# Renyi entropy of the wormholes

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The (elementwise) positive matrix

$$T_{bc;\alpha} = \sum_a d_a^\alpha N_{ba}^c \tag{1}$$

is normal as

$$\begin{align} T_{ab}T_{cb} &= \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{xa}^b N_{yc}^b \tag{2} \\ &= \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{x\bar{b}}^{\bar{a}} N_{y\bar{b}}^{\bar{c}} \tag{3} \\ &= \sum_x \sum_y d_x^\alpha d_y^\alpha \sum_b N_{\bar{y}b}^{c} N_{\bar{x}b}^{a} \tag{4} \\ &= T_{ba}T_{bc} \tag{5} \end{align}$$

Thus, The eigenvectors correspond to distinct eigenvalues must be orthogonal. Assume the eigenvalues and eigenvectors are $\{\lambda_i, \mathbf{v}_i\}$. The eigenvector $\mathbf{v}_m = d_i\mathbf{e}_i, v_m^2 = \mathcal{D}^2$ with all components are positive has the max eigenvalue

$$A_\alpha = \lambda_{m;\alpha} = \sum_a d_a^{\alpha+1}$$

In this notation, $\mathcal{D}^2 = A_1$.

$$\mathbf{e}_1 = \sum_i \frac{\mathbf{e}_1 \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}\mathbf{v}_i = \sum_i c_i \mathbf{v}_i, \quad c_m = \frac{\mathbf{e}_1 \cdot \mathbf{v}_m}{v_m^2} = d_1/\mathcal{D}^2 = 1/A_1 \tag{6}$$

Define

$$\begin{aligned}
B_\alpha &= (\mathbf{T}_\alpha^n)_{11} && (7)\\
&= \mathbf{e}_1^T \mathbf{T}_\alpha^n \mathbf{e}_1 && (8)\\
&= \sum_i c_i^2 v_i^2 \lambda_i^n && (9)\\
&= A_\alpha^n / A_1 \left[1 + \sum_{i \neq m} \frac{c_i^2 v_i^2}{c_m^2 v_m^2} \left(\frac{\lambda_i}{\lambda_m}\right)^n\right] && (10)
\end{aligned}$$

So, For $\alpha \in (0, +\infty)$,

$$\begin{aligned}
S_{\alpha,n} &= \frac{\log B_\alpha - \alpha(n-1)\log \mathcal{D}^2}{1-\alpha} && (11)\\
&= n\frac{\log A_\alpha - \alpha \log A_1}{1-\alpha} - \log A_1 + O(k^n), \quad k < 1 && (12)
\end{aligned}$$

Neglect the last term, we can verify from 12 that

$$\begin{aligned}
\lim_{\alpha \to 1} S_{\alpha,n} &= n\left(\log A_1 - \sum_a \frac{d_a^2 \log d_a}{A_1}\right) - \log A_1 && (13)\\
&= 2(n-1)\log \mathcal{D} - n\sum_a p_a \log d_a && (14)
\end{aligned}$$

In the $\alpha = 1$ case, there is no other eigenvector with nonzero nonzero eigenvalue except $d_a$ because $T_{bc} = d_b d_c$. So there is no $O(k^n)$ term in 12.