3. (Random subsets)

(a)

Proof. Let S be the set of integers in $[n]$ that was selected. Let h_i and t_i denote the event that the i_{th} coin is *head* or *tail* respectively. Since each each coin flip is independent for each element of the set, then the probability of generating such a set S is

$$Pr[S] = Pr\Big[\Big(\bigcap_{i\in S} h_i\Big)\bigcap\Big(\bigcap_{j\in[n]-S} t_j\Big)\Big] \tag{1}$$
$$= \prod_{i\in S} Pr[h_i] \times \prod_{j\in[n]-S} Pr[t_i] \tag{2}$$
$$= \prod_{i\in S} \frac{1}{2} \times \prod_{j\in[n]-S} (1-\frac{1}{2}) \tag{3}$$
$$= \frac{1}{2^n} \tag{4}$$

Since our choice of S is random and the final expression is independent of the selection of S, we have shown that any of the possible subsets is equally likely to be chosen.

(b) Solution: From the result of (a), we can do this subset selection using a process of selecting each number in sequence by flip two coins, for X_i and Y_i respectively. In order to get $X \subseteq Y$, we need to make sure for any $i \in [n]$, we shouldn't have the event $[X_i = 1]\bigcap[Y_i] = 0$. Since each selection is independent, the total probability of getting $X \subseteq Y$ is expressed as

$$Pr[X \subseteq Y] = Pr\Big[\bigcap_{i=1}^{n} !([X_i = 1] \cap [Y_i] = 0)\Big] \tag{5}$$
$$= \prod_{i=1}^{n} Pr[!([X_i = 1] \cap [Y_i] = 0)] \tag{6}$$
$$= \prod_{i=1}^{n} (1 - 1/4) \tag{7}$$
$$= \Big(\frac{3}{4}\Big)^n \tag{8}$$

(c) Solution: Similar to (b), we use the same sequential order to decide whether a number should be in the three sets. For each number $i \in [n]$, there are three possibilities in order to satisfy the requirement: 1) i is not selected in any of the three sets; 2) i is selected in 2 sets; 3) i is selected in three sets. Let the event S_i denote the event that number i satisfy any of the 3 conditions. $Pr[S_i] = \binom{3}{0}(1/2)^3 + \binom{3}{2}(1/2)^3 + \binom{3}{3}(1/2)^3 = 5/8$. Since each number is selected independently, the overall probability is

$$
\begin{aligned}
Pr[E] &= \prod_{i=1}^{n} Pr[S_i] && (9)\\
&= \prod_{i=1}^{n} \left(\frac{5}{8}\right) && (10)\\
&= \left(\frac{5}{8}\right)^n && (11)
\end{aligned}
$$