Collaborators: None

3. (Consecutive ones)

(a) Solution: Let R.V. R_2 denote the number of rolls until getting a pair of ones, and X_i be the random variable that denote the result at the i^{th} roll. Then by law of total probability and linearity of expectation, we have

$$\begin{aligned}\mathbf{E}[R_2] =&(1 - Pr[X_1])(\mathbf{E}[R_2] + 1)\\&+ (Pr[X_1 = 1](1 - Pr[X_2 = 1]))(\mathbf{E}[R_2] + 2)\\&+ (Pr[X_1 = 1]Pr[X_2 = 1])2\\=&\frac{k-1}{k}(\mathbf{E}[R_2]) + \frac{1}{k}\frac{k-1}{k}(\mathbf{E}[R_2] + 2) + \frac{1}{k^2}2\end{aligned} \tag{1}$$

If we solve (1), we can obtain $\mathbf{E}[R_2] = k^2 + k$.

(b) Solution: Similar to the last question, let R.V. R_3 denote the number of rolls until getting a triple of consecutive ones.

$$\begin{aligned}\mathbf{E}[R_3] =&(1 - Pr[X_1 = 1])(\mathbf{E}[R_3] + 1)\\&+ Pr[X_1 = 1](1 - Pr[X_2 = 1])(\mathbf{E}(R_3) + 2)\\&+ Pr[X_1 = 1]Pr[X_2 = 1](1 - Pr[X_3 = 1])(\mathbf{E}(R_3) + 3)\\&+ Pr[X_1 = 1]Pr[X_2 = 1]Pr[X_3 = 1]3\end{aligned} \tag{2}$$

We solve (2) for $\mathbf{E}[R_3] = k^3 + k^2 + k - 3$.