From d'Alembert to Lagrange

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1 Nowtonian

Assume $\boldsymbol{F}$ is active force and $\boldsymbol{R}$ is constraint force. Then for ith body we have Newton's Second Law:

$$\boldsymbol{F}_i^{\text{tot}} = \boldsymbol{F}_i + \boldsymbol{R}_i = m\boldsymbol{a}_i$$

With a virtual displacement $\delta \boldsymbol{x}_i$, we define:

$$\delta W_i = \boldsymbol{F}_i^{\text{tot}} \cdot \delta \boldsymbol{x}_i, \quad \delta K_i = m\boldsymbol{a}_i \cdot \delta \boldsymbol{x}_i$$

Then we have $\delta W_i = \delta K_i$. Define

$$\delta W = \sum_i \delta W_i, \quad \delta K = \sum_i \delta K_i$$

and obviously

$$\delta W = \delta K$$

As the constraints will not do work, we can rewrite

$$\delta W = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i$$

2 Lagrangian

Statics $\delta K = 0$ Principle of Virtual Work:

$$\frac{\delta W}{\delta q} = 0$$

i.e. Minimal Potential Energy

$$Q \stackrel{\text{def}}{=} -\frac{\delta V}{\delta q} = 0$$

Dynamics d'Alembert's Principle—Counterpart of Principle of Virtual Work:

$$\frac{\delta W}{\delta q} = \frac{\delta K}{\delta q}$$

i.e.

$$
\begin{aligned}
Q \stackrel{\text{def}}{=} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} &= m_i \boldsymbol{a}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} && (1)\\
&= m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} && (2)\\
&= m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q}\right) - m_i \boldsymbol{v} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial q}\boldsymbol{x}_i && (3)\\
&= m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v} \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial \dot{q}}\right) - m_i \boldsymbol{v} \cdot \frac{\partial}{\partial q}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i && (4)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}}\right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q} && (5)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} && (6)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) T && (7)
\end{aligned}
$$

For conservative or monogenic system,

$$Q = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) V \tag{8}$$

As a result of Eq. 7 and 8, if we define $L = T - V$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) L = 0$$