathbb{Z}$, the set $\{K^{i,j} | i + j = n, K^{i,j} \neq 0\}$ is a finite set.

Example 2.2.1. A complex K concentrated in a quadrant of the first (resp. the third) type (i.e there exist some $a, b \in \mathbb{Z}$ such that $K^{i,j} = 0$ if $i < a$

(resp. $i > a$) or $j < b$ (resp. $j > b$)) is biregular. but in general, a bicomplex concentrated in a quadrant of the second (resp. the fourth) type (i.e. there exist some $a, b \in \mathbb{Z}$ such that $K^{i,j} = 0$ if $i > a$ (resp. $i < a$) or $j < b$ (resp. $j > b$)), is not biregular.

Example 2.2.2. A bicomplex K concentrated in a horizontal strip of finite width (i.e. there exists some interval $[a, b]$ of $\mathbb{Z}$ such that $K^{i,j} = 0$ if $j \notin [a, b]$) is biregular. A bicomplex concentrated in a vertical strip of finite width is also biregular.

Let K be a biregular bicomplex, we define $\mathbf{s}K \in C(\mathcal{A})$ as follows

$$(\mathbf{s}K)^n = \bigoplus_{i+j=n} K^{i,j}, \quad d = d' + d'' : (\mathbf{s}K)^n \to (\mathbf{s}K)^{n+1}.$$

The complex $\mathbf{s}K$ is called the *simple complex associated to K*. Let $f = (f^{i,j}) : K \to L$ be a morphism of bicomplexes, we define $\mathbf{s}f : \mathbf{s}K \to \mathbf{s}L$ by $(\mathbf{s}f)^n = \bigoplus_{i+j=n} f^{i,j}$. The functor $\mathbf{s} : C^2(\mathcal{A})_{reg} \to C(\mathcal{A})$ is called *the associated simple complex functor* where $C^2(\mathcal{A})_{reg}$ denotes the full subcategory of $C^2(\mathcal{A})$ consisting of biregular bicomplexes. For any $a, b \in \mathbb{Z}$, we have $\mathbf{s}(K[a, b]) = (\mathbf{s}K)[a + b]$.

Proposition 2.3. *If $\mathcal{A}$ is an abelian category, then the functor $\mathbf{s} : C^2(\mathcal{A}_{reg}) \to C(\mathcal{A})$ is exact.*

Proof. Indeed, a finite direct sum of exact sequences is exact. □

2.4. Cone of a morphism. Let $u : L \to M$ be a morphism of complexes of an additive category $\mathcal{A}$. Then we obtain a naive bicomplex concentrated in columns of degree -1 and 0 with L and M filling in them respectively. It can be converted to a bicomplex K concentrated in the columns of the same degree, i.e.

$$K = \left(\begin{tikzcd} & \vdots & \vdots & \\ 0 \arrow[r] & L^1 \arrow[u] \arrow[r, "u"] & M^1 \arrow[u] \arrow[r] & 0 \\ 0 \arrow[r] & L^0 \arrow[u, "-d_L"] \arrow[r, "u"] & M^0 \arrow[u, "d_M"'] \arrow[r] & 0 \\ 0 \arrow[r] & L^{-1} \arrow[u, "-d_L"] \arrow[r, "u"] & M^{-1} \arrow[u, "d_M"'] \arrow[r] & 0 \\ & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd} \right)$$

where $d'' = -d_L$. The complex $C(u) = \mathbf{s}K$ is called the *cone* of u. By definition, we have

$$C(u)^n = L^{n+1} \bigoplus M^n$$

and the differential is given by

$$d = d_{C(u)} : C(u)^n \to C(u)^{n+1}, \quad d(x, y) = -d_L x + (ux + d_M y),$$

i.e.

$$d = \begin{pmatrix} -d_L & 0 \\ u & d_M \end{pmatrix}$$

Suppose from now on that $\mathcal{A}$ is abelian. By naive truncation, we have an exact sequence

$$0 \longrightarrow M \longrightarrow K \longrightarrow L[1,0] \longrightarrow 0$$

of bicomplexes where L and M are considered as the cones of the morphism $0 \to L$ and $0 \to M$ respectively. By Proposition 2.3, we get an exact sequence

$$0 \longrightarrow \mathbf{s}M \longrightarrow \mathbf{s}K \longrightarrow \mathbf{s}L[1,0] \longrightarrow 0\,,$$

that is, $0 \longrightarrow M \longrightarrow C(u) \longrightarrow L[1] \longrightarrow 0$.

Proposition 2.5. *The boundary morphism of the above complex $\delta = H^{n+1}(u) : H^n(L[1]) \to H^{n+1}(M)$.*

Proof. We can check this fact directly from definitions. $\square$

Corollary 2.6. *A morphism $u : L \to M$ of complexes in $\mathcal{A}$ is a quasi-isomorphism if and only if the cone of u is acyclic.*

Proof. By the above proposition, we have the exact sequence of cohomology

$$\cdots \longrightarrow H^i(M) \longrightarrow H^i(C(u)) \longrightarrow H^{i+1}(L) \xrightarrow{H^{i+1}(u)} H^{i+1}(M) \longrightarrow \cdots.$$

Then $C(u)$ is acyclic if and only if $H^i(C(u)) = 0$, that is, the morphism $H^i(L) \xrightarrow{H^i(u)} H^i(M)$ is an isomorphism for each $i \in \mathbb{Z}$. Thus the result follows. $\square$

Proposition 2.7. *Let*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] & M \arrow[r, "v"] \arrow[dr, "i"'] & N \arrow[r] & 0 \\ & & & C(u) \arrow[u, "\varphi"'] & \end{tikzcd} \tag{2.7.1}$$

be a commutative diagram of complexes in $\mathcal{A}$ where the row is exact, the morphism $i : M \to C(u)$ is the natural embedding, and $\varphi = (0, v) : C(u)^n \to N^n$. Then the following square

$$\begin{tikzcd} H^i(C(u)) \arrow[r, "H^i(-pr)"] \arrow[d, "H^i(\varphi)"'] & H^{i+1}(L) \arrow[d, equal] \\ H^i(N) \arrow[r, "\delta"] & H^{i+1}(L) \end{tikzcd} \tag{2.7.2}$$

is commutative, where $pr : C(u) \to L[1]$ is the natural projection, and δ is the boundary morphism of the long exact sequence of the upper row in (2.7.1).

Proof. Using Mitchell's embedding theorem, we may assume $\mathcal{A}$ is a full subcategory of a small category of modules over a ring. The advantage is that we could pick up elements. Consider the diagram

$$\begin{tikzcd}
 & & & H^i(C(u)) \arrow[d, "H^i(\varphi)"] & \\
 & H^iL \arrow[r] \arrow[d] & H^iM \arrow[r] \arrow[d] & H^iN \arrow[d] \arrow[dddll, dotted] & \\
 & L^i/B^iL \arrow[r, "\tilde{u}"] \arrow[d, "\bar{d_L}"] & M^i/B^iM \arrow[r, "\tilde{v}"] \arrow[d, "\overline{d_M}"] & N^i/B^iN \arrow[r] \arrow[d, "\overline{d_N}"] & 0 \\
0 \arrow[r] & Z^{i+1}L \arrow[r, "u"] \arrow[d] & Z^{i+1}M \arrow[r, "v"] \arrow[d] & Z^{i+1}N \arrow[d] & \\
 & H^{i+1}L \arrow[r] & H^{i+1}M \arrow[r] & H^{i+1}N &
\end{tikzcd}$$

where the dotted arrow is the boundary map δ. Let $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) \in Z^i(C(u))$ where $a \in L^{i+1}$, $b \in M^i$. Then $H^i(\varphi)([\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)]) = [v(b)]$ where the brackets denote cohomology class. Thus we can choose $\tilde{b} \in M^i/B^iM$ such that $\tilde{v}(\tilde{b}) = \widetilde{v(b)}$. Then $\overline{d_M}(\tilde{b}) = d_M(b) \in M^{i+1}$. There exists $l \in Z^{i+1}$ such that $u(l) = d_M(b)$. Then $(\delta H^i(\varphi))[\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)] = [l]$. On the other hand, $d_{C(u)} = \begin{pmatrix} -d_{L[1]} & 0 \\ u & d_M \end{pmatrix}$. Since $d_{C(u)} \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) = 0$, we obtain $u(a) = -d_M(b)$. Thus $a = -l$. It follows that $H^i(-pr)\,[\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)] = [-a] = [l] = (\delta H^i(\varphi))[\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)]$. Finally the diagram (2.7.2) commutes. $\square$

Corollary 2.8. *φ is a quasi-isomorphism.*

Proof. By Propositions 2.5 and 2.7, we get the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^iL \arrow[r, "H^i(u)"] \arrow[d, equal] & H^iM \arrow[r] \arrow[d, equal] & H^i(C(u)) \arrow[r, "H^i(-pr)"] \arrow[d, "H^i(\varphi)"] & H^{i+1}L \arrow[r, "H^i(u)"] \arrow[d, equal] & H^{i+1}M \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^iL \arrow[r, "H^i(u)"] & H^iM \arrow[r, "H^i(v)"] & H^iN \arrow[r, "\delta"] & H^{i+1}L \arrow[r, "H^i(u)"] & H^{i+1}M \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows. Then the result follows from *the five lemma*(Lemma 1.6). $\square$

Remark. By duality, there exists a natural morphism $\psi : L \to C(v)[-1]$ such that the diagram

$$\begin{tikzcd} H^i(C(v)) & H^i(N) \arrow[l, "H^i(-i_2)"'] \arrow[d, equal] \\ H^{i+1}(L) \arrow[u, "H^i(\psi)"] & H^i(N) \arrow[l, "\delta"'] \end{tikzcd}$$

is commutative, where $i_2 : N \to C(v)$ is the inclusion. The morphism $\psi : L \to C(v)[-1]$ is also a quasi-isomorphism.

Proposition 2.9. *Let $u : L \to M$ be a morphism of biregular bicomplexes. If u induces a quasi-isomorphism $u^{i,\bullet} : L^{i,\bullet} \to M^{i,\bullet}$ on each column (resp. u induces a quasi-isomorphism $u^{\bullet,j} : L^{\bullet,j} \to M^{\bullet,j}$ for each row), then $\mathbf{s}u : \mathbf{s}L \to \mathbf{s}M$ is a quasi-isomorphism.*

Proof. We have to show $H^n(\mathbf{s}u) : H^n(\mathbf{s}L) \to H^n(\mathbf{s}M)$ is an isomorphism for all n. For a fixed n, as L and M are biregular, only finite many components of L and M contribute to $H^n(\mathbf{s}L)$ and $H^n(\mathbf{s}M)$. Thus we may assume that $L^{i,j}$ and $M^{i,j}$ are 0 except for finitely many $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. In particular, there is some interval $[a,b]$ of $\mathbb{Z}$ such that $L^{i,\bullet} = M^{i,\bullet} = 0$ for any $i \notin [a,b]$, so u has the following shape

$$\begin{tikzcd} 0 \arrow[r] & L^{a,\bullet} \arrow[r] \arrow[d] & L^{a+1,\bullet} \arrow[r] \arrow[d] & \cdots \arrow[r] & L^{b,\bullet} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M^{a,\bullet} \arrow[r] & M^{a+1,\bullet} \arrow[r] & \cdots \arrow[r] & M^{b,\bullet} \arrow[r] & 0 \end{tikzcd}$$

We prove the result by induction on $b-a$. If $b-a=0$, since

$$\mathbf{s}L = L^{a,\bullet}[-a], \quad \mathbf{s}M = M^{a,\bullet}[-a]$$

the conclusion follows. Now assume that the result holds for $b-a<n$. For $b-a=n$, using the naive truncations, we get the following commutative diagram of bicomplexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L^{\geq a+1,\bullet} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L^{a,\bullet}[-a,0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M^{\geq a+1,\bullet} \arrow[r] & M \arrow[r] & M^{a,\bullet}[-a,0] \arrow[r] & 0 \end{tikzcd}$$

(a sign is involved in the identification of $L/L^{\geq a+1,\bullet}$ (resp. $M/M^{\geq a+1,\bullet}$) with $L^{a,\bullet}[-a,0]$ (resp. $M^{a,\bullet}[-a,0]$)). Applying the functor $\mathbf{s}$, we get a commutative diagram of complexes with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{s}L^{\geq a+1,\bullet} \arrow[r] \arrow[d] & \mathbf{s}L \arrow[r] \arrow[d] & L^{a,\bullet}[-a] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbf{s}M^{\geq a+1,\bullet} \arrow[r] & \mathbf{s}M \arrow[r] & M^{a,\bullet}[-a] \arrow[r] & 0 \end{tikzcd}$$

By the induction hypothesis, the left and right vertical arrows are quasi-isomorphisms, so is $\mathbf{s}(u)$ by Corollary 1.12. □

Let L be a bicomplex. For any $i,j \in \mathbb{Z}$, we denote ${}'H^{i,j} = \frac{\mathrm{Ker}\ d'^{i,j}}{\mathrm{Im}\ d'^{i-1,j}}$ and ${}''H^{i,j} = \frac{\mathrm{Ker}\ d''^{i,j}}{\mathrm{Im}\ d''^{i,j-1}}$. The complex

$${}'H^i(L) = (\cdots \longrightarrow {}'H^{i,j-1} \longrightarrow {}'H^{i,j} \longrightarrow {}'H^{i,j+1} \longrightarrow \cdots)$$

is called the i-th *column of cohomology* and the complex

$${}''H^j(L) = (\cdots \longrightarrow {}''H^{i-1,j} \longrightarrow {}''H^{i,j} \longrightarrow {}''H^{i+1,j} \longrightarrow \cdots)$$

is called the j-th *row of cohomology.*

Proposition 2.10. *Let $u : L \to M$ be a morphism of biregular bicomplexes. If u induces a quasi-isomorphism on each row (resp. each column) of cohomology, then $\mathbf{s}u : \mathbf{s}L \to \mathbf{s}M$ is a quasi-isomorphism.*

Proof. We only prove the assertion in the case of row. The proof is similar to that of Proposition 2.9. The difference is that we should use canonical truncations instead of naive ones. Using notations as above, we assume the result holds for $b - a < n$. For $b - a = n$, by truncations we get the following commutative diagram of bicomplexes with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L_a \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \tau_{\geq a+1,\bullet}L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_a \arrow[r] & M \arrow[r] & \tau_{\geq a+1,\bullet}M \arrow[r] & 0 \end{tikzcd}$$

where L_a (resp. M_a) is the bicomplex $\left(0 \longrightarrow L^{a,\bullet} \xrightarrow{\alpha_L} B^{a+1,\bullet}L \longrightarrow 0\right)$ (resp. $\left(0 \longrightarrow M^{a,\bullet} \xrightarrow{\alpha_M} B^{a+1,\bullet}M \longrightarrow 0\right)$) with $L_{a,\bullet}$ (resp. $M_{a,\bullet}$) in the column of degree a and $B^{a+1,\bullet}L$ (resp. $B^{a+1,\bullet}M$) in the column of degree $a+1$ and α_L (resp. α_M) is induced by d'_L (resp. d'_M). Applying the functor $\mathbf{s}$, we get the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{s}(L_a) \arrow[r] \arrow[d] & \mathbf{s}(L) \arrow[r] \arrow[d] & \mathbf{s}(\tau_{\geq a+1,\bullet}L) \arrow[r] \arrow[d] & 0\,. \\ 0 \arrow[r] & \mathbf{s}(M_a) \arrow[r] & \mathbf{s}(M) \arrow[r] & \mathbf{s}(\tau_{\geq a+1,\bullet}M) \arrow[r] & 0 \end{tikzcd}$$

The right vertical morphism is quasi-isomorphism by the induction hypothesis. Since $C(\alpha_L) = \mathbf{s}L_a[a+1]$ (resp. $C(\alpha_M) = \mathbf{s}M_a[a+1]$), it is enough to verify the natural morphism $C_a(u) : C(\alpha_L) \to C(\alpha_M)$ is quasi-isomorphism. Consider the commutative diagram

$$\begin{tikzcd} {'H^a(L)} \arrow[r] \arrow[d] & C(\alpha_L)[-1] \arrow[d, "C_a(u)"] \\ {'H^a(M)} \arrow[r] & C(\alpha_M)[-1] \end{tikzcd} \quad ,$$

by Propositions 2.5 and 2.7, the two horizontal morphisms are quasi-isomorphisms. It follows that $C_a(u)$ is also a quasi-isomorphism since $'H(u)$ is a quasi-isomorphism. Then $\mathbf{s}u$ is a quasi-isomorphism. □

3 Homotopy category of complexes, triangulated categories

Definition 3.1. Let $\mathcal{A}$ be an additive category and $u, v : L \to M$ be morphisms of complexes in $\mathcal{A}$, we say that u, v are *homotopic* if there exists $h = \{h^i : L^i \to M^{i-1}, i \in \mathbb{Z}\}$ such that $v - u = dh + hd$. We then write $u \sim v$ and say that h is a *homotopy* from u to v.

Proposition 3.2. *The relation $\sim$ is an equivalence relation compatible with the group structure* (*i.e* $\{v \in \mathrm{Hom}(L, M) : v \sim 0\}$ *is a subgroup of* $\mathrm{Hom}(L, M)$). *Moreover, $\sim$ is compatible with the morphism of complexes, i.e. if $u : K \to L$, $v_1, v_2 : L \to M$, $w : M \to N$ and $v_1 \sim v_2$, then $v_1 u \sim v_2 u$ and $wv_1 \sim wv_2$.*

Proof. This is immediate. □

Definition 3.3. Let $\mathcal{A}$ be an additive category. The *homotopy category of complexes*, denoted by $K(\mathcal{A})$, is defined as follows:

$$Ob(K(\mathcal{A})) = Ob(C(\mathcal{A})),$$

$$\mathrm{Hom}_{K(\mathcal{A})}(L, M) = \mathrm{Hom}_{C(\mathcal{A})}(L, M)/\{w \in \mathrm{Hom}_{C(\mathcal{A})}(L, M) : w \sim 0\}$$

and the composition in $K(\mathcal{A})$ is naturally induced from the composition in $C(\mathcal{A})$.

The category $K(\mathcal{A})$ is an additive category, and the natural functor from $C(\mathcal{A})$ to $K(\mathcal{A})$ is additive.

Remark. (a). A morphism $u : L \to M$ in $C(\mathcal{A})$ becomes invertible in $K(\mathcal{A})$ if there exists $v : M \to L$ in $C(\mathcal{A})$ such that $uv \sim id_M$, $vu \sim id_L$. Such a u is called a *homotopy equivalence*.

(b). If $\mathcal{A}$ is abelian, then $u \sim v$ implies that $H^i(u) = H^i(v)$ for any $i \in \mathbb{Z}$. In particular, if u is a homotopy equivalence, then u is a quasi-isomorphism, but in general the converse is not true. Indeed, consider the diagram of $K(Mod(\mathbb{Z}))$

$$\begin{tikzcd}
L \arrow[d, "f"] & 0 \arrow[r] & 2\mathbb{Z} \arrow[r] \arrow[d, "f^0"] & 0 \arrow[r] \arrow[d] & 0 \\
M & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where L is concentrated in degree 0 and f^0 is the natural inclusion. Then f is a quasi-isomorphism, but not a homotopy equivalence since there does not exist $\mathbb{Z}$-linear map $g^0 : \mathbb{Z} \to 2\mathbb{Z}$ such that $g^0 \circ f^0 = \mathrm{Id}_{2\mathbb{Z}}$.

(c). We say that $L \in C(\mathcal{A})$ is *homotopic to zero* or *homotopically trivial* if $L \sim 0$ in $K(\mathcal{A})$. In other words, there exists $h : L^i \to L^{i-1}$ such that $\mathrm{Id}_L = d \circ h + h \circ d$. It is equivalent to saying that L breaks into splitting short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & Z^i \arrow[r] & L^i \arrow[r] & Z^{i+1} \arrow[r] & 0 .
\end{tikzcd}$$

That is, $L^i = Z^i \bigoplus Z^{i+1}$.

(d). Even if $\mathcal{A}$ is abelian, $K(\mathcal{A})$ is not abelian in general. For example, consider the morphism of $K(Mod(\mathbb{Z}))$ defined by the natural projection

$$\begin{tikzcd}
L \arrow[d, "f"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
M & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where L, M are concentrated in degree 0. Then f has no kernel in $K(\mathcal{A})$. Indeed, if such a kernel exist, it should be $K = (0 \longrightarrow 2\mathbb{Z} \longrightarrow 0)$ concentrated in degree 0 with the natural inclusion $i : K \to L$. Then in the diagram

$$\begin{tikzcd} N \arrow[d, "h"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] & \cdots, \\ L & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

we get $fh \sim 0$ in $C(\mathcal{A})$, i.e. $fh = 0$ in $K(\mathcal{A})$. But there does not exist any morphism $s : N \to K$ such that $s \sim h$. Thus K is not the kernel of f. Then $K(\mathcal{A})$ is not abelian. More general, if $\mathcal{A}$ is an additive category, a morphism $u : L \to M$ in $\mathcal{A} \subset K(\mathcal{A})$ has a kernel in $K(\mathcal{A})$ if and only if it has a kernel Z in $\mathcal{A}$ and Z is a direct summand of L. Indeed, if u has a kernel in $K(\mathcal{A})$, it should have a kernel in $\mathcal{A}$ and the kernel in $\mathcal{A}$ is also a kernel in $K(\mathcal{A})$. Assume Z together with $v : Z \to L$ is such a kernel. Consider the distinguished triangle $C(v) \xrightarrow{\alpha} Z[1] \xrightarrow{v[1]} L[1]$, we have $v[1]\alpha = 0$. Since $v[1]$ is a monomorphism, we have $\alpha = 0$, i.e. there exists a morphism $f : L \to Z$ in $\mathcal{A}$ such that $fv = \mathrm{Id}_Z$ in $\mathcal{A}$. It follows that Z is a direct summand of L and the converse is immediate.

3.4. Translation functor and triangle. Let $\mathcal{D}$ be an additive category. A *translation functor* on $\mathcal{D}$ is an additive automorphism $T : \mathcal{D} \to \mathcal{D}$. For example, if $\mathcal{A}$ is an additive category, then in $\mathcal{D} = K(\mathcal{A})$, the functor $L \mapsto TL = L[1]$ gives a translation functor. For $n \in \mathbb{Z}$ and L in $\mathcal{D}$, we set

$$T^n L = \begin{cases} T^n L & n > 0 \\ (T^{-1})^{-n} L & n < 0 \\ L & n = 0. \end{cases}$$

Usually we denote $T^n L$ by $L[n]$. For L, N in $\mathcal{D}$ we define *the group of morphisms of degree* n from L to M by

$$\mathrm{Hom}^n_{\mathcal{D}}(L, M) = \mathrm{Hom}_{\mathcal{D}}(L, M[n]) = \mathrm{Hom}_{\mathcal{D}}(L[-n], M).$$

A *triangle* in $\mathcal{D}$ is a sequence of morphisms

$$L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} L[1]$$

where $w \in \mathrm{Hom}^1(N, L)$. Sometimes we can write it as a diagram as follows

$$\begin{tikzcd} & N \arrow[dl, phantom] \\ L \arrow[ur, "+1"] \arrow[r, "u"] & M, \arrow[u, "v"'] \end{tikzcd}$$

or denote it by $L \xrightarrow{u} M \xrightarrow{v} N \longrightarrow$.

Let $\Delta = L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} L[1]$ and $\Delta' = L' \xrightarrow{u'} M' \xrightarrow{v'} N' \xrightarrow{w'} L'[1]$ be triangles. A *morphism from* Δ *to* Δ' is a triple (f, g, h) making the following diagram

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"] & M \arrow[r, "v"] \arrow[d, "g"] & N \arrow[r, "w"] \arrow[d, "h"] & L[1] \arrow[d, "f[1]"] \\ L' \arrow[r, "u'"] & M' \arrow[r, "v'"] & N' \arrow[r, "w'"] & L'[1] \end{tikzcd}$$

commute. The triangles in $\mathcal{D}$ form a category.

Definition 3.5. A *triangulated category* is an additive category $\mathcal{D}$ endowed with a translation functor $L \to L[1]$ and a set $\mathcal{T}$ of triangles of $\mathcal{D}$ called *distinguished (or exact) triangles*, satisfying the following properties (TR1)-(TR5)

(TR1) If $T \in \mathcal{T}$, and $T' \sim T$, then $T' \in \mathcal{T}$. Moreover for any $X \in \mathcal{D}$, then $(\, X \xrightarrow{\mathrm{Id}_X} X \xrightarrow{0} 0 \xrightarrow{0} X[1]\,) \in \mathcal{T}$.

(TR2) For any morphism $X \xrightarrow{u} Y$ in $\mathcal{D}$. There exists a triangle

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{T}$.

(TR3) (Rotation) $(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]) \in \mathcal{T}$ if and only if

$$(\, Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]\,) \in \mathcal{T}$$

(TR4) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$
$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

there exists a morphism $h : Z \to Z'$ such that the triple (f, g, h) is a morphism of triangles, i.e. the following diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

commutes.

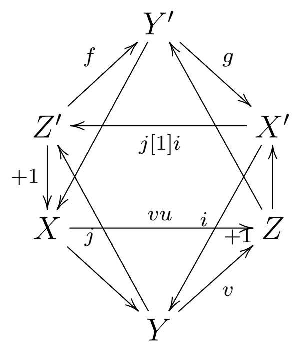
(TR5)(Octahedron) Given

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r] & Z' \arrow[r] & {} \\ Y \arrow[r, "v"] & Z \arrow[r] & X' \arrow[r] & {} \\ X \arrow[r, "vu"] & Z \arrow[r] & Y' \arrow[r] & {} \end{tikzcd}$$

in $\mathcal{T}$, there exist morphisms $Z' \xrightarrow{f} Y'$ and $Y' \xrightarrow{g} X'$ such that

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{j[1]i} Z'[1]$$

is distinguished, and in the diagram



(Id_X, v, f) is a morphism $XYZ' \to XZY'$ and (u, Id_Z, g) is a morphism $XZY' \to YZX'$.

Theorem 3.6. *Let $\mathcal{A}$ be an additive category, and $L \to L[1]$ be the translation functor on $K(\mathcal{A})$. Let $\mathcal{T}$ be the family of distinguished triangles of $K(\mathcal{A})$, defined by $T \in \mathcal{T}$ if and only if T is isomorphic to a triangle of the form*

$$L \xrightarrow{u} M \longrightarrow C(u) \xrightarrow{-pr} L[1],$$

where $C(u)$ is the cone of u, i (resp. pr) the canonical monomorphism (resp. epimorphism). Then $(K(\mathcal{A}), E \mapsto E[1], \mathcal{T})$ is a triangulated category.

Sketch of the proof. (See [K-S] or [V] for details):

(TR1):First we have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\mathrm{Id}_L"] & L \arrow[r] \arrow[dr] & 0 \arrow[r] & L[1] \\ & & C(\mathrm{Id}_L) \arrow[u] \arrow[ur] & \end{tikzcd}$$

For (TR1), it suffices to check that $C = C(\mathrm{Id}_L)$ is homotopically trivial. i.e. there exists $h : C^i \to C^{i-1}$ such that $\mathrm{Id} = hd + dh$. Write $C^{i-1} = L^{i-1} \bigoplus L^i$ and $C^i = L^i \bigoplus L^{i+1}$, one can set

$$h^i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : C^i = L^{i+1} \bigoplus L^i \to C^{i-1} = L^i \bigoplus L^{i-1}.$$

(TR4): Since the diagram

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[d, "g"] \\ L' \arrow[r, "u'"] & M' \end{tikzcd}$$

commutes in $K(\mathcal{A})$, we have a homotopy $s : gu \sim u'f$. Define

$$h^i = \begin{pmatrix} u^{i+1} & 0 \\ s^{i+1} & v^i \end{pmatrix} : L^{i+1} \bigoplus M^i \to L'^{i+1} \bigoplus M'^i,$$

one shows easily that the triple (f, g, h) gives a morphism of distinguished triangles

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & C(u) \arrow[r] \arrow[d, dotted, "h"] & L[1] \arrow[d, "f[1]"] \\ L' \arrow[r, "u'"] & M' \arrow[r] & C(u') \arrow[r] & L'[1]. \end{tikzcd}$$

(TR5): See [K-S]. □

Proposition 3.7. *Let $\mathcal{D}$ be a triangulated category and*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be a distinguished triangle. Then for any L in $\mathcal{D}$, the following sequences are exact:

(1) $\mathrm{Hom}(L, X) \to \mathrm{Hom}(L, Y) \to \mathrm{Hom}(L, Z)$,
(2) $\mathrm{Hom}(Z, L) \to \mathrm{Hom}(Y, L) \to \mathrm{Hom}(X, L)$.

Proof. First note that in a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$, the composition of any two consecutive morphisms is zero. Indeed by (TR3) (rotation) it is enough to check that $vu = 0$. By (TR1),

$$X \xrightarrow{\mathrm{Id}} X \xrightarrow{v} 0 \longrightarrow X[1]$$

is a distinguished triangle. By (TR4), we can complete the left square of the diagram into a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dotted] & X[1] \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & X[1] \end{tikzcd}$$

Then we get $vu = 0$. Next we check (1). By the above observation, we have $\mathrm{Im}\, u \subset \mathrm{Ker} v$. Let $v \in \mathrm{Hom}(L, Y)$ such that $vf = 0$. By (TR3) and (TR4) we can find $\tilde{f} : L \to X$ making the triple $(\tilde{f}, f, 0)$ a morphism of triangles.

$$\begin{tikzcd} L \arrow[r, "\mathrm{Id}"] \arrow[d, "\tilde{f}"] & L \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "0"] & L[1] \arrow[d, dotted, "\tilde{f}[1]"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & X[1] \end{tikzcd}$$

Thus we get $u\tilde{f} = 0$. This completes the proof of (1). The proof of (2) is similar, or one can use the fact that $\mathcal{D}^\circ$ is also a triangulated category. □

Remark. By (TR4), the sequences in (1), (2) give long exact sequences:
(1') $\cdots \mathrm{Hom}^n(L, X) \to \mathrm{Hom}^n(L, Y) \to \mathrm{Hom}^n(L, Z) \to \mathrm{Hom}^{n+1}(L, X) \cdots$,
(2') $\cdots \mathrm{Hom}^n(Z, L) \to \mathrm{Hom}^n(Y, L) \to \mathrm{Hom}^n(X, L) \to \mathrm{Hom}^{n+1}(Z, L) \cdots$.

Corollary 3.8. *Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & {} \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & {} \end{tikzcd}$$

be a morphism of distinguished triangles. Then if two of the morphisms f, g, h are isomorphisms, so is the third one. In particular, the third vertex of a distinguished triangle built on $f : X \to Y$ is unique up to isomorphism. Such a vertex is sometimes called a cone of f.

Proof. Applying the functor $\mathrm{Hom}(L, -)$ to the diagram, we get a morphism between two long exact sequences. The result follows from five lemma (1.6). □

Remark. The isomorphism above may be not unique. Note that for any X, Y, the triangle

$$X \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} X \bigoplus Y \xrightarrow{(0,1)} Y \xrightarrow{0}$$

is distinguished. Then in the diagram,

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X \oplus Y \arrow[r] \arrow[d, dotted, "f"] & Y \arrow[r, "0"] \arrow[d, equal] & X[1] \arrow[d, equal] \\ X \arrow[r] & X \oplus Y \arrow[r] & Y \arrow[r, "0"] & X[1] \end{tikzcd}$$

any

$$f = \begin{pmatrix} \mathrm{Id}_X & * \\ 0 & \mathrm{Id}_Y \end{pmatrix} : X \bigoplus Y \to X \bigoplus Y$$

gives a morphism of distinguished triangles.

Definition 3.9. Let $\mathcal{D}$ be a triangulated category. A functor $F : \mathcal{D} \to \mathcal{D}$ is called *triangulated* (or *exact*) if:
(1) F is additive;
(2) There is a functorial isomorphism $F(X[1]) \sim (FX)[1]$;
(3) F sends distinguished triangles into distinguished triangles.

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F induces $F : C(\mathcal{A}) \to C(\mathcal{B})$ and a triangulated functor $F : K(\mathcal{A}) \to K(\mathcal{B})$. This follows from the fact that $\mathrm{Cone}(F(u)) = F(\mathrm{Cone}(u))$.

Proposition 3.10. *Let $u : L \to M$ be a morphism in a triangulated category $\mathcal{D}$ and*

$$L \xrightarrow{u} M \longrightarrow N \longrightarrow$$

be a distinguished triangle. Then u is an isomorphism if and only if $N \sim 0$ in $\mathcal{D}$. In particular, any morphism $u : L \to M$ in $C(\mathcal{A})$ is a homotopy equivalence if and only if the cone of u is homotopically trivial.

Proof. By TR(1), we have a morphism of distinguished triangles,

$$\begin{tikzcd}
L \arrow[r, "u"] & M \arrow[r] & N \arrow[r] & L[1] \\
L \arrow[u, equal, "\mathrm{Id}_L"] \arrow[r, equal] & L \arrow[u, "u"] \arrow[r] & 0 \arrow[u] \arrow[r] & L[1] \arrow[u, equal]
\end{tikzcd}$$

Then the conclusion follows from Corollary 3.8. □

Proposition 3.11. *Let $L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w} L[1]$ be a distinguished triangle in $K(\mathcal{A})$, then we have a long exact sequence*

$$\cdots \longrightarrow H^iL \xrightarrow{H^i(u)} H^iM \xrightarrow{H^i(v)} H^iN \xrightarrow{H^i(w)} H^{i+1}L \longrightarrow \cdots.$$

Proof. Without loss of generality, we may assume the distinguished triangle is given by the cone $N = C(u)$ of a morphism $u : L \to M$ in $C(\mathcal{A})$, so that the corresponding w is just $-pr$. From the short exact sequence below

$$0 \longrightarrow M \xrightarrow{v} N \xrightarrow{-pr} L[1] \longrightarrow 0$$

we get that the sequence $H^iM \xrightarrow{H^i(v)} H^iN \xrightarrow{H^i(w)} H^{i+1}L$ is exact. By the axiom TR(3) of triangulated categories, we can rotate the distinguished triangle to obtain a new one, and hence, extend the exact sequence above to get the desired long exact sequence. □

4 Derived categories

Proposition 4.1. *Let $\mathcal{C}$ be a category and $S \subset Ar(\mathcal{C})$ be a subset of arrows. There exists a category $\mathcal{C}(S^{-1})$ and a functor $Q : \mathcal{C} \to \mathcal{C}(S^{-1})$ such that*

(1) *For any $s \in S$, $Q(s)$ is invertible;*

(2) *For any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is invertible for any $s \in S$, there exists a unique functor $\widetilde{F} : \mathcal{C}(S^{-1}) \to D$ such that $\tilde{F}Q = F$. Moreover, $(\mathcal{C}(S^{-1}), Q)$ is unique up to a unique isomorphism.*

Proof. The uniqueness is clear. Let us prove the existence. Let

$$Ob(\mathcal{C}(S^{-1})) = Ob(\mathcal{C}).$$

For X, Y in $\mathcal{C}$, let $H(X,Y) = \{X \to\leftarrow\to \cdots \to\leftarrow\to Y\}$ be the set of finite diagrams, where " $\leftarrow$ " is an element in S. Let " $\sim$ " be the equivalence relation on $H(X,Y)$ generated by the diagrams of the following type:
(1) $X \to \cdots \xrightarrow{f}\xrightarrow{g} \cdots \leftarrow \cdots Y \sim X \to \cdots \xrightarrow{gf} \cdots \leftarrow \cdots Y$;
(2) $X \to \cdots \xleftarrow{s}\xleftarrow{t} \cdots \to \cdots Y \sim X \to \cdots \xleftarrow{st} \cdots \to \cdots Y$;
(3) $X \to \cdots \xleftarrow{s}\xrightarrow{s} \cdots \to \cdots Y \sim X \to \cdots \xrightarrow{\mathrm{Id}} \cdots \to \cdots Y$ for any $s \in S$;
(4) $X \to \cdots \xleftarrow{s}\xrightarrow{f} \cdots \to \cdots Y \sim X \to \cdots \xrightarrow{g}\xleftarrow{t} \cdots \to \cdots Y$
if

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "s"'] & \cdot \arrow[d, "t"] \\ \cdot \arrow[r, "g"] & \cdot \end{tikzcd}$$

is commutative. Set $\mathrm{Hom}_{\mathcal{C}(S^{-1})} = H(X,Y)/\sim$. Define the composition of morphisms in $C(S^{-1})$ in the natural way and denote by Q the the natural functor $\mathcal{C} \to \mathcal{C}(S^{-1})$. Now it's easy to check that $(\mathcal{C}(S^{-1}), Q)$ solves the universal problem. $\square$

4.2. As in the case of rings of fractions, we would like to write an element

$$f \in \mathrm{Hom}_{\mathcal{C}(S^{-1})}(X,Y)$$

as a fraction $f = gs^{-1}$ or $t^{-1}h$, but it's not always possible. We will consider conditions on S which make it possible.

Definition 4.3. Let $\mathcal{C}$ be a category and $S \subset Ar(\mathcal{C})$. We say that S is a *multiplicative system* if:

(M1) For any $s : X \to Y$, $t : Y \to Z$ in S, then ts is in S. Moreover $\mathrm{Id}_L \in S$ for any L in $\mathcal{C}$.

(M2) Any diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \\ X' & \end{tikzcd}$$

with $s \in S$ can be completed into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "s"'] & Y \arrow[d, dotted, "t"] \\ X' \arrow[r, dotted, "u'"] & Y' \end{tikzcd}$$

with $t \in S$. Ditto with all the arrows reversed.

(M3) For any morphism $X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{t} Y'$ with $t \in S$ such that $tu = tv$, there exists some $s : X' \to X$ in S such that $us = vs$. Ditto with all arrows reversed.

Example 4.3.1. Let $\mathcal{A}$ be an abelian category and S be the set of quasi-isomorphisms in $K(\mathcal{A})$. Then S is a multiplicative system. (1) is trivial. Let $X' \xleftarrow{s} X \xrightarrow{u} Y$ be a diagram with $s \in S$. Using (TR2) and (TR4)we can construct the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, equal, "\mathrm{Id}_Z"] & X \arrow[r, "s"] \arrow[d, "u"] & X' \arrow[d, "v"] \\ Z \arrow[r, "uf"] & Y \arrow[r, "t"] & Y' \end{tikzcd}.$$

By hypothesis, s is a quasi-isomorphism, hence Z is acyclic (3.11), conversely the fact that Z is acyclic implies that t is a quasi-isomorphism (3.11), thus (2) follows. Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{s} Z$$

such that $sf = 0$ and $s \in S$, then by TR(2), $Y \to Z$ can be extended to a distinguished triangle

$$Y \xrightarrow{s} Z \longrightarrow M \longrightarrow$$

Since $sf = 0$, by (3.7'), there exists $f' : X \to M[-1]$ such that the diagram below commutes

$$\begin{tikzcd} M[-1] \arrow[r] & Y \\ X. \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$

Choose a distinguished triangle

$$X' \xrightarrow{t} X \xrightarrow{f'} M[-1] \xrightarrow{w},$$

then $ft = 0$, in fact $ft = t'f't = 0$. Moreover s is a quasi-isomorphism implies that M is acyclic, it follows that t is also a quasi-isomorphism.

Definition 4.4. A category I is called *filtering* if

(1) For any $i,\ j \in I$, there exists $k \in I$ and morphisms $i \to k$, $j \to k$.

(2) For any $i,\ j \in I$ and morphisms $u, v : i \to j$, there exists $k \in I$ and morphism $w : j \to k$ such that $wu = wv$.

Remark. Let $F : I \to \mathcal{S}ets$ be a functor, then

$$\varinjlim_{i \in I} F(i) = \bigsqcup_{i \in I} F(i)/\sim$$

where for any $x \in F(i)$ and $y \in F(j)$, we say $x \sim y$ if and only if there exists some $k \in I$ and morphisms $i \xrightarrow{u} k$ and $j \xrightarrow{v} k$ such that $F(u)(x) = F(v)(y)$. In particular if $F : I \to \mathcal{A}b \subset \mathcal{S}ets$, then

$$\varinjlim_{i \in I} F(i) = \bigoplus_{i \in I} F(i)/H,$$

where H is the subgroup of $\bigoplus_{i \in I} F(i)$ generated by $F(u)x - F(v)y$, for all $i \xrightarrow{u} j, j \xrightarrow{v} k \in Ar(I), x \in F(i), y \in F(j)$.

Proposition 4.5. *If $S \in Ar(\mathcal{C})$ is a multiplicative system, denote by I_Y the category $\{s : Y \to Y' \in S\}$, and $I^X = \{s : X' \to X \in S\}$. Then*
(a) *For any $X, Y \in \mathcal{C}$, $(I^X)^\circ$ and I_Y are both filtering.*
(b) *For any $X \in \mathcal{C}$, we have*

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X, Y) &= \varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}(X', Y) \\
&= \varinjlim_{Y \xrightarrow{t} Y' \in I_Y} Hom(X, Y') \\
&= \varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ, Y \xrightarrow{t} Y' \in I_Y} \mathrm{Hom}_{K(\mathcal{A})}(X', Y')
\end{aligned}$$

Proof. (a) For any $Y \xrightarrow{i'} Y', Y \xrightarrow{i''} Y'' \in I_Y$, by the axiom(M2) of multiplicative systems, we can complete these two morphisms to a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "i'"] \arrow[d, "i''"'] & Y' \arrow[d, "k'"] \\
Y'' \arrow[r, "k''"] & Y'''.
\end{tikzcd}$$

Let $j = k'i' : Y \to Y'''$, then the diagram commutes means that we have morphisms $i' \xrightarrow{k'} j, i'' \xrightarrow{k''} j$ in I_Y. On the other hand, let $i : Y \to Y', j : Y \to Y''$ be in I^Y, and $u, v : Y' \to Y''$, give morphisms between $i \to j$, then $ui = vi = j$, then by (M3) there exists $w : Y'' \to Y''' \in S$ such that $wu = wv$. Let $k = wj : Y \to Y'''$, by (M1) $k \in S$, and w give a morphism $j \to k$ such that $wu = wv : i \to k$ hence I^Y is filtering by definition. The proof $(I_X)^\circ$ is filtering is similar, thus (a) follows.

For (b), consider a category $\mathcal{D}$ as follows: the class of objects in $\mathcal{D}$ is the same as that in $\mathcal{A}$, and for any X, Y in $\mathcal{D}$, define

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = \varinjlim_{X' \xrightarrow{s} X \in (I_X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

By definition, a morphism in $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ can be represented by a triple (X', t, f), where

$$X' \in \mathcal{C}, \quad s \in \mathrm{Hom}_{\mathcal{C}}(X', X), \quad f \in \mathrm{Hom}_{\mathcal{C}}(X', Y)$$

and two triples (X', s', f'), (X'', s'', f'') are equivalent if and only if there exists a commutative diagram

$$\begin{tikzcd}
 & X & \\
X' \arrow[ur, "s'"] \arrow[dr, "f'"'] & X''' \arrow[l] \arrow[r] \arrow[u, "u"] & X'' \arrow[ul, "s''"'] \arrow[dl, "f''"] \\
 & Y &
\end{tikzcd}$$

with $u \in (I_X)^\circ$. The composition of $(X', s', f') \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $(Y', t, g) \in \mathrm{Hom}_{\mathcal{D}}(Y, Z)$ is defined as follows. Use the axioms of multiplicative system, we can find a commutative diagram:

$$\begin{tikzcd}
 & & X'' \arrow[dl, "t'"'] \arrow[dr, "h"] & & \\
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

with $t' \in S$, and we set

$$(Y', t, g) \circ (X', s, f) = (X'', st', gh)$$

One checks that the definition of composition doesn't depend on the choice of the representative. Moreover we have that $\mathcal{D}$ is an additive category and the natural functor $Q' : \mathcal{C} \to \mathcal{D}$ is additive. We claim that $(\mathcal{D}, Q')$ solves the same universal problem as $(C(S^{-1}, Q)$. Indeed, for any morphism $s : X \to Y \in S$, $Q'(s) = (X, \mathrm{Id}_X, s)$ is invertible in $\mathcal{D}$. Moreover, for any functor $F : \mathcal{C} \to \mathcal{A}$ with property that $F(s)$ is invertible for any $s \in S$, then we can define $F' : \mathcal{D} \to \mathcal{A}$ by $F'(u) = F(f)F(s)^{-1}$ for a morphism $u = (X', s, f) \in \mathrm{Hom}_{\mathcal{D}}(X', Y)$. One can check easily that F' is well-defined, and $F = F'Q'$. By the construction of $\mathcal{D}$, it's also clear that such a F' is unique, hence $(\mathcal{D}, Q')$ solves the same universal problem as $(C(S^{-1}), Q)$, in particular, we have a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X, Y) \xrightarrow{\sim} \varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}(X', Y).$$

The proof of the other statements in (b) is similar. This completes the proof of (b) $\square$

Remark. In the situation of 4.5, we say that S permits a *calculation of fractions* on both sides: we can write a morphism $f : X \to Y$ in $C(S^{-1})$ as a "fraction" $f = t^{-1}g$ or $f = hs^{-1}$.

Definition 4.6. $D(\mathcal{A}) = K(\mathcal{A})(\mathrm{qis}^{-1})$. Where qis denotes the set of quasi-isomorphism of $K(\mathcal{A})$.

Remark. We have $D(\mathcal{A}) = C(\mathcal{A})(\mathrm{qis}^{-1})$, where qis denotes the set of quasi-isomorphism of $C(\mathcal{A})$. Indeed, let Q be the composition of the two functors

$$C(\mathcal{A}) \xrightarrow{Q_1} K(\mathcal{A}) \xrightarrow{Q_2} D(\mathcal{A})$$

claim that $(D(\mathcal{A}), Q)$ solves the same universal problem as $C(\mathcal{A})(\mathrm{qis}^{-1})$. In fact we can show that $K(\mathcal{A}) = C(\mathcal{A})(S^{-1})$ (exercise), then the conclusion follows easily.

Remark. Though we have $D(\mathcal{A}) = C(\mathcal{A})(\mathrm{qis}^{-1})$, it is not convenient to use this as a definition. The reason is that qis, the set of quasi-isomorphisms in $C(\mathcal{A})$ is not a multiplicative system, so we cannot "calculate" a morphism in $D(\mathcal{A})$ by "fraction" in $C(\mathcal{A})$.

4.7. Let $\mathcal{A}$ be an abelian category, then we have the following natural functors:

$$C(\mathcal{A}) \longrightarrow K(\mathcal{A}) \xrightarrow{Q} D(\mathcal{A}).$$

Let $S = \{f \in \mathrm{Hom}_{K(\mathcal{A})}(X, Y) |\ f$ is a quasi-isomorphism $\}$, we know in 4.3.1 that S is a multiplicative system. Therefore morphisms in $D(\mathcal{A})$ can be defined as follows (4.5):

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{A})}(X, Y) &= \varinjlim_{Y \xrightarrow{t} Y' \in S} \mathrm{Hom}_{K(\mathcal{A})}(X, Y') \\
&= \varinjlim_{X' \xrightarrow{s} X \in S} \mathrm{Hom}_{K(\mathcal{A})}(X', Y) \\
&= \varinjlim_{X' \xrightarrow{s} X \in S, Y \xrightarrow{t} Y' \in S} \mathrm{Hom}_{K(\mathcal{A})}(X', Y')
\end{aligned}$$

The composition is the same as that given in the proof of 4.5. And the definition of morphisms in $D(\mathcal{A})$ also shows that $D(\mathcal{A})$ is an additive category and Q is an additive functor. Moreover, the translation functor

$$X \mapsto X[1]$$

on $K(\mathcal{A})$ gives a translation functor

$$X \mapsto X[1]$$

on $D(\mathcal{A})$. Define a distinguished triangle in $D(\mathcal{A})$ to the image by Q of a distinguished triangle $X \to Y \to Y \to Z \to X[1]$ in $K(\mathcal{A})$.

Proposition 4.8. *With the set of distinguished triangles defined above, $D(\mathcal{A})$ is a triangulated category. Moreover Q is triangulated (3.9). This triangulated category structure is called the canonical triangulated structure on $D(\mathcal{A})$.*

4.9. For $X, Y \in D(\mathcal{A})$, $n \in \mathbb{Z}$, we usually write $\mathrm{Ext}^n(X, Y)$ for $\mathrm{Hom}^n_{D(\mathcal{A})}(X, Y)$ (3.4).

For $i \in \mathbb{Z}$, the functor

$$H^i : K(\mathcal{A}) \to \mathcal{A}$$
$$X \mapsto H^i(X)$$

maps quasi-isomorphisms into isomorphisms, hence by the universal property of the derived categories, factors through $D(\mathcal{A})$, so we get a functor from $D(\mathcal{A})$ to $\mathcal{A}$,which is still denoted by $H^i : D(\mathcal{A}) \to \mathcal{A}$. Note that $H^i(X) = H^0(X[i])$.

Proposition 4.10. *Let* $L \xrightarrow{u} M \xrightarrow{v} N \xrightarrow{w}$ *be a distinguished triangle in* $D(\mathcal{A})$. *Then*

(a) For any object K *in* $D(\mathcal{A})$, *the sequences*

$$\cdots \to \operatorname{Ext}^n(K,L) \to \operatorname{Ext}^n(K,M) \to \operatorname{Ext}^n(K,N) \to \operatorname{Ext}^{n+1}(K,L) \to \cdots$$

and

$$\cdots \to \operatorname{Ext}^n(N,K) \to \operatorname{Ext}^n(M,K) \to \operatorname{Ext}^n(L,K) \to \operatorname{Ext}^{n+1}(N,K) \to \cdots$$

are exact.

(b) The sequence

$$\cdots \longrightarrow H^iL \xrightarrow{H^i(u)} H^iM \xrightarrow{H^i(v)} H^iN \xrightarrow{H^i(w)} H^{i+1}L \longrightarrow \cdots$$

is exact.

Proof. Part(a) is a particular case of 3.7'. For part(b), we may assume that the distinguished triangle is given by the cone $N = C(u)$ of a morphism $u : L \to M$ in $C(\mathcal{A})$, then the result follows from 3.11. □

Corollary 4.11. *A morphism* $u : L \to M$ *in* $D(\mathcal{A})$ *is an isomorphism if and only if* $H^i(u)$ *is an isomorphism for any* $i \in \mathbb{Z}$.

Proof. Only the sufficient part of this corollary is not trivial. Let N be a cone of u. From(3.11), we see that $H^iN = 0$ for any $i \in \mathbb{Z}$, which implies that $N = 0$, we have a morphism of distinguished triangles in $D(\mathcal{A})$

$$\begin{tikzcd} L \arrow[r, "u"] & M \arrow[r] & N \arrow[r] & . \\ L \arrow[u, equal, "\mathrm{Id}"] \arrow[r, equal] & L \arrow[u, "u"'] \arrow[r] & 0 \arrow[u, "0"] \arrow[r] & \end{tikzcd}$$

Both Id and 0 are isomorphism in $D(\mathcal{A})$, so u is also an isomorphism (3.8). □

We say that the sequence of functors $\{H^i\}_{i\in\mathbb{Z}}$ is a *conservative system*.

Proposition 4.12. *The functor* $R : \mathcal{A} \to D(\mathcal{A})$ *defined by the composition of* $\mathcal{A} \to C(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$ *is fully faithful, and its essential image is the full subcategory* $D^{[0,0]}(\mathcal{A})$ *of* $D(\mathcal{A})$ *consisting of complex* L *such that* $H^iL = 0$ *for any* $i \neq 0$.

Proof. It's easy to see R factors through $D^{[0,0]}(\mathcal{A})$, through a functor (we use the same notation here) $R : A \to D^{[0,0]}(\mathcal{A})$. To show that R is an equivalence of categories, we consider the functor

$$S : D^{[0,0]}(\mathcal{A}) \to \mathcal{A}$$

$$L \mapsto H^0 L$$

We show that S is a quasi-inverse of R. Obviously, $SR = \mathrm{Id}_{\mathcal{A}}$, so it remains to show $RS \sim eq\,\mathrm{Id}_{D(\mathcal{A})}$.

Let L be any object in $D^{[0,0]}(\mathcal{A})$, we have a natural quasi-isomorphism $s : L \to \tau_{\geqslant 0} L$ (1.8), which implies $\tau_{\geqslant 0} L \in D^{[0,0]}(\mathcal{A})$. Moreover, we have another quasi-isomorphism $t : R(H^0 L) = \tau_{\leqslant 0}(\tau_{\geqslant 0} L) \to \tau_{\geqslant 0} L$ (1.8). So we obtain a natural isomorphism between $RS(L)$ and L represented by the triple $(\tau_{\geqslant 0} L, s, t)$. □

From now on, we use the same notation L to indicate an object of $\mathcal{A}$ or $D(\mathcal{A})$.

***Remark*. 4.13.** (a) The truncation functor $\tau_{\geqslant n}(\tau_{\leqslant n}, \tau_{[a,b]})$ transforms quasi-isomorphisms into quasi-isomorphisms, hence induce functor from $D(\mathcal{A})$ to $D(\mathcal{A})$.

(b) If $u : L \to M$ is a morphism in $D(\mathcal{A})$ such that $H^i(u) = 0$ for all $i \in \mathbb{Z}$, in general, u is not zero. For example, consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

it defines(see below) an element $0 \neq e \in \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$, but $e : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}[1]$ induce 0 on H^i for all i.

(c) Let $u \in \mathrm{Hom}_{D(\mathcal{A})}(K, K)$ and assume

$$K \in D^{[a,b]}(\mathcal{A})$$

i.e $H^i(K) = 0$, $\forall i \notin [a, b]$, where $[a, b]$ is an interval of $\mathbb{Z}$. If $H^i(u) = 0$ for any $i \in \mathbb{Z}$, then $u^{b-a+1} = 0$. Indeed we can prove it by induction on $b - a$.

If $b - a = 0$, by (4.13), we may assume K is a complex concentrated in one degree, then $u = 0$ in this case (4.12).

Now let $b - a = k > 0$, we have a distinguished triangle in $D^{[a,b]}(\mathcal{A})$

$$\tau_{\leqslant a} K \xrightarrow{f} K \xrightarrow{g} \tau_{\geqslant a+1} K \longrightarrow$$

We have $\tau_{\leqslant a}K \in D^{[a,a]}(\mathcal{A})$ and $\tau_{\geqslant a+1}K \in D^{[a+1,b]}(\mathcal{A})$. Consider the following morphisms of distinguished triangles

$$\begin{tikzcd}
\tau_{\leqslant a}K \arrow[r, "f"] \arrow[d] & K \arrow[r, "g"] \arrow[d, "u^{b-a}"] \arrow[dl, dotted, "t"'] & \tau_{\geqslant a+1} \arrow[r] \arrow[d, "(\tau_{\geqslant a+1}u)^{b-a}=0"] & {} \\
\tau_{\leqslant a}K \arrow[r, "f"] \arrow[d] & K \arrow[r, "g"] \arrow[d, "u"] & \tau_{\geqslant a+1} \arrow[r] \arrow[d] & {} \\
\tau_{\leqslant a}K \arrow[r, "f"] & K \arrow[r, "g"] & \tau_{\geqslant a+1} \arrow[r] & {} .
\end{tikzcd}$$

By the induction hypothesis, $(\tau_{\geqslant a+1}u)^{b-a} = 0$, so we get $gu^{b-a} = 0$, hence u^{b-a} can factor through $\tau_{\leqslant a}K$, that is there exists $t : K \to \tau_{\leqslant a}K$ such that $u^{b-a} = ft$. Therefore $u^{b-a+1} = uft = 0$ by the $b - a = 0$ case.

4.14. For any $L, M \in \mathcal{A}$, let $\mathrm{Ext}(L, M)$ be the group of extensions of L by M. As a set, $\mathrm{Ext}(L, M)$ is the set of short exact sequences of the form $0 \to M \to E \to L \to 0$ modulo the following equivalence relation. A short exact sequence $0 \to M \to E \to L \to 0$ is said to be equivalent to $0 \to M \to E' \to L \to 0$ if and only if we have a commutative diagram

$$\begin{tikzcd}
 & & E \arrow[dd] \arrow[dr] & & \\
0 \arrow[r] & M \arrow[ur] \arrow[dr] & & L \arrow[r] & 0 \\
 & & E' \arrow[ur] & &
\end{tikzcd}$$

($E \to E'$ is therefore an isomorphism). Recall that $\mathrm{Ext}(L, M)$ is an abelian group, the addition of two extensions is defined as follows: given two extensions $\mathbf{E_1}, \mathbf{E_2}$ of L by M

$$\mathbf{E_1} : 0 \to M \to E_1 \to L \to 0$$

$$\mathbf{E_2} : 0 \to M \to E_2 \to L \to 0$$

take the direct sum of these two short exact sequences

$$0 \to M \bigoplus M \to E_1 \bigoplus E_2 \to L \bigoplus L \to 0.$$

Pulling it back by the diagonal morphism $L \to L \bigoplus L$, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \bigoplus M \arrow[r] \arrow[d] & E_1 \times_L E_2 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \bigoplus M \arrow[r] & E_1 \bigoplus E_2 \arrow[r] & L \bigoplus L \arrow[r] & 0.
\end{tikzcd}$$

Then, pushing-out the top row by the sum map $M \bigoplus M \to M$, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \bigoplus M \arrow[r] \arrow[d] & E_1 \times_L E_2 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & (E_1 \times_L E_2) \bigoplus_{M \oplus M} M \arrow[r] & L \bigoplus L \arrow[r] & 0.
\end{tikzcd}$$

Then we define the sum of $\mathbf{E_1}, \mathbf{E_2}$, which is denoted by $\mathbf{E_1} + \mathbf{E_2}$, as the bottom row of the above diagram. One checks that with the addition described above, $\mathrm{Ext}(L, M)$ is an abelian group, with 0 being the class of the trivial extension $0 \to M \to M \bigoplus L \to L \to 0$.

Proposition 4.15. *We have a natural isomorphism*

$$\mathrm{Ext}(L, M) \xrightarrow{\sim} \mathrm{Ext}^1(L, M)$$

Proof. Let

$$\mathbf{E}: \qquad 0 \longrightarrow M \xrightarrow{u} E \xrightarrow{v} L \longrightarrow 0$$

be an extension of L by M, consider the complex $C(u)$ and define the morphisms of complexes

$$\begin{tikzcd}
M[1] & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
C(u) \arrow[u, "p=p(1,0)"] \arrow[d, "s=s(0,v)"'] & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & M \arrow[r, "u"] \arrow[u, "1"] \arrow[d] & E \arrow[r, "v"] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
L & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & L \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

The fact that $\mathbf{E}$ is exact implies that s is a quasi-isomorphism, hence an isomorphism in $D(\mathcal{A})$. Hence we get a morphism

$$\varphi(\mathbf{E}) = p \circ s^{-1} : L \to M[1] \in \mathrm{Ext}^1(L, M)$$

by definition. It can be shown easily that $\varphi(\mathbf{E}) = \varphi(\mathbf{E}')$ if $\mathbf{E}$ and $\mathbf{E}'$ present the same extension class of M by L. So we get a well-defined map

$$\varphi : \mathrm{Ext}(L, M) \to \mathrm{Ext}^1(L, M)$$

$$E \mapsto \varphi(E)$$

Now we claim that φ is a morphism of abelian groups. In fact, first φ is functorial with respect to L and M. That is for any $f : L \to L'$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(L, M) \arrow[r, "\varphi"] & \mathrm{Ext}^1(L, M) \\
\mathrm{Ext}(L', M) \arrow[r, "\varphi"] \arrow[u, "\mathrm{Ext}(f,M)"] & \mathrm{Ext}^1(L', M), \arrow[u, "\mathrm{Ext}^1(f,M)"']
\end{tikzcd}$$

similarly, for $g : M \to M'$, a commutative diagram as follows

$$\begin{tikzcd}
\mathrm{Ext}(L, M) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ext}(L,g)"'] & \mathrm{Ext}^1(L, M) \arrow[d, "\mathrm{Ext}^1(L,g)"] \\
\mathrm{Ext}(L, M') \arrow[r, "\varphi"] & \mathrm{Ext}^1(L, M').
\end{tikzcd}$$

Moreover, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ext}(L, M) \times \mathrm{Ext}(L, M) \arrow[r, "\varphi"] \arrow[d, "\oplus"'] & \mathrm{Ext}^1(L, M) \times \mathrm{Ext}^1(L, M) \arrow[d, "\oplus"] \\
\mathrm{Ext}(L \bigoplus L, M \bigoplus M) \arrow[r, "\varphi"] & \mathrm{Ext}^1(L \bigoplus L, M \bigoplus M).
\end{tikzcd}$$

Hence, combining the commutative diagrams above together, we get

$$\begin{tikzcd}
\mathrm{Ext}(L, M) \times \mathrm{Ext}(L, M) \arrow[r, "\varphi"] \arrow[d, "\oplus"'] & \mathrm{Ext}^1(L, M) \times \mathrm{Ext}^1(L, M) \arrow[d, "\oplus"] \\
\mathrm{Ext}(L \bigoplus L, M \bigoplus M) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ext}(L\bigoplus L,\delta)"'] & \mathrm{Ext}^1(L \bigoplus L, M \bigoplus M) \arrow[d, "\mathrm{Ext}^1(L\bigoplus L,\delta)"] \\
\mathrm{Ext}(L \bigoplus L, M) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ext}(\Delta,M)"'] & \mathrm{Ext}^1(L \bigoplus L, M) \arrow[d, "\mathrm{Ext}^1(\Delta,M)"] \\
\mathrm{Ext}(L, M) \arrow[r, "\varphi"] & \mathrm{Ext}^1(L, M),
\end{tikzcd}$$

where $\delta : M \bigoplus M \to M$ and $\Delta : L \to L \bigoplus L$ are the sum map and the diagonal map respectively. From this, we see that φ is a group homomorphism.

Next, we construct an inverse

$$\psi : \mathrm{Ext}^1(L, M) \to \mathrm{Ext}(L, M)$$

of φ. Let $u \in \mathrm{Ext}^1(L, M)$ be represented by the diagram $(L \xleftarrow{s} L' \xrightarrow{f} M[1]) \in \mathrm{Ext}^1(L, M)$, where L' is a object in $D(\mathcal{A})$, $s : L' \to L$ is a quasi-isomorphism in $K(\mathcal{A})$ and $f \in \mathrm{Hom}_{K(\mathcal{A})}(L', M[1])$. Using the truncation $\tau_{[-1,0]} = \tau_{\leqslant 0} \circ \tau_{\geqslant -1}$, we may assume that L' is a complex concentrated in degree -1 and 0. Since

$$\begin{tikzcd} L' \arrow[d, "s"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & L'^{-1} \arrow[r, "d^{-1}"] \arrow[d] & L'^{0} \arrow[r] \arrow[d, "s^0"] & 0 \arrow[r] \arrow[d] & \cdots \\ L & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & L \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism, the following sequence is exact

$$0 \longrightarrow L'^{-1} \xrightarrow{d^{-1}} L'^{0} \xrightarrow{s^0} L \longrightarrow 0.$$

Pushing-out by f^{-1}, we obtain a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & L'^{-1} \arrow[r, "d^{-1}"] \arrow[d, "f^{-1}"] & L'^{0} \arrow[r, "s^0"] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r, "t^0"] & L \arrow[r] & 0, \end{tikzcd}$$

and define $\psi(s, f)$ as the class of the bottom exact sequence. It's easy to check that ψ is well-defined and is inverse to φ, so

$$\varphi : \mathrm{Ext}(L, M) \to \mathrm{Ext}^1(L, M)$$

is an isomorphism. □

4.16. Triangle associated to a short exact sequence Let $0 \to L \xrightarrow{u} M \xrightarrow{v} N \to 0$ be a short exact sequence of $\mathcal{A}$. We know (2.4) that we have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "u"] & M \arrow[r, "v"] \arrow[d, "i=(0,1)"'] & N \\ & C(u) \arrow[ur, "s=(0,v)"'] & \end{tikzcd}$$

with s a quasi-isomorphism, then

$$L \longrightarrow M \longrightarrow N \xrightarrow{w} L[1]$$

is a distinguished triangle, where $w = -pr \circ s^{-1}$. Note that with this sign convention, we have $H^i(w) = \delta$, where δ is the boundary of the long exact sequence of cohomology of $0 \to L \to M \to N \to 0$ (1.9).

Here are some special cases. Let $a \leqslant b$ be two integers, we have a short exact sequence

$$0 \to \tau_{\leqslant a} L \to \tau_{\leqslant b} L \to \tau_{\leqslant b} L/\tau_{\leqslant a} L \to 0.$$

As in $D(\mathcal{A})$ we have an isomorphism $\tau_{\leqslant b} L/\tau_{\leqslant a} L \sim \tau_{[a,b]} L$,then we get a distinguished triangle

$$\tau_{\leqslant a} L \to \tau_{\leqslant b} L \to \tau_{[a,b]} L \to$$

In particular, if $b = a + 1$, then we have a distinguished triangle

$$\tau_{\leqslant b-1} L \to \tau_{\leqslant b} L \to H^b L[-b] \to$$

since $\tau_{[b-1,b]} \sim H^b L[-b]$ in $D(\mathcal{A})$.

4.17. Some subcategories of $D(\mathcal{A})$

Let $*$ denote any one of the following symbols: +,-.b. We define full subcategories of $D(\mathcal{A})$ as follows:

$$D^+(\mathcal{A}) = \{L \in D(\mathcal{A}) | H^i L = 0, i << 0\}$$

$$D^-(\mathcal{A}) = \{L \in D(\mathcal{A}) | H^i L = 0, i >> 0\}$$

$$D^b(\mathcal{A}) = \{L \in D(\mathcal{A}) | H^i L = 0, i << 0 \quad or \quad i >> 0\}$$

We have a natural functor $K^*(\mathcal{A}) \to D^*(\mathcal{A})$, which maps quasi-isomorphisms to isomorphisms, hence we get a functor

$$S : K^*(\mathcal{A})(\text{qis}^{-1}) \to D^*(\mathcal{A}),$$

where qis is the set of quasi-isomorphisms in $K^*(\mathcal{A})$.

Proposition 4.18. *The functor S defined above is an equivalence of categories.*

For the proof of this proposition, we need some preparation.

Definition 4.19. Let I be a filtering category, $J \subset I$ is called *cofinal* if J is a full subcategory and for any $i \in I$, there exists a morphism $i \longrightarrow j$, where $j \in J$.

Note that if $J \subset I$ is cofinal, then J is again filtering.

Lemma 4.20. *If I is a filtering category, $F : I \longrightarrow \mathcal{S}ets$ be a functor, then the natural morphism*

$$\varinjlim_{j \in J} F(j) \sim \varinjlim_{i \in I} F(i)$$

is an isomorphism.

Proof. Since $J \subset I$ is a subcategory, we have a natural morphism

$$\varphi : \varinjlim_{j \in J} F(j) \longrightarrow \varinjlim_{i \in I} F(i)$$

what's more, since J is cofinal, for any $(i, x) \in \varinjlim_{i \in I} F(i)$ we can find some morphism $a : i \longrightarrow j$ with $j \in J$, then we can define

$$\psi : \varinjlim_{i \in I} F(i) \longrightarrow \varinjlim_{j \in J} F(j)$$

$$(i, x) \longmapsto (j, F(a)x)$$

It can be checked easily that ψ is well-defined and

$$\varphi\psi = \mathrm{Id}, \quad \psi\varphi = \mathrm{Id}\,.$$

□

Proof of 4.18. We will prove this proposition in the case of $* = +$. Let L, M be objects in $K^+(\mathcal{A})$. For any morphism $(M', t, f) \in \mathrm{Hom}_{D(\mathcal{A})}(L, M)$, where $M \xrightarrow{t} M'$ is a quasi-isomorphism in $K^+(\mathcal{A})$ and $f \in \mathrm{Hom}_{K^+(\mathcal{A})}(L, M')$, there exits some integer n such that $H^i M' \sim H^i M = 0$ for any $i < n$. So the natural morphism

$$t' : M' \longrightarrow \tau_{\geqslant n} M'$$

is a quasi-isomorphism in $K^+(\mathcal{A})$, so

$$(M', t, f) = (M'', t't, t'f) \in \mathrm{Hom}_{K^+(\mathcal{A})}(L, M),$$

hence by 4.20 S is fully faithful. Moreover, we can define a quasi-inverse R of S as follows:

$$R : D^+(\mathcal{A}) \to K^+(\mathcal{A})(\text{qis}^{-1})$$
$$L \mapsto \tau_{\geqslant n} L$$

where n is some integer such that $H^i L = 0$, for $i < n$. Now it can be shown easily that $R \circ S = id, S \circ R \sim eqid$. $\square$

4.21. Structure of $D^+(\mathcal{A})$

Recall that in an abelian category $\mathcal{A}$, an object $L \in \mathcal{A}$ is called projective (resp. injective) if $\text{Hom}_{\mathcal{A}}(L, -)$ (resp. $\text{Hom}_{\mathcal{A}}(-, L)$) is an exact functor(i.e takes short exact sequences to short exact sequences). For example, in the category of R-modules, where R is a ring, an R-module is projective if and only if it is a direct summand of a free R-module.

Definition 4.22. An abelian cateogry $\mathcal{A}$ is said to *have enough injectives* if for any object $L \in \mathcal{A}$, there exists a monomorphism $L \to L'$ with L injective.

Theorem 4.23. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $Mod(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules has enough injectives.*

Lemma 4.24. *Let R be a ring, M a left R-module. Then M is injective if and only if for any left ideal $I \subset R, \text{Hom}(R, M) \to \text{Hom}(I, M)$ is surjective. In particular, if R is a principal ideal domain, then M is injective if and only if M is divisible.*

Proof. The condition is of course necessary. Conversely, let M be a left R-module having the property that any morphism $I \to M$ can be extended to a morphism $R \to M$, where I is a left ideal, we need to show that M is injective. Given any monomorphism $L \to N$ and a morphism $\psi : L \to M$, by Zorn's lemma, there exists a maximal submodule N' of N such that one can extend ψ to it, we claim that $N = N'$. If not, choose an element $x \in N - N'$ and let $I = \{r \in R | rx \in N'\}$, it's easy to see that I is a left ideal of R, and we have a morphism

$$\psi' : I \to Rx \bigcap N' \to M.$$

By hypothesis, we can extend this morphism to R, which is still denoted by ψ'. Now we see that we can extend the morphism $N' \to M$ to

$$N' + Rx(\subset N) \to M : n' + rx \mapsto \psi(n') + \psi'(r)x,$$

which contradicts the choice of N'. $\square$

Lemma 4.25. *Let R be a ring. The the category $Mod(R)$ of left R-modules has enough injectives.*

Proof. First, we claim that, in the category of $\mathbb{Z}$-modules, we have enough injectives. Using the lemma above, as $\mathbb{Q}/\mathbb{Z}$ is divisible, $\mathbb{Q}/\mathbb{Z}$ is injective. Let L be any left $\mathbb{Z}$-module, take any element $x \in L$, there exists a morphism $\mathbb{Z}x \to \mathbb{Q}/\mathbb{Z}$ such that the image of x is nonzero in $\mathbb{Q}/\mathbb{Z}$. As $\mathbb{Q}/\mathbb{Z}$ is injective, this morphism can be extended to a morphism $f_x : L \to \mathbb{Q}/\mathbb{Z}$ such that $f_x(x) \neq 0$. On the other hand, we have a natural morphism of abelian groups

$$h : L \to \prod_{f:L\to\mathbb{Q}/\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$$
$$y \mapsto (f(y))_f.$$

As $f_x(x) \neq 0$, this morphism is injective, so any $\mathbb{Z}$-module can be embedded into an injective $\mathbb{Z}$-module, in other words, $Mod(\mathbb{Z})$ has enough injectives.

Now, let R be a ring, L be a left R-module, $\mathrm{Hom}_{\mathbb{Z}}(R, L)$ is an R-module, and we have a canonical injection $L \to \mathrm{Hom}_{\mathbb{Z}}(R, L)$. Choose an embedding of L into an injective $\mathbb{Z}$-module L', we get an embedding $L \to \mathrm{Hom}_{\mathbb{Z}}(R, L')$. On the other hand, by adjunction isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(N, L') \xrightarrow{\sim} \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(R, L'))$$

where N is an R-module, therefore $\mathrm{Hom}_{\mathbb{Z}}(R, L')$ is an injective R-module, which completes the proof. ☐

proof of theorem 4.23. Let $\mathcal{F}$ be a sheaf of left $\mathcal{O}_X$-module. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module, so there is an injection $\mathcal{F}_x \to I_x$, where I_x is an injective $\mathcal{O}_{X,x}$-module. For each point $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion, and let $\mathcal{G} = \prod_{x\in X} i_{x,*}(I_x)$, where $i_{x,*}$ is the direct image functor. Let $\mathcal{F} \hookrightarrow \mathcal{G}$ be the composition of the following two morphisms

$$\mathcal{F} \hookrightarrow \prod_{x\in X} i_{x,*}(\mathcal{F}_x) \hookrightarrow \prod_{x\in X} i_{x,*}(I_x) = \mathcal{G},$$

where the first morphism is given by $s \mapsto (s_x)_{x\in X}$. Moreover, let $\mathcal{H}$ be an $\mathcal{O}_X$-module, we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}) = \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, i_{x,*}(I_x)) = \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{H}_x, I_x).$$

hence $\mathcal{G}$ is an injective $\mathcal{O}_X$ module. This completes the proof. ☐

Now, the main purpose of this section is to prove the following theorem.

Theorem 4.26. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then the natural functor*

$$K^+(\mathcal{I}) \to D^+(\mathcal{A})$$

is an equivalence of categories, where $\mathcal{I}$ is the full subcategory of $\mathcal{A}$ consists of injectives in $\mathcal{A}$.

In fact, we will deduce 4.26 from the following theorem:

Theorem 4.27. *Let $\mathcal{A}' \subset \mathcal{A}$ be a full additive category of $\mathcal{A}$, such that for any object $L \in \mathcal{A}$, there exists a monomorphism $L \to L'$ with $L' \in \mathcal{A}'$. Then*

(1) For any $L \in D^+(\mathcal{A})$, there exists a quasi-isomorphism $L \to L'$ with $L' \in K^+(\mathcal{A}')$.

(2) The functor $K^+(\mathcal{A}')(\mathrm{qis}^{-1}) \to D^+(\mathcal{A})$, where qis denotes the set of quasi-isomorphisms of $K^+(\mathcal{A})$, is an equivalence of triangulated categories.

Let us show that 4.27 implies 4.26. Take $\mathcal{A}' = \mathcal{I}$, then from 4.27, we know that

$$K^+(\mathcal{I})(\mathrm{qis}^{-1}) \to D^+(\mathcal{A})$$

is an equivalence of categories. It remains to show that the functor

$$K^+(\mathcal{I}) \to K^+(\mathcal{J})(\mathrm{qis}^{-1})$$

is an equivalence of triangulated categories, which follows from the following lemma.

Lemma 4.28. *If $t : M \to M'$ is a quasi-isomorphism in $K^+(\mathcal{I})$, then t is a homotopy equivalence.*

Proof. By (3.8), we just have to show that $C(t)$ is homotopical trivial. Note that $C(t)$ is acyclic, so are reduced to showing that if $M \in K^+(\mathcal{I})$ is acyclic, then M is homotopically trivial. But it is easy to see that M breaks into short exact sequences such that all the components are in $\mathcal{I}$, hence the short exact sequences split. From this, we can construct the homotopy we need. $\square$

Proof of 4.27. It's easy to see that if the conclusion of (1) holds, for $M \in K^+(\mathcal{A})$ the category of quasi-isomorphisms $\{M \to M'$, where $M' \in K^+(\mathcal{A}')$

is cofinal in the category of all quasi-isomorphisms $\{M \to M''$, where $M'' \in K^+(\mathcal{A})$, and therefore a similar argument as the one in (e.g. 4.13) shows that

$$K(\mathcal{A}')(\mathrm{qis}^{-1}) \to D^+(\mathcal{A})$$

is an equivalence of categories. So we only need to prove (1).

Let L be any object in $D^+(\mathcal{A})$, we need to construct a quasi-isomorphism $L \to L'$ with $L' \in K^+(\mathcal{A}')$. Since there exists some integer a such that $L \to \tau_{\geqslant a} L$ is a quasi-isomorphism, we can assume that $L \in K^+(\mathcal{A})$. By shifting the degree, we may assume that $L \in K^{\geqslant 0}(\mathcal{A})$. Now we shall construct inductively a complex $L'_n \in K^{[0,n]}(\mathcal{A}')$ and a morphism $u_n : L \to L'_n$ such that: (a) for each i, u^i_n is a monomorphism; (b) it induces isomorphisms on $H^j(K) \to H^j(L'_n)$ for $i < n$ and a monomorphism $L^n/B^n \hookrightarrow L'_n/B'^n$. For $n < 0$, we can take $L'_n = 0$. Now assume we have constructed L'_n and a morphism $u'_n : L \to L'_n$ with the properties (a) and (b). Then consider the cocartesian diagram

$$\begin{tikzcd}
L'^n/B'^n \arrow[r] & L'^n/B'^n \bigoplus_{L^n/B^n} L^{n+1} (= \widetilde{L}^{n+1}) \\
L^n/B^n \arrow[u] \arrow[r] & L^{n+1} \arrow[u]
\end{tikzcd}$$

By assumption, there exists a monomorphism $\widetilde{L}^{n+1} \to L'^{n+1}$ with $L'^{n+1} \in \mathcal{A}'$, then we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & L'^n \arrow[r] & L'^n/B'^n \arrow[r] & \widetilde{L}^{n+1} \arrow[r] & L'^{n+1} \\
\cdots \arrow[r] & L^n \arrow[u] \arrow[r] & L^n/B^n \arrow[u] \arrow[r] & L^{n+1} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

with $L^{n+1} \to L'^{n+1}$ a monomorphism. Now

$$H^n L = \mathrm{Ker}(L^n/B^n \to L^{n+1}), H^n L'_{n+1} = \mathrm{Ker}(L'^n/B'^n \to \widetilde{L}^{n+1})$$

and

$$L^{n+1}/B^{n+1} \longrightarrow \mathrm{Coker}(L'^n \to \widetilde{L}^{n+1}) \longrightarrow L'^{n+1}/B'^{n+1}$$

is a composition of two monomorphism, hence is also a monomorphism. Applying the lemma below, we see that

$$H^n L \xrightarrow{\sim eq} H^n L'_{n+1}$$

That is just what we need. $\square$

Lemma 4.29. *Let*

$$\begin{tikzcd} A \arrow[r, "\varphi"] & A \bigoplus_S B \\ S \arrow[u, "\alpha"] \arrow[r, "\psi"] & B \arrow[u, "\beta"'] \end{tikzcd}$$

be a cocartesian diagram, assume ψ is a monomorphism, then we have an isomorphism

$$\mathrm{Ker}\alpha \xrightarrow{\sim eq} \mathrm{Ker}\beta$$

Proof. By the property of a cocartesian diagram, we know that φ is also a monomorphism, moreover we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & \mathrm{Coker}\,\psi \arrow[r] \arrow[d, "\sim eq"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\varphi"] & A \bigoplus_s B \arrow[r] & \mathrm{Coker}\,\varphi \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma (1.4), ψ induces an isomorphism

$$\mathrm{Ker}\alpha \xrightarrow{\sim eq} \mathrm{Ker}\beta.$$

□

5 Derived functors

5.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called *left exact (resp. right exact)* if for any exact sequence :

$$0 \to L' \to L \to L'' \to 0,$$

the sequence

$$0 \to F(L') \to F(L) \to F(L'')$$

(resp. $F(L') \to F(L) \to F(L'') \to 0$) is exact. F is called *exact* if it is both left and right exact.

Example 5.1.1. If $\mathcal{A}$ is an abelian category and P is an object of $\mathcal{A}$, the functor $\mathrm{Hom}(P, -)$ from $\mathcal{A}$ to the category $\mathcal{A}b$ of abelian groups is left exact, and it is exact if and only if P is projective. Similarly if Q is an object of $\mathcal{A}$, the functor $\mathrm{Hom}(-, Q)$ from $\mathcal{A}^\circ$ to $\mathcal{A}b$ is left exact and it is exact if and only if Q is injective.

Example 5.1.2. Let R be a ring, and L a right $R-$module. Then the functor $L \otimes_R -$ from left $R-$module to $\mathcal{A}b$ is right exact, and it is exact if and only if L is flat.

Consider the extension of F to $C(\mathcal{A})$. This is an additive functor F from $C(\mathcal{A})$ to $C(\mathcal{B})$, defined by

$$L = (\cdots \to L^i \xrightarrow{d} L^{i+1} \to \cdots) \mapsto FL = (\cdots \to FL^i \xrightarrow{F(d)} FL^{i+1} \to \cdots)$$

$$(u : L \to M) \mapsto (F(u) = (F(u_i)) : FL \to FM)$$

This functor F induces $F : C^*(\mathcal{A}) \to C^*(\mathcal{B})$, where $* = +, -, b$.

Then we have the following diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[d, "F"] & C^+(\mathcal{A}) \arrow[r] \arrow[d, "F"] & K^+(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "F"] & D^+(\mathcal{A}) \arrow[d, dotted, "\bar{F}"] \\
\mathcal{B} \arrow[r, hook] & C^+(\mathcal{B}) \arrow[r] & K^+(\mathcal{B}) \arrow[r, "Q"] & D^+(\mathcal{B})
\end{tikzcd},$$

in which $F : K(\mathcal{A}) \to K(\mathcal{B})$ is triangulated, and there does not exist, in general, a triangulated functor $\bar{F}$ making the right square commutative. In fact, such an $\bar{F}$ exists if and only if $F(u)$ is a quasi-isomorphism, where u is a quasi-isomorphism, or, equivalently, for all acyclic $L \in K^+(\mathcal{A})$, $F(L)$ is acyclic or $F : \mathcal{A} \to \mathcal{B}$ is exact.

5.2. A *right derived functor RF* of F is a triangulated functor

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

together with a morphism of functors $\varepsilon : QF \to RF \circ Q$ having the following universal property:

For any triangulated functor $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and morphism of functors $\eta : QF \to G \circ Q$, there exists a unique morphism $\alpha : RF \to G$ such that $\eta = \alpha \circ \varepsilon$.

If (RF, ε) exists, it is unique up to a unique isomorphism.

Theorem 5.3 (existence of RF). *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Assume that there exists a full additive subcategory $\mathcal{A}'$ of $\mathcal{A}$, such that*

(i). For all $E \in \mathcal{A}$, there exists $E' \in \mathcal{A}$, and a monomorphism $E \hookrightarrow E'$.

(ii). If $0 \to E' \to E \to E'' \to 0$ *is an exact sequence, and* $E',\ E \in \mathcal{A}'$*, then* $E'' \in \mathcal{A}'$*, and* $0 \to FE' \to FE \to FE'' \to 0$ *is exact.*
Then $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ *exists and for* $L \in D^+(\mathcal{A})$*, we have an isomorphism* $RF(L) \xleftarrow{\sim} F(L')$*, where* $L \to L'$ *is a quasi-isomorphism with* $L' \in K^+(\mathcal{A}')$*.*

Lemma 5.4. *Let* $\mathcal{A}'$ *be a subcategory of* $\mathcal{A}$ *satisfying the condition in 5.3. Suppose* $L \in K^+(\mathcal{A}')$*, and* L *is acyclic, then* FL *is acyclic.*

Proof. Suppose L is acyclic, then we have an exact sequence

$$0 \longrightarrow L^a \longrightarrow L^{a+1} \longrightarrow \cdots \longrightarrow L^i \longrightarrow L^{i+1} \longrightarrow \cdots$$

And for each L^i, we have a short exact sequence

$$0 \longrightarrow Z^i \longrightarrow L^i \longrightarrow Z^{i+1} \longrightarrow 0$$

It follows from (ii), by induction on i, that for all i, $Z^i \in \mathcal{A}'$ and the sequence

$$0 \longrightarrow FZ^i \longrightarrow FL^i \longrightarrow FZ^{i+1} \longrightarrow 0$$

is exact. Splicing short exact sequences, we get that FL is acyclic. □

Lemma 5.5. *Suppose* $L,\ L' \in K^+(\mathcal{A}')$*,* $s : L \to L'$ *is a quasi-isomorphism, then* Fs *is also a quasi-isomorphism.*

Proof. Consider the distinguished triangle

$$\begin{tikzcd} & M \arrow[dl] & \\ L \arrow[rr, "s"] & & L' \arrow[ul] \end{tikzcd}$$

where M is the cone of s. Then $M \in K^+(\mathcal{A}')$. Because s is a quasi-isomorphism, M is acyclic. By 5.4, $F(M)$ is acyclic. So Fs is a quasi-isomorphism. □

Let us prove the theorem. Consider the diagram:

$$\begin{tikzcd} K^+(\mathcal{A}') \arrow[r] \arrow[d, "F"] & K^+(\mathcal{A}')(\mathrm{Quis}^{-1}) \arrow[r, "\psi"] \arrow[d, dotted, "F"] & D^+(\mathcal{A}) \arrow[dl, dotted, "RF"] \\ K^+(\mathcal{B}) \arrow[r, "Q"] & D^+(\mathcal{B}) & \end{tikzcd}$$

where Quis is the set of quasi-isomorphisms in $K^+(\mathcal{A}')$. As F maps quasi-isomorphisms of $K^+(\mathcal{A}')$ into quasi-isomorphisms, there exists a functor $F : K^+(\mathcal{A}')(\mathrm{Quis}^{-1}) \to D^+(\mathcal{B})$ making the square commutative.

Recall that ψ is an equivalence (4.18). Let $\varphi : D^+(\mathcal{A}) \to K^+(\mathcal{A}')(\mathrm{Quis}^{-1})$ be a quasi-inverse to ψ. Define

$$RF = F \circ \varphi : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

i.e. $RF(L) = F(\varphi L)$. Now let us define a functorial morphism: $\varepsilon : Q(FL) \to RF(QL)$, for $L \in D^+(\mathcal{A})$.

For $L \in K^+(\mathcal{A})$, we have a functorial isomorphism $a(L) : L \to \varphi L$ with $\varphi L \in K^+(\mathcal{A}')$. By (4.18), we can write

$$a(L) = t^{-1}s : \; L \xrightarrow{s} M' \xleftarrow{t} \varphi L$$

where s, t are both quasi-isomorphisms and $M' \in K^+(\mathcal{A}')$. Then we have a diagram

$$\begin{tikzcd} FL \arrow[r, "Fs"] \arrow[rr, bend right, "\varepsilon(L)"'] & FM' & F\varphi L = RFL \arrow[l, "Ft"'] \end{tikzcd}$$

where Ft is a quasi-isomorphism because both M' and ϕL are in $K^+(\mathcal{A}')$. We define

$$\varepsilon(L) = (Ft)^{-1} \circ Fs \in \mathrm{Hom}_{D^+(\mathcal{B})}(FL, RFL).$$

One easily checks that $\varepsilon(L)$ doesn't depend on the choices and gives a map of functors $\varepsilon : QF \to RFQ$.

Let us verify the universal property. Let $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be a triangulated functor and $\eta : FQ \to QG$ be a morphism. The following diagram shows that η uniquely factors as $\eta = \alpha \circ \varepsilon$.

$$\begin{tikzcd} FL \arrow[r, "\varepsilon"] \arrow[d, "\eta"'] & RFL \arrow[r, equal] \arrow[dl, dotted, "\alpha"] & F(\varphi L) \arrow[d, "\eta(\varphi L)"] \\ G(L) \arrow[rr, "\simeq"] & & G(\varphi L) \end{tikzcd}$$

Finally, let $L \xrightarrow{u} L'$ be a quasi-isomorphism with $L' \in K^+(\mathcal{A}')$. Then the following commutative diagram

$$
\begin{tikzcd}
FL \arrow[r] \arrow[d] & RFL \arrow[d, "\simeq"', "RF(u)"] \\
FL' \arrow[dr, "F(\varepsilon(L'))"'] & RFL' \arrow[d, equal] \\
 & F(\varphi L')
\end{tikzcd}
$$

defines a canonical isomorphism $RF(u)^{-1} \circ F(\varepsilon(L')) : FL' \xrightarrow{\sim} RFL$ in $D^+(\mathcal{B})$.

Note that, for $a \in \mathbb{Z}$, $RF(D^{\geqslant a}(\mathcal{A})) \subset D^{\geqslant a}(\mathcal{B})$. This is because for $L \in D^{\geqslant a}(\mathcal{A})$, there exists a quasi-isomorphism $L \to L'$ with $L' \in K^{\geqslant a}(\mathcal{A}')$, such that $RFL \simeq FL' = (0 \to FL'^a \to \cdots)$

For $i \in \mathbb{Z}$, and $L \in D^+(\mathcal{A})$, we define $R^iFL = H^iRFL$. The functor

$$R^iF : D^+(\mathcal{A}) \to \mathcal{B}$$

is a "cohomological functor", meaning that if $L' \to L \to L'' \to$ is a distinguished triangle in $D^+(\mathcal{A})$, then we can get a long exact sequence

$$\cdots \to R^iFL' \to R^iFL \to R^iFL'' \to R^{i+1}FL' \to \cdots$$

coming from the distinguished triangle $RFL' \to RFL \to RFL'' \to$.

In particular, if $L \in D^{\geqslant 0}(\mathcal{A})$, then $R^iFL = 0$, $i < 0$. Moreover, for $L \in \mathcal{A}$, the natural map (given by $\varepsilon : FL \to RFL$)

$$FL \xrightarrow{\varepsilon} R^0FL = H^0RFL$$

is an isomorphism if and only if F is left exact.

Corollary 5.6. *If $\mathcal{A}$ has enough injectives, then $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, and for any $L \in D^+(\mathcal{A})$, if $L \to L'$ is a quasi-isomorphism with $L' \in K^+(\mathcal{A})$ and L'^i injective for all i, then $FL' \xrightarrow{\sim} RFL$.*

Indeed we can take for $\mathcal{A}'$ the full subcategory of $\mathcal{A}$ consisting of injectives (4.27).

Definition 5.7. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor (in other words, $FL \xrightarrow{\sim} R^0FL$). An object L of $\mathcal{A}$ is called F-*acyclic*, if $R^nFL = 0$ for any $n > 0$.

For example, objects of $\mathcal{A}'$ are $F-$acyclic.

Proposition 5.8. *Under the assumption of 5.3, the subcategory $\mathcal{A}_F = \{L \in \mathcal{A},\ L$ is $F-$acyclic$\}$ of $\mathcal{A}$ satisfies the properties (i), (ii) of 5.3.*

Proof. (i). We have $\mathcal{A}' \subset \mathcal{A}_F$, so the condition (i) is immediate.

(ii). Let $E' \to E \to E'' \to 0$ be an exact sequence in $\mathcal{A}$, and $E', E \in \mathcal{A}_F$, then $RFE' \to RFE \to RFE'' \to$ is a distinguished triangle, hence we get a long exact sequence

$$\cdots \to R^iFE' \to R^iFE \to R^iFE'' \to R^{i+1}FE \to \cdots .$$

Because E', E are both $F-$acyclic, $R^nFE = R^nFE' = 0$ for all $n > 0$. With the long exact sequence, we get $R^nFE'' = 0$ for all $n > 0$, so E'' is also $F-$acyclic, hence $E'' \in \mathcal{A}_F$.

From the long exact sequence, we also have an exact sequence $0 \to R^0FE' \to R^0FE \to R^0FE'' \to 0$, then the sequence $0 \to FE' \to FE \to FE'' \to 0$ is exact by definition. □

Corollary 5.9. *Suppose $L \to L'$ is a quasi-isomorphism with $L' \in K^+(\mathcal{A})$ and for any i, L'^i is $F-$acyclic, then we have a canonical isomorphism $FL' \to RFL$.*

Definition 5.10. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two abelian categories. A *left derived functor* of F is a functor

$$LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$$

together with $\varepsilon : LFQ \to QF$ having the following universal property: For any triangulated functor $G : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ together with a morphism $\eta : GQ \to QF$, there exists a unique morphism $\alpha : G \to LF$, such that $\eta = \varepsilon \circ \alpha$:

$$\begin{tikzcd} K^-(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "F"] & D^-(\mathcal{A}) \arrow[d, "LF"'] \arrow[d, bend left, "G"] \\ K^-(\mathcal{B}) \arrow[r, "Q"] & D^-(\mathcal{B}) \end{tikzcd}$$

Theorem 5.3' *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor, and $\mathcal{A}'$ a additive subcategory of $\mathcal{A}$, such that*

(i). For all $E \in \mathcal{A}$, there exists an epimorphism $E' \twoheadrightarrow E$ with $E' \in \mathcal{A}$

(ii). For any exact sequence $0 \to E' \to E \to E'' \to 0$ *with* $E,\ E'' \in \mathcal{A}'$*, then* $E' \in \mathcal{A}'$*, and* $0 \to FE' \to FE \to FE'' \to 0$ *is exact.*
Then $(LF : D^-(\mathcal{A}) \to D^-(\mathcal{B}),\ \varepsilon)$ *exists and for* $L \in D^+(\mathcal{A})$*, we have* $LFL \xrightarrow{\sim} FL'$*, where* $L' \to L$ *is a quasi-isomorphism with* $L' \in K^-(\mathcal{A}')$*.*

The proof is similar to that of 5.3.

Example 5.10.1. : Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{A} = Mod(\mathcal{O}_X)$. The category $\mathcal{A}'$ of flat $\mathcal{O}_X-$modules satisfies the conditions (i) and (ii) of 5.3' for any functor of the form $P \otimes_{\mathcal{O}_X} - : Mod(X) \to Mod(Y)$ with $P \in Mod(X)$.

Definition 5.11. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, and $\mathcal{A}'$ be a full additive subcategory of $\mathcal{A}$. We say $\mathcal{A}'$ is *right adapted* to F if $\mathcal{A}'$ satisfies the following condition (i)-(iii):

(i) For any $E \in \mathcal{A}$, there exists an object E' and a monomorphism $E \to E'$ with $E' \in \mathcal{A}'$.

(ii) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E \in \mathcal{A}'$, then $E'' \in \mathcal{A}'$.

(iii) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E, E'' \in \mathcal{A}'$, then the sequence

$$0 \to FE' \to FE \to FE'' \to 0$$

is exact.

Recall that if such a category $\mathcal{A}'$ exists, then $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists and for any $L \in D^+(\mathcal{A})$, we have $RF(L) \simeq F(L')$ where $L' \in K^+(\mathcal{A}')$ and $L \to L'$ is a quasi-isomorphism. For example, if $\mathcal{A}$ has enough injectives, then the category of injectives in $\mathcal{A}$ is right adapted to any $F : \mathcal{A} \to \mathcal{B}$.

Definition 5.11' Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories, $\mathcal{A}'$ a full additive subcategory of $\mathcal{A}$. We say $\mathcal{A}'$ is *left adapted* to F if

(i) For any $E \in \mathcal{A}$, there exists an object E', such that $E' \to E$ is an epimorphism.

(ii) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E, E'' \in \mathcal{A}'$, then $E' \in \mathcal{A}'$.

(iii) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E, E'' \in \mathcal{A}'$, then the sequence

$$0 \to FE' \to FE \to FE'' \to 0$$

is exact.

If $\mathcal{A}' \subset \mathcal{A}$ is left adapted to F, then $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ exists and for any $L \in D^-(\mathcal{A})$, we have $LF(L) \simeq F(L')$ where $L' \in K^-(\mathcal{A}')$ and $L' \to L$ is a quasi-isomorphism.

For example, if $\mathcal{A}$ has enough projectives, then the category of projectives is left adapted to any $F : \mathcal{A} \to \mathcal{B}$.

Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be two additive functors between abelian categories. We will now discuss when we can "state" $R(GF) = RG \circ RF$ and $L(GF) = LG \circ LF$.

Theorem 5.12. *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ *be additive functors between abelian categories. Assume there exist* $\mathcal{A}' \subset \mathcal{A}$ *right adapted to* F*,* $\mathcal{B}' \subset \mathcal{B}$ *right adapted to* G*, and* $F(\mathcal{A}') \subset \mathcal{B}'$*. Then* $\mathcal{A}'$ *is right adapted to* GF*,* RF*,* RG*,* $R(GF)$ *exist, and there is a canonical isomorphism*

$$RG \circ RF \simeq R(G \circ F).$$

Proof. For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

with $E', E, E'' \in \mathcal{A}'$, because $\mathcal{A}'$ is right adapted to F, then the sequence

$$0 \to FE' \to FE \to FE'' \to 0$$

is exact. Since $F(\mathcal{A}') \subset \mathcal{B}'$, the sequence

$$0 \to GFE' \to GFE \to GFE'' \to 0$$

is also exact. It follows that $\mathcal{A}'$ is right adapted to GF. For any $L \in D^+(\mathcal{A})$, there exists $L' \in K^+(\mathcal{A}')$ with $L \to L'$ a quasi-isomorphism. The isomorphism stated in the theorem is given by the following composition

$$\begin{tikzcd} RG \circ RF(L) & RG(FL') \arrow[l, "\sim"'] . \\ & G(FL') \arrow[u, "\simeq"'] \arrow[d, equal] \\ R(GF)(L) & GF(L') \arrow[l, "\sim"'] \end{tikzcd}$$

□

Example 5.12.1. Suppose $\mathcal{A}$ has enough injectives, $\mathcal{B}$ has enough injectives, G is left exact, and F transforms injective objects to $G-$acyclic objects. Then we can take $\mathcal{B}' = \mathcal{B}_G$, where $\mathcal{B}_G$ is the subcategory consisting of $G-$acyclic objects, so we get $R(GF) \simeq RG \circ RF$.

Theorem 5.12' *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be additive functors where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are abelian categories. Assume we have a full additive subcategory $\mathcal{A}'$ (resp. $\mathcal{B}'$) of $\mathcal{A}$ (resp. $\mathcal{B}$), which is left adapted to F (resp. G) and $F(\mathcal{A}') \subset \mathcal{B}'$. Then $\mathcal{A}'$ is left adapted to GF, $LF, LG, L(GF)$ exist, and there is a canonical isomorphism*

$$L(GF) \simeq LG \circ LF.$$

The proof is similar to that of 5.12.

6 The functors $R\Gamma, Rf_*, Lf^*, \otimes^L$

Let $(X, \mathcal{O}_X)$ be a ringed space and suppose $\mathcal{O}_X$ is commutative. Let $\mathcal{A} = Mod(X) = Mod(\mathcal{O}_X)$ be the category of sheaves of $\mathcal{O}_X-$modules. Then $\mathcal{A}$ is abelian and has enough injectives (4.23). Let $\mathcal{A}b$ be the category of abelian groups. Then the functor $\Gamma(X,-) : Mod \to \mathcal{A}b$; $E \mapsto \Gamma(X,E) = E(X)$ is additive and left exact, but it is not exact in general. The right derived functor $R\Gamma(X,-) : D^+(X) \to D^+(\mathcal{A}b)$ exists (Where $D^*(X) = D^*(Mod(X)), * = +, -, b$ or empty).

For each $L \in D^+(X)$, $n \in \mathbb{Z}$, $H^n(X, L) = H^n R\Gamma(X, L)$ is called the *n-th cohomology group* of X with values in L. For all $E \in Mod(X)$, $\Gamma(X, E) \xrightarrow{\sim} H^0(X, E)$. We have $R\Gamma(X, L) = \Gamma(X, L')$, where $L \to L'$ is a quasi-isomorphism with $L' \in K^+(X)$ and L'^i is injective for all i. It is easy to see that $L \in D^{\geqslant a}(X)$ implies $R\Gamma(X, L) \in D^{\geqslant a}(\mathcal{A}b)$.

Definition 6.1. $F \in Mod(X)$ is called *flasque* if for all $U \hookrightarrow X$ open, $F(X) \to F(U)$ is surjective.

Remark. If F is flasque, then $F|_U$ is flasque, for all open subset U of X.

Proposition 6.2. *(1). Suppose $0 \to F' \to F \to F'' \to 0$ is an exact sequence of $Mod(X)$ and F' is flasque, then $0 \to \Gamma(X, F') \to \Gamma(X, F) \to \Gamma(X, F'') \to 0$ is exact.*

(2). Under the same assumption as in (1), then if F', F are flasque, F'' is flasque.

(3). If F is injective, then F is flasque.

Corollary 6.3. *The subcategory of flasque sheaves is right adapted to $\Gamma(X, -)$. In particular, for $L \in D^+(X)$, $R\Gamma(X, L) \simeq \Gamma(X, L')$, where $L \to L'$ is a quasi-isomorphism with $L' \in K^+(X)$ and L'^i is flasque for all i.*

As $Mod(X)$ has enough injectives, 6.3 immediately follows from 6.2.

Proof of 6.2(1) and (2). (1). Let $s'' \in \Gamma(X, F'')$, we want to find $s \in \Gamma(X, F)$ such that the image of s in $\Gamma(X, F'')$ is s''. Order the set

$$\{(U, t) \mid U \hookrightarrow X \text{ open}, \Gamma(U, F) \ni t \mapsto s''|_U \in \Gamma(U, F'')\}$$

by $(U, t) < (U_1, t_1)$ if $U \subset U_1$ and t_1 extends t. This is an inductive ordered set. So, by Zorn's lemma, there exists a maximal (U, s), $U \hookrightarrow X$, $s \in \Gamma(U, F)$ and $s \mapsto s''|_U$.

Assume there exists $x \notin U$, then there exists an open neighborhood V of x and $t \in \Gamma(V, F)$, $t \mapsto s''|_V$. Hence $z := s|_{U\cap V} - t|_{U\cap V} \in \Gamma(U \cap V, F')$. Since F' is flasque, we can find $z_1 \in \Gamma(V, F')$ such that $z_1|_{U\cap V} = z$. Then s and $t + z_1$ agree on $U \cap V$, hence s extends to a section $\bar{s}$ of F on $U \cup V$ such that $\bar{s} \mapsto s''|U \cup V$. This contradicts the maximality of (U, s), and finishes the proof.

(2). Since $0 \to F' \to F \to F'' \to 0$ is exact and F, F' are flasque, by (1) we have the following commutative diagram whose rows are exact and the two left vertical maps are surjective.

$$\begin{tikzcd} 0 \arrow[r] & F'(X) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & F''(X) \arrow[r] \arrow[d, "\varphi"] & 0 . \\ 0 \arrow[r] & F'(U) \arrow[r] & F(U) \arrow[r] & F''(U) \arrow[r] & 0 \end{tikzcd}$$

So φ is surjective, hence F'' is flasque. □

Preliminary to the proof of (3):

Suppose that $U \overset{j}{\hookrightarrow} X$ is open and $Y := X - U \overset{i}{\hookrightarrow} X$ is the complementary closed subset. Then, for each $F \in Mod(U)$, we can consider the sheaf $j_! F \in Mod(X)$, which is associated to the presheaf:

$$V \mapsto \begin{cases} \Gamma(V, F) & \text{if } V \subset U \\ 0 & \text{if } V \not\subseteq U \end{cases}$$

where V is an open subset of X. This sheaf is called the *extension of F by zero*. It is easy to see that if $x \in U$, $(j_! F)_x = F_x$, and if $x \notin U$, $(j_! F)_x = 0$. In fact, $j_! F|_U = F$. For each $E \in Mod(X)$, it follows from the definition that

$$\mathrm{Hom}(F, j^* E) = \mathrm{Hom}(F, E|_U) = \mathrm{Hom}(j_! F, E),$$

hence $j_!$ is left adjoint to j^*. For $E \in Mod(X)$, we have a basic exact sequence:

$$0 \to j_! j^* E \to E \to i_* i^* E \to 0;$$

i.e.

$$0 \to j_!(E|_U) \to E \to i_*(E|_Y) \to 0.$$

Now, let's come back to the proof of 6.2.(3).

Assume F is injective on X, let $j : U \hookrightarrow X$ be an open subset of X. A section $s \in F(U) = \mathrm{Hom}(\mathcal{O}_U, j^* F)$ defines a map $s : j_! \mathcal{O}_U \to F$. Then by the injectivity of F, there exists $\bar{s} \in F(X)$ extending s, i.e. making the following diagram commutative:

$$\begin{tikzcd} j_! \mathcal{O}_U \arrow[r, "s"] \arrow[d, hook] & F . \\ \mathcal{O}_X \arrow[ur, dotted, "\bar{s}"'] & \end{tikzcd}$$

6.4. Godement's flasque resolution

For $x \in X$, let $i_x : \{x\} \to X$ denote the inclusion, let $F \in Mod(X)$. Then the canonical map $F \to \prod_{x \in X} i_{x*} i_x^* F = \mathcal{C}^0(F)$ $(s \mapsto (s_x)_{x \in X})$ is injective and $\mathcal{C}^0(F)$ is flasque. Therefore we get a flasque resolution of F :

$$0 \to F \to \mathcal{C}^0(F) \to \mathcal{C}^1(F) \to \mathcal{C}^2(F) \to \cdots$$

where $\mathcal{C}^{n+1}(F) = \mathcal{C}^0(\mathrm{Coker} : \mathcal{C}^{n-1}(F) \to \mathcal{C}^n(F))$. This resolution is called *Godement's canonical flasque resolution.* Then,

$$R\Gamma(X, F) \simeq \Gamma(X, \mathcal{C}(F)) = (\Gamma(X, \mathcal{C}^0(F)) \to \cdots \to \Gamma(X, \mathcal{C}^n(F)) \to \cdots)$$

More generally, for $F \in K^+(X)$, by the above flasque resolution we have a morphism of bi-complexes $F \to \mathcal{C}(F)$, where $\mathcal{C}(F)^{ij} = \mathcal{C}^j(F^i)$. This morphism induces a quasi-isomorphism on each column, hence a quasi-isomorphism $F \to \mathrm{s}\mathcal{C}(F) := F'$ and F'^n is flasque for all n. Hence we get an isomorphism $R\Gamma(X, F) \simeq \Gamma(X, \mathrm{s}\mathcal{C}(F))$.

(2) Rf_*

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces and $f : X \to Y$ a morphism of ringed spaces. Recall that the direct image functor, defined by $f_* : Mod(X) \to Mod(Y)$ $(f_*F)(V) = F(f^{-1}V)$ with $V \subset Y$, is additive and left exact. By definition, f_* transforms flasque sheaves into flasque sheaves, hence injective sheaves into flasque sheaves by 6.2. If Y consists of a single point, and $\mathcal{O}_Y = \mathbb{Z}$, then $f_* = \Gamma(X, -)$.

The right derived functor $Rf_* : D^+(X) \to D^+(Y)$ exists. For $F \in D^+(X), Rf_*F = f_*F'$ for $F \to F'$ a quasi-isomorphism with $F' \in K^+(X)$ and F'^i injective for all i. Flasque sheaves are acyclic for f_* (hence for $F \in D^+(X), Rf_*F \simeq f_*F'$ for $F \to F'$ a quasi-isomorphism with $F' \in K^+(X)$ and F'^i flasque for all i.)

By definition, $\Gamma(Y, f_*F) = \Gamma(X, F)$, in other words, the functor $\Gamma(X, -)$ is the composition:

$$\Gamma(X, -) = \Gamma(Y, -) \circ f_* : \begin{tikzcd} Mod(X) \arrow[r, "f_*"] \arrow[rr, bend right, "\Gamma(X,-)"'] & Mod(Y) \arrow[r, "\Gamma(Y,-)"] & \mathcal{A}b \end{tikzcd}$$

Then from 5.12, we deduce $R\Gamma(Y, Rf_*L) \simeq R\Gamma(X, L)$ for $L \in D^+(X)$.

Similarly, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of ringed spaces, then we have

$$R(gf)_* = Rg_* \circ Rf_* : D^+(X) \to D^+(Z).$$

Indeed the subcategory consisting of flasque sheaves on X is adapted to f_* for any $f : X \to Y$ and f_* transforms flasque sheaves into flasque sheaves.

(3) $\otimes^L$ **(and** $\mathrm{Tor_q}(-,-)$**)**

Let A be a commutative ring, then the functor

$$\mathrm{Mod}(A) \times \mathrm{Mod}(A) \to \mathrm{Mod}(A)$$

$$(E, F) \mapsto E \otimes_A F$$

is bi-additive and right exact in each argument.

If $E, F \in C(\mathcal{A})$, we get a naive bi-complex $(E^p \otimes F^q, d \otimes Id, Id \otimes d)$ and the associated bi-complex $(E \otimes F)^{\bullet,\bullet}$ with the diffirentials defined as follows:

$$d'(x \otimes y) = dx \otimes y$$

$$d''(x \otimes y) = (-1)^p x \otimes dy$$

where $x \otimes y \in E^p \otimes F^q$

The simple associated complex $\mathbf{s}(E \otimes F)^{\bullet,\bullet}$ is denoted $E \otimes F$. We have

$$(E \otimes F)^n = \bigoplus_{p+q=n} E^p \otimes F^q$$

$$d = d' + d''$$

The bi-additive functor

$$\begin{gathered} C(\mathcal{A}) \times C(\mathcal{A}) \to C(\mathcal{A}) \\ (E, F) \mapsto E \otimes F \end{gathered}$$

where $\mathcal{A} = Mod(A)$, extends to a bi-triangulated functor

$$\begin{gathered} K(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathcal{A}) \\ (E, F) \mapsto E \otimes F \end{gathered}$$

For fixed E, let us consider the functor

$$\begin{gathered} K(\mathcal{A}) \to K(\mathcal{A}) \\ F \mapsto E \otimes F \end{gathered}$$

We will define a left derived functor of it. For this we need some generalization of the definitions given in (5.2).

Definition 6.5. Let $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ be a triangulated functor. A right derived functor of F is a pair

$$(RF : D^+(\mathcal{A}) \to D^+(\mathcal{B}), \quad \varepsilon : F \to RF),$$

where RF is triangulated, satisfying the following universal property: For any triangulated functor $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and morphism of functors $\eta : QF \to G \circ Q$, there exists a unique morphism $\alpha : RF \to G$ such that $\eta = \alpha \circ \varepsilon$.

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "F"] & D^+(\mathcal{A}) \arrow[d, "RF"'] \arrow[d, bend left, "G"] \\ K^+(\mathcal{B}) \arrow[r, "Q"] & D^+(\mathcal{B}) \end{tikzcd}$$

Definition 6.5' Let $F : K^-(\mathcal{A}) \to K^-(\mathcal{B})$ be a triangulated functor. A left derived functor of F is a pair

$$(LF : D^-(\mathcal{A}) \to D^-(\mathcal{B}), \quad \varepsilon : LF \to F)$$

where LF is triangulated satisfying the following universal property: For any triangulated functor $G : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and morphism of functors $\eta : G \circ Q \to QF$, there exists a unique morphism $\alpha : G \to LF$ such that $\eta = \varepsilon \circ \alpha$.

$$\begin{tikzcd} K^-(\mathcal{A}) \arrow[r, "Q"] \arrow[d, "F"] & D^-(\mathcal{A}) \arrow[d, "LF"'] \arrow[d, bend left, "G"] \\ K^-(\mathcal{B}) \arrow[r, "Q"] & D^-(\mathcal{B}) \end{tikzcd}$$

Recall $\mathrm{Mod}(A)$ has enough projectives.

Lemma 6.6. *Let $\mathcal{A} = \mathrm{Mod}(A)$, $E \in C(\mathcal{A})$, and $F \in C^-(\mathcal{A})$. Assume F^i is projective for all i. Then if E or F is acyclic, then $E \otimes F$ is acyclic.*

Proof. (a). Assume F is acyclic, then F is homotopically trivial. So $E \otimes F$ is acyclic.

(b). Assume E is acyclic, $E \in K^-(\mathcal{A})$. Then $(E \otimes F)^{\bullet,\bullet}$ is biregular. For each q, we have $E^\bullet \otimes F^q$ is acyclic, so $E \otimes F$ is acyclic.

(c). General case. $E = \varinjlim \tau_{\leqslant n} E$, then $H^q(\varinjlim \tau_{\leqslant n} E \otimes F) = \varinjlim H^q(\tau_{\leqslant n} E \otimes F) = 0$. □

Proposition 6.7. *Let $E \in K(\mathcal{A})$, then the functor $K^-(\mathcal{A}) \to K(\mathcal{A})$ given by $F \mapsto E \otimes F$ has a left derived functor*

$$\begin{aligned} D^-(\mathcal{A}) &\to D(\mathcal{A}) \\ F &\mapsto E \otimes_A^L F \end{aligned},$$

calculated by $E \otimes_A^L F = E \otimes_A F'$ where $F' \in K^-(\mathcal{A})$ with F'^i projective and $F' \to F$ a quasi-isomorphism. Moreover, the functor $K(\mathcal{A}) \to D(\mathcal{A})$ given by $E \mapsto E \otimes^L F$ factors (uniquely) through $D(\mathcal{A})$ and gives a triangulated functor

$$\begin{aligned} D(\mathcal{A}) \times D^-(\mathcal{A}) &\to D(\mathcal{A}) \\ (E, F) &\mapsto E \otimes_A^L F \end{aligned}$$

sending $D^-(\mathcal{A}) \times D^-(\mathcal{A})$ to $D^-(\mathcal{A})$.

Remark 6.8. Let $\mathcal{P}$ be the full subcategory of $\mathcal{A}$ consisting of projective A-modules. For $E, F \in D^-(\mathcal{A})$, we have isomorphisms in $D(\mathcal{A})$

$$E \otimes_A^L F \simeq E \otimes F' \simeq E' \otimes F \simeq E' \otimes F',$$

where $E', F' \in K^-(\mathcal{P})$, $E' \to E$ and $F' \to F$ quasi-isomorphisms. (Actually, $E \otimes^L F \simeq E' \otimes^L F \simeq E' \otimes F'$, and by an analog of Lemma 6.7, $E' \otimes F' \simeq E' \otimes F$.)

Proposition 6.9. *(1). There is a canonical isomorphism*

$$E \otimes^L (F \otimes^L G) \simeq (E \otimes^L F) \otimes^L G$$

for $E \in D(\mathcal{A})$, $F, G \in D^-(\mathcal{A})$.

(2). There is a canonical isomorphism

$$E \otimes^L F \simeq F \otimes^L E$$

for $E, F \in D^-(\mathcal{A})$.

Proof. (1) Replace F, G by $F', G' \in K^-(\mathcal{P})$ such that there are quasi-isomorphisms $F' \to F$ and $G' \to G$. Apply then the isomorphism of complexes $E \otimes (F' \otimes G') \to (E \otimes F') \otimes G'$ given by $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

(2) Take quasi-isomorphisms $E' \to E$ and $F' \to F$ with $E', F' \in K^-(\mathcal{P})$. Then $E \otimes^L F \simeq E' \otimes F'$, $F \otimes^L E \simeq F' \otimes E'$. Apply the isomorphism of complexes $E' \otimes F' \to F' \otimes E'$ given by $x \otimes y \mapsto (-1)^{pq} y \otimes x$ for $x \in E'^p$, $y \in F'^q$. □

The isomorphism in (2) is called the Koszul isomorphism. The sign convention is adopted to get a morphism of complexes, and is called the Koszul rule.

Definition 6.10. For $E \in D(\mathcal{A})$, $F \in D^-(\mathcal{A})$, $n \in \mathbb{Z}$, define $\mathrm{Tor}_n(E, F) = \mathrm{Tor}_n^A(E, F) = H^{-n}(E \otimes_A^L F)$.

In particular, for $E, F \in \mathcal{A}$, $\mathrm{Tor}_0^A(E, F) = E \otimes_A F$ (by right exactness of $E \otimes_A \bullet$).

Definition 6.11. An A-module E is called *flat* if the functor

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{A} \\ F & \mapsto & E \otimes_A F \end{array}$$

is exact.

The exact sequences of cohomology shows that the following conditions are equivalent:

(i) E is flat;

(ii) For all $F \in \mathcal{A}$, $\mathrm{Tor}_1^A(E, F) = 0$;

(iii) For all $F \in \mathcal{A}$ and $q > 0$, $\mathrm{Tor}_q^A(E, F) = 0$.

Proposition 6.12. *Let $\mathcal{A}' \subset \mathcal{A}$ be the full subcategory of flat A-modules. Then $\mathcal{A}'$ is left adapted to the functors*

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{A} \\ F & \mapsto & E \otimes F \end{array}$$

for all $E \in \mathcal{A}$, i.e.,

(i) For all $F \in \mathcal{A}$, there exists an epimorphism $F' \to F$ with $F' \in \mathcal{A}'$.

(ii) If $F' \in \mathcal{A}$, $F, F'' \in \mathcal{A}'$ and

$$0 \to F' \to F \to F'' \to 0$$

exact, then $F' \in \mathcal{A}'$.

(iii) If the above sequence is exact with $F', F, F'' \in \mathcal{A}'$, then

$$0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$$

is exact.

Proof. (i) follows from the fact that projective modules are flat.

For (ii) and (iii), use long exact sequences and Koszul isomorphisms (6.9 (2)). □

Corollary 6.13. *For $E \in D(\mathcal{A})$, $F \in D^-(\mathcal{A})$, $E \otimes^L F \simeq E \otimes F'$ where $F' \to F$ is a quasi-isomorphism, $F' \in K^-(\mathcal{A})$ and F'^i flat for all i.*

Proof. Choose a quasi-isomorphism $P \to F'$ with $P \in K^-(\mathcal{A})$ and P^i projective for all i. Complete it into a distinguished triangle in $K(\mathcal{A})$. The corollary then follows from Lemma 6.7 with "projective" replaced by "flat", which we will prove later as Lemma 6.17 in more generality. □

As in 6.8, for $E, F \in D^-(\mathcal{A})$, $E' \to E$ and $F' \to F$ quasi-isomorphisms, $E', F' \in K^-(\mathcal{A})$, E'^i, F'^i flat for all i, we have

$$E \otimes^L F \simeq E \otimes F' \simeq E' \otimes F \simeq E' \otimes F'.$$

Note that for a commutative ring A with a multiplicative system S, $S^{-1}A$ is a flat A-module, but not a projective A-module in general (e.g., when A is a principal ideal domain which is not a field and $S^{-1}A$ is its fractional field).

6.14. Let $(X, \mathcal{O}_X)$ be a (commutative) ringed space. Write

$$\begin{aligned} Mod(X) &= Mod(\mathcal{O}_X), \\ C(X) &= C(Mod(X)), \\ K(X) &= K(Mod(X)), \\ D(X) &= D(Mod(X)). \end{aligned}$$

For $E, F \in Mod(X)$, define $E \otimes F = E \otimes_{\mathcal{O}_X} F$ to be the sheaf associated to the presheaf

$$U \mapsto E(U) \otimes_{\mathcal{O}(U)} F(U).$$

For $x \in X$, $(E \otimes_{\mathcal{O}_X} F)_x = E_x \otimes_{\mathcal{O}_{X,x}} F_x$. E is called *a flat $\mathcal{O}_X$-module* if the functor

$$\begin{array}{ccc} Mod(X) & \to & Mod(X) \\ F & \mapsto & E \otimes_{\mathcal{O}_X} F \end{array}$$

is exact.

For $E, F \in C(X)$, define the double complex $(E \otimes F)^{\bullet\bullet}$ as in 6.5 and $E \otimes F = \mathbf{s}(E \otimes F)^{\bullet\bullet}$, that is, $(E \otimes F)^n = \bigoplus_{p+q=n} E^p \otimes F^q$ and $d(a \otimes b) = da \otimes b + (-1)^p a \otimes db$ for $a \in E^p$, $b \in F^q$. The functor

$$\begin{array}{rcl} C(X) \times C(X) & \to & C(X) \\ (E, F) & \mapsto & E \otimes F \end{array}$$

defines a bi-triangulated functor $K(X) \times K(X) \to K(X)$.

Proposition 6.15. *For $E \in K(X)$, the functor*

$$\begin{array}{rcl} K^-(X) & \to & K(X) \\ F & \mapsto & E \otimes F \end{array}$$

has a left derived functor

$$\begin{array}{rcl} D^-(X) & \to & D(X) \\ F & \mapsto & E \otimes^L F, \end{array}$$

calculated as $E \otimes^L F = E \otimes F'$ for $F' \to F$ a quasi-isomorphism with $F' \in K^-(X)$ and F'^i flat for all i. Moreover, for $F \in D^-(X)$ fixed,

$$\begin{array}{rcl} K(X) & \to & D(X) \\ E & \mapsto & E \otimes^L F \end{array}$$

induces a triangulated functor $D(X) \to D(X)$. So we get a bi-triangulated functor

$$\begin{array}{rcl} D(X) \times D^-(X) & \to & D(X) \\ (E, F) & \mapsto & E \otimes^L F \end{array}$$

sending $D^-(X) \times D^-(X)$ to $D^-(X)$.

Proof. Imitate the proof in the case of modules over a ring, making use of 6.16(i) and 6.17 below. □

Lemma 6.16. *Let $\mathcal{A}' \subset Mod(X)$ be the full subcategory of flat modules. Then $\mathcal{A}'$ is left adapted to the functors*

$$\begin{array}{rcl} Mod(X) & \to & Mod(X) \\ F & \mapsto & E \otimes F \end{array}$$

for all $E \in Mod(X)$, i.e.,

(i) *For all $F \in Mod(X)$, there exists an epimorphism $F' \to F$ with $F' \in \mathcal{A}'$.*

(ii) *If $F' \in Mod(X)$, $F, F'' \in \mathcal{A}'$ and*

$$0 \to F' \to F \to F'' \to 0$$

exact, then $F' \in \mathcal{A}'$.

(iii) *If the above sequence is exact with $F', F, F'' \in \mathcal{A}'$, then*

$$0 \to E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$$

is exact.

Proof. For (i), define

$$E' = \bigoplus_{U,s} j_{U!}\mathcal{O}_U \xrightarrow{\sum_{U,s} \tilde{s}} E,$$

where the sums are taken for all open $U \subset X$ and $s \in \Gamma(U, E)$, $j_U : U \hookrightarrow X$ is the embedding, and $\tilde{s}$ is defined by the canonical isomorphism

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{O}_U, {j_U}^* E) & \xrightarrow{\sim} & \mathrm{Hom}(j_{U!}\mathcal{O}_U, E) \\ s & \mapsto & \tilde{s}. \end{array}$$

This is obviously an epimorphism, and E' is flat because $j_{U!}\mathcal{O}_U$ flat. (These facts are easily seen by taking stalks.)

For (ii) and (iii), we only need to use the corresponding results for modules over rings (6.12) and the fact that an $\mathcal{O}_X$-module M is flat if and only if M_x flat over $\mathcal{O}_{X,x}$ for all $x \in X$. $\square$

Remark. In general, there are not enough projectives in $Mod(X)$. In fact, if X is a locally noetherian Jacobson scheme with no isolated points, then every projective $\mathcal{O}_X$-module is zero. In addition, if X is a projective scheme over a field which does not have any isolated point, then every projective object in the category of quasi-coherent $\mathcal{O}_X$-modules is zero. See [Ga].

Lemma 6.17. *Let $E \in K(X)$, $F \in K^-(X)$ with F^i flat for all i. If E or F is acyclic, then $E \otimes F$ is, too.*

Proof. If E is acyclic, proceed as in 6.7. If F is acyclic, we show that $E \otimes F$ is acyclic. First, assume $E \in K^-(X)$. Now F is bounded above, acyclic and flat in each component, we can show by induction that it breaks into flat short exact sequences. Hence, by (ii) and (iii) of 6.16, $E^p \otimes F$ breaks into short exact sequences, and thus is acyclic, for all' p. Therefore, $E \otimes F = \mathbf{s}(E \otimes F)^{\bullet\bullet}$ is acyclic. For the general case, use $E = \varinjlim \tau_{\leq n} E$ and $E \otimes F = \varinjlim ((\tau_{\leq n} E) \otimes F)$. □

Proposition 6.18. *(1). There is a canonical isomorphism*

$$E \otimes^L (F \otimes^L G) \simeq (E \otimes^L F) \otimes^L G$$

for $E \in D(X)$, $F, G \in D^-(X)$.

(2). There is a canonical (Koszul) isomorphism

$$E \otimes^L F \simeq F \otimes^L E$$

for $E, F \in D^-(X)$.

The proof is similar to that of 6.9, projective modudes being replaced by flat ones.

6.19. The functor Lf^*. Let $f : X \to Y$ be a morphism of ringed spaces. The morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ induces a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, by which we can regard $\mathcal{O}_X$ as an $f^{-1}\mathcal{O}_Y$ module. Define $f^*E = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}E$. $f^* : Mod(Y) \to Mod(X)$ is a right exact additive functor left adjoint to $f_* : Mod(X) \to Mod(Y)$. f^* extends to $C(Y) \to C(X)$ and, in turn, defines a triangulated functor $K(Y) \to K(X)$.

Proposition 6.20. *The functor $f^* : Mod(Y) \to Mod(X)$ has a left derived functor $Lf^* : D^-(Y) \to D^-(X)$ calculated as $Lf^*(E) = f^*E'$ for $E' \to E$ a quasi-isomorphism with $E' \in K^-(Y)$ and E'^i flat for all i.*

Proof. It suffices to show that the full subcategory of $Mod(Y)$ consisting of flat $\mathcal{O}_Y$-modules is left adapted to f^* (5.10), that is, to check (i), (ii) and (iii) in the definition. (i) and (ii) have already been proved in 6.16, while (iii) follows from the following two facts easily seen by taking stalks: $f^{-1} : Mod(Y) \to Mod(f^{-1}(\mathcal{O}_Y))$ is exact; if M is a flat $\mathcal{O}_Y$-module, then $f^{-1}(M)$ is a flat $f^{-1}(\mathcal{O}_Y)$-module. □

Define $L^i f^* = H^i Lf^*$, $L_i f^* = L^{-i} f^*$.

Proposition 6.21. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of ringed spaces. Then*

$$Lf^*Lg^* \simeq L(gf)^*; \tag{6.21.1}$$

$$Lg^*(E \otimes^L F) \simeq Lg^*E \otimes^L Lg^*F \tag{6.21.2}$$

for $E, F \in D^-(Z)$;

$$\mathrm{Hom}_{D(X)}(Lf^*E, F) \simeq \mathrm{Hom}_{D(Y)}(E, Rf_*F) \tag{6.21.3}$$

for $E \in D^-(Y)$, $F \in D^+(X)$.

The last isomorphism is called the trivial duality.

Proof. The isomorphisms (6.21.1) and (6.21.2) follow from the fact that $g^*(M)$ is a flat $\mathcal{O}_Y$-module for a flat $\mathcal{O}_Z$-module M. The proof of (6.21.3) will be given in 7.6. □

7 $R\,\mathrm{Hom}$, $R\mathcal{H}om$, Ext^i, $\mathcal{E}xt^i$

7.1. The functor $R\,\mathrm{Hom}$. Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, $F : \mathcal{A}^\circ \to \mathcal{B}$ an additive functor. For $L \in C(\mathcal{A})$:

$$\cdots \longrightarrow L^{-i-1} \xrightarrow{d_L^{-i-1}} L^{-i} \longrightarrow \cdots$$

we define $F(L) \in C(\mathcal{B})$ to be the complex

$$\cdots \longleftarrow F(L)^{i+1} \xleftarrow{d_{F(L)}^{i}} F(L)^i \longleftarrow \cdots$$

where $F(L)^i = F(L^{-i})$ and $d^i_{F(L)} = (-1)^{i+1}F(d_L^{-i-1})$. For a morphism $u : L \to M$, we define $F(u) : F(M) \to F(L)$ by $F(u)^i = F(u^{-i})$. Thus we get a functor $C(\mathcal{A})^\circ \to C(\mathcal{B})$, which defines a triangulated functor $K(\mathcal{A})^\circ \to K(\mathcal{B})$. We still use F to denote them.

Example 7.1.1. The additive functor

$$\begin{array}{rcl} \mathcal{A}^\circ \times \mathcal{A} & \to & Ab \\ (L, M) & \mapsto & \mathrm{Hom}(L, M) \end{array}$$

is left exact in both arguments. For $L, M \in C(\mathcal{A})$, define a bicomplex of abelian groups $\mathrm{Hom}(L, M)^{\bullet\bullet}$ as follows: let the component of bidegree (p, q) be $\mathrm{Hom}(L, M)^{p,q} = \mathrm{Hom}(L^{-q}, M^p)$, the differentials $d'^{p,q} = \mathrm{Hom}(L^{-q}, d_L^p)$,

$$d''^{p,q} = (-1)^p(-1)^{q+1}\,\mathrm{Hom}(d_L^{-q-1}, M^p) : \mathrm{Hom}(L^{-q}, M^p) \to \mathrm{Hom}(L^{-q-1}, M^p).$$

Take $\mathrm{Hom}^\bullet(L, M) = \mathbf{s}(\mathrm{Hom}(L, M)^{\bullet\bullet}) \in C(Ab)$, where $\mathbf{s}$ is defined by

$$\begin{aligned}\mathrm{Hom}^n(L, M) &= \prod_{p+q=n} \mathrm{Hom}(L, M)^{p,q} = \prod_{p+q=n} \mathrm{Hom}(L^{-q}, M^p)\\ &= \prod_{p-q=n} \mathrm{Hom}(L^q, M^p).\end{aligned}$$

For $f \in \mathrm{Hom}^n(L, M)$, we have $f = (f^q)_{q\in\mathbb{Z}}$, $f^q : L^q \to M^{q+n}$, $df = d_M \circ f + (-1)^{n+1} f \circ d_L$.

Note that $\mathrm{Hom}(L, M)^{\bullet\bullet}$ is biregular if L or M is bounded or

$$L \in K^-(\mathcal{A}),\ M \in K^+(\mathcal{A}).$$

The functor

$$\begin{aligned} C(\mathcal{A})^\circ \times C(\mathcal{A}) &\to C(Ab)\\ (L, M) &\mapsto \mathrm{Hom}^\bullet(L, M)\end{aligned}$$

defines a bi-triangulated functor $K(\mathcal{A})^\circ \times K(\mathcal{A}) \to K(Ab)$.

Proposition 7.2. *Let $\mathcal{A}$ be an abelian category with enough injectives. For $L \in K(\mathcal{A})$, the functor*

$$\begin{aligned} K^+(\mathcal{A}) &\to K(Ab)\\ M &\mapsto \mathrm{Hom}^\bullet(L, M)\end{aligned}$$

has a right derived functor

$$\begin{aligned} D^+(\mathcal{A}) &\to D(Ab)\\ M &\mapsto R\,\mathrm{Hom}(L, M),\end{aligned}$$

calculated as $R\,\mathrm{Hom}(L, M) = \mathrm{Hom}^\bullet(L, M')$ for $M \to M'$ a quasi-isomorphism with $M' \in K^+(\mathcal{A})$, M'^i injective for all i. Moreover, for $M \in D^+(\mathcal{A})$ fixed,

$$\begin{aligned} K(\mathcal{A})^\circ &\to D(Ab)\\ L &\mapsto R\,\mathrm{Hom}(L, M)\end{aligned}$$

induces a triangulated functor $D(\mathcal{A}) \to D(Ab)$. So we get a bi-triangulated functor

$$\begin{aligned} R\operatorname{Hom} : D(\mathcal{A})^\circ \times D^+(\mathcal{A}) &\to D(Ab) \\ (L, M) &\mapsto R\operatorname{Hom}(L, M) \end{aligned}$$

sending $D^-(\mathcal{A})^\circ \times D^+(\mathcal{A})$ to $D^+(\mathcal{A})$.

Proof. Proceed as in 6.8, applying the following lemma. □

Lemma 7.3. *Let $\mathcal{A}$ be as in 7.2, $E \in C(\mathcal{A})$, $F \in C^+(\mathcal{A})$. Assume F^i injective for all i. Then if E or F is acyclic, then so is $\operatorname{Hom}^\bullet(E, F)$.*

Proof. Use $H^i \operatorname{Hom}^\bullet(E, F) \xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{A})}(E, F[i])$, which follows from Lemmas 7.4 and 7.5 below. (When F is acyclic or $E \in K^-(\mathcal{A})$, there is an alternate proof similar to the proof of 6.7.) □

Lemma 7.4. *Let $\mathcal{A}$ be an additive category, $E, F \in C(\mathcal{A})$. Then*

$$Z^0 \operatorname{Hom}^\bullet(E, F) = \operatorname{Hom}_{C(\mathcal{A})}(E, F),$$
$$B^0 \operatorname{Hom}^\bullet(E, F) = \{f \in \operatorname{Hom}_{C(\mathcal{A})}(E, F); f \simeq 0\},$$
$$H^0 \operatorname{Hom}^\bullet(E, F) = \operatorname{Hom}_{K(\mathcal{A})}(E, F).$$

Proof. For $f \in \operatorname{Hom}^0(E, F)$, $(df)^i = df^i - f^i d$. Thus $f \in Z^0 \operatorname{Hom}^\bullet(E, F)$ if and only if $f \in \operatorname{Hom}_{C(\mathcal{A})}(E, F)$. For $h \in \operatorname{Hom}^{-1}(E, F)$, $(dh)^i = dh^i + h^i d$. Thus $f \in B^0 \operatorname{Hom}^\bullet(E, F)$ if and only if $f \simeq 0$. □

Lemma 7.5. *Let $\mathcal{A}$ be an abelian category with enough injectives, $E \in K(\mathcal{A})$, $F \in K^+(\mathcal{A})$ such that F^i injective for all i. Then we have an isomorphism*

$$\operatorname{Hom}_{K(\mathcal{A})}(E, F) \xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{A})}(E, F).$$

Proof. By definition,

$$\operatorname{Hom}_{D(\mathcal{A})}(E, F) = \varinjlim_{s: F \to F'} \operatorname{Hom}_{K(\mathcal{A})}(E, F'),$$

where s runs through quasi-isomorphisms in $K(\mathcal{A})$. By cofinality, we can restrict to quasi-isomorphisms such that $F' \in K^+(\mathcal{A})$ and F'^i injective for all i. Note that in this case, s is actually an isomorphism in $K^+(\mathcal{A})$ (4.28). We then get

$$\operatorname{Hom}_{K(\mathcal{A})}(E, F) \xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{A})}(E, F).$$

□

7.6. We now give the proof of (6.21.3). We may assume $E \in K^-(Y)$, $F \in K^+(X)$, E^i flat and F^i injective, for all i. In this case, (6.21.3) becomes

$$\operatorname{Hom}_{D(X)}(f^*E, F) \simeq \operatorname{Hom}_{D(Y)}(E, f_*F).$$

By definition,

$$\operatorname{Hom}_{D(Y)}(E, f_*F) = \varinjlim_{s: E' \to E} \operatorname{Hom}_{K(Y)}(E, f_*F),$$

where s runs through quasi-isomorphisms in $K(Y)$. By cofinality, we can restrict to quasi-isomorphisms such that $E' \in K^-(Y)$ and E'^i flat for all i. By 7.4 and 7.5,

$$\begin{aligned}\operatorname{Hom}_{K(Y)}(E', f_*F) = H^0 \operatorname{Hom}^\bullet(E', f_*F) \simeq H^0 \operatorname{Hom}^\bullet(f^*E', F) \\ \simeq \operatorname{Hom}_{D(X)}(f^*E', F).\end{aligned}$$

Thus we get isomorphisms $\operatorname{Hom}_{K(Y)}(E', f_*F) \to \operatorname{Hom}_{D(X)}(f^*E, F)$ which commute with transition maps induced by morphisms between different s, and hence an isomorphism

$$\operatorname{Hom}_{D(Y)}(E, f_*F) \to \operatorname{Hom}_{D(X)}(f^*E, F).$$

7.7. Calculation of Ext^i**.** Let $\mathcal{A}$ be an abelian category with enough injectives. Recall that in 4.9, for $E, F \in D(\mathcal{A})$, we defined $\operatorname{Ext}^n(E, F)$ as $\operatorname{Hom}_{D(\mathcal{A})}(E, F[n])$.

Proposition 7.8. *With $\mathcal{A}$ as in 7.7, $E \in D(\mathcal{A})$, $F \in D^+(\mathcal{A})$, we have*

$$\operatorname{Ext}^n(E, F) \simeq H^n R\operatorname{Hom}(E, F).$$

Proof. Up to shifting, we may assume that $n = 0$. We have

$$H^0 R\operatorname{Hom}(E, F) \simeq H^0 \operatorname{Hom}^\bullet(E, F'),$$

for $F \to F'$ a quasi-isomorphism with F'^i injective for all i. The proposition then follows from Lemmas 7.4 and 7.5. □

Remark 7.9. If $\mathcal{A}$ has enough projectives, then

$$R\operatorname{Hom}(E, F) \simeq \operatorname{Hom}^\bullet(E', F) \simeq \operatorname{Hom}^\bullet(E', F')$$

for $E \in D^-(\mathcal{A})$, $F \in D^+(\mathcal{A})$, $E' \to E$ and $F \to F'$ quasi-isomorphisms, $E' \in K^-(\mathcal{A})$, $F' \in K^+(\mathcal{A})$, E'^i projective and F'^i injective for all i (by an analog of (the special cases of) Lemma 7.3).

7.10. The functor $R\mathcal{H}om$. Let $(X, \mathcal{O}_X)$ be a (commutative) ringed space. For $E, F \in Mod(X)$,

$$\mathcal{H}om_{\mathcal{O}_X}(E, F) : U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(E|U, F|U)$$

is a sheaf of $\mathcal{O}_X$-modules. Similar to 7.1.1, the functor $\mathcal{H}om_{\mathcal{O}_X} : Mod(X)^\circ \times Mod(X) \to Mod(X)$ induces a functor

$$\mathcal{H}om^\bullet : C(X)^\circ \times C(X) \to C(X)$$

($\mathcal{H}om^\bullet(E, F)^n = \prod_{p-q=n} \mathcal{H}om(E^q, F^p)$, $df = d_F \circ f + (-1)^{n+1} f \circ d_E$ for $f \in \mathcal{H}om^\bullet(E, F)^n$), which defines a bi-triangulated functor $K(X)^\circ \times K(X) \to K(X)$. We have a bi-triangulated functor

$$R\mathcal{H}om : D(X)^\circ \times D^+(X) \to D(X)$$

calculated as $R\mathcal{H}om(E, F) = \mathcal{H}om^\bullet(E, F')$ for $F \to F'$ a quasi-isomorphism, $F' \in K^+(X)$, F'^i injective for all i. (Applying

$$\Gamma(U, \mathcal{H}om(E, F)) = \mathrm{Hom}(E|U, F|U)$$

and 7.3, we get a variant of 7.3 with $\mathcal{H}om^\bullet$ instead of $\mathrm{Hom}^\bullet$.)

We have

$$R\Gamma(X, R\mathcal{H}om(E, F)) \simeq R\,\mathrm{Hom}(E, F) \tag{7.10.1}$$

for $E \in D^-(X)$, $F \in D^+(X)$; and

$$R\mathcal{H}om(E \otimes^L F, G) \simeq R\mathcal{H}om(E, R\mathcal{H}om(F, G)) \tag{7.10.2}$$

for $E, F \in D^-(X)$, $G \in D^+(X)$. Moreover, we have a canonical isomorphism

$$Rf_* R\mathcal{H}om(Lf^* E, F) \simeq R\mathcal{H}om(E, Rf_* F) \tag{7.10.3}$$

for $E \in D^-(Y)$, $F \in D^+(X)$, $f : X \to Y$ a morphism of ringed spaces. This isomorphism implies (6.21.3) (but actually the proof of (7.10.3) uses (6.21.3), see 7.12.)

7.11. The functor $\mathcal{E}xt^i$. for $E \in D(X)$, $F \in D^+(X)$, for all integers i, define $\mathcal{E}xt^i(E, F) = H^i(R\mathcal{H}om(E, F))$. We shall see later that these sheaves are related to the global Ext^i

$$a(U \mapsto \mathrm{Ext}^i(E|U, F|U))$$

by a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(E,F)) \Rightarrow \operatorname{Ext}^{p+q}(E,F).$$

7.12. Appendix: Proof of (7.10.3). Let $f : X \to Y$, $E \in D^-(Y)$, $F \in D^+(X)$ as in (7.10.3). We may assume $E \in K^-(Y)$, $F \in K^+(X)$ and E^i flat, F^i injective for all i. Then (7.10.3) becomes

$$f_* \mathcal{H}om^\bullet(f^*E, F) \simeq R\mathcal{H}om(E, f_*F)$$

by 7.13 below. We claim that the composition of morphisms in $D(Y)$

$$f_* \mathcal{H}om^\bullet(f^*E, F) \to \mathcal{H}om^\bullet(E, f_*F) \to R\mathcal{H}om(E, f_*F)$$

provides such an isomorphism, where the second map is the canonical map while the first map is the isomorphism induced by the canonical isomorphisms in $C(Ab)$

$$\begin{aligned}\Gamma(V, f_* \mathcal{H}om^\bullet(f^*E, F)) &= \operatorname{Hom}^\bullet((f^*E)|f^{-1}(V), F|f^{-1}(V)) \\ &= \operatorname{Hom}^\bullet(f^*(E|V), F|f^{-1}(V)) \xrightarrow{\sim} \operatorname{Hom}^\bullet(E|V, f_*(F|f^{-1}(V))) \\ &= \operatorname{Hom}^\bullet(E|V, (f_*F)|V) = \Gamma(V, \mathcal{H}om^\bullet(E, f_*F))\end{aligned}$$

for $V \subset Y$ open. Take a quasi-isomorphism $f_*F \to G$ with $G \in C^+(Y)$ and G^i injective for all i. The claim is then equivalent to that the composition of morphisms in $C(Y)$

$$f_* \mathcal{H}om^\bullet(f^*E, F) \xrightarrow{\sim} \mathcal{H}om^\bullet(E, f_*F) \to \mathcal{H}om^\bullet(E, G)$$

is a quasi-isomorphism. For this, it suffices to show for every open $V \subset Y$, the induced map of complexes of sections on Y

$$\operatorname{Hom}^\bullet(f^*(E|V), F|f^{-1}(V)) \to \operatorname{Hom}^\bullet(E|V, G|V)$$

is a quasi-isomorphism, that is, for all n,

$$H^n \operatorname{Hom}^\bullet(f^*(E|V), F|f^{-1}(V)) \to H^n \operatorname{Hom}^\bullet(E|V, G|V)$$

is an isomorphism. Since $F|f^{-1}(V)$ and $G|V$ are injective modules, the last map corresponds to the canonical map

$$\operatorname{Hom}_{D(X)}(f^*(E|V), F|f^{-1}(V)[n]) \to \operatorname{Hom}_{D(Y)}(E|V, f_*(F|f^{-1}(V))[n])$$

by 7.4 and 7.5, which is an isomorphism by (6.21.3).

Lemma 7.13. *Let $E \in Mod(X)$ be flat, $F \in Mod(X)$ be injective. Then $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ is injective.*

Proof. For all $M \in Mod(X)$, we have, by the definition of $M \otimes E$, isomorphisms

$$\mathrm{Hom}(M \otimes E, F) \xrightarrow{\sim} \mathrm{Hom}(M, \mathcal{H}om(E, F)),$$

which are functorial. Hence $\mathrm{Hom}(\bullet, \mathcal{H}om(E, F)) \simeq \mathrm{Hom}(\bullet \otimes E, F)$ is an exact functor, and the result follows. $\square$

8 Čech Cohomology

8.1. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of a topological space X, i.e. $U_i \subset X$ open for all $i \in I$ and $X = \cup_{i \in I} U_i$, and let F be a presheaf of abelian groups on X. Define the Čech complex $\check{C}(\mathcal{U}, F)$ as follows. Let

$$\check{C}^n(\mathcal{U}, F) = \prod_{(i_0, \cdots, i_n) \in I^{n+1}} F(U_{i_0 \cdots i_n}), \; U_{i_0 \cdots i_n} = \bigcap_{j=0}^{n} U_{i_j}.$$

An element $a \in \check{C}^n(\mathcal{U}, F)$ is called an *n-cochain of $\mathcal{U}$ with values in F*, and is written as

$$(i_0, \cdots, i_n) \mapsto a(i_0, \cdots, i_n) \in F(U_{i_0 \cdots i_n})$$

(or $(a_{i_0 \cdots i_n})$). Define the differential $d : \check{C}^n(\mathcal{U}, F) \to \check{C}^{n+1}(\mathcal{U}, F)$ by

$$(da)(i_0, \cdots, i_{n+1}) = \sum_{j=0}^{n+1} (-1)^j a(i_0, \cdots, \widehat{i_j}, \cdots, i_{n+1})|U_{i_0 \cdots i_{n+1}}. \tag{8.1.1}$$

Then $d \circ d = 0$ and we get a cochain complex of abelian groups

$$\check{C}(\mathcal{U}, F) = (\check{C}^0(\mathcal{U}, F) \to \check{C}^1(\mathcal{U}, F) \to \cdots).$$

Define the ith *Čech cohomology group* of $\mathcal{U}$ with values in F to be $\check{H}^i(\mathcal{U}, F) = H^i \check{C}(\mathcal{U}, F)$.

8.2. Sheafification For $V \subset X$ open, $\mathcal{U} \cap V = (U_i \cap V)_{i \in I}$ is an open covering of V. Then $V \mapsto \check{C}(\mathcal{U} \cap V, F|V)$ defines a complex of presheaves, and the associated complex of sheaves is denoted by $\check{\mathcal{C}}(\mathcal{U}, F)$. (If F is a sheaf, then $V \mapsto \check{C}^n(\mathcal{U} \cap V, F)$ defines a sheaf itself, and thus $\check{\mathcal{C}}^n(\mathcal{U}, F)(V) = C^n(\mathcal{U} \cap V, F)$, for all n.) The natural morphism $\varepsilon : F(V) \to \check{C}(\mathcal{U} \cap V, F)$ of complexes of abelian groups induces a morphism $\varepsilon : F \to \check{\mathcal{C}}(\mathcal{U}, F)$ of complexes of presheaves.

Theorem 8.3. *Let F be a sheaf on X. Then $\varepsilon : F \to \check{\mathcal{C}}(\mathcal{U}, F)$ is a quasi-isomorphism.*

Proof. It is enough to check that $F_x \to \check{\mathcal{C}}(\mathcal{U}, F)_x$ is a quasi-isomorphism for all $x \in X$, and, in turn, enough to check that for all $x \in X$, there exists an open neighborhood V of x such that $F(V) \to \check{\mathcal{C}}(\mathcal{U}, F)(V)$ is a quasi-isomorphism. To show the latter, we take $V \subset U_i$ for some $i \in I$, and apply the following lemma with $X, \mathcal{U}, F$ replaced respectively by $V, \mathcal{U} \cap V, F|V$. □

Lemma 8.4. *Suppose there exists $i \in I$ such that $U_i = X$. Then the map $\varepsilon : F(X) \to \check{C}(\mathcal{U}, F)$ is a homotopy equivalence.*

Proof. The morphism of complexes $\alpha : \check{C}(\mathcal{U}, F) \to F(X)$ defined by

$$\begin{aligned} \alpha^0 : \check{C}^0(\mathcal{U}, F) &\to F(X) \\ a &\mapsto a(i) \end{aligned}$$

satisfies $\alpha\varepsilon = \mathrm{Id}_{F(X)}$. To conclude the proof, we use the "canonical homotopy operator" k defined by

$$\begin{aligned} k^n : \check{C}^n(\mathcal{U}, F) &\to \check{C}^{n-1}(\mathcal{U}, F) \\ a &\mapsto k^n a, \end{aligned}$$

where

$$(k^n a)(i_0, \cdots, i_{n-1}) = a(i, i_0, \cdots, i_{n-1})$$

(this is well defined since $U_{ii_0\cdots i_{n-1}} = U_{i_0\cdots i_{n-1}}$). It remains to check

$$\mathrm{Id}_{\check{C}(\mathcal{U},F)} - \varepsilon\alpha = kd + dk. \tag{8.4.1}$$

In degree 0, (8.4.1) holds since

$$(k^1 d^0 a)(i_0) = (d^0 a)(i, i_0) = a(i_0) - a(i)|U_{i_0} = (\mathrm{Id}_{\check{C}^0(\mathcal{U},F)}\, a)(i_0) - (\varepsilon^0 \alpha^0 a)(i_0)$$

for all $a \in \check{C}^0(\mathcal{U}, F)$. In degree $n > 0$, (8.4.1) holds since

$$\begin{aligned} (k^{n+1} d^n a)(i_0, \cdots, i_n) &= (d^n a)(i, i_0, \cdots, i_n) \\ &= a(i_0, \cdots, i_n) + \sum_{j=0}^{n} (-1)^{j+1} a(i, i_0, \cdots, \widehat{i_j}, \cdots, i_n)|U_{i_0\cdots i_n} \\ &= (\mathrm{Id}_{\check{C}^n(\mathcal{U},F)}\, a)(i_0, \cdots, i_n) - (d^{n-1} k^n a)(i_0, \cdots, i_n) \end{aligned}$$

for all $a \in \check{C}^n(\mathcal{U}, F)$. □

Definition 8.5. An n-cochain $a \in \check{C}^n(\mathcal{U}, F)$ is called *alternate* if

(i) $a(i_0, \cdots, i_n) = 0$ if there exists $j < k$ such that $i_j = i_k$;

(ii) $a(i_{\sigma(0)}, \cdots, i_{\sigma(n)}) = \varepsilon(\sigma) a(i_0, \cdots, i_n)$, for all $\sigma \in \mathrm{Aut}(\{0, \cdots, n\}) = S_{n+1}$ ($\varepsilon(\sigma)$ denotes the signature of σ) if all i_j are distinct.

Let $\check{C}^{\mathrm{alt},n}(\mathcal{U}, F)$ be the subgroup of alternate n-cochains in $\check{C}^n(\mathcal{U}, F)$.

Lemma 8.6. *We have*

$$d\check{C}^{\mathrm{alt},n}(\mathcal{U}, F) \subset \check{C}^{\mathrm{alt},n+1}(\mathcal{U}, F).$$

Proof. Take $a \in \check{C}^{\mathrm{alt},n}(\mathcal{U}, F)$. Then, trivially, da satisfies (i) of 8.5. To show da satisfies (ii) of 8.5, it suffices to do so in the case when $\sigma \in \mathrm{Aut}(\{0, \cdots, n+1\})$ is of the form $\sigma = (j, j+1)$. Then,

$$\begin{aligned}
&(da)(i_0, \cdots, i_{j+1}, i_j, \cdots, i_{n+1}) \\
=&\sum_{k=0}^{j-1} (-1)^k a(i_0, \cdots, \widehat{i_k}, \cdots, i_{j+1}, i_j, \cdots, i_{n+1}) \\
&\quad + (-1)^j a(i_0, \cdots, \widehat{i_{j+1}}, i_j, \cdots, i_{n+1}) + (-1)^{j+1} a(i_0, \cdots, i_{j+1}, \widehat{i_j}, \cdots, i_{n+1}) \\
&\quad + \sum_{k=j+2}^{n+1} (-1)^k a(i_0, \cdots, i_{j+1}, i_j, \cdots, \widehat{i_k}, \cdots, i_{n+1}) \\
=&\sum_{k=0}^{j-1} (-1)^{k+1} a(i_0, \cdots, \widehat{i_k}, \cdots, i_j, i_{j+1}, \cdots, i_{n+1}) \\
&\quad + (-1)^j a(i_0, \cdots, i_j, \widehat{i_{j+1}}, \cdots, i_{n+1}) + (-1)^{j+1} a(i_0, \cdots, \widehat{i_j}, i_{j+1}, \cdots, i_{n+1}) \\
&\quad + \sum_{k=j+2}^{n+1} (-1)^{k+1} a(i_0, \cdots, i_j, i_{j+1}, \cdots, \widehat{i_k}, \cdots, i_{n+1}) \\
=&- (da)(i_0, \cdots, i_j, i_{j+1}, \cdots, i_{n+1}),
\end{aligned}$$

as desired. $\square$

Thus we get a complex $\check{C}^{\mathrm{alt}}(\mathcal{U}, F) \subset \check{C}(\mathcal{U}, F)$, which is called the alternate Čech complex. We have a commutative diagram of canonical homomorphisms:

$$\begin{tikzcd}
 & \check{C}^{\mathrm{alt},0}(\mathcal{U}, F) \arrow[r] \arrow[d, equal] & \check{C}^{\mathrm{alt},1}(\mathcal{U}, F) \arrow[r] \arrow[d] & \cdots \\
F(X) \arrow[ur] \arrow[r] & \check{C}^0(\mathcal{U}, F) \arrow[r] & \check{C}^1(\mathcal{U}, F) \arrow[r] & \cdots
\end{tikzcd}$$

If F is a sheaf, we sheafify it as in 8.2 and get a commutative diagram:

$$\begin{tikzcd} & \check{\mathcal{C}}^{\mathrm{alt}}(\mathcal{U}, F) \arrow[d] \\ F \arrow[ur, "\varepsilon"] \arrow[r, "\varepsilon"'] & \check{\mathcal{C}}(\mathcal{U}, F) \end{tikzcd} \tag{8.6.1}$$

Theorem 8.7. *The morphism $\varepsilon : F \to \check{\mathcal{C}}^{\mathrm{alt}}(\mathcal{U}, F)$ is a quasi-isomorphism.*

Proof. Use the same homotopy operator as in the proof of 8.3. □

Remark 8.8. (1). The alternate complex is more economical. Suppose $<$ is a total order on I. Then the restriction maps

$$\begin{array}{ccc} \check{C}^{\mathrm{alt},n}(\mathcal{U}, F) & \to & \prod_{\{i_0<\cdots<i_n\}} F(U_{i_0\cdots i_n}) \\ a & \mapsto & (a(i_0, \cdots, i_n)) \end{array}$$

defines an isomorphism of $\check{C}^{\mathrm{alt},n}(\mathcal{U}, F)$ to the complex $\check{C}(\mathcal{U}, <; F)$ defined by $\check{C}(\mathcal{U}, <; F)^n = \prod_{\{i_0<\cdots<i_n\}} F(U_{i_0\cdots i_n})$, with differential given by (8.1.1). Using this identification, we have, for example:

(i) For $\mathcal{U} = \{X\}$,

$$\check{C}^{\mathrm{alt}}(\mathcal{U}, F) = (\ \underset{\substack{\| \\ F(X)}}{\check{C}^{\mathrm{alt},0}(\mathcal{U}, F)} \to 0\ ).$$

(ii) For $\mathcal{U} = \{U, V\}$,

$$\check{C}^{\mathrm{alt}}(\mathcal{U}, F) = (\ \underset{\substack{\| \\ F(U) \oplus F(V)}}{\check{C}^{\mathrm{alt},0}(\mathcal{U}, F)} \longrightarrow \underset{\substack{\| \\ F(U \cap V)}}{\check{C}^{\mathrm{alt},1}(\mathcal{U}, F)} \to 0\ ).$$

(iii) For $\mathcal{U} = \{U_0, \cdots, U_n\}$,

$$\check{C}^{\mathrm{alt}}(\mathcal{U}, F) =$$
$$(\bigoplus_{i=0}^{n} F(U_i) \to \bigoplus_{i<j} F(U_i \cap U_j) \to \cdots \to F(U_0 \cap \cdots \cap U_n) \to 0).$$

(2). The morphisms ε in (8.6.1) are quasi-isomorphisms, so the vertical map of the triangle is a quasi-isomorphism, too. In fact, one can show that $\check{C}^{\mathrm{alt}}(\mathcal{U}, F) \to \check{C}(\mathcal{U}, F)$ is a quasi-isomorphism. (Exercise, see Godement's book [G] or Serre's paper [S].)

8.9. Suppose X has a ringed space structure and F is a sheaf of $\mathcal{O}_X$-modules. Then $\check{\mathcal{C}}(\mathcal{U}, F)$ and $\check{\mathcal{C}}^{\mathrm{alt}}(\mathcal{U}, F)$ are $\mathcal{O}_X$-modules in a natural way.

In what follows we shall work only with $\check{\mathcal{C}}^{\mathrm{alt}}(\mathcal{U}, F)$ (and often drop the superscript "alt").

Corollary 8.10. *Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{U}$ be an open covering of X. For a sheaf F of $\mathcal{O}_X$-modules, there is a natural morphism $\gamma : \check{C}(\mathcal{U}, F) \to R\Gamma(X, F)$ in $D(Ab)$ defined by the commutative diagram:*

$$
\begin{tikzcd}
\check{C}(\mathcal{U}, F) = \Gamma(X, \check{\mathcal{C}}(\mathcal{U}, F)) \arrow[r] \arrow[dr, dashed, "\gamma"'] & R\Gamma(X, \check{\mathcal{C}}(\mathcal{U}, F)) \\
& R\Gamma(X, F) \arrow[u, "\simeq"', "R\Gamma(X,\varepsilon)"]
\end{tikzcd}
$$

This induces homomorphisms $\check{H}^n(\mathcal{U}, F) \to H^n(X, F)$ for all n.

8.11. Let $F \in C^+(X)$. We define a bicomplex $\check{C}(\mathcal{U}, F)^{\bullet\bullet}$ of abelian groups as follows: let $\check{C}(\mathcal{U}, F)^{pq} = \check{C}^q(\mathcal{U}, F^p)$ and let $d'^{pq} : \check{C}^q(\mathcal{U}, F^p) \to \check{C}^q(\mathcal{U}, F^{p+1})$ be induced by $F^p \to F^{p+1}$,

$$d''^{pq} = (-1)^p d^q_{\check{C}(\mathcal{U}, F^p)} : \check{C}^q(\mathcal{U}, F^p) \to \check{C}^{q+1}(\mathcal{U}, F^p).$$

Let $\check{C}(\mathcal{U}, F)^{\bullet} = \mathbf{s}(\check{C}(\mathcal{U}, F)^{\bullet\bullet}) \in C(Ab)$. We can do the same for $\check{\mathcal{C}}$ and get $\check{\mathcal{C}}(\mathcal{U}, F)^{\bullet} \in C(X)$. The canonical morphism $\varepsilon : F \to \check{\mathcal{C}}(\mathcal{U}, F)^{\bullet}$ is still a quasi-isomorphism.

Theorem 8.12 (Leray). *Let $X, \mathcal{U}, F$ be as in 8.10, $\mathcal{U} = (U_i)_{i \in I}$. Suppose that for every nonempty finite subset J of I and every $q > 0$, $H^q(U_J, F) = 0$, where U_J is the intersection of the U_j's for $j \in J$. Then the map γ in 8.10 is an isomorphism.*

The proof is easy using 8.11, see Ex. 26.

Exercises

1. Show that in an additive category $\mathcal{A}$, for any two objects A, B of $\mathcal{A}$, there is a natural isomorphism $A \oplus B \xrightarrow{\sim} A \times B$.

2. Let A be a commutative ring and $\mathcal{A}$ be the category of pairs (E, E') of an A-module E and a submodule E'. Show that $\mathcal{A}$ is additive, but not abelian (hint : give an example of a morphism which is both an epimorphism and a monomorphism, but is not an isomorphism).

3. Show that in an abelian category $\mathcal{A}$ amalgamated sums (push-outs) and fibered products (pull-backs) exist. Show that the push-out of a monomorphism is a monomorphism, and the pull-back of an epimorphism is an epimorphism. Show that if (u', u, u'') is a map of short exact sequences, then the first square is cocartesian (resp. the last square is cartesian) if and only if u'' (resp. u') is an isomorphism.

4. Let $0 \to L' \to L \to L'' \to 0$ be a short exact sequence of complexes of an abelian category $\mathcal{A}$ and let M be the cone of u. Let $\varphi : M \to L''$ be the natural map, and $\delta : H^i L'' \to H^{i+1} L'$ be the coboundary map. Check that :

$$\delta H^i(\varphi) = -H^i(pr_1),$$

where $pr_1 : M \to L[1]$ is the first projection.

5. Let (L, d_1, d_2) be a *naïve* bicomplex ($d_1 d_2 = d_2 d_1$) of an abelian category $\mathcal{A}$. Let L_1 be the bicomplex with $d' = d_1$ and $d'' = (-1)^i d_2$ on $L^{i,\cdot}$ and let L_2 be the bicomplex with $d'' = d_2$ and $d' = (-1)^i d_1$ on $L^{\cdot,i}$. Define a canonical isomorphism between L_1 and L_2. (Hint : see Cartan-Eilenberg, and SGA 4 XVII for generalizations).

6. Let $u : K \to L$ be a map of bicomplexes of an abelian category $\mathcal{A}$. Assume K and L are biregular. Let $'H^n$ (resp. $''H^n$) denote the n-th column (resp. row) of cohomology. Show that if $'H^n(u)$ (resp. $''H^n(u)$) is a quasi-isomorphism for all n, then $\mathbf{s}u : \mathbf{s}K \to \mathbf{s}L$ is a quasi-isomorphism. (Hint : first reduce to the case where K and L are concentrated in bounded vertical (resp. horizontal) strips, then make a dévissage using the canonical truncations.) Show by an example that the conclusion becomes false if one drops the assumption of biregularity.

7. Let $\mathcal{P}$ be an additive full subcategory of an abelian category $\mathcal{A}$ such that all short exact sequences of $\mathcal{P}$ split (e. g. A the category of modules over a commutative ring A and $\mathcal{P}$ the category of projective modules over A). Let $u : K \to L$ be a quasi-isomorphism of complexes of $\mathcal{A}$ such that the

components of K and L belong to $\mathcal{P}$. Show that if K and L are bounded above (resp. bounded below), then u is a homotopy equivalence.

8. Let $\mathcal{A}$ be an additive category. Let $f : K \to L$ be a map of $K(\mathcal{A})$. Show that there exists a factorization $f = gf'$ in $C(\mathcal{A})$ with g an isomorphism in $K(\mathcal{A})$ and f' injective and split in each degree. Show that there exists a factorization $f = f''g$ in $C(\mathcal{A})$ with g an isomorphism in $K(\mathcal{A})$ and f'' surjective and split in each degree.

9. Let $\mathcal{A}$ be an additive category. Show that $K(\mathcal{A})$ is deduced from $C(\mathcal{A})$ by inverting homotopy equivalences (hint : use the *cyclinder object* $K = Cyl(L)$ defined by $K^i = L^i \oplus L^i \oplus L^{i+1}$ with differential d_K given by the matrix $d_K = \begin{pmatrix} d & 0 & Id \\ 0 & d & Id \\ 0 & 0 & -d \end{pmatrix}$ and the homotopy equivalence $s : K \to L$ given by $s(x, y, z) = -y + x$).

10. Check axiom (TR3) (*rotation*) in $K(\mathcal{A})$ ($\mathcal{A}$ an additive category).

Let $0 \to L \to M \to N \to 0$ be a *semi-split* exact sequence of $C(\mathcal{A})$, which means that for each i, $M^i = L^i \oplus N^i$ and the sequence is given by the natural injection and projection. Let $d_M = \begin{pmatrix} d_L & h \\ 0 & d_N \end{pmatrix}$ be the differential. Show that the triangle $L \to M \to N \to L[1]$, where the last map is given by h, is distinguished.

11. (a) Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

be a commutative square in $C(\mathcal{A})$ ($\mathcal{A}$ an additive category) and $w : Z = C(f) \to Z' = C(f')$ the map induced on the cones. Show that if f and f' are homotopy equivalences, then so is w.

(b) Show that if $f : L \to M$ is a morphism of $C(\mathcal{A})$, the cone of f is homotopically trivial if and only if f is a homotopy equivalence.

(Hint : use the structure of triangulated category of $K(\mathcal{A})$.)

12. Let $\mathcal{A}$ be an abelian category. Let $D^{\leq 0}(\mathcal{A})$ (resp. $D^{\geq 0}(\mathcal{A})$) be the full subcategory of $D(\mathcal{A})$ consisting of complexes K such that $H^i(K) = 0$ for $i > 0$ (resp. $i < 0$). Let $D^{\leq n}(\mathcal{A}) = D^{\leq 0}(\mathcal{A})[-n]$ (resp. $D^{\geq n}(\mathcal{A}) = D^{\geq 0}(\mathcal{A})[-n]$. Show the following properties :

(1) $D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A}) = \mathcal{A}$.

(2) For any $L \in D(\mathcal{A})$ there exists a distinguished triangle $L' \to L \to L'' \to$ with $L' \in D^{\leq -1}(\mathcal{A})$ and $L'' \in D^{\geq 0}(\mathcal{A})$.

(3) Show that, for any $K \in D^{\leq -1}(\mathcal{A})$ and $L \in D^{\geq 0}(\mathcal{A})$, $\mathrm{Hom}(K, L) = 0$.

13. Let $\mathcal{A}$ be an abelian category. For K, L in $\mathcal{A}$ define *two* natural isomorphisms from $\mathrm{Hom}_{D(\mathcal{A})}(K, L[1])$ to the usual group $\mathrm{Ext}^1(K, L)$ and compare them. More generally, discuss the *Yoneda description* of $\mathrm{Hom}(K, L[n])$ in terms of classes of exact sequences $E = (0 \to L \to E^0 \to \cdots \to E^{n-1} \to K \to 0)$.

14. Let D be a triangulated category. A *cross* in D is a diagram

(1)
$$\begin{tikzcd}
 & A' \arrow[d] & & \\
A \arrow[ur, "a"] \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl, "c"] & {} \\
 & C' \arrow[d] & & \\
 & {} & &
\end{tikzcd}$$

of D), where the triangles are commutative and the row and the column are distinguished triangles. Such a cross can be considered as an incomplete octahedron

(2)
$$\begin{tikzcd}
 & A' \arrow[d] \arrow[r, dotted] & M \arrow[r, dotted] \arrow[d, dotted] & {} \\
A \arrow[ur, "a"] \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[dl, "c"] & {} \\
 & C' \arrow[d] & & \\
 & {} & &
\end{tikzcd}$$

(cf. [BBD, 1.1.7.1]). Show that in general it is not possible to extend a cross (1) to an octahedron (2) (hint : consider cones on a and c).

15. Let $\mathcal{A}$ be an abelian category. Let $D^{[0,1]}(\mathcal{A})$ be the full subcategory of $D(\mathcal{A})$ consisting of complexes K such that $H^i(K) = 0$ for $i \notin [0, 1]$. Construct an equivalence of categories from $D^{[0,1]}(\mathcal{A})$ to the category C of triples (A, B, a) where A, B are objects of $\mathcal{A}$ and $a \in \mathrm{Ext}^2(A, B)$.

16. Let $\mathcal{A}$ be an abelian category, $[a,b]$ an interval of $\mathbb{Z}$ and L a complex of $\mathcal{A}$. Define quasi-isomorphisms

$$\tau_{\leq b}L/\tau_{\leq a-1}L \to \tau_{[a,b]}L \to \mathrm{Ker}(\tau_{\geq a}L \to \tau_{\geq b+1}L).$$

17. Let $\mathcal{A}$ be an abelian category and $u : E \to F$ a morphism of $\mathcal{A}$. Give a necessary and sufficient condition for u to have a kernel (resp. cokernel) in the category $K(\mathcal{A})$.

18. Consider a 9-diagram in $C(\mathcal{A})$ ($\mathcal{A}$ an abelian category), i. e. a short exact sequence of short exact sequences of complexes. Show that the horizontal and vertical boundary operators of the corresponding long exact sequences of cohomology anti-commute.

19. (Verdier) In a triangulated category, show that any commutative square can be completed into a 9-diagram ("distinguished triangle of distinguished triangles"), and that the corresponding degree 1 arrows anti-commute. (hint : use the octahedron axiom, see [BBD]). Show that a 9-diagram in $C(\mathcal{A})$ gives rise to a 9-diagram in $D(\mathcal{A})$ and recover the result of exercise 3.

20. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $\mathcal{A}$ having enough injectives, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. We say that F is of *(right) finite cohomological dimension* if there exists an integer d such that $R^qF(E) = 0$ for all $E \in \mathcal{A}$ and all $q > d$ (the smallest such d is then called the *(right) cohomological dimension* of F. Show that if F is of finite cohomological dimension, then $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ can be extended to a (triangulated) functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$, which is the right derived functor of $F : K(\mathcal{A}) \to K(\mathcal{B})$, and sends $D^*(\mathcal{A})$ to $D^*(\mathcal{B})$ for $* = -$ or b, cf. [RD, I 4.6, p. 42]. Examine generalizations and variants : (a) instead of assuming that $\mathcal{A}$ has enough injectives, assume the existence of an additive subcategory $\mathcal{A}'$ of $\mathcal{A}$ right adapted to F ; (b) left derived functors.

The rings of the ringed spaces considered below are assumed to be commutative.

21. Let X be a ringed space. Let E be a bounded above complex of $\mathcal{O}_X$-modules which are locally free of finite type, and let $F \in C^+(X)$. Show that the canonical map $\mathcal{H}om^{\cdot}(E,F) \to R\mathcal{H}om(E,F)$ is an isomorphism. Let $\check{E} := \mathcal{H}om^{\cdot}(E,\mathcal{O}_X)$. Deduce a canonical isomorphism $\check{E} \otimes^L F \xrightarrow{\sim} R\mathcal{H}om(E,F)$.

22. Let $f : X \to Y$ be a morphism of ringed spaces. Construct the following canonical isomorphisms :

$$Lf^*(E \otimes^L F) \simeq Lf^*E \otimes^L Lf^*F \tag{1}$$

$(E, F \in D^-(Y))$,

$$\text{(2)} \qquad R\mathcal{H}om(E \otimes^L F, G) \simeq R\mathcal{H}om(E, R\mathcal{H}om(F, G))$$

$(E, F \in D^-(Y), G \in D^+(Y)$,

$$\text{(3)} \qquad Rf_* R\mathcal{H}om(Lf^*E, F) \simeq R\mathcal{H}om(E, Rf_*F)$$

$(E \in D^-(Y), F \in D^+(X))$. Deduce from (3) isomorphisms

$$RHom(Lf^*E, F) \simeq RHom(E, Rf_*F),$$

$$Hom(Lf^*E, F) \simeq Hom(E, Rf_*F).$$

((2) is called the *Cartan* isomorphism, (3) the *trivial duality* isomorphism.)

23. Let X be a ringed space. Let $[a, b]$ be an interval of $\mathbb{Z}$. A complex $E \in D^-(X)$ is said to be of *tor-amplitude in* $[a, b]$ if E is isomorphic, in $D(X)$, to a complex concentrated in degrees in $[a, b]$ and having flat components. Show that this condition is equivalent to the following : for every $F \in Mod(X)$, $\mathcal{H}^q(E \otimes^L F) = 0$ for $q \notin [a, b]$.

We say that $E \in D^-(X)$ is of *finite tor-amplitude* (or *finite tor-dimension*) if there is an interval $[a, b]$ such that E is of tor-amplitude in $[a, b]$. Show that if $E' \to E \to E'' \to$ is a distinguished triangle of $D^-(X)$ and if two of its vertices are of finite tor-amplitude, so is the third one.

24. Let X be a ringed space, and $L \in D^+(X)$. Denote by L_0 the underlying complex of abelian sheaves. Show that the natural map $R\Gamma(X, L_0) \to R\Gamma(X, L)$ is an isomorphism in $D^+(Ab)$.

25. (*Verdier*) Let F be an $\mathcal{O}_X$-module on a ringed space X. Show that the following conditions are equivalent :

(i) F is flasque ;

(ii) for any space U étale over X (i. e. equipped with a continuous map to X which is a local homeomorphism), any open cover $\mathcal{U}$ of U and any $q > 0$, $\check{H}^q(\mathcal{U}, F) = 0$.

(Hints : for (i) ⇒ (ii), show first that the restriction of F to U is flasque ; for (ii) ⇒ (i), glue two copies of X along the given open subset.)

26. Let F be an $\mathcal{O}_X$-module on a ringed spaced X and let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. We assume that for every nonempty finite subset J of I and every $q > 0$, $H^q(U_J, F) = 0$, where U_J is the intersection of the U_j's for $j \in J$. Show that the canonical map

$$\check{H}^q(\mathcal{U}, F) \to H^q(X, F)$$

is an isomorphism (*Leray's theorem)*.

27. Let $X = \mathrm{Spec}(A)$ be an affine scheme and let $0 \to E \to F \to G \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. Show that if E and G are quasi-coherent, so is F (hint : define a canonical homomorphism $\Gamma(X, L)\tilde{} \to L$ for L an $\mathcal{O}_X$-module, which is an isomorphism if and only if L is quasi-coherent, and use Serre's theorem).

28. Let d be a nonnegative integer. A space X is said to be *of cohomological dimension* $\leq d$ if $H^q(X, F) = 0$ for all abelian sheaves F on X and all $q > d$ (in other words, the functor $\Gamma(X, -)$ is of (right) cohomological dimension $\leq d$ on the category of abelian sheaves). Prove the following theorem of Grothendieck : *Let X be a noetherian space of finite dimension d. Then X is of cohomological dimension $\leq d$.* Proceed in the following steps :

(1) Let X be a noetherian space, $(F_\lambda)_{\lambda \in L}$ a filtering inductive system of abelian sheaves on X, and F its inductive limit. Show that, for any open subset U of X, the natural map

$$\mathrm{colim}\, F_\lambda(U) \to F(U)$$

is an isomorphism. Deduce that any filtering inductive limit of flasque sheaves is flasque, and that for any filtering inductive system (F_λ) as above, and any $q \in \mathbb{Z}$, the natural map

$$\mathrm{colim}\, H^q(X, F_\lambda) \to H^q(X, F)$$

is an isomorphism.

(2) Let X be a noetherian space and F be a $\mathbb{Z}$-submodule of the constant sheaf $\mathbb{Z}_X$ (whose sections over an open subset U are the locally constant functions from U to $\mathbb{Z}$). Let $u : F \to \mathbb{Z}_X$ be the inclusion. For $x \in X$, let $n(x)$ be the nonnegative integer such that $\mathrm{Im}(u_x) = n(x)\mathbb{Z}$. Let $U = \{x \in X, n(x) \neq 0\}$, and for $n \geq 1$, $U_n = \{x \in X, 1 \leq n(x) \leq n\}$. Show that the U_n's form an increasing sequence of open subsets of X and that there exists an n such that $U = U_n$. Deduce that there exists a finite filtration of F of the form

$$0 = L_0 \subset \cdots \subset L_n = F,$$

where, for $i > 0$, L_i/L_{i-1} is the extension by zero of the constant sheaf $\mathbb{Z}$ on a locally closed subset of X.

(3) Let $i : Y \to X$ be a closed subset of a space X. Show that, for any abelian sheaf F on Y, $H^q(X, i_*F) \simeq H^q(Y, F)$.

(4) Prove the theorem by induction on d.

(a) By induction on the number of irreductible components, show that to prove the theorem, one may assume that X is irreducible. In particular, show that the theorem holds for $d = 0$.

(b) If $a = (s_i)$ is a finite set of sections of F (on open subsets U_i of X), let F_a be the subsheaf of F generated by these sections, i. e. by definition, the image of the corresponding map $\oplus (j_i)_! \mathbb{Z}_{U_i} \to F$ (where $j_i : U_i \to X$ is the inclusion). Show that F is the inductive limit of the F_a, a running through the (filtering ordered) set A of such finite sets. Using (1), show that it is enough to prove the theorem for the F_a's, then for the F_a's where a consists of a single element, and finally for the (abelian) subsheaves of the constant sheaf $\mathbb{Z}_X$.

(c) Using (2), show that it is enough to prove the theorem for sheaves which are constant on some locally closed subset of X and extended by zero. Conclude, using the irreducibility of X and the induction hypothesis.

29. Prove the following converse theorem of Serre for affine schemes : *Let X be a quasi-compact and separated scheme. Assume that for every quasi-coherent sheaf F on X and every $q > 0$, $H^q(X, F) = 0$. Then X is affine*. Proceed in the following steps.

(1) Show that X is affine if and only if there exists a finite set $(f_i)_{1 \leq i \leq r}$, $f_i \in A = \Gamma(X, \mathcal{O}_X)$, such that X_{f_i} is affine for all i and $\sum f_i A = A$. (Hint : for the "if" part, show first that, for $f \in A$, $\Gamma(X, \mathcal{O})_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{O})$, and consider the canonical map $X \to \operatorname{Spec} A$ given by the identity of A.)

(2) Let X be a quasi-compact, Kolmogoroff space (i. e. such that for any two distinct points x, y, there exists an open subset containing one of the two points and not the other one). Show that any nonempty closed subset of X contains a closed point.

(3) Using the vanishing assumption, prove that for every closed point x of X there exists $s \in A = \Gamma(X, \mathcal{O})$ such that $s(x) = 1$.

(4) Using (2), deduce that there exists a finite number of global sections f_i $(1 \leq i \leq r)$ of $\mathcal{O}_X$ such that X_{f_i} is affine and the union of the X_{f_i}'s is X. Using the vanishing assumption again, show that the f_i's generate A, and conclude.

30. Let $f : X \to Y$ be a morphism of schemes, with X noetherian of finite Krull dimension.

(1) Using Grothendieck's theorem, show that $f_* : \operatorname{Mod}(X) \to \operatorname{Mod}(Y)$ is of (right) finite cohomological dimension.

(2) Extend the natural map $E \otimes f_* F \to f_*(f^* E \otimes F)$ $(E \in \operatorname{Mod}(Y)$,

$F \in \mathrm{Mod}(X)$) to a natural map

$$\varphi : E \otimes^L Rf_* F \to Rf_*(Lf^* E \otimes^L F)$$

for $E \in D^-(Y)$, $F \in D(X)$.

(3) Show that, if $\mathcal{H}^i(E)$ is quasi-coherent for all i and F is in $D^-(X)$, then φ is an isomorphism (*projection formula*).

Chapter 2

Cohomology of Affine and Projective Morphisms

1 Serre's Theorem on Affine Schemes

Theorem 1.1 (Serre). *Let X be an affine scheme and let F be a quasi-coherent sheaf on X. Then $H^q(X, F) = 0$, for all $q > 0$.*

Lemma 1.2. *Let $\mathcal{U} = (U_i)_{i \in I}$, $I = \{0, \cdots, N\}$, be a finite open covering of $X = \operatorname{Spec} A$ by principal open sets $U_i = X_{f_i}$, $f_i \in A$. Then $\check{H}^q(\mathcal{U}, F) = 0$ for all $q > 0$, $\check{H}^0(\mathcal{U}, F) = F(X)$.*

Proof. Let $F = \widetilde{M}$ where $M \in Mod(A)$. Then

$$\check{C}(\mathcal{U}, F) =$$
$$\left(\bigoplus_{i=0}^{N} F(U_i) \to \bigoplus_{i<j} F(U_i \cap U_j) \to \cdots \to F(U_0 \cap \cdots \cap U_N) \to 0\right).$$

Note that $F(X) = M$, $F(U_{i_0 \cdots i_q}) = M_{f_{i_0} \cdots f_{i_q}}$ We want to show that the sequence

$$0 \to M \to \bigoplus_{i=0}^{N} M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_0 \cdots f_N} \to 0$$

is exact. This follows from the following sublemma and I.8.4. □

Sublemma 1.2.1. *Let A, f_i be the same as in the lemma, $L \in C(A)$. Then L is acyclic if and only if for all $i \in I$, L_{f_i} is acyclic.*

Proof. For $g \in A$, A_g is a flat A-module, thus $H^q(L)_g = H^q(L_g)$. In particular, $H^q(L_{f_i}) = H^q(L)_{f_i}$. The "only if" part then becomes obvious, and for the "if" part, we only need to note that for $E \in Mod(A)$, $E_{f_i} = 0$ for all i implies $E = 0$ (because $E_{f_i} = \Gamma(X_{f_i}, \widetilde{E})$). $\square$

The following lemma which can be seen as a variant of the classical Cartan's lemma ([G], II.5.9.2), was communicated to L. Illusie by A. Ogus.

Lemma 1.3. *Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{B}$ be a basis of open sets of X such that for all $V \in \mathcal{B}$, V is quasi-compact, $\emptyset \in \mathcal{B}$, and if $U, V \in \mathcal{B}$, $U \cap V \in \mathcal{B}$. Let $\mathcal{C}$ be the full subcategory of $Mod(X)$ consisting of all the modules F such that for every $U \in \mathcal{B}$ and every finite open covering $\mathcal{U}$ of U by elements of $\mathcal{B}$, $\check{H}^q(\mathcal{U}, F) = 0$ for all $q > 0$. Then $\mathcal{C}$ is right adapted to the functors $\Gamma(U, -)$ for all $U \in \mathcal{B}$ (I.5.10). In particular, if $F \in \mathcal{C}$ and $U \in \mathcal{B}$, then the map $\Gamma(U, F) \to R\Gamma(U, F)$ is an isomorphism, i.e., $H^q(U, F) = 0$ for all $q > 0$.*

Proof of Theorem 1.1 using the lemmas. Let $X = \operatorname{Spec} A$,

$$\mathcal{B} = \{X_f,\ f \in A\}.$$

Since $X_{fg} = X_f \cap X_g$, $\mathcal{B}$ satisfies the conditions of 1.3. Let $U \in \mathcal{B}$ and $\mathcal{U} = \{U_0, \cdots, U_N\}$ be a finite open covering of U with $U_i \in \mathcal{B}$, $i = 1, \cdots, N$. Then $U = X_f$ for some $f \in A$, and, for all i, we can take $g_i \in A$ such that $U_i = X_{fg_i}$. We then have $\check{H}^q(\mathcal{U}, F|U) = 0$ for all $q > 0$. (Replace X by X_f and apply Lemma 1.2) Thus we can apply the conclusion of 1.3. Take $U = X_1$, we get $H^q(X, F) = 0$, for all $q > 0$. $\square$

Proof of Lemma 1.3. Obviously, $\mathcal{C}$ is an additive subcategory, so we only need to check the following conditions:

(i) For all $E \in Mod(X)$, there exists a monomorphism $E \to F$ with $F \in \mathcal{C}$.

(ii) If $F', F \in \mathcal{C}$, $F'' \in Mod(X)$ and

$$0 \to F' \to F \to F'' \to 0$$

is exact, then $F'' \in \mathcal{C}$.

(iii) If the above sequence is exact with $F', F, F'' \in \mathcal{C}$, then

$$0 \to \Gamma(U, F') \to \Gamma(U, F) \to \Gamma(U, F'') \to 0$$

is exact for all $U \in \mathcal{B}$.

For (i), it suffices to show that F is flasque implies $F \in \mathcal{C}$, that is, for every $U \in \mathcal{B}$ and every finite open covering $\mathcal{U}$ of U by elements of $\mathcal{B}$, $\check{H}^q(\mathcal{U}, F) = 0$ for all $q > 0$. We may assume $X \in \mathcal{B}$, $U = X$. It then suffices to note that the quasi-isomorphism $F \to \check{\mathcal{C}}(\mathcal{U}, F)$ induces a quasi-isomorphism $\Gamma(X, F) \to \Gamma(X, \check{\mathcal{C}}(\mathcal{U}, F)) = \check{C}(\mathcal{U}, F)$, since both F and

$$\check{\mathcal{C}}^n(\mathcal{U}, F) = (V \mapsto \prod_{i_0 < \cdots < i_n} F(V \cap U_{i_0 \cdots i_n})) = \prod j_{i_0 \cdots i_n *} j_{i_0 \cdots i_n}{}^* F$$

are flasque.

For (ii) and (iii), we need the following sublemma.

Sublemma 1.3.1. *Suppose*

$$0 \to F' \to F \to F'' \to 0$$

is exact with $F' \in \mathcal{C}$. Then for all $U \in \mathcal{B}$,

$$0 \to \Gamma(U, F') \to \Gamma(U, F) \to \Gamma(U, F'') \to 0$$

is exact.

Proof. We may assume $X \in \mathcal{B}$, $U = X$. We only need to show that $\Gamma(X, F) \to \Gamma(X, F'')$ is an epimorphism. Take $s \in \Gamma(X, F'')$. There exists a finite open covering $\mathcal{U} = (U_i)_{i \in I}$ with $U_i \in \mathcal{B}$, and $s_i \in \Gamma(U_i, F)$ such that $s_i \mapsto s|U_i$. Let $t_{ij} = s_j|U_{ij} - s_i|U_{ij}$, then $t_{ij} \in \Gamma(U_{ij}, F')$. Since $t_{jk} - t_{ik} + t_{ij} = 0$, $(t_{ij}) \in \check{Z}^1(\mathcal{U}, F')$. By assumption, $\check{H}^1(\mathcal{U}, F') = 0$, and so $t_{ij} \in \check{B}^1(\mathcal{U}, F')$. Hence there exists $(t_i) \in \mathcal{C}^0(\mathcal{U}, F')$ with $t_i \in \Gamma(U_i, F')$ such that $t_{ij} = t_j|U_{ij} - t_i|U_{ij}$. Now put $\sigma_i = s_i - t_i$. Then the σ_i's glue and give $\sigma \in \Gamma(X, F)$ satisfying $\sigma \mapsto s$. □

We continue the proof of 1.3.

Proof of (ii). Take U as in the definition of $\mathcal{C}$. Consider the following commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & F'(U) \arrow[r] \arrow[d] & F(U) \arrow[r] \arrow[d] & F''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \check{C}(\mathcal{U}, F') \arrow[r] & \check{C}(\mathcal{U}, F) \arrow[r] & \check{C}(\mathcal{U}, F'') \arrow[r] & 0
\end{tikzcd}$$

The first row is exact by the sublemma. For all n,

$$0 \to \check{C}^n(\mathcal{U}, F') \to \check{C}^n(\mathcal{U}, F) \to \check{C}^n(\mathcal{U}, F'') \to 0$$

is nothing but

$$0 \to \prod F'(U_{i_0\cdots i_n}) \to \prod F(U_{i_0\cdots i_n}) \to \prod F''(U_{i_0\cdots i_n}) \to 0,$$

which is exact by the sublemma since $U_{i_0\cdots i_n} \in \mathcal{B}$ and $F' \in \mathcal{C}$. Thus the second row of the above diagram is also exact. Finally, note that for $M \in Mod(U)$, $M(U) \to \check{C}(\mathcal{U}, M)$ is a quasi-isomorphism if and only if for all $q > 0$, $\check{H}^q(\mathcal{U}, M) = 0$. Now $F', F \in \mathcal{C}$, the first two columns of the diagram are quasi-isomorphisms, and hence so is the third column. Therefore, $\check{H}^q(\mathcal{U}, F'') = 0$, for all $q > 0$.

(iii) has already been proved by the sublemma. □

Corollary 1.4. *Let $f : X \to Y$ be an affine morphism of schemes, $F \in Qcoh(X)$. Then for all $q > 0$,*

(1)

$$R^q f_* F = 0;$$

(2) We have a canonical isomorphism

$$H^q(X, F) \xrightarrow{\sim} H^q(Y, f_*F).$$

Proof. (1) The sheaf $R^q f_* F$ is the sheaf associated to the presheaf

$$V \mapsto H^q(f^{-1}(V), F)$$

on Y. For $V \subset Y$ affine, $f^{-1}(V)$ affine, and thus by Theorem 1.1,

$$H^q(f^{-1}V, F) = 0$$

for all $q > 0$.

(2) By (1), the canonical morphism $f_*F \to Rf_*F$ is an isomorphism in $D(Y)$. Applying $R\Gamma(Y, -)$ and using I.6.5, we get canonical isomorphisms

$$R\Gamma(Y, f_*F) \xrightarrow{\sim} R\Gamma(Y, Rf_*F) \simeq R\Gamma(X, F).$$

Passing to cohomology, we get the desired result. □

Corollary 1.5. *Let X be a separated scheme, $\mathcal{U} = (U_i)$ be a finite open covering by affine schemes, $F \in Qcoh(X)$. Then the canonical homomorphism $\check{H}^q(\mathcal{U}, F) \to H^q(X, F)$ is an isomorphism.*

Recall that a morphism of schemes $f : X \to Y$ is called separated if the diagonal map $\Delta : X \to X \times_Y X$ is a closed immersion. A scheme X is called separated if the canonical morphism $X \to \operatorname{Spec} \mathbb{Z}$ is separated. The intersection of two affine open subschemes of a separated scheme is still affine.

Proof. The conclusion is immediate from Leray's theorem (I.8.12) and 1.1. □

Corollary 1.6. *Let $f : X \to Y$ be a separated, quasi-compact morphism of schemes, $F \in Qcoh(X)$. Then $R^q f_* F \in Qcoh(Y)$, for all $q > 0$.*

Proof. We may assume Y affine, then we can find a finite affine open covering $\mathcal{U} = (U_i)_{i=1}^N$ of X, since f is quasi-compact. We have

$$R^q f_* F = a(V \mapsto H^q(f^{-1}(V), F)).$$

For any affine open $V \subset Y$, $f^{-1}(V)$ is separated, since f is seperated. By Corollary 1.5, we have $H^q(f^{-1}(V), F) = \check{H}^q(f^{-1}(V) \cap \mathcal{U}, F)$ because $f^{-1}(V) \cap U_{i_0 \cdots i_N}$ is affine for $N \geq 1$. It is then clear that $R^q f_* F = H^q(f_* \check{\mathcal{C}}(\mathcal{U}, F))$. By definition,

$$\check{\mathcal{C}}(\mathcal{U}, F)(W) = \\ (\oplus F(W \cap U_i) \to \oplus F(W \cap U_{ij}) \to \cdots \to F(W \cap U_{i_0 \cdots i_N}) \to 0)$$

for $W \subset X$ open. Hence $f_* \check{\mathcal{C}}(\mathcal{U}, F)$ is a complex of quasi-coherent sheaves on Y. Therefore, $R^q f_* F$ is quasi-coherent. □

2 Koszul complex and regular sequences

Let A be a commutative ring and E be an A-module. Then, for any A-morphism $u : E \to A$, we can define

$$K.(u) \in C^{\leq 0}(A)$$

as follows(here $C(A)$ denotes the category of complexes of A-modules):

$$K_n(u) = K.(u)^{-n} = \wedge^n E, \; n \geq 0;$$

$$d: K_n(u) \to K_{n-1}(u),\ d = \text{ the right interior product by } u;$$
$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i-1} u(x_i) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n.$$

It is easy to see $d^2 = 0$, $d = u: \ \wedge^1 E = E \to A$ and $d(a \wedge b) = da \wedge b + (-1)^p a \wedge db$, for $a \in \wedge^p E, b \in \wedge^q E$.

Let $E = E_1 \oplus E_2$, $u = u_1 + u_2 : \ E \to A$; $u_i : \ E_i \to A$, then $K.(u, E) = K.(u_1) \otimes K.(u_2)$, $\wedge^n E = \bigoplus_{p+q=n} \wedge^p E_1 \otimes \wedge^q E_2$, $d = d_1 \otimes 1 \oplus (-1)^* 1 \otimes d_2$. In particular, when $E = A^r, \mathrm{Hom}(A^r, \ A) = A^r$, for any $f = (f_1, \cdots, f_r) \in A^r$, $K.(f) = \bigotimes_{i=1}^{r} K.(f_i)$. For example, for $g \in A$, $K.(g) = (0 \to A \to A \to 0)$. Let (e_i) be the canonical basis of A^r, then $d(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \sum_{j=1}^{n} (-1)^j f_{i_j} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_n}$. Let $u: E \to A$ and M be an A-module, define $K.(u, M) = K.(u) \otimes_A M$, $d(x \otimes m) = dx \otimes m$. Let $I = u(E)$, then $H^0 K.(u, M) = M/IM$.

Definition 2.1. Let M be an A-module and $f = (f_1, \cdots, f_r) \in A^r$, f is called *M-regular* if for all $i > 0$

$$f_i : \ M/(\sum_{j<i} f_j M) \to M/(\sum_{j<i} f_j M)$$

is injective. When $M = A$, we just say f is *regular*.

Example 2.1.1. Let k be a commutative ring and $A = k[t_1, \cdots, t_m]$ be a polynomial ring, then for any $r \leq m$, $(t_1, \cdots, t_r)$ is regular, since $A/(t_1, \cdots, t_{i-1}) = k[t_i, \cdots, t_m]$.

Theorem 2.2 (Serre). *Let M be an A-module and $f = (f_1, \cdots, f_r) \in A^r$. Consider the conditions:*

(1). $K.(f, M) \to M/(f_1, \cdots, f_r)M$ is a quasi-isomorphism (i.e. $H^q K.(f, M) = 0,\ q < 0$);

(2). f is M-regular.

Then we have (2)$\Rightarrow$(1). And if A is noetherian, M is of finite type and $f_i \in \mathrm{rad}(A)$ (means the Jacobson radical), for all i, then (1)$\Rightarrow$(2) and (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3), where (3) is $H^{-1} K.(f, M) = 0$.

Lemma 2.3. *Let $L \in C(A)$ and $x \in A$, $K.(x) = (0 \to \ K.(x)^{-1}(= A) \xrightarrow{x} K.(x)^0(= A) \ \to 0)$. Then $K.(x) \otimes L \simeq C(\ L \xrightarrow{x} L\ ) = \mathrm{Cone}(x)$.*

Proof. $C^i = L^{i+1} \oplus L^i \simeq K.(x)^{-1} \otimes L^{i+1} \oplus K.(x)^0 \otimes L^i$ and a maps to $1 \otimes a$, b maps to $1 \otimes b$, for $a \in L^{i+1}$ and $b \in L^i$. Then $d_{K.(x)\otimes L}(1 \otimes a) = x \otimes a - 1 \otimes d_L a$ and $d_{K.(x)\otimes L}(1 \otimes b) = 1 \otimes d_L b$, hence $d_{K.(x)\otimes L}(1 \otimes a \oplus 1 \otimes b) = d_{C^i}(a \oplus b)$. So, $K.(x) \otimes L \simeq C(\ L \xrightarrow{x} L\ )$. $\square$

Then we have a distinguished triangle $L \to L \to K.(x) \otimes L \to L[1]$, so we can get a long exact sequence:

$$\cdots\ H^q(L) \xrightarrow{x} H^q(L)\ \to H^q(K.(x) \otimes L) \to H^{q+1}(L) \to \cdots .$$

From it we get the following short exact sequence:

$$\begin{aligned} 0 \to \mathrm{Coker}(x:\ H^q(L) \to H^q(L)) \to H^q(K.(x) \otimes L) \\ \to \mathrm{Ker}(x:\ H^{q+1}(L) \to H^{q+1}(L)) \to 0;\ (*) \end{aligned}$$

and this can be rewritten:

$$0 \to H^0K.(x, H^q(L)) \to H^q(K.(x) \otimes L) \to H^{-1}K.(x, H^{q+1}(L)) \to 0.(*)$$

Proof of 1.2.:
The implication of (1) $\Longrightarrow$ (3) is trivial. We prove (2)$\Rightarrow$(1) by induction on r.

For $r = 1$, $K.(f, M) = (0 \to\ M \xrightarrow{f} M\ \to 0)$ and the statement is trivial.

Assume now $r \geq 2$ and the statement proven for $m \leq r - 1$. Let

$$L = K.(f_1, \cdots, f_{r-1}, M) = K.(f_1, \cdots, f_{r-1}) \otimes M,$$

then $K.(f_r) \otimes L \simeq K.(f_1, \cdots, f_r, M)$. Hence we have the exact sequence:

$$0 \to H^0K.(f_r, H^q(L)) \to H^qK.(f_1, \cdots, f_r, M) \to H^{-1}K.(f_r, H^{q+1}(L)) \to 0.$$

We want to show $H^qK.(f_1, \cdots, f_r, M) = 0$, for all $q < 0$. When $q \leq -2$, it follows from the above exact sequence and the inductive assumption. When $q = -1$, it is also true, since $\mathrm{Ker}(f_r:\ M/(f_1, \cdots, f_{r-1})M \to M/(f_1, \cdots, f_{r-1})M) = 0$ for f is M-regular.

We also prove (3) $\Longrightarrow$ (2) by induction on r.

The case $r = 1$ is trivial. When $r \geq 2$, again let $L = K.(f_1, \cdots, f_{r-1}, M)$. First, we show $(f_1, \cdots, f_{r-1})$ is M-regular. By $(*)$, We have an inclusion:

$$H^q(L)/f_r H^q(L) \hookrightarrow H^qK.(f_1, \cdots, f_r, M)$$

When $q=-1$, $H^qK.(f_1,\cdots,f_r,M)=0$, hence $H^{-1}(L)=f_rH^{-1}(L)$. Since A is noetherian and M is of finite type, hence $H^{-1}(L)$ is finitely generated over A. So, because $f_r\in\mathrm{rad}(A)$, $H^{-1}(L)=0$. By induction, $(f_1,\cdots,f_{r-1})$ is M-regular. And by condition (3)

$$\mathrm{Ker}(f_r:\ M/(f_1,\cdots,f_r)M\to M/(f_1,\cdots,f_r)M)=0,$$

so $(f_1,\cdots,f_r)$ is M-regular.

Example 2.3.1. Let $A=k[t_1,\cdots,t_r]$, then $t=(t_1,\cdots,t_r)$ is regular. $H^qK.(t)=0$, for all $q<0$ and $H^0K.(t)=A/(t_1,\cdots,t_r)A=k$. This can also be seen by using $K.(t)=K.(k[t_1],t_1)\otimes_k\cdots\otimes_k K.(k[t_r],t_r)$.

Remark. (1). If A is nonnoetherian, let $M=A$, $f_i\in\mathrm{rad}(A)$, $(i=1,2)$. Then, it may happen that (f_1,f_2) is regular but (f_2,f_1) not.(See EGA IV 16 9.6.(ii) for an example.

(2). Let A be a noetherian ring, $f_i\notin\mathrm{rad}(A)(i=1,2)$. Then it may also happen that (f_1,f_2) is regular but (f_2,f_1) not. For example, assume B, C be fields and $A=B\times C$. Let $0\neq b\in B$ be and $f_1=(1,b)$, $f_2=(1,0)$, then (f_1,f_2) is A-regular but (f_2,f_1) not.

For any A-morphism $E\xrightarrow{u}A$ one can form the associated Koszul complex $K.(u)\in C^{\leq 0}(A)$. Similarly, for any A-morphism $A\xrightarrow{v}F$, we can also define a complex $K^\cdot(v)\in C^{\geq 0}(A)$ called the Koszul complex, too, as follows:

$$K^n(v)=\wedge^nF;\ d:\ K^n(v)\to K^{n+1},\ d(x)=v\wedge x.$$

Here we identity the morphism v with $v(1)\in F$. It is easy to check $d^2=0$ and d is the exterior product by v. For $F=F_1\oplus F_2$, $v=(v_1,v_2)$, $K^\cdot(v)=K^\cdot(v_1)\otimes K^\cdot(v_2)$. Let $f=(f_1,\cdots,f_r)\in A^r$, then we have two Koszul complex $K.(f)$ and $K^\cdot(f)$.

$$\begin{aligned}K.(f):0\to A\to A^r(=\wedge^{r-1}A^r)\to\cdots\to(\wedge^1A=)\ A^r\xrightarrow{f}A\\ \longrightarrow 0,\ f(a_1,\cdots,a_r)=\sum_i^r a_i\ ;\end{aligned}$$

$$\begin{aligned}K^\cdot(f):0\to A\xrightarrow{f}A^r\ (=\wedge^1A^r)\to\cdots\to(\wedge^{r-1}A=)A^r\to A\\ \longrightarrow 0,\ f(a)=(f_1a,\cdots,f_ra)\ .\end{aligned}$$

$K^\cdot(f)$ can be viewed as the naive dual of $K.(f)$. In fact, we have a canonical isomorphism

$$K^\cdot(f)[r]\simeq K.(f)$$

defined as follows:

For any $I = \{i_1 < \cdots < i_p\} \subset \{1, \cdots, r\}$, let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$, then $e_I \mapsto \varepsilon(J, I)e_J$, here $J = (j_1 < \cdots < j_{r-p})$ and $J \cup I = 1, \cdots, n$, $\varepsilon(J, I) = \operatorname{sign}(j_1, \cdots, j_{r-p}, i_1, \cdots, i_p)$.

Corollary 2.4. *Assume $(f_1, \cdots, f_r) \in A^r$ is regular and $B = A/(f_1, \cdots, f_r)A$. Then* $\operatorname{Ext}^q_A(B, A) = \begin{cases} 0 & q \neq r \\ B & q = r \end{cases}$ *and the class of the exact sequence $0 \to A \to A^r \to \cdots \to A^r \to A \to B \to 0$ given by $K.(f) \to B$ forms a basis of $\operatorname{Ext}^r_A(B, A)$.*

Proof. Since $K.(f) \to B$ is a quasi-isomorphism,

$$\operatorname{RHom}_A(B, A) = \operatorname{Hom}_A(K.(f), A) = K^{\cdot}(f) \simeq K.(f)[-r],$$

hence

$$\operatorname{Ext}^q_A(B, A) = \begin{cases} H^{q-r}K.(f) = 0 & q \neq r \\ H^0 K.(f) = B & q = r \end{cases}.$$

$\operatorname{Ext}^r(B, A)$ has a natural $B-$module structure and we only need to prove that the class of $K.(f) \to B$ forms a basis of it. Since $H^0(K.(f)) = B$, we have

$$\operatorname{Ext}^r(B, A) = \operatorname{Hom}_{D(A)}(B, A[r]) = \operatorname{Hom}_{D(A)}(K.(f), A[r]).$$

Because the components of $K.(f)$ are projective, hence

$$\operatorname{Hom}_{D(A)}(K.(f), A[r]) = \operatorname{Hom}_{K(A)}(K.(f), A[r])$$

and

$$\operatorname{Hom}_{K(A)}(K.(f), A[r]) = \operatorname{Coker}(\operatorname{Hom}(K.(f)^{-r+1}, A) \to \operatorname{Hom}(K.(f)^{-r}, N))$$
$$= H^r \operatorname{Hom}^{\cdot}(K.(f), A). \ (*)$$

The self-duality of the Koszul complex identifies the cokernel in $(*)$ to the cokernel of $d: \ K.(f)^{-1} \to K.(f)^0$, the class of the identity map of $K.(f)^{-r}$ corresponding to the class of $1 \in K.(f)^0$ in the cokernel B. So, the class $K.(f) \to B$ forms a basis of it. $\square$

Now, we generalize the above discussion to ringed spaces. Let $(X, \mathcal{O}_X)$ be a ringed space and $E \in Mod(X)$, then for any morphism $u: \ E \to \mathcal{O}_X$ define the Koszul complex $K.(u)$ by

$$(\cdots \longrightarrow \wedge^n E \xrightarrow{d} \wedge^{n-1} E \longrightarrow \cdots \longrightarrow E \xrightarrow{u} \mathcal{O}_X \longrightarrow 0),$$

where d is the right interior product by u.

Definition 2.5. Assume $Y \overset{i}{\hookrightarrow} X$ is the closed immersion defined by the ideal sheaf $I \subset \mathcal{O}_X$. Let $r \in \mathbb{N}$, we say that *i is regular of codimension r* if for all $x \in Y$, there exists an open neighborhood U of x, $U \subset X$ and an $\mathcal{O}_U$-module E locally free of rank r and an $\mathcal{O}_U$-linear map $u: \ E \to \mathcal{O}_U$, such that $K.(u)$ is acyclic in negative degrees and $I|_U = u(E) \subset \mathcal{O}_U$. In other words, there exists locally a sequence $(f_1, \cdots, f_r) \in {\mathcal{O}_U}^r$ such that $I|_U = (f_1, \cdots, f_r)$ and $K.(f) \to \mathcal{O}_U/I\mathcal{O}_U$ is a resolution. If X is locally noetherian, then it is also equivalent to saying that for every $x \in Y$, there exists an open neighborhood U of x such that I is defined by a sequence $f_1, \cdots, f_r$ of sections of $\mathcal{O}_X$ such that $(f_1)_x, \cdots, (f_r)_x \ \in \mathfrak{M}_{X,x}$ is a regular sequence.

For example, Let $A \twoheadrightarrow B = A/I$ and $Y = \operatorname{Spec} B \overset{i}{\hookrightarrow} X = \operatorname{Spec} A$, then if $I = (f_1, \cdots, f_r)$ and $(f_1, \cdots, f_r)$ is regular, then Y is regular of codimension r.

When $r = 1$ and $Y \overset{i}{\hookrightarrow} X$ is locally defined by $f = 0$ where $f \in \mathcal{O}_X$ is non-zero divisor, then we say Y is an effective Cartier divisor on X.

Corollary 2.6. *Assume $Y \overset{i}{\hookrightarrow} X$ is a regular immersion of codimension r, then $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$ is 0 for $q \neq r$, and $\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$ is a line bundle on Y.*

Proof. This follows directly from the calculation of

$$\operatorname{Ext}^q_{\Gamma(U,\ \mathcal{O}_X)}(\Gamma(U, \mathcal{O}_Y), \Gamma(U, \mathcal{O}_X)),$$

for $U = \operatorname{Spec}(A)$, $U \cap Y = \operatorname{Spec} B$, $B = A/(f_1, \cdots, f_r)A$ with $(f_1, \cdots, f_r)$ regular and $\Gamma(U, \mathcal{O}_Y) = \Gamma(U \cap Y, \mathcal{O}_Y)$. $\square$

Remark. One can show that $N_{Y/X} = I/I^2$ (called the canormal sheaf of i) is an $\mathcal{O}_Y$-module locally free of rank r (locally, $\overline{f_1}, \cdots, \overline{f_r} \in I/I^2$ form a basis) and there exists a canonical isomorphism (called *fundamental local isomorphism*):

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \simeq \omega_{Y/X}[-r], \ \omega_{Y/X} = (\wedge^r N_{Y/X})^\vee,$$

where $(-)^\vee$ means $\mathcal{H}om(-, \mathcal{O}_Y)$.

3 Cohomology of $\mathbb{P}^r$ with values in $\mathcal{O}_{\mathbb{P}^r}(n)$

Let A be a commutative ring and $r \geq 0$ an integer. Set $S = \operatorname{Spec} A$, $P = \mathbb{P}^r_S = \operatorname{Proj} B$ where $B = A[t_0, \cdots, t_r]$. Then $P = \bigcup_{i=0}^{r} U_i$, $U_i \xhookrightarrow{\text{open}} P$, $U_i = \operatorname{Spec} A[\frac{t_0}{t_i}, \cdots, \frac{t_r}{t_{i-1}}, \frac{t_r}{t_{i+1}}, \cdots, \frac{t_r}{t_i}] \simeq \mathbb{A}^r_S$. $\mathcal{O}_P(1)$ denoted briefly by $\mathcal{O}(1)$ is an invertible sheaf (line bundle). We have canonical sections $e_0, \cdots, e_r \in \Gamma(P, \mathcal{O}(1))$ defining an epimorphism $\mathcal{O}_P{}^{r+1} \xrightarrow{(e_i)} \mathcal{O}(1)$, and $U_i = \{x \in P \mid e_i \text{ gives a basis of } \mathcal{O}(1) \text{ at } x\}$, we have $\frac{e_i}{e_j}|_{U_i \cap U_j} = \frac{t_i}{t_j} \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. For any $n \in (Z)$, define

$$\mathcal{O}(n) = \begin{cases} \mathcal{O}(1)^{\otimes n} & n > 0 \\ \mathcal{O} & n = 0 \\ (\mathcal{O}(1)^{\vee})^{\otimes -n} & n < 0 \end{cases}$$

This line bundle $\mathcal{O}(n)$ is the quasi-coherent module associated to the graded B-module $B(n)$ (where the grading of $B(n)$ is defined by $B(n)_m = B_{m+n}$.) For any $f \in B_d$, set $B_{(f)} = (B_f)_0$, then $\Gamma(\operatorname{Spec} B_{(f)}, \mathcal{O}(n)) = (B_f)_n = \{\frac{a}{f^m} : \deg a - md = n\}$.

As $\Gamma(P, F)$ is a module over $\Gamma(S, \mathcal{O}_S) = A$ for any $F \in \mathit{Mod}(P)$, we have a functor $\Gamma(P, -) : \ \mathit{Mod}(P) \to \mathit{Mod}(A)$. Then we get the derived functor $\mathrm{R}\Gamma(P, -) : \ D^+(P) \to D^+(A)$ and $H^q(P, F) \in \mathit{Mod}(A)$.

Theorem 3.1. *(1). The canonical homomorphism (of graded A-algebras):* $B \to \bigoplus_{n \in \mathbb{Z}} \Gamma(P, \mathcal{O}(n))$ *such that* $t_i \mapsto e_i \in \Gamma(P, \mathcal{O}(1))$ *is an isomorphism, here the structure of graded A-algebra of* $\bigoplus_{n \in \mathbb{Z}} \Gamma(P, \mathcal{O}(n))$ *is defined by associating to* $s \in \Gamma(P, \mathcal{O}(p))$, $t \in \Gamma(P, \mathcal{O}(q))$ *the* $s \otimes t$.
(2).$H^q(P, \mathcal{O}(n)) = 0$ *for all* n *when* $0 < q < r$ *or* $q > r$.
(3). When $n \leq -r - 1$, $H^r(P, \mathcal{O}(n)) = \bigoplus A \frac{t^\alpha}{t_0 \cdots t_r}$, *where* $t^\alpha = t_0{}^{\alpha_0} \cdots t_r{}^{\alpha_r}$ *with* $\alpha_i \leq 0$ *and* $\sum \alpha_i - r - 1 = n$; *otherwise* $H^r(P, \mathcal{O}(n)) = 0$.

Corollary 3.2. $H^0(P, \mathcal{O}(n))$ *and* $H^r(P, \mathcal{O}(-n-r-1))$ *are free over A of rank* $\binom{n+r}{r}$. *In particular,* $H^r(P, \mathcal{O}(-r-1)) = A$, $H^0(P, \mathcal{O}(1)) = A^{r+1} = B_1$. $(e_i \mapsto t_i)$

Proof. Define $H^q(P, \mathcal{O}(*)) = \bigoplus_{n \in \mathbb{Z}} H^q(P, \mathcal{O}(n))$. By Serre's theorem, $H^q(P, \mathcal{O}(n)) \simeq$

$\breve{H}(\mathcal{U}, \mathcal{O}(n))$, where $\mathcal{U} = (U_i)_{0\le i\le r}$. Then $U_{i_0\cdots i_p} = \bigcap_{i=1}^{p} U_{i_j} = \mathrm{Spec}(B_{t_{i_0}\cdots t_{i_p}})_0$,

$$H^q(P, \mathcal{O}(*)) = \bigoplus_{n\in\mathbb{Z}} H^q(P, \mathcal{O}(n)) = H^q\check{C}(\mathcal{U}, \mathcal{O}(*))$$

here $\check{C}(\mathcal{U}, \mathcal{O}(*)) = \bigoplus_{n\in\ \mathbb{Z}} \check{C}(\mathcal{U}, \mathcal{O}(n))$ and

$$\check{C}^p(\mathcal{U}, \mathcal{O}(n)) = \bigoplus_{i_0<\cdots<i_p} \Gamma(U_{i_0\cdots i_p}, (O)(n)) = \bigoplus_{i_0<\cdots<i_p} (B_{t_{i_0}\cdots t_{i_p}})_n.$$

Hence $\check{C}(\mathcal{U}, \mathcal{O}(*)) = \bigoplus_{i_0<\cdots<i_p} \bigoplus_{n\in\mathbb{Z}} (B_{t_{i_0}\cdots t_{i_p}})_n$.

$$\check{C}(\mathcal{U}, \mathcal{O}(*)) :\ 0 \to \bigoplus_i B_{t_i} \to \bigoplus_{i<j} B_{t_i t_j} \to \cdots B_{t_0\cdots t_r} \to 0.$$

We also have $\check{C}(\mathcal{U}, \mathcal{O}(*)) = \bigcup_{n\ge 0} \check{C}_{-n}$, where

$$\check{C}_{-n} = (0 \to \bigoplus_i {t_i}^{-n} B \to \bigoplus_{i<j} (t_i t_j)^{-n} B \to \cdots (t_0\cdots t_r)^{-n} B \to 0)$$

and $\check{C}_{-n} \subset \check{C}_{-(n+1)} \subset \cdots$. Let

$$L_{-n} = (\ B \xrightarrow{d} \bigoplus {t_i}^{-n} B \xrightarrow{d} \cdots \xrightarrow{d} (t_0\cdots t_r)^{-n} B\ ),$$

then we have $L_{-n} \simeq K_{-n} = K^{\cdot}({t_0}^n, \cdots, {t_r}^n)$ by the isomorphism

$$\varphi_n :\ L_{-n}^{p+1} = \bigoplus (t_{i_0}\cdots t_{i_p})^{-n} B \longrightarrow \wedge^{p+1} B^{r+1}$$

sending the summand $(t_{i_0}\cdots t_{i_p})^{-n}B$ to $Be_{i_0}\wedge\cdots\wedge e_{i_p}$ by $b \mapsto (t_{i_0}\cdots t_{i_p})^n b$. Moreover, $\check{C}^p(\mathcal{U}, \mathcal{O}(*)) \xrightarrow{d} \check{C}^p(\mathcal{U}, \mathcal{O}(*))$ makes the following diagram commutative:

$$\begin{tikzcd} (t_{i_0}\cdots t_{i_p})^{-n} B \arrow[r, "d"] \arrow[d, "\varphi_n"] & \check{C}^{p+1} \arrow[d, "\varphi_n"] \\ K^{P+1} \arrow[r] & K^{p+2} \end{tikzcd}$$

where d sends $\frac{a_{i_0\cdots i_p}}{(t_{i_0}\cdots t_{i_p})^{-n}}$ to $\sum_{j=0}^{p}(-1)^j\frac{a_{i_0\cdots\widehat{i_j}\cdots i_p}}{(t_{i_0}\cdots\widehat{t_{i_j}}\cdots t_{i_p})^{-n}}$ which is send by φ_n to $\sum_{j=0}^{p}(-1)^j {t_{i_j}}^n a_{i_0\cdots\widehat{i_j}\cdots i_p}$. We know that:

$$H^q(K^{\cdot}_{-n}) = \begin{cases} 0 & q \neq r+1 \\ H^{r+1}(K^{\cdot}_{-n}) & q = r+1 \end{cases}$$

To calculate H^{r+1}, we note that

$$K^{\cdot}_{-n} = K^{\cdot}({t_0}^n, A[t_0]) \otimes_A \cdots \otimes_A K^{\cdot}({t_r}^n, A[t_r]),$$

where $K^{\cdot}({t_i}^n, A[t_i]) = (\ A[t_i] \xrightarrow{{t_i}^n} A[t_i]\ )$. Now,

$$H^1(K^{\cdot}({t_i}^n, A[t_i])) = A[t_i]/{t_i}^n A[t_i] = \bigoplus_{\alpha=0}^{n-1} {t_i}^{\alpha} A.$$

As $K^{\cdot}({t_i}^n) \to A[t_i]/{t_i}^n A[t_i][-1]$ is a quasi-isomorphism and the components of the complex are free of finite type over A, hence the tensor product

$$K^{\cdot}({t_0}^n, \cdots, {t_r}^n, B) \to \bigotimes_{i=0}^{r}(A[t_i]/{t_i}^n A[t_i][-1])$$

is also a quasi-isomorphism. So,

$$H^r K^{\cdot}({t_0}^n, \cdots, {t_r}^n, B) = \bigotimes_{i=0}^{r} A[t_i]/{t_i}^n A[t_i] = \bigoplus_{0\le\alpha_i<n} {t_0}^{\alpha_0}\cdots {t_r}^{\alpha_r} A.$$

So we get $H^0\check{C}_{-n} = B$, and as the augmented complex $0 \to\ B \to \check{C}_{-n}$ where B in degree 0 is isomorphic to L_{-n}, we get $H^q\check{C}_{-n} = 0$ for $0 < q < r$ and $H^r\check{C}_{-n} = \bigoplus_{0<\alpha_i\le n} t^{-\alpha}A$. These isomorphisms are compatible with gradings on both sides:

$H^r(P, \mathcal{O}(n))$ has a basis consists of the elements $\frac{t^{-\alpha}}{t_0\cdots t_r}$ for $-\sum_{\alpha_i}\alpha_i - r - 1 = n$, and

$$B_n = A[t_0, \cdots, t_r]_n \simeq H^0(P, \mathcal{O}(n)), n \geq 0,$$
$$B_1 = A^r = H^0(P, \mathcal{O}(1)),$$

$t_i \in B$, corresponding to $e_i \in H^0(P, \mathcal{O}(1))$. □

The class of the Čech cocycle $U_{0\cdots r} \mapsto \frac{1}{t_0\cdots t_r}$ forms a basis of $H^r(P, \mathcal{O}(-r-1))$.

4 Finiteness and vanishing theorems for projective morphisms

Let X be a locally noetherian scheme, $F \in Mod(X)$. Recall that F is called *coherent* if F is quasi-coherent and of finite type, or equivalently, for any affine open subset $U = \operatorname{Spec} A$ of X, $F|_U = \tilde{M}$ with M a finitely generated $A-$module. We have $Coh(X) \subset Qcoh(X) \subset Mod(X)$. Let A be a noetherian ring, $S = \operatorname{Spec} A$, $P = \mathbb{P}^r_s = \operatorname{Proj} B$, where $B = A[t_0, \cdots, t_r]$.

Proposition 4.1. *Let $F \in Coh(P)$. Then there exists $n_0 \geqslant 0$, such that for all $n \geqslant n_0$, there exists an epimorphism $\mathcal{O}_P(-n)^m \to F \to 0$.*

Let $(X, \mathcal{O}_X)$ be a ringed space. $E \in Mod(X)$, $s_i \in \Gamma(X, E), (i \in I)$. we say the family $\{s_i\}_{i \in I}$ generates E if $\mathcal{O}_X^{(I)} \to E, e_i \to s_i$ is an epimorphism, or we say that equivalently, for any $x \in X$, $(s_i)_x \in E_x$ generate E_x as an $\mathcal{O}_{X,x}-$module.

E is *generated by its global sections* if the family of all sections $s \in \Gamma(X, E)$ generates E.

***Remark*.** If X is quasi-compact, and E is of finite type, then E is generated by its global sections if and only if E is generated by a finite number of global sections.

So the proposition is equivalent to saying that there exists n_0, such that for all $n \geqslant n_0$, $F(n)$ is generated by its global sections.

Example 4.1.1. Let R be a ring, $E \in Qcoh(X)$, $X = \operatorname{Spec} R$. Then E is generated by its global sections. In fact, $E = \tilde{M}$ for some $M \in Mod(R)$, so an epimorphism $R^{(I)} \to M$ gives an epimorphism $\mathcal{O}_X^I \to \tilde{M}$.

Lemma 4.2. *Suppose X is a noetherian scheme, L is a line bundle on X, $f \in \Gamma(X, L)$, $X_f = \{x \in X, f(x) \neq 0\}$, where $f(x)$ is the image of f in $k \bigotimes_{\mathcal{O}_{X,x}} L$, which is an open subset of X. Let $E \in Coh(X)$. For any $s \in \Gamma(X_f, E)$, there exists $n \geqslant 0$ such that $s \otimes f^{\otimes n} \in \Gamma(X_f, E \otimes L^{\otimes n})$ extends to a section of $E \otimes L^{\otimes n}$ over X. If $t \in \Gamma(X_f, E \otimes L^{\otimes n})$,such that $t|X_f = 0$, then there exists $m \geqslant 0$ such that $t \otimes f^m = 0 \in \Gamma(X, E \otimes L^{\otimes m+n})$.*

Proof. Consider the inductive system

$$\cdots \to \Gamma(X, E \otimes L^{\otimes n}) \stackrel{\otimes f}{\to} \Gamma(X, E \otimes L^{\otimes n+1}) \to \cdots$$

we define

$$\Gamma(X,E)_f = \varinjlim_n \Gamma(X_f, E\otimes L^{\otimes n})$$

$$\begin{array}{ccc} \Gamma(X_f, E\otimes L^{\otimes n}) & \to & \Gamma(X_f, E) \\ t & \mapsto & t|_{X_f}\otimes f^{\otimes -n}\in \Gamma(X_f,E) \end{array}$$

Thus we get a morphism $\varphi : \Gamma(X,E)_f \to \Gamma(X_f,E)$. So the lemma is equivalent to saying φ is an isomorphism.

(a). Suppose $X = \operatorname{Spec} A$ with A noetherian, $L = \mathcal{O}_X$, $E = \tilde{M}$ for some finitely generated $M \in \mathit{Mod}(A)$. For $f \in A$, we have $X_f = \operatorname{Spec} A_f$. $\Gamma(X,E) = M$, so $\Gamma(X,E)_f = \varinjlim_f M \simeq M_f$. $\Gamma(X_f,E) = M_f$. So φ is an isomorphism.

(b). Suppose X is separated, $X = \bigcup_{i=0}^{n} U_i$ with $U_i = \operatorname{Spec} A_i$. $L|U_i = \mathcal{O}_{U_i}$. $U_i \cap U_j = \operatorname{Spec} A_{ij}$. Then we have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,E)_f \arrow[r] \arrow[d, dotted, "\varphi"] & \prod_i \Gamma(U_i,E)_f \arrow[r] \arrow[d] & \prod_{i,j} \Gamma(U_{i,j},E)_f \arrow[d] \\ 0 \arrow[r] & \Gamma(X_f,E) \arrow[r] & \prod_i \Gamma(U_i f,E) \arrow[r] & \prod_{i,j} \Gamma(U_{ijf},E) \end{tikzcd}$$

with the two right vertical maps are isomorphisms. By five lemma, we have φ is an isomorphism.

For the general case, cover $U_i \cap U_j$ by affine schemes U_{ijk}, and use the similar argument. □

Proof of 3.1.: For $F \in \mathit{Coh}(P)$, we want to find n_0 such that for all $n \geqslant n_0$ and all $i \in [0,r]$, $F(n)|U_i$ is generated by its sections. ($U_i = P_{t_i} = \operatorname{Spec} B_{(t_i)}$, where $t_i \in \Gamma(P, \mathcal{O}_P(1))$). For $F|_{U_i}$, there exists $h_{ij} \in \Gamma(U_i, F|_{U_i})$ generating $F|_{U_i}$ with $j \in J_i$ finite. The lemma implies that there exists $m_{ij} \geqslant 0$, such that $h_{ij}\otimes t_i^{m_{ij}}$ extends to a section of $F(n_0)$ over P. By taking a common multiple of all m_{ij}'s,we may assume they are equal to n_0, so, for $n \geqslant n_0$, $h_{ij}\otimes t_i^n$ extends to P to g_{ij} (as section of $F(n)$). These g_{ij} generate $F(n)|_{U_i}$ for all i, hence $F(n)$.

Let $S = \operatorname{Spec} A$, we say X is *projective* over S if there exists a closed immersion i making the following diagram commutative

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & \mathbb{P}^r_S \arrow[dl] \\ S & \end{tikzcd}$$

Suppose X/S is projective. A line bundle L on X is called *very ample* if there exists a close immersion $i : X \hookrightarrow \mathbb{P}^r_S = P$ such that $L \simeq i^*\mathcal{O}_P(1)$. L is called *ample* if there exists $n \geqslant 1$ such that $L^{\otimes n}$ is very ample.

Theorem 4.3. *Suppose X/S is projective, L is an ample bundle on X, and $F \in Coh(X)$. Then there exists $n_0 \geqslant 0$ such that for all $n \geqslant n_0$, $F(n)$ is generated by its global sections, where $F(n) = F \otimes L^{\otimes n}$.*

Proof. (1). Suppose L is very ample. $L = i^*\mathcal{O}_P(1)$. $i_*F \in Coh(P)$. Then the proposition implies that there exists n_0 such that for all $n \geqslant n_0$, there exists an epimorphism $\mathcal{O}_P(-n)^m \to i_*F \to 0$, then we have $\mathcal{O}_X(-n)^m \to i^*i_*F \to 0$.
(2). General case. There exists $m \geqslant 1$ such that $L' = L^{\otimes m}$ is very ample. By (1), we can find $n_0 \geqslant 0$ such that for all $n \geqslant n_0$ and all $0 \leqslant r < m$, $(F \otimes L^{\otimes r})L'^{\otimes m}$ is generated by its global sections. We claim that for $n \geqslant mn_0$, $F \otimes L^{\otimes n}$ is generated by global sections. We can write $n = md + r$ where $0 \leqslant r < m$, then $F \otimes L^{\otimes n} = F \otimes L^{\otimes r} \otimes L'^{\otimes d}$ with $d \geqslant n_0$. $\square$

Theorem 4.4 (finiteness theorem). *Suppose X/S is projective Then*
(1). There exists $d \geqslant 0$ such that for all $q > d$,and all $F \in Qcoh(X)$. $H^q(X, F) = 0$.
(2). For $F \in Coh(X)$, then $H^q(X, F)$is finitely generated over A for all q.

Proof. (1). Let $\mathcal{U} = (U_i)_{0 \leqslant i \leqslant d}$ be an open affine cover of X. Because X/S is separated, $U_{i_0} \cap \cdots \cap U_{i_p}$ is also affine, so $\check{H}^q(\mathcal{U}, F) = H^q(X, F) = 0$ for all $q > d$.
(2). First we reduce to the case $X = P$ by using the isomorphism $H^q(X, F) = H^q(P, i_*F)$. Then we use descending induction on q.

If $q >> 0$, the result is obvious. Suppose the finiteness is known in degree $\geqslant q + 1$ for any F. Consider the following exact sequence

$$0 \to G \to \mathcal{O}_P(-n)^m \to F \to 0$$

where $G \in Coh(P)$ and $n \geqslant 0$. Then we have

$$H^q(P, \mathcal{O}(-n)^m) \to H^q(P, F) \to H^{q+1}(P, G)$$

with $[H^q(P, \mathcal{O}(-n)^m)$ finitely generated over A by 2.1. By induction, we have that $H^{q+1}(P, G)$ is finitely generated over A, hence $H^q(X, F)$is finite generated over A for all q. $\square$

Theorem 4.5. *Suppose X/S is projective, L is an ample bundle on X, and $F \in Coh(X)$. Then there exists n_0 such that for all $n \geqslant n_0$ and all $q > 0$, $H^q(X, F(n)) = 0$ where $F(n) = F \otimes L^{\otimes n}$.*

Theorem 4.6 ((Vanishing)). *Let $S = \operatorname{Spec} A$, with A noetherian, let X be a projective scheme over S and let L be an ample line bundle on X. Then for all $\mathcal{F} \in Coh(X)$, there exists an integer n_0, such that $H^q(X, \mathcal{F}(n)) = 0$ for all $q > 0$ and $n \geq n_0$, where $\mathcal{F}(n) = \mathcal{F} \otimes L^{\otimes n}$.*

Proof. The proof is very similar to that of the finiteness theorem.

(a) Suppose L is very ample. $L = i^*\mathcal{O}_P(1)$, where $i : X \hookrightarrow \mathbb{P}^r_S = P$ is a closed immersion. Since

$$i_*\mathcal{F} \otimes \mathcal{O}_P(n) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}_P(n)) = i_*\mathcal{F}(n),$$

so $H^q(X, \mathcal{F}(n)) = H^q(P, (i_*\mathcal{F})(n))$. Since $i_*\mathcal{F} \in Coh(P)$, we may assume $X = P$.

We use descending induction on $q \geq 1$. We know that there exists an integer $N \geq 0$ such that for all $\mathcal{E} \in Qcoh(X)$ and $q > N$, $H^q(X, \mathcal{E}) = 0$. So the theorem holds for $q \gg 0$. Suppose the theorem holds in degree $\geq q+1$ for all $\mathcal{F} \in Coh(X)$. By Proposition 4.1, there exists an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_P(-m)^d \to \mathcal{F} \to 0$$

for some integers $m \geq 0$, $d \geq 0$. Then we get an exact sequence

$$H^q(P, \mathcal{O}(-m+n)^d) \to H^q(P, \mathcal{F}(n)) \to H^{q+1}(P, \mathcal{G}(n)) \to .$$

We know that the first term is zero for $n \geq m$, and by induction the third term is zero for n large enough, then the result follows.

(b) General case : By (a), we can choose m such that $L^{\otimes m} = L'$ is very ample, and choose n_0 such that $H^q(X, \mathcal{F} \otimes L^{\otimes i} \otimes L'^{\otimes n}) = 0$ for all $0 \leq i < m, n \geq n_0$ by (a). Then for all $n \geq mn_0$, we can rewrite $\mathcal{F}(n)$ in the form $\mathcal{F} \otimes L^{\otimes i} \otimes L^{\otimes md}$, for suitable $d \geq n_0, i \leq m$, hence $H^q(X, \mathcal{F}(n)) = 0$. $\square$

Recall that $X \xrightarrow{f} Y$ is called *proper* if f is of finite type, separated (i.e. $X \hookrightarrow X \times_Y X$ is a closed immersion), and universally closed. We have the following basic facts :

(1) $\mathbb{P}^r_S \longrightarrow S$ is proper ;
(2) Any closed immersion is proper ;
(3) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, if f and g are proper, then gf is proper ;
(4) If f is projective, f is proper.

Theorem 4.7 (Characterization of ampleness). *Let $S = \operatorname{Spec} A$, with A noetherian, Let X/S be proper and L be a line bundle on X, then the following three conditions are equivalent :*

(1) For all $\mathcal{F} \in Coh(X)$, there exists $n_0 \geq 0$, such that for all $n \geq n_0$, $\mathcal{F}(n) = \mathcal{F} \otimes L^{\otimes n}$ is generated by global sections ;

(2) There exists $m \geq 0$, and a closed immersion $i : X \hookrightarrow \mathbb{P}^r_S = P$, such that $L^{\otimes m} = i^*\mathcal{O}_P(1)$;

(3) For all $\mathcal{F} \in Coh(X)$, there exists n_0 such that for all $n \geq n_0$, $H^1(X, \mathcal{F} \otimes L^{\otimes n}) = 0$.

Remark. The implication (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) have been proved. So we only prove (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

Proof of (3) $\Rightarrow$ (1). (We reproduce the proof in [H], Chap III, Proposition 5.3) Let $x \in X$ be a closed point, and let $J_{\{x\}}$ be the ideal sheaf of the closed subscheme $\{x\} = \operatorname{Spec} k(x)$ of X. Then there is an exact sequence

$$0 \longrightarrow J_{\{x\}} \longrightarrow \mathcal{O}_X \longrightarrow k(x) \longrightarrow 0,$$

where $k(x) = i_{x*}\mathcal{O}_{X,x}$ with $i_x : \{x\} \longrightarrow X$ being the closed immersion. Tensoring with $\mathcal{F}$, we get an exact sequence

$$0 \longrightarrow J_{\{x\}}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes k(x) \longrightarrow 0,$$

where $J_{\{x\}}\mathcal{F}$ is the image of $J_{\{x\}} \otimes \mathcal{F} \longrightarrow \mathcal{F}$. Since L is a line bundle, it is flat, and we deduce an exact sequence

$$0 \longrightarrow J_{\{x\}}\mathcal{F} \otimes L^{\otimes n} \longrightarrow \mathcal{F} \otimes L^{\otimes n} \longrightarrow \mathcal{F} \otimes L^{\otimes n} \otimes k(x) \longrightarrow 0.$$

By the hypothesis, there exists an n_0 such that for all $n \geq n_0$, $H^1(X, J_{\{x\}}\mathcal{F} \otimes L^{\otimes n}) = 0$, so

$$\Gamma(X, \mathcal{F} \otimes L^{\otimes n}) \longrightarrow \Gamma(X, \mathcal{F} \otimes L^{\otimes n} \otimes k(x))$$

is surjective for all $n \geq n_0$. Use Nakayama's lemma on the local ring $\mathcal{O}_{X,x}$, we deduce that the stalk of $\mathcal{F} \otimes L^{\otimes n}$ at x is generated by global sections. Since it is a coherent sheaf, we conclude that there exists $s_1, \cdots, s_k \in \Gamma(X, \mathcal{F}(n))$, depending on n, generating $\mathcal{F} \otimes L^{\otimes n}$ in an open neighborhood U of x. In particular, there exists $n_1 \geq 0$ and a neighborhood V of x such that $\mathcal{O}(n_1) = \mathcal{O}_X \otimes L^{\otimes n_1}$ is generated by global sections in V.

For each $0 \leq r < n_1$, the above result gives a neighborhood U_r of x such that $\mathcal{F} \otimes L^{\otimes n_0 + r}$ is generated by global sections. Now let

$$U_x = V \cap U_0 \cap \cdots \cap U_{n_1 - 1}.$$

Since any sheaf of the form $\mathcal{F} \otimes L^{\otimes n}$ can be written as a tensor product

$$(\mathcal{F} \otimes L^{\otimes (n_0 + r)}) \otimes (L^{\otimes n_1})^m$$

for suitable $0 \leq r < n_1$ and $m \geq 0$, so over U_x, all of the sheaves $\mathcal{F} \otimes L^{\otimes n}$ for $n \geq n_0$ are generated by global sections.

Using the fact that X is noetherian, hence any open subset is quasi-compact (and therefore contains a closed point), we cover X by a finite number of the open sets U_x, we find N such that $\mathcal{F} \otimes L^{\otimes n}$ is generated by global sections over X, for all $n \geq N$.

Proof of (1) $\Rightarrow$ (2) : (See [H], Chap II, Theorem 7.6) Given any $x \in X$, let U be an open affine neighborhood of x such that $L|_U$ is free on U. Let Y be the closed set $X - U$, and let J_Y be its sheaf of ideals with the reduced induced scheme structure. Then J_Y is a coherent sheaf on X, so for some $n > 0$, $J_Y \otimes L^{\otimes n}$ is generated by global sections. Since $J_Y \otimes L^{\otimes n} \otimes k(x) \simeq k(x)$, there is a section $s \in \Gamma(X, J_Y \otimes L^{\otimes n})$ such that $s_x \notin \mathfrak{m}_x (J_Y \otimes L^{\otimes n})_x$. Let X_s be the open subset of X consisting of $y \in X$ such that $s(y) \neq 0$ (s viewed as a section of $L^{\otimes n}$), then $X_s \subset U$. Now U is affine, and $L|_U$ is trivial, so s induces an element $f \in \Gamma(U, \mathcal{O}_U)$, and then $X_s = U_f$ is also affine.

Thus we have shown that for any point $x \in X$, there is an $n > 0$ and a section $s \in \Gamma(X, L^{\otimes n})$ such that $x \in X_s$, and X_s is affine. Since X is quasi-compact, we can cover X by a finite number of such open affine subschemes, corresponding to sections $s_i \in \Gamma(X, L^{\otimes n_i})$, and we may assume that all n_i are equal to one n. Finally, since $L^{\otimes n}$ still satisfies condition (1), we may assume $n = 1$, i.e., there exist global sections $s_1, \cdots, s_k \in \Gamma(X, L)$ such that each $X_i = X_{s_i}$ is affine, and the X_i cover X. Moreover, if we let $B_i = \Gamma(X_i, \mathcal{O}_{X_i})$, then each B_i is a finitely generated A-algebra. So let $\{b_{ij} | j = 1, \cdots, k_i\}$ be a set of generators for B_i as an A-algebra. For each i, j, there is an integer

n_{ij} such that $s_i^{n_{ij}} b_{ij}$ extends to a global section $c_{ij} \in \Gamma(X, L^{\otimes n})$. We can take one n large enough to work for all i, j.

Now we define a morphism

$$\varphi : X \longrightarrow \mathbb{P}_A^{N-1} = \mathrm{Proj} A[\{x_i\}_{1 \leq i \leq n}; \{x_{ij}\}_{1 \leq j \leq r_i}]$$

such that $\varphi^* \mathcal{O}_P(1) = L^{\otimes n}$ and $\varphi^* x_i = s_i^n$, $\varphi^* x_{ij} = c_{ij}$. We show that f is a closed immersion. For each $i = 1, \cdots, k$, let $U_i \subset \mathbb{P}_A^N$ be the open subset $x_i \neq 0$. Then $\varphi^{-1}(U_i) = X_i$, and the corresponding map of affine rings

$$A[\{y_i\}; \{y_{ij}\}] \to B_i$$

is surjective, because $y_{ij} \mapsto c_{ij}/s_i^n = b_{ij}$, and we choose the b_{ij} so as to generate B_i as an A-algebra. Thus X_i is mapped onto a closed subscheme of U_i. It follows that φ gives an isomorphism of X with a closed subscheme of $\cup_{i=1}^k U_i \subseteq \mathbb{P}_A^N$, so φ is an immersion, hence a closed immersion, because φ is proper, X being proper over S. $\square$

Remark. Now we can give a more general definition of ampleness : let X be a noetherian scheme, L a line bundle on X. L is called *ample* if for all $\mathcal{F} \in Coh(X)$, there exists n_0 such that $\mathcal{F}(n) = \mathcal{F} \otimes L^{\otimes n}$ is generated by global sections for each $n \geq n_0$.

It follows from theorem 4.7 that L is ample if and only if there exists a basis of the topology of X of the form $\{X_s | s \in \Gamma(X, L^{\otimes n})\}$ with X_0 affine.

At the end of this section, we prove a generalization of the finiteness theorem.

Assume X is locally noetherian, $* = +, -, b$, let

$$D^*(X)_{coh} = \{E \in D^*(X) | H^i E \in Coh(X), \text{ for all } i\}.$$

Theorem 4.8. *Suppose S is locally noetherian, $f : X \to S$ is proper, then $Rf_* : D^+(X) \to D^+(S)$ sends $D^+(X)_{coh}$ to $D^+(S)_{coh}$.*

Remark. Theorem 4.8 is equivalent to the following proposition :

Proposition 4.9. Let $S = \mathrm{Spec}\, A$ be affine, noetherian, and f be proper, $\mathcal{F} \in Coh(X)$, then for all q, $H^q(X, \mathcal{F})$ is a finitely generated A-module.

Proof of the remark : Since

$$R^q f_* \mathcal{F} = H^q(X, \mathcal{F})\widetilde{\ } \in Coh(S) \qquad (*)$$

for any q, then $H^q(X, \mathcal{F})$ is finitely generated. Conversely, for $E \in D^+(X)_{coh}$, we want to prove $R^i f_* E \in Coh(S)$. This is a local question on S, so we may assume $S = \operatorname{Spec} A$, A is noetherian. By $(*)$ and the finiteness theorem, we get that $Rf_*E \in D^+(S)_{coh}$ for any $E \in Coh(X)$. We want to prove that $Rf_*E \in D^+(S)_{coh}$ for any $E \in D^+(X)_{coh}$. First we prove this is true for $E \in D^b(X)_{coh}$. Let $E \in D^{[a,b]}(X)_{coh}$, we use induction on $b - a$. We have a distinguished triangle

$$\tau_{\leq b-1} E \to E \to (H^b E)[-b] \to,$$

then we obtain

$$Rf_*(\tau_{\leq b-1} E) \to Rf_*E \to (Rf_*(H^b E))[-b] \to,$$

so by induction, $Rf_*E \in D^b(S)_{coh}$.
Now we let $E \in D^+(X)_{coh}$. We have the distinguished triangle

$$\tau_{\leq n} E \to E \to \tau_{\geq n+1} E \to,$$

since $R^n f_* E = R^n f_*(\tau_{\leq n} E)$, the conclusion follows. □

From this remark and the finiteness theorem, we know that theorem 4.8 holds when f is projective. And from the proof we know that to prove theorem 4.8, we only need prove it for all $\mathcal{F} \in D^{[0,0]}(X)_{coh} \simeq Coh(X)$.

Lemma 4.10 (Chow). *Let S be a noetherian scheme and let $f : X \to S$ be a proper morphism, then there exists a projective morphism $g : X' \to X$ such that fg is projective, and there exists an open dense subset U of X such that g induces an isomorphism from $g^{-1}(U)$ to U, i.e.,*

$$\begin{tikzcd}
g^{-1}(U) \arrow[r, hook] \arrow[d, "\wr"] & X' \arrow[d, "g"'] \arrow[dd, bend left, "fg"] \\
U \arrow[r, hook] & X \arrow[d, "f"'] \\
& S.
\end{tikzcd}$$

We don't prove this lemma, for the proof, one can see [EGA], II, 5.6.1. We only prove that Chow's lemma implies theorem 4.8.

Proof. First we may assume S is noetherian and by the remark we let $\mathcal{F} \in Coh(X)$. We use noetherian induction on all closed subsets $T \subseteq X$ satisfying $\mathrm{Supp}(\mathcal{F}) \subset T$. We have to show that : if $Rf_*\mathcal{F} \in D^+(S)_{coh}$ for all $\mathcal{F} \in Coh(X)$ satisfying $\mathrm{Supp}(\mathcal{F}) \subsetneqq T$, then $Rf_*\mathcal{F} \in D^+(S)_{coh}$ for all $\mathcal{F}$ with $\mathrm{Supp}\mathcal{F} \subseteq T$.

We may assume $T = X$. Consider the composition of morphisms $\mathcal{F} \to g_*g^*\mathcal{F} \to Rg_*(g^*\mathcal{F})$, let $\mathcal{G}$ be the cone of this morphism, we get a distinguished triangle

$$\mathcal{F} \to Rg_*(g^*\mathcal{F}) \to \mathcal{G} \to .$$

Since g is projective and $g^*(\mathcal{F}) \in Coh(X')$, by the remark, we get $Rg_*(g^*\mathcal{F}) \in D^+(X)_{coh}$, and hence $\mathcal{G} \in D^+(X)_{coh}$. Over U, we have $\mathcal{F}|_U \xrightarrow{\sim} Rg_*g^*\mathcal{F}|_U$, so $\mathcal{G}|_U = 0$, that is $H^i\mathcal{G}|_U = 0$, $\mathrm{Supp}(H^i\mathcal{G}) \subset X - U \subsetneqq X$, by noetherian induction assumption, we get $Rf_*\mathcal{G} \in D^+(S)_{coh}$. We have the distinguished triangle

$$Rf_*\mathcal{F} \to Rf_*Rg_*(g^*\mathcal{F}) = \mathcal{H} \to Rf_*\mathcal{G} \to,$$

since $Rf_*Rg_* = R(fg)_*$, and fg is projective, we have $\mathcal{H} \in D^+(S)_{coh}$, so $Rf_*\mathcal{F} \in D^+(S)_{coh}$. □

Corollary 4.11. *If $f : X \to S$ is proper and affine, f is finite.*

Recall that $f : X \to S$ is *affine* if Y can be covered by affine open subschemes $S_i = \mathrm{Spec}\, A_i (i \in I)$ such that each $f^{-1}(S_i) = \mathrm{Spec}\, B_i$ is affine. f is called finite if f is affine and each B_i is finitely generated as A_i-module.

Proof. We may assume $S = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, and have to show B is finitely generated as A-module. Let $\mathcal{F} = \mathcal{O}_X = \mathcal{O}_{\mathrm{Spec}\, B}$ in theorem 4.8, by the remark, $B = H^0(X, \mathcal{O}_X)$ is a finitely generated A-module. □

5 Hilbert Polynomial

Let A be an artinian ring, B a graded A-algebra, i.e. $B = \bigoplus_{i=0}^{\infty} B_i$, with $B_0 = A$. Suppose B is finitely generated by B_1 over A (that is, $B = A[t_1, \cdots, t_n]/I$ for some homogenous ideal I), let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded

B-module(thus we have $M_n = 0$ for $n \ll 0$, and M_n is finitely generated over A for all n and then has finite length), then the function $n \mapsto \lg_A M_n$ is a polynomial in n for $n \gg 0$ (See [A-M], Prop 11.4), i.e. there exists $P \in \mathbb{Q}[t]$ and n_0, such that for all $n \geq n_0$, $\lg_A(M_n) = P(n)$(this function is called *Hilbert polynomial* of M). We will give a cohomological interpretation of P in 5.10. It will rely on the following theorem.

Let $S = \operatorname{Spec} A$, X/S projective, and $\mathcal{F} \in Coh(X)$. By the finiteness theorem, $H^q(X, \mathcal{F})$ is finitely generated A-module for each q, hence has finite length, and there exists a $d \geq 0$ such that $H^q(X, \mathcal{F}) = 0$ for $q > d$. So we can define $\chi(X, \mathcal{F}) = \sum_{i=1}^{\infty}(-1)^i \lg_A H^i(X, \mathcal{F})$.

Theorem 5.1 (Hilbert-Serre). *Let X, S and $\mathcal{F}$ are as above, and L be a very ample line bundle on X, then*

(1) there exists $P_{\mathcal{F}} \in \mathbb{Q}[t]$, such that $\chi(X, \mathcal{F} \otimes L^{\otimes n}) = P_{\mathcal{F}}(n)$ for all n;

(2) $\deg P_{\mathcal{F}} = \dim \operatorname{Supp}(\mathcal{F})$, where $\operatorname{Supp}(\mathcal{F}) = \{x \in X | \mathcal{F}_x \neq 0\} \subset X$.

To prove this, we need the following result.

Theorem 5.2 (Hilbert syzygies theorem). *Suppose k is a field, $R = k[t_1, \cdots, t_n]$, M is a finitely generated graded R-module. Then there exists a resolution of M of the form*

$$0 \to L^{-n} \to L^{-n+1} \to \cdots \to L^0 \to M \to 0$$

with each L^i being free finitely generated as a graded R-module.

Recall that a graded R-module L is free finitely generated if and only if L admits a basis over R consisting of homogenous elements $x_1, \cdots, x_m$. This is also equivalent to saying that $L \simeq \bigoplus_{i=1}^{m} R(-d_i)$, where $d_i = \deg(x_i)$.

Lemma 5.3 (graded Nakayama's lemma). *Suppose R, k are as in 5.2, and M is a graded R-module such that $M_n = 0$ for $n \ll 0$, then $M \otimes_R k = 0$ implies $M = 0$.*

Proof. Replacing M by $M(d)$ we may assume $M_n = 0$ for all $n < 0$. Let $R_+ = \bigoplus_{n>0} R_n = \operatorname{Ker}(R \to k)$, then $M = R_+ M$. Suppose $M \neq 0$, choose $x \in M_d$, $x \neq 0$ such that d is minimal. Write $x = \sum a_i x_i$, where a_i, x_i are homogenous and $a_i \in R_+$, $\deg x_i \geq d$. But this implies $\deg x \geq d + 1$, hence a contradiction. $\square$

Lemma 5.4. *Let M be a finitely generated R-module, the two conditions are equivalent :*

(1) M is free finitely generated ;

(2) $\mathrm{Tor}_1^R(k, M) = 0$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To prove (2) $\Rightarrow$ (1), we choose a homogeneous basis of $M \otimes_R k$ as graded k-mod, lift it to M, we get

$$0 \to Z \to L \to M \to 0$$

such that $L \otimes_R k \simeq M \otimes_R k$. From the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(k, M) \to Z \otimes_R k \to L \otimes_R k \xrightarrow{\sim} M \otimes_R k \to 0,$$

$\mathrm{Tor}_1^R(k, M) = 0$ implies $Z \otimes_R k \hookrightarrow L \otimes_R k$ is injective, then $Z \otimes_R k = 0$, hence $Z = 0$ by graded Nakayama's lemma. □

Lemma 5.5 (Koszul). *The Koszul complex of $(t_1, \cdots, t_n)$ is a resolution of k, i.e., $K.(t_1, \cdots, t_n)$ is quasi-isomorphic to k, where*

$$K.(t_1, \cdots, t_n) = (0 \to \overset{n}{\wedge} R^n \to \cdots \to \overset{1}{\wedge} R^n \to R \to 0).$$

Proof. In deed, $(t_1, \cdots, t_n)$ is a regular sequence, then use Theorem 2.2. □

Proof of Theorem 5.2. Since R is noetherian, we have a resolution of M

$$0 \to L^{-n} \to L^{-n+1} \to \cdots \to L^0 \to M \to 0$$

with each term being a finitely generated graded R-module, and L^i free for all $i \geq -n+1$. Since

$$\begin{aligned} \mathrm{Tor}_1^R(R, L^{-n}) &= \mathrm{Tor}_{n+1}^R(k, M) \\ &= H^{-n-1}(K.(x_1, \cdots, x_n) \otimes_R M) \\ &= 0, \end{aligned}$$

L^{-n} is also free. □

Now we begin to prove theorem 5.1.

Proof of Theorem 5.1. (1) Write $A = \prod_{1\leq i\leq m} A_i$, where each A_i is artinian local, then $S = \coprod S_i$, with $S_i = \mathrm{Spec}\, A_i$, $X = \coprod X_i$, each X_i/S_i is projective. Let $L_i = L|_{X_i}$, $\mathcal{F}_i = \mathcal{F}|_{X_i}$, then each L_i ia very ample, and $\chi(X, \mathcal{F}(n)) = \sum_{i=1}^{m} \chi(X_i, \mathcal{F}_i \otimes L_i^{\otimes n})$. So we may assume A is local artinian. Let $k = A/\mathfrak{m}$, $\mathfrak{m}^N = 0$. Consider the $\mathfrak{m}$-adic filtration of $\mathcal{F}$, $0 \subset \mathfrak{m}^{i+1}\mathcal{F} \subset \cdots \subset \mathfrak{m}\mathcal{F} \subset \mathcal{F}$. From the exact sequence

$$0 \to \mathfrak{m}^{i+1}\mathcal{F}(n) \to \mathfrak{m}^i\mathcal{F}(n) \to gr^i_{\mathfrak{m}}\mathcal{F}(n) \to 0,$$

we get $\chi(X, \mathcal{F}(n)) = \sum_{i=0}^{N-1} \chi(X, gr^i_{\mathfrak{m}}\mathcal{F}(n))$, so it is enough to show the theorem for $gr^i_{\mathfrak{m}}\mathcal{F}$, so we may assume $\mathfrak{m}\mathcal{F} = 0$. We have a cartesian diagram :

$$\begin{tikzcd}
X' \arrow[r, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Spec}\, A = S.
\end{tikzcd}$$

Then $X' \to X$ is a closed immersion, therefore $\mathcal{F}' = i^*\mathcal{F}$ is coherent as an $O_{X'}$-module. Since X/S is projective, $X'/\mathrm{Spec}\, k$ is projective. If we let $L' = i^*L$, then L' is very ample over $\mathrm{Spec}\, k$ and $\mathcal{F}(n) = i_*\mathcal{F}'(n)$, where $\mathcal{F}'(n) = \mathcal{F}' \otimes_{\mathcal{O}_{X'}} L'^{\otimes n}$. Since $\chi(X, \mathcal{F}(n)) = \chi(X', \mathcal{F}')$, we may assume A is a field. Finally for a suitable r, $X \hookrightarrow P = \mathbb{P}^r_A = \mathrm{Proj} A[t_0, \cdots, t_r]$ is a closed subscheme, so by a similar argument, we may assume $X = P$.

Now $A = k$, $B = k[t_0, \cdots, t_r]$, and $X = P$. We need the following lemma.

Lemma 5.6. *Let $B = k[t_0, \cdots, t_r]$, $P = \mathrm{Proj} B$, $\mathcal{F} \in Coh(P)$. Then there exists a finitely generated graded B-module M such that $\widetilde{M} = \mathcal{F}$.*

Proof. Choose a presentation of $\mathcal{F}$ (by Prop 4.1)

$$\mathcal{O}_P(n_1)^{m_1} \xrightarrow{d} \mathcal{O}_P(n_0)^{m_0} \to \mathcal{F} \to 0,$$

let

$$L_0 = \bigoplus_{n\in Z} \Gamma(P, \mathcal{O}_P(n_0)^{m_0}(n))$$

$$L_1 = \bigoplus_{n\in Z} \Gamma(P, \mathcal{O}_P(n_1)^{m_1}(n))$$

be the graded B-modules associated to L_1 and L_2 respectively. Each L_i is a free finitely generated graded module for $i = 1, 2$, and $\widetilde{L_i} = \mathcal{O}_P(n_i)^{m_i}$. We have a canonical morphism $u : L_1 \to L_0$. Define $M = \operatorname{Coker} u$, we have the following commutative diagram

$$\begin{tikzcd}
\widetilde{L_1} \arrow[r] \arrow[d, "\simeq"] & \widetilde{L_0} \arrow[r] \arrow[d, "\simeq"] & \widetilde{M} \arrow[r] \arrow[d, dotted] & 0 \\
\mathcal{O}_P(n_1)^{m_1} \arrow[r] & \mathcal{O}_P(n_0)^{m_0} \arrow[r] & \mathcal{F} \arrow[r] & 0,
\end{tikzcd}$$

so we have an isomorphism $\widetilde{M} \to \mathcal{F}$. □

By this lemma, $\mathcal{F} = \widetilde{M}$ for some finitely generated graded B-module M. Let

$$0 \to L^{-r-1} \to \cdots \to L^0 \to M \to 0$$

be a resolution by free finitely generated B-module, apply the exact functor $' \sim'$, we get

$$0 \to \widetilde{L^{-r-1}} \to \cdots \to \widetilde{L^0} \to \mathcal{F} \to 0,$$

where each $\widetilde{L^i}$ is a finite sum of $\mathcal{O}_P(-d)'s$, as L^i is a finite sum of $B(-d)'s$. We know that $\chi(X, \mathcal{F}(n)) = \sum_{i=0}^{-r-1} (-1)^i \chi(X, \widetilde{L^i}(n))$, so we may assume $\mathcal{F} = \mathcal{O}_P(n)$, and part (1) of theorem 5.1 follows from the following lemma. □

Lemma 5.7. *Let* $\binom{x+r}{r} = \frac{(x+r)\cdots(x+1)}{r!} \in \mathbb{Q}[x]$, *then* $\chi(P, \mathcal{O}_P(n)) = \binom{n+r}{r}$.

Proof. We have proved that $H^q(P, \mathcal{O}_P(n)) = 0$ for all n when $q \neq 0, r$, then $\chi(\mathcal{O}_P(n)) = \dim_k H^0(P, \mathcal{O}_P(n)) + (-1)^r \dim_k H^r(P, \mathcal{O}_P(n))$, also we have

$$H^0(P, \mathcal{O}_P(n)) = \begin{cases} R_n & n \geq 0 \\ 0 & n < 0 \end{cases},$$

and

$$H^r(P, \mathcal{O}_P(n)) = \begin{cases} \bigoplus A \frac{t^\alpha}{t_0 \cdots t_r} & n \leq -r-1 \\ 0 & n > -r-1 \end{cases},$$

where $t^\alpha = {t_0}^{\alpha_0} \cdots {t_r}^{\alpha_r}$ with $\alpha_i \leq 0$ and $\sum \alpha_i - r - 1 = n$. Since $r \geq 1$, there are three cases :

(1) case $n \geq 0$, we have $H^r(P, \mathcal{O}_P(n)) = 0$, and $\dim_k H^0(P, \mathcal{O}_P(n)) = \dim_k B_n = \binom{n+r}{r}$;

(2) case $n \leq -r-1$, then $H^0(P, \mathcal{O}_P(n)) = 0$, and $\dim_k H^r(P, \mathcal{O}_P(n)) = \dim_k H^0(P, \mathcal{O}_P(-n-r-1)) = \binom{-n-1}{r} = (-1)^r \binom{n+r}{r}$;

(3) case $-r-1 < n < 0$, then the two terms are all zero, note that in this case $0 \leq n+r < r$, $\binom{n+r}{r} = 0$ by definition.

therefore the lemma holds. $\square$

We will now prove (2) of theorem 5.1, we first need recall some facts.

(1) Suppose A is noetherian, M is a finitely generated A-module, one defines $\mathrm{Ass}(M)$ as the set of $\mathfrak{p} \in \operatorname{Spec} A$ such that $\mathfrak{p}$ is the annihilator $\mathrm{Ann}(x)$ of some $x \in M$. We know $\mathfrak{p} \in \mathrm{Ass}(M) \Leftrightarrow \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}(M_{\mathfrak{p}})$. Now let X be a noetherian scheme, $\mathcal{F} \in \mathit{Coh}(X)$, for $x \in X$, we define $\mathrm{Ass}(\mathcal{F})$ as the set of points $x \in X$ such that $\mathfrak{m}_x \in \mathrm{Ass}(\mathcal{F}_x) \subset \operatorname{Spec} \mathcal{O}_{X,x}$. As in the affine case, $\mathrm{Ass}(\mathcal{F})$ is finite, and contains the maximal points of $\mathrm{Supp}(\mathcal{F})$. If $s \in \Gamma(X, \mathcal{O}_X)$, such that $s(x) \neq 0$ for all $x \in \mathrm{Ass}(\mathcal{F})$, then $\mathcal{F} \xrightarrow{s} \mathcal{F}$ is injective.

(2) If A is a noetherian ring and M is a finitely generated A-module, let $S = \operatorname{Spec}(A/\mathrm{Ann}(M))$. Then

$$\begin{aligned} \dim M &= \dim \mathrm{Supp} M = \dim S \\ &= \sup\nolimits_{x \in S} \dim M_x. \end{aligned}$$

For $\mathcal{F} \in \mathit{Coh}(X)$, we define the dimension of $\mathcal{F}$, $\dim \mathcal{F}$, as the dimension of the suppose of $\mathcal{F}$, $\mathrm{Supp}\, \mathcal{F}$. This is a closed subset of X_0. We have $\dim \mathcal{F} = \sup\limits_{x \in \mathrm{Supp}(\mathcal{F})} \dim \mathcal{F}_x$.

Lemma 5.8. *Let $Z \subset X$ be finite with X as in 5.1. Then there exists $n \geq 0$ and $f \in \Gamma(X, \mathcal{O}(n))$ such that $f(x) \neq 0$ for all $x \in Z$.*

Proof. As X is a closed subscheme of $\mathbb{P}^r_A$, we may assume $X = \mathbb{P}^r_A = \mathrm{Proj} B$, $B = A[t_0, \cdots, t_r]$. Each $x \in Z$ corresponds to some homogenous prime ideal $\mathfrak{p}_x \in \mathrm{Proj} B$. Since for each $x \in Z$, $B_+ = \bigoplus_{n>0} B_n$ is not contained in $\mathfrak{p}_x$, we can find a homogenous element $f \in B_n$ such that $f \notin \mathfrak{p}_x$ for all $x \in Z$ ([B] III.§1.4, Prop 8), then $f \in \Gamma(P, \mathcal{O}_P(n))$ and $f(x) \neq 0$ for all x. $\square$

Proof of theorem 5.2 (2). Use induction on $d = \dim \mathrm{Supp} \mathcal{F}$. Putting some scheme structure on $\mathrm{Supp}\, \mathcal{F}$, e.g., the reduced scheme structure, we may assume $X = \operatorname{Supp}\mathcal{F}$. When $d = 0$, that is X is zero dimensional, X is affine and $\Gamma(X, \mathcal{O}_X)$ is an artinian ring. Then for $n \geq 1$, $\mathcal{F}(n) = \mathcal{F}$, so

$$\chi(X, \mathcal{F}(n)) = \lg H^0(X, \mathcal{F}(n)) = \lg H^0(X, \mathcal{F}),$$

then $\chi(X, \mathcal{F}(n))$ is a constant, i.e., $\deg P_{\mathcal{F}} = 0$. Now assume (2) holds for all $\mathcal{F}$ satisfying $\deg P_{\mathcal{F}} \leq d - 1$. By the lemma above, there exists $f \in \Gamma(X, \mathcal{O}(m))$ such that $f(x) \neq 0$ for all $x \in \operatorname{Ass}(\mathcal{F})$, so we get an exact sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{F}(m) \to \mathcal{G} \to 0 \qquad (*).$$

Then we get the following exact sequence by taking stalks

$$0 \to \mathcal{F}_x \xrightarrow{f_x} \mathcal{F}_x(m) \to \mathcal{G}_x \to 0.$$

Because of the choice of f, we have $\operatorname{Supp}(\mathcal{G}) = \{x \in X | f(x) = 0\}$. Assume $d \geq 1$, then $\operatorname{Supp}\mathcal{G} \neq \varnothing$, moreover, we have the following lemma

Lemma 5.9. *Let A, X, L be as above, then for any $f \in \Gamma(X, L^{\otimes n})$, the set $V(f)$ of $x \in X$ such that $f(x) = 0$ meets every irreducible closed subset of X of positive dimension.*

Proof. Let Y be an irreducible closed subset of X not meeting $V(f)$, then $Y \subset X_f$ where X_f is open and affine over S. Thus Y/S is proper and affine, hence finite, which implies $\dim Y = 0$. $\square$

By dimension theory of noetherian local rings (See [A-M], Prop 11.3), we know $\dim \mathcal{G}_x = \dim \mathcal{F}_x - 1$ for all $x \in V(f)$. Then

$$\begin{aligned}\dim \mathcal{G} &= \sup_{\substack{f(x)=0 \\ x \text{ closed}}} \dim \mathcal{G}_x = \sup_{\substack{f(x)=0 \\ x \text{ closed}}} \dim \mathcal{F}_x - 1 \\ &\overset{(**)}{=} \sup_{x \text{ closed}} \dim \mathcal{F}_x - 1 = \dim \mathcal{F} - 1.\end{aligned}$$

Here for the equality $(**)$, we can choose such a irreducible closed subset T of X that $\dim T = 1$ and $\operatorname{codim}(T, X) = \dim X - 1$, then for any closed point $x \in T$, $\dim \mathcal{F}_x = \dim \mathcal{F}$, by lemma 5.9, $V(f) \cap T \neq \varnothing$, hence contains a closed point.

From the exact sequence $(*)$, we have

$$\chi(\mathcal{G}(n)) = \chi(\mathcal{F}(m+n)) - \chi(\mathcal{F}(n)) = P_{\mathcal{F}}(m+n) - P_{\mathcal{G}}(n).$$

By induction, $\deg P_{\mathcal{G}} = \dim \operatorname{Supp}\mathcal{G} = d - 1$, so $\deg P_{\mathcal{F}} = d$. $\square$

Corollary 5.10. *Let B be a graded A-algebra finitely generated by B_1 over $B_0 = A$, M be a finitely generated graded B-module, then* $\lg_A M_n = \chi(P, \mathcal{F}(n))$ *for $n \gg 0$, where $P = \mathrm{Proj} B$, $\mathcal{F} = \widetilde{M}$; moreover* $\deg P_{\mathcal{F}}(n) = \dim M - 1$.

Proof. We may assume B is the polynomial algebra $k[t_0, \cdots, t_n]$. Let

$$L^{-1} \to L^0 \to M \to 0$$

be a presentation of M. Applying the functor $' \sim '$, we get

$$\widetilde{L^{-1}} \to \widetilde{L^0} \to \mathcal{F} \to 0$$

and

$$\widetilde{L^{-1}}(n) \to \widetilde{L^0}(n) \to \mathcal{F}(n) \to 0.$$

For $n \gg 0$, we get the following exact sequence

$$\Gamma(P, \widetilde{L^{-1}}(n)) \to \Gamma(P, \widetilde{L^0}(n)) \to \Gamma(P, \mathcal{F}(n)) \to 0$$

For any graded B-module E, we define a canonical morphism $E_n \to \Gamma(P, \tilde{E}(n))$ given by $f \mapsto f/1 \in \Gamma(P_{(t)}, \tilde{E}(n))$, where $t \in B_1$. We have the following commutative diagram :

$$\begin{tikzcd}
L_n^{-1} \arrow[r] \arrow[d, "\wr"] & L_n^0 \arrow[r] \arrow[d, "\wr"] & M_n \arrow[r] \arrow[d, "u"] & 0 \\
\Gamma(P, \widetilde{L^{-1}}(n)) \arrow[r] & \Gamma(P, \widetilde{L^0}(n)) \arrow[r] & \Gamma(P, \mathcal{F}(n)) \arrow[r] & 0,
\end{tikzcd}$$

in which the two left vertical maps are isomorphisms. So u is an isomorphism for $n \gg 0$. On the other hand, we know that $H^q(P, \mathcal{F}(n)) = 0$ for $q > 0$ and $n \gg 0$, so $\chi(P, \mathcal{F}(n)) = \lg_A H^0(P, \mathcal{F}(n)) = \lg_A M_n$ for $n \gg 0$. $\square$

Remarks on the Riemann-Roch problem:

Let X be a proper schemes over a field k, and $\mathcal{F} \in Coh(X)$. We want to calculate

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}) \in \mathbb{Z}$$

In fact, this calculation, combined with vanishing theorem yields information on $H^0(X, \mathcal{F})$, which have geometric consequences.

(a) *The case of curves*

Suppose k is an algebraic closed field. X/k is a projective and smooth curve (i.e. For any $x \in X$ closed, the local rings $\mathcal{O}_{X,x}$ are discrete valuation rings.).

Let $\mathcal{F} = \mathcal{O}_X$, then $\chi(X, \mathcal{O}_X) = \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X)$. We have $\dim_k H^0(X, \mathcal{O}_X) = 1$ (because $H^0(X, \mathcal{O}_X)$ is a finite k-algebra contained in $K = k(\eta)$, where η is the generic point of X), on the other hand, $g = \dim_k H^1(X, \mathcal{O}_X)$ is the *genus* of X.

For $k = \mathbb{C}$, the set of rational points $X(\mathbb{C})$ is a Riemann surface. We conclude that $H^1(X, \mathcal{O}_X)$ is dual to $H^0(X, \Omega^1_X)$ (duality theorem()), $g = b_1/2$, $b_1 = \operatorname{rank} H^1(X(\mathbb{C}), \mathbb{Z}) = \operatorname{rank} \pi_1(X(\mathbb{C}))$. Suppose $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$, where $\mathcal{J}$ is a non zero ideal of $\mathcal{O}_X$. Then $\mathcal{J}$ is a line bundle, and the subscheme (a divisor) D defined by $\mathcal{J}$ is finite over k. We have $\mathcal{O}(D) = \mathcal{J}^{\otimes -1}$.Then Riemann's half of the Riemann Roch theorem says that $\chi(X, \mathcal{O}(D)) = \deg D + 1 - g$ (this is an exercise, use induction on $\deg D = \dim_k H^0(X, \mathcal{O}_D) = \sum_{x \in D} \dim_{k(x)} \mathcal{O}_{D,x}$; for $x \in D$, use the exact sequence $0 \longrightarrow \mathcal{O}_{D^1} \longrightarrow \mathcal{O}_D \longrightarrow k(x) \longrightarrow 0$). Roch's half of the Riemann Roch theorem is that $H^1(X, \mathcal{O}_X)$ is dual to $H^0(X, \Omega^1_X)$.

(b) *The case of surfaces*

For k algebraic closed, X/k regular, proper, and irreducible of dimension 2, M.Noether gave the formula that $\chi(X, \mathcal{O}_X) = \frac{c_1^2 + c_2}{12}$, where c_1^2 and c_2 are certain integers (Chern numbers) defined by intersection theory. On the other hand, for a divisor D on X, $\chi(X, \mathcal{O}(D)) - \chi(X, \mathcal{O})$ is a certain intersection invariant (see e.g. [H]).

(c) *The further development*

In 1956, Hirzebruch gave a general formula for any proper smooth scheme X/k, and any vector bundle $\mathcal{F}$ on X, (?) $\chi(X, \mathcal{F}) = \deg(ch(\mathcal{F}) \cdot \text{Todd } T_X)$, where T_X is the tangent bundle of X and ch and $Todd$ are certain intersection invariants involving Chern classes.

In 1957, Grothendieck gave a far reaching generalization of this formula for certain morphisms $X \longrightarrow Y$. Let us also mention that in 1963, Atiyah and Singer gave a formula for the index of an elliptic operator for smooth manifolds over $\mathbb{C}$, generalizing the Hirzebruch formula.

Chapter 3

Differential calculus, smooth and étale morphisms

1 Kähler differentials and derivations

Definition 1.1. Let A be a commutative ring, B be an A-algebra, $M \in Mod(B)$. A map $D : B \to M$ is called an A-*derivation* of B with values in M (or from B to M), if it satisfies the following two conditions:

1) D is A-linear;

2) for any $x, y \in B$, $D(xy) = xD(y) + yD(x)$.

We denote by $\mathrm{Der}_A(B, M)$ the set of A-derivations from B to M. For any $D \in \mathrm{Der}_A(B, M)$, $b \in B$, the map $bD : x \in B \mapsto bD(x)$ is an A-derivation, thus $\mathrm{Der}_A(B, M)$ is a B-module.

Definition 1.2. Let $B \otimes_A B \to B$ be a morphism defined as $x \otimes y \mapsto xy$. Denote by I the kernel of this map, and put $\Omega^1_{B/A} = I/I^2$. We call $\Omega^1_{B/A}$ the *Kähler differential module of* B/A. Note that $\Omega^1_{B/A}$ is a $B \otimes_A B$-module killed by I, so it is a B-module.

Define $B \to B \otimes_A B$: $b \mapsto b \otimes 1$ (resp.$b \mapsto 1 \otimes b$), so we have a left (resp.right) B-algebra structure on $B \otimes_A B$. It is easy to see that these two B-algebra structures induce the same B-algebra structure on $\Omega^1_{B/A}$. Put $P^1_{B/A} = B \otimes_A B/I^2$ (principal parts or 1-jets of B/A). We have an exact sequence: $0 \to \Omega^1_{B/A} \to P^1_{B/A} \to B \to 0$, which splits by the morphism j_1, j_2, where $j_1(b) = b \otimes 1 \mod I^2$ and $j_2(b) = 1 \otimes b \mod I^2$, so we have $P^1_{B/A} = B \oplus \Omega^1_{B/A}$.

Definition 1.3. Let A be a commutative ring, B be an A-algebra. Define a map $d_{B/A}\colon B \to \Omega^1_{B/A}$ by $b \mapsto 1 \otimes b - b \otimes 1 \mod I^2 = j_2(b) - j_1(b)$.

Proposition 1.4. *Let A be a commutative ring, B be an A-algebra. Then* $\Omega^1_{B/A} = B \cdot d_{B/A}(B)$

This follows from the following lemma:

Lemma 1.5. *For any $b_i \in B$, $x_i \in B$ $(1 \leq i \leq n)$, we have* $\sum b_i \otimes x_i = \sum (b_i \otimes 1)(1 \otimes x_i - x_i \otimes 1) + \sum b_i x_i \otimes 1 = \sum (1 \otimes x_i)(b_i \otimes 1 - 1 \otimes b_i) + \sum 1 \otimes b_i x_i$ *In particular, I is generated over B (for the left (resp. right) structure) by the elements of the form $1 \otimes x - x \otimes 1$.*

Theorem 1.6. *(1) We have* $d_{B/A}\colon B \to \Omega^1_{B/A} \in \mathrm{Der}_A(B, \Omega^1_{B/A})$

(2) For any $M \in Mod(B)$,

$$\mathrm{Hom}(\Omega^1_{B/A}, M) \longrightarrow \mathrm{Der}_A(B, M), \quad u \mapsto u d_{B/A} \qquad (*)$$

is an isomorphism.

Proof. (1) follows directly from the following formula

$$1 \otimes xy - xy \otimes 1 = (1 \otimes x)(1 \otimes y - y \otimes 1) + (1 \otimes y)(1 \otimes x - x \otimes 1).$$

(2) The injectivity of $(*)$ follows from **1.4**. For the surjectivity, we need a lemma.

Lemma 1.7. *If M is a B-module, the B-algebra $D_B(M) = B \oplus M$, where $(b_1 \oplus m_1)(b_2 \oplus m_2) = (b_1 b_2) \oplus (b_2 m_1 + b_1 m_2)$, we have an exact sequence of B-modules:*

$$0 \longrightarrow M \longrightarrow B \oplus M \xrightarrow{p} B \longrightarrow 0$$

where p, given by $p(b \oplus m) = b$, is a ring homomorphism. Then $\mathrm{Der}_A(B, M)$ *is identified to the set H of homomorphisms of A-algebras $f : B \to D_B(M)$ such that $p \circ f = Id$, by associating to D the homomorphism $f_D : x \mapsto x + D(x)$.*

Proof. The inverse map is given by $f \mapsto D_f$, $D_f(x) = f(x) - x$. □

Let us prove the surjectivity in (2). Let $D \in \mathrm{Der}_A(B, M)$. By the above lemma, D corresponds to a homomorphism of A-algebras from B to $D_B(M)$. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d] & B \otimes_A B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & D_B(M) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

where the middle vertical arrow is defined by: $x \otimes y \mapsto x f_D(y)$. And the left vertical arrow induces a map $u : I/I^2 \to M$ (because it maps I^2 to zero). So we have a map $u : \Omega^1_{B/A} \to M$, $ud_{B/A}(b) = u(1 \otimes b - b \otimes 1) = f_D(b) - b = D(b)$. This completes the proof. □

Definition 1.8. Let R be a commutative ring, E be an R-module. Recall that the symmetric algebra is the graded R-algebra $S_R(E) = \bigoplus_{n \geqslant 0} S^n_R(E) = (\bigoplus_{n \geqslant 0} (\otimes^n E))/T$, where T is the two sided ideal generated by elements of the form $x \otimes y - y \otimes x$ for some $x, y \in E$. In particular, $S^0_R(E) = R$, $S^1_R(E) = E$. $S_R(E)$ is sometimes denoted by $\mathrm{Sym}_R(E)$. This algebra satisfies the universal property $\mathrm{Hom}_{R-alg}(S_R(E), C) = \mathrm{Hom}_R(E, C)$, where C is an R-algebra, and the correspondence is defined by: $f \mapsto f|E = S^1_R(E)$.

Proposition 1.9. *Let A be a commutative ring, E be an A-module and $B = S_A(E)$. Then $\Omega^1_{B/A} \simeq B \otimes_A E$.*

Proof. We have a sequence of canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_B(\Omega^1_{B/A}, M) &= \{D \in \mathrm{Der}_A(B, M)\} \\
&= \{f \in \mathrm{Hom}_{A-\mathrm{alg}}(B, D_B(M)); f(x) = x + D(x)\} \\
&= \{u \in \mathrm{Hom}_{A-\mathrm{mod}}(E, B \oplus M); u(x) = x + D(x)\} \\
&= \mathrm{Hom}_A(E, M) \\
&= \mathrm{Hom}_B(B \otimes E, M)
\end{aligned}$$

So we have $B \otimes E \simeq \Omega^1_{B/A}$, and the correspondence is: $b \otimes x \mapsto bd_{B/A}x$.

□

Corollary 1.10. *Let $B = A[\{x_i\}_{i \in I}] = S_A(A^{(I)})$, where $\{x_i\}_{i \in I}$ is a basis of $A^{(I)}$. Then $\Omega^1_{B/A}$ is a free B-module with basis $\{d_{B/A}x_i\}_{i \in I}$.*

For $f \in B = A[\{x_i\}]$, we have $df = \sum_{i\in I} \frac{\partial f}{\partial x_i} dx_i$, where $\frac{\partial}{\partial x_i} : f \mapsto \frac{\partial f}{\partial x_i} \in \mathrm{Der}_A(B, B) = \mathrm{Hom}(\Omega^1_{B/A}, B)$. We have a natural pairing

$$\Omega^1_{B/A} \times \mathrm{Hom}(\Omega^1_{B/A}, B) \to B, \qquad (w, D) \mapsto < w, D >= D(w).$$

It is obvious that $< dx_i, \frac{\partial}{\partial x_j} >= \delta_{ij}$. Also we have $d(x_i^n) = nx_i^{n-1} dx_i$, $\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_i \partial x_j}$.

Definition 1.11. A *thickening of order 1* is a closed immersion $T_0 \overset{i}{\hookrightarrow} T$ with the ideal sheaf I such that $I^2 = 0$. More generally, a *thickening of order* n is a closed immersion defined by an ideal I such that $I^{n+1} = 0$.

Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of T, then $U \cap T_0 = \mathrm{Spec}(A_0)$ is also affine, and we have an exact sequence:

$$0 \to \mathcal{J} \to A \to A_0 \to 0.$$

Here $\mathcal{J}$ is an ideal of A such that $\mathcal{J}^2 = 0$ and $\widetilde{\mathcal{J}} = I|_U$. Note that $T_0 \hookrightarrow T$ is a homeomorphism on the underlying topological spaces and we have an exact sequence:

$$0 \to I \to \mathcal{O}_T \to \mathcal{O}_{T_0} \to 0.$$

Since $I^2 = 0$, I is an $\mathcal{O}_{T_0}$-module, and I is quasi-coherent on T_0.

Let $i : X \hookrightarrow Z$ be an immersion of schemes. Then i can be factorized as a closed immersion followed by an open immersion as in the following diagram.

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, hook, "j"'] & & Z \\ & U \arrow[ur, hook, "\text{open}"'] & \end{tikzcd}$$

Let $I \in Qcoh(U)$ be the ideal sheaf of the closed immersion and Z_1 be the scheme defined by $(|X|, \mathcal{O}_U/I^2)$. Then we have a factorization $X \overset{j}{\hookrightarrow} Z_1 \hookrightarrow U \overset{open}{\hookrightarrow} Z$. In the affine case, this corresponds to the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] & 0, \\ & & A_1 \arrow[ur] & & \end{tikzcd}$$

where $B = A/I,\ A_1 = A/I^2$. Z_1 is called the *first infinitesimal neighborhood* of X in Z. It is a thickening of order 1. We have an exact sequence:

$$0 \to I/I^2 \to \mathcal{O}_{Z_1} \to \mathcal{O}_X \to 0$$

The quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}_j = \mathcal{N}_{X/I} = I/I^2 \in Qcoh(X)$ is called the *conormal sheaf* of j.

Definition 1.12. Let $f:\ X \to Y$ be a morphism of schemes. Let $\Delta:\ X \to X \times_Y X$ be the diagonal map, this is an immersion. We put

$$\Omega^1_{X/Y} = \mathcal{N}_\Delta = \mathcal{N}_{X/X\times_Y X}$$

This is a quasi-coherent sheaf on X. If we have the following diagram:

$$\begin{tikzcd} U = \mathrm{Spec}\ B \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ V = \mathrm{Spec}\ A \arrow[r, hook] & Y \end{tikzcd}$$

where U(resp.V) is open in X(resp.V), then we get a commutative square.

$$\begin{tikzcd} \mathrm{Spec}\ (B \otimes_A B) \arrow[r, hook] & X \times_Y X \\ \mathrm{Spec}\ B \arrow[r, hook] \arrow[u] & X \arrow[u] \end{tikzcd}$$

where the left vertical arrow is defined by $B \otimes_A B \to B$, $b \otimes c \mapsto bc$. Denote by I the kernel of this map:

$$0 \to I \to B \otimes_A B \to B \to 0$$

Then we have

$$\Omega^1_{X/Y}|\ U = \widetilde{I/I^2} = \widetilde{\Omega^1_{B/A}}$$

Let $M \in Mod(X)$, a *Y-derivation* of $\mathcal{O}_X$ with values in M (or from $\mathcal{O}_X$ to M) is a map $D:\ \mathcal{O}_X \to M$, satisfying the following conditions:

(1) D is $f^{-1}(\mathcal{O}_Y)$-linear;

(2) for any $a, b \in \mathcal{O}_X(U)$, where U is an open set contained in X, we have $D(ab) = aD(b) + bD(a)$.

Let $d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y}$ defined by $d_{X/Y} = j_2^* - j_1^*$, where $j_1^* b = b \otimes 1 \ mod \ I^2$, $j_2^* b = 1 \otimes b \ mod \ I^2$.

Theorem 1.13. *(1)* $d_{X/Y} \in \mathrm{Der}_Y(\mathcal{O}_X, \Omega^1_{X/Y})$*;*

(2) For any $M \in Mod(X)$, $\mathrm{Hom}(\Omega^1_{X/Y}, M) \to \mathrm{Der}_Y(\mathcal{O}_X, M)$ $u \mapsto u \circ d_{X/Y}$ *is an isomorphism.*

Proof. For $M \in Qcoh(X)$, the bijectivity of (2) follows from **1.6** and the above discussion in **1.12**. For the general case, see **EGA IV 16.5.3**. □

Let

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"] & Y \end{tikzcd} \qquad (*)$$

be a commutative diagram. We deduce commutative diagrams.

$$\begin{tikzcd} X' \times_{Y'} X' \arrow[r, "g \times g"] & X \times_Y X \\ X' \arrow[u, "\Delta'"] \arrow[r, "g"] & X \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} Z_1' \arrow[r] & Z_1 \\ X' \arrow[u, "i'"] \arrow[r] & X \arrow[u, "i"'] \end{tikzcd}$$

From the last one, we get a canonical homomorphism: $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ and by adjunction, this corresponds to a homomorphism: $\Omega^1_{X/Y} \to g_*\Omega^1_{X'/Y'}$ where $d_{X/Y}(b) \mapsto d_{x'/Y'}(g_*b)$, for b a local section of $\mathcal{O}_X$, with image g_*b as a section of $g_*\mathcal{O}_{X'}$. In the affine case, these maps is induced by the following one:

$$\begin{array}{ccc} B' \otimes_B \Omega^1_{B/A} & \to & \Omega^1_{B'/A'} \\ b' \otimes d_{B/A} & \mapsto & b'dg(b) \end{array}$$

This homomorphism satisfies an obvious transitivity property for a composition of commutative squares.

Proposition 1.14. *If* $(*)$ *is cartesian, the canonical map* $g^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ *is an isomorphism.*

Proof. This is a local question, hence we may assume that all the schemes involved are affine. Then we get a commutative diagram of rings

$$\begin{tikzcd} B' & B \arrow[l, "g"'] \\ A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

with $A' \otimes_A B \simeq B'$. The sequence

$$0 \to I \to B \otimes_A B \to B \to 0$$

splits as a sequence of A-module. Hence by applying $\otimes_A A'$, we get an exact sequence

$$0 \to A' \otimes_A I \to B' \otimes_{A'} B' \to B' \to 0$$

Therefore we have $A' \otimes_A I \simeq I'$ and a surjection $A' \otimes_A I^2 \twoheadrightarrow I'^2$. Consider the diagram with the exact rows

$$\begin{tikzcd} & A' \otimes I^2 \arrow[r] \arrow[d] & A' \otimes I \arrow[r] \arrow[d] & A' \otimes I^2/I^2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I'^2 \arrow[r] & I' \arrow[r] & I'/I'^2 \arrow[r] & 0 \end{tikzcd}$$

The left vertical arrow is surjective and the middle one is bijective, so using the snake lemma we have $B' \otimes_B \Omega^1_{B/A} = A' \otimes_A I^2/I^2 \simeq I'/I'^2 = \Omega^1_{B'/A'}$ □

Proposition 1.15. *Let $f : X \to Y$, $g : Y \to S$ be two morphisms of schemes, then we have an exact sequence* $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$.

Proof. This is again a local question, hence we may assume $S = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$ are all affine. Then in this case, we need to show the following sequence is exact:

$$C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0\ .$$

So we only need to show the sequence below is exact for any C-module M:

$$0 \longrightarrow \operatorname{Hom}(\Omega^1_{C/B}, M) \longrightarrow \operatorname{Hom}(\Omega^1_{C/A}, M) \longrightarrow \operatorname{Hom}(C \otimes_B \Omega^1_{B/A}, M)\ .$$

This is equivalent to show that this sequence

$$0 \longrightarrow \operatorname{Der}_B(C, M) \longrightarrow \operatorname{Der}_A(C, M) \longrightarrow \operatorname{Der}_A(B, M)$$

is exact, which is followed from a direct calculation. □

Corollary 1.16. *Let*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d] \\ Y' \arrow[r, "h"] & Y \end{tikzcd}$$

be a commutative square. If it is cartesian, then we have a canonical isomorphism

$$f'^*\Omega^1_{Y'/Y} \oplus g^*\Omega^1_{X/Y} \simeq \Omega^1_{X'/Y}.$$

Proof. First we have an exact sequence as follows:

$$g^*\Omega^1_{X/Y} \longrightarrow \Omega^1_{X'/Y} \longrightarrow \Omega^1_{X'/X} \longrightarrow 0\ .$$

Using the canonical isomorphism $g^*\Omega^1_{X/Y} \simeq \Omega^1_{X'/Y'}$, we get a commutative diagram

$$\begin{tikzcd} g^*\Omega^1_{X/Y} \arrow[r] \arrow[d, "\simeq"'] & \Omega^1_{X'/Y} \arrow[r] \arrow[dl] & \Omega^1_{X'/X} \arrow[r] & 0\ . \\ \Omega^1_{X'/Y'} & & & \end{tikzcd}$$

This implies that $\Omega^1_{X'/Y} = g^*\Omega^1_{X/Y} \oplus \Omega^1_{X'/X}$, hence $\Omega^1_{X'/Y} \simeq g^*\Omega^1_{X/Y} \oplus f'^*\Omega^1_{Y'/Y}$ as we have $\Omega^1_{X'/X} \simeq f'^*\Omega^1_{Y'/Y}$. ☐

Corollary 1.17. *Let $A \to B$ be a ring extension, $S \subset B$ be a multiplicative system, then $S^{-1}\Omega^1_{B/A} \simeq \Omega^1_{S^{-1}B/A}$.*

Proof. This is because any A-derivation $D : B \to M$ can be extends to an A-derivation $D' : S^{-1}B \to S^{-1}M$ by defining $D'(b/s) = s^{-2}(sDb - bDs)$, for $b \in B,\ s \in S$. ☐

Corollary 1.18. *Let $f : X \to Y$ be a morphism of schemes, $x \in X,\ y = f(x)$ be two points, then we have a canonical isomorphism $(\Omega^1_{X/Y})_x \simeq \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}}$.*

Proof. This is a local question, hence we may assume $Y = \operatorname{Spec} A,\ X = \operatorname{Spec} B$ are affine. So we get the following isomorphisms

$$\left(\Omega^1_{B/A}\right)_x \simeq \left(\Omega^1_{B/A}\right) \otimes_B B_x \simeq \Omega^1_{B_x/A}$$

On the other hand, we have an exact sequence

$$B_x \otimes_{A_y} \Omega^1_{A_y/A} \longrightarrow \Omega^1_{B_x/A_y} \longrightarrow \Omega^1_{B_x/A} \longrightarrow 0\ .$$

Since $\Omega^1_{A_y/A} = 0$, so we have $\Omega^1_{B_x/A_y} \simeq \Omega^1_{B_x/A}$ which follows that $\Omega^1_{B_x/A_y} \simeq \left(\Omega^1_{B/A}\right)_x$. □

Proposition 1.19. *Let*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

be a commutative diagram with i a closed immersion defined by an ideal I. Let $\mathcal{N}_{X/Z} = I/I^2$ be conormal sheaf of i. Then $d_{Z/Y} : \mathcal{O}_Z \to \Omega^1_{Z/Y}$ induces an $\mathcal{O}_X$-linear map: $\mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y} = \Omega^1_{Z/Y}/I\Omega^1_{Z/Y}$ and the sequence

$$\mathcal{N}_{X/Z} \longrightarrow i^*\Omega^1_{Z/Y} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact. Moreover if $i_1 : X \to Z_1$, the first infinitesimal neighborhood of i, admits a restriction, then the sequence

$$0 \longrightarrow \mathcal{N}_{X/Z} \longrightarrow i^*\Omega^1_{Z/Y} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact and split.

Proof. For the first statement, we only need to show that the induce map $d : I \to i^*\Omega^1_{Z/Y}$ maps I^2 to 0. This is a local question, so we may assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$ are all affine, and $B = C/J$ for an ideal J of C. Then we have to show the map $J \to \Omega^1_{C/A}/J\Omega^1_{C/A}$ induced by $d_{C/A}$ maps J^2 to 0. Indeed, for any $a, b \in J$, $d_{C/A}(ab) = ad_{C/A}b + bd_{C/A}a \in J\Omega^1_{C/A}$, so $d_{C/A}(J^2) \subset J\Omega^1_{C/A}$, so $d_{C/A}$ induces a map $d : J/J^2 \to \Omega^1_{C/A}/J\Omega^1_{C/A}$ and d is C-bilinear (hence B-bilinear), because for any $a \in C$, $b \in J$, we have $d(ab) = ad(b) + bd(a) = ad(b)$.

Now in order to show the exactness of the sequence in the proposition, we may also focus on the affine case. So again, we assume that $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$ are all affine, and $B = C/J$ for an ideal J of C. Then the sequence corresponds to a sequence of B-modules

$$J/J^2 \longrightarrow B \otimes \Omega^1_{C/A} \longrightarrow \Omega^1_{B/A} \longrightarrow 0 \qquad (*)\,.$$

We only need to show that for any $M \in Mod(B)$,

$$0 \longrightarrow \operatorname{Hom}_B(\Omega^1_{B/A}, M) \longrightarrow \operatorname{Hom}_B(B \otimes_C \Omega^1_{C/A}, M) \longrightarrow \operatorname{Hom}_B(I \otimes_C B, M)$$

is exact. Using the universal property, this follows from the exactness of

$$0 \longrightarrow \mathrm{Der}_A(B, M) \longrightarrow \mathrm{Hom}_A(C, M) \longrightarrow \mathrm{Der}_C(I, M)\ ,$$

which can be checked directly.

For the last statement, assume we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[d] & Z_1 \arrow[l, dotted, bend left, "r"] \arrow[r, hook, "I^2"] \arrow[dl] & Z \arrow[dll] \\ Y & & \end{tikzcd}\ ,$$

where $i_1 : X \hookrightarrow Z_1$ is the first infinitesimal neighborhood of the closed immersion $i : X \hookrightarrow Z$ with I its ideal sheaf and $r : Z_1 \to X$ is a retraction of i_1. By the conclusion we proved just now we have a commutative diagram with exact rows

$$\begin{tikzcd} I/I^2 \arrow[r] \arrow[d, "\simeq"] & i^*\Omega^1_{Z/Y} \arrow[r] \arrow[d] & \Omega^1_{X/Y} \arrow[r] \arrow[d, "\simeq"] & 0\ . \\ I/I^2 \arrow[r] & i_1^*\Omega^1_{Z_1/Y} \arrow[l, dotted, bend left] \arrow[r] & \Omega^1_{X/Y} \arrow[r] & 0 \end{tikzcd}$$

Now we claim that the retraction r gives a split morphism of the bottom exact sequence. In fact, the retraction $r : Z_1 \to X$ gives a map $\mathcal{O}_{Z_1} \to I/I^2$ induced by $\mathrm{Id} - r^* i_1^*$. One can check that this is a Y-derivation, hence give a $\mathcal{O}_{Z_1}$-morphism $\Omega^1_{Z_1/Y} \to I/I^2$. By adjunction, we get $i_1^*\Omega^1_{Z_1/Y} \to I/I^2$ which splits the bottom exact sequence. It is easy to see that the bottom exact sequence splits implies that the top row also splits, this finishes the proof. $\square$

Corollary 1.20. *Let X be a Y-scheme locally of finite type (resp. locally of finite presentation), then $\Omega^1_{X/Y}$ is of finite type (resp.finte presentation). Moreover, if Y is locally noetherian, then $\Omega^1_{X/Y} \in Coh(X)$.*

Proof. We may assume that $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$ are affine, and B is an A algebra of finite type, hence we have a commutative diagram with exact row as follows:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A[t_1, \cdots, t_n] \arrow[r] & B \arrow[r] & 0\ . \\ & & A \arrow[u] \arrow[ur] & & \end{tikzcd}$$

So we have a exact sequence

$$I/I^2 \longrightarrow B \otimes_{A[t_1,\cdots,t_n]} \Omega^1_{A[t_1,\cdots,t_n]/A} \longrightarrow \Omega^1_{B/A} \longrightarrow 0\,.$$

We have seen before that $\Omega^1_{A[t_1,\cdots,t_n]/A} \simeq A[t_1,\cdots,t_n]^n$, so we have an epimorphism $B^n \to \Omega^1_{B/A}$, this implies that $\Omega^1_{B/A}$ is a finite B-module. Moreover, if we assume that B is of finite presentation as an A-algebra, then we may assume I is an ideal of $A[t_1,\cdots,t_n]$ of finite type. So as a $B = A[t_1,\cdots,t_n]/I$-module, I/I^2 is of finite type, using the exact sequence above again, we see that $\Omega^1_{B/A}$ is of finite presentation as an A-module. The last conclusion is a direct consequence of the previous one and the definition of coherent modules. □

Corollary 1.21. *Let k be a field, X be a k-scheme. Given any rational point $x \in X(k)$, denote by i_x the closed immersion $\{x\} \hookrightarrow X$, then we have an isomorphism $i_x^*\Omega^1_{X/k} = (\Omega^1_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k(x) \simeq m_x/m_x^2$, where $\Omega^1_{X/k} = \Omega^1_{X/\operatorname{Spec} k}$.*

Proof. Since $x \in X(k)$ is a rational point, we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k(x) \arrow[r, hook, "i_x"] \arrow[dr, "\simeq"] & X \arrow[d] \\ & \operatorname{Spec} k \arrow[ul, dotted, bend left, "x"] \end{tikzcd}\ .$$

Also, the rational point x gives a retraction of i_x. From here we get an exact sequence

$$0 \longrightarrow m_x/m_x^2 \longrightarrow \Omega^1_{X/k} \otimes k \longrightarrow \Omega^1_{k(x)/k}(\simeq 0) \longrightarrow 0\,,$$

which implies that $i_x^*\Omega^1_{X/k} = (\Omega^1_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k(x) \simeq m_x/m_x^2$.

□

Corollary 1.22. *Let k be a field, $k[\varepsilon]$ be a k-algebra defined by the relation $\varepsilon^2 = 0$. Denote by i the natural closed immersion $\operatorname{Spec} k \hookrightarrow k[\varepsilon]$ and choose a rational point $x \in X(k)$ of X, then we have an isomorphism $\mathcal{T}_x = \{t \in X(k[\varepsilon]) \mid xi = x\ \} \simeq (m_x/m_x^2)^\wedge$*

Proof. By definition we have

$$\begin{aligned} \mathcal{T}_x &= \{\ h \in \operatorname{Hom}_k(\mathcal{O}_{X_x}, k[\varepsilon] \mid \pi h = p\ \} \\ &= \operatorname{Der}_k(\mathcal{O}_{X,x}, k\varepsilon) \\ &= \operatorname{Hom}(\Omega^1_{Y/k}\otimes_{k(x)}, k) \\ &= (m_x/m_x^2)^\wedge \end{aligned}$$

Here $\pi : k[\varepsilon] \to k$ is the canonical map and $p : \mathcal{O}_{X,x} \to k$ is the morphism corresponding to the rational point x. $\square$

Proposition 1.23. *Let $S = \operatorname{Spec} A$, $P = \mathbb{P}^r_s$, then there is a canonical exact sequence.*

$$0 \longrightarrow \Omega^1_{P/S} \longrightarrow \mathcal{O}_P^{r+1}(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0$$

Proof. Let $B = A[t_0, \cdots, t_r]$, $L = B(-1)^{r+1} = \bigoplus_{0 \le i \le r} Be_i$, where $\deg e_i = 1$, then $\mathcal{O}_P(-1) = \widetilde{B(-1)}$, $\mathcal{O}_P^{r+1}(-1) = \widetilde{L}$. Hence we have an exact sequence

$$0 \longrightarrow M \longrightarrow L \xrightarrow{u} B \longrightarrow A \longrightarrow 0\,,$$

where u is defined as $(f_0, \cdots, f_r) \mapsto \sum f_i t_i$ and $M = \operatorname{Ker} u$. From here, we get a short exact sequence

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{L} \xrightarrow{v=\widetilde{u}} \widetilde{B} \longrightarrow 0\,.$$

Using the Koszul complex $K_\bullet(u)$, on can see immediately that $M =< e_i t_j - e_j t_i >$. This is a graded B-module such that $M_{(t_i)}$ is free over $B_{(t_i)} = A[(\frac{t_j}{t_i})]$ with basis $\frac{e_i t_j - e_j t_i}{t_i^2}$, $i \neq j$. On the other hand, we have

$$\Omega^1_{P/S}|U_i = (\Omega^1_{A[(\frac{t_j}{t_i})]/A})^\sim = \bigoplus_{0 \le j \le r, j \neq i} B_{(t_i)} d(\tfrac{t_j}{t_i}),$$

so we have a well-defined map

$$\begin{array}{rccc} \varphi_i : & M_{(t_i)} & \to & \Omega^1_{A[(\frac{t_j}{t_i})]/A} \\ & \dfrac{e_i t_j - e_j t_i}{t_i^2} & \mapsto & d(\dfrac{t_j}{t_i}) \end{array}$$

One can check that $\varphi_i | U_i \cap U_j = \varphi_j | U_i \cap U_j$, we get a global isomorphism φ.

$$\varphi : \widetilde{M} \simeq \Omega^1_{P/S},$$

hence the short exact sequence

$$0 \longrightarrow \Omega^1_{P/S} \longrightarrow \mathcal{O}_P^{r+1}(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0$$

$\square$

In fact, we have something more. From the previous proposition, we get some exact sequence, which is denoted by c_i, for each i $(1 \le i \le r)$:

$$0 \longrightarrow \Omega^i_{P/S} \xrightarrow{v} \bigwedge^i \mathcal{O}_P^{r+1}(-i) \xrightarrow{v} \cdots \xrightarrow{v} \mathcal{O}_P^{r+1}(-1) \xrightarrow{v} \mathcal{O}_P \xrightarrow{v} 0\,,$$

where $\Omega^i_{P/S} = \bigwedge^i \Omega^1_{P/S}$. So we get a quasi-isomorphism $\Omega^i_{P/S}[i] \to \sigma_{\ge -i}K^\bullet(v)$, in particular $\Omega^r_{P/S} \simeq \bigwedge^{r+1} \mathcal{O}_P^{r+1}(-r-1) = \mathcal{O}_P(-r-1)$ (by the exactness of Koszul complex). By the exactness of c_r, we have $c_r \in \mathrm{Ext}^r_{\mathcal{O}_P}(\mathcal{O}_P, \Omega^r_{P/S}) = \mathrm{Hom}(\mathcal{O}_P, \Omega^r_{P/S}[r]) = H^r(P, \Omega^r_{P/S}) = H^r(P, \mathcal{O}_P(-r-1)) \simeq A$. In fact, c_r gives a basis of $= H^r(P, \Omega^r_{P/S})$ over A, hence is called the "fundamental class". Similarly, c_i gives a nontrivial class of $H^i(P, \Omega^i_{P/S})$ and $H^j(P, \Omega^i_{P/S}) = 0,\ i \neq j$. Further more, we have $c_i = c_1^i$ given by the cup product defined as follows: let $a \in H^i(P, \Omega^i_{P/S})$, $b \in H^j(P, \Omega^j_{P/S})$ given by $a:\ \mathcal{O}_P \to \Omega^i_{P/S}[i]$ and $b:\ \mathcal{O}_P \to \Omega^j_{P/S}[j]$, then we have a map: $(a,b) \mapsto ab \in H^{i+j}(P, \Omega^{i+j}_{P/S})$ defined as the composition of $\mathcal{O}_P \otimes^L_{\mathcal{O}_P} \mathcal{O}_P \xrightarrow{a \otimes^L b} \Omega^i_{P/S}[i] \otimes \Omega^j_{P/S}[j] = \Omega^i \otimes \Omega^j[i+j]$.

Theorem 1.24. *Let $f : X \to Y$ be a morphism of schemes, $\Omega^r_{X/Y} = \bigwedge^i \Omega^1_{X/Y}$, then there exists a unique family of maps of abelian sheaves $d : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ such that*
(1) $d = f^{-1}(\mathcal{O}_Y)$-basis;
(2) $d \circ d = 0$;
(3)$d(a \wedge b) = da \wedge b + (-1)^i a \wedge db$, for any $a \in \Gamma(U, \Omega^i),\ b \in \Gamma(U, \Omega^j)$, where $U \subset X$ is an open set;
(4) $d = d_{X/Y}:\ \mathcal{O}_X \to \Omega^1_{X/Y}$.

Sketch of Proof. Since we have $\Gamma(U, \Omega^1_{X/Y}) = \mathcal{O}_X(U)d(\mathcal{O}_X(U))$ for any affine open subscheme $U \subset X$, hence an element of $\Gamma(U, \Omega^i_{X/Y})$ can be written as $\omega = \sum a db_1 \wedge \cdots \wedge db_i$ for some a and $b_i \in \mathcal{O}_X(U)$, so we have $d(adb_1 \wedge \cdots \wedge db_i) = da \wedge db_1 \wedge \cdots \wedge db_i$. Hence the uniqueness is clear.

For the existence: first since the uniqueness we proved just now, we may focus on the case that $X = \operatorname{Spec} B,\ Y = \operatorname{Spec} A$ are affine. In this case, $\Omega^1_{X/Y} = \left(\Omega^1_{B/A}\right)^\sim$ and $\Omega^\bullet_{X/Y} = \left(\bigwedge \Omega^1_{B/A}\right)^\sim$. So, we only need to construct an antiderivation $D : \bigwedge \Omega^1_{B/A} \to \bigwedge \Omega^1_{B/A}$ of degree 1 such that (1) $D(b) = d(b)$ for any $b \in B$ and (2) $D(a \cdot db_1 \wedge db_2 \cdots \wedge db_s) = da \wedge db_1 \cdots db_s$ for any $a, b_1, \cdots b_s \in B$. We first treat a special case, that is when $B = A[\{t_i\}_{i \in I}]$

where I is an index set imposed with a total order. In this situation, we have known that $\{dt_i\}_i \in I$ forms a B-basis of the free B-module $\Omega^1_{B/A}$. Hence $\{ dt_{i_1} \wedge dt_{i_2} \cdots \wedge dt_{i_r} \mid i_1 < i_2 < \cdots < i_r \}$ is a B-basis of $\bigwedge \Omega^1_{B/A}$. Then any element $\omega \in \bigwedge \Omega^1_{B/A}$ can be written uniquely as follows

$$\omega = \sum_{i_1 < \cdots < i_r, r < \infty} a dt_{i_1} \wedge \cdots dt_{i_r}$$

and so we define

$$D\omega = \sum_{i_1 < \cdots < i_r, r < \infty} da \wedge dt_{i_1} \wedge \cdots dt_{i_r} = \sum_{i_1 < \cdots < i_r, r < \infty} \sum_j \frac{\partial a_j}{\partial t_j} dt_j \wedge dt_{i_1} \cdots \wedge dt_{i_r}.$$

One can check that indeed such a definition gives an antiderivation of $\bigwedge \Omega^1_{B/A}$ of order 1 satisfying (1) and (2). For the general case, see EGA IV, 16.6.2.

$\square$

Remark. The morphism d defined in the previous theorem is called *exterior derivation*. Using this construction, we get a complex of abelian sheaves $\Omega^\bullet_{X/Y} = (\cdots \to 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^i_{X/Y} \to \cdots)$, which is called *de Rham complex of* X/Y. The cohomology group of this complex $H^\bullet(X, \Omega^\bullet_{X/Y}) = H^\bullet_{dR}(X/Y)$ is called *de Rham cohomology*.

2 Smooth unramified étale morphisms

Definition 2.1. Let $f : X \to Y$ be a morphism of schemes, then f is called *formally smooth* (resp. *unramified*, resp. *étale*), if and only if for any diagram of the form $(*)$ with i a thickening of order 1, locally on T_0, there exists at least(resp. at most, resp. exactly) one g making the diagram commutes. It is equivalent to say that $\mathcal{H}om(T_0, X) \to \mathcal{H}om(T_0, X)$ is surjective(resp. injective, resp. isomorphism).

$$\begin{tikzcd}
 & & X \arrow[d, "f"] \\
T_0 \arrow[r, hook, "i"] \arrow[urr, "g_0"] & T \arrow[r] \arrow[ur, dotted, "g"] & Y
\end{tikzcd} \qquad (*)$$

f is called *smooth*(resp. *unramified*, resp. *étale*), if and only if f is formally smooth (resp. unramified resp. étale) and locally of finite presentation.

Remark. (1) If $f : X \to Y$, $g : Y \to Z$ are formally smooth, then gf is formally smooth. The same holds in the case that f, g are ormally unramified (resp. formally étale).

(2) Formal smoothness (resp. smoothness) is stable under base change, that is if $f : X \to Y$ is formal smooth (resp. smooth), and

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

is a cartesian diagram, then f' is also formally smooth (resp. smooth).

(3) Let $X = \bigcup_{i\in I} U_i$, U_i is open, U_i/Y is smooth for any i, then X/Y is smooth.

(4) If locally there exists g on T_0, then in the étale case, we can extend g globally, and such extension is uniquely.

(5) In the previous definition, the condition on the thickening can be replaced by any thickening of order n for any positive $n \in \mathbb{Z}$. Indeed, given a thickening $T_0 \hookrightarrow T$ of order n, we can construct a chain of thickening such that the consecutive two are a thickening of order 1 as follows

$$T_0 \hookrightarrow T_1 \hookrightarrow \cdots \hookrightarrow T_{n-1} \hookrightarrow T$$

where $T_i = (|T|, \mathcal{O}_T/I^{i+1})$. Then using the commutative diagram below

$$\begin{tikzcd} T_0 \arrow[rr, hook, "I"] \arrow[dr, hook] & & T, \\ & T_1 \arrow[dr, hook] & \\ & & T_2 \arrow[uu, hook, dotted, bend right] \end{tikzcd}$$

we can reduce to the previous case.

We give some examples of smooth morphism.

Example 2.1.1. Consider the commutative diagram:

$$\begin{tikzcd} & & \mathbb{A}^n_S \arrow[d, "f"] \\ T_0 \arrow[r, hook, "i"] \arrow[urr, "g"] & T \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

Then f is smooth since by the following diagram

$$\begin{tikzcd}
\Gamma(T_0, \mathcal{O}_{T_0})^n \arrow[d, equal] & \Gamma(T, \mathcal{O}_T)^n \arrow[l] \arrow[d, equal] \\
\mathrm{Hom}_S(T_0, \mathbb{A}^n_S) & \mathrm{Hom}_S(T, \mathbb{A}^n_S) \arrow[l]
\end{tikzcd}$$

there exists a morphism h making the diagram commute.

Corollary 2.2. *The morphism $f : \mathbb{P}^n_S \to S$ is a smooth morphism.*

Theorem 2.3. *1) A morphism $f : X \to Y$ is unramified if and only if $\Omega^1_{X/Y} = 0$.*

2) If $f : X \to Y$ is smooth, $\Omega^1_{X/Y}$ is locally free of finite type.

3) Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes.

a) Assume that f is smooth, then the sequence

$$0 \longrightarrow f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \qquad (*)$$

is exact and locally split.

a')Assume f étale, then $f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$

b) Assume that $g \circ f$ is smooth, and $(*)$ is exact and locally split, then f is smooth.

b')Assume $g \circ f$ smooth, and $f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$, then f is étale.

4) Consider a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\
Y &
\end{tikzcd}$$

where i is a closed immersion with ideal $\mathcal{I}$, and let $\mathcal{N}_{X/Z} = \mathcal{I}/\mathcal{I}^2$. Then

a) Assume that f is smooth, then the sequence

$$0 \longrightarrow \mathcal{N}_{X/Z} \xrightarrow{d_{Z/Y}} i^*\Omega^1_{Z/Y} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \qquad (**)$$

is exact and locally split.

a') Assume that f is étale, then $\mathcal{N}_{X/Z} \xrightarrow{\sim} i^*\Omega^1_{Z/Y}$.

b) Assume that g is smooth. If $(**)$ is exact and locally split, then f is smooth.

b') Assume that g is smooth. If $\mathcal{N}_{X/Z} \xrightarrow{\sim} i^*\Omega^1_{Z/Y}$, then f is étale.

Lemma 2.4. *Consider a commutative diagram:*

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[r, hook, "i"] \arrow[urr, "g_0"] & T \arrow[r] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & Y \end{tikzcd}$$

where i is a thickening of order 1 with ideal $\mathcal{I}$, and $g_1 i = g_2 i = g_0$. Then $\boxed{g_2^* - g_1^*} : \mathcal{O}_X \to g_{0*}\mathcal{O}_T$ factors through $g_{0*}\mathcal{I}$, and

$$g_2^* - g_1^* \in \mathrm{Der}_Y(\mathcal{O}_X, g_{0*}\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, g_{0*}\mathcal{I}) = \mathrm{Hom}_{T_0}(g_0^*\Omega^1_{X/Y}, \mathcal{I});$$

the homomorphism $\Omega^1_{X/Y} \to g_{0*}\mathcal{I}$ corresponding to the derivation $g_2^* - g_1^*$ sends $d_{X/Y}(a)$ to $g_2^*(a) - g_1^*(a) \in g_{0*}\mathcal{I}$.

Proof. We may assume all schemes are affine. Then we have the following commutative diagram:

$$\begin{tikzcd} & & B \arrow[dll, "g_0"'] \arrow[dl, shift right, "g_1"'] \arrow[dl, shift left, "g_2"] \\ C_0 & C \arrow[l, "\mathcal{I}"] & A. \arrow[l] \arrow[u] \end{tikzcd}$$

Define $\varphi : B \to C$, $b \mapsto \varphi(b) = g_2(b) - g_1(b)$. We need to verify that $\varphi \in \mathrm{Der}_A(B, I_{[B]})$. In fact, $\varphi(ab) = a\varphi(b)$, for all $a \in A$. And for $x, y \in B$,

$$\begin{aligned} \varphi(xy) =& g_2(xy) - g_1(xy) \\ =& g_2(x)(g_2(y) - g_1(y)) + g_1(y)(g_2(x) - g_1(x)) \\ =& x\varphi(y) + y\varphi(x). \end{aligned}$$

□

Proof of Theorem 2.3 (1). Consider the commutative diagram:

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[urr, "g_0"] & T \arrow[r] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & Y \end{tikzcd}$$

Assume $\Omega^1_{X/Y} = 0$, we want to show $g_1 = g_2$. However $g_2^* - g_1^* \in \mathrm{Hom}_{T_0}(g_0^*\Omega^1_{X/Y}, \mathcal{I}) = 0$, hence $g_1 = g_2$.

Conversely, consider the commutative diagram:

$$\begin{tikzcd}
 & & X \arrow[d] \\
X \arrow[r, hook] \arrow[urr, "Id"] \arrow[dr, "\triangle"'] & (X \times_Y X)_1 \arrow[r] \arrow[ur, shift left, "p_1"] \arrow[ur, shift right, "p_2"'] \arrow[d, hook] & Y \\
 & (X \times_Y X) &
\end{tikzcd}$$

where $(X \times_Y X)_1$ is the first infinitesimal neighborhood of the diagonal $\triangle$. As f is unramified, we have $0{=}p_2^* - p_1^* \in \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \Omega^1_{X/Y})$. On the other hand, $p_2^* - p_1^* = d_{X/Y} : \mathcal{O}_X \to \Omega^1_{X/Y}$ which corresponds to the identity $Id : \Omega^1_{X/Y} \to \Omega^1_{X/Y}$. So $Id_{\Omega^1_{X/Y}} = 0_{\Omega^1_{X/Y}}$, which implies that $\Omega^1_{X/Y} = 0$. $\square$

For the proof of Theorem 2.3 (2), we need some preliminaries.

Definition 2.5. Let $f : X \to Y$ be a morphism, $\mathcal{I} \in Qcoh(X)$. A Y-*extension* of X by $\mathcal{I}$ is a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] & X' \arrow[dl] \\
Y &
\end{tikzcd}$$

where i is a thickening of order 1 defined by the ideal $\mathcal{I}$.

An *isomorphism of Y-extensions*

$$(\, X \overset{i'}{\hookrightarrow} X' \,) \xrightarrow{\sim} (\, X \overset{i''}{\hookrightarrow} X'' \,)$$

is a Y-morphism $a : X' \to X$", such that $ai' = i''$ and a induces the identity map on $\mathcal{I}$, i.e., the following diagrams

$$\begin{tikzcd}
 & & \mathcal{O}_{X'} \arrow[dr] & & \\
0 \arrow[r] & \mathcal{I} \arrow[ur] \arrow[dr] & & \mathcal{O}_X \arrow[r] & 0 \\
 & & \mathcal{O}_{X''} \arrow[uu, "a^*"] \arrow[ur] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X' \arrow[dd] \arrow[dr] & \\
X \arrow[ur, hook, "i'"] \arrow[dr, hook, "i''"'] & & Y \\
 & X'' \arrow[ur] &
\end{tikzcd}$$

commute. Note that a^* is an isomorphism.

Remark. Given a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{A} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}(\mathcal{O}_X) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

with exact row, where $\mathcal{A}$ is an $f^{-1}(\mathcal{O}_X)$-algebra, p is a homomorphism of $f^{-1}(\mathcal{O}_X)$-algebras, and $\mathcal{I}^2 = 0$. One can show $(|X|, \mathcal{A})$ is a scheme X', such that $\mathcal{A} = \mathcal{O}_{X'}$ and X' is a Y-extension of X by $\mathcal{I}$.

Definition 2.6. $\mathrm{Ext}_Y(X,\mathcal{I})$=$\{$isomorphism classes of Y-extensions$\}$.

One can endow $\mathrm{Ext}_Y(X,\mathcal{I})$ with the structure of an abelian group with 0 element being the class of the trivial Y-extension, i.e. X' defined by $\mathcal{O}_X{}' = \mathcal{O}_X \oplus \mathcal{I} = D(\mathcal{I})$(the *dual number* algebra on $\mathcal{I}$), and $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_{X'}$ defined by $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \to \mathcal{O}_{X'}$ with canonical morphisms. The addition in $\mathrm{Ext}_Y(X,\mathcal{I})$ is defined as follows: given two elements $e_1, e_2 \in \mathrm{Ext}_Y(X,\mathcal{I})$:

$$e_1 : class\ of\ (0 \to \mathcal{I} \to \mathcal{O}_{X_1} \to \mathcal{O}_X \to 0)$$

$$e_2 : class\ of\ (0 \to \mathcal{I} \to \mathcal{O}_{X_2} \to \mathcal{O}_X \to 0),$$

we construct the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{O}_{X_1} \times_{\mathcal{O}_X} \mathcal{O}_{X_2} \arrow[r] \arrow[d] \arrow[dd, bend left] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I} \otimes \mathcal{I} \arrow[r] \arrow[ur] \arrow[d] & \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \arrow[r] & \mathcal{O}_X \oplus \mathcal{O}_X \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X_3} \arrow[uur] & &
\end{tikzcd}$$

where $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{O}_X$ is the diagonal map, $\mathcal{I} \oplus \mathcal{I} \to \mathcal{I}$ is the sum map, $\mathcal{O}_{X_3} = \mathcal{I} \bigoplus_{\mathcal{I}\oplus\mathcal{I}} (\mathcal{O}_{X_1} \times_{\mathcal{O}_X} \mathcal{O}_{X_2})$. Then $e_1 + e_2$ is the class of the extension $(0 \to \mathcal{I} \to \mathcal{O}_{X_3} \to \mathcal{O}_X \to 0)$. One shows that $e_1 + e_2$ does not depend on the choices and that we thus obtain a structure of abelian group on $\mathrm{Ext}_Y(X,\mathcal{I})$ as desired. The proof is similar to the construction of structure of abelian group on the set of isomorphism classes $\mathrm{Ext}(L, M)$ of L by M in an abelian category.

Proof of Theorem 2.3 (4)(a). Consider the commutative diagram:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "i"] & Z_1 \arrow[r, hook] \arrow[dl] & Z \arrow[dll] \\
Y & &
\end{tikzcd}$$

where Z_1 is the first infinitesimal neighborhood of X in Z. As f is smooth, locally there exists $r \in \mathrm{Hom}_Y(Z_1, X)$, such that $r \circ i_1 = \mathrm{Id}$, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
 & & X \arrow[d, "f"] \\
X \arrow[r, hook, "i_1"] \arrow[urr, "Id"] & Z_1 \arrow[r] \arrow[ur, dashed, "r"] & Y.
\end{tikzcd}$$

Therefore r gives a map:

$$\varphi : i^*\Omega^1_{Z/Y} \to \mathcal{N}_{X/Z}$$

$$(da)^- \mapsto (-r^* i_1^* a + a)^- = \varphi(\overline{a}), \quad a \in \mathcal{O}_Z$$

where $(\ )^-$ represents a class mod $\mathcal{I}$. The map φ is inverse to $\mathcal{N}_{X/Z} \to i^*\Omega^1_{X/Y}$. So

$$0 \to \mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0$$

is exact and locally split. $\square$

Lemma 2.7. *Let $f : X \to Y$ be smooth, $\mathcal{I} \in Qcoh(X)$. Following 2.3 (4)(a), define* $\varphi : \mathrm{Ext}_Y(X, \mathcal{I}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{I})$ *by*

$$\textit{class of} \left(\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & X' \arrow[dl] \\ Y & \end{tikzcd} \right) \mapsto \textit{class of} \left(0 \to \mathcal{I} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0 \right).$$

Then φ is an isomorphism.

Proof. We easily checks that φ is a well defined group homomorphism. Define

$$\psi : \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{I}) \to \mathrm{Ext}_Y(X, \mathcal{I})$$

in the following way. Given an exact sequence

$$0 \longrightarrow \mathcal{I} \xrightarrow{u} E \xrightarrow{v} \Omega^1_{X/Y} \longrightarrow 0\,,$$

we form the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r, "\binom{0}{u}"] \arrow[dr] & \mathcal{O}_X \oplus E \arrow[r, "Id\oplus v"] & \mathcal{O}_X \oplus \Omega^1_{X/Y} \arrow[r] & 0 \\
& & \mathcal{O}_{X'} \arrow[u, "p"] \arrow[r, "q"] & \mathcal{O}_X \arrow[u, "Id+d_{X/Y}=p_2^*"'] \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{O}_{X'} = (\mathcal{O}_X \oplus E) \times_{\mathcal{O}_X \oplus \Omega^1_{X/Y}} \mathcal{O}_X$. We define

$$\psi(E) = \left(\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & X' \arrow[dl] \\ Y & \end{tikzcd} \right).$$

It sufficient to check that $\varphi\psi = \mathrm{Id}$, $\psi\varphi = \mathrm{Id}$.

The fact that $\psi\varphi = \mathrm{Id}$ is clear. We verify $\varphi\psi(E) = E$. Note that $p - q$ induces a Y-derivation: $\mathcal{O}_{X'} \to E$, and hence a morphism $\alpha : i^*\Omega^1_{X'/Y} \to E$. Considering the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[r] & \mathcal{I} \arrow[r] & E \arrow[r] & \Omega^1_{X/Y} \arrow[r] & 0\,, \\
\varphi\psi(E) : & 0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[u, "\mathrm{Id}"] & i^*\Omega^1_{X'/Y} \arrow[r] \arrow[u, "\alpha"] & \Omega^1_{X/Y} \arrow[r] \arrow[u, "\mathrm{Id}"] & 0
\end{tikzcd}$$

we conclude that α is an isomorphism. □

Lemma 2.8. *Let X be a scheme, $E \in Qcoh(X)$ be of finite type. Assume that for any $F \in Qcoh(X)$, $\mathcal{E}xt^1_{\mathcal{O}_X}(E, F) = 0$, then E is locally free.*

Proof. Locally we can write

$$0 \to F \to L \to E \to 0 \quad (*)$$

with L free of finite type. So $F \in Qcoh(X)$. Then by the assumption, (*) locally splits, which implies that E is locally free. □

Proof of Theorem 2.3 (2). For any $\mathcal{I} \in Qcoh(X)$, by 2.7, we get

$$\operatorname{Ext}_Y(X,\mathcal{I}) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega_{X/Y},\mathcal{I}),$$

so

$$\mathcal{E}xt_Y(X,\mathcal{I}) \xrightarrow{\sim} \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_{X/Y},\mathcal{I}),$$

where $\mathcal{E}xt_Y(X,\mathcal{I})$ denotes the sheaf associated to $U \mapsto \operatorname{Ext}_Y(U,\mathcal{I}|_U)$. As f is smooth, locally any Y-extension of X by $\mathcal{I}$ admits a Y-retraction, i.e. any Y-extension of X by $\mathcal{I}$ is (locally) trivial. So $\mathcal{E}xt_Y(X,\mathcal{I}) = 0 = \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_{X/Y},\mathcal{I})$, then $\Omega^1_{X/Y}$ is locally free of finite type by 2.8. $\square$

Lemma 2.9. *Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms of schemes with f affine. Let $\mathcal{I} \in Qcoh(X)$. Then the following sequence is exact.*

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Der}_Y(\mathcal{O}_X,\mathcal{I}) \arrow[r] & \operatorname{Der}_S(\mathcal{O}_X,\mathcal{I}) \arrow[r, "\alpha"] & \operatorname{Der}_S(\mathcal{O}_Y, f_*\mathcal{I}) \arrow[dll, "\partial"'] \\
& \operatorname{Ext}_Y(X,\mathcal{I}) \arrow[r, "\gamma"] & \operatorname{Ext}_S(X,\mathcal{I}) \arrow[r, "\beta"] & \operatorname{Ext}_S(Y, f_*\mathcal{I})
\end{tikzcd} \qquad (*)$$

where α, β, ∂ are defined as follows:

(1) For any $D \in \operatorname{Der}_Y(\mathcal{O}_X,\mathcal{I})$, define $\alpha(D) : \mathcal{O}_Y \to f_*\mathcal{O}_X \xrightarrow{f_*D} f_*I$.

(2) For any $D \in \operatorname{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$, define

$$\partial(D):\ \begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \oplus \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 , \\
& & f^{-1}(\mathcal{O}_Y) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

where $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \oplus \mathcal{I}$ corresponds to $(f^*, D) : \mathcal{O}_Y \to f_*\mathcal{O}_X \oplus f_*\mathcal{I}$.

(3) For a class $E \in \operatorname{Ext}_S(X,\mathcal{I})$ as follows:

$$E:\ \begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 , \\
& & f^{-1}g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

define

$$\beta(E):\ \begin{tikzcd}
0 \arrow[r] & f_*\mathcal{I} \arrow[r] & \mathcal{O}_{Y'} \arrow[r] & \mathcal{O}_Y \arrow[r] & 0 \\
& & f^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

by the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{I} \arrow[r] \arrow[dr] & f_*\mathcal{O}_{X'} \arrow[r] \arrow[dr, phantom, "\square"] & f_*\mathcal{O}_X \arrow[r] & 0 \\
& & \mathcal{O}_{Y'} \arrow[u] \arrow[r] & \mathcal{O}_Y \arrow[u] \arrow[r] & 0 \\
& & g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

where the upper row is exact, since f is affine.

Proof. We only prove the exactness at $\mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$, $\mathrm{Ext}_Y(X, I)$, $\mathrm{Ext}_S(X, \mathcal{I})$.

(a) Exactness at $\mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$. Assume that $D \in \mathrm{Der}_S(\mathcal{O}_X, \mathcal{I})$, then $\alpha(D) : \mathcal{O}_Y \to f_*\mathcal{O}_X \overset{f_*D}{\to} f_*\mathcal{I}$. Define $(1, D) : \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{I}$, it is easy to verify that the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \oplus \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "{(1,D)}"'] & 0 \\
& & f^{-1}(\mathcal{O}_Y) \arrow[u, "{(f^*,\alpha(D))}"] \arrow[ur, "f^*"'] & &
\end{tikzcd}$$

is commutative. So $\partial \circ \alpha(D)$ is trivial in $\mathrm{Ext}_Y(\mathcal{O}_X, \mathcal{I})$. And hence $\mathrm{Im}(\alpha) \subset \mathrm{Ker}(\partial)$.

Assume $D \in \mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$, and $\partial(D) = 0$. Then there exists a morphism $\varphi = (1, D_1) : \mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{I}$ making the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \oplus \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "{(1,D_1)}"'] & 0 \\
& & f^{-1}(\mathcal{O}_Y) \arrow[u, "{(f^*,D)}"] \arrow[ur, "f^*"'] & &
\end{tikzcd}$$

commutes, which implies that D_1 is a S-derivation and $\alpha(D_1) = D$. Hence $D_1 \in \mathrm{Der}_S(\mathcal{O}_X, \mathcal{I})$ and then $\mathrm{Ker}(\partial) \subset \mathrm{Im}(\alpha)$. So the sequence is exact at $\mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$.

(b) Exactness at $\mathrm{Ext}_Y(X, I)$. Assume $D \in \mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I})$. Then clearly, the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_X \oplus \mathcal{I} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend right, "{(1,0)}"'] & 0 \\
 & & f^{-1}(\mathcal{O}_Y) \arrow[u, "{(f^*,D)}"] & & \\
 & & f^{-1}g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

commutes. Hence $\gamma \circ \partial(D)$ is trivial in $\mathrm{Ext}_S(X, \mathcal{I})$. So $\mathrm{Im}\, \partial \subset \mathrm{Ker}(\gamma)$.

Let E defined by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}(\mathcal{O}_Y) \arrow[u, "u"] \arrow[ur] & &
\end{tikzcd}$$

be a Y-extension whose image in $\mathrm{Ext}_S(\mathcal{O}_X, \mathcal{I})$ is trivial. Then there exists an $\mathcal{O}_S$-homomorphism $r : \mathcal{O}_X \to \mathcal{O}_{X'}$ such that $p \circ r = \mathrm{Id}$. Then we can write $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I}$ and $r = (\mathrm{Id}, D)$, and $D : \mathcal{O}_X \to \mathcal{I}$ is an S-derivation. Then $u = (f^*, D)$, which shows that the class of E is $\partial(D)$. So $\mathrm{Ker}(\gamma) \subset \mathrm{Im}\, \partial$.

(c) Exactness at $\mathrm{Ext}_S(Y, f_*\mathcal{I})$. Assume that E defined by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}(\mathcal{O}_Y) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

is an element in $\mathrm{Ext}_Y(X, \mathcal{I})$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{I} \arrow[r] \arrow[dr] & f_*\mathcal{O}_{X'} \arrow[r] \arrow[dr, phantom, "\square"] & f_*\mathcal{O}_X \arrow[r] & 0 \\
 & & \mathcal{O}_{Y'} \arrow[u] \arrow[r, "p"'] & \mathcal{O}_Y \arrow[r] \arrow[u] \arrow[ul, bend left] & 0 \\
 & & g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

By the property of fiber product, there exists $r : \mathcal{O}_Y \to \mathcal{O}_{Y'}$, such that $p \circ r = \mathrm{Id}$, making the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{I} \arrow[r] & \mathcal{O}_{Y'} \arrow[r, "p"'] & \mathcal{O}_Y \arrow[r] \arrow[l, bend right, "r"'] & 0 \\
 & & g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

commutes. Hence $\beta \circ \gamma(E) = 0$

Assume

$$\begin{tikzcd}
E: 0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

and $E \in \mathrm{Ker}(\beta)$. Then there exists $r : \mathcal{O}_Y \to \mathcal{O}_{Y'}$, making the diagram

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{I} \arrow[r] \arrow[dr] & f_*\mathcal{O}_{X'} \arrow[r] & f_*\mathcal{O}_X \arrow[r] & 0 \\
 & & \mathcal{O}_{Y'} \arrow[r] \arrow[u, "p"] \arrow[ur, phantom, "\square"] & \mathcal{O}_Y \arrow[r] \arrow[u] \arrow[l, bend left, "r"] & 0 \\
 & & g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

commutes. Hence we can define

$$\psi : f^{-1}(\mathcal{O}_Y) \stackrel{f^{-1}r}{\to} f^{-1}(\mathcal{O}_{Y'}) \stackrel{f^{-1}p}{\to} f^{-1}f_*\mathcal{O}_{X'} \to \mathcal{O}_{X'},$$

making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O}_{X'} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
 & & f^{-1}(\mathcal{O}_Y) \arrow[u, "\psi"] \arrow[ur] & & \\
 & & f^{-1}g^{-1}(\mathcal{O}_S) \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

commutes, which implies $E \in \mathrm{Im}(\gamma)$. So the sequence is exact at $\mathrm{Ext}_S(Y, f_*\mathcal{I})$. □

Proof of Theorem 2.3 (3)(a). It is a local problem, so we may assume that $X = \mathrm{Spec}(C)$, $Y = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ are affine, so that $X \to Y \to S$ corresponds to $A \to B \to C$. We will show that the sequence

$$0 \to C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0 \quad (*)$$

is exact and split. It is equivalent to showing that $\mathrm{Hom}((*), I)$ is exact, for any $I \in \mathrm{Mod}(C)$. By 2.9,

$$0 \to \mathrm{Der}_B(C, I) \to \mathrm{Der}_A(C, I) \to \mathrm{Der}_A(B, I_{[B]}) \to \mathrm{Ext}_Y(X, I)$$

is exact. By 2.3 (2), $\Omega^1_{X/Y}$ is locally free of finite type, hence $\Omega^1_{C/B}$ is projective of finite type. So $\mathrm{Ext}^1_C(\Omega^1_{C/B}, I) = 0$, and $\mathrm{Ext}_Y(X, \mathcal{I}) = 0$. So $\mathrm{Hom}((*), \mathcal{I})$ is exact. □

Lemma 2.10. *$f : X \to Y$ is a morphism. Then the following conditions are equivalent:*
(a) f is formally smooth.
(b) For any open subset $U \subset X$, and any $\mathcal{I} \in Qcoh(U)$, $\mathcal{E}xt_Y(U, \mathcal{I}) = 0$

Proof. (a)⇒ (b) is clear. For (b)⇒(a), we will show there exists g making the following diagram commutes.

$$\begin{tikzcd}
& & X \arrow[d, "f"] \\
T_0 \arrow[urr, "g_0"] \arrow[r] & T \arrow[r] \arrow[ur, dashed, "g"] & Y
\end{tikzcd}$$

We may assume all schemes are affine. Then the above diagram corresponds to the following diagram:

$$\begin{tikzcd}
& & C \arrow[dll] \arrow[dl, dashed] \\
R_0 & R \arrow[l] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
& & E \arrow[r, "h_2"] \arrow[d, "h_1"] \arrow[dr, phantom, "\square"] & C \arrow[r] \arrow[d, "g_0"] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[ur] & R \arrow[r] & R_0 \arrow[r] & 0 \\
& & & B \arrow[ul] \arrow[u] &
\end{tikzcd}$$

where $E = R \times_{R_0} C$. By the property of fiber product, we get a Y-extension of X by I,

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & E \arrow[r] & C \arrow[r] & 0\,. \\ & & B \arrow[u] \arrow[ur] & & \end{tikzcd}$$

Since $\mathcal{E}xt_Y(X, I) = 0$, then $0 \to I \to E \to C \to 0$ splits, so there exists $s : C \to E$, such that $h_2 \circ s = \mathrm{Id}$. Then $g = h_1 \circ s$ is the required morphism. $\square$

Proof of Theorem 2.3 (3)(b). As $g \circ f$ is smooth, then $\mathcal{E}xt_S(X, \mathcal{I}) = 0$. We may assume X, Y, S are affine. Using 2.9, we get an exact sequence:

$$\mathrm{Der}_S(\mathcal{O}_X, \mathcal{I}) \xrightarrow{\alpha} \mathrm{Der}_S(\mathcal{O}_Y, f_*\mathcal{I}) \longrightarrow \mathrm{Ext}_Y(\mathcal{O}_X, \mathcal{I}) \longrightarrow 0. \quad (*)$$

By the exactness and local splitting of the sequence:

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0,$$

we get that α is surjective. This implies $\mathcal{E}xt_Y(X, \mathcal{I}) = 0$. Together with 2.10, we conclude that f is smooth.

When $f^*\Omega^1_{Y/S} \xrightarrow{\sim} \Omega^1_{X/S}$, then $\Omega^1_{X/Y} = 0$, and hence f is étale, using Theorem 2.3 (1). $\square$

Proof of Theorem 2.3 (4)(b). By 2.10, it is sufficient to show that $\mathcal{E}xt_Y(X, J) = 0$, for any $J \in Qcoh(X)$.

Suppose $X \overset{j}{\hookrightarrow} X'$ is a Y-extension of X by J. Since g is smooth, there exists $h : X' \to Z$, extending i. We have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, hook, "j"] \arrow[dd, "f"'] & & Z \arrow[ddll, bend left, "g"] \\ & X' \arrow[ur, dashed, "h"] \arrow[dl] & \\ Y & & \end{tikzcd}$$

Let $i_1 : X \to Z_1$ be the first infinitesimal neighborhood of X in Z. We have a commutative diagram with exact rows

$$\begin{tikzcd} \mathcal{N} \arrow[r, "d_{Z/Y}"] \arrow[d, "\mathrm{Id}"] & i^*\Omega^1_{Z/Y} \arrow[r] \arrow[d] & \Omega^1_{X/Y} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0\,, \\ \mathcal{N} \arrow[r, "d_{Z_1/Y}"] & i_1^*\Omega^1_{Z_1/Y} \arrow[r] & \Omega^1_{X/Y} \arrow[r] & 0 \end{tikzcd} \quad (*)$$

Let R (resp. R_1) denote the set of retractions of $d_{Z/Y}$ (resp. $d_{Z_1/Y}$). By the composition with the middle vertical arrow of $(*)$ we get a map $R_1 \to R$, one can show it is an isomorphism. So a splitting of

$$0 \to \mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y} \to \Omega^1_{X/Y} \to 0$$

gives a derivation $D : \mathcal{O}_{Z_1} \to \mathcal{I}$ and a retraction $r : Z_1 \to X$, such that $\mathrm{Id} - r^* \circ i_1^* = D$, and $r \circ h_1$ retracts j. Therefore $\mathcal{E}xt_Y(X, J) = 0$.

If $\mathcal{N}_{X/Z} \to i^*\Omega^1_{Z/Y}$ is an isomorphism, $\Omega^1_{X/Y} = 0$ and f is étale. $\square$

This completes the proof of 2.3.

Corollary 2.11. *The morphism $f : X \to Y$ is smooth if and only if locally X is étale over $\mathbb{A}^n_Y$. More precisely, suppose f is smooth, and let $s_1, \cdots, s_n \in \Gamma(X, \mathcal{O}_X)$, such that $(ds_1, \cdots, ds_n)$ is a basis of $\Omega^1_{X/Y}$ over $\mathcal{O}_X$, where $d = d_{X/Y}$ (such a system exists locally by 2.3 (1)). Then the morphism $s : X \to \mathbb{A}^n_Y$ given by $(s_1, \cdots, s_n)$ is étale.*

Proof. We have a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & \mathbb{A}^n_Y \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

If X is étale over $\mathbb{A}^n_Y$, then X is smooth over Y by the stability of smoothness.

For the second assertion, note that by definition of s, the map $\mathcal{O}_X^n = s^*\Omega^1_{\mathbb{A}^n_Y/Y} \to \Omega^1_{X/Y}$ is an isomorphism. Apply 2.3 (3)(b), we get that s is étale. $\square$

Lemma 2.12. *Let A be a local ring, $k = A/\mathfrak{m}$ be its residue field. Let $E, F \in \mathrm{Mod}(A)$, E be of finite type and F be projective. Let $u : E \to F$ be a homomorphism. Then the following conditions are equivalent:*

(1) u is injective and split. (i.e. there exists $v : F \to E$, such that $v \circ u = \mathrm{Id}$)

(2) $u \otimes k : E \otimes k \to F \otimes k$ is injective.

Proof. (1) $\Rightarrow$ (2) is clear. We only prove (2) $\Rightarrow$ (1).

(a) Assume E is free of finite type. By hypothesis, since F is projective and $E \to E \otimes k$ is surjective, there exists $v : F \to E$ such that $vu \otimes k = \mathrm{Id}_{E\otimes k}$.

Then $\det(vu) \in 1 + \mathfrak{m}$, which implies that $v \circ u$ is an isomorphism. Thus u is injective and split.

(b) General case. There exists L free of finite type, such that $w : L \to E$ is surjective, and $L \otimes k \xrightarrow{\sim} E \otimes k$. Then $uw \otimes k$ is injective. By case (a), we get that $u \circ w$ is injective and split. Then w is an isomorphism, and thus u is injective and split. $\square$

Corollary 2.13 (Jacobian criterion). *Suppose we have a commutative diagram:*

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

with g smooth. Let $\mathcal{I}$ be the ideal of the closed immersion i. Let x be a point of X, then f is smooth at x if and only if there exist sections $\{s_i\}_{1 \le i \le r}$ of $\mathcal{I}$ around x, such that $((s_i)_x)$ generate $\mathcal{I}_x$, and $d_{Z/Y}(s_i) \otimes k(x) \in \Omega^1_{Z/Y} \otimes k(x)$ are linearly independent.

In particular, for $Z = \mathbb{A}^n_Y$, f is smooth at x if and only if f can be defined by $s_1 = \cdots = s_r = 0$ locally around x, where the s_i's are sections of $\mathcal{O}_Z$, such that

$$\mathrm{rk}_{k(x)}(\partial s_i/\partial t_j)_{\substack{1\le i\le r \\ 1 \le j \le n}}(x) = r,$$

where $(\partial s_i/\partial t_j)(x) \in k(x)$.

Proof. First we prove the necessity. If f is smooth at x, then the sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Z/Y} \otimes \mathcal{O}_X \to \Omega^1_{X/Y} \to 0$$

is exact and split around x. This implies the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \otimes k(x) \arrow[r] & \Omega^1_{Z/Y} \otimes k(x) \arrow[r] & \Omega^1_{X/Y} \otimes k(x) \arrow[r] & 0 \\ & s_i \otimes k(x) \arrow[r, mapsto] & d_{X/Y} s_i \otimes k(x) & & \end{tikzcd}$$

is exact. Pick up (s_i) such that $s_i \otimes k(x)$ is a basis of $\mathcal{I}_x/\mathcal{I}_x^2 \otimes k(x)$. Using Nakayama's lemma, we get that (s_i) is a minimal system of generators of $\mathcal{I}_x$.

Then we prove the sufficiency. We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{I}_x/\mathcal{I}_x^2 \otimes k(x) \arrow[r, "d_{Z/Y}\otimes k(x)"] & \Omega^1_{Z/Y} \otimes k(x) \\
k(x)^r \arrow[u, "(s_i)"] \arrow[ur, hook, "ds_i\otimes k(x)"'] &
\end{tikzcd}$$

Since $k(x)^r \longrightarrow \Omega^1_{Z/Y} \otimes k(x)$ is injective, then $k(x)^r \xrightarrow{\sim} \mathcal{I}_x/\mathcal{I}_x^2 \otimes k(x)$. Hence by 2.12, $d_{Z/Y}$ is injective and split around x. Then we get the conclusion by 2.3 (4)(b). $\square$

Corollary 2.14. *Suppose we have a commutative diagram:*

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\
Y &
\end{tikzcd}$$

with f, g smooth at x, and i a closed immersion. Then there exists an open subset $U \subset Z$ containing x, such that the diagram:

$$\begin{tikzcd}
U \cap X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \subset Z \arrow[d, "h"] \\
Y[t_1 \cdots t_n] = \mathbb{A}^n_Y \arrow[r, hook] & \mathbb{A}^{n+r}_Y = Y[t_1 \cdots t_{n+r}]
\end{tikzcd}$$

is cartesian, with h étale.

Proof. Let $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal of i. Choose sections $f_1, \cdots, f_r$ of $\mathcal{I}$ around x, and local sections $g_1, \cdots, g_n$ of $\mathcal{O}_Z$ around x, such that $df_1 \otimes k(x), \cdots, df_r \otimes k(x), dg_1 \otimes k(x), \cdots, dg_n \otimes k(x)$ is a basis of $\Omega^1_{Z/Y} \otimes k(x)$, and $(f_1)_x, \cdots, (f_r)_x$ generate $\mathcal{I}_x$. By Nakayama's lemma, $df_1, \cdots, df_r, dg_1, \cdots, dg_n$ give a basis of $\Omega^1_{Z/Y}$ around x. Hence we have a cartesian diagram:

$$\begin{tikzcd}
U \cap X \arrow[r] \arrow[d] & U \arrow[d, "h"] \\
\mathbb{A}^n_Y \arrow[r, hook] & \mathbb{A}^{n+r}_Y
\end{tikzcd}$$

where $h : U \longrightarrow \mathbb{A}^{n+r}_Y$ is defined by $g_1, \cdots, g_n, f_1, \cdots, f_r$, and $\mathbb{A}^n_Y \hookrightarrow \mathbb{A}^{n+r}_Y$ by $(t_1, \cdots, t_n) \mapsto (t_1, \cdots, t_n, 0, \cdots, 0)$. So h is étale, by 2.11. $\square$

3 Smoothness, flatness and regularity

We recall the definition of regular local rings.

Let A be a noetherian local ring. $\mathfrak{m}$ be the maximal ideal, k be the residue field. Then $d = \dim A \leq \text{rank}_k\, \mathfrak{m}/\mathfrak{m}^2$. A is *regular* if and only if the following equivalent conditions hold:

(1) $d = \text{rk}_k \mathfrak{m}/\mathfrak{m}^2$.

(2) There exists $x_1, \cdots, x_d \in \mathfrak{m}$ generating $\mathfrak{m}$.

(3) $gr_{\mathfrak{m}}(A) \simeq S_k(\mathfrak{m}/\mathfrak{m}^2) \simeq k[t_1, \cdots, t_d]$.

A sequence $(x_1, \cdots, x_d) \in A^d$ is called *a regular system of parameters* if $x_1, \cdots, x_d$ generate $\mathfrak{m}$, i.e., $(\overline{x_1}, \cdots, \overline{x_d})$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$, where $\overline{x_i}$ is the image of x_i in k.

Proposition 3.1. *Let A be a regular local ring, $\mathfrak{m}$ be the maximal ideal,* $\dim A = d$. *Let I be an ideal contained in $\mathfrak{m}$, $B = A/I$. Then the following two conditions are equivalent:*

(1) B is regular.

(2) There exists a regular system of parameters $(x_1, \cdots, x_d)$ of A such that $I = \sum_{i=1}^{r} x_i A$

Proof. (2) $\Rightarrow$ (1). We assume $(x_1, \cdots, x_r)$ is part of a regular system of parameters of A, then $\dim B = d - r$ as $(x_1, \cdots, x_r)$ is part of a system of parameters ([EGA0] IV 16.3.7). Let $\mathfrak{n} = \mathfrak{m}/I$ be the maximal ideal of B, then we have an exact sequence:

$$0 \to (\mathfrak{m}^2 + I)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0 \quad (*)$$

Since $x_1, \cdots, x_r$ generate I and their images are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, $\dim_k(\mathfrak{m}^2 + I)/\mathfrak{m}^2 = r$. So $\dim \mathfrak{n}/\mathfrak{n}^2 = d - r = \dim B$, which implies that B is regular.

(1) $\Rightarrow$ (2): We assume B is regular. Suppose $\dim B = \dim \mathfrak{n}/\mathfrak{n}^2 = d - r$. Using the exact sequence $(*)$, we get $\dim(\mathfrak{m}^2 + I)/\mathfrak{m}^2 = r$. Take $x_1, \cdots, x_r \in I$, such that the images of $x_1, \cdots, x_r$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent. Choose $x_{r+1}, \cdots, x_d \in \mathfrak{m}$, such that $(x_1, \cdots, x_d)$ is a regular system of parameters of A. Let $I' = \sum_{i=1}^{r} x_i A \subset I$. Then we have an exact sequence

$$0 \to I/I' \to A/I' \to A/I \to 0.$$

By (2)$\Rightarrow$(1), A/I' is regular and $\dim A/I' = d - r = \dim A/I$. As A/I is regular, A/I is a domain, hence I/I' is prime, but since $\dim A/I' = \dim A/I$, then $I = I'$. $\square$

Theorem 3.2 (Serre). *A is a regular local ring of dimension d if and only if the global (homological) dimension $\operatorname{gl}\dim(A)$ is equal to d.*

Corollary 3.3. *If A is a regular local ring, and $\mathfrak{p} \in \operatorname{Spec} A$, then $A_\mathfrak{p}$ is regular.*

Proof. Let J be an ideal in $A_\mathfrak{p}$. Then $J = I_\mathfrak{p}$, for some $I \subset A$. As A is regular, $\operatorname{Ext}^i_A(A/I, M) = 0$ for $i > d$, and any $M \in \operatorname{Mod}(A)$. Thanks to the isomorphism

$$\operatorname{Ext}^i_A(A/I, M)_\mathfrak{p} \xrightarrow{\sim} \operatorname{Ext}^i_{A_\mathfrak{p}}(A_\mathfrak{p}/I_\mathfrak{p}, M_\mathfrak{p}),$$

we have $\operatorname{Ext}^i_{A_\mathfrak{p}}(A_\mathfrak{p}/I_\mathfrak{p}, M_\mathfrak{p}) = 0$ for $i > d$, which (by 3.2) implies that $A_\mathfrak{p}$ is regular. $\square$

Corollary 3.4. *Let X be noetherian. If $\mathcal{O}_{X,x}$ is regular at all closed point x, then $\mathcal{O}_{X,x}$ is regular for all x.*

Proof. X is noetherian and hence quasi-compact. For any $y \in X$, there exists a closed point $x \in \overline{\{y\}}$. So we may assume $X = \operatorname{Spec} A$. Since every closed point corresponds to a maximal ideal, $A_\mathfrak{m}$ is regular for any maximal ideal $\mathfrak{m}$. Therefore $A_\mathfrak{p} = (A_\mathfrak{m})_\mathfrak{p}$ is regular by 3.3, where $\mathfrak{p} \subset \mathfrak{m}$. $\square$

Definition 3.5. Let X be a scheme. X is called *regular* if X is locally noetherian and $\mathcal{O}_{X,x}$ is regular for any $x \in X$.

Remark. If X is regular, then the connected components of X are the irreducible components of X and any component is open (cf. [EGAI] 6.1.10).

Recall that if X/k is of finite type, where k is a field, then $x \in X$ is closed if and only if $[k(x) : k] < \infty$ by Hilbert Nullstellensatz.

Let X/k be integral and of finite type. Let η be the generic point of X. Then $\dim X = \operatorname{tr}\deg_k k(\eta) = \dim \mathcal{O}_{X,x}$ if x is a closed point.

Proposition 3.6. *Let X/k be of finite type. Then the following conditions are equivalent:*

(1) X/k is étale.

(2) $\Omega^1_{X/k} = 0$, i.e., X is unramified.

(3) $X = \operatorname{Spec} A$, where $A = \prod_{i=1}^n K_i$, K_i/k is finite separable extension.

Proof. (1)$\Rightarrow$ (2) is trivial.

(2)$\Rightarrow$ (3) We may assume X is affine. Let $X = \operatorname{Spec} A$. We have to show that if $\overline{k}$ is an algebraic closure of k, $A \otimes \overline{k} = \overline{k}^N$.

Let $Z = \operatorname{Spec}(A \otimes \overline{k})$, x be a closed point in Z. Then $k(x) = \overline{k}$, $Z = X \otimes \overline{k}$. $\Omega^1_{X/k} = 0$ implies $\Omega^1_{Z/\overline{k}} = 0$. Since $x \in Z(\overline{k})$ (x is rational), $\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \Omega^1_{Z/\overline{k}} \otimes k(x)$ implies $\mathfrak{m}_x/\mathfrak{m}_x^2 = 0$, and hence $\mathfrak{m}_x = 0$. So $\mathcal{O}_{Z,x} = k(x) = k$.

(3)$\Rightarrow$ (1) We may assume $X = \operatorname{Spec} K$, K/k is finite separable. Then $K = k[T]/(f)$, where $f'(x) \neq 0$, $\{x\} = \operatorname{Spec} K$. Apply Jacobian criterion to

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, hook, "i"] \arrow[d] & \operatorname{Spec} k[T] \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd}$$

we get X/k is smooth, and hence X/k is étale. $\square$

Theorem 3.7. *Let k be a field, and X/k be of finite type.*

(1) If X is smooth over k, then X is regular. Moreover, if X is integral, then $\operatorname{rk}\Omega^1_{X/k} = \dim X$.

(2) If k is perfect, and X is regular, then X/k is smooth.

Proof. (1) We have to check $\mathcal{O}_{X,x}$ is regular for all closed point $x \in X$. Let $x \in X$ be a closed point, we have $[k(x) : k] < \infty$.

We may assume that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & Z = \mathbb{A}_k^{n+r} \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd}$$

where i is a closed immersion of ideal $\mathcal{I}$. Pick up $(f_i)_{1 \le i \le r}$, such that $\mathcal{I}_x = \sum_{i=1}^{r} (f_i)_x \mathcal{O}_{Z,x}$, and $d_{Z/k} \otimes k(x)$ are linearly independent. Let $\mathfrak{m} = \mathfrak{m}_{Z,x}$. Since the following diagram commutes

$$\begin{tikzcd} \mathcal{I}/\mathcal{I}^2 \otimes k(x) \arrow[r, hook, "d_{Z/k}"] \arrow[dr, hook, "\varphi"'] & \Omega^1_{Z/k} \otimes k(x) \\ & \mathfrak{m}/\mathfrak{m}^2 \arrow[u, "d_{Z/k}"'] \end{tikzcd}$$

where $\varphi : f_i \otimes k(x) \mapsto (f_i)_x \mod \mathfrak{m}^2$, the $(f_i)_x \mod \mathfrak{m}^2$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, so $((f_1)_x, \cdots, (f_r)_x)$ is part of a regular system of parameters of $\mathcal{O}_{Z,x}$. Then $\mathcal{O}_{X,x} = \mathcal{O}_{Z,x}/\mathcal{I}_x$ is regular, by 3.1.

Since X is smooth over k, the sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Z/k} \otimes \mathcal{O}_X \to \Omega^1_{X/k} \to 0$$

is exact. As $\mathrm{rk}(\Omega^1_{Z/k} \otimes \mathcal{O}_X) = n + r$ and $\mathrm{rk}(\mathcal{I}/\mathcal{I}^2) = r$, $\mathrm{rk}\Omega^1_{X/k} = n = \dim X$.

(2) We will apply Jacobian criterion to

$$\begin{tikzcd} x \in X \arrow[r, hook, "i"] \arrow[d] & Z = \mathbb{A}^{n+r}_k \arrow[dl] \\ \operatorname{Spec} k & \end{tikzcd}$$

where x is any closed point, and $\mathcal{I}$ is the ideal of the closed immersion i. It is sufficient to show

$$d \otimes k(x) : \mathcal{I}/\mathcal{I}^2 \otimes k(x) \to \Omega^1_{Z/k} \otimes k(x)$$

is injective.

Since k is perfect, $k(x)/k$ is separable and hence $\Omega^1_{k(x)/k} = 0$, by 3.6.

Consider the exact sequences:

$$\mathcal{I}/\mathcal{I}^2 \otimes k(x) \to \Omega^1_{Z/k} \otimes k(x) \to \Omega^1_{X/k} \otimes k(x) \to 0 \quad (*)$$

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega^1_{X/k} \otimes k(x) \to \Omega^1_{k(x)/k} \to 0 \quad (**)$$

The sequence (*) implies $\dim \Omega^1_{Z/k} \otimes k(x) \geq n$, and the sequence (**) implies $\dim \Omega^1_{Z/k} \otimes k(x) \leq n$. So $\dim \Omega^1_{Z/k} \otimes k(x) = n$, and hence $d \otimes k(x)$ is injective. So X/k is smooth at x, using the Jacobian criterion. $\square$

Corollary 3.8. *Let k be a field, and X/k be of finite type. Then the following conditions are equivalent:*

(1) X/k is smooth.

(2) For any extension k'/k, $X \otimes k'$ is regular.

(3) There exists a perfect extension k'/k, such that $X \otimes k'$ is regular.

Proof. We only proof (3)⇒(1). Let $X' = X \otimes k'$. Since k' is perfect, X' is smooth over k' by 3.7. Since X/k is of finite type, there exists some n and a closed immersion $i : X \hookrightarrow \mathbb{A}^n_k$. Using base change, we have a similar closed immersion: $i' : X' \to \mathbb{A}^n_{k'}$. For any $x \in X$, let $x' \in X'$ be an inverse image of x of the canonical map $X' \to X$. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the ideal of i (resp. i'), let $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$, $\mathcal{N}' = \mathcal{I}'/\mathcal{I}'^2$. As X'/k' is smooth,

$$d_{\mathbb{A}^n_{k'}/k'} \otimes k(x') : \mathcal{N}' \otimes k(x') \to \Omega_{\mathbb{A}^n_{k'}/k'} \otimes k(x')$$

is injective by 2.3(4)(a) and 2.12. Since $k \to k'$ is flat, one can show that $\mathcal{N}' \otimes k(x) \cong \mathcal{N} \otimes k(x')$, and $i'^*\Omega^1_{\mathbb{A}^n_{k'}/X'} \otimes k(x') = i^*\Omega^1_{\mathbb{A}^n_k/X} \otimes k(x')$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{N}' \otimes k(x') \arrow[r, "d_{\mathbb{A}^n_{k'}/k'} \otimes k(x')"] & i'^*\Omega^1_{\mathbb{A}^n_{k'}/X} \otimes k(x') \\
\mathcal{N} \otimes k(x) \arrow[u] \arrow[r, "d_{\mathbb{A}^n_k/k} \otimes k(x)"] & i^*\Omega^1_{\mathbb{A}^n_k/X} \otimes k(x) \arrow[u]
\end{tikzcd}.$$

with the vertical arrows injective, which follows that $d_{\mathbb{A}^n_k/k} \otimes k(x)$ is injective. Therefore X is smooth over k by 2.12 and 2.10(4)(b). □

Theorem 3.9. *Let $f : X \to Y$ be locally of finite presentation. Then the following conditions are equivalent:*

(1) f is smooth.

(2) f is flat, and for any $y \in Y$, X_y/y is smooth.

Lemma 3.10. *Let A be an artinian local ring, with the maximal ideal $\mathfrak{m}$ and the residue field $k = A/\mathfrak{m}$. Let E be an A-module, then $E \otimes k = 0$ implies $E = 0$.*

Proof. Since A is an artinian local ring, there exist $n \in \mathbb{N}$, such that $\mathfrak{m}^n = 0$. Then $E \otimes k = 0$ implies

$$E = \mathfrak{m}E = \mathfrak{m}^2 E = \cdots \mathfrak{m}^n E = 0.$$

□

Lemma 3.11. *Let A be an artinian local ring, M be an A-module. Then the following conditions are equivalent:*

(1) M is free.

(2) M is flat.

Proof. (1)$\Rightarrow$ (2) is clear.

(2)$\Rightarrow$ (1): Let $\mathfrak{m}$ be the maximal ideal of A and k be the residue field. As M is flat, $\mathrm{Tor}_1(M,k)=0$. Let $(x_\alpha)_{\alpha\in I}$ be the elements of M whose images in $M/\mathfrak{m}M$ form a k-basis. Let P be a free A-module with basis $(e_\alpha)_{\alpha\in I}$ and ϕ be the homomorphism from P into M which maps e_α to x_α. By 3.10, ϕ is surjective. Let N be its kernel. The exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

gives rise to the exact sequence:

$$\mathrm{Tor}_1(P,k)=0 \to \mathrm{Tor}_1(M,k)=0 \to N/\mathfrak{m}N \to P/\mathfrak{m}P \xrightarrow{\overline{\phi}} M/\mathfrak{m}M \to 0$$

As $\overline{\phi}$ is bijective, $N/\mathfrak{m}N$=0 and hence N is zero. $\square$

Lemma 3.12. *Let A, B be noetherian local rings, $A\to B$ be a local morphism. Let E, F be finitely generated B-modules, and F be flat over A. Assume $u\otimes k : E\otimes k\to F\otimes k$ is injective, then u is injective, and* Coker u *is flat over A.*

Proof. (Raynaud) Let $A_n = A/\mathfrak{m}^{n+1}$, $E_n = E\otimes A_n$, $F_n = F\otimes A_n$. First we show that $u_n : E_n \to F_n$ is injective and split. Since F_n flat over A_n, F_n is free over A_n, by 3.11. Take a basis of $E_n\otimes k$, lift its image in $F_n\otimes k$ into a part of basis of F_n, which forms a free submodule L', making the following diagram commutes:

$$\begin{tikzcd} L' \arrow[d, "\varphi"'] \arrow[dr, hook] & \\ E_n \arrow[r] & F_n \end{tikzcd}$$

where φ is defined in the obvious way. We have φ is surjective by Nakayama's lemma, and hence φ is an isomorphism. So the sequence

$$0 \to E_n \xrightarrow{u_n} F_n \to \mathrm{Coker}(u_n) \to 0$$

is injective and split. Then F_n is flat over A_n implies $\mathrm{Coker}(u_n)$ is flat over A_n for any n. Consider the commutative diagram

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d] & F \arrow[d] \\ \varinjlim E_n = \hat{E} \arrow[r, hook] & \hat{F} = \varinjlim F_n \end{tikzcd}$$

where $E \to \hat{E}$ and $F \to \hat{F}$ are injective ([B] III §5, Proposition 2). So $E \to F$ is injective. Therefore $\mathrm{Coker}(u)$ is flat over A ([B] III §5, Theorem 1). $\square$

Lemma 3.13. *Let $A \to B$ be a local morphism, with $(A, \mathfrak{m}_A)$, $(B, \mathfrak{m}_B)$ noetherian local rings, $f \in \mathfrak{m}_B$. Let M be a finitely generated B-module. If $M/f^{n+1}M$ is flat over A for any $n \geq 0$, then M is flat over A.*

Proof. It is sufficient to show for any $N' \hookrightarrow N$, where N', N are finitely generated A-modules, $u : M \otimes_A N' \to M \otimes_A N$ is injective.

As $M \otimes_A N'$ is finitely generated B-module, it is separated for the f-adic topology. Let $x \in \mathrm{Ker}(u)$. For any $n \geq 0$, the map $M/f^{n+1}M \otimes_A N' \to M/f^{n+1}M \otimes_A N$ is injective, by the assumption that $M/f^{n+1}M$ is flat. Then we deduce that $x \in f^{n+1}(M \otimes_A N')$ from the commutative diagram:

$$\begin{tikzcd} M \otimes_A N' \arrow[r] \arrow[d] & M \otimes_A N \arrow[d] \\ M/f^{n+1}M \otimes_A N' \arrow[r] & M/f^{n+1}M \otimes_A N \end{tikzcd}$$

Thus $x \in \bigcap_n f^{n+1}(M \otimes_A N') = 0$. So u is injective, and hence M is flat over A. $\square$

Proposition 3.14. *Let $A \to B$ be a local morphism, with $(A, \mathfrak{m}_A)$, $(B, \mathfrak{m}_B)$ noetherian local rings, $k = A/\mathfrak{m}_A$. Let M be a finitely generated B-module, $f_1, \cdots, f_r \in \mathfrak{m}_B$. Then the following conditions are equivalent:*

(1) M is flat over A, and $(f_1 \otimes k, \cdots, f_r \otimes k)$ is $(M \otimes k)$-regular.

(2) $(f_1, \cdots, f_r)$ is M-regular, and $M/\sum_{i=1}^{r} f_i M$ is flat over A

Proof. (1)⇒(2) By induction on r, we may reduce the case to $r = 1$. By assumption, $f \otimes k : M \otimes k \to M \otimes k$ is injective, and M is flat over A. Thus f is injective and M/fM is flat, by 3.12.

(2)⇒(1) By induction on r, we may reduce to the case $r = 1$. Consider the exact sequence:

$$0 \to M \xrightarrow{f} M \to M/fM \to 0 \quad (*)$$

M/fM is flat over A, M/fM is free over A by 3.11, and hence $(*)$ splits. So $f \otimes k : M \otimes k \to M \otimes k$ is injective. It remains to show M is flat over A. Consider the exact sequence:

$$0 \to M/fM \xrightarrow{f^n} M/f^{n+1}M \to M/f^nM \to 0,$$

by induction on r, we get that $M/f^{n+1}M$ is flat for any n. Hence M is flat over A by 3.13. $\square$

Recall that a closed immersion $i : Y \to X$ of locally noetherian schemes is *regular* at $x \in Y$ if the ideal $\mathcal{I}$ of i can be locally defined by $f_1, \cdots, f_r$ at x, such that $(f_i)_x$ is a regular sequence in $\mathcal{O}_{X,x}$ (this is equivalent to saying that the Koszul complex $K_\bullet(f_i)$ is a resolution of $\mathcal{O}_Y$ around x).

Corollary 3.15. *Consider the following commutative diagram:*

$$\begin{tikzcd}
x \in Y_s \arrow[r, hook] \arrow[d, hook, "i_s"'] & Y \arrow[d, hook, "i"] \\
X_s \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
\operatorname{Spec} k(s) = s \arrow[r, hook] & S
\end{tikzcd}$$

where i is a closed immersion, S is locally noetherian, $X \to S$ is locally of finite type. Then the following conditions are equivalent:

(1) i_s is regular at x, and X is flat over S at x (i.e. $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module).

(2) Y is flat over S at x, and i is regular at x.

Proof. Apply 3.14 to $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, $M = \mathcal{O}_{X,x}$. $\square$

Corollary 3.16. *i is regular and Y is flat if and only if X is flat over S and i_s is regular for any $s \in S$.*

Proof of Theorem 3.9. (1)$\Rightarrow$ (2): Assume f is smooth, we need to prove (a): X_y is regular for any $y \in Y$, (b): f is flat. (a) is trivial. So it sufficient to prove (b). We have the commutative diagram

$$\begin{tikzcd}
x \in X \arrow[r, hook, "i"] \arrow[d] & Z = \mathbb{A}_Y^{n+r} \arrow[dl] \\
y \in Y &
\end{tikzcd}$$

where $x \in X_y$. Let $\mathcal{I}$ be the ideal of i, let $f_1, \cdots, f_r$ be local sections of $\mathcal{I}$ at x such that $(f_i)_x$ is a minimal system of generators of $\mathcal{I}_x$ (i.e. $f_i \otimes k(x)$ is a basis of $\mathcal{N} \otimes k(x)$. Define the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \otimes k(x) \arrow[r] \arrow[dr, hook] & \Omega^1_{Z/Y} \otimes k(x) \arrow[r] & \Omega^1_{X/Y} \otimes k(x) \arrow[r] & 0 , \\
& & \mathfrak{m}_{X_y,x}/\mathfrak{m}^2_{X_y,x} \arrow[u, "d_{Z/Y}"'] & &
\end{tikzcd}$$

the images of $f_i \otimes k(x)$ are linearly independent in $\mathfrak{m}_{X_y,x}/\mathfrak{m}^2_{X_y,x}$. Hence $(f_i)_x$ is part of a regular system of parameters of $\mathcal{O}_{X_y,x}$, in particular, $(f_i)_x$ form a regular sequence in $\mathcal{O}_{X_y,x}$. So f_i form a regular sequence and X is flat over Y at x, by 3.14.

(2)⇒(1):exercise. □

End of the proof of Theorem3: To show that (2) implies (1), we must show that for any point $x \in X$, f is smooth at x. Let $y = f(x)$. Since the problem is local on X, we may assume X is embedded in some $Z = \mathbb{A}^{n+r}_Y$ with ideal $\mathcal{I}$.

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] & X \arrow[r, "i"] \arrow[d, "f"] & Z = \mathbb{A}^{n+r}_Y \arrow[dl] \\
y \arrow[r] & Y &
\end{tikzcd}$$

Then we have an exact sequence:

$$0 \to \mathcal{I}_x \to \mathcal{O}_{Z,x} \to \mathcal{O}_{X,x} \to 0.$$

Since f is flat, applying $\otimes k(y)$, one gets an exact sequence:

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_{Y,y}} k(y) \to \mathcal{O}_{Z_y,x} \to \mathcal{O}_{X_y,x} \to 0.$$

Suppose $g_1, \cdots, g_r$ generate $\mathcal{I} \otimes k(y)$ and $dg_1(x), \cdots, dg_r(x)$ are linearly independent on $k(y)$ in $\Omega^1_{Z_y/y} \otimes k(x) = \Omega^1_{Z/Y} \otimes k(x)$. Lift $g_1, \cdots, g_r$ to $f_1, \cdots, f_r$ in $\mathcal{I}_x$, then $df_1(x), \cdots, df_r(x)$ are linearly independent on $k(x)$. By Nakayama's Lemma, $\mathcal{I}_x$ is generated by $f_1, \cdots, f_r$. Applied the Jacobian criterion, it follows that f is smooth at x. This completes the proof of Theorem

Remark. Let $f : X \to Y$ be a smooth morphism. Then $\Omega^1_{X/Y}$ is locally free of finite type. For a point x in X, let $y = f(x)$ and $\Omega^1_{X_y/y} = \Omega^1_{X/Y} \otimes \mathcal{O}_{X_y}$, then the integer

$$\mathrm{rk}_{k(x)} \Omega^1_{X/Y} \otimes k(x) = \mathrm{rk}_{k(x)} \Omega^1_{X_y/y}$$

is called the *relative dimension* of f at x. This is a locally constant function of x. By classical dimension theory, it's just the dimension of the irreducible component of the X_y containing x. Obviously, for f to be étale , it's necessary and sufficient that f is smooth with relative dimension 0.

Corollary 3.17. *A morphism $f : X \to Y$ is étale if and only if f is of finite presentation, flat and $\Omega^1_{X/Y} = 0$.*

Proof. The "only if" part, is clear. Conversely, by Theorem 3.9, we only need to show that $X_y \to y$ is smooth. But since $\Omega^1_{X/Y} \otimes k(y) = \Omega^1_{X_y/y} = 0$, by Proposition 3.6, $X_y \to y$ is étale, in particular smooth. $\square$

Corollary 3.18. *Consider the following diagram:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd} ,$$

where f, g are smooth, and i is a closed immersion. Then i is a regular immersion.

Proof. For a point x in X, let U be an affine open neighborhood of x in Z, such that we have the following cartesian diagram :

$$\begin{tikzcd} X \cap U \arrow[r, hook] \arrow[d] & U \arrow[d, "h"] \\ \mathbb{A}^n_Y \arrow[r] & \mathbb{A}^{n+r}_Y = \operatorname{Spec} \mathcal{O}_Y[t_1, \cdots, t_{n+r}] \end{tikzcd} .$$

where h is étale. Then $h^*(t_i) = f_i \in \Gamma(U, \mathcal{O}_U)\,(1 \leqslant i \leqslant n + r)$, and $\mathbb{A}^n_Y$ is the linear subspace with equations $t_1 = \cdots = t_r = 0$. Thus $X \cap U$ is the closed subscheme in U defined by the ideal $I = (f_1, \cdots, f_r)$. For i to be regular at x, it suffices that $f_1, \cdots, f_r$ is a regular sequence, i.e. $\epsilon : K.(f_1, \cdots, f_r) \to \mathcal{O}_{X \cap U}$ is a quasi-isomorphism, where $K_\bullet(f_1, \cdots, f_r)$ is the Kozsul complex. But the quasi-isomorphism $\epsilon_0 : K_\bullet(t_1, \cdots, t_r) \to \mathcal{O}_{\mathbb{A}^n}$, remains a quasi-isomorphism by tensoring it with $\mathcal{O}_U$, since $h : U \to \mathbb{A}^{n+r}_Y$ is flat by Theorem 3.9, and $\epsilon_0 \otimes \mathcal{O}_U = \epsilon$. $\square$

Before stating the next corollary, we first recall some basic definitions in linear algebra. Let k be a field, and E be a finite dimensional k-vector space.

Let V and W be two subspaces of E, then we say V and W are *transversal* in E, if $E = V + W$. In this case we have

$$\operatorname{codim}(V \cap W, E) = \operatorname{codim}(V, E) + \operatorname{codim}(W, E),$$

(i.e. there exists a decomposition of $E = V' \oplus (V \cap W) \oplus W'$ such that $V' \oplus (V \cap W) = V$ and $(V \cap W) \oplus W' = W$).

Corollary 3.19 (Transversality). *Let X be a scheme over S, and Y, Z be two closed subschemes of X. Then we have the following cartesian diagram:*

$$\begin{tikzcd} t \in Y \cap Z := Y \times_X Z \arrow[r, "h"] \arrow[d, "k"'] & Y \arrow[d, "i"] \\ Z \arrow[r, "j"] & X \end{tikzcd}$$

Suppose that X, Y, Z are smooth over S, and t is a point in $Y \cap Z$. From the natural maps: $N_{Y/X} \to i^*\Omega^1_{X/S}$ and $j^*\Omega^1_{X/S} \to \Omega^1_{Z/S}$, it follows that

$$h^*N_{Y/X} \to h^*i^*\Omega^1_{X/S} = k^*j^*\Omega^1_{X/S} \to k^*\Omega^1_{Z/S}.$$

After tensoring $k(t)$, one gets a canonical map:

$$N_{Y/X} \otimes k(t) \to \Omega^1_{Z/S} \otimes k(t). \tag{3.19.1}$$

Then the following conditions are equivalent:
(1) The canonical map (3.19.1) *is injective.*
(2) The analogous canonical map $N_{Z/X} \otimes k(t) \to \Omega^1_{Y/S} \otimes k(t)$ is injective.
(3) $T_{Y/S} \otimes k(t)$ and $T_{Z/S} \otimes k(t)$ are transversal in $T_{X/S} \otimes k(t)$.
When (1)-(3) are satisfied at t, then they are satisfied in a neighborhood of t and the natural map

$$N_{Y \cap Z/X} \to N_{Y \cap Z/Y} \bigoplus N_{Y \cap Z/Z}$$

is an isomorphism. In particular, for any $s \in S$,

$$\operatorname{codim}_t((Y \cap Z)_s, X_s) = \operatorname{codim}_t((Y \cap Z)_s, Y_s) + \operatorname{codim}_t((Y \cap Z)_s, Z_s).$$

The proof of this corollary is quite elementary, we leave it as an exercise.

4 Smoothness and Deformations

Let G be a sheaf of groups on a space X. A left G-*sheaf* on X is a sheaf E equipped with a map of sheaves: $G \times E \longrightarrow E$, satisfying:
(i) For any open subset U of X, $g(ha) = (gh)(a)$ for $g, h \in G(U)$, $a \in E(U)$;
(ii) $ea = a$ for any $a \in E(U)$, where $e \in G(U)$ is the neutral element.
A G-*morphism* (a G-*equivariant* morphism) between G-sheaves is a map $u : E \to F$ commuting with the action of G, such that $u(ga) = gu(a)$ for $g \in G(U)$ and $a \in E(U)$.

Definition 4.1. A G-*torsor* on X (or a *torsor under* G on X) is a G-sheaf E having the following properties:
(1) For any open subset U of X and $a, b \in E(U)$, there exists a unique $g \in G(U)$ such that $ga = b$;
(2) For any $x \in X$, E_x is not empty, or equivalently, there exists an open covering (U_i) of X, such that $E(U_i) \neq \emptyset$.

Example 4.1.1. Let $X = \{pt\}$(the space with one point), then G is just a group, and a G-sheaf E is just a G-set. E is a G-torsor if and only if E is an affine space under G, i.e. E is nonempty and for any $a, b \in E$, there exists a unique $g \in G$ such that $ga = b$.
When E is a G-torsor, if one chooses $a_0 \in E$, then the map $G \to E$ given by $g \mapsto ga_0$ is an isomorphism of G-sets.

Remark. Let E be a G-sheaf on X. Assume E satisfies 4.1 (1). Then if for some U open in X, $E(U) \neq \emptyset$, then by taking some $a \in E(U)$, one obtains an isomorphism of G-sheaves $G|_U \to E|_U$ given by $g \mapsto g(a|_V)$, where V is an open subset of U and $g \in G(V)$.
Hence for a G-sheaf E, E is a G-torsor if and only if E is locally G-isomorphic to G acting on itself by left translations. And any G-equivariant morphism of G-torsors $u : E \to F$ is actually an isomorphism.

Definition 4.2. Let E be a G-torsor. E is called *trivial* if E is isomorphic to G acting on itself by left translations.

Note that E is trivial if and only if $E(X)$ is nonempty.

4.3. Cohomology Class of a Torsor (Commutative Case) Let G be a sheaf of abelian groups on X, E be a G-torsor on X, and we will write the action of G on E additively, i.e. "$g + a$" instead of "ga".

Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X, and $s_i \in E(U_i)$. Denote $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0\cdots i_p}$. There exists a unique $g_{ij} \in G(U_{ij})$ such that

$$s_j|_{U_{ij}} - s_i|_{U_{ij}} = g_{ij}.$$

Then

$$(g_{ij} - g_{ik} + g_{jk})|_{U_{ijk}} = 0,$$

i.e. $(g_{ij}) \in \check{Z}^1(\mathcal{U}, G) = Z^1\check{C}(\mathcal{U}, G)$. Let $c(E) \in H^1(X, G)$ be the image of (g_{ij}) in $H^1(X, G)$ by the canonical map:

$$\check{Z}^1(\mathcal{U}, G) \to \check{H}^1(\mathcal{U}, G) \to H^1(X, G).$$

Proposition 4.4. *$c(E)$ does not depend on the choice of $(\mathcal{U}, s_i)$. Moreover $E \mapsto c(E)$ induces a bijection:*

$$Tors(X, G) \xrightarrow{\sim} H^1(X, G),$$

where $Tors(X, G)$ is the set of isomorphism classes of G-torsors on X. The class of trivial torsors corresponds to the zero element in $H^1(X, G)$. In particular, a G-torsor E is trivial if and only if $c(E) = 0$.

4.5. Preliminary on Čech Cohomology: We have seen before that there is a natural map $\check{H}(\mathcal{U}, G) \to H^n(X, G)$, and we want to look at it more closely. A covering $\mathcal{V} = (V_j)_{j\in J}$ of X is said to refine $\mathcal{U} = (U_i)_{i\in I}$, if there exists a map $\varphi : J \to I$ such that $V_j \subset U_{\varphi(j)}$ for any $j \in J$. Then φ induces a natural map: $\varphi^* : \check{C}^n(\mathcal{U}, G) \to \check{C}^n(\mathcal{V}, G)$ given by

$$(\varphi^* a)_{j_0\cdots j_n} = a_{\varphi(j_0)\cdots\varphi(j_n)}|_{V_{j_0\cdots j_n}}.$$

It's easily checked that φ^* is actually a map of complexes between $\check{C}(\mathcal{U}, G)$ and $\check{C}(\mathcal{V}, G)$, and that for two maps φ, ψ from J to I, the resulting maps φ^*, ψ^* are homotopic, i.e.

$$\varphi^* - \psi^* = hd + dh,$$

where $h : \check{C}^n(\mathcal{U}, G) \to \check{C}^{n-1}(\mathcal{V}, G)$ is given by

$$(h(s))_{j_0\cdots j_{n-1}} = \sum_{k=0}^{n-1} (-1)^k s_{\psi(j_0)\cdots\psi(j_k)\varphi(j_k)\cdots\varphi(j_{n-1})}|_{V_{j_0\cdots j_{n-1}}}.$$

Thus one gets a well defined (independent of φ) map:

$$\rho_{\mathcal{U}\mathcal{V}} : \check{H}^n(\mathcal{U}, G) \to \check{H}^n(\mathcal{V}, G).$$

This $\rho_{\mathcal{U}\mathcal{V}}$ makes the following diagram commute:

$$\begin{tikzcd} \check{H}^n(\mathcal{U}, G) \arrow[rr, "\rho_{\mathcal{U}\mathcal{V}}"] \arrow[dr] & & \check{H}^n(\mathcal{V}, G) \arrow[dl] \\ & H^n(X, G) & \end{tikzcd} .$$

Hence one gets a natural map:

$$\varinjlim \check{H}^n(\mathcal{U}, G) \to H^n(X, G) \tag{4.5.1}$$

where $\mathcal{U}$ runs through the open covering of X.

Lemma 4.6. *The map (4.5.1) is bijective for $n = 0, 1$.*

Proof. For $n = 0$, it follows from $\check{H}^0(\mathcal{U}, G) = H^0(X, G) = \Gamma(X, G)$. Suppose that $n = 1$. Take an exact sequence

$$0 \to G \to L \xrightarrow{p} M \to 0 \tag{4.6.1}$$

with L flasque. Then one gets

$$0 \to C^\bullet(\mathcal{U}, G) \xrightarrow{u} C^\bullet(\mathcal{U}, L) \to C^\bullet(\mathcal{U}, M). \tag{4.6.2}$$

Denote by $D^\bullet(\mathcal{U})$ the cokernel of u. Then one gets a long exact sequence:

$$0 \to \Gamma(X, G) \to \Gamma(X, L) \xrightarrow{\varphi_{\mathcal{U}}} H^0(D^\bullet(\mathcal{U})) \to \check{H}^1(\mathcal{U}, G) \to 0$$

since L is flasque. Now in the diagram

$$\begin{tikzcd} \Gamma(X, L) \arrow[r, "\varphi_{\mathcal{U}}"] \arrow[dr, "\eta"'] & H^0(D^\bullet(\mathcal{U})) \arrow[d, "\psi_{\mathcal{U}}"] \\ & \Gamma(X, M) \end{tikzcd} ,$$

note that $\psi_{\mathcal{U}}$ is injective, $H^1(X, G) = \operatorname{Coker} \eta$ and $\check{H}^1(\mathcal{U}, G) = \operatorname{Coker} \varphi_{\mathcal{U}}$. Then by Snake Lemma, one obtains

$$0 \to \check{H}^1(\mathcal{U}, G) \to H^1(X, G) \to \operatorname{Coker} \psi_{\mathcal{U}} \to 0.$$

By passing to the limit, we see that it only remains to show that

$$\varinjlim \operatorname{Coker} \psi_{\mathcal{U}} = \operatorname{Coker}(\varinjlim H^0(D^\bullet(\mathcal{U})) \to \Gamma(X, M)) = 0.$$

Since (4.6.1) is exact, for any $s \in \Gamma(X, M)$ there exists an open covering $\mathcal{U} = (U_i)_{i \in I}$, such that $s|_{U_i} = p(t_i)$, where $t_i \in \Gamma(U_i, L)$. Hence $\psi_{\mathcal{U}}(p(t_i)) = s$, this shows that the morphism $\varinjlim H^0(D^\bullet(\mathcal{U})) \to \Gamma(X, M)$ is surjective. Our conclusion follows from it. $\square$

Proof of Proposition (sketch): (a) First, we verify that the map $E \mapsto c(E)$ (denoted by φ) does not depend on the choice of $\mathcal{U}$. Suppose there are two coverings, say $\mathcal{U}_1 = (U_{1i})_{i \in I_1}$ and $\mathcal{U}_2 = (U_{2i})_{i \in I_2}$, and a torsor E with Čech cocycles (g_{1ij}) and (g_{2ij}) respectively. Then one can find a third covering $\mathcal{V}$ of X, which is a common refinement of $\mathcal{U}_1$ and $\mathcal{U}_2$. It's easily checked that in (g_{1ij}) and (g_{2ij}) has the same image in $\check{H}^1(\mathcal{V}, G)$.
(b) We give a map $\psi : H^1(X, G) \to Tors(X, G)$ inverse to φ. Suppose there is an element ξ in $H^1(X, G)$ represented by a Čech cocycle (g_{ij}) for some open covering $\mathcal{U} = (U_i)_{i \in I}$ of X. We can associate to it a torsor under G in the following way:

$$U \longmapsto E(U) = \{(s_i \in G(U_i \cap U)_{i \in I}) | s_j|_{U_ij \cap U} - s_i|_{U_ij \cap U} = g_{ij}|_{U_{ij} \cap U}\}. \quad (4.6.3)$$

One verifies that E is a G-sheaf and $E|_{U_i} \simeq G|_{U_i}$. This E is actually a G-torsor. Then we define $\psi(\xi)$ to be the isomorphism class of E. One can verify that it does not depend on the representing Čech cocycle.
Immediately we note that the class of $\psi(\xi)$ in $H^1(X, G)$ is the class of (g_{ij}). This proves $\varphi\psi = Id$. Conversely, given a torsor F, let $c(F)$ be its corresponding cohomology class. Then it is represented by a Čech cocycle (g_{ij}) for some open covering $\mathcal{U} = (U_i)_{i \in I}$, i.e. $t_i|_{U_{ij}} - t_j|_{U_{ij}} = g_{ij}$, where $t_i \in F(U_i)$. Then according to the above construction, $\psi(c(F))$ is represented by a torsor E defined in (4.6.3). Hence for an element $s = (s_i)$ in $E(U)$, one has

$$s_i|_{U_ij \cap U} + t_i|_{U_ij \cap U} = s_j|_{U_ij \cap U} + t_j|_{U_ij \cap U}.$$

This shows that $(s_i|_{U_ij \cap U} + t_i|_{U_ij \cap U})$ paste together to give a section in $F(U)$. This gives a well defined G-equivariant map $u : E \to F$. Hence it follows that $\psi\varphi = Id$.

$\square$

Remark. Since $H^1(X,G) = \mathrm{Ext}^1(\mathbb{Z}_X, G)$, where $\mathbb{Z}_X$ is the constant $\mathbb{Z}$-sheaf on X, one gets another description of the correspondence between cohomology classes and torsors. Let

$$0 \longrightarrow G \longrightarrow L \xrightarrow{p} \mathbb{Z}_X \longrightarrow 0$$

be an element in $\mathrm{Ext}^1(\mathbb{Z}_X, G)$, then we relate it to the torsor defined by $E = p^{-1}(1)$, where $1 \in \Gamma(X, \mathbb{Z}_X)$.

Example 4.6.1. For $G = \mathcal{O}_X^*$, an $\mathcal{O}_X^*$-torsor is just an invertible sheaf (or a line bundle) on X. Let $\mathcal{L}$ be a line bundle on X, and $(e_i \in \mathcal{L}(U_i))_{i\in I}$ a local basis of $\mathcal{L}$.Then there exist $g_{ij} \in \Gamma(X, \mathcal{O}_X^*)$ such that $e_j = g_{ij}e_i$ on U_{ij}. The map $\mathcal{L} \mapsto (g_{ij})$ gives an isomorphism

$$Pic(X) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^*),$$

where $Pic(X)$ is the set of isomorphism classes of $\mathcal{O}_X^*$ torsors on X.

Theorem 4.7. *Consider the diagram*

$$\begin{tikzcd} & & X \arrow[d, "f"] \\ T_0 \arrow[r, "i"] \arrow[urr, "g_0"] & T \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd} , \tag{4.7.1}$$

where f is smooth, and i is a first-order thickening with ideal sheaf $\mathcal{I}$.
(a) There exists an obstruction

$$o(i, g_0) \in \mathrm{Ext}^1(g_0^*\Omega^1_{X/Y}, \mathcal{I}),$$

whose vanishing is necessary and sufficient for the existence of a global Y-morphism $g : T \to X$ extending g_0.
(b) If $o(i, g_0) = 0$, then the set of extensions of g_0 is an affine space under $\mathrm{Hom}(g_0^*\Omega^1_{X/Y}, \mathcal{I})$.

Proof. First note that since $\Omega^1_{X/Y}$ is locally free of finite type,

$$G := R\mathcal{H}om(g_0^*\Omega^1_{X/Y}, \mathcal{I}) = \mathcal{H}om(g_0^*\Omega^1_{X/Y}, \mathcal{I}) \simeq g_0^*T_{X/Y} \otimes \mathcal{I}$$

hence

$$\mathrm{Ext}^i(g_0^*\Omega^1_{X/Y}, \mathcal{I}) = H^i(T_0, g_0^*T_{X/Y} \otimes \mathcal{I}) = H^i(T_0, G).$$

We also note that an extension g of g_0 is completely determined by its corresponding morphism $g^* : \mathcal{O}_X \to g_*\mathcal{O}_T$. Let E be the sheaf on T_0 given by

$$U_0 \longmapsto \{g^* | g \in \mathrm{Hom}_Y(U, X), g_0 = g \circ i\}$$

where U is the open subscheme of T corresponding to U_0. Then E is a G-sheaf: for any g^* in $E(U_0)$ and D in $G(U_0) = \mathrm{Der}_Y(X, g_{0*}\mathcal{I})$, $g^* + D$ is also in $E(U_0)$. Actually, E is a G-torsor, since locally there exist extensions of g_0(because f is smooth), and for any g_1^*, g_2^* in $E(U_0)$, $g_1^* - g_2^* \in G(U)$. Thus one gets a cohomology class $c(E) \in H^1(T_0, G)$, which is the desired obstruction $o(i, g_0)$. When $o(i, g_0) = 0$, $E \simeq G$ as a G-sheaf, hence $E(T_0)$, the set of global extensions of g_0, is an affine space under $G(T_0) = H^0(g_0^*\Omega^1_{X/Y}, \mathcal{I})$. □

Corollary 4.8. *In the diagram (4.7.1), if T is affine, then the obstruction $o(i, g_0)$ vanishes, hence global extensions of g_0 exists.*

Proof. Indeed, $g_0^*T_{X/Y} \otimes \mathcal{I}$ is a quasi-coherent sheaf on T_0, and

$$H^1(T_0, g_0^*T_{X/Y} \otimes \mathcal{I}) = 0$$

since T_0 is affine. □

Definition 4.9. Let $Y_0 \to Y$ be a first-order thickening with ideal $\mathcal{I}$, and $f_0 : X_0 \to Y_0$ be a smooth morphism. A *deformation* of X_0 over Y is a flat morphism $f : X \to Y$ such that $X_0 = Y_0 \times_Y X$, such that one has the following cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r, "\mathcal{J}"] \arrow[d, "f_0"] & X \arrow[d, "f"] \\ Y_0 \arrow[r, "\mathcal{I}"] & Y \end{tikzcd} \tag{4.9.1}$$

with f flat.

Remark. (a) If f is a deformation of f_0, then f is smooth according to Theorem 3.9.
(b) In the cartesian diagram (4.9.1), if f_0 is flat , then for f to be flat it is necessary and sufficient that $f_0^*\mathcal{I} \backsimeq \mathcal{J}$ by the flatness criterion.

In the sequel, we sometimes use the phrase *"smooth lifting"* (or *"lifting"* for short) instead of "deformation". For a given smooth $f_0 : X_0 \to Y_0$ and thickening $Y_0 \to Y$, an isomorphism between two liftings, $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, is a morphism $a : X_1 \to X_2$, such that the following diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[rr, "a"] \arrow[drr, "f_1"] & & X_2 \arrow[d, "f_2"] \\ Y_0 \arrow[rrr] & & & Y \end{tikzcd}$$

commutes and $f_1 = f_2 \circ a$. We note that it's just a map of Y-extensions of X_0 by $f_0^*\mathcal{I}$, hence automatically an isomorphism from X_1 to X_2.

Theorem 4.10. *Let $f_0 : X_0 \to Y_0$ be a smooth morphism, and $i : Y_0 \to Y$ a first-order thickening with ideal sheaf $\mathcal{I}$.*
(a) There exists an obstruction

$$o(f_0, i) \in \mathrm{Ext}^2(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I}),$$

whose vanishing is necessary and sufficient for the existence of a lifting of X_0 over Y.
(b) When $o(f_0, i) = 0$, the set of isomorphism classes of liftings is an affine spaces under the group $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$.
(c) The group of automorphisms of a lifting X is naturally identified with $\mathrm{Hom}(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$.

Proof. First of all, we note that since $\Omega^1_{X_0/Y_0}$ is locally free of finite type, there is, for each $i \in \mathbb{Z}$, a canonical isomorphism

$$\mathrm{Ext}^i(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I}) \backsimeq H^i(X_0, G), \tag{4.10.1}$$

where $G := \mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I}) \backsimeq T_{X_0/Y_0} \otimes f_0^*\mathcal{I}$.
Assertion (c) is a special case of 4.7 (b), the identification associates with an automorphism u of X the "derivation"$u - Id_X$.
For assertion (b), if X_1 and X_2 are two liftings of X_0, consider the sheaf

$$E : U_0 \longmapsto \{a : X_1|_{U_1} \to X_2 \text{ extending } Id_{U_0}\},$$

where U_0 is a open subscheme of X_0, and U_1 the corresponding open subscheme in X_1. As in the proof of 4.7, E is a torsor under

$$\mathcal{H}om(\Omega^1_{X_1/Y} \otimes \mathcal{O}_{X_0}, f_0^*\mathcal{I}) = \mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$$

Let $E(X_1, X_2) \in \mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$ be the obstruction to the existence of a global isomorphism $X_1 \simeq X_2$ according to 4.7. Fix X_1, then one checks the map $X \longmapsto E(X_1, X)$ gives a bijection between isomorphism classes of liftings of X_0 and $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$. We note that if X_0 is affine, then $\mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$ is a quasi-coherent sheaf on X_0. Thus $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$ vanishes by (4.10.1), hence all liftings of X_0 over Y are isomorphic.

Sketch of proof of (a): First we claim that for a point $x \in X_0$, there exists an open neighborhood U_0 of x, such that there exists a lifting U of U_0 over Y. We may assume that $Y = \mathrm{Spec}\, A$ is affine. Then X_0 is also affine with ring $A_0 = A/I$. By the Jacobian criterion, there is an open neighborhood V_0 of x, such that one has a commutative diagram

$$\begin{tikzcd} x \in V_0 \arrow[r, hook, "i_0"] \arrow[d] & \mathbb{A}^{n+r}_{Y_0} = \mathrm{Spec}\, A_0[t_1, \cdots, t_{n+r}] \arrow[dl] \\ Y_0 & \end{tikzcd}$$

where i_0 is a closed immersion with ideal $J = (g_1, \cdots, g_r)$, such that $dg_1(x), \cdots, dg_r(x)$ are linearly independent. Hence there exists an open neighborhood U_0 of x in V_0, such that $dg_1(y), \cdots, dg_r(y)$ are linearly independent for any $y \in U_0$. Now choose $\tilde{g}_1, \cdots, \tilde{g}_r$ to be the lifting of $g_1, \cdots, g_r$ in $A[t_1, \cdots, t_{n+r}]$. Let $V = V(\tilde{g}_1, \cdots, \tilde{g}_r)$ be the closed subscheme of A^{n+r}_Y, and U be the open subscheme of V corresponding to U_0. Then $d\tilde{g}_1(y), \cdots, d\tilde{g}_r(y)$ are linearly independent for any $y \in U$. Again by the Jacobian criterion, U is smooth over Y, hence a lifting of U_0 over Y.
Secondly we prove (a) under the assumption that X_0 is separated. Choose an affine open covering $\mathcal{U} = ((U_i)_0)_{i \in I}$ of X_0, such that for each $i \in I$, we have a lifting U_i of $(U_i)_0$ over Y. Since X_0 is assumed to be separated, each $(U_{ij})_0 = (U_i)_0 \cap (U_j)_0$ is affine. Consequently by (b), there exists an isomorphism $g_{ji} : U_j|_{(U_{ij})_0} \tilde{\rightarrow} U_i|_{(U_{ij})_0}$, which is completely determined by the induced map

$$f_{ji} = g_{ji}^* : \mathcal{O}_{U_i}|_{(U_{ij})_0} \to \mathcal{O}_{U_j}|_{(U_{ij})_0}$$

of $\mathcal{O}_Y$-algebras. On the triple intersection $(U_{ijk})_0 = (U_i)_0 \cap (U_j)_0 \cap (U_k)_0$, the automorphism $f_{ijk} = f_{ik} f_{kj} f_{ji}$ differs from the identity by a Čech 2-cochain $c_{ijk} = f_{ijk} - Id_{\mathcal{O}_{U_i}} : \mathcal{O}_{U_i} \to f_0^*\mathcal{I}$ of the sheaf $\mathcal{H}om(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I})$. One verifies that $c = (c_{ijk})$ is actually a Čech 2-cocycle for the covering $\mathcal{U}$, i.e.

$$c_{jkl} - c_{ikl} + c_{ijl} - c_{ijk} = 0 \qquad (4.10.2)$$

for any $i, j, k, l \in I$. Note that from $c_{jkl} = f_{jl} f_{lk} f_{kj} - Id_{\mathcal{O}_{U_j}}$, we get

$$f_{lk} = f_{lj}(Id_{\mathcal{O}_{U_j}} + c_{jkl}) f_{jk} = f_{lj} f_{jk} + c_{jkl}.$$

Hence it follows that

$$\begin{aligned} c_{ikl} &= f_{il} f_{lk} f_{ki} - Id_{U_i} = f_{il} f_{lj} f_{jk} f_{ki} + c_{jkl} - Id_{U_i} \\ c_{jkl} - c_{ikl} &= -f_{il} f_{lj} f_{jk} f_{ki} + Id_{U_i} \end{aligned}$$

Similarly we also have

$$c_{ijl} - c_{ijk} = f_{il} f_{lj} f_{jk} f_{ki} - Id_{U_i}.$$

Combined the above two formulas, (4.10.2) follows. Hence (c_{ijk}) is a Čech 2-cocycle. It determines a cohomology class in $\check{H}^2(\mathcal{U}, G) \tilde{\to} H^2(X_0, G)$, since X is assumed to be affine. This class vanishes if and only if there exists a 1-cochain $h = (h_{ij})$ such that $c_{ijk} = h_{jk} - h_{ik} + h_{ij}$. Then one defines

$$f'_{ij} = f_{ij} - h_{ij} : \mathcal{O}_{U_j}|_{(U_{ij})_0} \to \mathcal{O}_{U_i}|_{(U_{ij})_0},$$

and one can verify that $f'_{ik} = f'_{ij} f'_{jk}$. Hence the corresponding maps $g'_{ij} : U_i|_{(U_{ij})_0} \tilde{\to} U_j|_{(U_{ij})_0}$ glue on the triple intersections. Therefore one gets a global lifting X of X_0 over Y. □

Corollary 4.11. *Let $f_0 : X_0 \to Y_0$ be an étale morphism, and $i : Y_0 \to Y$ a first-order thickening with ideal sheaf $\mathcal{I}$. Then there exists a unique lifting $f : X \to Y$ of X_0 over Y, and f is necessarily étale.*

Proof. Since f_0 étale, $\Omega^1_{X_0/Y_0} = 0$ and $\mathrm{Ext}^2(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I}) = 0$, thus liftings exist. Moreover $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^*\mathcal{I}) = 0$ there all liftings are isomorphic. For the lifting, say $f : X \to Y$, from

$$\Omega^1_{X/Y} \otimes \mathcal{O}_{Y_0} = \Omega^1_{X_0/Y_0} = 0$$

it follows that $\Omega^1_{X/Y} = 0$ since $\mathcal{I}^2 = 0$. Hence f is necessarily étale. □

Corollary 4.12. *Let $f_0 : X_0 \to Y_0$ be a smooth morphism, and $i : Y_0 \to Y$ a first-order thickening with ideal sheaf $\mathcal{I}$. If X_0 is affine, there exists a unique lifting of X_0 over Y.*

Indeed, this follows from $H^2(X_0, T_{X_0/Y_0} \otimes f_0^*\mathcal{I}) = 0$ and $H^1(X_0, T_{X_0/Y_0} \otimes f_0^*\mathcal{I}) = 0$ since $T_{X_0/Y_0} \otimes f_0^*\mathcal{I}$ is a quasi-coherent sheaf.

Corollary 4.13. *Let $f_0 : X_0 \to Y_0$ be a smooth proper morphism with relative dimension 1, and $i : Y_0 \to Y$ a first-order thickening with ideal $\mathcal{I}$. If moreover Y is affine, then there always exists a lifting of X_0 over Y.*

Proof. First we note that

$$H^q(X_0, T_{X_0/Y_0} \otimes f_0^*\mathcal{I}) = \Gamma(Y_0, R^q f_{0*}(T_{X_0/Y_0} \otimes f_0^*\mathcal{I})).$$

By Zariski's main theorem, for any $q > 1$,

$$R^q f_{0*}(T_{X_0/Y_0} \otimes f_0^*\mathcal{I}) = 0.$$

Hence the obstruction $o(f_0, i) \in H^2(X_0, T_{X_0/Y_0} \otimes f_0^*\mathcal{I})$ vanishes. $\square$

5 Serre-Grothendieck Global Duality Theorem

For simplicity, we just discuss the locally neotherian case. Thus all schemes are presumed to be locally noetherian unless otherwise stated.

5.1. The $f^!$ Functor (a) Let $i : Y \to X$ be a closed immersion. Given a complex F in $D^+(X)$, define

$$i^! F := R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y$$

i.e. $i_* i^! F = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)$. It gives a functor from $D^+(X)$ to $D^+(Y)$. If

$$Z \xrightarrow{j} Y \xrightarrow{i} X$$

is a composition of closed immersions, then $j^! i^! = (ij)^!$, i.e. for any $F \in D^+(X)$

$$R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Z, R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y)|_Z = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, F)|_Z.$$

To prove it, we may assume that each $F^i (i \in \mathbb{Z})$ is an injective $\mathcal{O}_X$-module. Then $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y$ and each $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F^i)|_Y$ is an injective $\mathcal{O}_Y$-module. Hence we have

$$\begin{aligned} j^! i! F &= \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)|_Y)|_Z = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F))|_Z \\ &= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_Y, F)|_Z = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, F)|_Z = (ij)^! F. \end{aligned}$$

(b) Let $f : X \to Y$ be a smooth morphism with relative dimension d, then $\omega_{X/Y} = \Omega^d_{X/Y}$ is a line bundle. Define a functor $f^! : D^+(Y) \to D^+(X)$ by

$$f^! F := f^* F \otimes^L \omega_{X/Y}[d]$$

for an element $F \in D^+(X)$.
(c) Let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

be a commutative diagram with i a closed immersion and g smooth. One can define a functor $i^! g^!$ from $D^+(X)$ to $D^+(Y)$. The following theorem ensures that it is independent on the choice of i and g.

Theorem 5.2. *Suppose we have a commutative diagram*

$$\begin{tikzcd} Z'' \arrow[dr, "g''"'] & X \arrow[l, "i''"'] \arrow[r, "i'"] \arrow[d, "f"] & Z' \arrow[dl, "g'"] \\ & Y & \end{tikzcd} \tag{5.2.1}$$

where i' and i'' are closed immersions and g' and g'' smooth. Then there is a natural isomorphism

$$a(i', i'') : i'^! g'^! \backsimeq i''^! g''^!$$

satisfying the transitive formula:

$$a(i_2, i_3) \circ a(i_1, i_2) = a(i_1, i_3)$$

for any triple $(i_1, g_1), (i_2, g_2), (i_3, g_3)$.

We say that these $a(i', i'')$ form a transitive system. In order to prove this theorem we need some preparation.

Proposition 5.3. *Let $i : Y \to X$ be a regular immersion of codimension r with X noetherian. Let $\mathcal{I}$ be the ideal sheaf of i, and $N_{Y/X} = \mathcal{I}/\mathcal{I}^2$.*
(1) $N_{Y/X}$ is locally free of rank r.
(2) $\wedge^q N_{Y/X} \backsimeq \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ for any $q \in \mathbb{Z}$. In particular, $\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \backsimeq \wedge^r N_{Y/X}$ is a line bundle on Y.

(3) (a) $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \backsimeq \omega_{Y/X}[-r]$, *where* $\omega_{Y/X}$ *is a line bundle on* Y.
(b) $\omega_{Y/X} \backsimeq (\wedge^r N_{Y/X})^\vee$.
(4) For $F \in D^+(X)$, *there exists a functorial isomorphism*

$$i^! F \backsimeq Li^* F \otimes^L_{\mathcal{O}_Y} \omega_{Y/X}[-r]$$

i.e. $i_* i^! F = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F) \backsimeq i_*(Li^* F \otimes^L_{\mathcal{O}_Y} \omega_{Y/X}[-r])$.
In particular

$$\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, F) \backsimeq \mathcal{T}or^{\mathcal{O}_X}_{r-q}(\mathcal{O}_Y, F) \otimes \omega_{Y/X}.$$

Proof. From the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

one gets

$$N_{Y/X} = \mathcal{I}/\mathcal{I}^2 \backsimeq \mathcal{T}or^{\mathcal{O}_X}_1(\mathcal{O}_Y, \mathcal{O}_Y).$$

But since i is regular, locally one has a Koszul complex $\mathcal{K}_\bullet(f_1, \cdots, f_r) = (0 \to \mathcal{O}_X \to \cdots \to \mathcal{O}_X^r \to \mathcal{O}_X)$, which is a resolution of $\mathcal{O}_Y$. Hence locally

$$\mathcal{T}or^{\mathcal{O}_X}_1(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{H}^{-1}(\mathcal{K}_\bullet(f) \otimes_{\mathcal{O}_X} \mathcal{O}_Y) = \mathcal{O}_Y^r.$$

This proves (1).
In the Appendix, we see that $\mathcal{T}or^{\mathcal{O}_X}_*(\mathcal{O}_Y, \mathcal{O}_Y)$ carries a graded anti-commutative $\mathcal{O}_X$-algebra structure. When i is regular, locally one can take the Koszul complex $\mathcal{K}_\bullet(f)$ to calculate $\mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{O}_Y, \mathcal{O}_Y)$ explicitly. One obtains

$$\mathcal{T}or^{\mathcal{O}_X}_q(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^q(\mathcal{O}_Y^r),$$

and that the canonical anti-commutative $\mathcal{O}_X$-algebra homomorphism

$$\wedge^* \mathcal{T}or^{\mathcal{O}_X}_1(\mathcal{O}_Y, \mathcal{O}_Y) \longrightarrow \mathcal{T}or^{\mathcal{O}_X}_*(\mathcal{O}_Y, \mathcal{O}_Y).$$

is an isomorphism. But $\mathcal{T}or^{\mathcal{O}_X}_1(\mathcal{O}_Y, \mathcal{O}_Y) \backsimeq N_{Y/X}$, hence the assertion (2) of our theorem follows.
For the assertion (3)(a), $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$ can be computed locally using Koszul complexes

$$\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) = \mathcal{H}^q(\mathcal{K}^\bullet(f)) = \mathcal{H}^q(\mathcal{K}_\bullet(f)[-r])$$

which is 0 when $q \neq r$ and $\mathcal{O}_Y$ when $q = r$. Hence $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)[r]$ is a line bundle.
Before proceeding to the proof of assertion (4) and (3)(a), we state the following two lemmas, which are useful in the sequel.

Lemma 5.4. *Let $i : Y \hookrightarrow X$ be a regular closed immersion. Then one has*

$$F \otimes^L_{\mathcal{O}_X} i_*G \backsimeq i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G),$$

for $F \in D^+(X)$ and $G \in D^b(Y)$.

Proof. Note that $Li^* = \mathcal{O}_Y \otimes^L_{\mathcal{O}_X}$, hence (by the existence of a Koszul resolution) i^* is of finite cohomological dimension when i is regular. Li^* make sense on $D^+(X)$.
First we have to define a map $F \otimes^L i_*G \to i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G)$. But there is a natural map between $Li^*(F \otimes^L i_*G)$ and $Li^*F \otimes^L_{\mathcal{O}_Y} G$ defined by

$$Li^*(F \otimes^L i_*G) \to Li^*F \otimes^L Li^*(i_*G) \to Li^*F \otimes^L_{\mathcal{O}_Y} G,$$

where the last map is given by the natural map $Li^*i_*G \to G$. This gives the desired map $F \otimes^L i_*G \to i_*(Li^*F \otimes^L_{\mathcal{O}_Y} G)$ by the adjointness of Li^* and i_*. To show this is an isomorphism, by canonical truncations (using the fact that i^* is of finite cohomology dimension), we may assume that $F \in D^b(X)$. Replacing F by F', where $F' \to F$ is quasi-isomorphism with F'^i flat, we may assume F^i flat. Then

$$F \otimes^L i_*G = F \otimes i_*G \backsimeq i_*(i^*F \otimes G) = i_*(Li^*F \otimes^L G).$$

□

Lemma 5.5. *Let $F \in D^+(X)$ and $L, M \in D^b(X)_{perf}$, where $D^b(X)_{perf}$ means the subcategory of $D^b(X)$ consisting of perfect complexes on X. Then one has a natural isomorphism:*

$$F \otimes^L_{\mathcal{O}_X} R\mathcal{H}om(L, M) \backsimeq R\mathcal{H}om(L, F \otimes^L M). \tag{5.5.1}$$

Proof. Note that $R\mathcal{H}om(L, M) \in D^b(X)_{perf}$, and $F \otimes^L M \in D^+(X)$, hence both sides of 5.5.1 make sense. Defining the map

$$K := F \otimes^L_{\mathcal{O}_X} R\mathcal{H}om(L, M) \longrightarrow R\mathcal{H}om(L, F \otimes^L M)$$

is equivalent to giving a map $L \otimes^L K \to F \otimes^L M$. Since the problem is in $D(X)$, we may assume, for all $i \in \mathbb{Z}$, L^i, F^i are flat and that M^i are injective. Then one has the natural map

$$\begin{aligned} L \otimes^L K = F \otimes L \otimes \mathcal{H}om(L, M) &\longrightarrow F \otimes M = F \otimes^L M \\ f \otimes x \otimes u &\longmapsto f \otimes u(x). \end{aligned}$$

This defines the map (5.5.1). To show that it is an isomorphism, we may assume that L, M are both strictly perfect since the problem is local. Then (5.5.1) becomes

$$F \otimes \mathcal{H}om(L, M) \to \mathcal{H}om(L, F \otimes M).$$

By canonical truncations we assume furthermore that L is concentrated in degree 0, and finally $L = \mathcal{O}_X$. Then the conclusion becomes obvious. □

Now we can return to the proof of 5.3 (4). We need to show that

$$i_* i^! F = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F) \backsimeq i_*(Li^* F \otimes^L_{\mathcal{O}_Y} \omega_{Y/X}[-r]) \tag{5.5.2}$$

for each $F \in D^+(X)$. By Lemma 5.4, the right hand side of 5.5.2 is just

$$F \otimes^L_{\mathcal{O}_X} i_* \omega_{Y/X}[-r] = F \otimes^L_{\mathcal{O}_X} R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X).$$

Applying Lemma 5.5 one obtains $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)$, which is exactly the left hand side of (5.5.2).
Finally for assertion 5.3(3)(b), one sets $F = \mathcal{O}_Y$ in (5.5.2) and applies $\mathcal{H}^0$. Then one gets

$$\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \otimes \omega_{Y/X} \backsimeq \mathcal{H}^0(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)).$$

But clearly $\mathcal{H}^0(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)) = \mathcal{O}_Y$, and by 5.3(2),$\mathcal{T}or_r^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \backsimeq \wedge^r N_{Y/X}$, thus $\omega_{Y/X} \backsimeq (\wedge^r N_{Y/X})^\vee$. □

Lemma 5.6. *Consider a cartesian diagram*

$$\begin{tikzcd} Y' \arrow[r, "i'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ X' \arrow[r, "i"] & X \end{tikzcd}$$

where i is closed immersion and f is flat. Then one has $g^* i^! \backsimeq i'^! f^*$.

Proof. We have to show that for any $F \in D^+(X)$, there is a natural isomorphism $i'_* g^* i^! F \backsimeq i'_* i^! f^* F$. But the left hand side is

$$i'_* g^* i^! F = i'_*(g^* i^* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F)) = f^* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, F);$$

where the right hand side is

$$i'_* i^! f^* F = R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_{Y'}, f^* F) = R\mathcal{H}om_{\mathcal{O}_{X'}}(f^* \mathcal{O}_Y, f^* F).$$

Then our conclusion follows from the following lemma. □

Lemma 5.7. *Let X, Y be locally noetherian, and $f : X \to Y$ be a flat morphism. Then*

$$f^* R\mathcal{H}om(L, M) \tilde{\to} R\mathcal{H}om(f^*L, f^*M)$$

for $M \in D^+(Y)$ and $L \in D^b(Y)_{coh}$, where $D^b(Y)_{coh}$ is the subcategory of $D^b(Y)$ consisting of complexes with coherent cohomology.

Proof. Replacing M by an injective resolution, we may assume that $M^i (i \in \mathbb{Z})$ is injective. Then one gets

$$f^* R\mathcal{H}om(L, M) = f^* \mathcal{H}om(L, M) \to \mathcal{H}om(f^*L, f^*M) \to R\mathcal{H}om(f^*L, f^*M).$$

This defines the map. To show it is an isomorphism, we may assume that Y is noetherian and affine, since the problem is local. Then there is a quasi-isomorphism $L' \to L$, where each $L'^i (i \in \mathbb{Z})$ is free of finite type and $L'^i = 0$ when i is sufficiently large. Hence finally one reduces to prove it for $L = \mathcal{O}_Y$. This is trivial. □

Having these preparations, we can return to the proof of Theorem 5.2.

Proof of Theorem 5.2. Consider diagram 5.2.1. Let $Z''' = Z' \times_Y Z''$, then one can complete the diagram 5.2.1 as follows:

$$\begin{tikzcd} & Z''' \arrow[dl] \arrow[dr] & \\ Z'' \arrow[dr, "g''"'] & X \arrow[l, "i''"'] \arrow[r, "i'"] \arrow[u, "i"] \arrow[d, "f"] & Z' \arrow[dl, "g'"] \\ & Y & \end{tikzcd}$$

where i is the map determined by (i', i''). In general, i is not a closed immersion, but only an immersion, i.e. a composition of a closed immersion with an open immersion:

$$X \xrightarrow{closed} Z \xrightarrow{open} Z'''.$$

Thus one can replace Z''' by Z, and consider the diagram

$$\begin{tikzcd} & Z \arrow[d, "h'"] \\ X \arrow[ur, "i"] \arrow[r, "i'"] & Z' \end{tikzcd}$$

where i and i' are both closed immersions, and h' is smooth. If one can show that $i'^! \backsimeq i^! h'^!$, then one gets

$$i'^! g'^! \backsimeq i^! h'^! g'^! = i^! h''^! g''^! \backsimeq i''^! g''^!.$$

This gives the desired functor isomorphism.
Let $X' = X \times_{Z'} Z$, then one get the following cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "p"] & Z \arrow[d, "h'"] \\ X \arrow[u, bend left, "s"] \arrow[ur, "i"] \arrow[r, "i'"] & Z' \end{tikzcd}$$

where s is the section of X determined by (Id_X, i)(hence $ps = Id_X$). Notice that j is a closed immersion as the base change of i', hence so is s. But p is smooth(because h' is smooth), thus it follows that s is actually a regular closed immersion by 3.18.
Now suppose the relative dimension of h' is d, then for an arbitrary $F \in D^+(Z')$, one has

$$i^! h'^! F = s^! j^! h'^! F = s^! j^! (h'^* F \otimes \omega_{Z/Z'}[d])$$

But

$$\begin{aligned} j^!(h'^* F \otimes \omega_{Z/Z'}[d]) &= R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}'_X, h'^* F \otimes \omega_{Z/Z'}[d])|_{X'} \\ &= (R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_{X'}, h'^* F) \otimes \omega_{Z/Z'}[d])|_{X'} \\ &= R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_{X'}, h'^* F)|_{X'} \otimes \omega_{X'/X}[d] \\ &= j^! h'^* F \otimes \omega_{X'/X}[d]. \end{aligned}$$

Hence

$$\begin{aligned} i^! h'^! F = s^! j^! h'^! F &= s^!(j^! h'^* F \otimes \omega_{X'/X}[d]) \\ &= s^!(p^* i'^! F \otimes \omega_{X'/X}[d]), \end{aligned}$$

where the third equality is according to Lemma 5.6. Hence it only remains to show: $s^!(p^* M \otimes \omega_{X'/X}[d]) = M$ for any $M \in D^+(X)$. By 5.3 (4), the left hand side of the above formula is just

$$Ls^*(p^* M) \otimes^L \omega_{X/X'}[-d] \otimes s^* \omega_{X'/X}[d] = M \otimes \omega_{X/X'} \otimes s^* \omega_{X'/X},$$

Since $ps = Id_X$, we have $Ls^*p^* = Id$. But the conormal sheaf $N_{X/X'} \backsimeq s^*\Omega^1_{X'/X}$, and by 5.3 (3)(b) it follows that

$$\omega_{X/X'} \backsimeq (\wedge^d N_{X/X'})^\vee = (s^*\omega_{X'/X})^\vee,$$

and hence

$$M \otimes \omega_{X/X'} \otimes s^*\omega_{X'/X} = M.$$

This completes the proof. $\square$

Appendix 5.8 *(the Algebra Structure on $\mathcal{T}or_*^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$).* Let $i : Y \hookrightarrow X$ be a closed immersion. Then $\mathcal{T}or_*^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) = \bigoplus_{q=0}^r \mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ carries a natural structure of graded anti-commutative $\mathcal{O}_X$-algebra.
Consider the morphism φ given by:

$$\begin{tikzcd}
(\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y) \otimes^L_{\mathcal{O}_X} (\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y) \arrow[r, "\varphi"] \arrow[d, "\sigma_{23}"] & \mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y \\
(\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y) \otimes^L_{\mathcal{O}_X} (\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y) \arrow[ur, "\pi \otimes^L \pi"'] &
\end{tikzcd}$$

where σ_{23} is the permutation between the second and the third tensor component, and

$$\pi : \mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{O}_Y$$

is the morphism induced by the multiplication $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{O}_Y$. Now denote $\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{O}_Y$ by E, the $\mathcal{O}_X$-algebra structure on $\mathcal{T}or_*^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$ derives easily from the composition of the two natural map

$$H^*(E) \otimes H^*(E) \longrightarrow H^*(E \otimes^L E) \longrightarrow H^*(E) \qquad .$$

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \otimes \mathcal{T}or_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \longrightarrow \mathcal{T}or_{i+j}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y)$$

Locally this can be illustrated as follows: The map $\mathcal{O}_X \to \mathcal{O}_Y$ corresponds to a surjection of rings $A \twoheadrightarrow B$. Choose a quasi-isomorphism $P^\bullet \to B$ with P^i a projective A-module for all $i \in \mathbb{Z}$. Then consider the following diagram

$$\begin{tikzcd}
P^\bullet \otimes P^\bullet \arrow[r] \arrow[d, dotted, "\varphi"'] & B \otimes B \arrow[r] \arrow[d] & 0 \, . \\
P^\bullet \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

There exists a map $\varphi : P^\bullet \otimes P^\bullet \to P^\bullet$ making the diagram commute, since $P^\bullet \otimes P^\bullet$ is a projective resolution of $B \otimes B$. By classical homological algebra any two such maps are homotopic. Hence the multiplication map $\pi : B \otimes_A^L B \to B$ can be calculated as $\varphi : P^\bullet \otimes P^\bullet \to P^\bullet$, which is uniquely determined in the derived category.
Then the morphism

$$B \otimes_A^L B \otimes_A^L B \otimes_A^L B \to B \otimes_A^L B$$

is given by

$$P^\bullet \otimes P^\bullet \otimes P^\bullet \otimes P^\bullet \to P^\bullet \otimes P^\bullet \tag{5.8.1}$$

$$x \otimes y \otimes z \otimes w \longmapsto (-1)^{pq} yz \otimes xw) \tag{5.8.2}$$

where $y \in P^p$ and $z \in P^q$. Taking cohomology, one gets

$$\mathrm{Tor}_i^A(B,B) \otimes \mathrm{Tor}_j^A(B,B) \to H^{i+j}(P^\bullet \otimes P^\bullet \otimes P^\bullet \otimes P^\bullet) \to \mathrm{Tor}_{i+j}^A(B,B). \tag{5.8.3}$$

Now we check that this map endows $\mathrm{Tor}_*^A(B,B)$ an anti-commutative A-algebra structure, i.e.
(1) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for any $\alpha, \beta, \gamma \in \mathrm{Tor}_*^A(B,B)$.
(2) $\alpha\beta = (-1)^{ij}\beta\alpha$ for $\alpha \in \mathrm{Tor}_i^A(B,B)$ and $\beta \in \mathrm{Tor}_j^A(B,B)$.

(1) follows from a direct computation using 5.8.2. In order to prove (2), we introduce a kind of “permutation” morphisms. Denote by $P^{\bullet\otimes n}$ the n-copies tensor product of $P^\bullet$. Let η be a permutation of the set $\{1, 2, \cdots, n\}$. We say that (i,j) with $1 \leqslant i < j \leqslant n$ is a *permutated pair*, if $\eta^{-1}(i) > \eta^{-1}(j)$. Define a morphism of complexes $\sigma_\eta : P^{\bullet\otimes n} \to P^{\bullet\otimes n}$ by

$$x_1 \otimes \cdots \otimes x_n \mapsto (-1)^{s(x,\eta)} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)},$$

where

$$s(x,\eta) = \sum_{\text{all permutated pairs } (i,j)} \mathrm{deg} x_i \cdot \mathrm{deg} x_j.$$

Example 5.8.1. When $n = 4$ and $\eta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$, the map σ_η is given by

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \mapsto (-1)^{p_1(p_2+p_3+p_4)+p_3p_4} x_2 \otimes x_4 \otimes x_3 \otimes x_1,$$

where $p_i (1 \leqslant i \leqslant 4)$ is the degree of x_i.

One can checks that this σ_η is indeed a morphism of complexes, and $\sigma_{\eta_1\eta_2} = \sigma_{\eta_1}\sigma_{\eta_2}$ for any permutations η_1, η_2.
In particular when $n = 2$, we denote by σ the nontrivial permutation $\sigma_{(12)}$. Then we note that the diagram

$$\begin{tikzcd} P^\bullet \otimes P^\bullet \arrow[r, "\varphi"] \arrow[d, "\sigma"] & P^\bullet \\ P^\bullet \otimes P^\bullet \arrow[ur, "\varphi"] & \end{tikzcd} \tag{5.8.4}$$

is commutative up to homotopy. When $n = 4$, the map defined in (5.8.2) can be represented as $(\varphi \otimes \varphi)\sigma_{(23)}$.
We claim that, in order to prove (2), it suffices to show that

$$(\varphi \otimes \varphi)\sigma_{(23)}\sigma_{(13)(24)} \backsimeq (\varphi \otimes \varphi)\sigma_{(23)}, \tag{5.8.5}$$

where "$\backsimeq$" means the homotopy equivalence. Because when passing to cohomology, the right hand side of (5.8.5) is the multiplication $(\alpha, \beta) \mapsto \alpha\beta$ defined in (5.8.3); and the left hand side is the map $(\alpha, \beta) \mapsto (-1)^{\deg\alpha\cdot\deg\beta}\beta\alpha$. But (5.8.5) is equivalent to

$$(\varphi \otimes \varphi)\sigma_{(23)}\sigma_{(13)(24)}\sigma_{(23)}^{-1} \backsimeq \varphi \otimes \varphi.$$

And one has $\sigma_{(23)}\sigma_{(13)(24)}\sigma_{(23)}^{-1} = \sigma_{(12)(34)}$, hence

$$\begin{aligned}(\varphi \otimes \varphi)\sigma_{(23)}\sigma_{(13)(24)}\sigma_{(23)}^{-1} &= (\varphi \otimes \varphi)\sigma_{(12)(34)} \\ &= (\varphi\sigma) \otimes (\varphi\sigma) \backsimeq \varphi \otimes \varphi,\end{aligned}$$

where the last equivalence is according to (5.8.4). This completes the proof.

Definition 5.9. A morphism of schemes $f : X \to Y$ is smoothable if it can be decomposed as $f = gi$

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

where i is a closed immersion and g is a smooth morphism.

In this case, $i^! g^! : D^+(Y) \to D^+(X)$ depends only on f, and we denote it by $f^!$.

Definition 5.10. A morphism of S-schemes $f : X \to Y$ if S-smoothable if there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[d, "f"'] & {} \arrow[dl, "g"] \arrow[d] \\
Y \arrow[d] & {} \arrow[dl] \\
S &
\end{tikzcd}$$

with i a closed immersion and g a smooth morphism such that the parallelogram is Cartesian.

Let $f : X \to Y$ and $g : Y \to Z$ be S-smoothable morphisms. Then there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[d, "f"'] & & X_1 \arrow[r, hook, "i'"] \arrow[dll, "f_2"'] \arrow[dd] & W \arrow[dll, "h"'] \arrow[d] \\
Y \arrow[r, hook, "i"] \arrow[d, "g"'] & Y_1 \arrow[dl, "g_2"] \arrow[d] & & T \arrow[dll] \arrow[dl] \\
Z \arrow[d] & Y_2 \arrow[dl, "g_1"] & {} \arrow[dll, "f_1"] & \\
S & & &
\end{tikzcd}$$

with f_1, g_1 smooth, $X \to X_1$, $i : Y \to Y_1$ closed immersions such that all the parallelograms are Cartesian (and thus f_2, g_2, h are smooth, i' is a closed immersion.) It follows that $X \to X_1 \xrightarrow{i'} W$ is a closed immersion, and the morphism $W \xrightarrow{h} Y_1 \xrightarrow{g_2} Z$ is the base change of the smooth morphism $T \to Y_2 \xrightarrow{g_1} S$. Hence gf is S-smoothable. By Lemma 5.6, $f_2^! i^! \simeq i'^! h^!$ (cf. the proof of 5.2), and thus $(gf)^! \simeq f^! g^!$.

5.11. Trace map. We proceed to define a natural transformation of functors $\mathrm{Tr}_f : Rf_* f^! \to \mathrm{Id}$ in certain cases.

(a) Let $i : Y \to X$ be a closed immersion. For $E \in D^+(X)$, define Tr_i to be the morphism

$$i_* i^! E \simeq R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Y, E) \to R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, E) \simeq E$$

induced by $\mathcal{O}_X \to i_*\mathcal{O}_Y$.

(b) Let $X = \mathbb{P}^r_Y$, $f : X \to Y$ be the projection. We have a canonical exact sequence

$$0 \to \Omega^1_{X/Y} \xrightarrow{v} \mathcal{O}_X^{r+1}(-1) \xrightarrow{u} \mathcal{O}_X \to 0,$$

which is locally the exact sequence constructed in 1.24. The Koszul complex of u is

$$0 \to \wedge^{r+1}(\mathcal{O}_X^{r+1})(-r-1) \to \cdots \to \mathcal{O}_X^{r+1}(-1) \to \mathcal{O}_X \to 0. \tag{5.11.1}$$

It follows from what we have mentioned there (without proof) that each sequence

$$0 \to \Omega^i_{X/Y} \xrightarrow{\wedge^i v} \wedge^i(\mathcal{O}_X^{r+1})(-i) \to \cdots \to \mathcal{O}_X^{r+1}(-1) \to \mathcal{O}_X \to 0, i \geq 0,$$

is exact. In particular, both (5.11.1) and

$$0 \to \Omega^r_{X/Y} \to \wedge^r(\mathcal{O}_X^{r+1})(-r) \to \cdots \to \mathcal{O}_X^{r+1}(-1) \to \mathcal{O}_X \to 0 \tag{5.11.2}$$

are exact, and we have a canonical isomorphism $\Omega^r_{X/Y} \simeq \mathcal{O}_X(-r-1)$. These facts follow from the following lemma.

Lemma 5.12. *Let $(X, \mathcal{O}_X)$ be a ringed space,*

$$0 \to F \xrightarrow{v} E \xrightarrow{u} \mathcal{O}_X \to 0 \tag{5.12.1}$$

be an exact sequence of locally free sheaves of finite ranks. Then the Koszul complex of u

$$K_\bullet(u) = (0 \to \wedge^n E \xrightarrow{d_n} \wedge^{n-1}E \to \cdots \to E \xrightarrow{d_1 = u} \mathcal{O}_X \to 0) \tag{5.12.2}$$

(where $n = \operatorname{rank} E$) is acyclic and each sequence

$$0 \to \wedge^i F \xrightarrow{\wedge^i v} \wedge^i E \xrightarrow{d} \wedge^{i-1}E \cdots \to E \xrightarrow{d} \mathcal{O}_X \to 0 \tag{5.12.3}$$

is exact. Hence $\wedge^i v$ induces an isomorphism $\wedge^i F \to B^{-i-1}K_\bullet(u)$, $i \geq 0$. In particular, taking $i = n-1$, we get an isomorphism $\wedge^{n-1}F \to \wedge^n E$ such that

$$\begin{tikzcd} \wedge^{n-1}F \arrow[d] \arrow[dr, "\wedge^{n-1}v"] & \\ \wedge^n E \arrow[r, "d_n"'] & \wedge^{n-1}E \end{tikzcd}$$

commutes, which coincides with the isomorphism $\wedge^{n-1}F \to \wedge^n E$ given by taking the highest exterior power of (5.12.1) and locally defined by $u(b)a \mapsto b \wedge (\wedge^{n-1}v)(a)$, $a \in \wedge^{n-1}F(U), b \in E(U)$.

Proof. We may assume $E = \mathcal{O}_X \oplus F$ and u is the projection. Then

$$d_i : \wedge^i F \oplus (\mathcal{O}_X \otimes \wedge^{i-1} F) = \wedge^i E \to \wedge^{i-1} E = \wedge^{i-1} F \oplus (\mathcal{O}_X \otimes \wedge^{i-2} F)$$

is induced by

$$a \oplus (1 \otimes b) \mapsto b \oplus 0.$$

It can be checked directly that (5.12.3) is exact. Take $i = n$, we get the acyclicality of (5.12.2). The remainder of the lemma is then obvious. $\square$

We define $\mathrm{Tr}_f : Rf_*\omega[r] \to \mathcal{O}_Y$, where $\omega = \omega_{X/Y} = \Omega^r_{X/Y}$, as follows. The class of (5.11.1) corresponds to a morphism $c : \mathcal{O}_X \to \omega[r]$ in $D(X)$. In fact, the morphism from the first row to the second row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \omega \arrow[r] \arrow[d] & 0 & & \\
0 \arrow[r] & \wedge^r(\mathcal{O}_X^{r+1})(-r) \arrow[r] & \cdots \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \\
& & 0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

is a quasi-isomorphism, and c is just the inverse of it in $D(X)$ composed with the morphism from the third row to the second. Since $Rf_*(\mathcal{O}_X^q(-i)) = 0$ for $1 \le i \le r$ and for all q, $Rf_*[0 \to \wedge^r(\mathcal{O}_X^{r+1})(-r) \to \cdots \to \mathcal{O}_X^{r+1}(-1) \to 0] = 0$. Hence Rf_*c is an isomorphism. We define Tr_f to be the inverse of the composition of isomorphisms

$$\mathcal{O}_Y \xrightarrow{\sim} Rf_*\mathcal{O}_X \xrightarrow{Rf_*c} Rf_*\omega[r],$$

where the first morphism is the canonical map $\mathcal{O}_Y \to f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$, which is an isomorphism by Section II.3.

When Y is affine, the image of c under the morphism

$$\mathrm{Hom}_{D(X)}(\mathcal{O}_X, \omega[r]) \simeq H^r(X, \omega) \simeq H^0(Y, Rf_*\omega[r]) \xrightarrow{H^0(Y, \mathrm{Tr}_f)} H^0(Y, \mathcal{O}_Y)$$

is 1.

For $E \in D^+(Y)$, define Tr_f by

$$Rf_*f^!E = Rf_*(f^*E \otimes \omega[r]) \simeq E \otimes^L Rf_*\omega[r] \xrightarrow{E \otimes^L \mathrm{Tr}_f} E,$$

where the isomorphism in the middle is the projection isomorphism ([I], 3.2).

(c) The general case. Let $f : X \to Y$ be a morphism which can be factorized as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & \mathbb{P}^r_Y \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

where i is a closed immersion and g is the projection. This is the case when, e.g., f is projective and Y has an ample line bundle. (Then, $X \simeq \mathbf{Proj}\, B$ with B a quasi-coherent sheaf of graded $\mathcal{O}_Y$-algebras generated by B_1, B_1 of finite type. Up to replacing $\oplus B_n$ by $\oplus(B_n \otimes \mathcal{L}^{\otimes m})$, where $\mathcal{L}$ is an ample line bundle on Y, we may assume that we have an epimorphism $\mathcal{O}_Y^{r+1} \to B_1$. Thus we get an epimorphism $S(\mathcal{O}_Y^{r+1}) \to B$, and a closed immersion $\mathbf{Proj}\, B \hookrightarrow \mathbf{Proj}\, S(\mathcal{O}_Y^{r+1}) = \mathbb{P}^r_Y$.)

Define $\mathrm{Tr}_f = \mathrm{Tr}_g(Rg_* \mathrm{Tr}_i\, g^!)$. More specifically, for $E \in D^+(Y)$, define Tr_f by the composition

$$Rf_*f^!E \simeq Rg_*i_*i^!g^!E \xrightarrow{Rg_* \mathrm{Tr}_i(g^!E)} Rg_*g^!E \xrightarrow{\mathrm{Tr}_g} E.$$

This does not depend on the embedding, and is compatible with composition and flat base change. (Proof omitted.)

5.13. The duality theorem.

Let $f : X \to Y$ be a projective morphism with Y noetherian, $\dim Y < \infty$, Y having ample line bundle. Then the condition (c) above holds and so $\dim X < \infty$. Hence, by Ex. I.30, f_* has finite cohomological dimension. It follows that Rf_* extends to a functor $D(X) \to D(Y)$ (sending $D^-(X) \to D^-(X)$ and $D^b(X) \to D^b(Y)$) (Ex. I.20).

For $E, F \in Mod(X)$, define a canonical morphism

$$f_*\,\mathcal{H}om(E,F) \to \mathcal{H}om(f_*E, f_*F)$$

as follows. For $U \subset Y$ open, an element in $\Gamma(U, f_*\,\mathcal{H}om(E,F))$ is a morphism $E|f^{-1}(U) \to F|f^{-1}(U)$. It induces homomorphisms $\Gamma(f^{-1}(V), E|f^{-1}(U)) \to \Gamma(f^{-1}(V), F|f^{-1}(U))$ for all $V \subset U$ open, which determine a morphism $f_*E|U \to f_*F|U$, that is, an element in $\Gamma(U, \mathcal{H}om(f_*E, f_*F))$.

For $E, F \in C(X)$, we get a morphism of complexes

$$f_*\,\mathcal{H}om^\bullet(E,F) \to \mathcal{H}om^\bullet(f_*E, f_*F).$$

For $E \in D(X)$, $F \in D^+(X)$, take quasi-isomorphisms $F \to F'$, $E \to E'$ with $F' \in C^+(X)$, F'^i injective, E'^i f_*-acyclic (I.5.7), for all i. Then $R\mathcal{H}om(E,F) \simeq \mathcal{H}om^\bullet(E',F')$. Observe that $\mathcal{H}om^i(E',F')$ is flasque for all i. In fact, for any $L, M \in Mod(X)$, M injective, we have $\mathcal{H}om(L,M)$ is flasque. For an open embedding $j : U \hookrightarrow X$, any morphism $L|U \to M|U$ can be extended to L as M is injective:

$$\begin{tikzcd} 0 \arrow[r] & j_!j^*L \arrow[r] \arrow[d] & L \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

We define a morphism

$$Rf_*R\mathcal{H}om(E,F) \to R\mathcal{H}om(Rf_*E, Rf_*F)$$

by composition of canonical morphisms

$$\begin{aligned} Rf_*R\mathcal{H}om(E,F) &\simeq \mathcal{H}om^\bullet(E',F') \to \mathcal{H}om^\bullet(E',F') \\ &\to R\mathcal{H}om^\bullet(f_*E', f_*F') \simeq R\mathcal{H}om(Rf_*E, Rf_*F)^{\cdot} \end{aligned}$$

For $L \in D(X)$, $M \in D^+(Y)$, define $\theta_f(L,M)$ (sometimes abbreviated θ_f) to be the composition

$$\begin{aligned} Rf_*R\mathcal{H}om(L, f^!M) &\to R\mathcal{H}om(Rf_*L, Rf_*f^!M) \\ &\xrightarrow{R\mathcal{H}om(Rf_*L, \mathrm{Tr}_f)} R\mathcal{H}om(Rf_*L, M), \end{aligned}$$

where the first map is the canonical map defined above.

Theorem 5.14 (Grothendieck). *For $L \in D^-(X)_{coh}$, $M \in D^+(Y)_{coh}$, the morphism θ_f is an isomorphism.*

Proof. $f : X \to Y$ can be factorized as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & P \arrow[r, equal] \arrow[dl, "g"] & \mathbb{P}^r_Y \\ Y & & \end{tikzcd}$$

where i is a closed immersion and g is the projection. Then it is easily seen that $\theta_f(L,M) = \theta_g(Ri_*L, M)(Rg_*\theta_i(L, g^!M))$, with $Ri_*L \in D^-(P)_{coh}$ and $g^!M \in D^+(P)_{coh}$, so it is enough to check that θ_i, θ_g are isomorphisms.

Let $L \in D^-(X)_{coh}$, $M \in D^+(P)_{coh}$. To show θ_i is an isomorphism, we may assume, by $\tau_\leq$, induction and "way out functor", that $L \in Coh(X)$. We may assume P affine. Then we can write

$$L \simeq (\cdots \to L^{-1} \to L^0),$$

with L^i free of finite type. Using $\sigma_\geq$, we may assume $L = \mathcal{O}_X$. Then θ_i is nothing but the canonical isomorphism

$$i_* R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, i^! M) = i_* i^! M \to R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_X, M).$$

Therefore, we may assume that $f : X = \mathbb{P}^r_Y \to Y$ is the projection. Using $\tau_\leq$, we may assume L is concentrated in degree 0, that is, $L \in Coh(X)$. Then there is an exact sequence

$$\cdots \to \mathcal{O}_X(-n_1)^{m_1} \to \mathcal{O}_X(-n_0)^{m_0} \to L \to 0$$

with all $n_i > r+1$. Using $\sigma_\geq$, we may assume $L = \omega(-d)$ with $d \geq 0$ (where $\omega = \Omega^r_{X/Y} \simeq \mathcal{O}_X(-r-1)$).

Then, we have isomorphisms

$$\begin{aligned} Rf_* R\mathcal{H}om(L, f^! M) = Rf_* R\mathcal{H}om(\omega(-d), f^* M \otimes \omega)[r] &\simeq Rf_*(f^* M)(d)[r] \\ \simeq M \otimes^L Rf_* \mathcal{O}_X(d)[r] &\simeq M \otimes^L f_* \mathcal{O}_X(d)[r], \end{aligned}$$

where the last but second isomorphism is the projection formula (Ex. I.30), and isomorphisms

$$\begin{aligned} R\mathcal{H}om(Rf_* L, M) = R\mathcal{H}om(Rf_* \omega(-d), M) &\simeq \mathcal{H}om^\bullet(R^r f_* \omega(-d)[-r], M) \\ &\simeq M \otimes \mathcal{H}om(R^r f_* \omega(-d), \mathcal{O}_Y)[r], \end{aligned}$$

where we have used the fact that $R^r f_* \omega(-d)$ is a locally free sheaf of finite type. We have to check

$$\theta_f : f_* \mathcal{O}_X(d) \to \mathcal{H}om(R^r f_* \omega(-d), \mathcal{O}_Y)$$

is an isomorphism, that is, the pairing

$$f_* \mathcal{O}_X(d) \otimes R^r f_* \omega(-d) \to \mathcal{O}_Y$$

is perfect. For $V = \operatorname{Spec} A \subset Y$, the pairing

$$\Gamma(V, f_* \mathcal{O}_X(d)) \times \Gamma(V, R^r f_* \omega(-d)) \to \Gamma(V, \mathcal{O}_Y)$$

is given by

$$(t^a, \frac{1}{t^b t_0 \cdots t_r}) \mapsto \begin{cases} 0, \text{ if } a \neq b, \\ 1, \text{ if } a = b, \end{cases}$$

where $\sum a_i = \sum b_i = d$, and thus is a perfect pairing. $\square$

Applying $R\Gamma$ to θ_f, we get an isomorphism

$$R\operatorname{Hom}(L, f^! M) \xrightarrow{\sim} R\operatorname{Hom}(Rf_* L, M)$$

in $D(Ab)$. Applying H^i, we get $\operatorname{Ext}^i(L, f^! M) \xrightarrow{\sim} \operatorname{Ext}^i(Rf_* L, M)$.

In the remainder of this section, we suppose $Y = \operatorname{Spec} k$, $f : X \to Y$ projective. Then $K_X = f^! \mathcal{O}_Y \in D^+(X)$ is called a *dualizing complex* on X. By the remark above, $\operatorname{Ext}^i(L, K_X) \simeq \operatorname{Ext}^i(R\Gamma(X, L), k) = \operatorname{Hom}(H^{-i}(X, L), k)$, hence the following corollary.

Corollary 5.15. *Let X/k be projective, $L \in D^-(X)_{coh}$. Then there is a perfect pairing of finite dimensional k-vector spaces between $H^j(X, L)$ and $\operatorname{Ext}^{-j}(L, K_X)$.*

We first consider the case when X/k is smooth.

Corollary 5.16 (Serre). *Let X/k be projective, smooth, purely of dimension d. Then $K_X = \omega_X[d]$. Hence there is a perfect pairing between $H^j(X, L)$ and $\operatorname{Ext}^{d-j}(L, \omega_X)$. In particular, for L locally free of finite type, $H^j(X, L)$ is dual to $H^{d-j}(X, \check{L} \otimes \omega_X)$, where $\check{L} = \mathcal{H}om(L, \mathcal{O}_X)$.*

Proof. We only need to prove the last assertion. For that, $R\mathcal{H}om(L, \omega_X) = \check{L} \otimes \omega_X$, so $\operatorname{Ext}^n(L, \omega_X) = H^n R\Gamma(X, R\mathcal{H}om(L, \omega_X)) = H^n(X, \check{L} \otimes \omega_X)$. $\square$

In fact, the pairing is given by the natural pairing followed by Tr:

$$H^j(X, L) \otimes H^{d-j}(X, \check{L} \otimes \omega) \to H^d(X, \omega) \xrightarrow{\operatorname{Tr}} k.$$

When $d = 1$, we get "Roch's half" of the Riemann-Roch theorem, which claims that for L a line bundle, $H^1(X, L)$ is dual to $H^0(X, \check{L} \otimes \omega_X)$.

Corollary 5.17. *Let X/k be projective, smooth, purely of dimension d. Then $H^j(X, \Omega^i_X)$ is dual to $H^{d-j}(X, \Omega^{d-i}_X)$.*

Let $h^{ij} = \dim_k H^k(X, \Omega^i)$. These numbers h^{ij} are called the *Hodge numbers* of X. Then $h^{ij} = h^{d-i,d-j}$. Let $h^n = \dim H^n(X, \Omega^\bullet_{X/k})$. When $\mathrm{char}(k) = 0$, the Hodge degeneration theorem implies $h^n = \sum_{i+j=n} h^{ij}$. When $\mathrm{char}(k) = p > 0$, we might have $h^n < \sum_{i+j=n} h^{ij}$. See e.g. L. Illusie, Frobenius and Hodge Degeneration, in [B-D-I-P].

Corollary 5.18. *Let X be projective over an algebraically closed field k, smooth, connected, of dimension $d > 2$, and let $Y \subset X$ be an effective Cartier divisor such that $\mathcal{O}_X(Y) = I^{\otimes -1}$ is ample, where I is the ideal of Y. Then Y is connected. In particular, Y is irreducible if it is smooth.*

Proof. Let $Y_n = V(I^{n+1})$. Then we have a short exact sequence

$$0 \to \mathcal{O}_X(-(n+1)Y) \to \mathcal{O}_X \to \mathcal{O}_{Y_n} \to 0,$$

and hence a long exact sequence

$$H^0(X, \mathcal{O}_X(-(n+1)Y)) \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{Y_n}) \to H^1(X, \mathcal{O}_X(-(n+1)Y)).$$

The first term is 0, the second term is k, and the fourth term is dual to $H^{d-1}(X, \omega_X(-(n+1)Y))$, which is 0 for $n >> 0$ by Serre's vanishing theorem (II.4.7) since $d - 1 > 1$. Then the third term must be k, and so $|Y| = |Y_n|$ is connected. $\square$

Next, we discuss K_X in general.

Proposition 5.19. *Let X/k be projective with* $\dim X = n$. *Then $K_X \in D^{[-n,0]}(X)_{coh}$.*

Proof. We have

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & P \arrow[r, equal] \arrow[dl, "g"] & \mathbb{P}^r_Y \\ \operatorname{Spec} k & & \end{tikzcd}$$

with i a closed immersion. $i_* K_X = R\mathcal{H}om_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P)[N]$, so it is enough to show $\mathcal{E}xt^{i+N}_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) = 0$ for $i \notin [-n, 0]$, that is,

$$\mathcal{E}^j = \mathcal{E}xt^j_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P) = 0$$

for $j < N - n$ or $j > N$. This holds for $j > N$ since for all $x \in X$, $\mathcal{E}xt^j_{\mathcal{O}_P}(\mathcal{O}_X, \omega_P)_x = \mathrm{Ext}^j_{\mathcal{O}_{P,x}}(\mathcal{O}_{X,x}, \omega_{P,x})$, where $\omega_{P,x} \simeq \mathcal{O}_{P,x}$ is regular of dimension $\leq N$. Note that for $q >> 0$, $\mathcal{E}^j(q)$ is generated by global sections. It then suffices to show for a fixed $j < N-n$, $\Gamma(P, \mathcal{E}^j(q)) = 0$ for $q >> 0$. By the following lemma, $\Gamma(P, \mathcal{E}xt^j(\mathcal{O}_X, \omega_P)(q)) = \mathrm{Ext}^j_P(\mathcal{O}_X, \omega_P(q))$, which is dual to $H^{N-j}(P, \mathcal{O}_X(-q)) = H^{N-j}(X, \mathcal{O}_X(-q)) = 0$ since $N - j > n = \dim X$. □

Lemma 5.20. *For fixed $E, F \in Coh(P)$ and fixed l, $H^0(P, \mathcal{E}xt^l(E, F)(q)) = \mathrm{Ext}^l(E, F(q))$ for $q >> 0$.*

Proof. We have biregular spectral sequences

$$E_2^{ij}(q) = H^i(P, \mathcal{E}xt^j(E, F)(q)) \Rightarrow Ext^{i+j}(E, F(q)),$$

which concentrate in the first quadrant. By Serre's vanishing theorem (II.4.7), there exists q_0 such that for all $q \geq q_0$, $j \leq l$, $i > 0$, $E_2^{ij}(q) = 0$. Hence $d_r^{0l} = 0$, for all $r \geq 2$. It follows that

$$\mathrm{Ext}^l(E, F(q)) \simeq E_\infty^{0l}(q) = E_2^{0l}(q) = H^0(P, \mathcal{E}xt^l(E, F)(q))$$

. □

Let A be a local ring with residue field k, M be an A-module. The depth of M is

$$\begin{aligned} \operatorname{depth}_A M &= \sup\{n \mid \text{there exists } M\text{-regular sequence } (t_1, \cdots, t_n),\ t_i \in A\} \\ &= \inf\{m \mid \mathrm{Ext}^m_A(k, M) \neq 0\}. \end{aligned}$$

The depth of A is its depth as an A-module. A is called Cohen-Macaulay if its depth is equal to $\dim A$. A scheme X is called Cohen-Macaulay if all its local rings are Cohen-Macaulay.

Proposition 5.21. *Let X/k be projective. Suppose X is Cohen-Macaulay and all irriducible components have dimension n. Then $K_X \in D^{[-n,-n]}(X)$, and so $K_X \simeq \omega_X^\circ[n]$ with $\omega_X^\circ = H^{-n}(K_X)[n]$.*

Proof. By the proof of Proposition 5.19, we only need to show for all $j > N - n$, $x \in X$,

$$\mathrm{Ext}^j_{\mathcal{O}_{P,x}}(\mathcal{O}_{X,x}, \omega_{P,x}) = 0,$$

which follows from the equation

$$\begin{aligned}
\operatorname{proj}\dim_{\mathcal{O}_{P,x}} &= \dim \mathcal{O}_{P,x} - \operatorname{depth}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} \quad \text{by [EGAIV], 0, 17.3.4} \\
&= \dim \mathcal{O}_{P,x} - \operatorname{depth}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} \\
&= \dim \mathcal{O}_{P,x} - \dim \mathcal{O}_{X,x} = N - n.
\end{aligned}$$

□

The sheaf ω_X° in the proposition is called the dualizing sheaf for X in [H].

X is Cohen-Macaulay if, e.g., there is a regular k-immersion i of X into a projective space over k.

$$\begin{tikzcd}
& X \arrow[r, hook, "i"] \arrow[d, "f"'] & P \arrow[r, equal] \arrow[dl, "g"] & \mathbb{P}_k^N \\
\operatorname{Spec} k \arrow[r, equal] & S & &
\end{tikzcd}$$

In this case, we even have ω_X° is a line bundle. Indeed,

$$\begin{aligned}
f^! \mathcal{O}_Y &\simeq i^! g^! \mathcal{O}_Y = i^! \omega_P[N] \\
&\simeq i^* \omega_P \otimes \omega_{X/P}[-(N-n)][N] \quad \text{by Proposition 5.3} \\
&= i^* \omega_P \otimes \omega_{X/P}[n],
\end{aligned}$$

and hence $\omega_X^\circ = i^* \omega_P \otimes \omega_{X/P}$.

6 Spectral Sequences

6.1. The spectral sequences of a filtered complex. Let $\mathcal{A}$ be an abelian category. A filtered complex (K, F^p) in $\mathcal{A}$ is a complex K of $\mathcal{A}$ endowed with a decreasing filtration by subcomplexes

$$K \cdots \supset F^p K \supset F^{p+1} K \supset \cdots.$$

Denote by $\operatorname{gr}_F^p(K)$ the quotient complex $F^p K / F^{p+1} K$.

Problem. We want to relate $H^n(K)$ to $H^m(\operatorname{gr}_F^p(K))$. For the inclusion $F^p K \to K$ we denote $\operatorname{Im}(H^n(F^p K) \to H^n(K))$ by $F^p(H^n(K))$. In particular, we want to understand the relationship between $\operatorname{gr}_F^p(H^n(K))$ and $H^n(\operatorname{gr}_F^p(K))$.

Example 6.1.1. Let K be a complex of $\mathcal{A}$, K' a subcomplex of K. Define a filtered complex (K, F^p) as follow: $F^0K = K \supset F^1K = K' \supset F^2K = 0$
We have the short exact sequence:

$$0 \to K' \to K \to K'' \to 0$$

and $\mathrm{gr}^1_F(K) = K', \mathrm{gr}^0_F(K) = K''$.
From the long exact sequence:

$$\cdots \longrightarrow H^{n-1}(K'') \xrightarrow{\delta} H^n(K') \longrightarrow H^n(K) \longrightarrow H^n(K'') \xrightarrow{\delta} H^{n+1}(K') \longrightarrow \cdots$$

we get a short exact sequence:

$$0 \to \mathrm{Coker}(H^{n-1}\mathrm{gr}^0_F(K) \to H^n\mathrm{gr}^1_F(K)) \to H^n(K) \to \mathrm{Ker}(H^n\mathrm{gr}^0_F(K) \to H^n\mathrm{gr}^1_F(K)) \to 0$$

i.e. we have

$$F^1H^n(K) = \mathrm{Coker}(H^{n-1}\mathrm{gr}^0_F(K) \to H^n\mathrm{gr}^1_F(K))$$

$$\mathrm{gr}^0_F H^n(K) = \mathrm{Ker}(H^n\mathrm{gr}^0_F(K) \to H^n\mathrm{gr}^1_F(K))$$

6.2. Now we consider the general case. We follow the approach of [C-E]. Let (K, F^p) be a filtered complex. For $\infty \geq q \geq p \geq -\infty$, let $K(p,q) = F^pK/F^qK$, $F^{-\infty}K = K$, $F^{\infty}K = 0$. With these notations, we get $\mathrm{gr}^p_F(K) = K(p, p+1)$, $K/F^qK = K(-\infty, q)$ and $F^pK = K(\infty, p)$.
For integers $p \leq q \leq r$, we have a short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & K(q,r) \arrow[r] & K(p,r) \arrow[r] & K(p,q) \arrow[r] \arrow[ll, bend left, "\delta"] & 0 \end{tikzcd} \qquad (*)$$

We denote this sequence $(*)$ by $K(p,q,r)$. It defines a distinguished triangle in $\mathcal{D}$, with $\delta : K(p,q) \to K(q,r)[1]$. Let $n = p+q$ and $r \geq 1$, denote $H^n(p, p+r) = H^n(F^pK/F^{p+r}K)$ and $E_1^{p,q} = H^n(p, p+1) = H^n(\mathrm{gr}^p_F(K))$.
For a fixed r, consider the triangles defined by $(p, p+1, p+r)$, and $(p+r+1, p, p+1)$, and define

$$Z_r^{p,q} = \mathrm{Ker}(H^n(p,p{+}1) \xrightarrow{\delta} H^{n+1}(p{+}1,p{+}r)) = \mathrm{Im}(H^n(p,p{+}r) \to H^n(p,p{+}1)) \subset E_1^{p,q}$$

$$B_r^{p,q} = \mathrm{Im}(H^{n-1}(p{-}r{+}1,p{+}1) \xrightarrow{\delta} H^n(p,p{+}1)) = \mathrm{Ker}(H^n(p,p{+}1) \to H^n(p{-}r{+}1,p)) \subset E_1^{p,q}$$

We have $Z_1^{p,q} = E_1^{p,q}$ and $B_1^{p,q} = 0$

Theorem 6.3. *(1) We have a chain of inclusions* $0 = B_1 \subset \cdots \subset B_r \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_s \subset \cdots \subset Z_1 = E_1$

(2) In the diagram:

$$\begin{tikzcd} H^n(p,p+1) \arrow[r, "\delta"] & H^{n+1}(p+1,p+r+1) \\ B_{r+1} \arrow[r, hook] & H^{n+1}(p+r,p+r+1) \arrow[u] \end{tikzcd}$$

we have $\mathrm{Im}(\delta) \subset \mathrm{Im}(B_{r+1} \to H^{n+1}(p+1,p+r+1))$*, and* δ *induces an isomorphism*

$$Z_r^{p,q}/Z_{r+1}^{p,q} \underset{\simeq}{\xrightarrow{\delta}} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}$$

(3) Let $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$ *and denote by* d_r *the composition:*

$$\begin{tikzcd} E_r^{p,q} = Z_r/B_r \arrow[r, two heads] \arrow[rrr, bend right=10] & Z_r/Z_{r+1} \arrow[r, "\delta", "\simeq"'] & B_{r+1}/B_r \arrow[r, hook] & Z_r/B_r = E_r^{p+r,q-r+1} \end{tikzcd}$$

we have $d_r \cdot d_r = 0$ *and* $H^{p,q}(E_r) = E_{r+1}^{p,q}$ *i.e.* $H(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}) = Z/B = E_{r+1}^{p,q}$

(4) Let $F^pH^n(K) = \mathrm{Im}(H^n(F^pK) \to H^n(K))$*, in the diagram:*

$$\begin{tikzcd} Z_\infty \subset H^n(p,p+1) \arrow[r, "can"] & H^n(-\infty,p+1) \\ F^pH^n(K) \arrow[r, hook] & H^n(K) \arrow[u] \end{tikzcd}$$

we have $can(Z_\infty) \subset \mathrm{Im}(F^pH^n(K) \to H^n(K) \to H^n(-\infty,p+1))$*, and an isomorphism*

$$Z_\infty^{p,q}/B_\infty^{p,q} \xrightarrow{\sim} gr_F^p(H^n(K))$$

Proof. (1) To show $Z_{r+1} \subset Z_r$, we consider the morphism of short exact sequences $K(p,p+1,p+r+1) \to K(p,p+1,p+r)$ which gives the following commutative diagram:

$$\begin{tikzcd} K(p,p+q) \arrow[r, "\delta"] \arrow[d, equal] & K(p+1,p+r+1)[1] \arrow[d] \\ K(p,p+q) \arrow[r, "\delta"] & K(p+1,p+r)[1] \end{tikzcd}$$

Thus we get $Z_{r+1} \subset Z_r$. Similarly we have $B_r \subset B_{r+1}$.
To show $B_r \subset Z_s$ for all r, s, we need the following lemma:

Lemma 6.4. *For integers $p \le q \le r \le s$, $K(p,q) \xrightarrow{\delta} K(q,r)[1] \xrightarrow{\delta} K(r,s)[2]$ we have $\delta \cdot \delta = 0$.*

Proof. We have the diagram:

$$\begin{tikzcd}
& K(q,r)[1] \arrow[dl, "\delta_2"'] \arrow[dr, phantom, "(1)"] \arrow[d, phantom, "(2)"'] & & K(p,q) \arrow[ll, "\delta_1"'] \arrow[dl, "\delta_3"] \\
K(r,s)[1] \arrow[rr] & & K(q,s)[1] \arrow[ul, "u"] &
\end{tikzcd}$$

In which (1) is commutative and (2) is distinguished. We get $\delta_2 \cdot \delta_1 = \delta_2 \cdot u \cdot \delta_3 = 0$ □

By the above lemma we get the composition:

$$H^{n-1}(p-r+1,p) \xrightarrow{\delta} H^n(p,p+1) \xrightarrow{\delta} H^{n+1}(p+1,p+s)$$

to be 0. This gives the injection of $B_r \hookrightarrow Z_s$.

(2) We need a very useful lemma the proof of which is left as an exercise.

Lemma 6.5 (C-E,XV1.1). *Suppose we have a commutative diagram with the bottom row exact:*

$$\begin{tikzcd}
& M \arrow[d, "c"] \arrow[dr, "d"] & \\
L' \arrow[ur] \arrow[r, "a"] & L \arrow[r, "b"] & L''
\end{tikzcd}$$

Then b induces an isomorphism $\operatorname{Im} c / \operatorname{Im} a \xrightarrow{\sim} \operatorname{Im} d$

The morphism of short exact sequences $K(p,p+r,p+r+1) \to K(p,p+1,p+r+1)$ gives a commutative diagram:

$$\begin{tikzcd}
H^n(p,p+r) \arrow[r, "\epsilon"] \arrow[d, "u_1"'] & H^{n+1}(p+r,p+r+1) \\
H^n(p,p+1) \arrow[r, "\delta"] & H^{n+1}(p+1,p+r+1)
\end{tikzcd}$$

; Secondly, we have a commutative diagram with exact row

$$\begin{tikzcd}
& H^n(p,p+r) \arrow[d, "u_1"] \arrow[dr, "\varphi"] & \\
H^n(p,p+r+1) \arrow[ur] \arrow[r, "v"] & H^n(p,p+1) \arrow[r, "\delta"] & H^{n+1}(p+1,p+r+1)
\end{tikzcd}$$

; we have $\operatorname{Im} u_1 = Z_r^{p,q}$ and $\operatorname{Im} v = Z_{r+1}^{p,q}$, and the lemma shows $Z_r^{p,q}/Z_{r+1}^{p,q} \xrightarrow{\sim} \operatorname{Im}\varphi$; Finally we have a commutative diagram with exact row

$$\begin{tikzcd}
 & H^n(p,p+r) \arrow[d, "\epsilon"] \arrow[dr, "\varphi"] & \\
H^n(p+1,p+r) \arrow[ur] \arrow[r, "\epsilon'"] & H^{n+1}(p+r,p+r+1) \arrow[r, "u_2"] & H^{n+1}(p+1,p+r+1)
\end{tikzcd}$$

we have $\operatorname{Im}\epsilon = B_{r+1}^{p+r,q-r+1}$ and $\operatorname{Im}\epsilon' = B_r^{p+r,q-r+1}$, and the lemma shows $B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \xrightarrow{\sim} \operatorname{Im}\varphi$. Composing the two isomorphisms we get $Z_r/Z_{r+1} \underset{\simeq}{\xrightarrow{\delta}} B_{r+1}/B_r$.

(3) Obviously we have $\mathrm{Ker} d_r = Z_{r+1}^{p,q}/B_r^{p,q}$ and $\operatorname{Im} d_r = B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}$. This implies that $d_r^{p+r,q-r+1} \circ d_r^{p+r,q-r+1} = 0$ and $H(E_r) = E_{r+1}$.

(4) The morphism of exact sequences $K(-\infty, p+1, \infty) \to K(-\infty, p, \infty)$ gives a commutative diagram:

$$\begin{tikzcd}
H^n(p,\infty) \arrow[r, "c"] \arrow[d, "u"'] & H^n(-\infty,\infty) = H^n(K) \\
H^n(p,p+1) \arrow[r, "can"] & H^n(\infty,p+1)
\end{tikzcd}$$

Secondly, we have a commutative diagram with exact row:

$$\begin{tikzcd}
 & H^nF^pK = H^n(p,\infty) \arrow[d, "u"] \arrow[dr, "a"] & \\
H^n(-\infty,p) \arrow[ur, "\delta"] \arrow[r, "\delta"] & H^n(p,p+1) \arrow[r, "can."] & H^n(-\infty,p+1)
\end{tikzcd}$$

where $\operatorname{Im} u = Z_\infty$ and again by the CE lemma we get $Z_\infty/B_\infty \xrightarrow{\sim} \operatorname{Im} a$. Similarly from the diagram:

$$\begin{tikzcd}
 & H^nF^pK = H^n(p,\infty) \arrow[d, "c"] \arrow[dr, "a"] & \\
H^n(p+1,\infty) \arrow[ur] \arrow[r] & H^n(K) \arrow[r, "d"] & H^n(-\infty,p+1)
\end{tikzcd}$$

we get $F^pH^n(K)/F^{p+1}H^n(K) \xrightarrow{\sim} \operatorname{Im} a$.By composing the two isomorphisms together we get $\mathrm{gr}^p H^(K) \xrightarrow{\sim} Z_\infty/B_\infty$ $\square$

Definition 6.6. A *spectral sequence* in $\mathcal{A}$, denoted as $E_a^{p,q} \Rightarrow (H^n, F)$ for $0 \le a \in \mathbb{Z}$ and usually $a = 1$ or 2, consists of the following data:

(1) A family of objects H^n in $\mathcal{A}$ for all $n \in \mathbb{Z}$, and a decreasing filtration:

$$H^n = F^0H^n \supset F^1H^n \supset \cdots \supset F^pH^n \supset F^{p+1}H^n \supset \cdots$$

(2) A family of objects $E_r^{p,q}$ in $\mathcal{A}$ for all p, q in $\mathbb{Z}$ and for $r \ge a$; and a family of morphisms $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \cdot d_r = 0$.

(3) A family of isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \xrightarrow{\sim} E_{r+1}^{p,q}$

(4) For $s \ge a$ and $r \ge a$, define

$$0 = B_a \subset B_{a+1} \subset \cdots \subset B_r \subset \cdots \subset Z_s \subset \cdots \subset Z_a = E_a$$

inductively as follows:

$$B_{a+1}^{p,q} = \mathrm{Im}(d_a) \subset E_r^{p,q}\ , \qquad Z_{a+1}^{p,q} = \mathrm{Ker}(d_a) \subset E_r^{p,q}$$

We have $Z_{a+1}^{p,q}/B_{a+1}^{p,q} = E_{a+1}^{p,q}$, define $Z_{a+2}^{p,q}/B_{a+1}^{p,q}$ by $\mathrm{Ker}(d_{a+1})$ and $B_{a+2}^{p,q}/B_{a+1}^{p,q}$ by $\mathrm{Im}(d_{a+1})$, by pulling-back we get $B_{a+1} \subset B_{a+2} \subset Z_{a+2} \subset Z_{a+1}$. Inductively we define B_{r+1}/B_r to be $\mathrm{Im}(d_r)$, and Z_{r+1}/Z_r to be $\mathrm{Ker} d_r$, and then by pulling-back we have $B_r \subset Z_r$.

(5) Two objects in $\mathcal{A}$, $B_\infty^{p,q} \subset Z_\infty^{p,q} \subset E_a^{p,q}$ such that $B_r^{p,q} \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset Z_s^{p,q}$, for all s, r, and an isomorphism

$$\beta : E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} \xrightarrow{\sim} \mathrm{gr}_F^p(H^n)$$

The spectral sequence associated to a filtered complex constructed in 6.3 is a example, with $a = 1$.

The term $E_a^{\cdot,\cdot}$ is called the *initial term*, the filtered term H^n, F the *abutment* of the spectral sequence. We usually give a picture of a spectral sequence by plotting the term $E_r^{p,q}$ at the point of coordinates (p, q) in the plane. The differential d_r corresponds to a generalized "knight's jump"

$$\begin{tikzcd} \cdot \arrow[r, "r"] \arrow[dr, "d_r"'] & {} \arrow[d, "r-1"] \\ & \cdot \end{tikzcd}$$

6.7. Let $E_a^{p,q} \Rightarrow (H^n, F)$ and $E_a'^{p,q} \Rightarrow (H'^n, F)$ be two spectral sequences. A morphism of spectral sequences

$$\begin{tikzcd} E_a^{p,q} \Rightarrow (H^n, F) \arrow[d, "u"] \\ E_a^{p,q} \Rightarrow (H^n, F) \end{tikzcd}$$

consists of morphisms $u_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ and $u^n : H^n \to H'^n$ satisfying the following conditions:
(i) The diagram:

$$\begin{tikzcd} H^n \arrow[r, "u^n"] & H'^n \\ F^pH^n \arrow[u] \arrow[r] & F^pH'^n \arrow[u] \end{tikzcd}$$

commutes, so u^n induces a morphism $\gamma : \mathrm{gr}H^n \to \mathrm{gr}H'^n$.
(ii) The diagram:

$$\begin{tikzcd} E_r \arrow[r, "u_r"] \arrow[d, "d_r"] & E_r' \arrow[d, "d_r"] \\ E_r \arrow[r, "u_r"] & E_r' \end{tikzcd}$$

commutes.
((iii) Morphisms $Z_\infty \to Z'_\infty$ and $B_\infty \to B'_\infty$ which induces a morphism $epsilon : E_\infty \to E'_\infty$ so that the following diagram commutes:

$$\begin{tikzcd} E_\infty \arrow[r, "\epsilon"] \arrow[d, "beta"', "\simeq"] & E'{}_\infty \arrow[d, "\simeq"', "\beta"] \\ \mathrm{gr}H^n \arrow[r, "\gamma"] & \mathrm{gr}H'^n \end{tikzcd}$$

(iv) The following diagram commutes.

$$\begin{tikzcd} H(E_r) \arrow[r] \arrow[d, "\simeq"', "\alpha_r"] & H(E_r') \arrow[d, "\alpha_r"', "\simeq"] \\ E_{r+1} \arrow[r] & E'_{r+1} \end{tikzcd}$$

In this way spectral sequences of $\mathcal{A}$ form an *Additive Category.*

Definition 6.8. A spectral sequence $E_a^{p,q} \Rightarrow (H^n, F)$ is called *biregular* if it satisfies the following properties:
(i) For any pair (p, q), there exists an integer r_0 such that $B_r^{p,q} = B_{r_0}^{p,q}$ and $Z_r^{p,q} = Z_{r_0}^{p,q}$ for all $r \geq r_0$
(ii) For all n, $(F^pH^n)_{p\in\mathbb{Z}}$ is a finite filtration, i.e. $F^pH^n = H^n$ for p sufficiently small and $F^qH^n = 0$ for q sufficiently large.

Example 6.8.1. A filtered complex (K, F^p) is said to be *regular* if for all n, the filtration (F^pK^n) is finite. The spectral sequence $E_1^{p,q} = H^{p,q}(\mathrm{gr}^p K) \Rightarrow (H^n, K)$ is then biregular.

Proposition 6.9. *Let $u : (E_a^{p,q} \Rightarrow (H^n, F)) \to (E_a'^{p,q} \Rightarrow (H'^n, F))$ be a morphism of biregular spectral sequences. If for some $r \geq a$, $u_r^{p,q} : E_r^{p,q} \to E_r'^{p,q}$ is an isomorphism for all p, q, then $u^n : H^n \to H'^n$ is an isomorphism for all n.*

Proof. We have the following diagram:

$$\begin{tikzcd}
E_{r_0}^{p,q} \arrow[r, "\sim"] \arrow[d, equal] & E_{r_0}'^{p,q} \arrow[d, equal] \\
E_{\infty}^{p,q} \arrow[r, "\sim"] \arrow[d, equal] & E_{\infty}'^{p,q} \arrow[d, equal] \\
\mathrm{gr}^p H^n \arrow[r, "\sim"] & \mathrm{gr}^p H'^n
\end{tikzcd}$$

As the filtrations of H^n and H'^n are finite ,so we get the conclusion. □

Definition 6.10. A spectral sequence $E_a^{p,q} \Rightarrow (H^n, F)$ is said to *degenerate at E_{r_0}*, if it is biregular and $d_r = 0$, for all $r \geq r_0$.

Proposition 6.11. *Let E be a biregular spectral sequence in $\mathcal{A}$ where $\mathcal{A}$ is the category of modules of finite length over some ring R. Then $(E_a^{p,q} \Rightarrow (H^n, F))$ degenerates at E_{r_0} if and only if $\sum_{p+q=n} lgE_{r_0}^{p,q} = lgH^n$, for all n.*

Proof. We have $\cdots \leq \mathrm{lg}E_{r+1}^{p,q} \leq \mathrm{lg}E_r^{p,q} \leq \cdots$ and $\mathrm{lg}E_{\infty}^{p,q} \leq \mathrm{lg}E_r^{p,q}$. For N sufficiently large,we have $E_N^{p,q} = E_{\infty}^{p,q}$, so

$$\mathrm{lg}H^n = \sum_{p+q=n} \mathrm{lg}(\mathrm{gr}^p(H^n)) = \sum_{p+q=n} \mathrm{lg}E_{\infty}^{p,q} = \sum_{p+q=n} \mathrm{lg}E_N^{p,q} \leq \sum_{p+q=n} \mathrm{lg}E_{r_0}^{p,q}$$

E degenerates at E_{r_0} if and only if $Z_r = E_r = E_{r+1}$, for all $r \geq r_0$,which holds if and only if $\mathrm{lg}E_{\infty}^{p,q} = \mathrm{lg}E_r^{p,q} = \mathrm{lg}E_{r_0}^{p,q}$, for all $r \geq r_0$. Thus the conclusion follows. □

6.12. Spectral sequences of a bicomplex. Let $\mathcal{A}$ be an additive category. Let $K = K^{\bullet,\bullet} = (K^{p,q}; d', d")$ be a bicomplex of $\mathcal{A}$. Then $d'^2 = d"^2 = (d' + d")^2 = 0$. We call the complex $(K^{\bullet,q}, d')$ the q-th row complex and the complex $(K^{p,\bullet}, d")$ the p-th column complex. Recall that K is biregular if for all n, the set $\{(p,q)|p+q=n, K^{p,q}\neq 0\}$ is finite. For K biregular, then the simple complex $sK \in C(\mathcal{A})$ associated to K is

$$sK^n = \bigoplus_{p+q=n} K^{p,q},\ d = d' + d".$$

For any double complex K, there are two filtrations on $K^{\bullet,\bullet}$:

$$F'^p K^{i,j} = \begin{cases} K^{i,j}, & i \geq p; \\ 0, & i < p. \end{cases}$$

$$F"^p K^{i,j} = \begin{cases} K^{i,j}, & j \geq p; \\ 0, & j < p. \end{cases}$$

We now assume K is regular.

(1). The first spectral sequence. We have

$$(F'^p(sK) = s(F'^p K),\ \mathrm{gr}^p_{F'}(sK) = (K^{p,\bullet})[-p].$$

The spectral sequence of (sK, F') is

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p_{F'} K) \Rightarrow H^{p+q}(sK).$$

Since $E_1^{p.q} = H^{q+p}(K^{p,\bullet}[-p]) = {}"H^q(K^{p,\bullet})$ and $d_1 : {}"H^q(K^{p,\bullet}) \to {}"H^q(K^{p+1,\bullet})$ is induced by d', then

$$E_2^{p,q} = {}'H^p{}"H^q(K^{\bullet,\bullet})$$

and

$${}'FH^n(sK) = \mathrm{Im}(H^n(sF'^p K) \to H^n(sK)).$$

(2). The second spectral sequence. Similar to the first case, we have

$$E_1^{p,q} = {}'H^q(K^{\bullet,p}),\ E_2^{p,q} = {}"H^{p\prime}H^q(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(sK)$$

and

$${}"FH^n(sK) = \mathrm{Im}(H^n(sF"^p K) \to H^n(sK)).$$

Remark. If F'(resp. $F"$) is biregular, then these spectral sequences are biregular.

Proposition 6.13. *Let $u : K^{\bullet,\bullet} \to L^{\bullet,\bullet}$ be a morphism of biregular complexes which hence induces $su : sK \to sL$. Then*

(a). If u induces a quasi-isomorphism on each row(resp. column), then su is a quasi-isomorphism.

(b). If u induces a quasi-isomorphism on each cohomology row(resp. column), then su is a quasi-isomorphism.

Proof. Exercise. □

6.14. Spectral sequences of hypercohomology. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor. Assume cA has enough injectives. Then $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is well defined. For $K \in K^+(\mathcal{A})$, then $RT(K) = T(K')$ where $K \overset{quis}{ra} K'$, $K' \in K(\mathcal{A})$ with K'^p injective. We write $R^nT(K) = H^nRT(K)$.

Theorem 6.15 (Cartan-Eilenberg). *One can construct spectral sequences:*

(1) $'E_1^{p,q} = R^qT(K^p) \Rightarrow R^nT(K)$, where d_1 is induced by $d : K^p \to K^{p+1}$. This is called the first spectral sequence of hypercohomology of K for T.

(2) $"E_2^{p,q} = R^pT(H^qK) \Rightarrow R^nT(K)$, which is called the second spectral sequence of hypercohomology.

Moreover, (1) and (2) are biregular. (1) is functorial in $K \in C(\mathcal{A})$, and in $K \in K^+(\mathcal{A})$ from E_2 on; (2) is functorial in $K \in D^+(\mathcal{A})$. The abutment filtration of (1) is

$$F^pR^nT(K) = \mathrm{Im}(R^nT(K^{\geq p}) \to R^nT(K)), \quad K^{\geq p} = \sigma_{\geq p}K.$$

The abutment filtration of (2) is

$$F^pT^n(K) = \mathrm{Im}(R^nT(\tau_{\leq n-p}K) \to R^nK)$$

where $\tau_{\leq m}K = (\cdots \to K^{m-1} \to Z^m \to 0)$.

We need to construct the injective Cartan-Eilenberg resolution to prove the theorem.

Lemma 6.16. *Consider a diagram*

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r, dotted] & \bullet \arrow[r, dotted] & I" \arrow[r] & 0 \\
0 \arrow[r] & L' \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, dotted] & L" \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

with exact rows and columns and I', $I"$ injective, then we can complete the diagram.

Proof. This is an easy exercise. $\square$

From the above Lemma, one has

Lemma 6.17. *Let $0 \to L' \to L \to L" \to 0$ be an exact sequence. Let $L' \to I'^{\bullet}$ and $L" \to I"^{\bullet}$ be injective resolutions. Then there exists an injective resolution $L \to I^{\bullet}$ to complete the diagram*

$$\begin{tikzcd} 0 \arrow[r] & I'^{\bullet} \arrow[r, dotted] & I^{\bullet} \arrow[r, dotted] & I"^{\bullet} \arrow[r] & 0 \\ 0 \arrow[r] & L' \arrow[r] \arrow[u] & L \arrow[r] \arrow[u, dotted] & L" \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

Lemma 6.18. *let $K \in C(\mathcal{A})$. Then there exists an exact sequence*

$$0 \to K \to [M^{\bullet,0} \to M^{\bullet,1} \to \cdots]$$

where $M^{\bullet,\bullet}$, such that

$$0 \to K^p \to M^{p,0} \to M^{p,1} \to \cdots$$

is an injective resolution of K^p, and

$$0 \to Z^pK \to Z^pM^{\bullet,0} \to Z^pM^{\bullet,1} \to \cdots$$

$$0 \to B^pK \to B^pM^{\bullet,0} \to B^pM^{\bullet,1} \to \cdots$$

$$0 \to H^pK \to' H^pM^{\bullet,0} \to' H^pM^{\bullet,1} \to \cdots$$

are all injective resolutions. This resolution is called Cartan-Eilenberg resolution.

Proof. We first apply Lemma 6.17 to the exact sequence

$$0 \to B^pK \to Z^pK \to H^pK \to 0$$

and then apply it to the exact sequence

$$0 \to Z^pK \to K^p \to B^{p+1}K \to 0.$$

$\square$

Remark. (1). Suppose that

$$K \to K^{'\bullet,\bullet}, \qquad L \to L^{'\bullet,\bullet}$$

are Cartan-Eilenberg resolutions. Then any map $f : K \to L$ can be lifted to

$$f' : K^{'\bullet,\bullet} \to L^{'\bullet,\bullet}.$$

Any two liftings f', $f"$ are related by vertical homotopy

$$s : K^{'p,q} \longrightarrow L^{'p,q-1} f" - f' = ds + sd, \quad sd' + d's = 0.$$

(2). Suppose $f \sim g : K \to L$, and $k : K \to L$ is a homotopy of f and g: $dk + kd = g - f$. If $f' : K' \to L'$ and $g' : K' \to L'$ lift f and g respectively. Then there exists a homotopy $s = s' + s"$ such that

$$g' - f' = ds + sd, \; s'd" + d"s' = 0, \; s"d' + d's" = 0.$$

Proof of Theorem 6.15. Suppose $K \in C^+(\mathcal{A})$. Let $K \to M^{\bullet,\bullet}$ be an injective Cartan-Eilenberg resolution of K. Note that $K \to sM^{\bullet,\bullet}$ is a quasi-isomorphism, and

$$(sM)^n = \prod_{p+q=n} M^{p,q} \text{ injective},$$

Then

$$RT(K) = T(sM^{\bullet,\bullet}) = s(TM^{\bullet,\bullet}).$$

(1). The 1st spectral sequence of hypercohomology is just the 1st spectral sequence of $TM^{\bullet,\bullet}$, with

$$'E_1^{p,q} =' H^q T(M^{p,\bullet}) = R^q T(K^p),$$

and d_1 induced by $d : K^p \to K^{p+1}$. Since

$$K^{\geq p} = \sigma_{\geq p} K^\bullet \xrightarrow{quis} {'F^p} sM^{\bullet,\bullet},$$

then

$$RT(K^{\geq p}) = T'F^p(sM^{\bullet,\bullet}) = \mathbb{F}^p(sTM^{\bullet,\bullet})$$

and $F^p R^n T(K) = \mathrm{Im}(R^n T(K^{\geq p}) \to R^n TK)$.

(2). The second spectral sequence of hypercohomology is the second spectral sequence of $TM^{\bullet,\bullet}$, with

$$E_2^{p,q} = {"H^p} {'H^q}(TM^{\bullet,\bullet}).$$

Since ${}'H^q(TM^{\bullet,j}) = T'H^q(M^{\bullet,j})$, and all the components, boundaries and cycles of $M^{\bullet,j}$ are injectives,

$$''H^{p\prime}H^q(TM^{\bullet,\bullet}) = R^pT(H^qK).$$

The abutment filtration of the 2nd spectral sequence needs to use "décalage" of Deligne. We leave it as an exercise. □

Corollary 6.19 (Spectral sequence of a composite functor). *Suppose*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C},$$

where F, G are additive functors of abelian categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and $F(\text{injective}) = G-\text{acyclic}$. Thus

$$\begin{tikzcd} D^+(\mathcal{A}) \arrow[r, "RF"] \arrow[rr, bend right, "R(GF)"'] & D^+\mathcal{B} \arrow[r, "RG"] & D^+(\mathcal{C}). \end{tikzcd}$$

Then there exists, for $K \in D^+(\mathcal{A})$, a functorial, biregular spectral sequence

$$E_2^{p,q} = R^pGR^qF(K) \Rightarrow (GF)(K).$$

Proof. Take the second spectral sequence of hypercohomology of $RF(K)$ for G, then $E_2^{p,q} = R^pG(H^q(RF(K))) = R^pGR^qF(K)$ and $R^nG(RFK) = H^n(RG(RF(K))) = H^n(R(GF)(K)) = R^n(GF)(K)$, thus

$$E_2^{p,q} = R^pGR^qF(K) \Rightarrow (GF)(K).$$

The abutment is

$$\begin{aligned} F^pR^n(GF)(K) &= \mathrm{Im}(R^n(GF)(\tau_{\leq n-p}K) \to R^n(GF)(K)) \\ &= \mathrm{Im}(R^G(\tau_{\leq n-p}RFK) \to R^n(GF)K). \end{aligned}$$

□

We give three applications for Cartan-Eilenberg's Theorem and the corollary.

6.20. Hodge-de Rham spectral sequence. Let k be a field and X/k be a proper smooth scheme. Then $\Omega^\bullet_{X/k} \in C(X,k)$. The first spectral sequence of hypercohomology of $\Omega^\bullet_{X/k}$ for

$$T = \Gamma(X,-) : Mod(X,k) \to V(k) = \{\text{finite dimensional } k\text{-vector spaces}\}$$

is

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \Omega_X^\bullet) = H_{dR}^{p+q}(X/k),$$

d_1 is induced by $d : \Omega^p \to \Omega^{p+1}$. This spectral sequence is biregular and concentrated on the first quadrant. For every r and for every p, q, $E_r^{p,q}$ is finite dimensional over k, therefore $H_{dR}^n(X/k)$ is finite dimensional over k. This spectral sequence is called the Hodge to de Rham spectral sequence. Let

$$h^n = h^n(X/k) = \dim_k H_{dR}^n(X/k), h^{p,q} = h^q(X, \Omega_X^p) = \dim_k H^q(X, \Omega_X^p).$$

From the Hodge to De Rham spectral sequence, we always have

$$h^n \leq \sum_{p+q=n} h^{p,q}.$$

If $k = \bar{k}$, X is connected and $\dim X = d$, then one can show that $H^d(X, \Omega_X^d) = H_{dR}^{2d}(X/k)$ and $h^{p,q} = h^{d-p,d-q}$.

One would like to show the Hodge to de Rham spectral sequence degenerates at E_1, which means that $d_r = 0$ for all $r \geq 1$, in particular, $E_1 = E_\infty$. This is equivalent to show that for all n,

$$h^n = \sum_{p+q=n} h^{p,q}.$$

Theorem 6.21 (Hodge+Deligne). *If* $\operatorname{char} k = 0$*, then the Hodge to de Rham spectral sequence degenerates at* E_1

Theorem 6.22 (Deligne-Illusie). *If* $\operatorname{char} k = p > 0$*,* k *perfect and* $\dim X \leq p$ *and* X *is liftable to the second Witt ring* $W_2(k)$*, then the Hodge to de Rham spectral sequence degenerates at* E*.*

6.23. Leray spectral sequence. Let $f : X \to Y$ be a morphism of ringed spaces. Then $\Gamma(Y, f_*(-)) = \Gamma(X, -)$. We have a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* K) \Rightarrow H^{p+q}(X, K).$$

6.24. Local to global spectral sequence of Ext**.** Let X be a ringed space, $L \in D^-(X)$, $M \in D^+(X)$. Then

$$\mathrm{R}\mathcal{H}om(L, M) \in D^+(X),\ \mathcal{E}xt^n(L, M) = H^n(\mathrm{R}\mathcal{H}om(L, M)).$$

$$\mathrm{RHom}(L, M) = R\Gamma(X, \mathrm{R}\,\mathcal{H}om(L, M)) \in D^+(Ab),$$

$$\mathrm{Ext}^n(L, M) = H^n(\mathrm{RHom}(L, M)) = H^n(X, \mathrm{R}\,\mathcal{H}om(L, M)).$$

The second spectral sequence of hypercohomology for $\Gamma(X, -)$ then gives

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(L, M)) \Rightarrow \mathrm{Ext}^{p+q}(L, M).$$

Chapter 4

Grothendieck's Comparison and Existence Theorems in Formal Geometry

1 Locally Noetherian Formal Schemes

Definition 1.1. An *adic noetherian ring* is a noetherian ring which is separated and complete in the I-adic topology for some ideal $I \subset A$, i.e. $A \simeq \varprojlim A/I^{n+1}$.

Set $A_n = A/I^{n+1}$ and $X_n = \operatorname{Spec} A_n$. It is clear that the X_n's have the same underlying topological space and one obtains an increasing sequence of thickenings of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$:

$$X_0 = \operatorname{Spec} A/I \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} A$$

$$\mathcal{O}_{X_o} \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{O}_{X_n} \twoheadleftarrow \mathcal{O}_{X_{n+1}} \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{O}_{\operatorname{Spec} A}$$

Associated to this sequence is the *formal spectrum* of A, denoted by $\operatorname{Spf}(A) = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, where $\mathfrak{X} = |X_0|$ as topological spaces and $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_{X_n}$ is a sheaf of topological rings.

For any open subset $U \subset \mathfrak{X}$, define the section $\Gamma(U, \mathcal{O}_{\mathfrak{X}})$ to be the topological ring $\varprojlim \Gamma(U, \mathcal{O}_{X_n})$ where $\Gamma(U, \mathcal{O}_{X_n})$ is endowed with the discrete topology. For instance, given $f \in A$, denote by f_0 the image of f modulo I in A_0, then the section over the open set $\mathfrak{D}(f) = \mathfrak{X}_f = \operatorname{Spec}(A_0)_{f_0}$, where f_0 is invertible, is the completed fraction ring $A_{\{f\}} = \varprojlim \Gamma(\mathfrak{X}_f, O_{X_n}) = \varprojlim S_f^{-1} A \otimes_A (A/I^n)$.

Remark 1.2. (1) As a topological space, $\mathfrak{X} = \operatorname{Spf} A$ depends only on A as a topological ring. It doesn't change if one replaces I by some *ideal of definition*, namely, an ideal J such that $J \supset I^p \supset J^q$ for some positive integer p and q. The space $\mathfrak{X}$ is the subspace of $\operatorname{Spec} A$ consisting of *open* prime ideals, and $\mathcal{O}_{\mathfrak{X}} = \varprojlim \widetilde{(A/J)}$ where J runs through the ideals of definition of A.

(2) $\mathfrak{X} = \operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A$ being a closed subspace, $\mathfrak{X}$ contains all the closed points of $\operatorname{Spec} A$ and thus every open subset of $\operatorname{Spec} A$ containing $\mathfrak{X}$ coincides with $\operatorname{Spec} A$ itself, resulting from $I \subset \mathrm{Rad}A$.

Definition 1.3. An *affine noetherian formal scheme* is a topologically ringed space isomorphic to $(\mathfrak{X} = \operatorname{Spf} A, \mathcal{O}_{\mathfrak{X}})$ for some adic notherian ring A. A *local noetherian formal scheme* is a topologically ringed space covered by affine noetherian formal schemes, namely, every point lies in a neighborhood which is an affine noetherian formal scheme. Morphisms between local noetherian formal schemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ are those morphisms $(f, f^{\#})$ between ringed spaces that are local and *continuous*, i.e. for every point $x \in \mathfrak{X}$ the map $\mathcal{O}_{\mathfrak{Y}, f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ is local and for any affine open subscheme $V \subset \mathfrak{Y}$, the homomorphism between topological rings $\Gamma(V, \mathcal{O}_{\mathfrak{Y}}) \to \Gamma(f^{-1}(V), \mathcal{O}_{\mathfrak{X}})$ is continuous.

As in the case of usual schemes, for any local noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and affine formal scheme $(\mathfrak{Y} = \operatorname{Spf} A, \mathcal{O}_{\mathfrak{Y}})$,

$$\operatorname{Hom}(\mathfrak{X}, \mathfrak{Y}) = \operatorname{Hom}_{cont}(A, \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}))$$

where $\operatorname{Hom}_{cont}$ stands for the set of continuous ring homomorphisms.

Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine noetherian formal scheme and I an ideal of definition of A. With any A-module of finite type M is associated an $\mathcal{O}_{\mathfrak{X}}$-module $M^{\Delta} = \varprojlim \tilde{M}_n$ where $\tilde{M}_n$ is the coherent module on $\operatorname{Spec} A_n$ associated to the A_n-module M_n. Immediately derived from the definition is the following

Proposition 1.4. $\Gamma(\mathfrak{X}, M^{\Delta}) = \varprojlim M_n = M$. *For $f \in A$, $\mathfrak{X}_f \subset \mathfrak{X}$, $\Gamma(\mathfrak{X}_f, M^{\Delta}) = (M_f)^{\wedge}$ is the I-adic completion of the fractional module M_f. The functor $M \to M^{\Delta}$ is exact where M ranges over the category of finitely generated A-modules. The map* $(\operatorname{Hom}_A(M, N))^{\wedge} \to \mathcal{H}om(M^{\Delta}, N^{\Delta})$ *induced by $v \mapsto v^{\Delta}$ is an isomorphism. In particular, any homomorphism $u : M^{\Delta} \to N^{\Delta}$ is uniquely determined by its global section v via $u = v^{\Delta}$.*

Recall that given a ringed space $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module E is locally of finite type if for every $x \in X$ there exists a neighborhood $U \supset x$ and an epimorphism $\mathcal{O}_U^r \to E|_U \to 0$; it is locally of finite presentation if the exact sequence above can be extended to $\mathcal{O}_U^m \to \mathcal{O}_U^n \to E|_U \to 0$. If $\mathcal{O}_X$ is coherent, an $\mathcal{O}_X$-module E is coherent if and only if it is finitely generated and locally of finite presentation.

Proposition 1.5. *Let $\mathfrak{X}$ be a locally noetherian formal scheme, then*

(1) *$\mathcal{O}_{\mathfrak{X}}$ is coherent;*

(2) *given $E \in Mod(\mathfrak{X})$, E is coherent if and only if for every affine open piece $U = \operatorname{Spf} A \subset \mathfrak{X}$, there exists M an A-module of finite type such that $E|_U = M^{\Delta}$.*

The set of coherent $\mathcal{O}_{\mathfrak{X}}$-modules will be denoted by $Coh(\mathfrak{X})$ in later sections.

Proof. (1) The question may be reduced to local case, and one may assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Actually for any epimorphism $\mathcal{O}_{\mathfrak{X}}^r \to \mathcal{O}_{\mathfrak{X}}$, the kernel of the corresponded map $v = \Gamma(\mathfrak{X}, u) : A^r \twoheadrightarrow A$ is of finite type because A is noetherian itself, namely there exists a exact sequence of the form $A^s \xrightarrow{w} A^r \xrightarrow{v} A \to 0$. The exactness of the functor $^{\Delta}$ implies an exact sequence $\mathcal{O}_{\mathfrak{X}}^s \xrightarrow{w^{\Delta}} \mathcal{O}_{\mathfrak{X}}^r \xrightarrow{v^{\Delta}} \mathcal{O}_{\mathfrak{X}} \to 0$.

(2) The part of $\Leftarrow$ is clear. For the $\Rightarrow$ part, it suffices to show that $E = M^{\Delta}$ for some A-module M of finite type. Put $E_n = E \otimes \mathcal{O}_{X_n} \in Coh(X_n)$, $M_n = \Gamma(X_n, E_n)$, and let $M = \varprojlim \Gamma(X_n, E_n)$. Thus

$$E = \varprojlim E_n = \varprojlim \tilde{M}_n = M^{\Delta}$$

is coherent. □

1.6. Ideals of Definition and Inductive Limits

Definition 1.7. For a local noetherian formal scheme $\mathfrak{X}$, an ideal of definition of $\mathfrak{X}$ is a coherent ideal sheaf $\mathfrak{I} \in Coh(\mathfrak{X})$ such that the formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ has the same underlying topological space as $\mathfrak{X}$.

Proposition 1.8. *Let $\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}$ be a local noetherian scheme.*

(1) *An ideal sheaf $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ if and only if for any affine open subscheme $U = \operatorname{Spf} A \subset \mathfrak{X}$, $\mathfrak{I}|_U = I^{\Delta}$ where $I \subset A$ is*

an ideal of definition for the topological ring $A = \varprojlim A/J^n$, *with* $I \supset J^p \supset I^q$ *for some positive integer* p *and* q.

(2) (Similar to the case of schemes) *Ideals of definition exist and there exist a* largest *one* $\mathfrak{T}$ *such that* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{T})$ *is reduced. On any affine open subscheme* $U = \operatorname{Spf} A$, $\mathfrak{T} = N^{\Delta}$, *where* $N = \{a \in A : A^n \to 0 \text{ as } n \to \infty\}$ *is the ideal of topological nilpotent elements which coincides with the inverse image of the nilpotent radical of* A *via* $A = \varprojlim A/I^n$. *Any ideal of definition is contained in* $\mathfrak{T}$.

Remark 1.9. Given a noetherian formal scheme $\mathfrak{X}$, two ideals of definition of X, say $\mathfrak{I}$ and $\mathfrak{J}$, gives a chain $\mathfrak{I} \supset \mathfrak{J}^p \supset \mathfrak{I}^q$ for some positive integer p and q.

Fix an ideal of definition $\mathfrak{I}$ of $\mathfrak{X}$. For $n \in \mathbb{N}$, the ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$ is a locally noetherian scheme, denoted X_n. One obtains an increasing chain of thickenings

$$X. = (X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots) \tag{1.9.1}$$

whose inductive limit, in the category of locally noetherian formal schemes, is $\mathfrak{X}$: the thickenings induce the identity map on the underlying topological spaces, which are all equal to $|\mathfrak{X}|$, and it is clear that $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_{X_n}$ as sheaves of topological rings, where $\Gamma(U, \mathcal{O}_{X_n})$ is endowed with the discrete topology for any open subset $U \subset X_n$. Let $J_n = \operatorname{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0})$ be the ideal of X_0 in X_n. Then for integers $m \leq n$, the ideal of X_m in X_n is J_n^{m+1}, and in particular $J_n^{n+1} = 0$. J_1 is a coherent module on X_0 and $J_n = \mathfrak{I}/\mathfrak{I}^{n+1}$.

Converse to the argument above is the following

Proposition 1.10. *Consider a sequence of ringed spaces 1.9.1 satisfying*

(1) X_0 *is a locally noetherian scheme;*

(2) *the underlying maps of topological spaces are homeomorphisms and, using them to identify the underlying spaces, the maps of sheaves of rings* $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ *are surjective;*

(3) *setting* $J_n = \operatorname{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_0})$, *then for* $m \leq n$, $\operatorname{Ker}(\mathcal{O}_{X_n} \to \mathcal{O}_{X_m}) = J_n^{m+1}$;

(4) J_1 *is a coherent* $\mathcal{O}_{X_0}$*-module.*

Then the topologically ringed space $\mathfrak{X} = (X_0, \varprojlim \mathcal{O}_{X_n})$ *is a locally noetherian formal scheme, and* $\mathfrak{I} := \varprojlim J_n = \operatorname{Ker}(\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{X_0})$ *is an ideal of definition of* $\mathfrak{X}$ *and* $\mathfrak{I}^{n+1} = \operatorname{Ker}(\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{X_n})$.

Proof. The verification is straightforward as reduced to the affine case. Assume $X_0 = \operatorname{Spec} A_0$ is affine, one checks easily that each X_n is affine noetherian of ring $A_n = \Gamma(X_n, \mathcal{O}_{X_n})$. Then $A = \Gamma(X_0, \mathcal{O}_{\mathfrak{X}}) = \varprojlim A_n$ is separated and complete. $\mathfrak{X} = \operatorname{Spf} A$ is an affine noetherian formal scheme. □

Let $\mathfrak{X}$ be a locally noetherian formal scheme, $\mathfrak{I}$ an ideal of definition of $\mathfrak{X}$. Consider the corresponding chain of thickenings as above. For $m \leq n$, denote by $u_{mn} : X_m \to X_n$ and $u_n : X_n \to X_0$ the canonical morphisms. Given a coherent module E on $\mathfrak{X}$, then $E_n := u_n^* E$ is a coherent module on X_n and these modules form an inverse system, with $\mathcal{O}_{X_n}$-linear transition maps $E_n \to E_m$ inducing isomorphisms $u_{mn}^* E_n \xrightarrow{\sim} E_m$ and $E = \varprojlim E_n$.

Conversely, let $F. = (F_n, f_{mn})$ be an inverse system of $\mathcal{O}_{X_n}$-modules, with $\mathcal{O}_{X_n}$-linear transition maps $f_{mn} : F_n \to F_m$ for $m \leq n$. $F.$ is said to be coherent if each F_n is $\mathcal{O}_{X_n}$-coherent and the transition maps f_{mn} induce isomorphisms $u_{mn}^* F_n \xrightarrow{\sim} F_m$. If $F.$ is coherent and $F := \varprojlim F_n$ is the corresponding $\mathcal{O}_{\mathfrak{X}}$-module, then F is coherent and $F.$ is canonically isomorphic to the inverse system $(u_n^* F)$. The functor $\mathit{Coh}(\mathfrak{X}) \to \mathit{Coh}(X.)$, sending E to the system $(u_n^* E)$ from the category of coherent sheaves on $\mathfrak{X}$ to the category $\mathit{Coh}(X.)$ of coherent inverse systems (F_n) is an equivalence. For $E = \varprojlim E_n \in \mathit{Coh}(\mathfrak{X})$ as above, the *support* of E is, as E is coherent, closed and coincides with that of E_0. By a special case of flatness criterion, E is flat, or equivalently locally free of finite type, if and only if E_n is locally free of finite type for all n.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally noetherian formal schemes, $\mathfrak{J}$ an ideal of definition of $\mathfrak{Y}$. Since $\mathfrak{J} \subset \mathfrak{T}_{\mathfrak{Y}}$, the continuity of f implies that the ideal $f^*(\mathfrak{J})\mathcal{O}_{\mathfrak{X}}$ is contained in $\mathfrak{T}_{\mathfrak{X}}$. Fix an ideal of definition $\mathfrak{I}$ such that $f^*(\mathfrak{J})\mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}$ and consider the inductive systems $X.$, $Y.$ defined by $\mathfrak{I}$ and $\mathfrak{J}$ respectively. Then, since $f^*(\mathfrak{J}^{n+1})\mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}^{n+1}$, f induces a morphism of inductive systems

$$f. : X. \to Y. \tag{1.10.1}$$

i.e. morphisms of schemes $f_n : X_n \to Y_n$ such that the squares

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, "f_m"'] & X_n \arrow[d, "f_n"] \\ Y_m \arrow[r] & Y_n \end{tikzcd} \tag{1.10.2}$$

are commutative and $f = \varinjlim f_n$, characterized by making the squares

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[d, "f_n"'] & \mathfrak{X} \arrow[d, "f"] \\ Y_n \arrow[r, "u_n"] & \mathfrak{Y} \end{tikzcd} \tag{1.10.3}$$

commutative. It is easily checked [EGA I 10.6.8] that $f \to f.$ defines a bijection from the set of morphisms $\{f \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{Y}) : f^*(\mathfrak{J})\mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}\}$ to the set of morphisms of the type $f. : X. \to Y.$.

The above results is summarized as follows:

Proposition 1.11. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a locally noetherian formal scheme.*

(1) *the functor $Coh(\mathfrak{X}) \to Coh(X.)$, sending E to the inverse system $(u_n^* F)$, is an equivalence.*

(2) *given morphism of locally noetherian formal schemes $f : \mathfrak{X} \to \mathfrak{Y}$, and assume that $\mathfrak{J} \subset \mathfrak{T}_{\mathfrak{Y}}$ is an ideal of definition of $\mathfrak{Y}$ and $f^*(\mathfrak{J})\mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}$ where $\mathfrak{I}$ is an ideal of definition for $\mathfrak{X}$. Then $f = \varprojlim f_n$ where the f_n's are characterized by the diagram*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, "f_n"'] & \cdots \arrow[r] & \mathfrak{X} \arrow[d, "f"] \\ Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & \cdots \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

and the X_n's (resp. the Y_n's) are thickenings defined by the ideal $\mathfrak{I}$ (resp. $\mathfrak{J}$). The map sending $f : \mathfrak{X} \to \mathfrak{Y}$ to $f. : X. \to Y.$ is bijective.

In general, $f^*(\mathfrak{J})\mathcal{O}_(\mathfrak{X})$ is not an ideal of definition of $\mathfrak{X}$. When this is the case, f is called an *adic morphism* and $\mathfrak{X}$ an $\mathfrak{Y}$-adic formal scheme. One can then take $\mathfrak{I} = f^*(\mathfrak{J})\mathcal{O}_{\mathfrak{X}}$ and the squares 1 are *cartesian*. Conversely any morphism of inductive systems 1.10.1 such that the squares 1 are cartesian define an adic morphism from $\mathfrak{X}$ to $\mathfrak{Y}$.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism and E a coherent sheaf on $\mathfrak{X}$. Then the following conditions are equivalent:

(1) E is *flat* over $\mathfrak{Y}$ (or $\mathfrak{Y}$-flat), i.e. for every point $x \in \mathfrak{X}$, the stalk E_x is flat over $\mathcal{O}_{\mathfrak{Y}, f(x)}$;

(2) with the notations of 1.10.3, $E_n = u_n^* E$ is Y_n-flat for all $n \leq 0$;

(3) E_0 is Y_0-flat and the natural epimorphism

$$\mathrm{gr}^n \mathcal{O}_{\mathfrak{Y}} \underset{\mathrm{gr}^0 \mathcal{O}_{\mathfrak{Y}}}{\otimes} \mathrm{gr}^0 E \longrightarrow \mathrm{gr}^n E$$

is an isomorphism for all n, where the associated graded module is taken with respect to the $\mathfrak{J}$-adic filtration.

This is a consequence of the flatness criterion, and when the equivalent conditions are satisfied for $E = \mathcal{O}_{\mathfrak{X}}$, f is said to be *flat*.

1.12. Formal Completion Let X be a locally noetherian scheme, and X' a closed subset of the underlying topological space of X. Choose a coherent ideal $I \subset \mathcal{O}_X$ such that the closed subscheme of X defined by I has X' as its underlying space. Such ideals do exist, and there is, in fact, a largest one, consisting of the local sections of $\mathcal{O}_X$ vanishing on X' for which X' has the reduced scheme structure. Consider the inductive system of locally noetherian schemes, all having X' as the underlying space,

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$$

where X_n is the closed subscheme of X defined by I^{n+1}. It satisfies the conditions of Proposition 1.10 and therefore the inductive limit $X_{/X'} := \varinjlim X_n$ is a locally noetherian formal scheme, having X' as the underlying space, called the *formal completion of* X *along* X', sometimes denoted by $\hat{X}$.

It is easily checked that $X_{/X'}$ does not depend on the choice of the ideal I. Actually $\mathcal{O}_{\hat{X}} = \varprojlim \mathcal{O}_X/J$ where J runs through all the coherent ideals of $\mathcal{O}_X$ such that the support of $\mathcal{O}_X/J$ is X' and on any noetherian open subset of X, the powers of I form a cofinal system. If X is affine, $X = \operatorname{Spec} A$ and $I = \tilde{J}$, then $\hat{X} = \operatorname{Spf} \hat{A}$, with $\hat{A} = \varprojlim A/J^n$

The canonical immersion $i_n : X_n \hookrightarrow X$ defines a morphism of ringed spaces

$$i = i_X : \hat{X} \to X \tag{1.12.1}$$

which is flat and for any coherent sheaf F on X, the natural map

$$i^*F \to F_{/X'} := \varprojlim i_n^*F \tag{1.12.2}$$

is an isomorphism. When $X = \operatorname{Spec} A$ and $F = \tilde{M}$, M being an A-module of finite type, then $F_{/X'} = M^{\Delta}$.

The assertion above follows from Krull's theorem: if A is noetherian and J an ideal of A, then the J-adic completion $\hat{A}$ is (faithfully) flat over A, and for any A-module of finite type, $\hat{M} = M \otimes_A \hat{A}$. One writes then $\hat{F}$ for $F_{/X'}$ when no confusion arises. Note that if F is not coherent, 1.12.2 is not in general an isomorphism. One checks easily that the kernel of the adjunction map

$$F \to i_* i^* F \tag{1.12.3}$$

consists of sections of F that vanish in a neighborhood of X'.

Let $f : X \to Y$ be a morphism of locally noetherian schemes, X' (resp. Y') a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Choose coherent ideals $J \subset \mathcal{O}_X$, $I \subset \mathcal{O}_X$, defining closed subschemes with underlying spaces X' and Y' resppectively and such that $f^*(I)\mathcal{O}_X \subset J$. Then f induces a morphism of inductive systems $f. : X. \to Y.$, and thus a morphism

$$\hat{f} : X_{/X'} \to Y_{/Y'} \tag{1.12.4}$$

which does not depend on the choice of J and I, called the *extension* of f to the completions $X_{/X'}$ and $Y_{/Y'}$. This morphism sits in a commutative square

$$\begin{tikzcd} X_{/X'} \arrow[r, "i_X"] \arrow[d, "\hat{f}"] & X' \arrow[d, "f"] \\ Y_{/Y'} \arrow[r, "i_Y"] & Y \end{tikzcd} \tag{1.12.5}$$

When $X' = f^{-1}(Y')$, one may take $J = f^*(I)\mathcal{O}_X$, all the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\ Y_n \arrow[r] & Y \end{tikzcd}$$

are cartesian, hence the same holds for the square and therefore $\hat{f}$ is an *adic* morphism.

2 The Comparison Theorem

Let $f : X \to Y$ be a morphism of locally noetherian schemes, Y' a closed subscheme of locally noetherian schemes, let Y' be a closed subset of Y, $X' = f^{-1}(Y')$. Write $\hat{X} = X_{/X'}$ and $\hat{Y} = Y_{/Y'}$. For $F \in Mod(X)$ the square

$$\begin{tikzcd} X_{/X'} \arrow[r, "i_X"] \arrow[d, "\hat{f}"] & X' \arrow[d, "f"] \\ Y_{/Y'} \arrow[r, "i_Y"] & Y \end{tikzcd}$$

defines base change maps

$$i^*R^qf_*F \longrightarrow R^q\hat{f}_*(i^*F)$$

for all $q \in \mathbb{Z}$, which are maps of $\mathcal{O}_Y$-modules. When F is coherent, then i^*F can be identified with $\hat{F} = F_{/X'}$ and similarly $i^*R^qf_*F$ with $(R^qf_*F)_{/Y'}$ if R^qf_*F is coherent, which is the case when F is coherent and f is proper (or f is of finite type and the support of F is *proper* over Y, i.e. (see [EGA II 5.4.10]) the there is a closed subscheme of X, proper over Y and with Supp$F$ as the underlying space, by the finiteness theorem for proper morphisms [EGA III 3.2.1, 3.2.4]. In this case 2 can be rewritten as

$$(R^qf_*F)^\wedge \longrightarrow R^q\hat{f}_*\hat{F}$$

On the other hand the squares 1.10.3, with $\mathfrak{X} = \hat{X}$ and $\mathfrak{Y} = \hat{Y}$, define $\mathcal{O}_{Y_n}$-linear base change maps

$$u_n^*R^q\hat{f}_*\hat{F} \longrightarrow R^q(f_n)_*F_n$$

where $F_n = u_n^*\hat{F} = i_n^*F$, following the notation of 1.12.1. By adjunction, these maps can be viewed as $\mathcal{O}_{\hat{Y}}$-linear maps

$$R^q\hat{f}_*\hat{F} \longrightarrow R^q(f_n)_*F_n$$

hence define $\mathcal{O}_{\hat{Y}}$-linear maps

$$R^q\hat{f}\hat{F} \longrightarrow \varprojlim R^q(f_n)_*F_n$$

Note that the base change map 2 is defined more generally for $F \in D^+(X, \mathcal{O}_X)$, as induced on the sheaves $\mathcal{H}^q$ from the base change map in $D^+(\hat{Y}, \mathcal{O}_{\hat{Y}})$

$$i^*Rf_*F \longrightarrow R\hat{f}i^*F$$

Theorem 2.1. *Let $f : X \to Y$ be a morphism of finite type between noetherian schemes, Y' a closed subset of Y, $X' = f^{-1}(Y')$, $\hat{f} : \hat{X} \to \hat{Y}$ the extension of f to the formal completions of X and Y along X' and Y' respectively. Let F be a coherent sheaf on X whose support is proper over Y. Then the canonical maps $(R^qf_*F)^\wedge \to R^q\hat{f}_*\hat{F}$ and $R^q\hat{f}\hat{F} \longrightarrow \varprojlim R^q(f_n)_*F_n$ are topological isomorphisms for all $q \in \mathbb{Z}$.*

Remark 2.2. (a) It follows from the assumption of 2.1 on f that for any $F \in D^+(X, \mathcal{O}_X)$ such that for any i, $\mathcal{H}^i F$ is coherent and properly supported over Y, the base change map $i^* Rf_* F \longrightarrow R\hat{f} i^* F$ is an isomorphism. Noting that the natural functor from the bounded derived category $D^b(Coh(X))$ of coherent sheaves on X to the full subcategory $D^b(X)_{coh}$ of $D^b(X) := D^b(Mod(X))$ is an equivalence [SGA 6-II 2.2.2.1], where $D^b(X)$ consists of the complexes with coherent cohomology, one can extend the isomorphism 2 of 2.1 to the case $F \in D^b(X)_{coh}$

(b) Grothendieck's original approach, though not published, is guessed to consists of two steps: (1) proof in the case where f is projective, using descending induction on q; (2) proof in the general case by reducing to the projective case via Chow's Lemma and noetherian induction. The proof given in [EGA III 4.1.7 4.1.8] follows an argument due to Serre.

By considering a closed subscheme Z of X whose underlying space is the support of F, 2.1 is reduced to the case where f is *proper*. And it is easily seen that the theorem is reduced to the following special case:

Corollary 2.3. *Under the assumption of* 2.1, *suppose that* $Y = \operatorname{Spec} A$, *with* A *a noetherian ring,* I *an ideal of* A *such that* $\operatorname{Supp}(\mathcal{O}_Y/\mathcal{I}) = Y'$, *where* $\mathcal{I} = \tilde{I}$. *Set* $Y_n = \operatorname{Spec}(A/I^{n+1}$, $X_n = Y_n \underset{Y}{\times} X$, $F_n = i_n^* F = F/\mathcal{I}^{n+1}F$. *Then for all* $q \in \mathbb{Z}$ *the natural maps*

$$\varphi_q : H^q(X, F)^\wedge \longrightarrow \varprojlim H^q X, F_n$$

defined by the composition of 2 and 2, and

$$\psi_q : H^q(\hat{X}, \hat{F}) \longrightarrow \varprojlim H^q(X, F_n)$$

defined by 2, are topological isomorphisms.

The proof of the corollary, which also appears in [EGA III 4.1.7], uses two ingredients: (a) the Artin-Rees Lemma and the Mittag-Leffler Conditions, mainly elementary commutative and homological algebra; (b) the finiteness theorem for proper morphisms ([EGA III 3.2]), especially a *graded* variant [EGA III 3.3.2]. A brief revision of (a) and (b) will be given before the proof of the theorem is presented.

2.4. ***Artin-Rees and Mittag-Leffler*** Let A be an noetherian ring, I an ideal of A and M a finitely generated A-module endowed with a descending filtration by submodules $(M_n)_{n\in\mathbb{Z}}$. The filtration $(M_n)_{n\in\mathbb{Z}}$ is called I-good if it is *exhaustive*, i.e. $M_{n_0} = M$ for some n_0, and it satisfies the following two conditions:

(i) $IM_n \subset M_{n+1}$ for all $n \in \mathbb{Z}$, namely, M is a filtered module over the ring A filtered by the I-adic filtration;

(ii) $M_{n+1} = IM_n$ when n is large enough.

All I-good filtrations define on M the same topology, namely the I-adic topology, filtering M by $M_n = I^{n+1}M$ for $n \geq 0$.

Assume the condition (i) holds. Consider the associated graded ring $A' := \operatorname{gr} A = \oplus_{n\in\mathbb{N}} I^n$, sometimes written $\oplus I^n t^n$ where t is an indeterminate, to make clear that $I^n = I^n t^n$ is the n-th component of A', and the graded module associated to M is $M' = \operatorname{gr} M = \oplus_{n\in\mathbb{N}} M_n = \oplus M_n t^n$. A basic observation [B, III, §3, th.1] is that the condition (ii) is equivalent to

(ii)' M' is finitely generated over A'.

Since A' is noetherian, this immediately implies the classical *Artin-Rees Theorem*: for any submodule $N \subset M$, the filtration on N induced by the I-adic filtration of M is I-good, namely there exists $n_0 \in \mathbb{N}$ such that $(I^{n+n_0}M) \cap N = I^n(I^{n_0}M \cap N)$ for all $n \in \mathbb{N}$.

Let A be a ring, $M. = (M_n, u_{mn})$ be a projective system of A-modules, indexed by $\mathbb{N}$. The terminology below will be useful:

(1) $M.$ is *strict* if the transition maps $u_{mn} : M_n \to M_m$ are all surjective;

(2) $M.$ is *essentially zero* if for each m there exists $n \geq m$ such that $u_{mn} = 0$, i.e. the *pro-object* defined by $M.$ is zero.

(3) $M.$ satisfies the *Mittag-Leffler Condition* (ML for short) if for each m there is an $n \geq m$ such that $\operatorname{Im} u_{mn'} = \operatorname{Im} u_{mn}$ in M_m for all $n' \geq n$.

It is sometimes useful to consider the following stronger conditions :

$(2)'$ $M.$ is *Artin-Rees zero* (AR zero for short) if there exists an integer $r \geq 0$ such that $u_{n,n+r} = 0$ for all n;

$(3)'$ $M.$ satisfies the *Artin-Rees-Mittag-Leffler condition* (ARML for short) if there exists an integer $r \geq 0$ such that $\operatorname{Im} u_{mn} = \operatorname{Im} u_{m,m+r}$ for all m and all $n \geq m + r$.

The following facts about the Mittag-Leffler conditions are found in [EGA 0_{III} 13]

(a) If $M.$ is essentially zero, then $\varprojlim M_n = 0$;

(b) The functor $M. \mapsto \varprojlim M_n$ is left exact; and for any exact sequence of inverse system of A-modules, say

$$0 \longrightarrow L. \longrightarrow M. \longrightarrow N. \longrightarrow 0$$

the sequence

$$0 \longrightarrow \varprojlim L_n \longrightarrow \varprojlim M_n \varprojlim N_n \longrightarrow 0$$

is exact whenever $L.$ satisfies the ML condition.

(2)Mittag-Leffler condition and projective limits.

Let A be a ring. Let $\mathrm{Mod}(A_\bullet)$ denote the category of projective systems of A-modules indexed by $\mathbb{N}$, $E_\bullet = (E_0 \leftarrow E_1 \leftarrow \cdots)$, with $E_n \in \mathrm{Mod}(A)$, $u_{mn} : E_n \to E_m$ $(m \leq n)$. For $E_\bullet \in \mathrm{Mod}(A_\bullet)$, the projective limit of $E_\bullet$ is the A-module $\varinjlim E_\bullet = \varinjlim E_n = \{(x_n) | u_{mn}(x_n) = x_m\}$.

Definition 2.5. Let $E_\bullet \in \mathrm{Mod}(A_\bullet)$, then

(1) E is *strict* if u_{mn} is surjective for any $m \leq n$.

(2) E satisfies the *Mittag-Leffler condition* (*ML* for short) if for any m, there exists $n \geq m$, such that for any $p \geq n$, $u_{mn}(E_n) = u_{mp}(E_p)$.

Remark. (1) If $E_\bullet$ is strict, then it satisfies ML.

(2) Let $E_\bullet \in \mathrm{Mod}(A_\bullet)$, then for any fixed n, $u_{np}(E_p) \subset E_n$ decreases with p. Define $E'_n = \bigcap_{p \geq n} u_{np}(E_p) \subset E_n$ (E'_n is called a *universal image*). Let $E = \varinjlim E_n$, define $u_n : E \to E'_n$ in the obvious way, then $u_n(E) \subset E'_n$, $u_{mn}(E'_n) \subset E'_m$, and $\varinjlim E'_n \xrightarrow{\sim} \varinjlim E_n$.

If E satisfies ML, it means that for any fixed n, $\{u_{np}(E_p)\}_p$ is stationary. In particular E' is strict.

We have similar definitions of "ML" and "strict" in the category of sets.

Proposition 2.6. *Let*

$$0 \to L_\bullet \xrightarrow{f} M_\bullet \xrightarrow{g} N_\bullet \to 0$$

be an exact sequence of $A_\bullet$-modules. Then the sequence

$$0 \to \varinjlim L_n \to \varinjlim M_n \to \varinjlim N_n$$

is exact, and if $L_\bullet$ satisfies ML, then $\varinjlim M_n \to \varinjlim N_n$ is surjective.

Proof. The first assertion is immediate. Assume $L_\bullet$ satisfies ML. Let

$$z = (z_n) \in \varinjlim N_n, \quad E_n = g_n^{-1}(z_n).$$

Let $E_\bullet = (E_0 \leftarrow E_1 \leftarrow \cdots)$ be the projective system of sets induced by $M_\bullet$, and denote by $v_{mn} : E_n \to E_m$ the transition map for $n \geq m$. As $L_\bullet$ satisfies ML, and E_n is an affine space under L_n, $E_\bullet$ satisfies ML, and hence E' is strict. As $\{v_{np}(E_p)\}_p$ is stationary for any n, $E'_n = \bigcap_{p \geq n} u_{np}(E_p) \neq \varnothing$, hence $\varinjlim E_n \cong \varinjlim E'_n \neq \varnothing$. So there exist $(y_n) \in \varinjlim E_n$, such that g maps (y_n) to (z_n). $\square$

Definition 2.7. Let $L_\bullet = ((L_n)_{n\in\mathbb{N}}, u_{mn}) \in \mathrm{Mod}(A_\bullet)$, then $L_\bullet$ *satisfies uniform ML* (or *AR ML*) if there exists $r \geq 0$, such that for any $n \geq 0$, and any $p \geq n+r$, $\operatorname{Im} u_{n,n+r} = \operatorname{Im} u_{n,p}$.

See [SGA5, V] for a detailed discussion of this notion.

Definition 2.8. Let $L_\bullet \in \mathrm{Mod}(A_\bullet)$, $L_\bullet$ is called *essentially zero* if for any $m \geq 0$, there exists $n \geq m$, such that $u_{mn} : L_n \to L_m$ is the zero map.

Remark. $L_\bullet$ is essentially zero implies $\varinjlim L_n = 0$.

Definition 2.9. Let $L_\bullet \in \mathrm{Mod}(A_\bullet)$, $L_\bullet$ is called *uniformly essentially zero* (or *AR zero*) if there exist $r \geq 0$, such that $u_{n,n+r} = 0$, for any n.

Lemma 2.10. *Let $n \in \mathbb{N}$, define*

$$\varepsilon_n^* : \mathrm{Mod}(A_\bullet) \to \mathrm{Mod}(A) \quad E_\bullet \mapsto E_n,$$

then ε_n^* has a right adjoint functor $\varepsilon_{n*} : \mathrm{Mod}(A) \to \mathrm{Mod}(A_\bullet)$ which is defined as follows:

$$\varepsilon_{n*}(F) = (0 \leftarrow \cdots \leftarrow 0 \leftarrow F \xleftarrow{\mathrm{Id}} \cdots \xleftarrow{\mathrm{Id}} F \leftarrow \cdots)$$

Proof. We need to verify $\mathrm{Hom}(\varepsilon_n^*(E_\bullet, F) \xrightarrow{\sim} \mathrm{Hom}(E_\bullet, \varepsilon_{n*}(F))$. Define

$$\varphi : \mathrm{Hom}(\varepsilon_n^*(E_\bullet), F) \to \mathrm{Hom}(E_\bullet, \varepsilon_{n*}(F))$$

in the following way. Given $f : E_n \to F$, then $\varphi(f)$ is defined by the diagram

$$\begin{tikzcd}
E_\bullet : \arrow[d] & E_0 \arrow[d] & \cdots \arrow[l] & E_{n-1} \arrow[l, "u_{n-1,n}"'] \arrow[d] & E_n \arrow[l] \arrow[d, "f"'] & E_{n+1} \arrow[l, "u_{n,n+1}"'] \arrow[d, "f\circ u_{n,n+1}"] & \cdots \arrow[l] & E_p \arrow[l, "u_{p-1,p}"'] \arrow[d, "f\circ u_{np}"] & \cdots . \arrow[l] \\
\varepsilon_{n*}(F) : & 0 & \cdots \arrow[l] & 0 \arrow[l] & F \arrow[l] & F \arrow[l, "\mathrm{Id}"'] & \cdots \arrow[l, "\mathrm{Id}"'] & F \arrow[l, "\mathrm{Id}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Define

$$\psi : \mathrm{Hom}(E_\bullet, \varepsilon_{n*}(F)) \to \mathrm{Hom}(\varepsilon_n^*(E_\bullet), F)$$
$$(f_i)_{i\geq 0} \mapsto f_n.$$

Easy to verify $\varphi \circ \psi = \mathrm{Id}$, $\psi \circ \varphi = \mathrm{Id}$. $\square$

Remark. As ε_n^* is exact, this implies ε_{n*} maps injectives to injectives.

Lemma 2.11. *There exist enough injectives in* $\mathrm{Mod}(A_\bullet)$*, which are injective in each degree and strict.*

Proof. Let $E_\bullet \in \mathrm{Mod}(A_\bullet)$, then we can choose for each n an injective

$$\varepsilon_n^* E = E_n \hookrightarrow I_n,$$

where I_n is injective. And hence we have injectives

$$E \hookrightarrow \prod_{n\in\mathbb{N}} \varepsilon_{n*}(\varepsilon_n^* E_\bullet) \hookrightarrow \prod_{n\in\mathbb{N}} \varepsilon_{n*} I_n.$$

Let $F_\bullet = \prod\limits_{n\in\mathbb{N}} \varepsilon_{n*} I_n$, then $F_n = \prod\limits_{p\leq n} I_p$, is injective. And $F_\bullet$ is strict. □

Remark. (1)$\varinjlim F_n = \prod I_n$, in particular, is injective.

(2)Thanks to 2.11, we can define the derived functor

$$R\varinjlim : D^+(A_\bullet) \to D^+(A).$$

For any $E_\bullet \in D^+(A_\bullet)$, define $R^q \varinjlim E_\bullet = H^q R \varinjlim E_\bullet$. Then $\varinjlim E_\bullet = R^0 \varinjlim E_\bullet$.

Proposition 2.12. *(a)For any* $E_\bullet \in \mathrm{Mod}(A_\bullet)$*, any* $q > 1$*,* $R^q \varinjlim E = 0$*.*

(b)If E satisfies ML, then $R^q \varinjlim E_\bullet = 0$*, for any* $q > 0$*.*

Proof. We first show(b). We have an exact sequence

$$0 \to E_\bullet \to F_\bullet \to G_\bullet \to 0$$

where $F_\bullet$ is injective and strict, and hence $G_\bullet$ is strict. Consider the long exact sequence of cohomology

$$0 \to \varinjlim E_\bullet \to \varinjlim F_\bullet \to \varinjlim G_\bullet \to R^1 \varinjlim E_\bullet \to R^1 \varinjlim F_\bullet \to \cdots$$

As E satisfies ML, by 2.6, $R^1 \varinjlim E_\bullet \to R^1 \varinjlim F_\bullet$ is injective. Since $R^1 \varinjlim F_\bullet = 0$, $R^1 \varinjlim E_\bullet = 0$. By induction on $q \geq 1$, we get $R^q \varinjlim E = 0$, for all $q \geq 1$.

Then we show (a). For any $E_\bullet \in \mathrm{Mod}(A_\bullet)$ we have an exact sequence

$$0 \to E_\bullet \to F_\bullet \to G_\bullet \to 0$$

with $F_\bullet$ injective and strict, and hence $G_\bullet$ strict. Apply (b) to $G_\bullet$, we have $G_\bullet$ $\varinjlim$-acyclic, which implies the conclusion. □

We have the following generalization. Let $(X, \mathcal{O}_X)$ be a ringed space. Let

$$\mathrm{Mod}(X_\bullet) = \{E_0 \leftarrow \cdots \leftarrow E_m \xleftarrow{u_{mn}} E_n \leftarrow \cdots\}$$

denote the category of inverse systems of $\mathcal{O}_X$-modules. For $E_\bullet \in \mathrm{Mod}(X_\bullet)$, $E_\bullet$ is called *strict* if u_{mn} is surjective for any $m \le n$. $E_\bullet$ is said to satisfy the *Mittag-Leffler condition* (*ML* for short) if for any m, there exists $n \ge m$, such that for any $p \ge n$, $u_{mn}(E_n) = u_{mp}(E_p)$. Define $\varepsilon_n^*(E_\bullet) = E_n$,

$$\varepsilon_{n*}(F) = (0 \leftarrow \cdots \leftarrow 0 \leftarrow F \xleftarrow{\mathrm{Id}} \cdots \xleftarrow{\mathrm{Id}} F \leftarrow \cdots).$$

Using the adjoint functors $(\varepsilon_n^*, \varepsilon_{n*})$ we see again that there exist enough injectives in $\mathrm{Mod}(X_\bullet)$ whose components are injective and which are strict. We can define a derived functor:

$$R\varinjlim : D^+(X_\bullet) \to D^+(X).$$

For any $L_\bullet \in D^+(X_\bullet)$, define $R^q \varinjlim L_\bullet = H^q R \varinjlim L_\bullet$. Then $\varinjlim L_\bullet = R^0 \varinjlim L_\bullet$.

Proposition 2.13. *Let $T : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(\mathbb{Z})$ be the functor defined by $T(E_\bullet) = \Gamma(X, \varinjlim E_n) = \varinjlim \Gamma(X, E_n)$. Then we have a commutative diagram:*

$$\begin{tikzcd}
D^+(X_\bullet) \arrow[r, "R\varinjlim"] \arrow[d, "R\Gamma"'] \arrow[dr, "RT"] & D^+(X) \arrow[d, "R\Gamma"] \\
D^+(\mathbb{Z}_\bullet) \arrow[r, "R\varinjlim"] & D^+(\mathbb{Z})
\end{tikzcd}$$

Proof. For any $E_\bullet \in \mathrm{Mod}(X_\bullet)$,

$$\Gamma(E_\bullet) = (\cdots \leftarrow \Gamma(X, E_n) \leftarrow \cdots) \in \mathrm{Mod}(\mathbb{Z}_\bullet).$$

If $E_\bullet = \prod \varepsilon_{n*}(I_n)$, where I_n is injective for any n, then $\Gamma(E_\bullet) = \prod \varepsilon_{n*}\Gamma(X, I_n)$ is acyclic for $\varinjlim$, So we get $R\varinjlim R\Gamma = RT$.On the other hand, $\varinjlim E_\bullet = \prod I_n$ is injective, so we get $R\Gamma R \varinjlim = RT$.Hence

$$R\varinjlim \circ R\Gamma = R(T) = R\Gamma \circ R\varinjlim .$$

□

For further discussion of $R\varinjlim$, see [N],[J].

Theorem 2.14. *Let X be a scheme, $F_\bullet \in Qcoh(X_\bullet)$. Assume $F_\bullet$ is strict, and for any $i \in \mathbb{Z}$, $H^i(X, F_\bullet) = (\cdots \leftarrow H^i(X, F_n) \leftarrow \cdots)$ satisfies ML, then for any q, the natural map*

$$H^q(X, \varprojlim_n F_n) \to \varprojlim_n H^q(X, F_n)$$

is an isomorphism.

Proof. By 2.13, $R\Gamma(X, R\varprojlim F_\bullet) = R\varprojlim R\Gamma(X, F_\bullet)$, where $R^q \varprojlim F_\bullet$ is the sheaf associated to the presheaf $(U \mapsto R^q \varprojlim \Gamma(U, F_n))$. If U is affine, by Serre, $\Gamma(U, F_n)$ is strict. If U is affine, then $R^q \varprojlim \Gamma(U, F_n) = 0$, for any $q > 0$. Hence $R^q \varprojlim F_n = 0$, for any $q > 0$, which implies $\varprojlim F_n \xrightarrow{\sim} R\varprojlim F_\bullet$. Let $F = \varprojlim F_\bullet$. Consider the spectral sequence

$$E_2^{pq} = R^p \varprojlim H^q(X, F_n) \Rightarrow H^{p+q}(X, F) \quad (*)$$

As $H^q(X, F_\bullet)$ satisfies ML, by 2.12, $E_2^{pq} = 0$ for any $p > 0$. Then $(*)$ degenerates at E_2, and $H^q(X, F) \xrightarrow{\sim} E_2^{0,q} = \varprojlim H^q(X, F_\bullet)$. □

Proof of **??**. Consider the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{I}^{n+1} F \to F \to F_n \to 0,$$

namely

$$H^q(\mathcal{I}^{n+1}) \to H^q(F) \to H^q(F^n) \to H^{q+1}(\mathcal{I}^{n+1} F),$$

where $H^q(-) = H^q(X, -)$. Let

$$R_n = \mathrm{Ker}(H^q(F) \to H^q(F_n)) = \mathrm{Im}(H^q(I^{n+1} F) \to H^q(F)),$$

$$Q_n = \mathrm{Ker}(H^{q+1}(\mathcal{I}^{n+1} F) \to H^{q+1}(F)) = \mathrm{Im}(H^q(F_n) \to H^{q+1}(\mathcal{I}^{n+1} F).$$

The main points are the following:

(1) For all q, the descending filtration R_n of $H^q(F)$ is I-good.

(2) $Q_\bullet$ is AR zero.

(3) For all q, $H^q(F_\bullet)$ satisfies ML.

Let's first show that (1),(2),(3) imply the conclusion. Consider the exact sequence

$$0 \to H^q(F)/R_n \to H^q(F_n) \to Q_n \to 0.$$

By (2) we have $\varinjlim Q_\bullet = 0$, using the left exactness of the functor $\varinjlim$, we get an isomorphism:

$$\varinjlim H^q(F)/R_n \xrightarrow{\sim} \varinjlim H^q(F_n).$$

By (1) the map

$$H^q(F)\hat{} = \varinjlim_n H^q(F)/\mathcal{I}^{n+1}H^q(F) \to H^q(F)/R_n$$

is an isomorphism, so we get

$$H^q(F^n)\hat{} \xrightarrow{\sim} \varinjlim H^q(F_n).$$

Thanks to (3), the assumptions of 2.14 are satisfied, therefore

$$H^q(\hat{X}, \hat{F}) = H^q(\hat{X}, \varinjlim F_n) = H^q(X, \varinjlim F_n) = \varinjlim H^q(X, F_n)$$

.

Proof of (1) Consider the graded module $\bigoplus H^q(\mathcal{I}^{n+1}F)$ over the graded ring $\bigoplus I^n$, it is finitely generated (by the graded variant of the finiteness theorem, applied to $\mathcal{I}F$).The exact sequence

$$\bigoplus_{n\in\mathbb{N}} H^q(\mathcal{I}^{n+1}F) \to \bigoplus_{n\in\mathbb{N}} R_n \to 0$$

implies $\bigoplus_{n\in\mathbb{N}} R_n$ is finitely generated over $\bigoplus_{n\in\mathbb{N}} I^n$, hence (R_n) is I-good.

Proof of (2) Let $B = \bigoplus I^n$.By the finiteness theorem again, $\bigoplus_n H^{q+1}(\mathcal{I}^{n+1}F)$ is finitely generated over B. Since B is noetherian, $\bigoplus Q_n$ as a sub-B-module of $\bigoplus_n H^{q+1}(\mathcal{I}^{n+1}F)$ is also finitely generated, and therefore there exists $r \geq 0$ such that $Q_{n+1} = IQ_n$ for all $n \geq r$. Since Q_r, as a quotient of $H^q(F_k)$ is killed by I^{k+1}(as an A-module), each Q_n is therefore killed by I^{r+1}(as an A-module). For any $a \in I^p$, the composition of the multiplication by a from $H^{q+1}(I^{n+1}F)$ to $H^{q+1}(I^{p+n+1}F)$ with the transition map from $H^{q+1}(I^{p+n+1}F)$ to $H^{q+1}(I^{n+1}F)$ is the multiplication by a in $H^{q+1}(I^{n+1}F)$. Since $Q_{n+r+1} = I^{r+1}Q_n$ for any $n \geq r$, it follows that, for any $n \geq r$, the transition map $Q_{n+r+1} \to Q_n$ is zero, and hence, if $s = 2r+1$, for all n, the transition map $Q_{n+s} \to Q_s$ is zero.

Proof of (3) Consider the exact sequence

$$0 \to H^q(F)/R_n \to H^q(F_n) \to Q_n \to 0$$

As $H^q(F)/R_n$ is strict and Q_n is AR zero, they both satisfies ARML, then the middle term satisfies ARML using the following lemma, whose proof is elementary. □

Lemma 2.15. *Let*

$$0 \to L'_\bullet \to L_\bullet \to L''_\bullet$$

be an exact sequence in $\mathrm{Mod}(A_\bullet)$*. If* $L_\bullet$ *satisfies ML (resp. ARML), so does* L''*, and if* $L'_\bullet$ *and* $L''_\bullet$ *satisfies ML (resp. ARML), then* $L_\bullet$ *satisfies ML(resp. ARML).*

Corollary 2.16 (theorem on formal functions). *Let* $f : X \to Y$ *be a proper morphism of noetherian schemes. Let* y *be a point of* Y*,* $\mathfrak{m} = \mathfrak{m}_{Y,y}$*,* $X_y = X \times_Y \operatorname{Spec} k(y)$ *be the fiber of* f *at* y*. Let* F *be a coherent sheaf of* X*, then the natural map*

$$(R^q f_*(F)_y)\hat{} = \varinjlim R^q f_*(F)_y/\mathfrak{m}^{n+1} R^q f_*(F)_y \to \varinjlim H^q(X_y, F/\mathfrak{m}^{n+1}F)$$

is an isomorphism.

Proof. If y is closed, this is a special case of the comparison theorem. The general case can be reduced to this one by base change. In fact, consider the following commutative diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ y \arrow[r] & \operatorname{Spec} \mathcal{O}_{Y,y} \arrow[r, "g"] & Y \end{tikzcd}$$

As g is flat, we have the base change isomorphism $g^* R^q f_* F \xrightarrow{\sim} R f'_*(h_* F)$, hence

$$R^q f_*(F)_Y)\hat{} = (g^* R^q f_*(F))_y\hat{} \xrightarrow{\sim} (R f'_*(h^* F)_y)\hat{}$$

□

The following special cases in which the assumptions are those of 2.16

Corollary 2.17. $f_*(F)_y\hat{} \to \varinjlim H^0(X_y, F/\mathfrak{m}^{n+1}F)$ *is an isomorphism.*

Corollary 2.18. *Assume* $\dim X_y = r$*, then for any* $q > r$*, there exists an open set* U *contains* y*, such that* $R^q f_*(F)|_U = 0$*.*

Proof. Since

$$R^q f_*(F)_y \hookrightarrow R^q f_*(F)_y\hat{} = \varinjlim H^q(X_y, F/\mathfrak{m}^{n+1}F) = 0,$$

$R^q f_*(F)_y = 0$. As $R^q f_*(F)$ is coherent, there exists an open set U contains y, such that $R^q f_*(F)|_U = 0$. □

2.19. Stein factorization and Zariski's main theorem

Let Y be a scheme and let B be a quasi-coherent $\mathcal{O}_Y$-algebra. Let $Z = \operatorname{Spec} B$. Recall that for any commutative diagram of schemes

$$\begin{tikzcd} & Z = \operatorname{Spec} B \arrow[d, "g"] \\ X \arrow[ur, "f'"] \arrow[r, "f"'] & Y \end{tikzcd} ,$$

the natural morphism

$$\operatorname{Hom}_Y(X, Z) \to \operatorname{Hom}_{\mathcal{O}_Y}(g_*\mathcal{O}_Z, f_*\mathcal{O}_X) \quad (*)$$

is a bijection. In particular, let Y be a locally noetherian scheme, $f : X \to Y$ a proper morphism. Then $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-algebra, the identity map $\operatorname{Id} : f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ corresponds by $(*)$ to a morphism $f' : X \to Y'$ making the following diagram commutes:

$$\begin{tikzcd} & Y' = \operatorname{Spec} f_*\mathcal{O}_X \arrow[d, "g"] \\ X \arrow[ur, "f'"] \arrow[r, "f"'] & Y \end{tikzcd} \quad (**)$$

We have $g_*(f_*\mathcal{O}_X) = g_*\mathcal{O}_{Y'} \xrightarrow{\sim} f_*\mathcal{O}_X$. It follows that the adjoint map $\mathcal{O}_{Y'} \to f'_*\mathcal{O}_X$ is an isomorphism. $(**)$ is called *Stein factorization* of f.

Corollary 2.20 (Zariski's connectedness theorem). *Under the assumptions above, f' has connected, nonempty fibers, (i.e. for any $y' \in X'$, $f'^{-1}(y') \neq \varnothing$, and connected.)*

Proof. We may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$. As

$$\mathcal{O}_{Y,y}\hat{} = f_*(\mathcal{O}_X)_y\hat{} \xrightarrow{\sim} H^0(X_y, \varinjlim \mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X)$$

is a local ring, X_y is connected and nonempty. (If X_y had $n \geq 2$ components, then $H^0(X_y, \varinjlim \mathcal{O}_X/\mathfrak{m}^{n+1}\mathcal{O}_X)$ would be a product of $n \geq 2$ non zero rings, which is impossible since $\mathcal{O}_{Y,y}\hat{}$ is local.) □

Lemma 2.21. *Let A be a local noetherian ring with the residue field $k = A/\mathfrak{m}$, k'/k be a field extension. Then there exists a local noetherian ring A', flat over A, with the residue field $A'/\mathfrak{m}' = k'$.*

Proof. In the case $[k' : k] < \infty$, we reduce to the case $k' = k(y) = k[T]/(f)$, where f is the minimal polynomial of y. Lift f to a monic polynomial $F \in A[T]$, then $A' = A[T]/(F)$ is just the required local ring.

For general case, see [EGA0] III 10.3.1. □

Remark. In the situation of 2.20, the fibers of f' are geometrically connected. This means that for any $y' \in Y'$ and any $y'' = \operatorname{Spec} k(y') \to y'$ (or equivalently, any $y'' \to y'$ with $[k(y'') : k(y'')] < \infty$), the fiber $X_{y'} \times_{y'} y''$ is connected.

Proof of the remark. We may assume $Y' = Y$, i.e. $f_*\mathcal{O}_X = \mathcal{O}_Y$. By base change, using $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$ we may assume Y local, let k' be a finite extension of k. By 2.21, choose $Y' \to Y$ flat with Y' local, with the residue field k'. Base changing by $Y' \to Y$, we get the result. □

Corollary 2.22. *In the situation of 2.20, $\pi_0(X_y) = |g^{-1}(y)|$, where $\pi_0(X_y)$ is the set of connected components of X_y, and $|g^{-1}(y)|$ denotes the underlying finite set of $g^{-1}(y)$.*

Corollary 2.23. *Let $f : X \to Y$ be a proper and surjective morphism of integral schemes, with Y normal. Let ζ be the generic point of X, $\eta = f(\zeta)$ be the generic point of Y. Assume that the generic fiber X_η of f is geometrically connected. Then all fibers of f are geometrically connected.*

Proof. The hypothesis on generic fiber means that the algebraic closure K' of $K = k(\eta)$ in $k(\zeta)$ is a finite radiciel extension of K. Let y be a point of Y, we want to show X_y is geometrically connected. Since $\mathcal{O}_{Y,y}$ is normal and K' is radiciel over K, the normalization A of $\mathcal{O}_{Y,y}$ in K' is a local ring and the residue field extension is radiciel. Since A contains $(f_*\mathcal{O}_X)_y$, $(f_*\mathcal{O}_X)_y$ is a local ring and $(f_*\mathcal{O}_X)_y/\mathfrak{m}_y$ is radiciel over $k(y)$. Therefore X_y is geometrically connected. □

Remark. We can give a simpler argument in the case f is birational. Then $k(\zeta) = k(\eta)$. We have the commutative diagram:

$$\begin{tikzcd} f_*(\mathcal{O}_X)_y \arrow[r, hook] & k(\eta) \\ \mathcal{O}_{Y,y} \arrow[u] \arrow[r, hook] & k(\eta) \arrow[u, equal] \end{tikzcd}$$

where $\mathcal{O}_{Y,y} \to f_*(\mathcal{O}_X)_y$ is finite and $\mathcal{O}_{Y,y}$ normal, which implies $f_*(\mathcal{O}_X)_y = \mathcal{O}_{Y,y}$.

Corollary 2.24. *Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. Consider the Stein factorization of f*

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

(a) Let x be a point of X, $y = f(x)$, $y' = f'(x)$, then x is isolated (i.e. both open and closed) in its fiber $f^{-1}(y)$ if and only if $f'^{-1}(y') = \{x\}$.

(b) Let $U = \{x \in X | x \text{ isolated in } f^{-1}(f(x))\}$, then U is open in X, $U' = f'(U)$ is open in Y', f' induces an isomorphism $f' : U \xrightarrow{\sim} U'$ and $U = f'^{-1}(U')$.

Proof. (a) As $g^{-1}(y)$ is finite and discrete, x is isolated in $f^{-1}(y)$ if and only if x is isolated in $f'^{-1}(y')$. So we may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$. By Zariski's connectedness theorem, $f^{-1}(y)$ is connected and nonempty. So $f^{-1}(y) = \{x\}$ if and only if x is isolated in its fiber.

(b) For any $x \in U$, $f'^{-1}(y') = \{x\}$. So $f' : U \to U'$ is bijective as a map of sets and $f'^{-1}(U') = U$. We may assume $f_*\mathcal{O}_X = \mathcal{O}_Y$ (replace Y by Y'). It is enough to show U is open and f is a local isomorphism. It is enough to show that for any $x \in U$, there exists an open neighborhood T of x such that $T \subset X$ and $f : T \xrightarrow{\sim} f(T)$. Let $V = \operatorname{Spec} B$ be an affine neighborhood of x, such that $f(V) \subset W$, where $W = \operatorname{Spec} A$ is an affine neighborhood of $y = f(x)$. We know that $f^{-1}(y) = \{x\}$. On the other hand, f is closed implies $f(X - V)$ is closed. As $f^{-1}(y) = \{x\}$, $y \notin f(X - V)$, we can find $s \in A$, such that $W_s \cap f(X-V) = \varnothing$, where $W_s = \operatorname{Spec} A_s$, i.e. $f^{-1}(W_s) \subset V$. Then $f^{-1}(W_s) \subset V_s$, in fact, $f^{-1}(W_s) = V_s$. As $f_*\mathcal{O}_X = \mathcal{O}_Y$, $f'_*\mathcal{O}_{V_s} = \mathcal{O}_{W_s}$, which implies $A_s = B_s$. Therefore $V_s \subset U$ and $f : V_s \xrightarrow{\sim} W_s$. $\square$

Corollary 2.25. *Let $f : X \to Y$ be a proper morphism of locally noetherian schemes. Suppose f is quasi-finite (i.e. $f^{-1}(y)$ is finite for any y), then f is finite.*

Proof. In this case $U = X$, and in the Stein factorization of f

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

f' is an isomorphism. $\square$

Corollary 2.26 (Zariski's main theorem). *Let $f : X \to Y$ be a quasi-finite morphism, with Y locally noetherian. Suppose f can be compactified into*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & P \arrow[dl] \\ Y & \end{tikzcd}$$

where j is an open immersion and $P \to Y$ is proper (e.g. f is quasi-projective). Then f can be factored as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

with g finite and i an open immersion.

Proof. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, hook] \arrow[ddd, "f"'] & & P \arrow[ddd, "h"] \\ & U \arrow[ur, hook] \arrow[d] & \\ & h(U) \arrow[dr, hook] & \\ Y & & Y' \arrow[ll, "g"] \end{tikzcd}$$

where U is the set of points of P isolated in their fibers, $P \to Y' \to Y$ is the Stein factorization of $P \to Y$. We have $U \xrightarrow{\sim} h(U)$, and $U \to P$ and

$h(U) \to Y'$ are open immersions. It remains to show $j(X) \subset U$, i.e. for any $x \in X$, x is isolated in $(gh)^{-1}(y)$, where $y = f(x)$. As f is quasi-finite, x is isolated in $f^{-1}(y)$. Since $X \hookrightarrow P$ is open, x is open in $(gh)^{-1}(y)$. As $[k(x) : k(y)] < \infty$, x is closed in $(gh)^{-1}(y)$. □

Exercise. Let A be a henselian noetherian local ring, $S = \operatorname{Spec} A$, s be a closed point, X be a proper scheme over S. Then the natural map

$$\pi_0(X_s) \to \pi_0(X)$$

is a bijection.

3 Grothendieck's existence theorem

Let $Y = \operatorname{Spec} A$ be an affine scheme, where A is a noetherian ring, I is an ideal of A, and $A = \varinjlim A/I^{n+1}$. The problem which is addressed in this section is the following: given a proper adic noetherian $\hat{Y}$-formal scheme $\mathcal{Z}$, when can we assert the existence of a proper scheme Z over Y, whose I-adic completion $\hat{Z} = \varprojlim Z_n$, where $Z_n = Z \times_Y Y_n$, is isomorphic to $\mathcal{Z}$?

The strategy is try to embed $\mathcal{Z}$ in the completion $\hat{P}$ of some projective space P over Y, then try to algebraize the ideal of $\mathcal{Z}$ in $\hat{P}$. We first consider this second problem.

Theorem 3.1. *Consider a commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spf}(A) = \hat{Y} \arrow[r] & Y = \operatorname{Spec} A \end{tikzcd}$$

where $A = \hat{A} = \varinjlim A/I^{n+1}$, X is separated and of finite type over Y. Then the functor

$$E \in Coh(X) \mapsto \hat{E} = i^*(E) \in Coh(\hat{X})$$

defines an equivalence of categories:

$$\{E \in Coh(X) | Supp(E) \text{ proper over} Y\} \xrightarrow{\approx} \{F \in Coh(\hat{X}) | Supp(F) \text{ proper over} \hat{Y}\}$$

(here $Supp(F) = Supp(F_0) \subset X_0$).

We will give the proof in the case f is proper. In this case, the statement $E \mapsto \hat{E}$ gives an equivalence ("GAGA style") $Coh(X) \overset{\approx}{\to} Coh(\hat{X})$. For the general case see [EGAIII] or [T].

Proposition 3.2. *Let A be a noetherian ring, I be an ideal of A. Consider the commutative diagram*

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spf}(A) = \hat{Y} \arrow[r] & Y = \mathrm{Spec}\, A \end{tikzcd}$$

where f is proper. Let F, G be coherent sheaves on X, then for any q, $\mathrm{Ext}^q_{\mathcal{O}_X}(F,G)$ is finitely generated over A, and the natural map $\mathrm{Ext}^q_{\mathcal{O}_X}(F,G) \to \mathrm{Ext}^q_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{G})$ induces an isomorphism

$$\mathrm{Ext}^q_{\mathcal{O}_X}(F,G) \to \mathrm{Ext}^q_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{G})$$

($\hat{F} = i^*F$, $\hat{G} = i^*G$).

Proof. We have

$$\mathrm{Ext}^q(F,G) = H^q\,\mathrm{RHom}(F,G) = H^q(X, \mathrm{RHom}(F,G)).$$

As $\mathrm{R}\mathcal{H}om(F,G) \in D^+(X)_{coh}$ (i.e. $Ext^i(F,G) \in Coh(X)$, for any i), by the finiteness theorem,$H^q(X, \mathrm{R}\mathcal{H}om(F,G))$ is finitely generated over A. We have a spectral sequence

$$E_2^{ij} = H^i(X, Ext^j(F,G)) \Rightarrow \mathrm{Ext}^{i+j}(F,G).$$

Consider the map

$$R\Gamma(X, \mathrm{R}\mathcal{H}om(F,G) \to R\Gamma(\hat{X}, i^*\,\mathrm{R}\mathcal{H}om(F,G))$$

defined by $i : \hat{X} \to X$. As i is flat, and $F, G \in Coh(X)$,

$$i^*\,\mathrm{R}\mathcal{H}om(F,G) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om(i^*F, i^*G) = \mathrm{R}\mathcal{H}om(\hat{F},\hat{G}),$$

in particular,

$$i^*Ext^q(F,G) = Ext^q(F,G)\hat{}\ \xrightarrow{\sim} Ext^q(\hat{F},\hat{G}).$$

Therefore we have a map of spectral sequences

$$\begin{array}{ccc} H^p(X, Ext^q(F,G))\hat{} & \xrightarrow{(**)} & H^p(\hat{X}, Ext^q(\hat{F},\hat{G})) \\ \Downarrow & & \Downarrow \\ \mathrm{Ext}^{p+q}(F,G)\hat{} & \xrightarrow{(*)} & \mathrm{Ext}^{p+q}(\hat{F},\hat{G}) \end{array}$$

As $Ext^q(\hat{F},\hat{G}) = Ext^q(F,G)\hat{}$, $(**)$ is an isomorphism by the comparison theorem.So $(*)$ is an isomorphism. □

Remark. The conclusion holds for X separated and of finite type over Y, and $Supp(F) \cap Supp(G)$ proper over Y (see [EGAIII]).

Proof of 3.1. (In the case X proper over Y)

(1) *Proof of full faithfulness.* Let F, G be coherent sheaves on X Since $\mathrm{Hom}(F,G)$ is finitely generated over A and $A = \hat{A}$, $\mathrm{Hom}(F,G) = \mathrm{Hom}(F,G)\hat{}$. By 3.2 for $q = 0$, the natural map

$$\mathrm{Hom}(F,G) \to \mathrm{Hom}(\hat{F},\hat{G})$$

is an isomorphism. This proves that the $(-)\hat{}$ functor is fully faithful.

(2) *Proof of essential surjectivity.* (a) Consider the projective case. Assume X is projective over Y. Let L be an ample line bundle on X, $\hat{L}$ is an ample line bundle on $\hat{X}$. For M on X (resp. $\hat{X}$), $M(n) = M \otimes L^{\otimes n}$ (resp. $M \otimes \hat{L}^{\otimes n}$). Let $E \in Coh(\hat{X})$. By the following lemma, we have a presentation

$$\mathcal{O}_{\hat{X}}(-m_1)^{r_1} \xrightarrow{u} \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \to E \to 0.$$

By the full faithfulness, there exists a unique $v : \mathcal{O}_X(-m)^{r_1} \to \mathcal{O}_X(-m_0)^{r_0}$, such that $\hat{v} = u$. Let $F = \mathrm{Coker}(v)$, then by the exactness of $(-)\hat{}$, $E = \hat{F}$. □

Lemma 3.3. *Let E be a coherent sheaf on $\hat{X}$, then there exists $m \geq 0$, $r \geq 0$, and a epimorphism*

$$\mathcal{O}_{\hat{X}}(-m)^r \to E \to 0.$$

Proof. Let $\mathcal{I} = I^{\triangle} \subset \mathcal{O}_{\hat{Y}}$, $\mathcal{O}_{\hat{Y}}/\mathcal{I} = \mathcal{O}_{Y_0}$, $E = (E_n)$, $E_n = E/\mathcal{I}^{n+1}E$. We have $\mathrm{gr}^k_{\mathcal{I}}E = \mathcal{I}^k E/\mathcal{I}^{k+1}E \in Coh(X_0)$. Let $M = \mathrm{gr}_{\mathcal{I}}E = \bigoplus_{n \geq 0} \mathrm{gr}^n_{\mathcal{I}}E$. This is a graded module over $f_0^*(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_Y)$, where $\mathrm{gr}_{\mathcal{I}}\mathcal{O}_Y = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$. Since

$$\mathrm{gr}\mathcal{I}^0 E \otimes_{\mathrm{gr}^0_{\mathcal{I}}\mathcal{O}_Y} \mathrm{gr}\mathcal{O}_Y \to \mathrm{gr}_{\mathcal{I}}E, \quad E/\mathcal{I}E \otimes_{\mathcal{O}_{Y_0}} \mathcal{I}^n/\mathcal{I}^{n+1} \mapsto \mathcal{I}^n E/\mathcal{I}^{n+1}E$$

is surjective, M is of finite type over $f_0^*(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_Y)$, hence corresponds to a coherent module M' on X', where X' is defined by the following cartesian diagram

$$\begin{tikzcd} X_0 \arrow[d, "f_0"'] \arrow[dr, phantom, "\square"] & \operatorname{Spec} f_0^*(\mathrm{gr}_{\mathcal{I}}\mathcal{O}_Y) = X' \arrow[l] \arrow[d, "f'"] \\ Y_0 & Spec\mathrm{gr}_{\mathcal{I}}\mathcal{O}_Y = Y' \arrow[l] \end{tikzcd}$$

(with f_0 and f' proper). Since L is ample, $L' = L_0 \otimes \mathcal{O}_{X'}$ is ample, where $L_0 = L \otimes \mathcal{O}_{X_0}$. Apply Serre's vanishing theorem to f', M' and L', there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ and any $q > 0$, $R^q f'_* M'(n) = 0$. Since $R^q f'_*(M'(n)) = (\bigoplus_{k\geq 0} R^q f_{0*}\mathrm{gr}^k_{\mathcal{I}} E(n))$, it follows that for any $k \geq 0$, any $n \geq n_0$, any $q > 0$, $R^q f_{0*}\mathrm{gr}^k_{\mathcal{I}} E(n) = 0$. Apply the $\Gamma(X_0, -)$ to the following exact sequence

$$0 \to \mathrm{gr}^k_{\mathcal{I}} E(n) \to E_{k+1}(n) \to E_k(n) \to 0,$$

we get an exact sequence

$$\Gamma(X_0, E_{k+1}(n)) \to \Gamma(X_0, E_k(n)) \to H^1(X_0, \mathrm{gr}^k_{\mathcal{I}} E(n)).$$

For any $n \geq n_0$, $H^1(X_0, \mathrm{gr}^k_{\mathcal{I}} E(n)) = 0$, and hence

$$\Gamma(\hat{X}, \hat{E}(n)) = \varinjlim_k \Gamma(X_0, E_k(n)) \twoheadrightarrow \Gamma(X_0, E_0(n)).$$

Choose $m \geq n$, such that $E_0(m)$ is generated by a finite number $s_1, \cdots, s_r$ of global sections. Lift these sections to $t_i \in \Gamma(\hat{X}, \hat{E}(m))$, we get $\mathcal{O}^r_{\hat{X}} \to \hat{E}(m)$, $\mathcal{O}^r_{\hat{X}}(-m) \to \hat{E}$, such that $u_0 = u \otimes \mathcal{O}_{X_0} : \mathcal{O}^r_{X_0}(-m) \to E_0$ given by the s_i's is surjective. Recall $I \subset \mathrm{Rad}(A)$, so by Nakayama's lemma, u is surjective. $\square$

Let A be a noetherian ring, I be an ideal of A, and $A = \varprojlim A/I^{n+1}$. Suppose $Y = \operatorname{Spec} A$, and X/Y is a proper morphism. $\hat{Y} = Spf(A)$. Then we have

Theorem 3.4. *The functor $F \mapsto \hat{F}$ from the category of coherent sheaves on X whose support is proper over Y to the category of coherent sheaves on $\hat{X}$ whose support over $\hat{Y}$ is an equivalence.*

Proof. Essentially surjectivity (have been proved)

Case (i): Projective case (have been proved);

Case(ii): General case: We use noetherian induction on X. Assume that for all closed subschemes T of X distinct of X, and all $E \in Coh(\hat{T})$, whose support is proper over $\hat{Y}$ is *algebraizable*, i.e., there exists $F \in Coh(T)$ with proper support over Y, such that $E = \hat{F}$, and we want to show that every $E \in Coh(\hat{X})$ whose support is proper is algebraizable.

By Chow's lemma, there exists a projective and surjective morphism $g : Z \to X$ such that fg is projective, and there exists an open dense subset U of X such that g induces an isomorphism from $g^{-1}(U)$ to U, that is, we have the following commutative diagram:

$$\begin{tikzcd} g^{-1}(U) \arrow[r, hook] \arrow[d, "\wr"] & X' \arrow[d, "g"] \arrow[dd, bend left, "fg"] \\ U \arrow[r, hook] & X \arrow[d, "f"] \\ & S. \end{tikzcd}$$

For $E \in Coh(\hat{X})$, we have the following canonical exact sequence

$$0 \to K \to E \to \hat{g}_*\hat{g}^*E \to C \to 0 \qquad (*).$$

We will show the following points:

(1) $\hat{g}_*\hat{g}^*E$ is algebraizable ;

(2) Let $\mathcal{J}$ be the ideal of $\mathcal{O}_X$ defining $T = X - U$, with $T = T_{red}$, then there exits a $N \geq 0$ such that $\hat{\mathcal{J}}^N C = \hat{\mathcal{J}}^N K = 0$;

(3)Let T' be the closed subscheme of X defined by the ideal sheaf $\mathcal{J}^N$, then $C, K \in Coh(\hat{T'})$, and C, K are algebraizable ;

(4) The category of algebraizable coherent sheaves on X is stable under kernel, cokernel and extension.

We claim that $(1) - (4)$ imply the theorem. In fact, we can write $(*)$ to two short exact sequences:

$$0 \to K \to E \to H \to 0,$$

$$0 \to H \to \hat{g}_*\hat{g}^*E \to C \to 0,$$

then by $(1), (3), (4)$, H is algebraizable, and by $(3), (4)$ we get that E is algebraizable.

Now we only need to prove conditions $(1) - (4)$.

For (1), Since $\hat{g}^*E \in Coh(\hat{Z})$ and g is projective, by case (i), we know $\hat{g}^*E = \hat{F}$ for some $F \in Coh(Z)$. By the comparison theorem, $\hat{g}_*\hat{F} = (\widehat{g_*F})$, and by the finiteness theorem, $g_*F \in Coh(X)$, thus we get (1).

To Prove (2), we may work locally on $\hat{X}$, assume $\hat{X} = \mathrm{Spf}(B)$, where B is an IB-adic noetherian ring. Then $E = \hat{F}$ for a coherent sheaf F on $\operatorname{Spec} B$, and we have an exact sequence corresponding to $(*)$

$$0 \to K' \to F \to g_*g^*F \to C' \to 0,$$

where $\hat{K'} = K$, $\hat{C'} = C$. Since $g|_U$ is an isomorphism, $C'|_U = K'|_U = 0$, then C' and K' are killed by a positive power of J, and we get (2).

In view of (2), (3) follows from the noetherian induction assumption.

In (4), for $u : E_1 \to E_2$ with $E_1,\ E_2 \in Coh(\hat{X})$ and $E_1 = \hat{F}_1$, $E_2 = \hat{F}_2$. By full faithfulness, there exists $u : F_1 \to F_2$ such that $u = \hat{v}$. So

$$\widehat{\mathrm{Ker} v} = \mathrm{Ker} u,\ \ \widehat{\mathrm{Coker}\, v} = \mathrm{Coker}\, u.$$

And the stability under extension follows from the isomorphism

$$\mathrm{Ext}^1(F, G) \xrightarrow{\sim} \mathrm{Ext}^1(\hat{F}, \hat{G}).$$

□

3.5. Algebraizable of closed formal scheme

Let $\mathfrak{X}$ be a locally noetherian formal scheme. A closed formal scheme of $\mathfrak{X}$ is a formal subscheme $(\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}})$ such that $\mathfrak{Z} = \operatorname{Supp} \mathcal{O}_{\mathfrak{Z}}$, $\mathcal{O}_{\mathfrak{Z}} = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}$ with $\mathcal{J}$ being a coherent ideal of $\mathfrak{X}$. If $\mathfrak{X} = \mathrm{Spf}(A)$, then $\mathfrak{Z} = \mathrm{Spf}(A/\mathcal{J})$ for some ideal of A. If $\mathcal{I}$ is an ideal of definition of $\mathfrak{X}$, and $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$, so that $\mathfrak{X} = \varinjlim X_n$. Let Z_n be the closed subscheme of X_n such that $\mathcal{O}_{Z_n} = \mathcal{O}_{\mathfrak{Z}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}(X_n)$, then we have

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathfrak{Z} \arrow[d, hook] \\
X_0 \arrow[r, hook] & \cdots \arrow[r, hook] & X_n \arrow[r, hook] & \cdots \arrow[r, hook] & \mathfrak{X}
\end{tikzcd}$$

and $\mathfrak{Z} = \varinjlim Z_n$. Conversely, given a morphism of inductive systems as follows,

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, hook, "i_0"] \arrow[rr, phantom, "\square"] & \cdots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, hook, "i_n"] & \cdots \\
X_0 \arrow[r, hook] & \cdots \arrow[r, hook] & X_n \arrow[r, hook] & \cdots
\end{tikzcd}$$

such that $i_n's$ are closed immersions, and each square is cartesian, then $\mathfrak{Z} = \varinjlim Z_n \hookrightarrow \mathfrak{X}$ is a closed formal scheme of $\mathfrak{X}$.

Corollary 3.6. *Suppose $A = \hat{A}$, $Y = SpecA$, and X/Y is proper. Then the map $Z \mapsto \hat{Z}$ is a bijection from the set of closed subschemes of X to the set of closed subschemes of $\hat{X}$.*

Proof. We only need prove surjectivity. Let $\mathfrak{Z}$ be a closed formal subscheme of $\hat{X}$. It corresponds to a coherent quotient

$$\mathcal{O}_{\hat{X}} \xrightarrow{u} \mathcal{O}_Z \to 0.$$

By3.4, there exists a unique coherent $\mathcal{O}_X$-module F such that $\hat{F} = \mathcal{O}_{\mathfrak{Z}}$, and a unique $v : \mathcal{O}_X \to F$ such that $\hat{v} = u$. Since u_0 is surjective, so is v, hence $F = \mathcal{O}_Z$ for a closed subscheme Z of X such that $\hat{Z} = \mathfrak{Z}$. □

3.7. Algebraizable of finite morphism Let $\mathfrak{X}$ be a locally noetherian formal scheme, and $\mathfrak{Z} \to \mathfrak{X}$ be a morphism of locally noetherian formal schemes. f is called finite if f is an adic map and $f_0 : Z_0 \to X_0$ is finite. We have

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathfrak{Z} \arrow[d, hook] \\ X_0 \arrow[r, hook] & \cdots \arrow[r, hook] & X_n \arrow[r, hook] & \cdots \arrow[r, hook] & \mathfrak{X}, \end{tikzcd}$$

Obviously f is finite if and only if f is adic and for all n, f_n is finite. This is also equivalent to saying that for every open subset $U = \mathrm{Spf}(A) \subset \mathfrak{X}$, $\mathfrak{Z}|_U = f^{-1}(U) = \mathrm{Spf}(B)$, where B is a finite A-algebra and IB-adic, I being a n ideal of definition of A. To see the second equivalence, one reduce to affine case, see $[B]III, \S 2, \text{no.}11$. We have that if $\mathfrak{X} = \hat{X}$, and Z is a finite scheme over X, then $\hat{Z}$ is finite over $\hat{X}$.

Corollary 3.8. *Suppose $Y = \operatorname{Spec} A$, $A = \hat{A}$, and X/Y is proper. Then the functor $Z \mapsto \hat{Z}$ is an equivalence from the category of finite X-schemes to the category of finite $\hat{X}$-formal schemes.*

Proof. Full faithfulness: let B, C be finite $\mathcal{O}_X$-algebras, and $u : \hat{B} \to \hat{C}$ is a map of $\mathcal{O}_{\hat{X}}$-algebras. By full faithfulness for modules, one can find

$v \in \mathrm{Hom}_{\mathcal{O}_C-mod}(B, C)$ such that $u = \hat{v}$. v is automatically a map of $\mathcal{O}_X$-algebras. In fact, we need check that the following two diagrams

$$\begin{tikzcd} B \otimes B \arrow[r] \arrow[d, "v\otimes v"] & B \arrow[d, "v"] & (*) \\ C \otimes C \arrow[r] & C \end{tikzcd}$$

and

$$\begin{tikzcd} B \arrow[rr, "v"] & & C & (**) \\ & \mathcal{O}_X \arrow[ul, "\varepsilon"] \arrow[ur, "\eta"'] \end{tikzcd}$$

are commutative, where η, ε are canonical morphisms as $\mathcal{O}_X$-algebras. We have that $(*)^\wedge$ and $(**)^\wedge$ are commutative, by full faithfulness, so are $(*)$ and $(**)$.

Essential surjectivity: let $\mathfrak{B}$ be a finite $\mathcal{O}_{\hat{X}}$-algebra, then by 3.4, there exists a coherent $\mathcal{O}_X$-module B such that $\hat{B} = \mathfrak{B}\mathfrak{B} = \hat{B}$ as $\mathcal{O}_{\hat{X}}$-module. The maps $\mathfrak{B} \otimes \mathfrak{B} \to \mathfrak{B}$ and $\mathcal{O}_{\hat{X}} \to \mathfrak{B}$ giving the algebra structure on $\mathfrak{B}$ is uniquely algebraized to maps $B \otimes B \to B$ and $\mathcal{O}_X \to B$. So we get a coherent $\mathcal{O}_X$-algebra B, such that $\hat{B} = \mathfrak{B}$. $\square$

Corollary 3.9. *Let Y be as above. Then the functor $X \to \hat{X}$ from the category of all proper Y-schemes to the category of all proper $\hat{Y}$-formal schemes is fully faithful.*

Proof. Let X_1, X_2 be proper schemes over Y, we want to show that

$$Hom_Y(X_1, X_2) \to Hom_{\hat{Y}}(\hat{X}_1, \hat{X}_2)$$

is bijective. If $f : X_1 \to X_2$ is a Y-morphism, we have the following diagram

$$\begin{tikzcd} & X_1 \times_Y X_2 \arrow[dl, shift left, "pr_1"] \arrow[dr, "pr_2"] & \\ X_1 \arrow[ur, shift left, "\Gamma_f"] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

where $\Gamma_f : X_1 \to X_1 \times_Y X_2$ is the graph morphism of f. It is a closed immersion and pr_1 induces an isomorphism $\Gamma_f(X_1) \xrightarrow{\sim} X_1$. Conversely, for any closed subscheme $\Gamma \subset X_1 \times_Y X_2$ such that $pr_1 : \Gamma \xrightarrow{\sim} X_1$, $\Gamma = \Gamma_f$ where $f = pr_2 \circ pr_1^{-1} \in \mathrm{Hom}_Y(X_1, X_2)$.

On the other hand, apply Corollary 3.8 to $X_1 \times_Y X_2$, we have that the set of all closed subschemes of $X_1 \times X_2$ is bijective to the set of all closed formal subschemes of $(X_1 \times_Y X_2)^\wedge = \hat{X}_1 \times \hat{X}_2$, and $pr_1 : \Gamma \xrightarrow{\sim} X_1$ if and only if $\widehat{pr_1} : \hat{\Gamma} \xrightarrow{\sim} \hat{X}$. So we get the correspondence

$$\begin{array}{c} \{''\textit{graph like}'' \text{ closed formal subschemes of } X_1 \times X_2\} \\ \updownarrow \\ \{''\textit{graph like}'' \text{ closed formal subschemes of } \hat{X}_1 \times \hat{X}_2\}, \end{array}$$

hence the conclusion. □

Theorem 3.10 (Grothendieck, 1959). *Let $A = \hat{A}$ be a complete noetherian ring, $Y = \operatorname{Spec} A$. Let $fx = \varprojlim X_n$ be a proper, adic formal $\hat{Y}$-scheme, where $X_n = \mathfrak{X} \times_{\hat{Y}} Y_n$. Let L be a line bundle on $\mathfrak{X}$ such that $L_0 = L \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_0}$ is ample. Then $\mathfrak{X}$ is algebraizable, and if X is a proper Y-scheme such that $\mathfrak{X} = \hat{X}$, then there exists an unique line bundle M on X such that $L = \hat{M}$. Moreover, X is projective over Y, and M is ample.*

Proof. (??)The proof is not clear.

$S = \mathrm{gr}_I \mathcal{O}_Y = \bigoplus_{n \in \mathbb{N}} I^n / I^{n+1}$ can be viewed as an $\mathcal{O}_{Y_0}$-algebra.

$$\begin{tikzcd} X_0 \arrow[d, "f_0"] \arrow[dr, phantom, "\square"] & \operatorname{Spec} f_0^* S = \tilde{X} \arrow[l] \arrow[d, "\tilde{f}"'] \\ Y_0 & \operatorname{Spec} S = \tilde{Y} \arrow[l] \end{tikzcd}$$

$\mathcal{O}_{X_0}(1) = L_0$ is ample, $\tilde{L} = \phi^* L_0$ is ample.

By Serre-Grothendieck vanishing theorem, there exists n_0 such that for all $n \geq n_0$ and $q > 0$,

$$R^q \hat{f}_*(\tilde{E}(n)) = 0.$$

Since

$$\begin{aligned} R^q \hat{f}_*(\tilde{E}(n)) &= \bigoplus_{n \in \mathbb{N}} R^q f_{0*} \mathrm{gr}_I^k E(n) \\ &= \bigoplus_{n \in \mathbb{N}} H^q(X_0, \mathrm{gr}_I^k E(n)), \end{aligned}$$

and the exact sequence

$$0 \to \operatorname{gr}_I^k E(n) \to E_{k+1}(n) \to E_k(n) \to 0,$$

so in particular, for $n \geq n_0$,

$$\Gamma(X_0, E_{k+1}(n)) \to \Gamma(X_0, E_k(n))$$

is surjective, hence

$$\Gamma(\mathfrak{X}, E(n)) = \varprojlim_k \Gamma(X_k, E_k(n)) \to \Gamma(X_), E_0(n))$$

is surjective for all $n \geq n_0$. Apply this to $E = \mathcal{O}_{\mathfrak{X}}$, then there exists n_0 such that for all $n \geq 0$, $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}(n)) \to \Gamma(X_0, \mathcal{O}_{X_0}(n))$ is surjective. Choose $n \geq n_0$ such that $\mathcal{O}_{X_0}(n)$ is very ample corresponding to

$$\begin{tikzcd} X_0 \arrow[r, hook, "i_0"] \arrow[dr] & P_0 = \mathbb{P}^r_{Y_0} \arrow[d] \\ & Y_0 \end{tikzcd}$$

where i_0 is a close immersion. $i_0^*\mathcal{O}_{P_0}(1) = \mathcal{O}_{X_0}(n)$. $E_0(n_0)$ is generated by a finite number of global sections, lifting these sections to $H^0(\hat{X}, E(n_0))$, we find a map

$$u : \mathcal{O}^r_{\hat{X}} \to \mathcal{O}_{\mathfrak{X}}(n)$$

such that $u_0 = u \otimes \mathcal{O}_{X_0} : \mathcal{O}^r_{X_0} \to E_(n)$ is surjective. By Nakayama's lemma, $u_k = u \otimes \mathcal{O}_{X_n}$ is surjective for all k.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "i_0"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, "i_k"] & \cdots \arrow[r] & \mathfrak{X} \arrow[r] \arrow[d, "i"] & X \arrow[d, "j"] \\ \mathbb{P}^r_{Y_0} = P_0 \arrow[r] & \cdots \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & \hat{P} \arrow[r] & P = \mathbb{P}^r_Y, \end{tikzcd}$$

Since i_0 ia a closed immersion, i_k is a closed immersion for all k, thus $\mathfrak{X}$ is a closed formal subscheme of $\hat{P}$. By 3.6, $\mathfrak{X}$ is algebraizable, i.e., $\mathfrak{X} = \hat{X}$ for some closed subscheme of P. Then by 3.4, there exists a unique line bundle M on X such that $L = \hat{M}$, and

$$\begin{aligned} (M^{\otimes n})^\wedge &= \hat{M}^{\otimes n} \\ &= \mathcal{O}_{\hat{X}}(n) \\ &= \hat{j}^*\mathcal{O}_{\hat{P}}(1) \\ &= (j^*\mathcal{O}_P(1))^\wedge, \end{aligned}$$

by 3.4 again, $M^{\otimes n} \simeq j^*\mathcal{O}_P(1)$, then $M^{\otimes n}$ is very ample, hence M is ample. $\square$

4 Application to lifting problems

Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Let $T = \operatorname{Spec} A$, $S_n = \operatorname{Spec} A_n$, $S = \operatorname{Spf}(\hat{A})$ where $A_n = A/\mathfrak{m}^{n+1}$. Given a scheme X_0 proper over $S_0 = \operatorname{Spec} k$, we have the following diagram

$$\begin{tikzcd} X_0 \arrow[d] \\ S_0 \arrow[r] & \cdots \arrow[r] & S_n \arrow[r] & \hat{S} \arrow[r] & S \arrow[r] & T. \end{tikzcd}$$

There are three problems:

Pb1: Find a proper flat lifting of X_0 to T;

Pb2: Find a proper flat lifting of X_0 to S, for X/S proper, flat, such that $X \times_S S_0 = X_0$;

Pb3: Find $\mathfrak{X}$ proper, flat over $\hat{S}$ lifting X_0. Try to lift X_0 to an inductive system of (proper and flat) schemes X_n such that $X_{n+1} \times_{S_{n+1}} S_n = X_n$. For flatness, X_n/S_n is flat for all n, if and if only in the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathfrak{X} \arrow[d] \\ S_0 \arrow[r] & \cdots \arrow[r] & S_n \arrow[r] & S_{n+1} \arrow[r] & \cdots \arrow[r] & S \end{tikzcd},$$

X_n/S_n flat for all n.

Suppose X_n has been constructed, X_n is flat, proper over S_n lifting X_0. We want to find X_{n+1} lifting X_0 to S_{n+1}. Encounters an obstruction $o(X_n, i_n)$, where $i_n : S_n \to S_{n+1}$, in some global cohomology group of X_0. For example, if X_n is smooth,

$$\begin{aligned} o(X_n, i_n) &\in H^2(X_n, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes_{\mathcal{O}_{S_0}} T_{X_0/S_0}) \\ &= H^2(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes_{\mathcal{O}_{S_0}} T_{X_0/S_0}) \\ &= H^2(X_0, T_{X_0/S_0}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}. \end{aligned}$$

Pb: Assume X_0 has been lifted to $\mathfrak{X}$ proper, flat over $\hat{S}$, algebraize this $\mathfrak{X}$, find X proper, flat over S such that $\hat{X} = \mathfrak{X}$ (Note that if $\hat{X} = \mathfrak{X}$, X is proper over S, then X will be flat over S).

Pb4: Lift L_0 to L on $\mathfrak{X}$, where L_0 is a line bundle on X_0. Suppose L_0 has been lifted to L_n, we want to lift L_n to L_{n+1} on X_{n+1}. Encounters an obstruction

$$o(L_n, i_n) \in H^2(X_0, \mathcal{O}_{X_0}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}.$$

4.1. Deformation of vector bundles Let $i : X_0 \to X$ be a thickening of order one, defined by an ideal I of square zero. Let E_0 be a vector bundle on X_0. We want to find a vector bundle E on X such that $\mathcal{O}_{X_0} \otimes E = E/IE = E_0$. More precisely, by a lifting of E_0 to X, we mean a pair of a vector bundle E on X and an $\mathcal{O}_X$linear map $E \to i_* E_0$ such that $i^* E \xrightarrow{\sim} E_0$.

Suppose $E \in Mod(X)$ is a lifting of E_0 to X, we have a short exact sequence

$$0 \to IE \to E \to i_* E_0 \to 0,$$

since E is flat over X, $I \otimes E_0 \xrightarrow{\sim} IE$, then we get

$$0 \to I \otimes E_0 \to E \to i_* E_0 \to 0.$$

Proposition 4.2. *Let $i : X_0 \to X$ be as above.*

(a) Let E, F be vector bundles on X, $E_0 = i^* E$, $F_0 = i^* F$, and $u_0 : E_0 \to F_0$ be a $\mathcal{O}_{X_0}$-linear map. There exists an obstruction

$$o(u_0, i) \in Ext^1_{\mathcal{O}_{X_0}}(E_0, I \otimes F_0)$$

such that $o(u_0, i) = 0$ if and only if there exists $u : E \to F$ such that $u \otimes \mathcal{O}_{X_0} = u_0$. When $o(u_0, i) = 0$, the set of extensions u is an affine space under $Ext^0(E_0, I \otimes F_0)$.

Note: $Ext^i_{\mathcal{O}_{X_0}}(E_0, I \otimes F_0) = H^i(X_0, I \otimes \mathcal{H}om(E_0, F_0))$.

(b) Let E_0 be a vector bundle on X_0. There is an obstruction

$$o(E_0, i) \in Ext^2(E_0, I \otimes F_0) = H^2(X_0, I \otimes \mathcal{E}nd(E_0)$$

such that $o(E_0, i) = 0$ if and only if there exists a vector bundle E lifting E_0. When $o(E_0, i) = 0$, the set of isomorphisms of E is an affine space under $Ext^1(X, I \otimes \mathcal{E}nd(E_0))$, and the group of automorphisms of E is identified by $a \mapsto a - Id$ with $Ext^0(E_0, I \otimes \mathcal{E}nd(E_0))$.

Proof. (a) We want to find u such that the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes E_0 \arrow[r] \arrow[d, "Id\otimes u_0"'] & E \arrow[r] \arrow[d, dashed, "u"'] & E_0 \arrow[r] \arrow[d, "u_0"'] & 0 \\
0 \arrow[r] & I \otimes F_0 \arrow[r] & F \arrow[r] & F_0 \arrow[r] & 0.
\end{tikzcd}$$

Denote M by the pushout of E and $I \otimes F_0$ over $I \otimes E_0$, similarly let N be the pullback of F and E_0 over F_0, we get

$$\begin{tikzcd} I \otimes E_0 \arrow[r] \arrow[d] & E \arrow[r, "f"] \arrow[d] & E_0, \\ I \otimes F_0 \arrow[r] & M \arrow[ur, "\alpha"] & \end{tikzcd}$$

and

$$\begin{tikzcd} & N \arrow[r] \arrow[d] & E_0 \arrow[d] \\ I \otimes F_0 \arrow[ur, "\beta"] \arrow[r, "g"] & F \arrow[r] & F_0, \end{tikzcd}$$

where $\alpha = (f, 0)$, $\beta = (0, g)$. One can easily check that existence of u is equivalent to that

$$[0 \to E \otimes F_0 \to M \to F_0 \to 0] - [0 \to I \otimes F_0 \to N \to F_0 \to 0] = 0$$

in $\mathrm{Ext}^1(E_0, I \otimes F_0)$, denote by $o(u_0, i)$, then it is the desired one.

When $o(u_0, i) = 0$, fix one extension v, we have that the composite morphism

$$E \xrightarrow{u-v} F \longrightarrow F_0$$

is zero, so can be factored as

$$\begin{tikzcd} & E \arrow[dl] \arrow[d] \\ I \otimes F_0 \arrow[r, "u-v"] & F. \end{tikzcd}$$

Thus we get a group action

$$u \longmapsto u - v$$

under which the set of extensions is an affine space.

(b) For simplicity, we assume that X_0 (or X, this is equivalent) is *separated*. Choose $\mathcal{U} = (U_i)_{i \in S}$, $(E_i)_{i \in S}$ such that $\mathcal{U}$ is an affine open cover of X_0 and E_i is a vector bundle on $X|_{U_i}$ extending $E|_{U_i}$. Since X_0 is separated, $U_{ij} = U_i \cap U_j$ is affine, so by (a) one can find an isomorphism $g_{ij} : E_i|_{U_{ij}} \xrightarrow{\sim} E_j|_{U_{ij}}$ inducing the identity on $E_0|_{U_{ij}}$. On $U_{ijk} = U_i \cap U_j \cap U_k$, $c_{ijk} = g_{ij} g_{ik}^{-1} g_{jk} \in \mathrm{Aut}(E_j|U_{ijk})$. $c_{ijk} - Id \in H^0(U_{ijk}, I \otimes \mathcal{E}nd(E_0))$, and

$(c_{ijk}) = d((h_ij))$ for some $h_{ij} \in H^0(U_{ij}, I \otimes \mathcal{E}nd(E_0))$ if and only if the (h_{ij}) can be modified into a gluing data for the (E_i), i.e., (h_{ij}) can be replaced by $g'_{ij} = g_{ij} + h_{ij}$ such that $g'_{ij} g'^{-1}_{ik} g'_{jk} = Id$. Then gluing (E_i), we get global E extending E_0. Thus $d(c_{ijk}) = o(E_0, i) \in H^2(X, I \otimes \mathcal{E}nd(E_0))$ is the desired obstruction, and it does not depend on the choices. If E_1 and E_2 are two extendings of E_0 over X, then by (a) the isomorphisms from E_1 to E_2 form a torsor under $I \otimes \mathcal{E}nd(E_0)$. □

Remark 4.3. (1) If L is a line bundle, then $\mathcal{O}_{X_0} \xrightarrow{\sim} \mathcal{E}nd(L_0)$, so $o(L_0, i) \in H^2(X_0, I)$.

(2) Let L_0, M_0 be line bundles, then

$$o(L_0 \otimes M_i, i) = o(L_0, i) + o(M_i, i).$$

Corollary 4.4. *Let A be a complete local noetherian ring, with maximal ideal $\mathfrak{m}$ and residue field k. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{Spf}(A) = \varinjlim S_n$, where $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$, $S_0 = \operatorname{Spec} k = s$. Let $\mathfrak{X}$ be a $\mathfrak{m}$-adic formal $\hat{S}$-scheme, and proper, flat over $\hat{S}$.*

(a) If X/S is proper and $\hat{X} = \mathfrak{X}$, then X is flat over S.

(b) If $H^2(X_0, \mathcal{O}_{X_0}) = 0$, then any line bundle L_0 on X_0 can be lifted to a line bundle L on X, and L is unique up to (non unique) isomorphism if $H^1(X_0, \mathcal{O}_{X_0}) = 0$. Moreover, if L_0 is ample, then any lifting L is ample and X is projective.

Proof. (a) Each $X_n = S_n \times_S X$ is flat over S_n ($=S_n \times_{\hat{S}} \mathfrak{X}$), so by flatness criterion, for all $x \in X_0 = X_s$, $\mathcal{O}_{X,x}$ is flat over A. By openness of flatness introduced later, X is flat over S on an open neighborhood U of X_s, since X/S is proper, we get $U = X$.

Theorem 4.5 (openness of flatness). *Let Y be a locally noetherian scheme, $f : X \to Y$ be locally of finite type, $\mathcal{F} \in Coh(X)$. Then the set of $x \in X$ such that $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$ is open in X.*

Proof. Uses flatness criterion and the following theorem. □

Theorem 4.6 (Generic flatness). *Let $f : X \to Y$ be of finite type, where Y is a locally noetherian integral scheme, $\mathcal{F} \in Coh(X)$. Then there exists an open nonempty subset V of Y such that $\mathcal{F}|_{f^{-1}(V)}$ is flat over V.*

Proof. See [EGA], IV, 6.9.1. □

(b)

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, "f_n"] \arrow[dr, phantom, "\square"] & X_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & {} \\ S_0 \arrow[r] & \cdots \arrow[r] & S_n \arrow[r] & S_{n+1} \arrow[r] & {} \end{tikzcd}$$

We want to lift L_0 to $\mathfrak{X}$. Assume L_0 has been lifted into L_n on X_n.

$$\begin{aligned} o(L_n, i_n) &\in H^2(X_n, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes \mathcal{O}_{X_n}) \\ &= H^2(X_0, \mathcal{O}_{X_0}) \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+1} \\ &= 0, \end{aligned}$$

so we get $\mathcal{L}$ on $\mathfrak{X}$ lifting L_0, $\mathfrak{X} = \hat{X}$. And by the existence theorem, $\mathcal{L} = \hat{L}$ for some $L \in \mathit{Coh}(X)$. When L_0 is ample, L is ample.

Uniqueness of lifting in case $H^1(X_0, \mathcal{O}_{X_0}) = 0$: Suppose L, M be two liftings of L_0, we will find an isomorphism $L \xrightarrow{\sim} M$ inducing identity on L_0. Suppose we have got $u_n : L_n \xrightarrow{\sim} M_n$, since $H^1(X_0, \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \otimes \mathcal{O}_{X_0}) = 0$, we get an isomorphism $u_{n+1} : L_{n+1} \xrightarrow{\sim} M_{n+1}$ lifting u_n. □

Corollary 4.7. *Let S be as in 4.4, X_0/S_0 be proper and smooth. Then*

(a) If $H^2(X_0, T_{X_0/S_0}) = 0$, then there exists a proper and flat formal scheme $\mathfrak{X}$ lifting X_0.

(b) If moreover $H^2(X_0, \mathcal{O}_{X_0}) = 0$, and X_0 is projective, then there exists a proper and smooth scheme X/S lifting X_0.

Proof. (a) We have the diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & X_n \arrow[d] & \\ S_0 \arrow[r, hook] & \cdots \arrow[r, hook] & S_n \arrow[r, hook] & S_{n+1}. \end{tikzcd}$$

Suppose X_n/S_n is a smooth lifting X_0, then

$$\begin{aligned} o(X_n, i_n) &\in H^2(X_n, T_{X_n/S_n} \otimes_{\mathcal{O}_{S_n}} \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) \\ &= H^2(X_0, T_{X_0/S_0} \otimes_k \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}) \\ &= H^2(X_0, T_{X_0/S_0}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} \\ &= 0. \end{aligned}$$

so we have $\mathfrak{X} = \varinjlim X_n$ is a lifting of X_0.

Note: If $H^1(X_0, T_{X_0/S_0}) = 0$, then the lifting $\mathfrak{X}$ is unique up to isomorphism (unique isomorphism if $H^0(X_0, T_{X_0/S_0}) = 0$).

(b) Take L_0 being ample on X, lifting it to $\mathcal{L}$ on $\mathfrak{X}$, then algebraizing it by 3.10, we get $\mathfrak{X} = \hat{X}$, $\mathcal{L} = \hat{L}$, and L is ample over X.

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S_0 \arrow[r, hook] & S \end{tikzcd}$$

where X/S is projective and flat. Since X/S is flat, X_0/S_0 is smooth, X is smooth at every point $x \in X_0 = X_s$, then X is smooth over S in an open neighborhood U of X_s, hence $U = X$ since X/S is proper. $\square$

4.8. Curves Let Y be a locally noetherian scheme. By a curve over Y we mean a morphism $f : X \to Y$ which is flat, separated and of finite type, with relative dimension 1.

Corollary 4.9. *Suppose $X_0/S_0 = \operatorname{Spec} k$ is a proper, smooth, geometrically connected curve of genus g. Then there exists a proper smooth curve X/S with geometrically connected fibers of genus g lifting X_0.*

Proof. From
$$H^2(X_0, T_{X_0/S_0}) = 0, \quad H^2(X_0, \mathcal{O}_{X_0}) = 0,$$
we get a projective smooth lifting X/S, with fibers smooth of dim 1. Have to show $f_*\mathcal{O}_X = \mathcal{O}_S$, where $F : X \to S$ is the structure morphism.

Lemma 4.10. *Suppose $f : X \to Y$ is proper an flat, and we have the following diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[dl, "g"'] \arrow[dr, phantom, "\square"] & X'' \arrow[l] \arrow[dl, "g'"] \\ Y & Y' \arrow[l, "h"] & \end{tikzcd},$$

where $X' = \operatorname{Spec} f_*\mathcal{O}_X \to Y$ is the stein factorization of $X \to Y$. Assume the fibres of f are geometrically reduced. Then g is finite étale and $g_*\mathcal{O}_{X'}$ commutes with any base change, i.e., $h^*g_*\mathcal{O}_{X'} = g'_*\mathcal{O}_{X''}$. In particular, f is cohomologically flat in degree zero, and the following conditions are equivalent:

(i) $f_*\mathcal{O}_Y = \mathcal{O}_Y$;

(ii) the fibers of f are geometrically connected.

Proof. See [SGA 1], X, 1.2. □

Now we want to prove the fibers are of genus g. In fact, since $Rf_*\mathcal{O}_X$ is perfect, of tor-amplitude in $[0,1]$, by lemma 4.10, and that S is locally noetherian, one can prove that $R^1f_*(\mathcal{O}_X)$ is locally of finite rank g and commutes with base change. In general, let $E \in D(S)$ be a perfect complex of tor-amplitude in $[0,1]$, and assume for some $s \in S$, the canonical map

$$\alpha^0(s) : k(s) \otimes H^0(E) \to H^0(k(s) \otimes^L E)$$

is surjective. Then $\alpha^0(s)$ is an isomorphism, and $H^0(E)$, $H^1(E)$ are locally free of finite type around s (and commute with base change). □

Note: $f_*\Omega^1_{X/S}$ is locally free of rank g, since by Grothendieck's duality theorem, we have

$$\begin{tikzcd} f_*\Omega^1_{X/S} \otimes R^1f_*\mathcal{O}_X \arrow[r] \arrow[dr] & R^1f_*\Omega^1_{X/S} \arrow[d, "\simeq"] \\ & \mathcal{O}_S. \end{tikzcd}$$

Proper smooth curves in positive characteristic can be lifted to characteristic zero: if k is of characteristic $p > 0$, there exists a complete discrete ring A which is flat over $\mathbb{Z}_p$ with residue field $k = A/pA$ (Cohen ring of k). Denote by $K = \mathrm{Frac}(A)$. When k is perfect, we can take $A = W(k) = \{(a_0, a_1, \cdots)|\ a_i \in k\}$, the Witt vectors on k.

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} K \arrow[l, hook'] \\ X_0 \arrow[u] \arrow[r] & X \arrow[u] & X_k \arrow[u] \arrow[l, hook'] \end{tikzcd} \quad (??)$$

4.11. Surfaces Let X be a locally noetherian scheme. By a *étale cover* of X we mean a finite and étale morphism $Y \to X$. A morphism $Y' \to Y$ of étale covers is defined as an X-morphism from Y' to Y. Denote by $\mathrm{Et}(X)$ the category of finite étale covers of X. Suppose X is connected and fix a geometric point $\overline{x}$ of X, i.e., a morphism $\operatorname{Spec} k(\overline{x}) \to \operatorname{Spec} k(x)$, with $k(\overline{x})$ a separably closed field. The functor

$$F_x : Y \longmapsto Y(\overline{x}) = Y_{\overline{x}}$$

associating to an étale cover Y of X the finite set of its points over $\mathcal{O}$, is called fiber functor. We define the *fundamental group* of X at $\mathcal{O}$ to be the group of automorphisms of F_x, $\mathrm{Aut}(F_x)$.

Corollary 4.12. *Let A be a complete local noetherian ring, with residue field k. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{Spf}(A) = \varinjlim S_n$, where $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$. Let X be a proper scheme over S. Then the inverse image functor*

$$\mathrm{Et}(X) \longrightarrow \mathrm{Et}(X_s)$$

where $s = S_0 = \operatorname{Spec} k$, is an equivalence. So if X_0 is connected (so that X is connected), and $\overline{x}$ is a generic point of X_0 (hence of X), then the natural homomorphism

$$\pi_1(X_0, \overline{x}) \to \pi_1(X, \overline{x})$$

is an isomorphism.

Proof. Let $\hat{X}$ be the formal completion along X_s, so that $\hat{X} = \varinjlim X_n$, where $X_n = S_n \times_S X$, then we have

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & \hat{X} & \xrightarrow{j} & X \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & \hat{S} & \longrightarrow & S. \end{array}$$

Consider the natural morphisms

$$Et(X) \xrightarrow{j^*} \mathrm{Et}(\hat{X}) \xrightarrow{i^*} \mathrm{Et}(X_s)\ ,$$

where $j^* : Y \mapsto \hat{Y}$ and $i^* : y \mapsto y_s$. We claim that i^* is an equivalence. In fact, let Y_0/X_0 be finite é tale. Suppose that Y_n lifting Y_0 and étale over X_n exists and is unique. Since Y_0/X_0 is étale, $T_{Y_0/X_0} = 0$, then

$$H^2(Y_0, T_{Y_0/X_0}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = H^1(Y_0, T_{Y_0/X_0}) \otimes \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2} = 0,$$

we can find Y_{n+1} lifting Y_0, étale over X_{n+1}, and is unique. Thus get a unique $\mathfrak{Y}$ lifting $\mathfrak{X}$. To prove the full faithfulness, consider a Y-morphism $i_0 : Z_0 \to Y_0$, by a similar argument, $o(u_n, i_n) = 0$, where $u_n : Z_n \to Y_n$, so u_0 can be lifted.

For j^*, it is fully faithful. It remains to prove j^* is essentially surjective. Let $\mathfrak{Y}$ be a étale cover of $\hat{X}$, by 3.8, there exists a unique scheme Y finite over X such that $\hat{Y} = \mathfrak{Y}$. Since Y_n/X_n is étale for all $n \geq 0$, then Y/X is étale at all points $x \in Y_s$, therefore Y/X is étale since X/S is proper. $\square$

4.13. Abelian varieties In 1961, Serre constructed an example. Let X_0/k be a proper smooth scheme, where k is an algebraically closed of characteristic $p > 0$. For any local integral domain A with residue field k, then there is no proper, smooth lifting of X_0 to A.

In 1962, Lang, Raynaud and Szpiro constructed examples of proper smooth surfaces X_0/k such that the spectral sequence

$$E_1^{ij} = H^j(X_0, \Omega_{X_0/k}) \Rightarrow H_{dR}^{i+j}(X_0/k)$$

does not generate at E_1. Then by a theorem of Deligne-Illusie, X_0 can't be lifted to $W_2(k) = W(k)/p^2W(k)$.

4.14. K3 surfaces Here are some results in the positive direction. For example K3 surfaces. For a K3 surface over an algebraically closed field k, we mean a proper, smooth, connected surface X_0 such that

$$\Omega^2_{X_0/k} \simeq \mathcal{O}_{X_0}, \quad \text{and} \quad H^1(X_0, \mathcal{O}_{X_0}) = 0.$$

More precisely, we have the following result, due to Rudakov-Shafarevich and Deligne:

Theorem 4.15. *Let A be a complete local noetherian ring, with maximal ideal $\mathfrak{m}$ and residue field k which is algebraically closed. Let $S = \operatorname{Spec} A$, $\hat{S} = \operatorname{Spf}(A) = \varinjlim S_n$, where $S_n = \operatorname{Spec} A/\mathfrak{m}^{n+1}$. Let X_0 be a K3 surface over k. Then there exists a proper and smooth formal scheme $\mathfrak{X}$ over $\hat{S}$ lifting X_0.*

5 Serre's Example

Let k be an algebraically closed field with characteristic $p > 0$. Let $P_0 = \mathbb{P}_k^n$ be the projective space. In this section, we will construct a smooth complete intersection $Y_0 = V(h_1, \cdots, h_{n-r}) \subset P_0$ of dimension r, together with a free action of a finite group G such that $X_0 = Y_0/G$ has the following property: Let A be a complete local integral noetherian ring with residue field k, and whose fraction field is of characteristic 0. Then there exists no formal flat scheme $\mathcal{Y}$ over $Spf(A)$ lifting Y_0.

First, recall that we have a natural identification(see [H] II.7.1)

$$\operatorname{Aut}_k P_0 \xrightarrow{\sim} PGL_{n+1}(k) = GL_{n+1}(k)/k^*$$
$$g \mapsto (x = (x_0, \cdots, x_n) \mapsto gx).$$

Let G be a finite group. A homomorphism $\rho_0 : G \to PGL_{n+1}(k)$ gives an action of G on P_0. For each $g \in G$, we have a morphism $T_g : P_0 \to P_0$ given by $x \mapsto gx$ on the point $x \in P_0(T)$, where T is a k-scheme. Define a closed subscheme of P_0

$$Fix(g) = \Gamma_g \times_{P_0 \times P_0} \Delta \subset P_0,$$

where Γ_g is the graph of T_g and Δ is the diagonal of $P_0 \times P_0$. Then we see that for any k-scheme T, $x \in P_0(T)$ belongs to $Fix(g)(T)$ if and only if $gx = x$. Let $Q_0 = \bigcup_{g \in G, g \neq e} Fix(g)$. This is a closed subset of P_0.

Proposition 5.1. *Let $r \geqslant 1$ be an integer. Assume* $\dim Q_0 < n - r$. *Then there exists an integer $d_0 \geqslant 1$, such that for any $d = md_0$ $(m \geqslant 1)$ there exists a smooth complete intersection $Y_0 \subset P_0$ of dimension r defined by $V(h_1, \cdots, h_{n-r})$ $(h_i \in \Gamma(P_0, \mathcal{O}_{P_0}(d)))$, such that Y_0 is stable under the action of G on P_0 and G acts freely on Y_0.*

Proof. Since G acts admissibly on P_0, i.e. P_0 is the union of affine open subsets stable under G, $Z_0 = P_0/G$ exists and one has $\mathcal{O}_{Z_0} = (f_{0*}\mathcal{O}_{P_0})^G$, where f_0 is the natural projection $P_0 \to Z_0$ (see [SGA1 5.1.8]). Moreover since Z_0 is normal, and by [EGA II 6.6.4], Z_0 is projective. Hence we get a closed immersion $i : Z_0 \to \mathbb{P}^s_k$. Composed with the m-uple embedding of $\mathbb{P}^s_k$ in $\mathbb{P}^N_k$, where $N = \begin{pmatrix} s+m \\ s \end{pmatrix} - 1$, we obtain $i_m : Z_0 \to \mathbb{P}^N_k$ and the following diagram:

$$\begin{tikzcd} & P_0 \arrow[d, "f_0"] & & \\ f_0(Q_0) \arrow[r] & Z_0 \arrow[r, "i"] & \mathbb{P}^s_k \arrow[r] & \mathbb{P}^N_k \end{tikzcd}$$

We have $f_0^*(i^*\mathcal{O}_{\mathbb{P}^s_k}(1)) = \mathcal{O}_{P_0}(d_0)$ for some $d_0 \in \mathbb{Z}$ (as $Pic(P_0) = \mathbb{Z}$), then $f_0^* i_m^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{P_0}(md_0)$. Since f_0 is finite, hence $f^*(i^*\mathcal{O}_{\mathbb{P}^s_k}(1))$ is ample and we have $d_0 > 0$. Since $\dim f_0(Q_0) = \dim Q_0 < n - r$, by Bertini's theorem (see [J 6.11]) there exists a linear surface $L_0 \subset P' = \mathbb{P}^N_k$ of codimension $n - r$ defined by $V(g_1, \cdots, g_{n-r})(g_i \in \Gamma(P', \mathcal{O}_{P'}(1)))$, such that $L_0 \cap f_0(Q_0) = \emptyset$ and L_0 intersects $U_0 = Z_0 - f_0(Q_0)$ transversally. In particular, $X_0 := L_0 \cap Z_0 = L_0 \cap U_0$ is smooth. Let $h_i = (i_m f_0)^* g_i \in \Gamma(P_0, \mathcal{O}_{P_0}(md_0))$ and $Y_0 = V(h_1, \cdots, h_{n-r})$. We claim that this Y_0 satisfies our requirement. Since $f_0 : f_0^{-1}(U_0) \to U_0$ is an étale cover with group G (see [SGA1 5.2.3]) and Y_0 is a complete intersection in $f_0^{-1}(U_0)$, $f_0 : Y_0 \to X_0$ is also an étale cover with group G. In particular, G acts freely on Y_0. $\square$

Proposition 5.2. *Assume $r \geqslant 3$ and $d \geqslant 2$ ($d = md_0$ as in Proposition 5.1) or $r = 2$ and $p|d$, $p \nmid n+1$. Let $f_0 : Y_0 \to X_0 = Y_0/G$ as in the proof of Proposition 5.1. Let A be a complete local noetherian ring with reside field k. Let $\mathcal{X}/Spf(A)$ be a flat formal lifting of X_0/k. Then $\mathcal{X}$ is algebraizable, i.e. there exists a unique proper and smooth scheme $X/\operatorname{Spec}(A)$, such that $\widehat{X} = \mathcal{X}$. Moreover X is projective and the representation ρ_0 of G lifts to A, i.e. there exists a homomorphism $\rho : G \to PGL_{n+1}(A)$ such that the following diagram commutes.*

$$\begin{tikzcd} G \arrow[r, "\rho_0"] \arrow[dr, "\rho"'] & PGL_{n+1}(k) \\ & PGL_{n+1}(A) \arrow[u] \end{tikzcd}$$

In order to prove this proposition we need some preliminaries.

Lemma 5.3. *Let $P = \mathbb{P}^n_k$ and $Y = V(h_1, \cdots, h_{n-r})$ (where $h_i \in \Gamma(P, \mathcal{O}_P(d_i))$) be a complete intersection of dimension $r \geqslant 1$. Here "complete intersection" means that*

$$h : \bigoplus_{i=1}^{n-r} \mathcal{O}_P(d_i) \xrightarrow{(h_1,\cdots,h_{n-r})} \mathcal{O}_P$$

is a regular morphism, i.e. the Koszul complex

$$K^\bullet = (0 \to K^{-(n-r)} \to \cdots \to K^{-1} \xrightarrow{h} K^0 \to 0),$$

where $K^0 = \mathcal{O}_P$, $K^{-1} = \bigoplus_{i=1}^{n-r} \mathcal{O}_P(d_i)$ and $K^{-i} = \wedge^i K^{-1}$, is quasi-isomorphic to $\mathcal{O}_Y$ by the natural augmentation. Then $H^0(Y, \mathcal{O}_Y) = k$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < r$.

Proof. Note that $H^*(Y, \mathcal{O}_Y) = H^*(P, \mathcal{O}_Y) = H^*(P, K^\bullet)$. Hence we have a natural spectral sequence $E_1^{p,q} = H^q(P, K^p)$ converging to $H^{p+q}(Y, \mathcal{O}_Y)$, where $K^{-p} = \bigoplus_{i_1 < \cdots < i_p} \mathcal{O}_P(-d_{i_1} - \cdots - d_{i_p})$. From the fact that $E_1^{p,q} = 0$ for $0 < q < n$, it follows that $H^i(Y, \mathcal{O}_Y) = 0$ for all $i \in (0, r)$. And from

$$E_1^{p,0} = \begin{cases} k & p = 0 \\ 0 & p < 0 \end{cases},$$

we obtain that $H^0(Y, \mathcal{O}_Y) = k$. □

Remark 5.4. (a) For $H^r(Y,\mathcal{O}_Y)$, we have the following exact sequence:

$$0 \to H^r(Y,\mathcal{O}_Y) \to E_1^{-(n-r),n} \to E_1^{-(n-r)+1,n} \to \cdots \to E_1^{0,n} \to 0,$$

where $E^{-p,n} = H^n(P,K^{-p}) = \bigoplus_{i_1<\cdots<i_p} H^n(P,\mathcal{O}_P(-d_{i_1}-\cdots-d_{i_p}))$. Through the analysis of the dimension of $E_1^{\bullet,n}$, we obtain that $H^r(Y,\mathcal{O}_Y)=0$ if and only if $\sum d_i \leqslant n$.
(b) Similarly, we can obtain $H^i(Y,\mathcal{O}_Y(1))=0$ for $0<i<r$. And if $d_i \geqslant 2$ for any i, then $H^0(Y,\mathcal{O}_Y(1)) = k^{n+1}$.

Now we recall some base change formula. Suppose that in the cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd},$$

X and Y are noetherian separated schemes, and that f is proper. Let F be a coherent sheaf on X, flat over Y. Then there is a base change isomorphism:

$$Lg^*Rf_*F \xrightarrow{\sim} Rf'_*(h^*F).$$

We can thus obtain a natural map

$$g^*R^qf_*F \to H^q(Lg^*Rf_*F) = R^qf'_*(h^*F)). \tag{5.4.1}$$

F is called *cohomologically flat in degree q* if (5.4.1) is an isomorphism for any base change $Y' \to Y$.
Now let y be a point in Y, denote the map

$$k(y)\otimes R^qf_*F \to H^q(X_y, k(y)\otimes_{\mathcal{O}_Y} F).$$

by α^q. We have the following convenient criterion of cohomological flatness.

Lemma 5.5. *Assume $H^{q+1}(X_y,F_y)=0$, then in a neighborhood of y, α^q is an isomorphism and F is cohomologically flat in degree q. Moreover, if α^{q-1} is surjective, then in a neighborhood of y, R^qf_*F is locally free of finite type and of formation compatible with base change.*

Proof. Exercise (or see Trieste notes). □

Having these lemmas in hand, we can come to the proof of Proposition 5.2.

Proof. (1) Since Y_0/X_0 is étale, Y_0 lifts uniquely (up to a unique isomorphism) to Y_m/X_m étale for all $m \geqslant 1$. Hence we get a formal étale cover $\mathcal{Y}$ of $\mathcal{X}$. Similarly suppose that the action of G on Y_0 has been lifted to an action on Y_m: $G \times Y_m \to Y_m$. Then it extends uniquely $G \times Y_{m+1} \to Y_{m+1}$, since $Y_m \to X_m$ is étale. This is a group action automatically by uniqueness. Hence finally we get a free action of G on $\mathcal{Y}/\mathcal{X}$, which makes $\mathcal{Y}/\mathcal{X}$ an étale Galois cover.

Remark. For each $n \geqslant 0$, the diagram

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_n \arrow[d, "p_n"] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A_n \end{tikzcd},$$

where $A_n = A/\mathfrak{m}^n$, is cartesian. Since $\dim Y_0 = r > 1$, by Lemma 5.3 we have $H^0(Y_0, \mathcal{O}_{Y_0}) = k$ and $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$. And by Lemma 5.5, it follows that $p_{n*}\mathcal{O}_{Y_n} = A_n$, hence $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \varprojlim H^0(Y_n, \mathcal{O}_{Y_n}) = A$.

(2) Consider the natural ample line bundle $L_0 = \mathcal{O}_{Y_0}(1)$ on Y_0 induced by $\mathcal{O}_{P_0}(1)$. We want to lift it to a line bundle $\mathcal{L}$ on $\mathcal{Y}$.
(a) Case $r \geqslant 3$. We have a chain of thickenings:

$$Y_0 \hookrightarrow \cdots \hookrightarrow Y_m \hookrightarrow Y_{m+1} \hookrightarrow \cdots \hookrightarrow \mathcal{Y}$$

As $H^2(Y_0, \mathcal{O}_{Y_0}) = 0$ and $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ by Lemma 5.3, the obstruction to lifting L_0 to Y_m vanishes. We get a unique (up to non-unique isomorphisms) line bundle L_m on Y_m, hence finally a line bundle $\mathcal{L}$ on $\mathcal{Y}$ (unique up to isomorphisms), which lifts L_0.
(b) When $r = 2$, it no longer holds that $H^2(Y_0, \mathcal{O}_{Y_0}) = 0$ (see 5.4 (a)). But we have the following argument due to Mumford. Let $\mathcal{I}$ be the ideal of the closed immersion $i : Y_0 \to P_0$. Then we have a natural exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{P_0} \to \Omega^1_{Y_0} \to 0.$$

Then $\Omega^2_{Y_0} = \Omega^n_{P_0} \otimes (\wedge^{n-2}\mathcal{I}/\mathcal{I}^2)^\vee$. But via the Koszul complex, it follows that

$$\wedge^{n-r}\mathcal{I}/\mathcal{I}^2 = \mathcal{T}or^{\mathcal{O}_{P_0}}_{n-r}(\mathcal{O}_{Y_0}, \mathcal{O}_{Y_0}) = \mathcal{O}_{Y_0}(-(n-r)d).$$

Hence we get $\varepsilon_0 : \Omega^2_{Y_0} \backsimeq \mathcal{O}_{Y_0}(N) \backsimeq L_0^{\otimes N}$, where $N = (n-1)d - n - 1$. By the assumption that $p|d$ and $p \nmid n+1$, it follows that $p \nmid N$.

Now consider the immersion $i_m : Y_m \to Y_{m+1}$ and assume that L_0 has been lifted to L_m on Y_m and the isomorphism ε_0 lifted to $\varepsilon_m : \Omega^2_{Y_m} \tilde{\to} L_m^{\otimes N}$. Let's show that this isomorphism extends to $m+1$. Since $\Omega^2_{Y_{m+1}}$ lifts $\Omega^2_{Y_m}$, we have

$$0 = o(\Omega^2_{Y_m}, i_m) = o(L_m^{\otimes N}, i_m) = N \cdot o(L_m, i_m).$$

Since $p \nmid N$, it follows that $o(L_m, i_m) = 0$, i.e. L_m can be lifted to L_{m+1} over Y_{m+1}. But since $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$ by 5.3, ε_0 can be lifted to an isomorphism $\varepsilon_0 : \Omega^2_{Y_{m+1}} \tilde{\to} L_{m+1}^{\otimes N}$.
Combining (a) and (b), we get (in all cases) an ample line bundle $\mathcal{L}$ on $\mathcal{Y}$ lifting L_0. By Grothendieck's Existence Theorem, $\mathcal{Y}$ is algebraizable, i.e. there exists a projective and flat (hence smooth) scheme $Y/\operatorname{Spec} A$ with an ample line bundle L, such that $\widehat{Y} = \mathcal{Y}$ and $\widehat{L} = \mathcal{L}$.
(3) Algebraization of $\mathcal{Y}/\mathcal{X}$. Denote by M_0 the line bundle $\wedge^{|G|} f_{0*} L_0$ on X_0, where $|G|$ denotes the cardinality of G. Since L_0 is ample, by [EGA II 6.5.1] it follows that M_0 is ample too. Clearly, for each $m \geqslant 1$, the line bundle $M_m = \wedge^{|G|} f_{m*} L_m$ lifts M_0, and the line bundle $\mathcal{M} = \varprojlim M_m$ lifts L_0 to $\mathcal{X}$. By Grothendieck's Existence Theorem, there is a projective, flat (hence smooth) scheme $X/\operatorname{Spec} A$ with an ample line bundle M, such that $\widehat{X} = \mathcal{X}$ and $\widehat{M} = \mathcal{M}$. And by full faithfulness of the functor $Z \to \widehat{Z}$, there is a morphism f making the following diagram cartesian:

$$\begin{tikzcd} \mathcal{Y} = \widehat{Y} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathcal{X} = \widehat{X} \arrow[r] & X \end{tikzcd} .$$

For the same reason the action of G on $\widehat{Y}$ comes from an action of G on Y:

$$\begin{aligned} G \times Y &\tilde{\longrightarrow} Y \times_X Y \\ (g, y) &\longmapsto (y, gy). \end{aligned}$$

And moreover, $M = \wedge^{|G|} f_* L$. By the comparison theorem, we have $\widehat{M} = \wedge^{|G|} \widehat{f}_*(\widehat{L}) = \wedge^{|G|} (f_* L)^\wedge$.
(4) Lifting of ρ_0. Since we have assumed $d \geqslant 2$, it follows from Remark 5.4 that $H^0(Y_0, \mathcal{O}_{Y_0}(1)) = k^{n+1}$ and $H^1(Y_0, \mathcal{O}_{Y_0}) = 0$. By Lemma 5.5, we get $H^0(Y_m, \mathcal{O}_{Y_m}(1)) = A_m^{n+1}$, hence $H^0(Y, \mathcal{O}_Y(1)) = A^{n+1}$.
For any $g \in G$ there is a natural isomorphism $a(g)_0 : g^* L_0 \tilde{\to} L_0$ on Y_0 induced

by the isomorphism $g^*\mathcal{O}_{P_0}(1)\tilde{\rightarrow}\mathcal{O}_{P_0}(1)$ given by the action of g on P_0. Now suppose that $a(g)_0$ has been lifted to $a(g)_m : g^*L_m\tilde{\rightarrow}L_m$, then the obstruction

$$o(a(g)_m, i_m) \in H^1(Y_0, \mathcal{O}_{Y_0}) \otimes_k \mathfrak{m}^{m+1}/\mathfrak{m}^{m+2} = 0.$$

Therefore $a(g)_m$ can be lifted, uniquely up to automorphisms of L_m, to an isomorphism $a(g)_{m+1} : g^*L_{m+1} \to L_{m+1}$. Hence finally it can be lifted to an isomorphism $\widehat{a(g)} : g^*\mathcal{L} \to \mathcal{L}$. Algebraizing $\widehat{a(g)}$, we get $a(g) : g^*L \to L$, unique up to automorphisms of L. For $g, h \in G$, $a(gh) = a(g)a(h)$ and $a(e) = Id_L$ up to automorphism of L. But

$$\mathrm{Aut}(L) = (\mathrm{End}(L))^* = H^0(Y, \mathcal{E}nd(L))^* = H^0(Y, \mathcal{O}_Y)^* = A^*,$$

(since $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = A$). From the isomorphism $a(g) : g^*L \to L$, we get

$$\begin{tikzcd} H^0(Y,L) \arrow[r, "\delta"] \arrow[dr, "\check{\rho}(g)"'] & H^0(Y, g^*L) \arrow[d, "H^0(P_0, a(g))"] \\ & H^0(Y,L) \end{tikzcd} ,$$

where δ is given by the functoriality of H^0. Note $H^0(Y, L) = A^{n+1}$ and that $\check{\rho}(g)$ lifts $\check{(\rho)}_0(g)$ up to an element of A^*, where $\check{\rho}_0(g) : H^0(Y, L_0) = k^{n+1} \to H^0(Y, L_0) = k^{n+1}$ is obtained similarly via the action of g on P_0. Hence finally we get a representation $\rho : G \longrightarrow PGL_{n+1}(A)$, which lifts $\rho_0 : G \longrightarrow PGL_{n+1}(k)$. □

Now we give explicit constructions of G and its representation ρ_0. Let n and r be integers with $1 \leqslant r < n$, and let G be a group of type $(p, \cdots, p)$ of order p^s, i.e. $G \backsimeq \mathbb{F}_p^s$. Moreover we suppose that $p \geqslant n + 1$ and $s \geqslant n$. We choose an injective homomorphism of $\mathbb{F}_p$-vectors spaces $h : G \to k$ (since k is infinite dimensional over $\mathbb{F}_p$). Let $N \in M_{n+1\times n+1}(k)$ be the nilpotent matrix given by

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots} \\ 0 & \multicolumn{3}{c}{\dots\dots\dots} & 1 \\ 0 & \multicolumn{3}{c}{\dots\dots\dots} & 0 \end{pmatrix}.$$

For $g \in G$, set

$$\begin{aligned} \widetilde{\rho_0}(g) &= exp(h(g)N) \\ &= 1 + h(g)N + \cdots + \frac{h(g)^n N^n}{n!}, \end{aligned}$$

then $\widetilde{\rho}(g) \in GL_{n+1}(k)$. Let $\rho_0(g)$ be the image of $\widetilde{\rho}(g)$ in $PGL_{n+1}(k)$, then we get a representation

$$\rho_0 : G \longrightarrow PGL_{n+1}(k). \tag{5.5.1}$$

It is easily seen that $\rho_0 g = Id$ if and only if $h(g) = 0$. Hence ρ_0 is faithful, since h is injective. For $g \neq e$ in G, $Fix(g)$ consists of only one point $(1, 0, \cdots, 0)$. Thus $\dim Q_0 = 0$ $(< n - r)$, so that the condition of 5.1 is satisfied.

Proposition 5.6. *Assume that $p > n+1$ and $s \geqslant n+1$. Let A be an integral local ring with residue field k and field of fractions K of characteristic zero. If ρ_0 is the representation defined in 5.5.1, then there is no homomorphism $\rho : G \longrightarrow PGL_{n+1}(A)$ lifting ρ_0.*

Proof. Assume that such a lifting ρ exists. Then ρ is necessarily faithful (since ρ_0 is faithful), and so is the composition, still denoted by ρ, $G \longrightarrow PGL_{n+1}(K)$. Note that we have a central extension of groups

$$1 \to \mu_{n+1}(K) \to SL_{n+1}(K) \to H \to 1, \tag{5.6.1}$$

where H the canonical image of $SL_{n+1}(K)$ in $PGL_{n+1}(K)$. Since G is finite, after an extension of scalars, we may assume that the image of ρ lies in H. Pulling back the extension 5.6.1 by G, we obtain an extension of G by $\mu_{n+1}(K)$ denoted by E. E corresponds to an element in $H^2(G, \mu_{n+1}(K))$. Since $p \nmid |\mu_{n+1}(K)|$, we have $H^2(G, \mu_{n+1}(K)) = 0$ and in particular the extension E splits. Hence the representation ρ lifts to a (faithful) representation $\rho' : G \longrightarrow SL(V)$, where $V = K^{n+1}$. After some extension of scalars again, we may assume that K contains μ_p, the group of pth roots of unit. As G is commutative and K is of characteristic zero, V decomposes into a sum

$$V = \bigoplus_{i=1}^{n+1} V_i$$

of sub-representations of dimension 1. Let $\chi_i : G \to \mu_p \subset K^*$ be the corresponding character of V_i. Then one has $\prod_{i=1}^{n+1} \chi_i = 1$. Each kernel H_i of χ_i is a hyperplane in G. Since by assumption $s = \dim_{\mathbb{F}_p} G \geqslant n+1$, the kernel of ρ'

$$Z = \mathrm{Ker}\rho' = \bigcap_{i=1}^{n+1} H_i = \bigcap_{i=1}^{n} H_i$$

will be at least of dimension 1 over $\mathbb{F}_p$, in particular not zero. This contradicts the faithfulness of ρ'. □

Combining all the results above, we have obtained the following theorem.

Theorem 5.7. *Let r and n be integers with $2 \leqslant r < n$ and $n+1 < p$. Let $G = \mathbb{F}_p^s$ with $s \geqslant n+1$. Then there exists a smooth, projective complete intersection $Y_0 = V(h_1, \cdots, h_{n-r}) \subset P_0 = \mathbb{P}_k^n$ of dimension r (and of multi-degree $(d, \cdots, d)$), endowed with a free action of G, and such that the projective smooth scheme $X_0 = Y_0/G$ has the following property. Let A be a complete, integral, noetherian local ring with residue field k and field fractions of characteristic zero. Then there exists no formal scheme $\mathcal{X}$ flat over $Spf\, A$, lifting X_0.* A fortiori*, there exists no proper smooth lifting of X_0 over* $\operatorname{Spec} A$.

Proof. Let ρ_0 be the representation defined in (5.5.1), through which G acts on P_0. The existence of such a complete intersection Y_0 is guaranteed by Proposition 5.1. Assume that there exists a lifting $\mathcal{X}$ of X_0 over $Spf(A)$. Then by Proposition 5.2, we get a representation $\rho : G \longrightarrow PGL_{n+1}(A)$ lifting ρ_0. But Proposition 5.6 claims that such a ρ cannot exist. Hence we get a contradiction. □

Remark. Serre has improved the above results as follows.
(a) One can take $s = 2$, i.e. $G = \mathbb{F}_p \times \mathbb{F}_p$, in the theorem.
(b) Suppose A is a complete noetherian local ring with residue field k. If there exists a flat formal lifting $\mathcal{X}$ of X_0 over $Spf(A)$, then it is necessary that $pA = 0$.

For details of this remark see Trieste notes.

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