Randomness in Computing

Lecture 4

CS 537

Last time
- Randomized min-cut algorithm
- Amplification
- Random variables

Discussions
- Random variables. Expectation.

Today
- Bernoulli and binomial RVs
- Jensen's inequality
- Conditional expectation

Review question: peer grading

We have n students in the class.

Everybody submits one problem, then problems get randomly permuted, and each student gets one problem to grade.

What is the expected number of students grading their own problems?

Bernoulli random variables

- A Bernoulli random variable with parameters p:
$$\begin{cases} 1 & \text{with probability } p; \\ 0 & \text{otherwise.} \end{cases}$$
- The expectation of a Bernoulli R.V. X is
$$\mathbb{E}[X] = p \cdot 1 + (1 - p) \cdot 0 = p.$$

Binomial random variables

- A binomial random variable with parameters n and p, denoted $\text{Bin}(n, p)$, is the number of HEADS in n tosses of a coin with bias p.
- Lemma. The probability distribution of $X = \text{Bin}(n, p)$ is
$$\Pr[X = j] = \binom{n}{j} p^j (1-p)^{n-j}$$
for all $j = 0,1, \ldots, n$.
- Lemma. The expectation of $X = \text{Bin}(n, p)$ is
$$\mathbb{E}[X] = np.$$

CS 537 Question

Throw m balls into n bins.

Let X be the number of balls that land into bin 1.

(Recall that Bin(n,p) is the binomial distribution, i.e., the distribution of the number of HEADS in n tosses of a coin with bias p.)

Then the distribution of X is

A. Bin (n, m)

B. Bin $(m, 1/n)$

C. Bin $\left(m, \frac{n-1}{n}\right)$

D. a binomial distribution, but none of the above.

E. not a binomial distribution.

Question

Throw m balls into n bins. Let Y be the number of empty bins.

Compute $\mathbb{E}[\mathrm{Y}]$.

A. 1

B. $n/2$

C. $\left(1-\frac{1}{n}\right)^m$

D. $n\left(1-\frac{1}{n}\right)^m$

E. None of the above.

Product of independent RVs

- Theorem. For any two **independent** random variables X and Y on the same probability space,
$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

- Note. The equality does not hold, in general, for dependent random variables.

Example. We toss two coins.

Let X= number of HEADS, Y= number of TAILS.

Calculate $\mathbb{E}$[X], $\mathbb{E}$[Y] and $\mathbb{E}$[XY].

Jensen's inequality

- Exercise: Let X be the length of a side of a square chosen from [99] uniformly at random. What is the expected value of the area?

  Solution: Find $\mathbb{E}[X^2]$.

  $\mathbb{E}[X^2] =$

- Comparison. $(\mathbb{E}[X])^2 =$

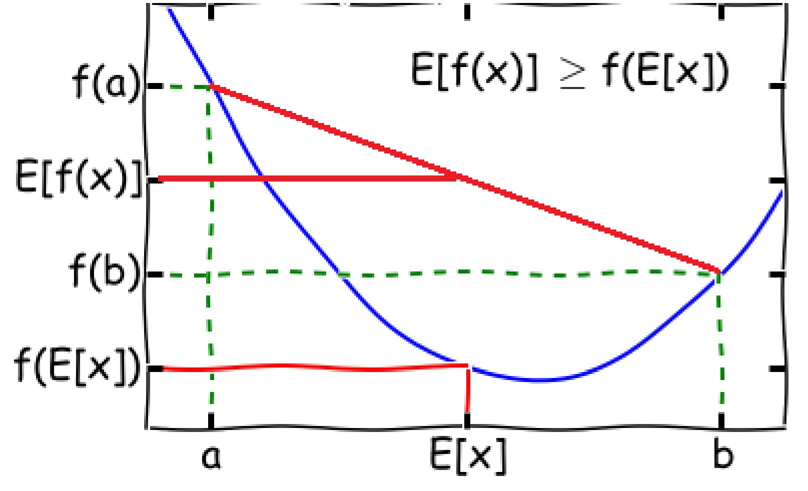
- In general, $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$

Proof: Let $\mu = \mathbb{E}[X]$. Consider $Y = (X - \mu)^2$.

Jensen's inequality

- A function $f:\mathbb{R}\to\mathbb{R}$ is convex if, for all x, y and all $\lambda \in [0,1]$,
$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

- Jensen's inequality. If f is a convex function and X is a random variable, then
$$\mathbb{E}[f(X)] \ge f(\mathbb{E}[X]).$$



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Review question

For arbitrary random variables X and Y, by linearity of expectation:

A. $\mathbb{E}[\mathrm{X}+\mathrm{Y}]=\mathbb{E}[X]+\mathbb{E}[Y]$

B. $\mathbb{E}[a\mathrm{X}+b\mathrm{Y}]=a\mathbb{E}[X]+b\mathbb{E}[Y]$ for all $a, b \in \mathbb{R}$.

C. $\mathbb{E}[\mathrm{XY}]=\mathbb{E}[X]\,\mathbb{E}[Y]$

D. Both A and B are correct.

E. A,B and C are correct.

Conditional expectation: definition

- For any random variable X and event E, the conditional expectation of X given E is

$$\mathbb{E}[X|E] = \sum_i i \cdot \Pr[X = i|E],$$

where the sum is over all possible values *i* taken by X.

Exercise

You roll two dice. Let X_1 be the value on the first die, X_2 be the value on the second die, and $X = X_1 + X_2$.

Calculate $\mathbb{E}[X_1 \mid X = 5]$

Linearity of conditional expectation

- Theorem. For all random variables X and Y and all events A,

$$\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A].$$

Also, for all $c \in \mathbb{R}$,

$$\mathbb{E}[cX \mid A] = c \cdot \mathbb{E}[X \mid A].$$

Exercise

You roll two dice. Let A be the event that you got no sixes.

Let X_1 be the value on the first die, X_2 be the value on the second die, and $X = X_1 + X_2$.

Calculate $\mathbb{E}[X \mid A]$

Recall

Let A be an event and let $E_1, \dots, E_n$ be mutually disjoint events whose union is Ω.

- Law of total probability.

$$\Pr[A] = \sum_{i \in [n]} \Pr[A \cap E_i] = \sum_{i \in [n]} \Pr[A \mid E_i] \cdot \Pr[E_i].$$

Law of total expectation

Let X be a random variable over sample space Ω and let $E_1, \ldots, E_n$ be mutually disjoint events whose union is Ω. Then

$$\mathbb{E}[X] = \sum_{i \in [n]} \mathbb{E}[X \mid E_i] \cdot \Pr[E_i].$$

Conditional expectation: definition

- For random variables X and Y,

  the conditional expectation of X given Y,
  denoted $E[X|Y]$,
  is a random variable that depends on Y.
  Its value, when $Y = y$, is $E[X\,|Y = y]$.

- Example: Let N be the number you get when you roll a die. You roll a fair coin N times and get H heads.

  Find $E[H|N]$.

$$E[H|N = n] = \frac{n}{2}.$$

$$E[H|N] = \frac{N}{2}.$$