2. (MaxCut)

(a) Let X_{ij} denote the Bernoulli random variable whether the edge $\{i, j\}$ is in the cut set S, where $\{i, j\} \in E$, $i < j$ and $m = |E|$ is the total number of edges in the graph. Then $X = \sum_{\{i,j\}\in E} X_{ij}$. For R.V. $\{i, j\}$, since it is a Bernoulli random variable, $\mathbf{E}(X_{ij}) = Pr[X_{ij} = 1] = 1/2$. Thus,

$$
\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}\Big[\sum_{\{i,j\}\in E} X_{ij}\Big] && (1)\\
&= \sum_{\{i,j\}\in E} \mathbf{E}[X_{ij}] && (2)\\
&= \sum_{\{i,j\}\in E} Pr[X_{ij} = 1] && (3)\\
&= \sum_{\{i,j\}\in E} \frac{1}{2} && (4)\\
&= \frac{1}{2}m && (5)
\end{aligned}
$$

It is easy to show that $OPT \leq m$ since the size of max cut cannot exceed the total number of edges in the graph, thus $\mathbf{E}(X) = \frac{1}{2}m \geq \frac{OPT}{2}$.

(b)

Proof. Let R.V. Y denote the number of edges not in the cut set S, then $Y = m - X$.

$$
\begin{aligned}
Pr[X \geq 0.49OPT] &= Pr[Y \leq m - 0.49OPT] && (6)\\
&= 1 - Pr[Y \geq m - 0.49OPT] && (7)\\
&\geq 1 - \frac{\mathbf{E}(Y)}{m - 0.49OPT} && (8)\\
&= 1 - \frac{m}{2(m - 0.49OPT)} && (9)\\
&\geq 1 - \frac{m}{2(0.51m)} && (10)\\
&= 1 - \frac{1}{2 * 0.51} = \frac{1}{51} && (11)
\end{aligned}
$$

From (7) to (8), we have used Markov Inequality. From (8) to (9), we have used the fact that $\mathbf{E}(Y) = m - \mathbf{E}(m) = 0.5m$. And from (9) to (10), we used the obvious fact that $OPT \leq m$.

(c) Similar to (a), we write $X = \sum_{\{i,j\}\in E} X_{ij}$.

$$Var[X] = \mathbf{E}[X^2] - \mathbf{E}(X)^2 \tag{12}$$

$$= \mathbf{E}\Big[\big(\sum_{\{i,j\}\in E} X_{ij}\big)^2\Big] - (0.5m)^2 \tag{13}$$

$$= \mathbf{E}\Big[\sum_{\{i,j\}\in E} X_{ij}^2 + \sum_{\{i,j\}\in E, \{i,j\}\neq\{m,n\}} \sum_{\{m,n\}\in E} X_{ij}X_{mn}\Big] - (0.5m)^2 \tag{14}$$

$$= \sum_{\{i,j\}\in E} \mathbf{E}\big(X_{ij}^2\big) + \sum_{\{i,j\}\in E, \{i,j\}\neq\{m,n\}} \sum_{\{m,n\}\in E} \mathbf{E}[X_{ij}X_{mn}] - 0.25m^2 \tag{15}$$

$$= m/2 + \binom{m}{2}\frac{1}{4} - 0.25m^2 = m/4 \tag{16}$$

From (14) to (15), we used linearity of expectation, and from (15) to (16), we used the fact that X_{ij}^2 follows the same distribution as X_{ij}, thus $\mathbf{E}(X_{ij}^2) = \mathbf{E}(X_{ij})$ and that $\mathbf{E}(X_{ij}X_{mn}) = 1/4$ if the two edges are different edges and there are $\binom{m}{2}$ such pairs of edges.

(d)

Proof.

$$p = Pr[X \leq 0.49OPT] \tag{17}$$

$$= 1 - Pr[Y \geq m - 0.49OPT] \tag{18}$$

$$= 1 - Pr[Y - 0.5m \geq 0.5m - 0.49OPT] \tag{19}$$

$$\geq 1 - Pr[|Y - \mathbf{E}(Y)| \geq 0.5m - 0.49OPT] \tag{20}$$

$$\geq 1 - \frac{Var[Y]}{(0.5m - 0.49OPT)^2} \tag{21}$$

$$\geq 1 - \frac{m}{4(0.01m)^2} \tag{22}$$

$$= 1 - \frac{2500}{m} = 1 - 2500 \cdot \frac{1}{|E|} \tag{23}$$

$$p = \Omega(1 - 2500 \cdot \frac{1}{|E|}) \tag{24}$$

$$= 1 - O(1/|E|) \tag{25}$$

From (19) to (20), we used the fact that $0.5 - 0.49OPT > 0$. From (20) to (21), we used Chebyshev's inequality. From (21) to (22), we used $Var[Y] = Var[m - X] = Var[X]$ and $OPT \leq m$.