Invariance of Euler-Lagrange Equations

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E-L is deducted from the Hamilton's principle

$$\delta S = \delta \int L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) d t = 0$$

It is easy to find that, for $L' = L + \mathrm{d} f(\boldsymbol{q}, t)/\mathrm{d}t$ or change of variables $\boldsymbol{q} \to \boldsymbol{Q}$, the min of δS will not change. Here we want to prove it the hard way—using E-L equations.

The original E-L Equations are:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} \right) L = 0$$

1 Commutator $\left[\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\partial}{\partial q_i} \right] f(\boldsymbol{q}, t) = 0$

Proof

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\partial}{\partial q_i} \right] f(\boldsymbol{q}, t) = \left[\dot{q}_i \frac{\partial}{\partial q_i} + \frac{\partial}{\partial t}, \frac{\partial}{\partial q_i} \right] f(\boldsymbol{q}, t) = 0$$

2 Condition $L' = L + \frac{\mathrm{d} f(\boldsymbol{q}, t)}{\mathrm{d}t}$

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} \right) L' &= \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i} \right) \frac{\mathrm{d}f}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial q_i} f - \frac{\partial}{\partial q_i} \frac{\mathrm{d}}{\mathrm{d}t} f \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\partial}{\partial q_i} \right] f \\
&= 0
\end{aligned}$$

3 Condition $\boldsymbol{q} \to \boldsymbol{Q}$

We we change generalized coordinates $\boldsymbol{q} \to \boldsymbol{Q}$, the Lagrangian:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \to L'(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = L\big[\boldsymbol{q}(\boldsymbol{Q}, t), \dot{\boldsymbol{q}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t), t\big]$$

We want to prove:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{Q}_i} - \frac{\partial}{\partial Q_i}\right) L' = 0$$

$$\begin{aligned}
\text{LHS} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \dot{q}_j}{\partial \dot{Q}_i}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial q_j}{\partial Q_i}\frac{\partial}{\partial q_j} - \frac{\partial \dot{q}_j}{\partial Q_i}\frac{\partial}{\partial \dot{q}_j}\right) L && (1)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial q_j}{\partial Q_i}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial q_j}{\partial Q_i}\frac{\partial}{\partial q_j} - \frac{\partial \dot{q}_j}{\partial Q_i}\frac{\partial}{\partial \dot{q}_j}\right) L && (2)\\
&= \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial Q_i} q_j\right)\frac{\partial}{\partial \dot{q}_j} + \frac{\partial q_j}{\partial Q_i}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial q_j}{\partial Q_i}\frac{\partial}{\partial q_j} - \left(\frac{\partial}{\partial Q_i}\frac{\mathrm{d}}{\mathrm{d}t} q_j\right)\frac{\partial}{\partial \dot{q}_j}\right] L && (3)\\
&= \left(\left[\frac{\mathrm{d}}{\mathrm{d}t}, \frac{\partial}{\partial Q_i}\right] q_j\right)\frac{\partial}{\partial \dot{q}_j} L + \frac{\partial q_j}{\partial Q_i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}\right) L && (4)\\
&= 0 && (5)
\end{aligned}$$