Homework 4 – Due Friday, February 14, 2020 <u>at noon</u>

Submit solutions to all problems on separate sheets. They will be graded by different people.

Page limit   You can submit **at most** 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder   Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises   Please practice on exercises in Chapter 2 of Mitzenmacher-Upfal.

Problems

1. (**Geometric distribution**)

    (a) (**In parallel**) Two faulty machines M_1 and M_2 are repeatedly run synchronously in parallel (i.e., both machines execute one run, then both machine execute a second run, and so on.) On each run, M_1 fails with probability p_1 and M_2 fails with probability p_2, all failure events are independent. Let the random variables X_1 and X_2 denote the number of runs until the first failure of M_1 and M_2, respectively. Let $X = \min\{X_1, X_2\}$ denote the number of runs until the first failure of *either* machine. What distributions do X_1, X_2 and X have? Give a formal justification for your answer for X.

    (b) (**007 style**) James Bond is imprisoned in cell with three *possible* ways to escape: an air-conditioning duct, a sewer pipe, and the door (which is unlocked). The air-conditioning duct leads to him on a two-hour trip after which he falls through a trap door on his head, much to the amusement of his captors. The sewer pipe is similar, but takes five hours to traverse. Each fall produces temporary amnesia (in fiction, people get this a lot) and he is returned to his cell immediately after his fall. Assume he always chooses one of these three exits with probability 1/3. On average, how long does it take before he realizes that the door is unlocked and he just escapes?

2. (**Fish**) In the ocean around Boston there are n kinds of fish. Each catch comes uniformly at random from the n kinds.

    (a) What is the expected number of fish you must catch to get all n kinds?

    (b) If you catch $2n$ fish, what is the expected number of kinds of fish that you did not get?

    (c) If you catch $3n$ fish, what is the expected number of kinds of fish that you got exactly once?

    (d) What is the expected number of fish you must catch to get $n/2$ kinds. (You may assume that n is even.)

    You discovered that one of the kinds of fish can reproduce asexually. One fish of this kind produces one offspring with probability p_1, two offsprings with probability p_2, and zero offsprings with probability $1 - p_1 - p_2$.

(e) You caught one fish of this kind and put it in your fish tank. Assuming no fish die, what is the expected number of fish your tank will eventually have? For which values of p_1 and p_2 is it bounded?

3. (**Consecutive ones**) You have a k-sided fair die. The sides of the die are labeled $1, 2, \ldots, k$.

   (a) You roll the die until you get a **pair** of consecutive ones. What is the expected number of rolls? (*Hint:* A discussion problem from February 5 might be helpful.)

   (b) You roll the die until you get a **triple** of consecutive ones. What is the expected number of rolls?

4. (**Selling a plane**) You built a plane and would like to sell it to the buyer who makes the best offer. You received responses from n potential buyers who asked to see the plane. They will come to your factory one at a time and name their price. Each insists on an immediate decision from you whether you accept their price or not. You can compare each offer to all the previous offers (assume there are no ties), but you don't know what subsequent buyers will offer. Assume that buyers come in a random order, chosen uniformly at random from all $n!$ possible orderings. You want to maximize the probability of choosing the best offer.

   Consider the following strategy. You do not accept any of the first m offers, only collect information about how much buyers value your plane. After seeing buyer m, you change your strategy: you accept the first offer that is better than all the previous ones. Your goal in this problem (broken down into parts) is to find the best value of m as a function of n.

   (a) Suppose $m < i \leq n$. What is the probability that the best among the first $i-1$ offers is in the first m offers?

   (b) Let B be the event that you choose the best offer. Let B_i be the event that offer i is the best and you choose it. Determine $\Pr[B_i]$.

   (c) Show that $\Pr[B] = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}$.

   (d) Show that $\frac{m}{n} \ln \frac{n}{m} \leq \Pr[B] \leq \frac{m}{n} \ln \frac{n-1}{m-1}$.

   (e) Show that $\frac{m}{n} \ln \frac{n}{m}$ is maximized when $\frac{m}{n} = \frac{1}{e}$. Give a lower bound on $\Pr[B]$ when $m = \frac{n}{e}$.

5* (**Optional, no collaboration**) Konstantinos asks you to sort the homework assignments you graded in alphabetical order. You are trying to convince him that the assignments are already nearly sorted. You pull out a set of assignments (of your choice) in order they appear, and Konstantinos checks that they are indeed in alphabetical order. Suppose there are n assignments, and they are actually in a uniformly random order. Let L be the largest number of assignments for which the check that Konstantinos performs passes. (For example, if the assignments are in order Feng, Ali, Palak, Will, Dina, Miles, Iden, then L is 3: Ali, Dina, and Iden appear in the correct order, but there is no larger set of assignments that are in alphabetical order.) Prove that the expectation of L is at least $\sqrt{n}$. (For simplicity, you may assume that $\sqrt{n}$ is an integer, and don't actually play this trick on Konstantinos!)