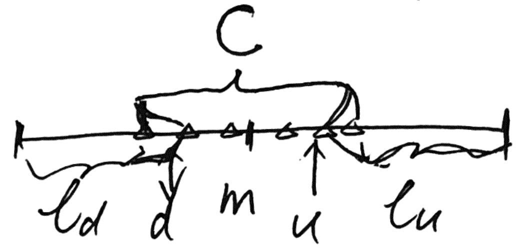
Randomized Median Algorithm



① Sample elements from S: u, d s.t. $d \le m \le u$

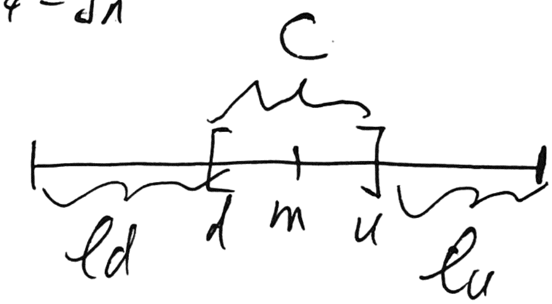
② $C = \{x \in S : d \le x \le u\}$ is small

Main Analysis Tool: Tail bounds on Binomial RVs.

$\mathcal{E}_1 : Y_1 = |\{r \in R : r \le m\}| < \frac{1}{2}n^{\frac{3}{4}} - \sqrt{n}$

$\mathcal{E}_2 : Y_2 = |\{r \in R : r \ge m\}| < \frac{1}{2}n^{\frac{3}{4}} - \sqrt{n}$

$\mathcal{E}_3 : |C| > 4n^{\frac{3}{4}}$



Claim: $\ell_u \Rightarrow \mathcal{E}_2$

Pf: Suppose $\ell_u > \frac{n}{2}$. Then the $(\frac{1}{2}n^{\frac{3}{4}} + \sqrt{n})$th smallest element of R is smaller than m.
Therefore element in R above m is the

$$Y_2 \le |R| - (\tfrac{1}{2}n^{\frac{3}{4}} + \sqrt{n}) = n^{\frac{3}{4}} - (\tfrac{1}{2}n^{\frac{3}{4}} + \sqrt{n})$$
$$= \tfrac{1}{2}n^{\frac{3}{4}} - \sqrt{n} \quad \boxtimes$$

Claim $\Pr[\text{FAIL}] \le \Pr[\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3]$.
$\le \Pr[\mathcal{E}_1] + \Pr[\mathcal{E}_2] + \Pr[\mathcal{E}_3]$

Pf (in book). $\Pr[\mathcal{E}_1], \Pr[\mathcal{E}_2]$

Lemma 3.14 $\Pr[\mathcal{E}_3] \le \frac{1}{2}n^{-\frac{1}{4}}$

Pf: $\mathcal{E}_{31}$: $\ge 2n^{\frac{3}{4}}$ elements in C above median.
$\mathcal{E}_{32}$: $\ge 2n^{\frac{3}{4}}$ elements in C below median.

$\Pr[\mathcal{E}_3] \le \Pr[\mathcal{E}_{31} \cup \mathcal{E}_{32}] \le \Pr[\mathcal{E}_{31}] + \Pr[\mathcal{E}_{32}]$.

WTS $\Pr[\mathcal{E}_{31}] \le \frac{1}{4} n^{-\frac{1}{4}}$ 2-19-Discussion 2/2

(a) If $\mathcal{E}_{31}$

(b) u is at least the $(\frac{1}{2}n + 2n^{\frac{3}{4}})$th largest in S, so.

(c) R has at least $X \geq \frac{1}{2}n^{\frac{3}{4}} - \sqrt{n}$ samples among the $\frac{1}{2}n - 2n^{\frac{3}{4}}$ largest elements of S.

$$X = \sum_{i=1}^{n^{\frac{3}{4}}} X_i$$

$$\Pr[X_i = 1] = \frac{\frac{1}{2}n - 2n^{\frac{3}{4}}}{n} = \frac{1}{2} - 2n^{-\frac{1}{4}}$$

$$X \sim \mathrm{Bin}(n^{\frac{3}{4}}, \frac{1}{2} - 2n^{-\frac{1}{4}})$$

$$\mathbb{E}[X] = \frac{1}{2}n^{\frac{3}{4}} - 2\sqrt{n}.$$

$$\begin{aligned}
\mathrm{Var}[X] &= n^{\frac{3}{4}}(\tfrac{1}{2} - 2n^{-\frac{1}{4}})(\tfrac{1}{2} + 2n^{-\frac{1}{4}}) \\
&= n^{\frac{3}{4}}(\tfrac{1}{4} - 4n^{-\frac{1}{2}}) \\
&= \tfrac{1}{4}n^{\frac{3}{4}} - 4n^{\frac{1}{4}} \le \tfrac{1}{4}n^{\frac{3}{4}}
\end{aligned}$$

$$\begin{aligned}
\Pr[\mathcal{E}_{31}] &\le \Pr[X \geq \tfrac{1}{2}n^{\frac{3}{4}} - \sqrt{n}] \\
&= \Pr[X \geq \tfrac{1}{2}n^{\frac{3}{4}} - \sqrt{n} + (-\sqrt{n} + \sqrt{n})] \\
&= \Pr[X - (\tfrac{1}{2}n^{\frac{3}{4}} - 2\sqrt{n}) \geq \sqrt{n}] \\
&\le \Pr[|X - \mathbb{E}[X]| \geq \sqrt{n}]. \qquad \text{Chebyshev.} \\
&\le \frac{\mathrm{Var}[X]}{(\sqrt{n})^2} \le \frac{\frac{1}{4}n^{\frac{3}{4}}}{n} = \frac{1}{4}n^{-\frac{1}{4}} \qquad \boxtimes
\end{aligned}$$