

2. Homework Assignments, 10 points

a) Solution:

Proof. We can prove this by induction. When $k = 1$, it is obvious that the probability of working on it is 100%; When $k = 2$, the probability of switching to second homework is 50% and of staying on homework 1 is also 50%. Thus, the statement we need to prove is true for $k = 1, 2$. Now, for induction, we assume that the statement is true for $k \geq 2$, that is, the probability of working on any of the k homework is $1/k$. Now, when the $(k + 1)$ th homework arrives, there is a probability of $\frac{1}{k+1}$ that you will switch to it no matter which homework you are working on right now. Thus, by law of total probability:

$$\begin{aligned} Pr[(\text{working on homework } (k + 1))] &= \left(\sum_{i=1}^k Pr[(\text{work on homework } i)] \cdot \right. \\ &\quad \left. Pr[(\text{switching to homework } (k + 1)) \mid (\text{work on homework } i)] \right) \\ &= \sum_{i=1}^k \frac{1}{i} \cdot \frac{1}{k + 1} = \frac{1}{k + 1} \end{aligned}$$

Now let's also show that the probability of staying in any of the first k homework, i , is also $\frac{1}{k+1}$:

$$\begin{aligned} Pr[(\text{staying on homework } (i))] &= \left(1 - Pr[(\text{switching to homework } (k + 1)) \mid (\text{work on homework } i)] \right) \cdot \\ &\quad Pr[(\text{work on homework } i)] \\ &= \left(1 - \frac{1}{k + 1} \right) \cdot \frac{1}{k} \\ &= \frac{1}{k + 1} \end{aligned}$$

We have shown that when a new homework $k + 1$ is assigned, the probability of working on each homework is still equal. Thus, by induction, we have shown that you are equally likely to work on any homework assigned so far. \square

b) Solution:

Let k be the total number of homeworks assigned so far. and let $Pr_k[i]$ be the probability of working on homework i when total number of homeworks assigned so far is k . It is obvious that $Pr_1[1] = 1$, and $Pr_2[1] = Pr_2[2] = 1/2$.

According to the rules specified, when homework k is assigned, the probability of switching to it is $1/2$ no matter which homework you are working on now. Thus, by law of total probability, for any $k \geq 2$, we have

$$Pr_k[k] = 1/2, Pr_k[1] = Pr_k[2]$$

and

$$Pr_k[i] = 2 \cdot Pr_k[i - 1]$$

for i and k satisfying $3 \leq i \leq k$. Of course

$$\sum_{i=1}^k Pr_k[i] = 1$$

The following distribution is the only distribution satisfying all the requirements and is thus the distribution we want to obtain

When $k = 1$,

$$Pr_1[1] = 1$$

When $k \geq 2$,

$$Pr_k[i] = \begin{cases} \frac{1}{2^{k-1}} & i = 1, 2 \\ \frac{1}{2^{k-i+1}} & 3 \leq i \leq k \end{cases}$$