CSCI-C 241 HW #7 Assigned Friday, February 25th, 2022
Instructor: Haixu Tang Due Friday, March 4th, 2022

For the toy models on this assignment, the universe will be some set of letters. Some of the letters will be vowels (A, a, E, e, I, i, ...) and some will be consonants (B, b, D, d, F, f, G, g, ...) Some will be in uppercase (A, B, D, E, ...), and some will be in lowercase (a, b, d, e, ...). Some will be from the first half of the alphabet (A-M), and some will be from the second half (N-Z).

To avoid confusion, I'll try to avoid letters where it's hard to tell the difference between upper and lowercase, and you should do the same (e.g., don't use C or O or X). Also, for this assignment, we will treat Y as a vowel.

Use the following interpretations for the predicate symbols.

$V(x)$: "x is a vowel."
$L(x)$: "x is lowercase."
$F(x)$: "x is from the first half of the alphabet."

When asked to translate a formula into a "natural-sounding" English sentence, remember that you can't use any variables in the sentence (so nothing like "For every letter x..."), and your sentence must be unambiguous (so nothing like "Every vowel is not...").

1. For these problems, you will be asked to rewrite formulas using equivalence rules so that they are in a particular format. In particular, you will be asked to rewrite a formula so that no quantifier or connective is inside the scope of a negation, or you will be asked to rewrite a formula so that *all* other connectives and quantifiers are inside the scope of a negation.

   In case you're still unclear about what I mean by "the scope of the negation", here are a few examples. In the formula $\neg \exists x\big(A(x) \wedge B(x)\big)$, the quantifier $\exists x$ and the connective $\wedge$ are both *inside* the scope of the negation $\neg$. And in the formula $\exists x\big(\neg P(x) \wedge \neg Q(x)\big)$, the quantifier $\exists x$ and the connective $\wedge$ are *outside* the scope of both negations. And finally, in a formula like $\forall x \neg\big(R(x) \vee S(x)\big)$, the quantifier $\forall x$ is *outside* the scope of the negation, but the connective $\vee$ is *inside* the scope of the negation.

   (a) Rewrite the formula $\neg \forall x\big(F(x) \to V(x)\big)$ so that no quantifiers or connectives are inside the scope of a negation.(2 points)

   (b) Rewrite the formula $\neg \exists x\big(F(x) \wedge L(x)\big)$ so that no quantifiers or connectives are inside the scope of a negation.(2 points)

   (c) Rewrite the formula $\forall x\big(F(x) \to \neg V(x)\big)$ so that all other connectives and quantifiers are inside the scope of the negation.(2 points)

   (d) Rewrite the formula $\exists x\big(F(x) \wedge \neg V(x)\big)$ so that all other connectives and quantifiers are inside the scope of the negation.(2 points)

2. For the questions in this problem, consider the following models:

$\mathcal{M}_1$:	e	A	i	E
$\mathcal{M}_2$:	e	A	j	T
$\mathcal{M}_3$:	B	d	j	T
$\mathcal{M}_4$:	E	A	j	T
$\mathcal{M}_5$:	e	a	j	T
$\mathcal{M}_6$:	e	A	I	T
$\mathcal{M}_7$:	e	a	i	u
$\mathcal{M}_8$:	E	A	B	T

(a) Which of the above models ($\mathcal{M}_1$-$\mathcal{M}_8$) satisfy the formula $\neg\forall x\big(V(x)\to L(x)\big)$? **Hint**: This is the same as asking for models that do *not* satisfy $\forall x\big(V(x)\to L(x)\big)$.(4 points)

(b) Translate $\neg\forall x\big(V(x)\to L(x)\big)$ into a natural-sounding English sentence.(2 points)

(c) Which of the above models ($\mathcal{M}_1$-$\mathcal{M}_8$) satisfy the formula $\forall x\big(V(x)\to\neg L(x)\big)$? **Hint**: This is not the same as the formula in the previous question.(4 points)

(d) Translate $\forall x\big(V(x)\to\neg L(x)\big)$ into a natural-sounding English sentence.(2 points)

(e) Which of the above models ($\mathcal{M}_1$-$\mathcal{M}_8$) satisfy the formula $\neg\exists x\big(V(x)\wedge L(x)\big)$? (4 points)

(f) Translate $\neg\exists x\big(V(x)\wedge L(x)\big)$ into a natural-sounding English sentence. (2 points)

(g) Which of the above models ($\mathcal{M}_1$-$\mathcal{M}_8$) satisfy the formula $\exists x\big(V(x)\wedge\neg L(x)\big)$? (4 points)

(h) Translate $\exists x\big(V(x)\wedge\neg L(x)\big)$ into a natural-sounding English sentence? (2 points)

3. Translate the following English sentences into formulas of First-Order Logic.

   (a) No vowel is from the first half of the alphabet.(2 points)

   (b) Not every lowercase letter is from the first half of the alphabet.(2 points)

   (c) There is a vowel that is not from the first half of the alphabet.(2 points)

   (d) There aren't any vowels that are from the first half of the alphabet.(2 points)

   (e) Every letter that is *not* a vowel is from the first half of the alphabet.(2 points)

4. For this problem, I'm going to ask you to partition a collection of formulas into groups based on logical equivalence. In other words, every formula in each group you list must be equivalent to every other formula in the group. For example, if I asked you to do this for the propositional logic formulas $A\to B$, $A\wedge B$, $A\vee B$, $\neg A\vee B$, $B\wedge A$, and $\neg B\to\neg A$, you would write:(10 points)

   - $A\to B$, $\neg A\vee B$, $\neg B\to\neg A$
   - $A\wedge B$, $B\wedge A$
   - $A\vee B$

   Collect the following First-Order Logic formulas into groups based on logical equivalence:

$$
\begin{array}{ll}
\exists x \neg\big(V(x) \to L(x)\big) & \forall x \neg\big(V(x) \to L(x)\big) \\
\forall x \neg\big(V(x) \land L(x)\big) & \neg \exists x\big(V(x) \land L(x)\big) \\
\forall x\big(\neg V(x) \lor \neg L(x)\big) & \forall x\big(V(x) \to \neg L(x)\big) \\
\neg \forall x\big(V(x) \to L(x)\big) & \exists x\big(V(x) \land \neg L(x)\big)
\end{array}
$$

5. If you are asked to create a model and you think no model exists, say so, and explain why not. For this problem, we'll be considering models where the universe is some collection of shapes, with arrows drawn between some of the shapes. We will use the following interpretation for the predicate symbols:

   $S(x)$: x is solid.
   $C(x)$: x is a circle.
   $P(x,y)$: x points to y.

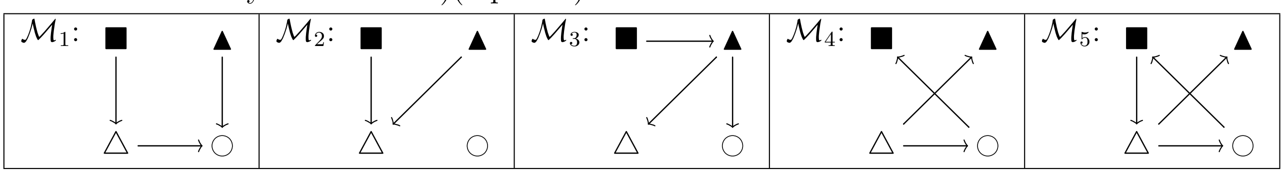
   (a) Create a model which satisfies $\forall x \exists y P(x,y)$, but not $\exists y \forall x P(x,y)$.(3 points)

   (b) Create a model which satisfies $\exists y \forall x P(x,y)$, but not $\forall x \exists y P(x,y)$.(3 points)

   (c) Create a model which satisfies $\forall x \exists y P(x,y)$, but not $\forall x \exists y P(y,x)$.(3 points)

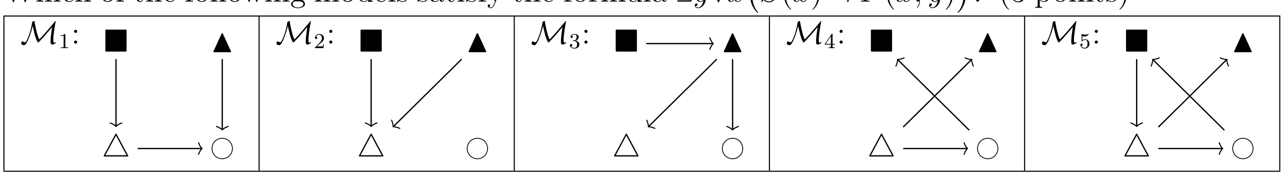
   (d) Create a model which satisfies $\forall x \exists y P(x,y)$ and $\forall x \exists y P(y,x)$, but not $\forall x \forall y P(x,y)$.(3 points)

   (e) Which of the following models satisfy the formula $\forall x\big(S(x) \to \exists y P(x,y)\big)$? (As always, there may be more than one correct answer, and you need to list *all* of the models that satisfy the formula.)(3 points)



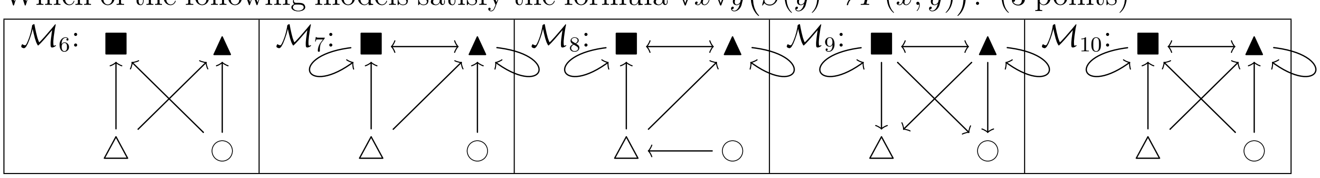
   (f) Translate $\forall x\big(S(x) \to \exists y P(x,y)\big)$ into a natural-sounding English sentence. (As always, there cannot be any variables in your translation.) (3 points)

   (g) Which of the following models satisfy the formula $\exists y \forall x\big(S(x) \to P(x,y)\big)$? (3 points)



   (h) Translate $\exists y \forall x\big(S(x) \to P(x,y)\big)$ into a natural-sounding English sentence. (3 points)

   (i) Which of the following models satisfy the formula $\forall x \forall y\big(S(y) \to P(x,y)\big)$? (3 points)



   (j) Translate $\forall x \forall y\big(S(y) \to P(x,y)\big)$ into a natural-sounding English sentence.(3 points)

   (k) Create a model that satisfies $\forall x \exists y\big(S(y) \land P(x,y)\big)$ but not $\forall x \forall y\big(S(y) \to P(x,y)\big)$.(3 points)

(l) Translate $\forall x \exists y \big(S(y) \land P(x, y)\big)$ into a natural-sounding English sentence(3 points)

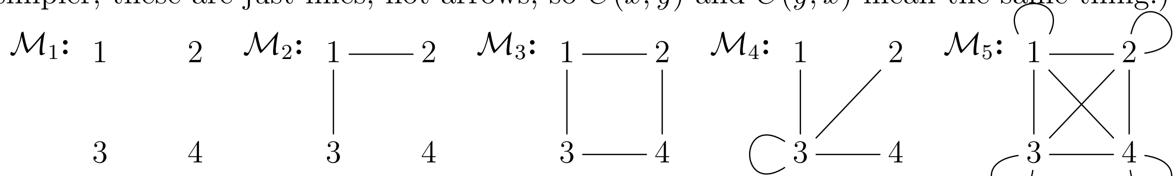
(m) Translate "Some circle points to all the solid shapes," into a first-order logic formula.(3 points)

(n) Translate "There is a circle that all solid shapes point to," into a first-order logic formula.(3 points)

(o) Translate "Every solid shape has a circle that points to it," into a first-order logic formula.(3 points)

(p) Translate "Every solid shape has some circle that it points to," into first-order logic.(3 points)

6. In each part of this problem, the universe will consist of four numbers $\{1, 2, 3, 4\}$. To complete the toy models, I'll be drawing lines connecting some of the numbers. $C(x, y)$ means "x is connected to y." Consider the following toy models. (To make things simpler, these are just lines, not arrows, so $C(x, y)$ and $C(y, x)$ mean the same thing.)



   (a) Which of the above models satisfy the formula $\forall x \exists y \neg C(x, y)$?(3 points)

   (b) Translate $\forall x \exists y \neg C(x, y)$ into an English sentence that doesn't use variables. (Hint: You might find it easier to use some equivalence laws to rewrite the formula before translating.)(3 points)

   (c) Which of the above models satisfy the formula $\forall x \neg \exists y C(x, y)$?(3 points)

   (d) Translate $\forall x \neg \exists y C(x, y)$ into an English sentence that doesn't use variables.(3 points)

   (e) Which of the above models satisfy the formula $\neg \forall x \exists y C(x, y)$?(3 points)

   (f) Translate $\neg \forall x \exists y C(x, y)$ into an English sentence that doesn't use variables.(3 points)

7. For this problem, the universe will be some set of English words, and we will use the following translations for our predicate symbols:

   $R(x)$: x starts with the letter 'r'.
   $T(x)$: x starts with the letter 't'.
   $F(x)$: x has five letters.
   $L(x, y)$: x is longer than y.
   $S(x, y)$: x shares a letter with y.

   So $L(\text{enormous}, \text{tiny})$, $S(\text{cat}, \text{tiny})$, $S(\text{cat}, \text{tan})$,and $S(\text{cat}, \text{cat})$ are all true, but $L(\text{tiny}, \text{enormous})$, $L(\text{cat}, \text{cat})$, and $S(\text{cat}, \text{dog})$ are all false.

(a) Consider the sentence "There is a word that doesn't share a letter with all 5-letter words." Which of the following five models satisfy this sentence? (3 points)

$\mathcal{M}_1$:	proof	valid	bug	set	
$\mathcal{M}_2$:	proof	valid	logic	set	not
$\mathcal{M}_3$:	proof	valid	logic	pun	not
$\mathcal{M}_4$:	proof	valid	logic	set	program
$\mathcal{M}_5$:	proof	valid	logic	not	program
$\mathcal{M}_6$:	proof	logic	not	program	

(b) Translate the sentence "There is a word that doesn't share a letter with all 5-letter words," into a formula of first-order logic. (3 points)

(c) Consider the sentence "There is a word that doesn't share a letter with *any* 5-letter words." Which of the following five models satisfy this sentence? (3 points)

$\mathcal{M}_1$:	proof	valid	bug	set	
$\mathcal{M}_2$:	proof	valid	logic	set	not
$\mathcal{M}_3$:	proof	valid	logic	pun	not
$\mathcal{M}_4$:	proof	valid	logic	set	program
$\mathcal{M}_5$:	proof	valid	logic	not	program
$\mathcal{M}_6$:	proof	logic	not	program	

(d) Translate the sentence "There is a word that doesn't share a letter with *any* 5-letter words," into a formula of first-order logic. (3 points)

(e) Translate the English sentence "No word that starts with 't' is longer than any of the words that start with 'r' into a formula of first-order logic. (3 points)

(f) Consider the formula $\exists x \exists y \big(F(y) \land \neg S(x,y)\big)$. Which of the following five models satisfy this formula? (3 points)

$\mathcal{M}_1$:	proof	valid	bug	set	
$\mathcal{M}_2$:	proof	valid	logic	set	not
$\mathcal{M}_3$:	proof	valid	logic	pun	not
$\mathcal{M}_4$:	proof	valid	logic	set	program
$\mathcal{M}_5$:	proof	valid	logic	not	program
$\mathcal{M}_6$:	proof	logic	not	program	

(g) Translate the formula $\exists x \exists y \big(F(y) \land \neg S(x,y)\big)$ into an English sentence that doesn't have any variables in it. (3 points)

(h) Consider the formula $\forall x \exists y \big(T(y) \land \neg L(x,y)\big)$. Use logical equivalence rules to rewrite this formula with the negation between the quantifiers. (3 points)

(i) Consider the formula $\forall x \exists y \big(T(y) \land \neg L(x,y)\big)$. Use logical equivalence rules to rewrite this formula with the negation before/outside all of the quantifiers. (3 points)

(j) Consider the formula $\forall x \exists y \big(T(y) \land \neg L(x,y)\big)$. Which of the following models satisfy this formula? (3 points)

$\mathcal{M}_7$:	up	set	truth	transitivity	
$\mathcal{M}_8$:	up	truth	computer	transitivity	
$\mathcal{M}_9$:	up	top	over	truth	computer
$\mathcal{M}_{10}$:	top	over	truth	computer	
$\mathcal{M}_{11}$:	top	truth	computer	commutativity	

Hint: You can use any of the three equivalent versions of the formula to help you understand the formula. Use whichever one makes the most sense to you.

(k) Translate the formula $\forall x \exists y \big(T(y) \wedge \neg L(x, y)\big)$ into an English sentence that doesn't use any variables. (3 points)