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Practice Problems for Midterm

- Do not open this exam booklet until you are directed to do so. Read all the instructions on this page.
- When the exam begins, write your name on every odd page of this exam booklet.
- This exam contains 5 problems, some with multiple parts. You have 75 minutes to earn 137 points.
- This exam booklet contains 8 pages, including this one. At the end, there is one sheet with extra space to be used if you run out of space on any question. One extra sheet of scratch paper is attached. Please detach it before turning in your exam.
- This exam is closed-book and closed-devices. You may use one handwritten $8\frac{1}{2} \times 11$ crib sheet.
- Do not waste time and paper rederiving facts that we have studied. It is sufficient to cite known results.
- Do not spend too much time on any one problem. Read them all through first and attack them in the order that allows you to make the most progress.
- You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Name: ______________________ ID: __________

Problem	1	2	3	4	5	6	Total
Points							137
Grade							

Problem 1 (Multiple choice). For each item below, fill in the circles for all answers that fit. (A question could have multiple correct choices or none at all.) **You do not need to justify.**

- The algorithm from class for permutation routing on the hypercube uses randomness to choose

  ⓐ priorities for packets;

  ⓑ the order in which to "fix" the bits in the Bit Fixing algorithm;

  ⓒ intermediate destinations for all packets;

  ⓓ the ordering of packets in queues.

- The randomized median-finding algorithm from class

  ⓐ outputs fail if the number of elements it has to sort before performing partitioning is too large;

  ⓑ outputs fail if the number of elements it has to sort after performing partitioning is too large;

  ⓒ outputs fail if the number of elements that are smaller than all the elements it has to sort (after performing partitioning) is too large;

  ⓓ never outputs fail, but can return an incorrect element with small probability.

- Basic-Karger algorithm (one iteration of the Min-Cut algorithm)

  ⓐ chooses an edge to contract uniformly at random among remaining edges;

  ⓑ makes $m-1$ edge contractions, where m is the number of edges in the input graph;

  ⓒ outputs each possible cut with a positive probability, but it could be a different probability for different cuts;

  ⓓ could choose to contract edges such that a subset of edges contracted by the algorithm forms a cycle.

Problem 2 (Balls into Bins). Suppose m balls are thrown independently and uniformly at random into m bins. Let $k = \lceil 4 \ln m \rceil$ and assume m is divisible by k. Now merge the first k bins into one new bin, do the same for the next k original bins, and so forth, creating m/k new bins. Give the best lower bound you can on the probability that all the new bins are nonempty.

Problem 3 (Variance). Does there exist a random variable X taking real values such that

$$\Pr[X = 4] = \Pr[X = 10] \geq 1/3$$

and $\mathrm{Var}(X) = 1$? Prove your answer is correct.

Problem 4 (Amplifying Success Probability). Let $\mathcal{A}$ be a randomized algorithm that approximates some function $f(x)$ as follows: for all values x, it produces the answer in the range $(1 \pm \varepsilon) f(x)$ with probability at least $2/3$. Consider the algorithm $\mathcal{B}$ that outputs the median of t independent executions of algorithm $\mathcal{A}$ on the same input, where t is a parameter. You are given $\delta \in (0, 1/3)$. What value should you choose for t, so that algorithm $\mathcal{B}$, for all input x, produces the answer in the range $(1 \pm \varepsilon) f(x)$ with probability at least $1 - \delta$?

Give the best bound on t you can, using Big-O notation.

Problem 5 (Collisions). We throw m balls into n bins uniformly and independently at random. A *3-way collision* is a triple of balls that end up in the same bin. Calculate the expected number of 3-way collisions. For which m does the expected number of 3-way collisions exceed 1? (Give an asymptotic bound on m as a function of n.)

Repeat the analysis above for k-way collisions for a small constant k.