Geometrical Optics in Continuum

Pèijùn Zhū

November 15, 2016

1 Fermat's Principle

Fermat's Principle is

$$\delta s = \delta \int n \mathrm{d}\ell = 0 \tag{1}$$

If we choose the path connecting two points to be

$$\boldsymbol{r} = \boldsymbol{r}(t), \quad t_0 \le t \le t_1 \tag{2}$$

We can write

$$s = \int n \frac{\mathrm{d}\ell}{\mathrm{d}t} \mathrm{d}t, \quad \frac{\mathrm{d}\ell}{\mathrm{d}t} = |\dot{\boldsymbol{r}}| \tag{3}$$

The Lagrangian about parameter t is

$$L = n \frac{\mathrm{d}\ell}{\mathrm{d}t} = n(\boldsymbol{r})|\dot{\boldsymbol{r}}| \tag{4}$$

then the Euler-Lagrange Equation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{\boldsymbol{r}}} - \frac{\partial}{\partial \boldsymbol{r}} \right) L = 0 \tag{5}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = n\hat{\theta}, \quad \hat{\theta} = \frac{\partial |\dot{\boldsymbol{r}}|}{\partial \dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} / |\dot{\boldsymbol{r}}| \tag{6}$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial L}{\partial \boldsymbol{r}} = \frac{\mathrm{d}\ell}{\mathrm{d}t} \nabla n \tag{7}$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\ell} = \nabla n \tag{8}$$

2 Examples

2.1 Parallel Gradient

$$n(x, y, z) = f(x), \quad \nabla n = f'(x)\hat{x} \tag{9}$$

It is obvious that p_y, p_z are conserved.

2.2 Central Gradient

For a spherical symmetric distribution

$$n(\boldsymbol{r}) = f(r), \quad \nabla n = f'(r)\hat{r} \tag{10}$$

Define $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$,

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}\ell} &= \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\ell} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\ell} && (11)\\
&= \hat{\theta} \times \boldsymbol{p} + \boldsymbol{r} \times \nabla n && (12)\\
&= \hat{\theta} \times n\hat{\theta} + \boldsymbol{r} \times f'(r)\hat{r} && (13)\\
&= 0 && (14)
\end{aligned}$$

So $\boldsymbol{L}$ is conserved.