2. (Exercise 4.10 from MU)

(a) Solution: Let R.V. X_i be the amount of money the casino lost in game i. $\mathbf{E}(X_i) = (3-1)\frac{4}{25} + (100-1)\frac{1}{200} + (1)(1 - \frac{4}{25} - \frac{1}{200}) = -0.02$. In expectation, in each game, the casino will make 0.02 dollars. Let X denote the total amount of loss the casino has in the first 1 million games, i.e. $X = \sum_{i=1}^{1,000,000} X_i$ with $\mathbf{E}[X] = 1,000,000 \cdot (-0.02) = -20,000$. Using Theorem 4.12, $a = -1, b = 99$, we can set $\epsilon = 0.03$,

$$Pr\Big[\frac{1}{1,000,000}\sum_{i=1}^{1,000,000} X_i - \mu \geq \epsilon\Big] \leq e^{-2n\epsilon^2/(b-a)} \tag{1}$$

$$Pr\Big[\frac{1}{1,000,000}X - (-0.02) \geq 0.03\Big] \leq e^{-0.18} \tag{2}$$

$$Pr\Big[X \geq 10,000\Big] \leq e^{-0.18} = 0.84 \tag{3}$$

$$\tag{4}$$

(b) Solution: Since X_i are mutually independent,

$$\begin{aligned}
\mathbf{E}[e^{tX}] &= \mathbf{E}\Big[\exp\big\{t\sum_{i=1}^{1,000,000} X_i\big\}\Big] &(5)\\
&= \mathbf{E}\Big[\prod_{i=1}^{1,000,000} \exp\big\{tX_i\big\}\Big] &(6)\\
&= \prod_{i=1}^{1,000,000} \mathbf{E}[e^{tX_i}] &(7)\\
&= \prod_{i}^{1,000,000} \sum_{X_i \in \{-1,2,99\}} Pr[X_i]e^{tX_i} &(8)\\
&= \prod_{i}^{1,000,000} \Big(\frac{4}{25}e^{2t} + \frac{1}{200}e^{99t} + \frac{167}{200}e^{-t}\Big) &(9)
\end{aligned}$$

(c) Solution:

$$\begin{aligned}
Pr[X \geq 10,000] &= Pr[e^{tX} \geq e^{10,000t}] &(10)\\
&\leq \frac{\mathbf{E}(e^{tX})}{e^{10,000t}} &(11)\\
&= \frac{\prod_i^{1,000,000}\Big(\frac{4}{25}e^{2t} + \frac{1}{200}e^{99t} + \frac{167}{200}e^{-t}\Big)}{e^{10,000t}} &(12)\\
&= \frac{\prod_i^{1,000,000}\Big(\frac{4}{25}e^{2\cdot 0.0006} + \frac{1}{200}e^{99\cdot 0.0006} + \frac{167}{200}e^{-0.0006}\Big)}{e^{10,000\cdot 0.0006}} &(13)\\
&\approx 0.000160646 &(14)
\end{aligned}$$

where we have used $t = 0.0006$ and also results from (9). Notice this is a march tighter bound compared to the bound in (a).