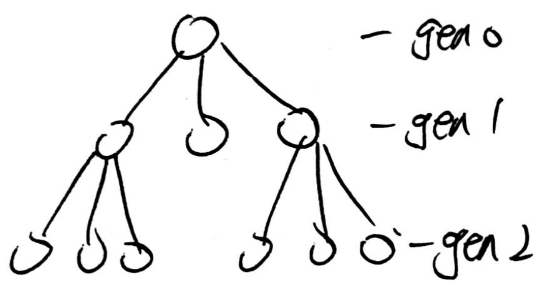
Law of Total Expectation :

Branching Process :



Let Y_i = # recursive calls in gen i.
for $i = 0, 1, 2 \ldots$

Y = total number of recursive calls.

$\mathbb{E}[Y] = \sum_{i=1}^{\infty} \mathbb{E}[Y_i]$

$\mathbb{E}[Y_0] = 1$; $\mathbb{E}[Y_1] = np$

$Y_i \sim Bin(n, p)$

$\mathbb{E}[Y_2] = \mathbb{E}[\mathbb{E}[Y_2 | Y_1]]$ ---- $(*)$

$\cancel{\mathbb{E}[Y_2 | Y_2]} =$

$\mathbb{E}[Y_2 | Y_1 = y_1] \sim Bin(y_1 n, p)$.

$= y_1 np$

$(*) = \mathbb{E}_{Y_1}[y_1 np]$

$= (np)^2$

$\mathbb{E}$

$\ldots \quad \mathbb{E}[Y_i | Y_{i-1}] = Y_{i-1} \cdot np$.

$\mathbb{E}[\mathbb{E}[Y_i | Y_{i-1}]] = \mathbb{E}[Y_{i-1} \cdot np]$

$= \mathbb{E}[Y_{i-1}] \cdot np$

$= (np)^{i}$

Axiom, Lemma.

$E[Y] = \sum_{i=0}^{\infty} (np)^i$ $\quad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ $(x<1)$ geometric series.

$$= \begin{cases} \frac{1}{1-np} & \text{if } np<1 \\ \text{unbounded.} \end{cases}$$

Geometric Random Variables

Lemma. $Geom(p)$. number of tosses till it lands on HEADS.

$$Pr[X=n] = (1-p)^{n-1}p \quad \text{for all } n = 1, 2, \dots$$

Lemma $X = Geom(p)$

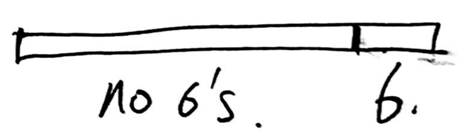
$E[X] = \frac{1}{p}$ (~~to be~~ proof to be done)

$E[X] = \frac{1}{1/6} = 6$. Expected number of 1's.

$5 \times \frac{1}{5} = 1$

$E[E[\dots]] = E[$

$E[X] = E[E[X|N]]$



$E[X|N=n] = \frac{n-1}{5}$

where $N \sim Geom(\frac{1}{6})$.

$\Rightarrow E[X] = E[\frac{N-1}{5}] = \frac{6-1}{5} = 1$

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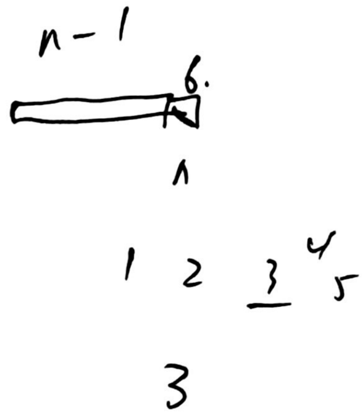
E.G. Sum (S) until you see a 6



$E[S]$.

$$E[S] = E[E[S|N]]$$

$$E[S|N=n] = 3\times(n-1)+6$$

$$= 3n+3.$$

$$E[E[S|N=n]] = 3\cdot 6+3 = 21$$

Coupon Collector's Problems n coupons.

X_i # of boxes while you had $i-1$ different coupons.

$\cancel{E[X_i]} =$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X_i \sim Geom\Big(\underbrace{\frac{n-i+1}{n}}_{p_i}\Big) \qquad E[X_i] = \frac{1}{p_i} = \frac{n}{n-i+1}$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}_{H_n}$$

$$\simeq n(\ln n + \theta(1))$$

$$\simeq n\ln n + \theta(n).$$

H_n Harmonic number. $H(n)$ nth Harmonic number.

Lemma. $\cancel{H(n)}$

$$\ln n \le H(n) \le \ln n + 1$$

Proof intuition: $\int_1^n \frac{1}{x}\,dx = \ln n$