Randomness in Computing

CS
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LECTURE 9

Last time

- Variance, covariance
- Chebyshev's inequality
- Variance of Binomial and Geometric RVs
- **Today**
- Median of a RV
- Computing the median of an array

Median of a random variable

- A value m is the median of a random variable X **if** $\Pr[X \leq m] \leq 1/2$ **and** $\Pr[X \geq m] \leq 1/2$.

- Example 1: X is uniform over $x_1, \ldots, x_{2k+1}$, where $x_1 < \cdots < x_{2k+1}$. What is the median?
- Example 2: X is uniform over $x_1, \ldots, x_{2k}$, where $x_1 < \cdots < x_{2k}$. Find all medians.

Median and mean: another view

- Theorem. For a random variable X with a finite expectation μ and a finite median m,

1. the expectation μ is the value of c that minimizes the expression
$$E[(X-c)^2];$$
2. the median m is a value of c that minimizes the expression
$$E[|X-c|].$$

Median and mean are close

- Theorem. For a random variable X with expectation μ, median m, and standard deviation σ,
$$|\mu - m| \le \sigma.$$

Chebyshev's Inequality

- Theorem. For a random variable X and $a > 0$,

$$\Pr[|X - E[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

- Alternatively: Then, for all $t > 1$,

$$\Pr[|X - E[X]| \geq t \cdot \sigma[X]] \leq \frac{1}{t^2}.$$

- Example 1: $X \sim \mathrm{Bin}(n, 1/2)$.

  Bound $\Pr\left[X > \frac{3n}{4}\right]$ using Markov and Chebyshev.

- Example 2: Coupon Collector Problem.

  Bound $\Pr[X > 2nH_n]$ using Markov and Chebyshev.