

Energy Density for Electrostatic Field

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$$E_{\pm} = \frac{kq_{\pm}}{r_{\pm}^3} \mathbf{r}_{\pm}, \quad \mathbf{r}_{\pm} = \mathbf{r} \pm a\hat{z}$$

We want to prove

$$\frac{kq_+q_-}{2a} = \int \epsilon \mathbf{E}_+ \cdot \mathbf{E}_- dV = \frac{kq_+q_-}{4\pi} \int \frac{\mathbf{r}_+ \cdot \mathbf{r}_-}{r_+^3 r_-^3} dV$$

We choose $a = 1$, so:

$$\int \frac{\mathbf{r}_+ \cdot \mathbf{r}_-}{r_+^3 r_-^3} dV = \iiint \frac{r^2 - 1}{\sqrt{(r^2 + 1)^2 - 4r^2 \cos^2 \theta}}^3 r^2 \sin \theta dr d\theta d\phi \quad (1)$$

$$= 2\pi \int_0^\infty dr \int_{-1}^1 dx \frac{r^2(r^2 - 1)}{[(r^2 + 1)^2 - 4r^2 x^2]^{3/2}}, \quad x = \cos \theta \quad (2)$$

$$= 2\pi \int_0^\infty dr \frac{r^2(r^2 - 1)}{(r^2 + 1)^3} \int_{-1}^1 [1 - k^2 x^2]^{-3/2} dx, \quad k = \frac{2r}{r^2 + 1} \quad (3)$$

$$= 2\pi \int_0^\infty dr \frac{r^2(r^2 - 1)}{(r^2 + 1)^3} \frac{2}{\sqrt{1 - k^2}} \quad (4)$$

$$= 4\pi \int_0^\infty \frac{dr}{(r + 1/r)^2} \frac{r^2 - 1}{|r^2 - 1|} \quad (5)$$

$$= -4\pi \left[\int_0^1 \frac{dr}{(r + 1/r)^2} + \int_\infty^1 \frac{dr}{(r + 1/r)^2} \right] \quad (6)$$

$$= -4\pi \left[\int_0^1 \frac{dr}{(r + 1/r)^2} + \int_0^1 \frac{d(1/r')}{(r' + 1/r')^2} \right], \quad r' = 1/r \quad (7)$$

$$= -4\pi \int_\infty^2 dt/t^2 \quad (8)$$

$$= 2\pi \quad (9)$$