2. (Improved Randomized Min-Cut Algorithm)

(a) Solutions: If we run the algorithm twice, then the number of edge contractions will be two times the number of contractions in a single run, $2(n-2)$.

The probability of missing a min cut in both runs will be bounded by

$$\Big(1 - \frac{2}{n(n-1)}\Big)^2$$

Thus, the probability of finding a min cut with two runs will be bounded by

$$1 - \Big(1 - \frac{2}{n(n-1)}\Big)^2$$

(b) Solutions: When we contract the graph from n vertices down to k vertices, the number of contraction is $n-k$. For the second step, the number of contractions is $l(k-2)$. So the total number of contraction is $(n-k)+l(k-2)$.

Assume that the number of edges in the min-cut set C is m. Let E_i be the event that the edge contracton contracted in iteration i is not in C, adn let $F_i = \cap_{j+1}^{i} E_j$. In the first contraction, the probability of not choosing an edge in the min-cut set C to contract

$$Pr[E_1] = Pr[F_1] \geq 1 - \frac{2m}{nm} = 1 - \frac{2}{n}.$$

Conditioned on the event that in the first contraction, the edge contracted is not in C, we can continue the analysis,

$$Pr[E_2|F_1] \geq 1 - \frac{m}{m(n-1)/2} = 1 - \frac{2}{n-1}.$$

Similarly,

$$Pr[E_i|F_{i-1}] \geq 1 - \frac{m}{m(n-i+1)/2} = 1 - \frac{2}{n-i+1}.$$

Thus, the probability that the edges in the set C have never been contracted until we are left with k vertices is

$$\begin{aligned} Pr[F_{n-k}] &= Pr[E_{n-k}|F_{n-k-1}] \cdot Pr[E_{n-k-1}|F_{n-k-2}] \cdots Pr[E_2|F_1] \cdot Pr[F_1] \\ &\geq \prod_{i=1}^{n-k} \left(1 - \frac{2}{n-i+1}\right) = \prod_{i=1}^{n-k} \left(\frac{n-i-1}{n-i+1}\right) \\ &= \frac{k(k-1)}{n(n-1)} \end{aligned}$$

Similarly for the second stage, for each copy of the l problems, we can apply the same analysis and bound the probability of finding the min-cut set of size m by

$$\frac{2}{k(k-1)}$$

Thus, the totally probability of finding the min-cut set after the two steps is bounded by

$$\frac{k(k-1)}{n(n-1)} \left(1 - (1 - \frac{2}{k(k-1)})^l\right)$$

(c) Solutions: The problem can be formulated as the following optimization problem,

$$\begin{aligned}
&\max_{k,l} \ \frac{k(k-1)}{n(n-1)}\left(1-(1-\frac{2}{k(k-1)})^{l}\right) \\
&\text{s.t. } (n-k)+l(k-2)=2(n-2)
\end{aligned}$$