Randomness in Computing

CS 537

LECTURE 13

Last time

- Finished routing on hypercube
- Balls into bins model

Today

- Poisson distribution
- Poisson approximation

The number of empty bins

m balls into n bins

- The probability that bin 1 is empty is

$\boxed{\text{for } x \leq 1/2}$

$\boxed{e^{-x-x^2} \leq 1 - x \leq x^{-x}}$

- Expected number of empty bins

X = the number of empty bins

$X_i =$

$\mathbb{E}[X] =$

The number of bins with r balls

***m* balls into *n* bins, *r* is a small constant**

- The probability p_r that bin 1 has r balls is

$p_r =$

Poisson random variables

- A Poisson random variable with parameters μ is given by the following distribution on $j = 0,1,2,\ldots$

$$\Pr[X = j] = \frac{e^{-\mu}\mu^j}{j!}$$

- Check that probabilities sum to 1:

Taylor expansion: $\boldsymbol{e^x} = \sum_{\boldsymbol{j=0}}^{\infty} \frac{\boldsymbol{x^j}}{\boldsymbol{j!}}$

$\sum_{j=0}^{\infty} \Pr[X = j] = \sum_{j=0}^{\infty} \frac{e^{-\mu}\mu^j}{j!} =$

- The expectation of a Poisson R.V. X is

$\mathbb{E}[X] =$

$\mathrm{var}[X] = \mu$ (See Ex. 5.5)

Independent Poisson RVs

Theorem

Let X and Y be independent Poisson RVs with means μ_X and μ_Y.

Then $X + Y$ is a Poisson RV with mean $\mu_X + \mu_Y$.

Chernoff Bounds for Poisson RVs

Theorem. Let X be a Poisson RV with mean μ.

- (upper tail, additive) If $x > 0$, then
$$\Pr[X \geq \mu + x] \leq \frac{e^{-\mu}(e\mu)^x}{x^x}.$$
- (lower tail, additive) If $x < \mu$, then
$$\Pr[X \leq x] \leq \frac{e^{-\mu}(e\mu)^x}{x^x}.$$
- (upper tail, multiplicative) For any $\delta > 0$,
$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$
- (lower tail, multiplicative) For any $\delta \in (0,1)$,
$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

Poisson Distribution is Limit of Binomial Distribution

Theorem

Let $X_n \sim \text{Bin}(n, p)$, where p is a function of n and $\lim_{n\to\infty} np = \mu$, a constant independent of n.

Then, for all fixed k,

$$\lim_{n\to\infty} \Pr[X_n = k] = \frac{e^{-\mu}\mu^k}{k!}.$$

- Applies to balls-and-bins fi $m = nc$.

The Poisson Approximation

- The Balls-and-Bins model has dependences.
- E.g. if Bin 1 is empty, then Bin 2 is less likely to be empty.
- The Poisson Approximation gets rid of dependencies.
- (on the board).