10/10

Peilun Dai

1. Probability Review

Thirteen cards are drawn without replacement from a standard deck of 52 cards. What is the probability that:

a) they are all spades?

$$Pr[all\ spades] = 1 \Big/ \binom{52}{13} \quad (1)$$
$$\approx 1.57 \times 10^{-12} \quad (2)$$

b) they are all black?

$$Pr[all\ black] = \binom{26}{13} \Big/ \binom{52}{13} \quad (3)$$
$$= 0.00001637854 \quad (4)$$

c) they are not all of one color, given that none of the cards is an ace?

$$Pr[not\ of\ one\ color \mid none\ is\ ace] = \left(\binom{48}{13} - \binom{24}{13} - \binom{24}{13}\right) \Big/ \binom{48}{13} \quad (5)$$
$$= 0.9999741236 \quad (6)$$

d) none of the cards is an ace and none is a heart?

$$Pr[none\ ace\ and\ none\ heart] = \binom{36}{13} \Big/ \binom{52}{13} \quad (7)$$
$$= 0.00363896103 \quad (8)$$

e) there are 5 cards of one suit and 8 card of another suit?

$$Pr[five\ of\ one\ suit\ and\ eight\ of\ another\ suit] = \left(\binom{4}{1}\binom{13}{5} \times \binom{3}{1}\binom{13}{8}\right) \Big/ \binom{52}{13} \quad (9)$$
$$= 0.00003130079 \quad (10)$$

$\frac{9}{10}$

Collaborators: None?

2. Homework Assignments, 10 points

a) Solution:

Proof. We can prove this by induction. When $k = 1$, it is obvious that the probability of working on it is 100%; When $k = 2$, the probability of switching to second homework is 50% and of staying on homework 1 is also 50%. Thus, the statement we need to prove is true for $k = 1, 2$. Now, for induction, we assume that the statement is true for $k \geq 2$, that is, the probability of working on any of the k homework is $1/k$. Now, when the $(k+1)th$ homework arrives, there is a probability of $\frac{1}{k+1}$ that you will switch to it no matter which homework you are working on right now. Thus, by law of total probability:

$$Pr[(working\ on\ homework\ (k+1))] = \Big(\sum_{i=1}^{k} Pr[(work\ on\ homework\ i)] \cdot$$

$$Pr[(switching\ to\ homework\ (k+1)) \mid (work\ on\ homework\ i)]\Big)$$

$$= \sum_{i=1}^{k} \frac{1}{i} \cdot \frac{1}{k+1} = \frac{1}{k+1}$$

this is actomatically $\frac{1}{k+1}$ from question i)

$\frac{5}{6}$

Now let's also show that the probability of staying in any of the first k homework, i, is also $\frac{1}{k+1}$:

$$Pr[(staying\ on\ homework\ (i))] = \Big(1 - Pr[(switching\ to\ homework\ (k+1)) \mid (work\ on\ homework\ i)]\Big) \cdot$$

$$Pr[(work\ on\ homework\ i)]$$

$$= \Big(1 - \frac{1}{k+1}\Big) \cdot \frac{1}{k}$$

$$= \frac{1}{k+1}$$

independent

Switching to k+1th already implies that you are not on ith anymore.

$Pr[\text{doing } i\text{th}] = Pr[\text{not switching to } k{+}1\text{th}] \cdot Pr[\text{previously doing } i\text{th}] = (1 - \frac{1}{k+1}) \cdot \frac{1}{k}$

We have shown that when a new homework $k+1$ is assigned, the probability of working on each homework is still equal. Thus, by induction, we have shown that you are equally likely to work on any homework assigned so far. □

b) Solution:

Let k be the total number of homeworks assigned so far. and let $Pr_k[i]$ be the probability of working on homework i when total number of homeworks assigned so far is k. It is obvious that $Pr_1[1] = 1$, and $Pr_2[1] = Pr_2[2] = 1/2$.

According to the rules specified, when homework k is assigned, the probability of switching to it is $1/2$ no matter which homework you are working on now. Thus, by law of total probability, for any $k \geq 2$, we have

$$Pr_k[k] = 1/2, Pr_k[1] = Pr_k[2]$$

and

$$Pr_k[i] = 2 \cdot Pr_k[i-1]$$

for i and k satisfying $3 \leq i \leq k$. Of course

$$\sum_{i=1}^{k} Pr_k[i] = 1$$

The following distribution is the only distribution satisfying all the requirements and is thus the distribution we want to obtain

When $k = 1$,

$$Pr_1[1] = 1$$

$\checkmark$

$\frac{4}{4}$

When $k \geq 2$,

$$Pr_k[i] = \begin{cases} \frac{1}{2^{k-1}} & i = 1, 2 \\ \frac{1}{2^{k-i+1}} & 3 \leq i \leq k \end{cases}$$