Randomness in Computing

CS 537

LECTURE 10

Last time

- Chernoff Bounds

Today

- Hoeffding Bounds
- Applications of Chernoff-Hoeffding Bounds
- Estimating a Parameter
- Set Balancing

Sums of independent RVs

Chernoff Bound (Upper Tail). Let $X_1, \dots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for any $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1]$,

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}.$$

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Sums of independent RVs

Chernoff Bound (Lower Tail). Let $X_1, \dots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for any $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

- (easier to use) for any $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le e^{-\mu\delta^2/2}.$$

Sums of independent RVs

Chernoff Bound (Both Tails). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- for any $\delta \in (0,1)$,

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}.$$

Exercise 2

- The Halting Problem Team wins each hockey game they play with probability 1/3. Assuming outcomes of the games are independent, derive an upper bound on the probability that they have a winning season in n games.

- The Halting Problem Team hires a new coach, and critics revise their probability of winning each game to 3/4. Derive an upper bound on the probability they suffer a losing season.

Exercise 3

We throw n balls uniformly and independently into n bins.

Let Y_1 be the number of balls that fell into bin 1.

Determine m such that $\Pr[Y_1 > m] \le \frac{1}{n^2}$.

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Sums of independent RVs

Hoeffding Bound. Let $X_1, \ldots, X_n$ be independent RVs with $\mathbb{E}[X_i] = \mu_0$ and $\Pr[a \leq X_i \leq b] = 1$. Let $X = X_1 + \cdots + X_n$. Then

- (upper tail) $\Pr[X \geq \mu_0 n + \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$.
- (lower tail) $\Pr[X \leq \mu_0 n - \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$

Application: Estimating a parameter

- **Unknown:** probability p that a feature occurs in the population.
- Obtain an estimate by taking n samples
- $X \sim \text{Bin}(n, p)$
- Suppose $X = \tilde{p}n$.
- A $1 - \gamma$ confidence interval for parameter p is $[\tilde{p} - \delta, \tilde{p} + \delta]$ such that $\Pr[p \in [\tilde{p} - \delta, \tilde{p} + \delta]] \geq 1 - \gamma$.
- Find a tradeoff between γ, δ and n.

Application: Estimating a parameter

- A $1-\gamma$ confidence interval for parameter p is $[\tilde{p}-\delta, \tilde{p}+\delta]$ such that $\Pr[p \in [\tilde{p}-\delta, \tilde{p}+\delta]] \geq 1-\gamma$.
- Find a tradeoff between γ, δ and n.

Solution: $\mathbb{E}[X] = np$

- Suppose $p \notin [\tilde{p}-\delta, \tilde{p}+\delta]$
- Case 1: $p < \tilde{p}-\delta$. Then $\tilde{p} > \quad p+\delta$
- Case 2: $p > \tilde{p}+\delta$. Then $\tilde{p} < \quad p-\delta$
- $\gamma = 2 \cdot e^{-2\delta^2 n}$

Application: Set Balancing

- **Given:** an $n \times m$ matrix A with 0-1 entries
- **Definition:** $||(x_1, \dots, x_n)||_\infty = \max_{i \in [n]} |x_i|$
- **Find:** $b \in \{-1,1\}^m$ minimizing $\left||Ab|\right|_\infty$

n features

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

m subjects

Partition subjects into two groups, so that each feature is balanced.

Application: Set Balancing

- Algorithm: Choose each b_i u.a.r. from $\{-1,1\}$.
- Theorem. $\Pr\left[\left|\left|Ab\right|\right|_\infty \geq \sqrt{4m\ln n}\right] \leq 2/n$.