CS 537: *Randomness in Computing*
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Solutions to Homework 1

1. (Probability review) ***Thirteen cards are drawn (without replacement) from a standard deck of 52 cards. What is the probability that:***

   (a) ***They are all spades?***

   Let S_i be the event that the i-th card is a spade. After we've drawn i spades and no others, there are $13 - i$ spades in the deck of $52 - i$ cards. By the product rule:

$$\begin{aligned}\Pr\left[\bigcap_{i=1}^{13} S_i\right] &= \Pr[S_1] \cdot \Pr[S_2|S_1] \cdots \Pr[S_{13}|S_{12} \cap \cdots \cap S_1] \\ &= \frac{13}{52} \cdot \frac{12}{51} \cdots \frac{1}{40} \\ &= \frac{13!(52-13)!}{52!} = \frac{1}{\binom{52}{13}}.\end{aligned}$$

   Alternatively, we can obtain the final expression directly by observing that there are $\binom{52}{13}$ ways to choose 13 cards out of 52, all of these possibilities have equal probability, and exactly one of them corresponds to 13 cards being spades.

   (b) ***They are all black.***

   Let B_i be the event that the i-th card is black. After we've drawn i black cards, there are $26 - i$ black cards remaining in a deck of $52 - i$ cards. By the product rule:

$$\begin{aligned}\Pr\left[\bigcap_{i=1}^{13} B_i\right] &= \Pr[B_1] \cdot \Pr[B_2|B_1] \cdots \Pr[B_{13}|B_{12} \cap \cdots \cap B_1] \\ &= \frac{26}{52} \cdot \frac{25}{51} \cdots \frac{14}{40} \\ &= \frac{26!(52-13)!}{13! \cdot 52!} = \frac{\binom{26}{13}}{\binom{52}{13}}.\end{aligned}$$

   Alternatively, we can obtain the final expression directly by observing that there are $\binom{52}{13}$ ways to choose 13 cards out of 52, all of these possibilities have equal probability, and exactly $\binom{26}{13}$ of them correspond to picking the 13 cards out of 26 black cards.

   (c) ***They are not all of one color, given that none of the cards is an ace?***

   Let C be the event that the cards are all of the same color, and $\overline{A}$ the event that none of them is an ace. We want to compute

$$\Pr[\overline{C}|\overline{A}] = 1 - \Pr[C|\overline{A}],$$

   and it may be easier to compute the latter term. There are $\binom{48}{13}$ ways to choose 13 cards out of the 48 non-ace cards, and all of these possibilities have the same probability. Now we calculated in how many of these possibilities all 13 cards have the same color. There are 2 ways to choose the color, and for both of these choices, there are $\binom{24}{13}$ ways to choose 13 cards out of the 24 non-ace cards of that color. Therefore,

$$\Pr[C|\overline{A}] = \frac{2 \cdot \binom{24}{13}}{\binom{48}{13}} = \frac{2 \cdot 24! \cdot 35!}{11! \cdot 48!} = \frac{23! \cdot 35!}{11! \cdot 47!}.$$

The probability of the complement is

$$\Pr[\overline{C}|\overline{A}] = 1 - \frac{23! \cdot 35!}{13! \cdot 47!}.$$

(d) ***None of the cards is an ace and none is a heart?***

Let E_i be the event that card i is neither an ace nor a heart. Before drawing and cards, there are four aces and thirteen hearts in the deck, but one ace is also a heart so there are 16 cards we must avoid. By the product rule,

$$\begin{aligned}\Pr\left[\bigcap_{i=1}^{13} E_i\right] &= \Pr[E_1] \cdots \Pr\left[E_{13} \middle| \bigcap_{i=1}^{12} E_i\right] \\ &= \frac{36}{52} \cdot \frac{35}{51} \cdots \frac{24}{40} \\ &= \frac{36! \cdot 39!}{23! \cdot 52!} = \frac{\binom{36}{13}}{\binom{52}{13}}.\end{aligned}$$

Alternatively, we can obtain the final expression directly by observing that there are $\binom{52}{13}$ ways to choose 13 cards out of 52, all of these possibilities have equal probability, and exactly $\binom{36}{13}$ of them corresponds to 13 cards being chosen the 36 cards that are neither aces nor hearts.

(e) ***There are 5 cards of one suit and 8 card of another suit?***

As before, there are $\binom{52}{13}$ ways to choose 13 cards out of 52. Next we calculate in how many of them we get 5 cards of one suit and 8 cards of another suit. There are 4 ways to choose the 5-card suit, and 3 ways to choose the 8-card suit out of the remaining suits. There are $\binom{13}{5}$ ways to choose 5 cards from the first suit and $\binom{13}{8}$ ways to choose 8 cards of the second suit. The probability of the event in question is:

$$\frac{4 \cdot 3 \cdot \binom{13}{5} \cdot \binom{13}{8}}{\binom{52}{13}} = 12 \cdot \left(\frac{13!}{5! \cdot 8!}\right)^2 \cdot \frac{13! \cdot 39!}{52!}.$$

2. **(Homework assignments)**

(a) ***You start working on the first homework as soon as it is assigned to you. Every time a new homework is assigned, you switch to working on it with a certain probability and keep working on your current homework with the remaining probability. Specifically, when homework k is assigned, you switch to working on this homework with probability $1/k$. Prove that you are equally likely to work on any homework assigned so far. (In other words, the homework you are working on is uniformly distributed over all homework assignments so far.)***

We will prove that right after homework k is assigned, you are equally likely to be working on any homework assigned so far, that is, the probability that you are working on homework i is $1/k$ for $i = 1, \ldots, k$.

We prove that statement by induction on k.

(Base Case). You start working on the first homework as soon as it is assigned to you. That is, for $k = 1$, you work on homework 1 with probability $\frac{1}{k} = \frac{1}{1} = 1$, as required.

(Inductive Step). Suppose the statement holds for some $k = n$, that is, after homework n has been assigned, the probability of you working on homework i is $1/n$ for all $i = 1, \ldots, n$.

We will prove that the statement also holds for $k = n+1$, that is, after homework $n+1$ is assigned, the probability of you working on homework i is $\frac{1}{n+1}$ for all $i = 1, \ldots, n+1$.

Indeed, after homework $n+1$ is assigned, you will will switch to work on the new homework with probability $\frac{1}{n+1}$. That is, the statement we need to prove holds for $i = n+1$. You will keep working on the old homework with probability $1 - \frac{1}{n+1} = \frac{n}{n+1}$. By inductive hypothesis, the old homework is uniformly distributed between homeworks 1 to n. That is, each of them has probability $\frac{1}{n} \cdot \frac{n}{n+1} = \frac{1}{n+1}$ of being worked on by you at this point. That is, the statement also holds for all $i = 1, \ldots, n$.

To conclude, the probability of you working on homework i after homework $n+1$ is assigned is $\frac{1}{n+1}$ for all $i = 1, \ldots, n+1$. Thus, the statement we wanted to prove holds by induction.

(b) ***Suppose that your friend has a similar strategy, except that when the kth homework is assigned, she switches to working on it with probability 1/2. Describe the distribution of the homework assignment she is working on.***

Whenever a new homework is assigned, she will work on it with probability $1/2$ and stay on the current one with probability $1/2$. Therefore, the last homework assigned will always have probability $1/2$ and the previous homeworks will half their probability with each new homework assignment. Hence, after n homeworks have been assigned, she will be working on the i^{th} homework, for $i = 2, \ldots, n$ with probability:

$$\Pr\left[\text{She is working on homework } i\right] = \left(\frac{1}{2}\right)^{n-i+1}.$$

The first homework will have probability:

$$\Pr\left[\text{She is working on homework } 1\right] = \left(\frac{1}{2}\right)^{n-1}.$$