4. (Random Children)

(a) Solution: Let C, B, G denote the RVs the total number of children, the total number of boys, and the total number of girls the couple will have respectively. Then $C \sim Geom(1/2)$, and $\mathbb{E}(C) = 1/(1/2) = 2$. Since the couple will end until they have a girl, thus, $\mathbb{E}(G) = 1$. $\mathbb{E}(B) = \mathbb{E}(C) - \mathbb{E}(G) = 2 - 1 = 1$. Thus, both the expected number of girls and boys are 1.

(b) Solution: If the probability of having a girl is only 0.4. Then $C \sim Geom(0.4)$, and $\mathbb{E}(C) = 1/(0.4) = 2.5$, and $\mathbb{E}(G) = 1$. Then $\mathbb{E}(B) = \mathbb{E}(C) - \mathbb{E}(G) = 2.5 - 1 = 1.5$.

(c) Solution: Let C_1, B_1, G_1 denote the RVs the total number of children, the total number of boys, and the total number of girls the couple will have respectively following the new rule and p the probability of having a girl.

We can construct other variables C_2, B_2 and G_2 which denote the additional children the couple would have if they follow the old rule.

$$C = C_1 + C_2 \tag{1}$$
$$B = B_1 + B_2 \tag{2}$$
$$G = G_1 + G_2 \tag{3}$$
$$C_1 = B_1 + G_1 \tag{4}$$
$$C_2 = B_2 + G_2 \tag{5}$$
$$C = B + G \tag{6}$$

Note that $C_2 \sim Geom(p)$ happens with probability $(1-p)^k$ which is denote by $Pr[C > k]$ otherwise $C_2 = 0$.

$$\begin{aligned}\mathbb{E}(B_1) &= \mathbb{E}(B) - Pr[C > k]\mathbb{E}(B_2) \\ &= (1/p - 1) - (1-p)^k(1/p - 1) \\ &= \left[1 - (1-p)^k\right](1/p - 1)\end{aligned} \tag{7}$$

Similarly, for the expected number of girls, G_1, under the new rule,

$$\begin{aligned}\mathbb{E}(G_1) &= \mathbb{E}(G) - Pr[C > k]\mathbb{E}(G_2) \\ &= 1 - (1-p)^k 1 \\ &= 1 - (1-p)^k\end{aligned} \tag{8}$$

If we plug in the value $p = 0.5$ to (7) and (8) respectively, we obtain the expected number of boys to be $1 - (1/2)^k$, and the expected number of girls to be $1 - (1/2)^k$.

(d) We plug in the value $p = 0.4$ to (7) and (8) respectively, we obtain the expected number of boys to be $1.5(1 - 0.6^k)$, and the expected number of girls to be $1 - 0.6^k$.