

Jacobi Identity for Classical Possion Bracket

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Definition of Classical Possion Bracket:

$$[f, g] = \sum_{i=1,2,\dots,n} \frac{\partial(f, g)}{\partial(p_i, q_i)} = \sum_{i=1,2,\dots,n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

If we define $p_{-i} \stackrel{\text{def}}{=} q_i$, and $I = \{\pm 1, \pm 2, \dots, \pm n\}$, we can rewrite it:

$$[f, g] = \sum_{i \in I} \text{sgn}(i) \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_{-i}}$$

Jacobi Identity is $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$. With the new notation, we can rewrite it as

$$\sum_{\vec{fgh}} [f, [g, h]] = \sum_{\vec{fgh}} \sum_{i \in I} \text{sgn}(i) \frac{\partial f}{\partial p_i} \frac{\partial}{\partial p_{-i}} \left(\sum_{j \in I} \text{sgn}(j) \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial p_{-j}} \right) \quad (1)$$

$$= \sum_{\vec{fgh}} \sum_{i, j \in I} \text{sgn}(ij) \left(\frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_{-j}} \frac{\partial^2 g}{\partial p_{-i} \partial p_j} + \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial p_{-i} \partial p_{-j}} \right) \quad (2)$$

$$= \sum_{\vec{fgh}} \sum_{i, j \in I} \text{sgn}(ij) \left(\frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_{-j}} \frac{\partial^2 g}{\partial p_{-i} \partial p_j} + \frac{\partial h}{\partial p_i} \frac{\partial f}{\partial p_j} \frac{\partial^2 g}{\partial p_{-i} \partial p_{-j}} \right) \quad (3)$$

$$= \sum_{\vec{fgh}} \sum_{i, j \in I} \text{sgn}(ij) \left(\frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_{-j}} \frac{\partial^2 g}{\partial p_{-i} \partial p_j} + \frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_j} \frac{\partial^2 g}{\partial p_{-i} \partial p_{-j}} \right) \quad (4)$$

$$= \sum_{\vec{fgh}} \sum_{i, j \in I} \text{sgn}(ij) \left(\frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_{-j}} \frac{\partial^2 g}{\partial p_{-i} \partial p_j} - \frac{\partial f}{\partial p_i} \frac{\partial h}{\partial p_{-j}} \frac{\partial^2 g}{\partial p_{-i} \partial p_j} \right) \quad (5)$$

$$= 0 \quad (6)$$