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For any edge e, let Y_e = # routes that pass through edge e.

$P_i = (e_1 \dots e_k)$

Then $X \leq Y_{e_1} + Y_{e_2} + \dots \cancel{+} Y_{e_k}$

Lecture 12: Finishing Permutation routing on Hypercube. Birthday Paradox, balls and bins model

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Routing Permutations

Phase 1 : Route each packet to a uniformly random destination (independently) using bit fixing and FIFO queues.

Main combinatorial lemma : Let $P_i = (e_1, e_k)$ be the path of packet i. Let s be the set of packets ($\neq i$) that use P_i then the delay of i is $\leq |S|$.

Lemma : Consider any packet i.

It fails to reach its designation in phase 1 $\leq 3n$ steps w.p $\frac{1}{N^2}$.

$\boxed{\text{Proof}}$: Let X be the # of packets (other than i) that use at least 1 edge of P_i.

For each edge e Let Y_e = # of routes that pass through e

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$\boxed{k \leq n}$

Then $X \leq \sum_{j=1}^{k} Y_{e_j}$

$\mathbb{E}[X] \leq \sum \mathbb{E}[Y_{e_j}] = k\,\mathbb{E}[Y_{e_1}]$

<u>Method 2:</u> to calculate $\mathbb{E}[Y_e]$

Consider $e = (x_1 \dots x_n, x_1 \dots \bar{x}_d \dots x_n)$ in dimension d.

Only packets with sources $* \dots * x_d \dots x_n$ can traverse e (there are 2^{d-1} packets).

- To traverse e, such a packet must have designation: $x_1 \dots x_d * \dots *$ which happens with probability $\frac{1}{2^d}$.

Thus. $\mathbb{E}[Y_e] = 2^{d-1} \cdot \frac{1}{2^d} = \frac{1}{2}$

$\mathbb{E}[X] \leq \frac{k}{2} \leq \frac{n}{2}$

$K =$ R.V denote the length of path of packet.

Actually: $\mathbb{E}[X] = \mathbb{E}[\underbrace{\mathbb{E}[X \mid K]}_{\leq \frac{K}{2}}]$.

$\leq \mathbb{E}[\frac{K}{2}]$

$= \frac{K}{4}$

Date . . . No. $\mu_L \leq \mu \leq \mu_H$

By Chernoff,
$$\Pr[X \geq 2n] = \Pr\left[X \geq \underbrace{(1+7)\frac{n}{4}}_{(1+\delta)\mu_H}\right]$$

$$\leq \left(\frac{e^7}{(1+7)^{1+7}}\right)^{n/4} \leq \left(\frac{e}{8}\right)^{\frac{8n}{4}} \leq \frac{1}{2^{2n}} = \frac{1}{N^2}.$$

By a union bound
over all N packets, the probability that
at least 1 packet fails to complete
phase 1 in 3n steps is at most

$$N \cdot \frac{1}{N^2} = \frac{1}{N}.$$

Balls into bins model

m balls are thrown into n bins.
each ball falls into a uniformly random bin.

Q1 Is it more likely that there is a collision
or no collision?
(Birth day Problem).

Q2 How many balls are in the fullest bin?
(Load balancing).

Q3 How many bins are empty?

Q4 What does the distribution of balls in bins
look like.

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Birthday Problem

$n = 365$ bins (days)
For which m is the probability of collision $\frac{1}{2}$

E_i = event that ball i falls into an empty bin.

$$Pr[\text{no collision}] = Pr[E_1 \cap E_2 \cap \dots E_m]$$
$$= Pr[E_1]\, Pr[E_2 | E_1]\, Pr[E_3 | E_1, E_2] \dots Pr[E_m | E_1 \cap \dots E_{m-1}]$$
$$= \cancel{1(n-1)(n-}$$
$$= 1 \cdot \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) \dots \left(\frac{n-(m-1)}{n}\right)$$
$$= 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{m-1}{n}\right)$$
$$\leq e^{-\frac{1}{n}} e^{-\frac{2}{n}} \dots e^{-\frac{(m-1)}{n}}$$
$$= e^{-\frac{1}{n}(1+2+\dots+m-1)}$$
$$= e^{-\frac{1}{n}\sum_{i=1}^{m-1} i} = e^{-\frac{1}{n} \cdot \frac{m(m-1)}{2}} \approx e^{\frac{m^2}{2n}}$$
$$m = \Omega(\sqrt{n})$$

Bucket sort

n integer from range $[r]$
If $r \leq n$, we can sort in time $O(n)$

- Use possible values as buckets
- keep a linked list for buckets
- make a pass over our list and put

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each element in the correct bucket,
- concatenate the lists.

If $r > n$?

Theorem: Suppose that n divides r.
if we choose n integers u.i.d from range r. then can sort in expected $O(n)$ time

3/5/20 Lecture 13: Poisson distribution, Poisson approximation

m balls into n bins.

The probability that bin 1 is empty:

$$\left(1-\frac{1}{n}\right)^m \approx (5)\ e^{-\frac{m}{n}}$$

Probability p_r that bin 1 has r balls is

$$P_r = \binom{m}{r}\left(\frac{1}{n}\right)^r\left(1-\frac{1}{r}\right)^{m-r}.$$

$$= \frac{1}{r!}\left(\frac{m}{n}\cdot\frac{m-1}{n}\cdots\frac{m-r+1}{n}\right)\left(1-\frac{1}{n}\right)^{m-r}.$$

$$\approx \frac{1}{r!}\left(\frac{m}{n}\right)^r\cdot e^{-\frac{m}{n}}.$$

$$P_r \approx \frac{\mu^r\cdot e^{-\mu}}{r!} \quad \text{where } \mu = \frac{m}{n}$$