Energy of Electrostatic Field in a Triangle

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1 Energy of a triangle

For a triangle ABC, assume the points have potential $\varphi_a, \varphi_b, \varphi_c$. As shown in Fig. 1, $AB' \perp AC$, $AC' \perp AB$, combine the components of $\nabla\varphi$, we have

$$(\nabla\varphi)^2 = \frac{1}{\sin^2 A}\left(\frac{\varphi_{ab}^2}{c^2} + \frac{\varphi_{ac}^2}{b^2} - \frac{2\varphi_{ab}\varphi_{ac}\cos A}{bc}\right)$$

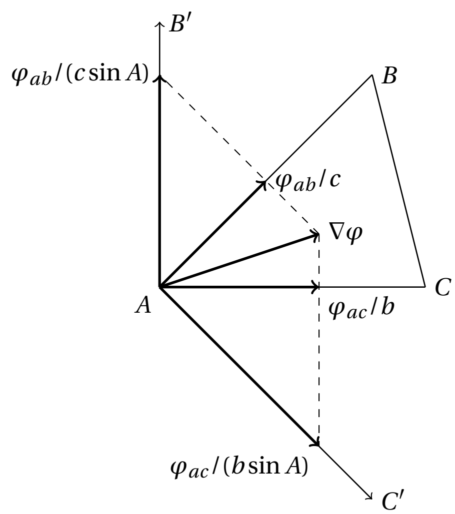

Figure 1: Using contra-variant bases to evaluate $(\nabla\varphi)^2$

The area of triangle is

$$S = \frac{bc\sin A}{2}$$

The energy of this triangle is

$$E_\triangle \propto S(\nabla\varphi)^2 \tag{1}$$

$$\propto \frac{bc}{\sin A}\left(\frac{\varphi_{ab}^2}{c^2} + \frac{\varphi_{ac}^2}{b^2} - \frac{2\varphi_{ab}\varphi_{ac}\cos A}{bc}\right) \tag{2}$$

φ_a

$$\frac{\partial E_\triangle}{\partial \varphi_a} \propto \left(\frac{b}{c\sin A} - \cot A\right)\varphi_{ab} + \left(\frac{c}{b\sin A} - \cot A\right)\varphi_{ac} \tag{3}$$

$$\frac{b}{c\sin A} - \cot A = \frac{b - c\cos A}{c\sin A} = \frac{a\cos C}{a\sin C} = \cot C$$

$$\frac{c}{b\sin A} - \cot A = \cot B$$

3

$$\frac{\partial E_\triangle}{\partial \varphi_a} \propto \varphi_{ab}\cot C + \varphi_{ac}\cot B \tag{4}$$

2 Stationary point equation for energy minimum

Let E be the total energy and use i to mark all triangles containing A. B_i, C_i are other points in triangle i. In the stationary point, we have

$$\frac{\partial E}{\partial \varphi_a} = \sum_i \frac{\partial E_i}{\partial \varphi_a} \propto \sum_i \varphi_{ab_i}\cot C_i + \varphi_{ac_i}\cot B_i = 0$$

So,

$$\sum_i (\cot C_i + \cot B_i)\,\varphi_a = \sum_i \left(\varphi_{b_i}\cot C_i + \varphi_{c_i}\cot B_i\right)$$

$$\Rightarrow \varphi_a = \frac{\sum_i \varphi_{b_i}\cot C_i + \varphi_{c_i}\cot B_i}{\sum_i \cot C_i + \cot B_i}$$

Assume we know coordinates of a triangle ABC. To calculate $\cot A$, we define $\boldsymbol{b} = \overrightarrow{AB}, \boldsymbol{c} = \overrightarrow{AC}$.

$$\cot A = \frac{bc\cos A}{bc\sin A} = \frac{\boldsymbol{b}\cdot\boldsymbol{c}}{|\boldsymbol{b}\times\boldsymbol{c}|} = \frac{\boldsymbol{b}\cdot\boldsymbol{c}}{2S}$$

The denominator $2S$ is the same for a triangle.