Randomness in Computing

CS 537

LECTURE 5

Last time

- Bernoulli and binomial RVs
- Jensen's inequality
- Conditional expectation

Today

- Conditional expectation
- Branching process
- Geometric RVs
- Coupon collector problem

Conditional expectation: definition

- For random variables X and Y,

  the conditional expectation of X given Y,

  denoted $E[X|Y]$,

  is a random variable that depends on Y.

  Its value, when $Y = y$, is $E[X\ |Y = y]$.

- Example: Let N be the number you get when you roll a die. You roll a fair coin N times and get H heads.

  Find $E[H|N]$.

$$E[H|N = n] = n/2.$$

$$E[H|N] = N/2.$$

Law of total expectation: compact form

- Lemma. For any two random variables X and Y,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y] \Pr[Y = y] \,.$$

- In other words,

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big].$$

- Example: Let N be the number you get when you roll a die. You roll a fair coin N times and get H heads.

  Find $\mathbb{E}[H]$.

$$\mathbb{E}[H] = \mathbb{E}\big[\mathbb{E}[H|N]\big] = \mathbb{E}\left[\frac{N}{2}\right] = \frac{3.5}{2} = 1.75.$$

Law of total expectation: application

Branching Process: A program P tosses n coins with bias p and calls itself recursively for every HEADS.

If we call P once, what is total expected number of calls to P that will be generated?

Branching process

Geometric random variables

- A geometric random variable with parameter p, denoted Geom(p), is the number of tosses of a coin with bias p until it lands on HEADS.
- Lemma. The probability distribution of X =Geom(p) is
$$\Pr[X = n] = (1 - p)^{n-1} p$$
for all $n = 1,2, \dots$.
- Lemma. The expectation of X =Geom(p) is
$$\mathbb{E}[X] = 1/p.$$

Exercise

- What is the distribution of the number of rolls of a die until you see a 6?
- What is the expected number of rolls until you see a 6?

Exercise

- You roll a die until you see a 6. Let X be the number of 1s you roll. Compute E[X].
- Hint: Let N be the number of rolls.
- Solution: We will condition on N:

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|N]\big].$$

$$\mathbb{E}[X|N=n] = \frac{n-1}{5}$$

$$N \sim Geom(1/6)$$

$$\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|N]\big] = \mathbb{E}\left[\frac{N-1}{5}\right] = \frac{6-1}{5} = 1.$$

Exercise

- You roll a die until you see a 6. Let S be the sum of the rolls. Compute E[S].
- Hint: Let N be the number of rolls.
- Solution: We will condition on N:

$$E[S] = E\big[E[S|N]\big].$$

$$E[S|N = n] = 3(n - 1) + 6 = 3n + 3$$

$$N \sim Geom(1/6)$$

$$E[S] = E\big[E[S|N]\big] = E[3N + 3] = 3 \cdot 6 + 3 = 21$$

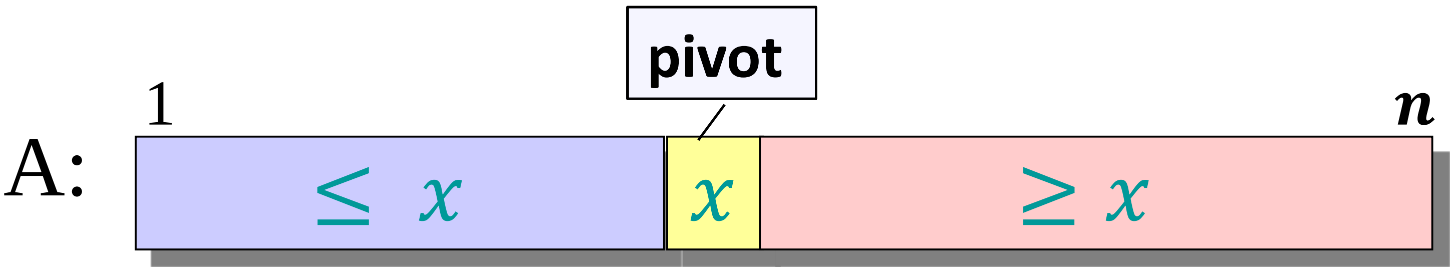
Coupon Collector's Problems

- There are n coupons.
- Each cereal box has 1 coupon chosen u.i.r.
- X = # of boxes bought until at least one copy of each coupon is obtained.
- Find $\mathbb{E}[X]$.

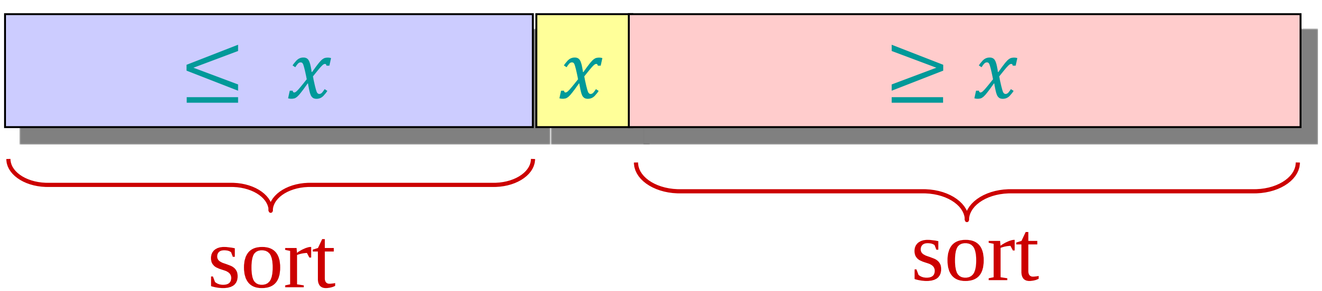
Solution: Let $X_i =$ # of boxes bought while you had exactly $i - 1$ different coupons.

Quicksort: divide and conquer

- Find a *pivot* element
- Divide: Find the correct position of the pivot by comparing it to all elements.



- Conquer: Recursively sort the two parts, resulting from removing the pivot.



Quicksort

QuickSort(array A, positive integers ℓ, r)

1. **if** $\ell < r$
2. **then** $p \leftarrow \mathrm{Partition}(\mathrm{A}, \ell, \mathrm{r})$
3. QuickSort $(\mathrm{A}, \ell, p-1)$
4. QuickSort $(\mathrm{A}, p+1, r)$

Initial call: QuickSort (A, 1, n)

Example of partitioning

6	10	13	5	8	3	2	11
i	j						

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Example of partitioning

6	10	13	5	8	3	2	11

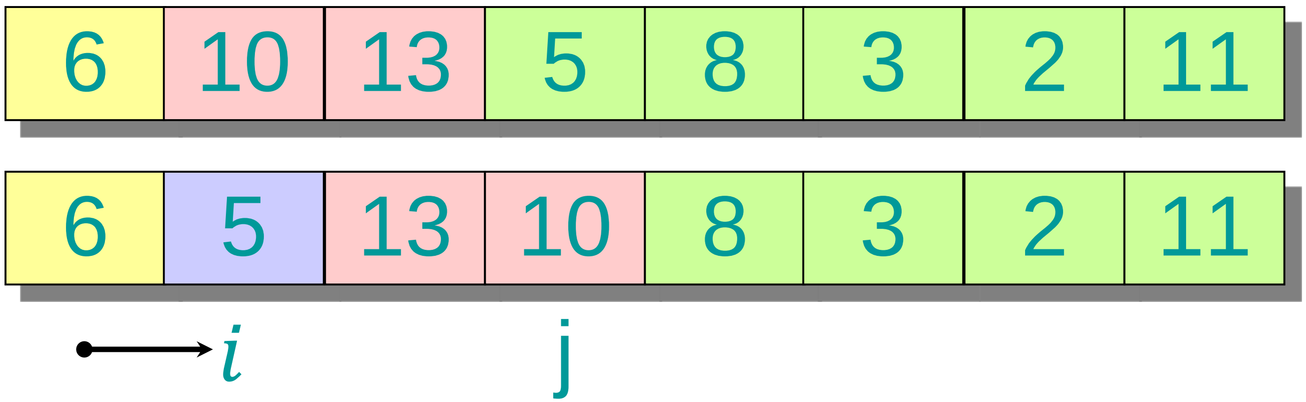
i j

Example of partitioning

6	10	13	5	8	3	2	11

i j

Example of partitioning



Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11
	i			j			

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Example of partitioning

6	10	13	5	8	3	2	11

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i j

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11

$\longrightarrow i$ $\qquad\qquad j$

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

Example of partitioning

6	10	13	5	8	3	2	11
6	5	13	10	8	3	2	11
6	5	3	10	8	13	2	11
6	5	3	2	8	13	10	11
			i			j	

Example of partitioning

6	10	13	5	8	3	2	11

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Example of partitioning

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i $\longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

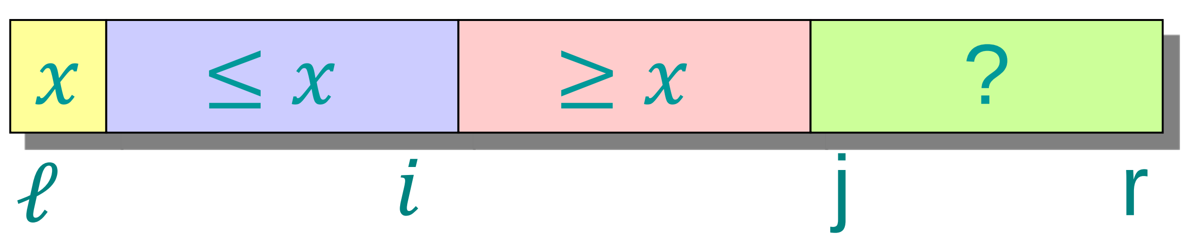
2	5	3	6	8	13	10	11

i

Partitioning algorithm

Partition (array A, positive integers ℓ, r)

```
x ← A[ℓ]          ///A[ℓ] becomes the pivot
i ← ℓ
for j = ℓ + 1 to r
    if A[j] ≤ x
        then i ← i + 1
             SWAP(A[i], A[j])
SWAP(A[ℓ], A[i])
return i
```



Exercise

How many comparisons does Quicksort perform on sorted array?

Answer:

$$(n-1)+(n-2)+\cdots+2+1=\frac{n(n-1)}{2}=\Omega(n^2)$$

How many comparisons does Quicksort perform if, in every iteration, the pivot splits the array into two halves?

Answer:

Let

$$C(n)=\Theta(n\log n)$$

Randomized Quicksort

BIG IDEA:

Partition around a ***random*** element.

- Analysis is similar when the input arrives in random order.
- But randomness in the input is unreliable.
- Rely instead on random number generator.

Analysis of Randomized Quicksort

- Theorem. If Quicksort chooses each pivot uniformly and independently at random from all possibilities then, for any input, the expected number of comparisons is

$$2n \ln n + O(n).$$