Renyi entropy of the wormholes

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Rényi entropy is defined as:

$$S_n(\alpha) := \frac{1}{1-\alpha} \log\left(\sum_{a_{1\to n}} \sum_{\mu=1}^{N} \left(\frac{p_{a_1 a_2 \cdots a_n}}{N}\right)^{\alpha}\right) \tag{1}$$

$$= \frac{1}{1-\alpha} \log\left(\frac{1}{\mathcal{D}^{2\alpha(n-1)}} \sum_{a_{1\to n}} N_{a_1 a_2 \cdots a_n} \prod_{i=1}^{n} d_{a_i}^{\alpha}\right) \tag{2}$$

$$= \frac{1}{1-\alpha}\left[\log\left(\sum_{a_{1\to n}} N_{a_1 a_2 \cdots a_n} \prod_{i=1}^{n} d_{a_i}^{\alpha}\right) - \alpha(n-1)\log \mathcal{D}^2\right] \tag{3}$$

$$= \frac{\alpha(n-1)\log \mathcal{D}^2 - \log t_n(\alpha)}{\alpha - 1} \tag{4}$$

We are using

$$t_n(\alpha) := \sum_{a_{1\to n}} N_{a_1 a_2 \cdots a_n} \prod_{i=1}^{n} d_{a_i}^{\alpha} \tag{5}$$

$$= \sum_{a_{1\to n}} \sum_{x_{1\to n-1}} N_{x_0 a_1}^{x_1} N_{x_1 a_2}^{x_2} \cdots N_{x_{n-1} a_n}^{x_n} \prod_{i=1}^{n} d_{a_i}^{\alpha}, \quad x_0 = x_n = 1 \tag{6}$$

$$= \sum_{x_{1\to n-1}} \prod_{i=1}^{n} \left(\sum_{a_{1\to n}} d_{a_i}^{\alpha} N_{x_{i-1} a_i}^{x_i}\right) \tag{7}$$

Define

$$T_{bc}(\alpha) := \sum_{a} d_a^{\alpha} N_{ba}^{c}$$

Then

$$t_n(\alpha) = \sum_{x_{1\to n-1}} \prod_{i=1}^{n} T_{x_{i-1} x_i}(\alpha) \left[\boldsymbol{T}^n(\alpha)\right]_{x_0 x_n} = \left[\boldsymbol{T}^n(\alpha)\right]_{11}$$

It's obvious that T is an elementwise positive matrix. Moreover, T is normal:

$$T_{ab}T_{cb} = \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{xa}^b N_{yc}^b \tag{8}$$

$$= \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{x\bar{b}}^{\bar{a}} N_{y\bar{b}}^{\bar{c}} \tag{9}$$

$$= \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{\bar{y}b}^{c} N_{\bar{x}b}^{a} \tag{10}$$

$$= T_{ba}T_{bc} \tag{11}$$

Thus, the eigenvectors correspond to distinct eigenvalues must be orthogonal. Assume the eigenvalues and orthonormal eigenvectors are $\boldsymbol{x}_i$ with eigenvalue $\lambda_i(\alpha)$. The eigenvector $\boldsymbol{x}_m := d_i \boldsymbol{e}_i / \mathcal{D}$ has the maximum eigenvalue because its all components are positive. The eigenvalue is

$$\lambda_{\max}(\alpha) = \sum_a d_a^{\alpha+1}, \quad \lambda_{\max}(1) = \mathcal{D}^2$$

We decompose $\boldsymbol{e}_1$ into $\boldsymbol{x}_i$ to calculate $[\boldsymbol{T}^n]_{11}$:

$$\boldsymbol{e}_1 = \sum_i c_i \boldsymbol{x}_i, \quad c_i = \boldsymbol{e}_1 \cdot \boldsymbol{x}_i \tag{12}$$

$$[\boldsymbol{T}_\alpha^n]_{11} = \boldsymbol{e}_1^T \boldsymbol{T}_\alpha^n \boldsymbol{e}_1 = \sum_i c_i^2 \lambda_i^n \tag{13}$$

$$= c_m^2 \lambda_{\max}^n \left[1 + \sum_{i \neq m} \frac{c_i^2}{c_m^2} \left(\frac{\lambda_i}{\lambda_{\max}} \right)^n \right] \tag{14}$$

$$= \frac{\lambda_{\max}^n}{\mathcal{D}^2} \left[1 + \Theta(k^n) \right], \quad k := \frac{\lambda_{\mathrm{sub}}}{\lambda_{\max}} < 1 \tag{15}$$

$$S_n(\alpha) = \frac{\alpha(n-1)\log \mathcal{D}^2 - \log[\boldsymbol{T}_\alpha^n]_{11}}{\alpha - 1} \tag{16}$$

$$= n \left[\frac{\alpha \log \lambda_{\max}(1) - \log \lambda_{\max}(\alpha)}{\alpha - 1} \right] - \log \mathcal{D}^2 + \Theta(k^n) \tag{17}$$

In the $\alpha = 1$ case, there is no other nontrivial eigenvector except $\boldsymbol{x}_m$ because $\boldsymbol{T} = \boldsymbol{x}_m \boldsymbol{x}_m^\mathsf{T}$. So the $\Theta(k^n)$ term in 17 reduce to zero.

$$S_n(1) = n \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=1} \Big[\alpha \log \lambda_{\max}(1) - \log \lambda_{\max}(\alpha) \Big] - \log \mathcal{D}^2 \tag{18}$$

$$= n \Big[\log \mathcal{D}^2 - \sum_a \frac{d_a^2 \log d_a}{\mathcal{D}^2} \Big] - \log \mathcal{D}^2 \tag{19}$$

$$= (n-1)\log \mathcal{D}^2 - n \sum_a p_a \log d_a \tag{20}$$