Max load

throw ~~m~~ n balls into n bins.

w.p $\leq \frac{1}{n}$. The bin with the most balls has more than $M = \frac{3 \ln n}{\ln \ln n}$.

Proofs: Let X_i be the # of balls in bin i,

$$\Pr[\max_i \{X_i\} \geq M] = \Pr\left[\bigcup_{i=1}^{n} (X_i \geq M)\right] \leq n \Pr[X_1 \geq M]. \quad \text{(union bound)}$$

$$\Pr[X_1 \geq M] \leq \binom{n}{M} \left(\frac{1}{n}\right)^M$$

$$\leq \frac{n^M}{M!} \cdot \frac{1}{n^M} = \frac{1}{M!} \leq \left(\frac{e}{M}\right)^M$$

$$\Pr[\max_i \{X_i\} \geq M] \leq n \left(\frac{e}{M}\right)^M.$$

$$= n \left(\frac{e \ln \ln n}{3 \ln n}\right)^{\frac{3 \ln n}{\ln \ln n}}.$$

$$\leq n \left(\frac{\ln \ln n}{\ln n}\right)^{\frac{3 \ln n}{\ln \ln n}}.$$

$$= e^{\ln n} \cdot e^{(\ln \ln \ln n - \ln \ln n)\frac{3 \ln n}{\ln \ln n}}.$$

$$= e^{\ln n \left(-2 + 3 \frac{\ln \ln \ln n}{\ln \ln n}\right)}$$

$$\leq \frac{1}{n} \text{ for large } n.$$

bucket sort

Assume $n = 2^m$ # of elems to sort $\in [0, 2^k]$ $(k \geq m)$.

1. put n elems into n buckets. $(O(n))$
2. sort within each bucket with insertion sort [Expected $O(n)$]
3. concatenate all buckets $(O(n))$

Example: $n = 4 = 2^2 \in$ range $(0, 2^4)$

0010
0001
1000
1111

4 buckets

00: 0001, 0010
01:
10: 1000
11: 1111

Time to sort bin j with X_j elements $\Rightarrow c(X_j)^2$.

Total time to sort.

$$\mathbb{E}\left[\sum_{j=1}^{n} c(X_j)^2\right] = c\sum_{j=1}^{n} \mathbb{E}(X_j^2) = cn\,\mathbb{E}(X_j^2)$$

$$= cn\left(2-\frac{1}{n}\right)$$

$$= 2cn - c = O(n)$$

$X_1 \sim \mathrm{Bin}\left(n, \frac{1}{n}\right)$.

$$\mathbb{E}(X_1^2) = \mathrm{Var}[X_1] - \mathbb{E}[X_1]^2.$$

$$= n\frac{1}{n}\left(1-\frac{1}{n}\right) + 1.$$

$$= 2 - \frac{1}{n}$$