3. (Generalization of Randomized Median Algorithm)

(a) Solutions: The main idea is to replace 1/2 with k/n. So we need the following modifications

- line 3: Let d be the $\left(\lfloor \frac{k}{n} n^{3/4} - \sqrt{n} \rfloor\right)$th smallest element in the sorted set R.
- line 4: Let u be the $\left(\lceil \frac{k}{n} n^{3/4} + \sqrt{n} \rceil\right)$th smallest element in the sorted set R.
- line 6: If $\ell_d > k$ or $\ell_u > n - k$ then FAIL.
- line 8: Output the $\left(k - \ell_d + 1\right)$th element in the sorted order of C.

(b) Solutions: We list the running time in big-O notation for each of the time consuming steps as follows:

- Choosing set R from A assuming $O(1)$ time element access for A: $O(n^{3/4})$.
- Sorting the set R: $O(n^{3/4} \log n^{3/4}) \approx O(n)$.
- Partition set A based on the values of d and u: $O(n)$.
- Sorting set C if $|C| \leq 4n^{3/4}$: $O(n^{3/4} \log n) \approx O(n)$.

Thus the total running time of the modified algorithm is $O(n)$.

(c) Solutions:

$$\begin{aligned}
&\mathcal{E}_1 : Y_1 = |\{r \in R | r \leq k\}| < \tfrac{k}{n} n^{3/4} - \sqrt{n} \\
&\mathcal{E}_2 : Y_2 = |\{r \in R | r \geq k\}| < (1 - \tfrac{k}{n}) n^{3/4} - \sqrt{n} \\
&\mathcal{E}_{3,1} : \text{at least } 2n^{3/4} \text{ elements of C are greater than the } k\text{th smallest element in } A \\
&\mathcal{E}_{3,2} : \text{at least } 2n^{3/4} \text{ elements of C are smaller than the } k\text{th smallest element in } A
\end{aligned} \tag{1}$$

(d) Solutions:

For $\mathcal{E}_1$, define a random variable X_i by

$$X_i = \begin{cases} 1 & \text{if the } i \text{ th sample is less than or equal to the } k\text{th smallest element.} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The X_i are independent. The probability that a sample is smaller than the kth smallest element is $Pr[X_i = 1] = k/n$. The event $\mathcal{E}_1$ is equivalent to

$$Y_1 = \sum_{i=1}^{n^{3/4}} X_i < \frac{k}{n} n^{3/4} - \sqrt{n} \tag{3}$$

Thus, $Pr[\mathcal{E}_1] = Pr[Y_1 < \frac{k}{n} n^{3/4} - \sqrt{n}]$.

For $\mathcal{E}_{3,1}$, let us bound the probability the first event occurs. If there are at least $2n^{3/4}$ elements of C above the kth smallest element, then the order of u in the sorted order of S was at least $k + 2n^{3/4}$ and thus the set R has at least $(1-k/n)n^{3/4}-\sqrt{n}$ samples among the $n-k-2n^{3/4}$ largest elements in A, the input array.

Let X be the number of samples among the $n-k-2n^{3/4}$ largest elements in A. Let $X=\sum_{i=1}^{n^{3/4}} X_i$.

$$X_i = \begin{cases} 1 & \text{if the } i \text{ th sample is among the } n-k-2n^{3/4} \text{ largest elements in } S, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then $Pr[\mathcal{E}_{3,1}] = Pr\big[X \geq (1-k/n)n^{3/4} - \sqrt{n}\big]$