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For any edge e, let Y_e = # routes that pass through edge e.

$P_i = (e_1 \dots e_k)$

Then $X \leq Y_{e_1} + Y_{e_2} + \dots \cancel{+} Y_{e_k}$

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Routing Permutations

Phase 1 : Route each packet to a uniformly random destination (independently) using bit fixing and FIFO queues.

Main combinatorial lemma : Let $p_i = (e_1, e_k)$ be the path of packet i. Let s be the set of packets ($\neq i$) that use p_i then the delay of i is $\leq |S|$.

Lemma : Consider any packet i.

It fails to reach its designation in phase 1 $\leq 3n$ steps w.p $\frac{1}{N^2}$.

$\boxed{\text{Proof}}$: Let X be the # of packets (other than i) that use at least 1 edge of P_i

For each edge e Let Y_e = # of routes that pass through e

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$\boxed{k \leq n}$

Then $X \leq \sum_{j=1}^{k} Y_{e_j}$

$\mathbb{E}[X] \leq \sum \mathbb{E}[Y_{e_j}] = k\,\mathbb{E}[Y_{e_1}]$

<u>Method 2:</u> to calculate $\mathbb{E}[Y_e]$

Consider $e = (x_1 \dots x_n, x_1 \dots \bar{x}_d \dots x_n)$ in dimension d.

Only packets with sources $* \dots * x_d \dots x_n$ can traverse e (there are 2^{d-1} packets).

- To traverse e, such a packet must have designation: $x_1 \dots x_d * \dots *$ which happens with probability $\frac{1}{2^d}$.

Thus $\mathbb{E}[Y_e] = 2^{d-1} \cdot \frac{1}{2^d} = \frac{1}{2}$

$\mathbb{E}[X] \leq \frac{k}{2} \leq \frac{n}{2}$

K = R.V denote the length of paths of packet.

Actually: $\mathbb{E}[X] = \mathbb{E}[\underbrace{\mathbb{E}[X \mid K]}_{\leq \frac{K}{2}}]$.

$\leq \mathbb{E}[\frac{K}{2}]$

$= \frac{K}{4}$

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$\mu_L \leq \mu \leq \mu_H$

By Chernoff,

$(1+\delta)\mu_H$

$$\Pr[X \geq 2n] = \Pr\left[X \geq (1+7)\frac{n}{4}\right]$$

$$\leq \left(\frac{e^7}{(1+7)^{1+7}}\right)^{n/4} \leq \left(\frac{e}{8}\right)^{\frac{8n}{4}} \leq \frac{1}{2^{2n}} = \frac{1}{N^2}.$$

By a union bound over all N packets, the probability that at least 1 packet fails to complete phase 1 in $3n$ steps is at most

$$N \cdot \frac{1}{N^2} = \frac{1}{N}.$$

Balls into bins model

m balls are thrown into n bins.
each ball falls into a uniformly random bin.

$\boxed{Q_1}$ Is it more likely that there is a collision or no collision?
(Birthday Problem).

$\boxed{Q_2}$ How many balls are in the fullest bin?
(Load balancing).

$\boxed{Q_3}$ How many bins are empty?

$\boxed{Q_4}$ What does the distribution of balls in bins look like.

Birthday Problem

$n = 365$ bins (days)
For which m is the probability of collision $\geq \frac{1}{2}$

E_i = event that ball i falls into an empty bin.

$Pr[\text{no collision}] = Pr[E_1 \cap E_2 \cap \ldots E_m]$

$= Pr[E_1]\, Pr[E_2 | E_1]\, Pr[E_3 | E_1, E_2] \ldots$

$Pr[E_m | E_1 \cap \ldots E_{m-1}]$

$= \cancel{1(n-1)(n-}$

$= 1 \cdot \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) \cdots \left(\frac{n-(m-1)}{n}\right)$

$= 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$

$\leq e^{-\frac{1}{n}} e^{-\frac{2}{n}} \cdots e^{-\frac{(m-1)}{n}}$

$= e^{-\frac{1}{n}(1+2+\ldots+m-1)}$

$= e^{-\frac{1}{n}\sum_{i=1}^{m-1} i} = e^{-\frac{1}{n} \cdot m\frac{(m-1)}{2}} \approx e^{\frac{m^2}{2n}}$

$m = \Omega(\sqrt{n})$

$\boxed{\text{Bucket sort}}$

n integer from range $[r]$
If $r \leq n$, we can sort in time $O(n)$

- Use possible values as buckets.
- Keep a linkedlist for buckets.
- make a pass over our list and put

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each element in the correct bucket,
- concatenate the lists.

If $r > n$?
Theorem: Suppose that n divides r.
if we choose n integers u.i.d from range r. then can sort in expected $O(n)$ time

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m balls into n bins.

The probability that bin 1 is empty:

$$\left(1-\frac{1}{n}\right)^{m} \approx (5) \; e^{-\frac{m}{n}}$$

Probability p_r that bin 1 has r balls is

$$P_r = \binom{m}{r}\left(\frac{1}{n}\right)^{r}\left(1-\frac{1}{r}\right)^{m-r}.$$

$$= \frac{1}{r!}\left(\frac{m}{n}\cdot\frac{m-1}{n}\cdots\frac{m-r+1}{n}\right)\left(1-\frac{1}{n}\right)^{m-r}.$$

$$\approx \frac{1}{r!}\left(\frac{m}{n}\right)^{r}\cdot e^{-\frac{m}{n}}.$$

$$P_r \approx \frac{\mu^r \cdot e^{-\mu}}{r!} \qquad \text{where } \mu = \frac{m}{n}$$