Lecture. 2/6.

Quick Sort.

-- Sorted ~~ar~~ array, choose first element.

$(n-1)\ (n-2)\ \cdots\ 1$

$$= \frac{n(n-1)}{2} = \Omega(n^2).$$

$(n-1)$

$C(n) = 2C(\frac{n}{2}) + n.$

$C(n) = \Theta(n\log n)$

choose pivot randomly

expected comparison

$2n\ln n + O(N)$

$$E[X] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2}{j-i+1}$$ substitute $k = j-i+1$

$$= \sum_{i=1}^{n-1}\sum_{k=2}^{n-i+1}\frac{2}{k}$$

$$= \sum_{k=2}^{n}\sum_{i=1}^{n+1-k}\frac{2}{k}$$

$$= 2\sum_{k=2}^{n}\frac{n+1-k}{k} = 2\sum_{k=2}^{n}\left(\frac{n+1}{k}-1\right)$$

$$= 2\left[(n+1)\left(\sum_{k=2}^{n}\frac{1}{k}\right) - (n-1)\right] = 2\left[(n+1)\sum_{k=1}^{n}\frac{1}{k} - 2n\right].$$

$$= 2(n+1)H(n) - 4n \approx 2n\ln n + \Theta(n) \quad \boxtimes$$

Markov's Inequality.

$X \geq 0$ - RV.

$a > 0$.

$$Pr[X \geq a] \leq \frac{E[X]}{a}$$

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proof: Let $a > 0$

Let I be the indicator RV for $X \geq a$.

$$I = \begin{cases} 1 & X \geq a \\ 0 & o/w \end{cases} \qquad (*)$$

Since $X \geq 0$, $I \leq \frac{X}{a}$ (for both cases)

$E[I] = Pr[I = 1] = Pr[X \geq a]$

$$Pr[X \geq a] = E[I] \underset{\text{by }(*)}{\leq} E\left[\frac{X}{a}\right] = \frac{E[X]}{a}. \qquad \boxtimes$$

Alternative Proof:

Let $a > 0$

Let A be the event that $X \geq a$.

$$E[X] = \sum_{\sigma \in \Omega} X(\sigma) \cdot Pr[\sigma]$$

$$= \underbrace{\sum_{\sigma \in A} X(\sigma) Pr[\sigma]}_{X(\sigma) \geq a} + \underbrace{\sum_{\sigma \in \Omega - A} X(\sigma) \cdot Pr[\sigma]}_{\geq 0}.$$

$$\geq a \sum_{\sigma \in A} Pr[\sigma] = a\, Pr[A].$$

$$= a\, Pr[X \geq a].$$

$$\Rightarrow Pr[X \geq a] \leq \frac{E[X]}{a}.$$

$$\Pr[X \geq b \cdot \mathbb{E}[X]] \leq \frac{1}{b}.$$

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

$$\Pr[X > \cdots] \leq \frac{2n\ln n + O(n)}{10n\ln n + O(n)} = \frac{1}{5}$$

<u>Bernoulli random walk</u> (Variance).

$$Var[X] = \mathbb{E}[(X-\mu)^2]$$

$$Var[X] = E[X^2] - \mu^2$$