8.0/10.

Peilun Dai

1. (Chess board)

0.5 (a) Solution: The events A and B are not independent. $P(A \cap B) = 0 \neq P(A)P(B) = 1/64 \times 1/64$.

2.5 (b) Solution: The events A and B are independent. $P(A \cap B) = 1/4 = P(A)P(B) = 1/2 \times 1/2$.

2.5 (c) Solution: The events A and B are independent. $P(A \cap B) = 16/64 = 1/4 = P(A)P(B) = 1/2 \times 1/2$.

2.5 (d) Solution: The events A, B and C are not independent. Assuming the orientation is fixed, and the first square is black, then all the squares located in even numbered rows as well as even numbered columns are black, thus

$$P(A \cap B \cap C) = 0$$

$$\begin{aligned} P(A)P(B)P(C) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} \\ &\neq P(A \cap B \cap C) \end{aligned}$$

The above proof is also true when we choose the orientation of the board such that the first square is white in which case $P(A \cap B \cap C) = 1 \neq P(A)P(B)P(C)$.

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2. (Multiple-choice test) 9.5/10

(a) Solution: Let K and $\overline{K}$ be the events that Adam knows and doesn't know the answer respectively, and events C and $\overline{C}$ be the events that Adam answer correctly and incorrectly respectively. Using Bayes's Rule:

$$\begin{aligned} Pr[K|C] &= \frac{Pr[C|K]Pr[K]}{Pr[C|K]Pr[K] + Pr[C|\overline{K}]Pr[\overline{K}]} && (1)\\ &= \frac{p \times 1}{p \times 1 + (1-p) \times 1/m} && (2)\\ &= \frac{mp}{(m-1)p+1} && (3) \end{aligned}$$

✓

(b) Solution: $Pr[K|C] = \frac{mp}{(m-1)p+1} = \frac{5\times0.6}{(5-1)\times0.6+1} \approx 0.88.$ ✓

(c) Solution: In addition to notations from (a), let E and N be the event that Bella can eliminate but two answers, and the event that Bella doesn't know the answer, then using Bayes's rule,

$$\begin{aligned} Pr[K|C] &= \frac{Pr[C|K]Pr[K]}{Pr[C|K]Pr[K] + Pr[C|E]Pr[E] + Pr[C|N]Pr[N]} && (4)\\ &= \frac{1 \times p_1}{1 \times p_1 + 0.5 \times p_2 + \frac{1}{m} \times (1-p_1-p_2)} && (5)\\ &= \frac{2mp_1}{2mp_1 + mp_2 + 2(1-p_1-p_2)} && (6) \end{aligned}$$

✓

(d) Suppose $m = 5$ and $p_2 = 0.1$, then (6) simplifies to $\frac{10p_1}{10p_1+2.3-2p_1}$, we can solve for $p_1 = 0.69$. ✓

Please write down what Pr[C|K], Pr[K], ... are in terms of p, m

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3. (Random subsets)

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(a)

Proof. Let S be the set of integers in $[n]$ that was selected. Let h_i and t_i denote the event that the i_{th} coin is *head* or *tail* respectively. Since each each coin flip is independent for each element of the set, then the probability of generating such a set S is

$$Pr[S] = Pr\Big[\Big(\bigcap_{i\in S} h_i\Big)\bigcap\Big(\bigcap_{j\in[n]-S} t_j\Big)\Big] \tag{1}$$
$$= \prod_{i\in S} Pr[h_i] \times \prod_{j\in[n]-S} Pr[t_i] \tag{2}$$
$$= \prod_{i\in S}\frac{1}{2} \times \prod_{j\in[n]-S}(1-\frac{1}{2}) \quad ✓ \tag{3}$$
$$= \frac{1}{2^n} \quad ✓ \tag{4}$$

Since our choice of S is random and the final expression is independent of the selection of S, we have shown that any of the possible subsets is equally likely to be chosen.

(b) Solution: From the result of (a), we can do this subset selection using a process of selecting each number in sequence by flip two coins, for X_i and Y_i respectively. In order to get $X \subseteq Y$, we need to make sure for any $i \in [n]$, we shouldn't have the event $[X_i = 1]\bigcap[Y_i] = 0$. Since each selection is independent, the total probability of getting $X \subseteq Y$ is expressed as

$$Pr[X \subseteq Y] = Pr\Big[\bigcap_{i=1}^{n} !([X_i = 1] \cap [Y_i] = 0)\Big] \tag{5}$$
$$= \prod_{i=1}^{n} Pr[!([X_i = 1] \cap [Y_i] = 0)] \tag{6}$$
$$= \prod_{i=1}^{n}(1 - 1/4) \tag{7}$$
$$= \Big(\frac{3}{4}\Big)^n \quad ✓ \tag{8}$$

(c) Solution: Similar to (b), we use the same sequential order to decide whether a number should be in the three sets. For each number $i \in [n]$, there are three possibilities in order to satisfy the requirement: 1) i is not selected in any of the three sets; 2) i is selected in 2 sets; 3) i is selected in three sets. Let the event S_i denote the event that number i satisfy any of the 3 conditions. $Pr[S_i] = \binom{3}{0}(1/2)^3 + \binom{3}{2}(1/2)^3 + \binom{3}{3}(1/2)^3 = 5/8$. Since each number is selected independently, the overall probability is

$$Pr[E] = \prod_{i=1}^{n} Pr[S_i] \tag{9}$$
$$= \prod_{i=1}^{n} \left(\frac{5}{8}\right) \tag{10}$$
$$= \left(\frac{5}{8}\right)^n \tag{11}$$

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4. (Verifying matrix multiplication)

(a) Solutions:

i. The only type of error it can make is that there is non-zero elements, but the algorithm accepts it. For a worst case input, there is only one non-zero element in the matrix D of size $n \times n$. Denote this event as E Thus, the probability of this happening $Pr[E]$ satisfies $Pr[E] \leq (n^2-1)/n^2$. Thus the best upper bound of accepts an incorrect answer is $(n^2-1)/n^2$ for a worst case input.

ii. Since there only a one-sided error, we can use k repeated independent tests to amplify the probability. Denote error for this multirun test as E_m,

$$
\begin{aligned}
Pr[E_m] &= (P[E])^k && (1)\\
&\leq \left(\frac{n^2-1}{n^2}\right)^k && (2)
\end{aligned}
$$

Since we want to solve for lower bound, we can use the equality $1-x \leq e^{-x}$ and solve for a lower bound fo k, with $(1-1/n^2)^k \leq (e^{-1/n^2})^k \leq 1/3$. The obtained lower bound is $(\ln 3)n^2$.

iii. For each run, we need to calculate D_{ij}, which is $O(n)$ due to the way how inner product works. And we need to do at least $(\ln 3)n^2$ runs, thus the total run time or this algorithm is n^3.

(b)

Proof. The only difference between the classroom case and this case we want to prove is the last step, when we select this random vector $\bar{r}$ sequentially. Assuming in the worst case, the element D_{ij} is non-zero. Let's select r_j last after selecting all other elements. If $A \cdot B = C$, then the probability $Pr[AB\bar{r} = C\bar{r}]$ is 1 no matter what r_j we choose. However, if D_{ij} is not zero, the probability $Pr[AB\bar{r} = C\bar{r}] \leq 1/2$ because when we select value of r_j from a uniform distribution of two values, the probability of selecting the only one possible value (which may not even exist) we select to make the equality true for this row is at most $1/2$. Similarly, in our case, since we are choosing values from v values $[0, 1, 2, ..., v-1]$ instead of two values, the probability of selecting the only possible value to make the quality holds is at most $1/v$, which is our new better bound than $1/2$.

(c) Solution: Denote the event that the multiplication equality is true is T and the event that the algorithm doesn't find any mistake after i runs is B_i. We know that $Pr[T] = Pr[\overline{T}] = 1/2$. For the first run of the test, we know that $Pr[B_1 \mid \overline{T}] \leq 1/v$ and $Pr[B_1 \mid T] = 1$. Thus, using Bayes' law, after the first run, the *posterior* probability is

$$
\begin{aligned}
Pr[T \mid B_1] &= \frac{Pr[T]Pr[B_1 \mid T]}{Pr[T]Pr[B_1 \mid T] + Pr[\overline{T}]Pr[B_1 \mid \overline{T}]} && (3)\\
&\geq \frac{1/2}{1/2 + 1/(2v)} = \frac{v}{v+1} && (4)
\end{aligned}
$$

Similarly for the second test, the *posterior* is $Pr[T \mid B_2] \geq \frac{v^2}{v^2+1}$. It is easy to show that after n runs without finding any mistake, the *posterior* $Pr[T \mid B_n] = \frac{v^n}{v^n+1}$.

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show all the work.
prove by induction.
see textbook page 12