From d'Alembert to Lagrange

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Statics Principle of Virtual Work (Minimal Potential Energy):

$$Q \stackrel{\text{def}}{=} -\frac{\delta V}{\delta q} = \frac{\delta W}{\delta q} = \frac{\boldsymbol{F}_i \cdot \delta \boldsymbol{x}_i}{\delta q} = 0$$

Dynamics d'Alembert's Principle—Counterpart of Principle of Virtual Work(We have neglected the work for constraints):

$$\frac{(\boldsymbol{F}_i - m_i \boldsymbol{a}_i) \cdot \delta \boldsymbol{x}_i}{\delta q} = 0$$

i.e.

$$\begin{aligned}
Q \stackrel{\text{def}}{=} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} &= m_i \boldsymbol{a}_i \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} && (1)\\
&= m_i \frac{\mathrm{d} \boldsymbol{v}_i}{\mathrm{d} t} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q} && (2)\\
&= m_i \frac{\mathrm{d}}{\mathrm{d} t}\left(\boldsymbol{v} \cdot \frac{\partial \boldsymbol{x}_i}{\partial q}\right) - m_i \boldsymbol{v} \cdot \frac{\mathrm{d}}{\mathrm{d} t} \frac{\partial}{\partial q} \boldsymbol{x}_i && (3)\\
&= m_i \frac{\mathrm{d}}{\mathrm{d} t}\left(\boldsymbol{v} \cdot \frac{\partial \dot{\boldsymbol{x}}_i}{\partial \dot{q}}\right) - m_i \boldsymbol{v} \cdot \frac{\partial}{\partial q} \frac{\mathrm{d}}{\mathrm{d} t} \boldsymbol{x}_i && (4)\\
&= \frac{\mathrm{d}}{\mathrm{d} t}\left(m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}}\right) - m_i \boldsymbol{v}_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial q} && (5)\\
&= \frac{\mathrm{d}}{\mathrm{d} t} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} && (6)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d} t} \frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}\right) T && (7)
\end{aligned}$$

For conservative force,

$$
\begin{aligned}
Q = -\frac{\partial V}{\partial q}, \quad \frac{\partial V}{\partial \dot{q}} = 0 \\
\Rightarrow \quad Q = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q} \right) V
\end{aligned} \tag{8}
$$

As a result of Eq. 7 and 8, if we define $L = T - V$, we have

$$
\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q} \right) L = 0
$$