Geometrical Optics in Continuum

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1 Fermat's Principle

Fermat's Principle is

$$\delta s = \delta \int n \mathrm{d}\ell = 0 \tag{1}$$

If we choose the path connecting two points to be

$$\boldsymbol{r} = \boldsymbol{r}(t), \quad t_0 \leq t \leq t_1 \tag{2}$$

We can write

$$s = \int n \frac{\mathrm{d}\ell}{\mathrm{d}t} \mathrm{d}t, \quad \frac{\mathrm{d}\ell}{\mathrm{d}t} = |\dot{\boldsymbol{r}}| \tag{3}$$

The Lagrangian about parameter t is

$$L = n \frac{\mathrm{d}\ell}{\mathrm{d}t} = n(\boldsymbol{r})|\dot{\boldsymbol{r}}| \tag{4}$$

then the Euler-Lagrange Equation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{r}}} - \frac{\partial}{\partial \boldsymbol{r}}\right) L = 0 \tag{5}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = n\hat{\tau}, \quad \hat{\tau} = \frac{\partial |\dot{\boldsymbol{r}}|}{\partial \dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}}/|\dot{\boldsymbol{r}}| = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\ell} \tag{6}$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\partial L}{\partial \boldsymbol{r}} = \frac{\mathrm{d}\ell}{\mathrm{d}t}\nabla n \tag{7}$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\ell} = \nabla n = \boldsymbol{F} \tag{8}$$

Generalized Momentum $\boldsymbol{p}$ is independent of the choice of t, and

2 Examples

2.1 Rectangular Coordinates

$$n(x, y, z) = f(x, y), \quad \boldsymbol{F} = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} \tag{9}$$

It is obvious that p_z are conserved.

2.2 Central Gradient

Define $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}\ell} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\ell} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\ell} \tag{10}$$

$$= \hat{\tau} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} \tag{11}$$

$$= \boldsymbol{r} \times \boldsymbol{F} \tag{12}$$

$\boldsymbol{L}$ is conserved for a spherical symmetric distribution

$$n(\boldsymbol{r}) = f(r), \quad \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{0} \tag{13}$$

L_z is conserved for

$$n(\boldsymbol{r}) = f(r, \theta), \quad (\boldsymbol{r} \times \boldsymbol{F})_z = 0 \tag{14}$$