4. (Selling a plane)

(a) Solution: Since the distribution of the sequence is uniform, by symmetry, we know that each of the first $i-1$ offers has an equal probability of being the highest among the first $i-1$ offers. Let $F_{m,i-1}$ denote the event that the best among the first $i-1$ offers is in the first m offers. Then

$$Pr[F_{m,i-1}] = m/(i-1)$$

(b) Solution: In order for B_i to happen, the largest among the first $i-1$ numbers must be in the first m numbers, otherwise, another number between $m+1$ and $i-1$ will be chosen instead of i. The second condition is that the i^{th} number must be the largest number in order for B_i to happen. The second event is denoted by L_i.

$$\begin{aligned} Pr[B_i] &= Pr[F_{m,i-1}]Pr[L_i] \\ &= \frac{m}{i-1}\frac{1}{n} = \frac{m}{n}\frac{1}{i-1} \end{aligned}$$

(c) Solution: Based on the strategy we are employing, we know that $Pr[B_i] = 0$ for $i \leq m$. And also note that the events B_i are mutually exclusive.

$$\begin{aligned} Pr[B] &= \sum_{i=1}^{n} Pr[B_i] \\ &= \sum_{i=1}^{m} Pr[B_i] + \sum_{j=m+1}^{n} Pr[B_j] \\ &= \sum_{i=m=1}^{n} Pr[B_i] = \frac{m}{n}\sum_{i=m+1}^{n}\frac{1}{i-1} \end{aligned}$$

(d) Solution: We use the approximation $H(n) = \sum_{i=1}^{n}\frac{1}{i} = \ln n + \Theta(1)$ and also that $H(x) - \ln x \geq H(y) - \ln y \geq 0$ for $x > y$. Note that $\lambda = \lim_{x\to\infty}(H(x) - \ln x) > 0$ is called Euler–Mascheroni constant. These are properties from the approximation of Harmonic Numbers the derivation of which will be omitted here (It involves using integration to get the approximation as covered in Discussion).

$$\begin{aligned} Pr[B] &= \frac{m}{n}\sum_{i=m+1}^{n}\frac{1}{i-1} \\ &= \frac{m}{n}\sum_{j=m}^{n-1}\frac{1}{j} = \frac{m}{n}\big(H(n-1) - H(m-1)\big) \geq \frac{m}{n}\big(\ln(n-1) - \ln(m-1)\big) = \frac{m}{n}\ln\frac{n-1}{m-1} \qquad (1) \\ Pr[B] &\geq \frac{m}{n}\sum_{j=m+1}^{n}\frac{1}{j} = \frac{m}{n}\big(H(n) - H(m)\big) \geq \frac{m}{n}\big(\ln n - \ln m\big) = \frac{m}{n}\ln\frac{n}{m} \qquad (2) \end{aligned}$$

Thus from (1) and (2), we have shown that

$$\frac{m}{n}\ln\frac{n}{m} \leq Pr[B] \leq \frac{m}{n}\ln\frac{n-1}{m-1}.$$

The equality is taken when $m = n$ which should not happen, and when $m, n \to \infty$.

(e) Solution: Let $x = m/n > 0$. So we can take the derivative of the function $f(x) = x \ln \frac{1}{x}$ and set it to 0.

$$f'(x) = -(\ln x + 1) = 0$$
$$\frac{m}{n} = x = \frac{1}{e}$$

When $\frac{m}{n} = \frac{1}{e}$ i.e. $m = \frac{n}{e}$, $\frac{m}{n} \ln \frac{n}{m}$ has a maximum value of $\frac{1}{e}$. A lower bound of $\frac{1}{e}$ is obtained for $Pr[B]$ when $m = \frac{n}{e}$.