1. (Using Chernoff/Hoeffding bounds)

(a) (**Amplification of the success probability**) Solutions: let t be the number of times we run the algorithm $\mathcal{A}$. Let X_i be the indicator variable that the i_{th} run produces a results that is outside the reasonable range. Let Y_i be the indicator random variable that the i_{th} run produces a result that falls on the left side of the range and let Z_i be an indicator random variable that the i_{th} run produces a result that falls to the right of the reasonable range. Assume we run the algorithm $\mathcal{A}$ t times and let $X = \sum_{i=1}^{t}$, $Y = \sum_{i=1}^{t}$ and $Z = \sum_{i=1}^{t}$, and note that $\mathbf{E}[X] \leq 1/3$, $X = Y + Z$, and $\mathbf{E}[Y] \leq \mathbf{E}[X]$ and $\mathbf{E}[Z] \leq \mathbf{E}[X]$. If our algorithm returns the middle number of these t runs as the results, then our algorithm fails if either $Y > t/2$ or $Z > t/2$. For simplicity, we use $Y \geq t/2$ and $Z \geq t/2$. Using Chernoff bounds,

$$
\begin{aligned}
Pr[Y > t/2] &\leq Pr[Y \geq t/2] && (1)\\
&= Pr[Y \geq (1 + 1/2)(t/3)] && (2)\\
&= Pr[Y \geq (1 + 1/2)\mathbf{E}[X]] && (3)\\
&\leq Pr[Y \geq (1 + 1/2)\mathbf{E}[Y]] && (4)\\
&\leq e^{-\mathbf{E}[Y](1/2)^2/3} && (5)\\
& && (6)
\end{aligned}
$$

Note that from (4) to (5), we have used Chernoff bounds. Similarly, for Z, we can derive,

$$Pr[Z > t/2] \leq e^{-\mathbf{E}[Z](1/2)^2/3}$$

Based on the information

Thus, Let F denote the event that the modified algorithm fails, using union bound, the fail probability is upper bounded by the sum of the above two probabilities, i.e.

$$
\begin{aligned}
Pr[F] &\leq Pr[Y > t/2] + Pr[Z > t/2] && (7)\\
&\leq e^{-\mathbf{E}[Y](1/2)^2/3} + e^{-\mathbf{E}[Z](1/2)^2/3} && (8)\\
&= e^{-\mathbf{E}[Y](1/2)^2/3} + e^{-(t/3-\mathbf{E}[Y])(1/2)^2/3} && (9)\\
&= e^{-tp_Y/12} + e^{-t(1/3-p_Y)/12} && (10)\\
&\leq \delta && (11)
\end{aligned}
$$

where $0 \leq p_Y \leq 1/3$ is the probability that a result falls to the left of the range. Without loss of generality, let's assume that $p_Y = 1/2p_X = (1/2)(1/3) = 1/6$, that is the probability of a sample falling on the left side of the range is same as the probability that it falls on the right side of the range. If we solve the above inequality (9) $\leq$ (10), we can get $t \geq 72 \ln \frac{2}{\delta}$. Thus means if we run the algorithm $\Theta(\log \frac{1}{\delta})$, i.e. total time $O(T(n) \log \frac{1}{\delta})$, and take the median of the results, the failure probability is less than δ, i.e. the probability of a good approximation is at least $1 - \delta$.

(b) (**Coronavirus resilience**)

Proof. From the given inequality,$s \geq \max\left\{\frac{2m}{\alpha}, \frac{8\ln(1\delta)}{\alpha}\right\}$, we also know that $s\alpha \geq 2m$ or $m \leq (\alpha s)/2$, which we will use later. Now let X_i be an indicator random variable that sample i is coronavirus-resilient and let $X = \sum_i^s X_i$ be the number of coronavirus-resilient people in the s samples. We want $Pr[X \geq m] \geq 1 - \delta$ where δ is a number between 0 and 1.

$$\begin{aligned} Pr[X \geq m] &= 1 - Pr[X < m] && (12) \\ &\geq 1 - Pr[X \leq m] && (13) \\ &= 1 - Pr\Big[X \leq \big(1 - (1 - \frac{m}{\mu})\mu\big)\Big] && (14) \\ &\geq 1 - e^{-\mu(1-m/\mu)^2/2} && (15) \end{aligned}$$

where $\mu = \mathbf{E}(X) = \alpha s$. One sufficient condition for $Pr[X \geq m] \geq 1 - \delta$ is that $1 - e^{-\mu(1-m/\mu)^2/2} \geq 1 - \delta$ or after simplifying,

$$\mu\big(1 - \frac{m}{\mu}\big)^2 \geq 2\ln\frac{1}{\delta} \tag{16}$$

Let's first assume that $s \geq \frac{2m}{\alpha} \geq \frac{8\ln(1\delta)}{\alpha}$, or $\mu \geq 2m \geq 8\ln(1/\delta)$.

$$\begin{aligned} \mu\big(1 - \frac{m}{\mu}\big)^2 &\geq 2m(1 - \frac{m}{2m})^2 && (17) \\ &= \frac{m}{2} && (18) \\ &\geq 2\ln(1/\delta) && (19) \end{aligned}$$

which shows that (16) is satisfied, thus we have a sufficient condition for the original inequality.

Then let's consider another possibility, $s \geq \frac{8\ln(1\delta)}{\alpha} \geq \frac{2m}{\alpha}$, or $\mu \geq 8\ln(1/\delta) \geq 2m$. Again (16) is a sufficient condition we want to prove.

$$\begin{aligned} \mu(1 - \frac{m}{\mu})^2 &\geq 8\ln(1/\delta)(1 - \frac{m}{8\ln(1/\delta)})^2 && (20) \\ &\geq 8\ln(1/\delta)(1 - \frac{m}{2m})^2 && (21) \\ &\geq 8\ln(1/\delta)1/4 = 2\ln(1/\delta) && (22) \end{aligned}$$

which also shows that the sufficient condition for our proof (16) is satisfied. Thus, in either case, as long as we have, $s \geq \max\left\{\frac{2m}{\alpha}, \frac{8\ln(1\delta)}{\alpha}\right\}$, our original claim is guaranteed.