3. (Jensen's Inequality)

(a) Solutions: If f is a concave function, then

$$\mathbb{E}(f(X)) \leq f(\mathbb{E}(X)) \tag{1}$$

This follows naturally from the original Jensen's inequality because if f is concave, then $-f$ will be convex, and we can apply the Jensen's equality to $-f$, we would obtain the equality for concave function f as above.

(b)

Proof. Let $G_n = \sqrt[n]{\prod_{i=1}^{n} x_i}$ be the geometric mean and $A_n = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the arithmetic mean of a collection of n positive real numbers $\{x_i\}$.

$$\begin{aligned}
\log A_n &= \log\Big(\frac{1}{n}\sum_{i=1}^{n} x_i\Big) && (2)\\
&\geq \frac{1}{n}\sum_{i=1}^{n}\log x_i && (3)\\
&= \sum_{i=1}^{n}\Big(\log x_i^{1/n}\Big) && (4)\\
&= \log\Big(\prod_{i=1}^{n} x_i^{1/n}\Big) && (5)\\
&= \log G_n && (6)\\
A_n &\geq G_n && (7)
\end{aligned}$$

From (2) to (3) we have used the inequality in (a) where $f(x) = \log x$. (7) is obtained when we take the exponential of $\log A_n$ and $\log G_n$ respectively.

(c)

Proof. let $f(x) = \sin x$ where $0 < x < \pi$. Because $f``(x) = -\sin x < 0$ when $0 < x < \pi$, $f(x)$ is concave in the interval $(0, \pi)$. We can apply the inequality in (a),

$$\begin{aligned}
\frac{1}{3}\Big(\sin A + \sin B + \sin C\Big) &\leq \sin\frac{1}{3}(A+B+C) && (8)\\
&\leq \sin 60^\circ && (9)\\
&\leq \frac{\sqrt{3}}{2} && (10)\\
\sin A + \sin B + \sin C &\leq \frac{3\sqrt{3}}{2} && (11)
\end{aligned}$$