1. (Random hats)

(a) Solution: Let X be the random variable that denote the number of pair of changes, and X_{ij} where $i < j$ be the indicator random variable that people i and j exchanged their hats. Then from linearity of expectation,

$$\begin{aligned}
\mathbf{E}(X) &= \mathbf{E}\Big(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}\Big) && (1)\\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbf{E}(X_{ij}) && (2)\\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{n}\frac{1}{n-1} && (3)\\
&= \frac{n(n-1)}{2}\frac{1}{n(n-1)} = \frac{1}{2} && (4)
\end{aligned}$$

(b) Solution: Since X_{ij} is a binary indicator Bernoulli random variable taking values from $\{0,1\}$, then the random variable X_{ij}^2 is also a Bernoulli random variable taking values from $\{0,1\}$, and $\mathbf{E}(X_{ij}^2) = Pr[X_{ij}^2 = 1] = Pr[X_{ij} = 1] = \frac{1}{n(n-1)}$.

$$\begin{aligned}
Var[X^2] &= \mathbf{E}[X^2] - \mathbf{E}[X]^2 && (5)\\
&= \mathbf{E}\Big[\Big(\sum_{i<j} X_{ij}\Big)^2\Big] - (1/2)^2 && (6)\\
&= \mathbf{E}\Big(\sum_{i<j} X_{ij}^2 + \sum_{i\neq m \text{ or } j\neq n} X_{ij}X_{mn}\Big) - 1/4 && (7)\\
&= \sum_{i<j} \mathbf{E}(X_{ij}^2) + \sum_{i\neq m \text{ or } j\neq n} \mathbf{E}(X_{ij}X_{mn}) - 1/4 && (8)\\
&= 1/2 + \sum_{i\neq j\neq m\neq n, i<j, m<n} \mathbf{E}(X_{ij}X_{mn}) - 1/4 && (9)\\
&= 1/2 + \binom{n}{2}\binom{n-2}{2} Pr[X_{ij} = 1 \;\&\&\; X_{mn} = 1] - 1/4 && (10)\\
&= 1/2 + \frac{n(n-1)}{2!}\frac{(n-2)(n-3)}{2!}\frac{1}{n(n-1)}\frac{1}{(n-2)(n-3)} - 1/4 && (11)\\
&= 1/2 + 1/4 - 1/4 && (12)\\
&= 1/2 && (13)
\end{aligned}$$

From (7) to (8) we have used linearity of expectation, and from (8) to (9) we have used the fact that if i, j and m, n are not four unique numbers, then $X_{ij}X_{mn}$ will be zero. Thus we have only kept the terms that are non-zero.