Randomness in Computing

CS 537

LECTURE 10

Last time

- Median of a RV
- Computing the median of an array

Today

- Chernoff Bounds

Tail Bounds So Far

- Markov. For a nonnegative random variable X and $a > 0$,
$$\Pr[X \geq a] \leq \frac{\mathrm{E}[X]}{a}.$$
- Chebyshev. For a random variable X and $a > 0$,
$$\Pr[|X - E[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

Sums of independent RVs

Chernoff Bound (Upper Tail). Let $X_1, \dots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = E[X]$. Then

- (stronger) for any $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1]$,

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}.$$

Sums of independent RVs

Chernoff Bound (Lower Tail). Let $X_1, \dots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = E[X]$. Then

- (stronger) for any $\delta \in (0,1)$,

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1)$,

$$\Pr[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2}.$$

Sums of independent RVs

Chernoff Bound (Both Tails). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = E[X]$. Then

- for any $\delta \in (0,1)$,

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}.$$

Exercise

- The Halting Problem Team wins each hockey game they play with probability 1/3. Assuming outcomes of the games are independent, derive an upper bound on the probability that they have a winning season in n games.
- The Halting Problem Team hires a new coach, and critics revise their probability of winning each game to 3/4. Derive an upper bound on the probability they suffer a losing season.

Exercise

- We throw n balls uniformly and independently into n bins.
  Let Y_1 be the number of balls that fell into bin 1.
  Determine m such that $\Pr[Y_1 > m] \leq \frac{1}{n^2}$.

Tail bounds so far

- Markov. For a nonnegative random variable X and $a > 0$,
$$\Pr[X \geq a] \leq \frac{\mathrm{E}[X]}{a}.$$
- Chebyshev. For a random variable X and $a > 0$,
$$\Pr[|X - E[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$
- Example 1: $X \sim \mathrm{Bin}(n, 1/2)$.

  Bound $\Pr\left[X > \frac{3n}{4}\right]$ using Markov and Chebyshev.
- Example 2: Coupon Collector Problem.

  Bound $\Pr[X > 2nH_n]$ using Markov and Chebyshev.