Energy Density for Electrostatic Field

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$$E_{\pm} = \frac{kq_{\pm}}{r_{\pm}^3}\boldsymbol{r}_{\pm}, \quad \boldsymbol{r}_{\pm} = \boldsymbol{r} \pm a\hat{z}$$

We want to prove

$$\frac{kq_+q_-}{2a} = \int \epsilon \boldsymbol{E}_+ \cdot \boldsymbol{E}_- \mathrm{d}V = \frac{kq_+q_-}{4\pi}\int \frac{\boldsymbol{r}_+ \cdot \boldsymbol{r}_-}{r_+^3 r_-^3}\mathrm{d}V$$

We choose $a = 1$, so:

$$\begin{aligned}
\int \frac{\boldsymbol{r}_+ \cdot \boldsymbol{r}_-}{r_+^3 r_-^3}\mathrm{d}V &= \iiint \frac{r^2-1}{\sqrt{(r^2+1)^2 - 4r^2\cos^2\theta}^3} r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi && (1)\\
&= 2\pi \int_0^\infty \mathrm{d}r \int_{-1}^1 \mathrm{d}x \frac{r^2(r^2-1)}{[(r^2+1)^2 - 4r^2x^2]^{3/2}}, \quad x = \cos\theta && (2)\\
&= 2\pi \int_0^\infty \mathrm{d}r\, \frac{r^2(r^2-1)}{(r^2+1)^3} \int_{-1}^1 [1-k^2x^2]^{-3/2}\mathrm{d}x, \quad k = \frac{2r}{r^2+1} && (3)\\
&= 2\pi \int_0^\infty \mathrm{d}r\, \frac{r^2(r^2-1)}{(r^2+1)^3} \frac{2}{\sqrt{1-k^2}} && (4)\\
&= 4\pi \int_0^\infty \frac{\mathrm{d}r}{(r+1/r)^2} \frac{r^2-1}{|r^2-1|} && (5)\\
&= -4\pi \left[\int_0^1 \frac{\mathrm{d}r}{(r+1/r)^2} + \int_\infty^1 \frac{\mathrm{d}r}{(r+1/r)^2}\right] && (6)\\
&= -4\pi \left[\int_0^1 \frac{\mathrm{d}r}{(r+1/r)^2} + \int_0^1 \frac{\mathrm{d}(1/r')}{(r'+1/r')^2}\right], \quad r' = 1/r && (7)\\
&= -4\pi \int_\infty^2 \mathrm{d}t/t^2 && (8)\\
&= 2\pi && (9)
\end{aligned}$$