Jacobi Identity for Classical Possion Bracket

Peijun Zhu

October 14, 2016

Definition of Classical Possion Bracket:

$$[f,g] = \sum_{i=1,2,\dots,n} \frac{\partial(f,g)}{\partial(p_i,q_i)} = \sum_{i=1,2,\dots,n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

If we define $p_{-i} \overset{\text{def}}{=} q_i$, and $I = \{\pm 1, \pm, 2, \dots, \pm n\}$, we can rewrite it:

$$[f,g] = \sum_{i\in I} \operatorname{sgn}(i) \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_{-i}}$$

Jacobi Identity is $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$. With the new notation, we can rewrite it as

$$\begin{aligned}
\sum_{\overrightarrow{fgh}} [f,[g,h]] &= \sum_{\overrightarrow{fgh}} \sum_{i\in I} \operatorname{sgn}(i) \frac{\partial f}{\partial p_i}\frac{\partial}{\partial p_{-i}} \left(\sum_{j\in I} \operatorname{sgn}(j) \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_{-j}} \right) && (1)\\
&= \sum_{\overrightarrow{fgh}} \sum_{i,j\in I} \operatorname{sgn}(ij) \left(\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_{-j}}\frac{\partial^2 g}{\partial p_{-i}\partial p_j} + \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_{-i}\partial p_{-j}} \right) && (2)\\
&= \sum_{\overrightarrow{fgh}} \sum_{i,j\in I} \operatorname{sgn}(ij) \left(\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_{-j}}\frac{\partial^2 g}{\partial p_{-i}\partial p_j} + \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_{-i}\partial p_{-j}} \right) && (3)\\
&= \sum_{\overrightarrow{fgh}} \sum_{i,j\in I} \operatorname{sgn}(ij) \left(\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_{-j}}\frac{\partial^2 g}{\partial p_{-i}\partial p_j} + \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_j}\frac{\partial^2 g}{\partial p_{-i}\partial p_{-j}} \right) && (4)\\
&= \sum_{\overrightarrow{fgh}} \sum_{i,j\in I} \operatorname{sgn}(ij) \left(\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_{-j}}\frac{\partial^2 g}{\partial p_{-i}\partial p_j} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_{-j}}\frac{\partial^2 g}{\partial p_{-i}\partial p_j} \right) && (5)\\
&= 0 && (6)
\end{aligned}$$