Randomness in Computing

CS 537

LECTURE 9

Last time

- Computing the median of an array

Today

- Chernoff Bounds

Tail Bounds So Far

- Markov. For a nonnegative random variable X and $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

- Chebyshev. For a random variable X and $a > 0$,

$$\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

Sums of independent RVs

- Bernoulli trials:
  $X_1, \ldots, X_n$ are mutually independent 0-1 RVs.
  $$\Pr[X_i = 1] = p$$
- Poisson trials (generalization):
  $X_1, \ldots, X_n$ are mutually independent 0-1 RVs.
  $$\Pr[X_i = 1] = p_i$$

≠ Poisson RVs

- Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then μ is
  $\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = p_1 + \cdots + p_n$
- Want to bound $\Pr[X \geq (1+\delta)\mu]$ for $\delta > 0$

  and $\Pr[X \leq (1-\delta)\mu]$ for $\delta \in (0,1)$

  in terms of μ and δ.

Obtaining the bounds

Ideas:

- Consider RV e^{tX}, where t is a parameter.
- Apply Markov for e^{tX}.
- Use independence of X_i (and hence e^{tX_i})
- Pick the value of t to get the best bound.

Aside:

- $\mathbb{E}[X^k]$ is called the k-th moment of X.
- $\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} \frac{t^k \, \mathbb{E}[X^k]}{k!}$ (power series)
- $\mathbb{E}[e^{tX}]$ is the moment-generating function of X.

Sums of independent RVs

Chernoff Bound (Upper Tail). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for any $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1]$,

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}.$$

Sums of independent RVs

Chernoff Bound (Lower Tail). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for any $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le e^{-\mu\delta^2/2}.$$

Sums of independent RVs

Chernoff Bound (Both Tails). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- for any $\delta \in (0,1)$,

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}.$$

Exercise

$$X \sim \text{Bin}\left(n, \frac{1}{2}\right) \qquad \Pr\left[X \geq \frac{3n}{4}\right] \leq ?$$

- Recall: $\mathbb{E}[X] =$, $\text{Var}[X] =$
- Markov: $\Pr[X \geq 3n/4] \leq$
- Chebyshev: $\leq 2/n$
- Chernoff: $\Pr[X \geq \quad] \leq e^{-}$

$$\Pr\left[X \leq \frac{n}{4}\right] \leq$$

$$\Pr\left[X \leq \frac{n}{2} - c\sqrt{n}\right] \leq \Pr\left[X \leq (1- \quad) \frac{n}{2}\right] \leq e^{-}$$

Exercise 2

- The Halting Problem Team wins each hockey game they play with probability 1/3. Assuming outcomes of the games are independent, derive an upper bound on the probability that they have a winning season in n games.

- The Halting Problem Team hires a new coach, and critics revise their probability of winning each game to 3/4. Derive an upper bound on the probability they suffer a losing season.