MATH 237 Online Calculus 3 for Honours Mathematics **Spring 2024**

Problem set and solutions

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1. A quadratic surface is the graph of a second-degree equation in three variables x, y, z

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, \cdots$ are constants. In particular, a two-variable scalar function $z = f(x, y)$ can determine a quadratic surface. For example, if $z = \sqrt{2x^2 - 4y^2}$. It determines a quadratic surface given by

$$-2x^2 + 4y^2 + z^2 = 0. \tag{1.1}$$

Now determine the quadratic surfaces given by the following two-variable functions and specify the domain and range.(Domain is the admissible values of (x, y) and range is the admissible values of z.)

(i) $f(x, y) = 2\sqrt{1 - x}$.

(ii) $f(x, y) = \frac{x^2}{4} + \frac{y^2}{2}$.

(iii) $f(x, y) = \frac{x^2}{4} - \frac{y^2}{2}$.

(i) The quadratic surface is given by

$$z^2 + 4x - 4 = 0.$$

Domain $= \{(x, y) : x \leqslant 1, y \in \mathbb{R}\} = (-\infty, 1] \times (-\infty, +\infty)$. Range $= [0, +\infty)$.

(ii) The quadratic surface is given by

$$x^2 + 2y^2 - 4z = 0$$

Domain $= \mathbb{R}^2$. Range $= [0, +\infty)$.

(iii) The quadratic surface is given by

$$x^2 - 2y^2 - 4z = 0$$

Domain $= \mathbb{R}^2$. Range $= (-\infty, +\infty)$.

2. Evaluate the limit or show the limit does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}.$$

(ii)

$$\lim_{(x,y)\to(0,0)} \exp\left(-\frac{1}{x^2 + y^2}\right).$$

(iii)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy}{\sqrt{x^2+y^2+z^2}}.$$

(iv)
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x^2+y^2+z^2}}.$$

(i) We use squeeze theorem:
$$\left|\frac{x^2-y^2}{\sqrt{x^2+y^2}}\right| \leqslant \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0, \quad (x,y)\to(0,0),$$
thus
$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0.$$

(ii) We use $\varepsilon-\delta$ language to formally prove it: for any $\varepsilon \in (0,1), \exists \delta = \sqrt{-\frac{1}{\ln\varepsilon}}$, such that when $0 < \sqrt{x^2+y^2} < \delta$, we have
$$\exp\left(-\frac{1}{x^2+y^2}\right) < \exp\left(-\frac{1}{\delta^2}\right) = \varepsilon,$$
thus
$$\lim_{(x,y)\to(0,0)} \exp\left(-\frac{1}{x^2+y^2}\right) = 0.$$

(iii) We use squeeze theorem:
$$\left|\frac{xy}{\sqrt{x^2+y^2+z^2}}\right| \leqslant \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leqslant \frac{1}{2}\left|\frac{x^2+y^2}{\sqrt{x^2+y^2}}\right| = \frac{1}{2}\sqrt{x^2+y^2} \to 0, \quad (x,y)\to(0,0),$$
thus
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy}{\sqrt{x^2+y^2+z^2}} = 0.$$

(iv) We use squeeze theorem:
$$\left|\frac{xyz}{\sqrt{x^2+y^2+z^2}}\right| \leqslant \left|\frac{xyz}{\sqrt{x^2+y^2}}\right| \leqslant \frac{1}{2}\left|\frac{x^2+y^2}{\sqrt{x^2+y^2}}z\right| = \frac{1}{2}\sqrt{x^2+y^2}|z| \to 0, \quad (x,y,z)\to(0,0,0),$$
thus
$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{\sqrt{x^2+y^2+z^2}} = 0.$$

3. Suppose a, b, c, d are positive numbers such that $(a, b) \neq (c, d)$. Show that

$$\lim_{(x,y)\to(0,0)} \frac{|x|^a + |y|^b}{|x|^c + |y|^d} \tag{1.2}$$

exists if and only if $a > c$ and $b > d$.

Sufficiency: If $a > c, b > d$ we need to show that

$$\lim_{(x,y)\to(0,0)} \frac{|x|^a + |y|^b}{|x|^c + |y|^d} = 0. \tag{1.3}$$

We use $\varepsilon - \delta$ language to prove it. For any $\varepsilon \in (0,1)$, we choose

$$\delta = \{\varepsilon^{\frac{1}{a-c}}, \varepsilon^{\frac{1}{b-d}}\}.$$

Then when $0 < \sqrt{x^2 + y^2} < \delta$, in particular $|x|, |y| < \delta$, we have

$$\begin{aligned}
&|x|^a + |y|^b - \varepsilon(|x|^c + |y|^d) \\
&= |x|^c(|x|^{a-c} - \varepsilon) + |y|^d(|y|^{b-d} - \varepsilon) \\
&< |x|^c(\delta^{a-c} - \varepsilon) + |y|^d(\delta^{b-d} - \varepsilon) \\
&\leqslant |x|^c(\varepsilon - \varepsilon) + |y|^d(\varepsilon - \varepsilon) \\
&= 0.
\end{aligned}$$

Therefore,

$$\frac{|x|^a + |y|^b}{|x|^c + |y|^d} < \varepsilon, \; 0 < \sqrt{x^2 + y^2} < \delta. \tag{1.4}$$

Necessity: We show that if $a \leqslant b$ or $b \leqslant d$ and $(a, c) \neq (b, d)$, then $\lim_{(x,y)\to(0,0)} \frac{|x|^a+|y|^b}{|x|^c+|y|^d}$ does not exist(DNE). In fact, if $a \leqslant c$ and $b \leqslant d$, we choose $|y| = k|x|^{\frac{a}{b}}$, then

$$\frac{|x|^a + |y|^b}{|x|^c + |y|^d} = \frac{1 + k^b}{|x|^{c-a} + k^b |y|^{d-b}} \to \begin{cases} \frac{1+k^b}{k^b}, \; a < c, b = d, \\ +\infty, \; a < c, b < d, \\ 1 + k^b, \; a = c, b < d. \end{cases}$$

Since the choice of $k > 0$ is arbitrary, we have $\lim_{(x,y)\to(0,0)} \frac{|x|^a+|y|^b}{|x|^c+|y|^d}$ does not exist. Other cases are similar. We have the following table exhibiting each case:

Limit / Comparison of b, d / Comparison of a, c	$b < d$	$b = d$	$b > d$
$a < c$	DNE; $\lvert y\rvert = k\lvert x^{\frac{a}{b}}\rvert$	DNE; $\lvert y\rvert = k\lvert x^{\frac{a}{b}}\rvert$	DNE; $\lvert y\rvert = k\lvert x^{\frac{a}{d}}\rvert$
$a = c$	DNE; $\lvert y\rvert = k\lvert x^{\frac{a}{b}}\rvert$	1	DNE; $\lvert y\rvert = k\lvert x^{\frac{a}{d}}\rvert$
$a > c$	DNE; $\lvert y\rvert = k\lvert x^{\frac{c}{b}}\rvert$	DNE; $\lvert y\rvert = k\lvert x^{\frac{c}{b}}\rvert$	0

4. Suppose a, b, c, d are positive numbers. Show that

$$\lim_{(x,y)\to(0,0)} \frac{|x|^a|y|^b}{|x|^c+|y|^d} \tag{1.5}$$

exists if and only if

$$\frac{a}{c}+\frac{b}{d}>1.$$

(Hint: use Young's inequality: suppose $x, y \geqslant 0$, then for any $p, q \geqslant 1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have $xy \leqslant \frac{x^p}{p}+\frac{y^q}{q}$)

Sufficiency: If $\frac{a}{c}+\frac{b}{d}>1$ we need to show that

$$\lim_{(x,y)\to(0,0)} \frac{|x|^a|y|^b}{|x|^c+|y|^d}\frac{|x|^a|y|^b}{|x|^c+|y|^d}=0. \tag{1.6}$$

We use Squeeze Theorem to prove it. In fact, since $\frac{a}{c}+\frac{b}{d}>1$ then there exist $p, q \geqslant 1$ with $\frac{1}{p}+\frac{1}{q}=1$ such that $\frac{a}{c}>\frac{1}{p}, \frac{b}{d}>\frac{1}{q}$. Then apply Young's inequality, we have

$$\begin{aligned}
&\frac{|x|^a|y|^b}{|x|^c+|y|^d}\\
&\leqslant \frac{|x|^{pa}/p+|y|^{bq}/q}{|x|^c+|y|^d}\\
&\leqslant \max\{\frac{1}{p},\frac{1}{q}\}\frac{|x|^{pa}+|y|^{bq}}{|x|^c+|y|^d} \to 0, (x,y)\to(0,0).
\end{aligned}$$

Note that we apply Question 3 above to get the limit converging to zero.
Necessity: We show that if $\frac{a}{c}+\frac{b}{d}\leqslant 1$ then $\lim_{(x,y)\to(0,0)} \frac{|x|^a|y|^b}{|x|^c+|y|^d}$ does not exist(DNE). In fact, we choose

$$|y|=k|x|^{\frac{c}{d}}, \tag{1.7}$$

we have

$$\frac{|x|^a|y|^b}{|x|^c+|y|^d}=|x|^{\frac{ad+bc-cd}{d}}\frac{k^b}{1+k^d}, \quad ad+bc-cd\leqslant 0$$

Since the choice of $k>0$ is arbitrary, we have $\lim_{(x,y)\to(0,0)} \frac{|x|^a+|y|^b}{|x|^c+|y|^d}$ does not exist.

5. Differentiability and Continuity.

   (i) State the definition: $f(x, y)$ is differentiable at (a, b).(See 5.1 in Möbius)

   (ii) Show that if $f(x, y)$ is differentiable at (a, b) then $f(x, y)$ is continuous at (a, b). (See 5.2 in Möbius)

   (iii) Show that if the partial derivatives f_x and f_y are continuous at (a, b) then $f(x, y)$ is differentiable at (a, b). (See Theorem 2, 5.3 in Möbius)

6. Show that $f(x, y) = |x^2 + y^2 - 1|$ is not differentiable at any (x_0, y_0) with $x_0^2 + y_0^2 = 1$.

We use the definition of differentiable functions. For any (x_0, y_0) with $x_0^2 + y_0^2 = 1$, if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$

does not exist, then we are done.
15 points: Calculation of $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, show that either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{|(x_0 + h)^2 + y_0^2 - 1| - |x_0^2 + y_0^2 - 1|}{h} = \lim_{h\to 0} \frac{|2x_0 h + h^2|}{h} = \begin{cases} 2|x_0|, h > 0 \\ -2|x_0|, h < 0. \end{cases} \tag{1.8}$$

Thus the limit exists only for $x_0 = 0$. Similar argument shows that $f_y(x_0, y_0)$ only exists for $y_0 = 0$. Since $x_0^2 + y_0^2 = 1$, x_0 and y_0 cannot be zero simultaneously. Thus either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

7. For the following functions, determine the differentiability at $(0,0)$:

(i) $f(x,y) = x(|y|-1)$.

(ii)

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, (x,y) \neq (0,0), \\ 0, (x,y) = (0,0). \end{cases}$$

(iii)

$$f(x,y) = \begin{cases} \frac{x^{100}y^2}{x^{100}+y^{98}}, (x,y) \neq (0,0), \\ 0, (x,y) = (0,0). \end{cases}$$

(Hint: Apply Problem 4)

(i) Calculation of f_x:

Note that $f(x,0) = -x$, thus $f_x(0,0) = -1$.

Calculation of f_y:

Note that $f(0,y) = 0$, thus $f_y(0,0) = 0$.

Differentiability:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x|y|}{\sqrt{x^2+y^2}} = 0$$

The limit is zero via squeeze theorem $\frac{|xy|}{\sqrt{x^2+y^2}} \leqslant |x| \to 0$. Therefore, it is differentiable.

(ii) Calculation of f_x:

Note that $f(x,0) = 0$, thus $f_x(0,0) = 0$.

Calculation of f_y:

Note that $f(0,y) = 0$, thus $f_y(0,0) = 0$.

Differentiability:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist by choosing $y = mx$.

(iii) One can show that f_x, f_y are continuous at $(0,0)$ and use 1. iii). For the differentiability, one can also use the definition.
Calculation of f_x and continuity:

$$f_x = \frac{100x^{99}y^{100}}{(x^{100}+y^{98})^2},$$

which is continuous at $(0,0)$ In fact,

$$\left|\frac{100x^{99}y^{100}}{(x^{100}+y^{98})^2}\right| = (\frac{100|x|^{99/2}y^{50}}{x^{100}+y^{98}})^2$$

which converges to 0 when $(x,y) \to (0,0)$ since $99/200 + 50/98 > 1$ via Problem 4 in Written assignment 1.
Calculation of f_y and continuity:

$$f_y = \frac{2x^{200}y - 96x^{100}y^{99}}{(x^{100}+y^{98})^2}$$

which converges to 0 when $(x,y) \to (0,0)$ since $\frac{2x^{200}y}{(x^{100}+y^{98})^2}$ tends to zero by $100/100 + \frac{1}{2\times 98} > 1$ and $\frac{96x^{100}y^{99}}{(x^{100}+y^{98})^2}$ tends to zero by $50/100 + \frac{99}{2\times 98} > 1$ via Problem 4 in Written assignment 1.
In summary f_x and f_y are continuous thus f is differentiable via 1. iii).
Remark: one can also show the differentiability via definition

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^{100}y^2}{(x^{100}+y^{98})\sqrt{x^2+y^2}} = 0.$$

One can use polar coordinate $x = r\cos\theta, y = \sin\theta$ to show the limit is 0.

8. Evaluate the limit or show the limit does not exist.

(i)

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3}.$$

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^3+y^6)^2}.$$

(iii) Suppose a,b,c,d are positive numbers and $\frac{a}{c} + \frac{b}{d} > 1$. Find

$$\lim_{(x,y)\to(0,0)} (|x|^c + |y|^d)^{|x|^a|y|^b}.$$

(i) Take $x = 0$, we get -1 and take $y = 0$ we get 1. Thus the limit does not exist.

(ii) Take $x = my^2$, we get $\frac{m^2}{(m^3+1)^2}$, thus the limit does not exist.

(iii) Note that

$$(|x|^c + |y|^d)^{|x|^a|y|^b} = \exp\big(|x|^a|y|^b \ln\big(|x|^c + |y|^d\big)\big) = \exp\left(\frac{|x|^a|y|^b}{|x|^c + |y|^d}(|x|^c + |y|^d)\ln\big(|x|^c + |y|^d\big)\right)$$

Since $\lim_{(x,y)\to(0,0)} \frac{|x|^a|y|^b}{|x|^c+|y|^d} = 0$ and $\lim_{(x,y)\to(0,0)}(|x|^c+|y|^d)\ln\big(|x|^c + |y|^d\big) = \lim_{t\to 0} t\ln t = 0$, we get the original limit is $\exp(0) = 1$ by limit theorem.

9. Suppose

$$f(x,y,z) = e^x\sqrt{y}z$$

(i) Find the gradient of f.

(ii) Find the linear approximation at the point $(0, 25, 1)$

(iii) Use the linear approximation above to estimate $e^{0.01} \times \sqrt{24.8} \times 1.02$.

(i) The gradient of f is given by

$$\nabla f(x,y,z) = (e^x\sqrt{y}z, \frac{e^x z}{2\sqrt{y}}, e^x\sqrt{y}).$$

(ii) The linear approximation at $(0, 25, 1)$ is given by

$$L(x,y,z) = f(0,25,1) + \nabla f(0,25,1)\cdot(x, y-25, z) = 5x + \frac{y}{10} + 5z - 2.5.$$

(iii) $e^{0.01} \times \sqrt{24.8} \times 1.02$ can be approximated by

$$f(0,25,1) + \nabla f(0,25,1)\cdot(0.01, -0.2, 0.02) = 5.13.$$

10. Suppose

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

(i) Is $f(x,y)$ continuous at $(0,0)$?

(ii) Calculate $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

(iii) State Clairaut's Theorem. Discuss why (ii) is not contradictive to Clairaut's Theorem.

(i) Note that

$$|xy\frac{x^2-y^2}{x^2+y^2}| \leqslant |xy\frac{x^2+y^2}{x^2+y^2}| = |xy| \to 0.$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and continuity follows.

(ii) We first calculate for $(x,y) \neq (0,0)$

$$f_x(x,y) = y\frac{x^2-y^2}{x^2+y^2} + y\frac{4x^2y^2}{(x^2+y^2)^2}.$$

In particular, for $x = 0$, we have

$$f_x(0,y) = -y,$$

thus by definition,

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0}\frac{-h}{h} = -1.$$

Similarly, for $(x,y) \neq (0,0)$

$$f_y(x,y) = x\frac{x^2-y^2}{x^2+y^2} - x\frac{4x^2y^2}{(x^2+y^2)^2}.$$

In particular, for $y = 0$, we have

$$f_y(x,0) = x,$$

thus by definition,

$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0}\frac{h}{h} = 1.$$

(iii) Clairaut's Theorem: If f_{xy} and f_{yx} are defined in some neighborhood of (a,b) and are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$. f_{xy} and f_{yx} are not continuous at $(0,0)$ so they can be different.

11. Suppose $f(x,y)$ is bounded on the disk

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}.$$

Moreover, f is homogeneous with order $k \geqslant 1$, i.e.,

$$\forall t \in \mathbb{R}, (x,y) \in \mathbb{R}^2, \quad f(tx,ty) = t^k f(x,y).$$

Find

$$\lim_{(x,y)\to(0,0)} f(x,y).$$

Use the polar coordinate

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$

where $r = \sqrt{x^2+y^2}$. Then we have

$$f(x,y) = f(r\cos\theta, r\sin\theta) = r^k f(\cos\theta, \sin\theta).$$

Note that by the boundedness condition $|f(\cos\theta,\sin\theta)| \leqslant M$, then by Squeeze theorem, we have

$$|f(x,y)| \leqslant r^k |f(\cos\theta,\sin\theta)| \leqslant Mr^k \to 0, \quad (x,y)\to(0,0). \tag{1.9}$$

12. Chain rules for multivariable functions.

(i) If $z = f(x,y)$ and f_x, f_y are differentiable. Let $x = r\cos\theta, y = r\sin\theta$, verify that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

(ii) If $z = f(x,y)$, $y = g(x)$ and $x = h(u,v)$, find $\frac{\partial z}{\partial u}$.

(iii) If $w = f(x,y,z)$, $x = g(y,z)$, $y = h(z)$, find $\frac{dw}{dz}$.
(Hint: See 6.4 in Möbius)

(i) Using Chain rules, we have

$$
\begin{aligned}
z_r &= \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
z_\theta &= \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta) \\
z_{rr} &= \frac{\partial^2 z}{\partial r^2} = \frac{\partial z_r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_r}{\partial y}\frac{\partial y}{\partial r} \\
z_{\theta\theta} &= \frac{\partial^2 z}{\partial \theta^2} = \frac{\partial z_\theta}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z_\theta}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial z_\theta}{\partial \theta} \\
\frac{\partial z_r}{\partial x} &= \frac{\partial}{\partial x}\left(f_x\cos\theta + f_y\sin\theta\right) = f_{xx}\cos\theta + f_{yx}\sin\theta \\
\frac{\partial z_r}{\partial y} &= \frac{\partial}{\partial y}\left(f_x\cos\theta + f_y\sin\theta\right) = f_{xy}\cos\theta + f_{yy}\sin\theta \\
\frac{\partial z_\theta}{\partial x} &= \frac{\partial}{\partial x}\left(f_x(-r\sin\theta) + f_y(r\cos\theta)\right) = -f_{xx}(r\sin\theta) + f_{yx}(r\cos\theta) \\
\frac{\partial z_\theta}{\partial y} &= \frac{\partial}{\partial y}\left(f_x(-r\sin\theta) + f_y(r\cos\theta)\right) = -f_{xy}(r\sin\theta) + f_{yy}(r\cos\theta) \\
\frac{\partial z_\theta}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(f_x(-r\sin\theta) + f_y(r\cos\theta)\right) = -f_x(r\cos\theta) - f_y(r\sin\theta)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
z_{rr} &= \frac{\partial^2 z}{\partial r^2} = \frac{\partial z_r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z_r}{\partial y}\frac{\partial y}{\partial r} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta, \\
\frac{1}{r}z_r &= \frac{1}{r}(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta), \\
\frac{1}{r^2}z_{\theta\theta} &= \frac{1}{r^2}(\frac{\partial z_\theta}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z_\theta}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial z_\theta}{\partial \theta}) \\
&= f_{xx}\sin^2\theta - 2f_{xy}\cos\theta\sin\theta + f_{yy}\cos^2\theta - \frac{1}{r}(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta).
\end{aligned}
$$

Therefore, we have the desired identity

$$z_{rr} + \frac{1}{r}z_r + \frac{1}{r^2}z_{\theta\theta} = z_{xx} + z_{yy}.$$

(ii) Using the Chain rule,

$$\frac{\partial z}{\partial u} = f_x(h(u,v), g(h(u,v)))h_u(u,v) + f_y(h(u,v), g(u,v))g_x(h(u,v))h_u(u,v)$$

(iii) Rewrite $w = f(x,y,z) = f(g(h(z),z),h(z),z)$, we have

$$
\begin{aligned}
\frac{dw}{dz} &= f_x(g(h(z),z),h(z),z)\frac{dg}{dz}(h(z),z) + f_y(g(h(z),z),h(z),z)\frac{dh}{dz} + f_z(g(h(z),z),h(z),z) \\
&= f_x(g(h(z),z),h(z),z)\left[g_y(h(z),z)h'(z) + g_z(h(z),z)\right] + f_y(g(h(z),z),h(z),z)h'(z) + f_z(g(h(z),z),h(z),z)
\end{aligned}
$$

13. Directional derivatives.

(i) State the definition: The directional derivative of $f(x,y)$ at a point (a,b) in the direction of a unit vector $\vec{u} = (u_1, u_2)$ with $u_1^2 + u_2^2 = 1$.

$$D_{\vec{u}} f(a,b) =$$

(ii) Prove the following important fact: suppose $f(x,y)$ is differentiable at (a,b) and $\nabla f(a,b) \neq (0,0)$, then the largest value of $D_{\vec{u}} f(a,b)$ is given by $\|\nabla f(a,b)\|$ and occurs when $\vec{u}$ is the unit vector in the direction of $\nabla f(a,b)$.

(iii) In what directions at the point $(2,1)$ does the directional derivative of the function $f(x,y) = xy$ equal to $\sqrt{\frac{5}{2}}$? Express your answer by giving the angle between the required directions and $\nabla f(2,1)$.

(i) Either one of the following can get 10 points:

$$D_{\vec{u}} f(a,b) = \frac{d}{ds} f(a + su_1, b + su_2)\Big|_{s=0} = \nabla f(a,b) \cdot \vec{u}$$

(ii)Proof: Since f is differentiable at (a,b) and $\|\vec{u}\| = 1$ we have

$$\begin{aligned} D_{\vec{u}} f(a,b) &= \nabla f(a,b) \cdot \vec{u} \\ &= \|\nabla f(a,b)\| \|\vec{u}\| \cos\theta \\ &= \|\nabla f(a,b)\| \cos\theta \end{aligned}$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$. Thus, $D_{\vec{u}} f(a,b)$ assumes its largest value when $\cos\theta = 1$, i.e., $\theta = 0$. Consequently, the largest value of $D_{\vec{u}} f(a,b)$ is $\|\nabla f(a,b)\|$ and occurs when $\vec{u}$ is in the direction of the gradient vector $\nabla f(a,b)$.
(iii) Since $\nabla f(x,y) = (y,x)$, we have

$$\nabla f(2,1) = (1,2).$$

Note that $\|\nabla f(2,1)\| = \sqrt{5}$ thus the angle between the required directions and $\nabla f(2,1)$ is $45°$ or $\frac{\pi}{4}$.

14. Taylor theorem for multivariable functions.

(i) State the definition: let $f(x,y)$ be a function of two variables. The second degree Taylor polynomial $P_{2,(a,b)}(x,y)$ of $f(x,y)$ at (a,b) is given by

$$P_{2,(a,b)}(x,y) =$$

(ii) State Taylor's Theorem for Functions of Two Variables and prove it. (See Theorem 2 of 8.2 in Möbius).

(iii) Let $f(x,y) = e^{x-4y}$. Use Taylor's Theorem to show that if $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$, the error in the linear approximation $L_{(1,1)}(x,y)$ is at most

$$\frac{e}{2}[5(x-1)^2 + 20(y-1)^2].$$

(i) The second degree Taylor polynomial $P_{2,(a,b)}$ of $f(x,y)$ at (a,b) is given by

$$\begin{aligned} P_{2,(a,b)}(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]. \end{aligned}$$

(ii) If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$ of (a,b), then for all $(x,y) \in N(a,b)$ there exists a point (c,d) on the line segment joining (a,b) and (x,y) such that

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where

$$R_{1,(a,b)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\right]$$

The proof is given in Theorem 2 of 8.2 in Möbius.
(iii) Note that the error in the linear approximation $L_{(1,1)}(x,y)$ is given by

$$\begin{aligned} R_{1,(1,1)}(x,y) &= \frac{1}{2}\left[f_{xx}(c,d)(x-1)^2 + 2f_{xy}(c,d)(x-1)(y-1) + f_{yy}(c,d)(y-1)^2\right] \\ &= \frac{e^{c-4d}}{2}\left[(x-1)^2 - 8(x-1)(y-1) + 16(y-1)^2\right], \end{aligned}$$

where $x \leqslant c \leqslant 1, y \leqslant d \leqslant 1$.(Up to here, 5 points)
Therefore we have $e^{c-4d} \leqslant e$ and use the fact that $8|(x-1)(y-1)| \leqslant 4(x-1)^2 + 4(y-1)^2$, we have

$$|R_{1,(1,1)}(x,y)| \leqslant \frac{e}{2}[5(x-1)^2 + 20(y-1)^2].$$

Remark: For (iii), a different and sharper bound is allowed. For example, the bound can be

$$\frac{e^{c-4d}}{2}\left[(x-1)^2 - 8(x-1)(y-1) + 16(y-1)^2\right] = \frac{e^{c-4d}}{2}(x-4y+3)^2 \leqslant \frac{e}{2}(x-4y+3)^2.$$

15. Suppose $F(x,y,z)$ is a differentiable function on $\mathbb{R}^3$. Its partial derivatives F_x, F_y, F_z are continuous and they satisfy

$$yF_x - xF_y + F_z \geqslant \alpha > 0, \forall (x,y,z) \in \mathbb{R}^3.$$

Calculate

$$\lim_{t \to +\infty} F(-\cos t, \sin t, t).$$

Using the chain rule, take the derivative of t we get

$$\frac{d}{dt}F(-\cos t, \sin t, t) = \sin t \cdot F_x(-\cos t, \sin t, t) + \cos t \cdot F_y(-\cos t, \sin t, t) + F_z(-\cos t, \sin t, t) \geqslant \alpha > 0 \tag{1.10}$$

where we have $x = -\cos t, y = \sin t, z = t$ and we apply the assumption.(Up to here 5 points). Then by fundamental theorem of calculus, we have

$$F(-\cos t, \sin t, t) - F(-1, 0, 0) = \int_0^t \frac{d}{ds}F(-\cos s, \sin s, s)ds \geqslant \alpha t,$$

thus for any $t > 0$, we have

$$F(-\cos t, \sin t, t) \geqslant F(-1, 0, 0) + \alpha t.$$

Let t tend to ∞ we have $\lim_{t\to+\infty} F(-\cos t, \sin t, t) = +\infty$.

16. Local Extrema and Critical Points

(i) State the definition of the critical point of two-variable functions. What are the three classes of critical points?

(ii) Suppose $f(x, y) = x^2 + y^2 + x^2y + 4$. Find all the critical points of $f(x, y)$.

(i) A point (a, b) in the domain of $f(x, y)$ is called a critical point of f if $\frac{\partial f}{\partial x}(a, b) = 0$ or $\frac{\partial f}{\partial x}(a, b)$ does not exist, and $\frac{\partial f}{\partial y}(a, b) = 0$ or $\frac{\partial f}{\partial y}(a, b)$ does not exist.
Local maximum point, local minimum point and saddle point.
(ii) By calculation, we have

$$\frac{\partial f}{\partial x}(x, y) = 2x + 2xy, \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 2y.$$

$\frac{\partial f}{\partial x}(x, y) = 0$ implies $x = 0$, or $y = -1$. $\frac{\partial f}{\partial y}(x, y) = 0$ implies $y = -\frac{x^2}{2}$. If $x = 0$ then $y = 0$; if $y = -1$ then $x = \sqrt{2}, -\sqrt{2}$. Therefore, there are three critical points:

$$(0, 0),\ (\sqrt{2}, -1),\ (-\sqrt{2}, -1).$$

17. The Second Derivative Test and convex functions of two variables.

(i) State the Theorem of Second partial derivatives test. Classify the critical points of $f(x, y) = x^2 + y^2 + x^2y + 4$.

(ii) State the definition of convex and strictly convex functions of two variables. Prove the following fact: Suppose $f(x, y) \in C^2$ is a convex function, then for every critical point (c, d) of $f(x, y)$, we have

$$f(x, y) \geqslant f(c, d), \quad \forall (x, y) \neq (c, d).$$

(See Möbius 9.3)

(i) Second partial derivatives test: suppose that $f(x,y) \in C^2$ in some neighborhood of (a,b) and that

$$f_x(a,b) = 0 = f_y(a,b)$$

If $Hf(a,b)$ is positive definite, then (a,b) is a local minimum point of f. If $Hf(a,b)$ is negative definite, then (a,b) is a local maximum point of f. If $Hf(a,b)$ is indefinite, then (a,b) is a saddle point of f. If $Hf(a,b)$ is semidefinite, then the test is inconclusive.
The Hessian matrix is given by

$$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2+2y & 2x \\ 2x & 2 \end{pmatrix}.$$

At $(0,0)$, the Hessian matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ which is positive definite, thus it is a local minimum.
At $(\sqrt{2},-1)$, the Hessian matrix is $\begin{pmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{pmatrix}$ which is indefinite(the eigenvalues are 4 and -2), thus it is a saddle point.
At $(-\sqrt{2},-1)$, the Hessian matrix is $\begin{pmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & 2 \end{pmatrix}$ which is indefinite(the eigenvalues are 4 and -2), thus it is a saddle point. (ii) Let $f(x,y)$ have continuous second partial derivatives. We say that f is convex if $Hf(x,y)$ is positive semi-definite for all (x,y) and that f is strictly convex if $Hf(x,y)$ is positive definite for all (x,y).

We use Taylor's theorem at (c,d):
If $f(x,y) \in C^2$ in some neighborhood $N(c,d)$ of (c,d), then for all $(x,y) \in N(c,d)$ there exists a point (u,v) on the line segment joining (c,d) and (x,y) such that

$$f(x,y) = f(c,d) + f_x(c,d)(x-a) + f_y(c,d)(y-b) + R_{1,(c,d)}(x,y)$$

where

$$R_{1,(c,d)}(x,y) = \frac{1}{2}\left[f_{xx}(u,v)(x-c)^2 + 2f_{xy}(u,v)(x-c)(y-d) + f_{yy}(u,v)(y-d)^2\right]$$

Note that f is a convex function, then

$$Hf(u,v) = \begin{pmatrix} f_{xx}(u,v) & f_{xy}(u,v) \\ f_{xy}(u,v) & f_{yy}(u,v) \end{pmatrix}$$

is positive semi-definite thus the quadratic form given by

$$\begin{pmatrix} x-c & y-d \end{pmatrix} Hf(u,v) \begin{pmatrix} x-c \\ y-d \end{pmatrix} = f_{xx}(u,v)(x-c)^2 + 2f_{xy}(u,v)(x-c)(y-d) + f_{yy}(u,v)(y-d)^2 \geqslant 0$$

Also note that (c,d) is a critical point thus $f_x(c,d) = f_y(c,d) = 0$. Finally we have

$$\begin{aligned} f(x,y) &= f(c,d) + f_x(c,d)(x-a) + f_y(c,d)(y-b) + R_{1,(c,d)}(x,y) \\ &= f(c,d) + R_{1,(c,d)}(x,y) \geqslant f(c,d). \end{aligned}$$

18. Extreme values and Lagrange multiplier.

(i) Find the maximum and minimum values of $f(x, y) = x + 2y$ on the disc $x^2 + y^2 \leqslant 4$.

(ii) Find the maximum value of $x + y + z$ on the surface

$$x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2 = 1.$$

(i) Since $f_x = 1, f_y = 2$, there is no critical point inside the disc. The maximum and minimum occur on the boundary $x^2 + y^2 = 4$. To find the maximum and minimum, suppose $x = 2\cos\theta, y = 2\sin\theta$, then we have $x + 2y = 2\cos\theta + 4\sin\theta$, then the maximum value is given by $\sqrt{2^2 + 4^2} = 2\sqrt{5}$ and the minimum value is given by $-2\sqrt{5}$.
(ii) We follow the standard Lagrange algorithm to find the maximum. Denote $f(x, y, z) = x + y + z, g(x, y, z) = x^2 + \frac{1}{4}y^2 + \frac{1}{9}z^2$. First set

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z),\ g(x, y, z) = 1.$$

The solution of the above equations is given by

$$x = \pm\frac{1}{\sqrt{14}},\ y = \pm\frac{4}{\sqrt{14}},\ z = \pm\frac{9}{\sqrt{14}}$$

Second, $\nabla g(x, y, z) = 0,\ g(x, y, z) = 1$, the solution does not exist. Finally, there is no edge point since the surface is closed.
Therefore, the maximum occurs at $x = \frac{1}{\sqrt{14}},\ y = \frac{4}{\sqrt{14}},\ z = \frac{9}{\sqrt{14}}$ thus the maximum value is $\sqrt{14}$.

19. Miscellaneous problems.

(i) Prove $ab \leqslant a\ln a - a + e^b$ for $a \geqslant 1, b \geqslant 0$.

(ii) Prove the following elementary inequality:

$$ab^2c^3 \leqslant 108\left(\frac{a+b+c}{6}\right)^6, \quad a, b, c > 0.$$

(Hint: Find the maximum value of $\ln x + 2\ln y + 3\ln z$ on the sphere $x^2 + y^2 + z^2 = 6r^2$.)

(i) Define the function $f(x,y) = x\ln x - x + e^y - xy$, our goal is to show that the global minimum of $f(x,y)$ on the region $\{x \geqslant 1, y \geqslant 0\}$ is 0.
To calculate the global minimum, we use the following approach: for any fixed $y_0 \geqslant 0$, we find the minimum of one-variable function $\varphi(x) := f(x, y_0)$. In fact,

$$\varphi'(x) = \ln x - y_0, \quad \varphi''(x) = \frac{1}{x} > 0.$$

Thus by second derivative test for one-variable function, $x = e^{y_0}$ is the only local minimum of $\varphi(x)$ thus is the global minimum. Therefore, for any fixed $y_0 \geqslant 0$,

$$\min_{x \geqslant 1} \varphi(x) = \varphi(e^{y_0}) = 0,$$

we have

$$\min_{y_0 \geqslant 0} \min_{x \geqslant 1} f(x, y_0) = \min_{y_0 \geqslant 0} 0 = 0.$$

(ii) We use Lagrange multiplier method. For any fixed $r > 0$, we set

$$\nabla(\ln x + 2\ln y + 3\ln z) = \lambda \nabla(x^2 + y^2 + z^2), \quad x^2 + y^2 + z^2 = 6r^2.$$

We get the solution $x = r, \quad y = \sqrt{2}r, \quad z = \sqrt{3}r$.
Next, we find $\nabla(x^2 + y^2 + z^2) = (2x, 2y, 2z) = 0$ and in this case $x^2 + y^2 + z^2 - 6r^2$ cannot be zero.
Finally, the edge point of $x^2 + y^2 + z^2 - 6r^2 = 0$ does not exist because it is a closed surface.
Therefore, note that as x approaches zero, the function $\ln x + 2\ln y + 3\ln z$ approaches $-\infty$. At the point $x = r, \quad y = \sqrt{2}r, \quad z = \sqrt{3}r$, it achieves global maximum, which means

$$\ln x + 2\ln y + 3\ln z \leqslant \ln\Big(r(\sqrt{2}r)^2(\sqrt{3}r)^3\Big) = \ln\Big(6\sqrt{3}r^6\Big) = \ln\Big(6\sqrt{3}(\frac{x^2+y^2+z^2}{6})^3\Big).$$

Taking the exponential on both sides, we have

$$xy^2z^3 \leqslant 6\sqrt{3}(\frac{x^2+y^2+z^2}{6})^3.$$

Finally taking square on both sides, and take $a = x^2, b = y^2, c = z^2$, we have

$$ab^2c^3 = x^2y^4z^6 \leqslant 6^2 \times 3(\frac{x^2+y^2+z^2}{6})^6 = 108(\frac{a+b+c}{6})^6.$$

Note that for part (i), using second derivative test for two-variable function is not possible. In fact,

$$\frac{\partial f}{\partial x} = \ln x - y, \quad \frac{\partial f}{\partial y} = e^y - x.$$

Therefore, on the region $\{x \geqslant 1, y \geqslant 0\}$, the critical points of f is given by

$$\mathcal{C}_f := \{y = \ln x, x \geqslant 1, y \geqslant 0\}.$$

The second derivative test fails. The Hessian matrix is given by

$$\begin{pmatrix} \frac{1}{x} & -1 \\ -1 & e^y \end{pmatrix}$$

which has determinant zero on the critical points $\mathcal{C}_f$. It is possible to do more detailed analysis to argue that it is a local minimum and also global minimum but the details are much more technical.

20. Partial derivatives.

(i) Determine the differentiability of the following function at $(0,0)$:

$$f(x,y) = \begin{cases} \frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & (x,y) \neq (0,0), \\ 0, \quad (x,y) = (0,0). \end{cases}$$

(ii) Suppose $z = f(r)$ and $r = \sqrt{x^2+y^2}$. Then use the chain rule to show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{d^2 z}{dr^2} + \frac{1}{r}\frac{dz}{dr}.$$

(i)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Using the definition of differentiability, we have

$$\left| \frac{f(x,y) - f(0,0) - f_x(0,0)x - f_y(0,0)}{\sqrt{x^2+y^2}} \right| = \frac{x^2y^2}{(x^2+y^2)^2}.$$

The limit does not exist, since we can choose $y = mx$ and by varying m we get different values as $(x,y) \to (0,0)$. Thus the function is not differentiable at $(0,0)$.
(ii) The calculation proceeds as follows:

$$\frac{\partial z}{\partial x} = \frac{dz}{dr}\frac{\partial r}{\partial x} = \frac{dz}{dr}\frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{dz}{dr}\frac{\partial r}{\partial y} = \frac{dz}{dr}\frac{y}{\sqrt{x^2+y^2}},$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{d^2 z}{dr^2}\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \frac{dz}{dr}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{d^2 z}{dr^2}\frac{x^2}{x^2+y^2} + \frac{dz}{dr}\frac{y^2}{(x^2+y^2)^{3/2}},$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{d^2 z}{dr^2}\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \frac{dz}{dr}\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{d^2 z}{dr^2}\frac{y^2}{x^2+y^2} + \frac{dz}{dr}\frac{x^2}{(x^2+y^2)^{3/2}}.$$

Then it is easy to see that we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{d^2 z}{dr^2} + \frac{1}{r}\frac{dz}{dr}.$$

21. Suppose $g(x, y, z) = 3\ln(x + e^{yz})$.

(i) Find the gradient of g.

(ii) Calculate the directional derivative of at $(0, 1, 0)$ in the direction from the point $(0, 1, 0)$ to the point $(5, 3, 3)$.

(i) The gradient is given by

$$\nabla g(x, y, z) = \left(\frac{3}{x + e^{yz}}, \frac{3ze^{yz}}{x + e^{yz}}, \frac{3ye^{yz}}{x + e^{yz}}\right)$$

(ii) Find the direction vector: $\vec{v} = (5, 3, 3) - (0, 1, 0) = (5, 2, 3)$. Next, normalize the vector

$$\begin{aligned} \overrightarrow{v^*} &= \frac{1}{\sqrt{5^2 + 2^2 + 3^2}}(5, 2, 3) \\ &= \frac{1}{\sqrt{38}}(5, 2, 3) \end{aligned}.$$

Note that $\nabla g(0, 1, 0) = (3, 0, 3)$.
Since g has continuous partial derivatives at $(0, 1, 0)$, it is differentiable at $(0, 1, 0)$. Thus, we can apply the Directional derivative theorem to get

$$\begin{aligned} D_{v*}g(0, 1, 0) &= \frac{1}{\sqrt{38}}(5, 2, 3) \cdot (3, 0, 3) \\ &= \frac{24}{\sqrt{38}}. \end{aligned}$$

22. Suppose $f(x,y) = \ln\big(-2\sin^2 x + 4\cos^2 y\big)$.

(i) Find the linearization at $(0,0)$, $L_{(0,0)}(x,y)$.

(ii) Find the second order Taylor polynomial at $(0,0)$, $P_{2,(0,0)}(x,y)$.

i) First we find $f(0,0) = \ln\left(-2\sin^2(0) + 4\cos^2(0)\right) = \ln(4)$ Next we find the gradient vector:

$$\nabla f = \left(\frac{-2\sin(2x)}{-2\sin^2 x + 4\cos^2 y}, -\frac{4\sin(2y)}{-2\sin^2 x + 4\cos^2 y}\right) \Rightarrow \nabla f(0,0) = (0,0)$$

Therefore, the linear approximation at $(0,0)$ is (10 points)

$$\begin{aligned} L_{(0,0)}(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) \\ &= \ln(4) + 0(x-0) + 0(y-0) \end{aligned}$$

ii) To set up the equation for the Taylor polynomial, we need to find the Hessian matrix. We begin by finding the second order derivatives:

$$\begin{aligned} f_{xx} &= \frac{(-2)\left[2\cos(2x)\left(-2\sin^2 x + 4\cos^2 y\right) - (-2)\sin^2(2x)\right]}{\left(-2\sin^2 x + 4\cos^2 y\right)^2} \\ f_{xy} &= f_{yx} = \frac{-8\sin(2x)\sin(2y)}{\left(-2\sin^2 x + 4\cos^2 y\right)^2} \\ f_{yy} &= \frac{-(4)\left[2\cos(2y)\left(-2\sin^2 x + 4\cos^2 y\right) + 4\sin^2(2y)\right]}{\left(-2\sin^2 x + 4\cos^2 y\right)^2} \end{aligned}$$

Then the Hessian matrix will be $Hf(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$. Using the Hessian matrix, we get (10 points)

$$P_{2,(0,0)}(x,y) = L_{(0,0)}(x,y) + \frac{1}{2}\left(f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\right)$$

Therefore,

$$P_{2,(0,0)}(x,y) = \underbrace{\ln(4)}_{\text{From part a)}} + \frac{1}{2}\big(\underbrace{-1}_{\text{From Hf(0,0)}} x^2 + \underbrace{0}_{\text{From Hf(0,0)}} xy + \underbrace{(-2)}_{\text{From Hf(0,0)}} y^2\big)$$

23. Miscellaneous problems.

(i) Suppose $\varphi(u)$ is a one-variable function such that for any $u \in \mathbb{R}$, $|\varphi(u)| \leqslant u^2$. Determine the differentiability of $f(x,y) = \varphi(|xy|)$ at $(0,0)$.

(ii) Suppose the second order partial derivatives of $f(x,y)$ exist. Moreover, we assume $f(x,y) > 0$ for any x, y. Then show that $f(x,y) = g(x)h(y)$ for some one-variable function g, h if and only if

$$f\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}.$$

(Hint: Consider $\varphi = \frac{\partial f}{\partial y}$ and try to calculate $\frac{\partial}{\partial x}(\frac{\varphi}{f})$.)

(i)

$$
\begin{aligned}
f(0,0) &= \varphi(0) = 0, \\
f_x(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0, \\
f_y(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0.
\end{aligned}
$$

Using the definition of differentiability and squeeze theorem, we have (2 points)

$$
\left|\frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)}{\sqrt{x^2+y^2}}\right| \leqslant \frac{|\varphi(|xy|)|}{\sqrt{x^2+y^2}} \leqslant \frac{|xy|^2}{\sqrt{x^2+y^2}} \leqslant |xy|^{\frac{3}{2}} \to 0.
$$

Therefore, f is differentiable at $(0,0)$.
(ii) Sufficiency. First, note that

$$
\frac{\partial}{\partial x}\left(\frac{\frac{\partial f}{\partial y}}{f}\right) = \frac{f\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}}{f^2} = 0,
$$

we have a function $\widetilde{h}$ of variable y, such that

$$
\frac{\frac{\partial f}{\partial y}}{f} = \widetilde{h}(y).
$$

Notice that $\frac{\frac{\partial f}{\partial y}}{f} = \frac{\partial}{\partial y}(\ln f)$, which implies that there exists a function $\widetilde{g}(x)$ such that

$$
\ln f = \int \widetilde{h}(y)dy + \widetilde{g}(x).
$$

Finally, taking exponential on both sides, we have

$$
f(x,y) = \exp\left(\int \widetilde{h}(y)dy\right)\exp(\widetilde{g}(x)).
$$

Take $g(x) = \exp(\widetilde{g}(x)), \quad h(y) = \exp\left(\int \widetilde{h}(y)dy\right)$ and we finish the proof.
Necessity. If $f(x,y) = g(x)h(y)$ for some one-variable function g, h, then

$$
f(x,y)\frac{\partial^2 f}{\partial x\partial y}(x,y) = h(y)\frac{dg}{dx}(x)g(x)\frac{dh}{dy}(y) = \frac{\partial f}{\partial x}(x,y)\frac{\partial f}{\partial y}(x,y).
$$

24. Polar coordinates, Cylindrical coordinates and Spherical coordinates

(i) Write down the transform equation from Cartesian to Polar, Cylindrical and Spherical coordinates and the other way around.

(ii) For the region $R = \{(x,y,z) : z \geqslant \sqrt{x^2+y^2}, x^2+y^2+z^2 \leqslant 2\}$ given in Cartesian coordinates, give a description in cylindrical coordinates and spherical coordinates.

(i)

Coordinate System	Coordinates	From Cartesian	To Cartesian	When to use
Polar	(r, θ)	$r = \sqrt{x^2 + y^2}$ $\tan(\theta) = \frac{y}{x}$	$x = r\cos(\theta)$ $y = r\sin(\theta)$	When there is symmetry about the origin in $2D$
Cylindrical	(r, θ, z)	$r = \sqrt{x^2 + y^2}$ $\tan(\theta) = \frac{y}{x}$ $z = z$	$x = r\cos(\theta)$ $y = r\sin(\theta)$ $z = z$	When there is symmetry about the z-axis in $3D$
Spherical	(ρ, φ, θ)	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\tan(\theta) = \frac{y}{x}$ $\cos(\varphi) = \frac{z}{\sqrt{x^2+y^2}}$	$x = \rho\sin(\varphi)\cos(\theta)$ $y = \rho\sin(\varphi)\sin(\theta)$ $z = \rho\cos(\varphi)$	When there is symmetry about the origin in $3D$

(ii)
Description in spherical coordinates: $\{(\rho, \varphi, \theta) : 0 \leqslant \varphi \leqslant \pi/4, \rho \leqslant \sqrt{2}\}$.
Description in cylindrical coordinates: $\big\{(z, r, \theta) : z \geqslant r, r^2 + z^2 \leqslant 2\big\}$.

25. Mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(i) Find the image of $D = \{(x, y) : 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 2\}$ under the mapping $T(x, y) = (x^2 - y^2, x^2 + y^2)$.

(ii) Use the linear approximation for mappings to approximate the image of $(0.01, 0.02)$ under the mapping $F(x, y) = (e^x + y, e^x - y)$.

(i) Set $u = x^2 - y^2, v = x^2 + y^2$, we have $x^2 = \frac{u+v}{2}, y^2 = \frac{v-u}{2}$. Since we have $0 \leqslant x^2 \leqslant 4, 0 \leqslant y^2 \leqslant 4$, the conditions on (u, v) is given by

$$\begin{aligned} 0 \leqslant u + v \leqslant 8 \\ 0 \leqslant v - u \leqslant 8. \end{aligned}$$

Thus it is a parallelogram in the (u, v)-plane with the four vertices given by $(0, 0), (4, 4), (0, 8), (-4, 4)$.
(ii) Use the approximation $F(x, y) = F(a, b) + DF(a, b)(\Delta x)$, with $\Delta x = \begin{pmatrix} x - a \\ y - b \end{pmatrix}$.

$$DF(x, y) = \begin{pmatrix} e^x & 1 \\ e^x & -1 \end{pmatrix}.$$

Finally, we have $F(0.01, 0.02) \approx F(0, 0) + DF(0, 0)\begin{pmatrix} 0.01 \\ 0.02 \end{pmatrix} = \begin{pmatrix} 1.03 \\ 0.99 \end{pmatrix}$.

26. Composite Mappings and the Chain Rule.

(i) State the Chain Rule in matrix form for mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(ii) Consider the maps F and G defined by

$$F(u,v) = \left(v + u^2, u\right), \quad G(x,y) = \left(e^x y, 2e^{-x} y\right)$$

Calculate the derivative $D(F \circ G)(0,1)$ of the composite map.

(i) Let F and G be mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. If G has continuous partial derivatives at (x,y) and F has continuous partial derivatives at $(u,v) = G(x,y)$, then the composite mapping $F \circ G$ has continuous partial derivatives at (x,y) and

$$D(F \circ G)(x,y) = DF(u,v)DG(x,y)$$

(ii) Note that

$$DF(u,v) = \begin{pmatrix} 2u & 1 \\ 1 & 0 \end{pmatrix}, \quad DG(x,y) = \begin{pmatrix} e^x y & e^x \\ -2e^{-x} y & 2e^{-x} \end{pmatrix}.$$

when $(x,y) = (0,1)$, we have $(u,v) = G(x,y) = (1,2)$. Therefore,

$$D(F \circ G)(0,1) = DF(1,2)DG(0,1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}.$$

27. See related discussion in Möbius 12.4, application.

Recall that any complex number $z \in \mathbb{C}$ can be written as $z = x + iy, x, y \in \mathbb{R}$. Here i is the imaginary unit satisfying $i^2 = -1$. There is a natural one-to-one correspondence of $\mathbb{C}$ and $\mathbb{R}^2$ defined by the map $\varphi : \mathbb{C} \to \mathbb{R}^2 : \; z = x + iy \mapsto (x, y)$ with the inverse map $\varphi^{-1} : \mathbb{R}^2 \to \mathbb{C} : \; (x, y) \mapsto x + iy$. Then any complex function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as a mapping F from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$F(x, y) := \varphi \circ f \circ \varphi^{-1}(x, y).$$

(i) Rewrite the famous Möbius transform $f(z) = \frac{az+b}{cz+d}$ as a mapping F from $\mathbb{R}^2$ to $\mathbb{R}^2$. Here a, b, c, d are complex numbers

$$a = a_1 + ia_2, \; b = b_1 + ib_2, \; c = c_1 + ic_2, \; d = d_1 + id_2.$$

(ii) What is the image of the open unit disc $\{z = x + iy \in \mathbb{C} : x^2 + y^2 < 1\}$ under Möbius transformation $f(z) = \frac{1+z}{1-z}$. (Hint: The answer is the right half plane. You need to explain why.)

(i) Plug in the expression of the complex numbers, we have

$$f(z) = \frac{(a_1 + ia_2)(x + iy) + (b_1 + ib_2)}{(c_1 + ic_2)(x + iy) + (d_1 + id_2)}$$

Expanding and separating into real and imaginary parts:

$$f(z) = \frac{(a_1x - a_2y + b_1) + i(a_2x + a_1y + b_2)}{(c_1x - c_2y + d_1) + i(c_2x + c_1y + d_2)}$$

Let:

$$u = c_1x - c_2y + d_1$$
$$v = c_2x + c_1y + d_2$$

Then: $f(z) = \frac{(a_1x - a_2y + b_1) + i(a_2x + a_1y + b_2)}{u+iv}$. To simplify this expression, we multiply the numerator and the denominator by the conjugate of the denominator:

$$f(z) = \frac{((a_1x - a_2y + b_1) + i(a_2x + a_1y + b_2))((c_1x - c_2y + d_1) - i(c_2x + c_1y + d_2))}{(c_1x - c_2y + d_1)^2 + (c_2x + c_1y + d_2)^2}$$

The real part u' and imaginary part v' of the resulting complex number can be written as functions of x and y. After simplifying the expression, the result will be in the form:

$$f(z) = u'(x, y) + iv'(x, y)$$

Thus, the Möbius transformation can be represented as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$(x, y) \mapsto (u'(x, y), v'(x, y))$$

where

$$u'(x, y) = \frac{(a_1x - a_2y + b_1)(c_1x - c_2y + d_1) + (a_2x + a_1y + b_2)(c_2x + c_1y + d_2)}{(c_1x - c_2y + d_1)^2 + (c_2x + c_1y + d_2)^2}$$

$$v'(x, y) = \frac{(a_2x + a_1y + b_2)(c_1x - c_2y + d_1) - (a_1x - a_2y + b_1)(c_2x + c_1y + d_2)}{(c_1x - c_2y + d_1)^2 + (c_2x + c_1y + d_2)^2}$$

(ii) For this example, $a_1 = b_1 = d_1 = 1, c_1 = -1, a_2 = b_2 = c_2 = d_2 = 0$. Therefore,

$$u'(x, y) = \frac{1 - x^2 - y^2}{(x - 1)^2 + y^2}, v'(x, y) = \frac{2y}{(x - 1)^2 + y^2}.$$

Note that $x^2 + y^2 < 1$, thus $u'(x, y) > 0$ and $v'(x, y)$ can be arbitrary real number. (Up to here you can get full score. A more detailed calculation is provided) We claim that the image is the right half plane, i.e, $\{(u, v) \in \mathbb{R}^2 : u > 0, v \in \mathbb{R}\}$. In fact, for any (u, v) in the right half plane, we need to find z inside the unit disk such that $\frac{1+z}{1-z} = w = u + iv$, and we have $z = \frac{w-1}{w+1}$. The existence of z is verified if we can show that $|z| = |\frac{w-1}{w+1}| < 1$. Since w is a point in the right half plane, the geometric meaning is that the distance to the point $(-1, 0)$ is larger than the distance to the point $(1, 0)$. This is true as long as $u > 0$:

$$|\frac{u - 1 + iv}{u + 1 + iv}| < 1 \iff u^2 + v^2 - 2u + 1 < u^2 + v^2 + 2u + 1 \iff u > 0.$$

28. Change of variable method: two dimensional.

(i) State the Change of Variable Theorem for double integrals.

(ii) Suppose a, b are two real numbers. Calculate

$$\iint_D (ax + by) dxdy$$

where D is the region in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = 0$.

(i) Let each of D_{uv} and D_{xy} be a closed bounded set whose boundary is a piecewise-smooth closed curve. Let

$$(x, y) = F(u, v) = (f(u, v), g(u, v))$$

be a one-to-one mapping of D_{uv} onto D_{xy}, with $f, g \in C^1$, and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ except for possibly on a finite collection of piecewise-smooth curves in D_{uv}. If $G(x, y)$ is continuous on D_{xy}, then

$$\iint_{D_{xy}} G(x, y) dxdy = \iint_{D_{\text{uv}}} G(f(u, v), g(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

(ii) Use the Polar coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

and it is straightforward to calculate $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$. Moreover, the regions are given by $D_{xy} = \{(x, y) : 0 \leqslant x, y \leqslant 2, x^2 + y^2 \leqslant 4\}$ and $D_{r\theta} = \{(r, \theta) : 0 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$. Using the Change of Variable Theorem for double integrals, we have

$$\iint_{D_{xy}} (ax + by) dxdy = \iint_{D_{r\theta}} (a\cos\theta + b\sin\theta) r^2 dr d\theta = \frac{8}{3}(a + b).$$

29. Change of variable method: three dimensional.

(i) State the Change of Variable Theorem for triple integrals.

(ii) Find the volume of ellipsoid using change of variable method:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a, b, c > 0.$$

(i) Let

$$x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w)$$

be a one-to-one mapping of D_{uvw} onto D_{xyz} with f, g, h having continuous partials, and

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0 \quad \text{on} \quad D_{uvw}$$

If $G(x, y, z)$ is continuous on D_{xyz}, then

$$\iiint_{D_{xyz}} G(x, y, z) dV = \iiint_{D_{\mathrm{urw}}} G(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV$$

(ii) Use the transformation

$$\begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$$

and it is straightforward to check $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$. The regions are $D_{xyz} = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\}$ and $D_{uvw} = \{(u, v, w) : u^2 + v^2 + w^2 \leqslant 1\}$. Then the volume of the ellipsoid is calculated as

$$\iiint_{D_{xyz}} dV = \iiint_{D_{uvw}} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV = abc \cdot \text{volume of unit ball} = \frac{4\pi}{3} abc.$$

Volume of unit ball is $\frac{4\pi}{3}$ and it can be calculated via Spherical coordinates:

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} r^2 \sin\varphi dr d\theta d\varphi = \frac{4\pi}{3}.$$

30. Evaluate the following triple integrals:

(i)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz dy dx.$$

(ii)

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{3}} dz dx dy.$$

(i) We use cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

it is straightforward to calculate $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$. The regions are given by $D_{xyz} = \{(x,y,z) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant \sqrt{1-x^2}, \sqrt{x^2+y^2} \leqslant z \leqslant \sqrt{2-x^2-y^2}\}$ and $D_{r\theta z} = \{(r,\theta,z) : 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2},\ r \leqslant z \leqslant \sqrt{2-r^2}\}$. The integral can be calculated as follows:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dzdydx = \frac{\pi}{2}\int_0^1 (\sqrt{2-r^2} - r)rdr = \pi\frac{\sqrt{2}-1}{3}.$$

(ii)We use cylindrical coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

it is straightforward to calculate $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$. The regions are given by $D_{xyz} = \{(x,y,z) : 0 \leqslant y \leqslant 1, 0 \leqslant x \leqslant \sqrt{1-y^2}, \sqrt{3(x^2+y^2)} \leqslant z \leqslant \sqrt{3}\}$ and $D_{r\theta z} = \{(r,\theta,z) : 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{2},\ \sqrt{3}r \leqslant z \leqslant \sqrt{3}\}$. The integral can be calculated as follows:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{3}} dzdxdy = \frac{\pi}{2}\int_0^1 \sqrt{3}(1-r)rdr = \frac{\sqrt{3}\pi}{12}.$$

31. Miscellaneous problems.

(i) Suppose D is the region surrounded by $z = 0, z = 1$ and $x^2 + \frac{1}{2}(y-z)^2 = 1$. Calculate

$$\iiint\limits_D (y-z)\arctan z \ dxdydz.$$

(Hint: Use transformation $x = u, y - z = \sqrt{2}v, z = w$)

(ii) Suppose $f(x,y)$ has continuous second partial derivatives. If $D = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, calculate

$$\iint\limits_D f_{xy}(x,y)dxdy, \quad \iint\limits_D f_{yx}(x,y)dxdy.$$

Briefly discuss how the above calculation can imply Clairaut's Theorem(For the statement, see Möbius 4.2).

(i) Use the transformation

$$\begin{cases} x = u \\ y - z = \sqrt{2}v \\ z = w \end{cases}$$

It is straightforward to calculate $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \sqrt{2}$. The regions are given by $D_{xyz} = \{(x,y,z) : 0 \leqslant z \leqslant 1, x^2 + \frac{1}{2}(y-z)^2 \leqslant 1\}$, and $D_{uvw} := \{(u,v,w) : 0 \leqslant w \leqslant 1, u^2 + v^2 \leqslant 1\}$. The integral can be calculated via change of variable method as follows:

$$\iiint\limits_{D_{xyz}} (y-z) \arctan z \; dxdydz = \iiint\limits_{D_{uvw}} 2v \arctan(w) dudvdw.$$

Note that by symmetry

$$\iint\limits_{u^2+v^2 \leqslant 1} vdudv = 0,$$

the above integral is 0.

(ii) Since f_{xy} and f_{yx} are continuous, recall that $D = \{(x,y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, calculate

$$\begin{aligned} &\iint\limits_{D} f_{xy}(x,y)dxdy = \iint\limits_{D} f_{xy}(x,y)dydx = \int_a^b \int_c^d \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}(x,y))dydx \\ &= \int_a^b \frac{\partial f}{\partial x}(x,d) - \frac{\partial f}{\partial x}(x,c)dx \\ &= f(b,d) - f(a,d) - (f(b,c) - f(a,c)) \\ &= f(a,c) + f(b,d) - f(a,d) - f(b,c). \end{aligned}$$

Similarly,

$$\begin{aligned} &\iint\limits_{D} f_{yx}(x,y)dxdy = \int_c^d \int_a^b \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(x,y))dxdy \\ &= \int_c^d \frac{\partial f}{\partial y}(b,y) - \frac{\partial f}{\partial y}(a,y)dy \\ &= f(b,d) - f(b,c) - (f(a,d) - f(a,c)) \\ &= f(a,c) + f(b,d) - f(a,d) - f(b,c). \end{aligned}$$

Therefore, $\iint_D f_{xy}(x,y)dxdy = \iint_D f_{yx}(x,y)dxdy = f(a,c) + f(b,d) - f(a,d) - f(b,c)$. . For any (x_0,y_0), we choose $D_n = [x_0, x_0 + \frac{1}{n}] \times [y_0, y_0 + \frac{1}{n}]$ and let n go to infinity, we have $f_{yx}(x_0,y_0) = f_{xy}(x_0,y_0)$ which implies Clairaut's theorem. (This explanation is optional and no point for this part.

32. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x$ on the curve defined by

$$y^2 + x^4 - x^3 = 0.$$

We apply the Lagrange Multiplier, we first solve

$$\nabla f(x,y) = \lambda \nabla g(x,y), g(x,y) = y^2 + x^4 - x^3 = 0.$$

Note that $\nabla f(x,y) = (1,0) = \lambda \nabla g(x,y), \nabla g(x,y) = (4x^3 - 3x^2, 2y)$, our solution is given by the following two points

$$(x,y) = (1,0), (0,0); \quad \lambda = 1$$

Second, we solve

$$\nabla g(x,y) = (0,0), g(x,y) = y^2 + x^4 - x^3 = 0.$$

The only solution is given by $(x,y) = (0,0)$. Finally, we find that the curve $y^2 + x^4 - x^3 = 0$ is closed and no end point exists.
In summary, we have two candidates $(1,0)$ and $(0,0)$ and the maximum is 1 which occurs at $(1,0)$ and the minimum is 0 which occurs at $(0,0)$.

33.

(i) Convert the following equations in Cartesian coordinates to spherical coordinates:

$$z = -\sqrt{x^2 + y^2}.$$

(ii) Convert the following equations in Cartesian coordinates to cylindrical coordinates:

$$z = \sqrt{5x^2 + 5y^2}.$$

(i) The spherical coordinates are given by

$$\begin{cases} x = \rho \sin(\varphi)\cos(\theta) \\ y = \rho \sin(\varphi)\sin(\theta) \\ z = \rho \cos(\varphi) \end{cases}$$

we have $\cos\varphi = -|\sin\varphi|$.**(15 points)**

(ii) The cylindrical coordinates are given by

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$$

we have $z = \sqrt{5}r$. **(15 points)**

34. Consider the map defined by

$$(u, v) = F(x, y) = (y + xy, y - xy).$$

(i) Show that F has an inverse map by finding F^{-1} explicitly.

(ii) Find the derivative matrices $DF(x, y)$ and $DF^{-1}(u, v)$ and verify that

$$DF(x, y)DF^{-1}(u, v) = I.$$

(iii) Verify that the Jacobians satisfy

$$\frac{\partial(x, y)}{\partial(u, v)} = \left[\frac{\partial(u, v)}{\partial(x, y)}\right]^{-1}.$$

For part (i), we need to solve

$$\begin{cases} u = y(1+x) \\ v = y(1-x) \end{cases} \tag{1.11}$$

and the solution is

$$\begin{cases} x = \frac{u-v}{u+v}, \\ y = \frac{u+v}{2}. \end{cases} \tag{1.12}$$

Therefore, the explicit expression is

$$F^{-1}(u,v) = (\frac{u-v}{u+v}, \frac{u+v}{2}),$$

which exists when $u + v \neq 0$ **(1 point)**.
For part (ii), the derivative matrix can be calculated directly using the definition

$$DF(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} y & 1+x \\ -y & 1-x \end{pmatrix},$$

$$DF^{-1}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial v}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{2v}{(u+v)^2} & \frac{-2u}{(u+v)^2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then we can direct verify using $x = \frac{u-v}{u+v}, y = \frac{u+v}{2}$:

$$\begin{aligned} DF(x,y)DF^{-1}(u,v) &= \begin{pmatrix} y & 1+x \\ -y & 1-x \end{pmatrix} \begin{pmatrix} \frac{2v}{(u+v)^2} & \frac{-2u}{(u+v)^2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2vy}{(u+v)^2} + \frac{1+x}{2} & \frac{-2uy}{(u+v)^2} + \frac{1+x}{2} \\ \frac{-2vy}{(u+v)^2} + \frac{1-x}{2} & \frac{2uy}{(u+v)^2} + \frac{1-x}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{v}{u+v} + \frac{u}{u+v} & \frac{-u}{u+v} + \frac{u}{u+v} \\ \frac{-v}{u+v} + \frac{v}{u+v} & \frac{u}{u+v} + \frac{v}{u+v} \end{pmatrix} = I. \end{aligned}$$

For part (iii),

$$\frac{\partial(x,y)}{\partial(u,v)} = det(DF^{-1}(u,v)) = \frac{1}{u+v}, \quad \frac{\partial(u,v)}{\partial(x,y)} = det(DF(x,y)) = 2y = u+v.$$

Therefore, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}.$$

35. Miscellaneous problems.

(i) Find the local maximum and minimum of the function $f(x, y) = (1 + e^y)\cos x - ye^y$ and the corresponding critical points if they exist.

(ii) Suppose $f(x, y), g(x, y)$ are two functions with continuous partial derivatives. Show that if for any $(x, y) \in \mathbb{R}^2$ we have

$$\frac{\partial f}{\partial x}(x,y)\frac{\partial g}{\partial y}(x,y) \neq \frac{\partial f}{\partial y}(x,y)\frac{\partial g}{\partial x}(x,y),$$

then the number of solutions of $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ in the region $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 100, 0 \leqslant y \leqslant 50\}$ is finite.

(Hint: Use the Inverse Mapping Theorem.
Additional hint: You can freely use the following property: if the region $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 100, 0 \leqslant y \leqslant 50\}$ is covered by infinitely many neighbors(open sets) $\{O_i : i \in I\}, |I| = +\infty$, i.e.,

$$\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 100, 0 \leqslant y \leqslant 50\} \subseteq \bigcup_{i \in I} O_i$$

then there are finitely many neighbors covering the region, i.e.,

$$\exists J \subseteq I, |J| < +\infty, \quad \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 100, 0 \leqslant y \leqslant 50\} \subseteq \bigcup_{i \in J} O_i$$

)

(i) We calculate the first derivatives

$$f_x(x,y) = -\sin x(1+e^y), \quad f_y(x,y) = (\cos x - 1 - y)e^y$$

and second derivatives

$$f_{xx}(x,y) = -\cos x(1+e^y), \quad f_{xy}(x,y) = -e^y \sin x, \quad f_{yy}(x,y) = (\cos x - 2 - y)e^y.$$

To find the critical points, we set $f_x = f_y = 0$, then we have a sequence of critical points given by

$$(x_n, y_n) = (n\pi, \cos n\pi - 1), \quad n = 0, \pm 1, \pm 2, \cdots$$

For even n, $(x_n, y_n) = (n\pi, 0)$, and we can show that $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is negative definite thus $(n\pi, 0)$ for n even are local maxima and the local maximum value is **(1 point)**

$$(1+e^0)\cos n\pi - 0e^0 = 2.$$

For odd n, $(x_n, y_n) = (n\pi, -2)$, and we can show that $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 1+e^{-2} & 0 \\ 0 & -e^{-2} \end{pmatrix}$ which is indefinite and via second derivative test, it is a saddle point.

In summary, we only have local maxima $(n\pi, 0)$ with n even integers, and the local maximum value is 2.

(ii) Construct a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ as $F(x,y) = (f(x,y), g(x,y))$. Denote the region $D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 100, 0 \leqslant y \leqslant 50\}$. Then the solution of $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ in D is given by the solution of $F(x,y) = (0,0)$ in D. By inverse mapping theorem, for any $(x_0, y_0) \in D$, we have a neighborhood $O(x_0, y_0)$ including (x_0, y_0) such that F is injective in $O(x_0, y_0)$. Then we have

$$D \subseteq \bigcup_{(x_0,y_0)\in D} O(x_0, y_0).$$

Since D is bounded and closed, there exist finitely many $(x_i, y_i), 1 \leqslant i \leqslant n$ such that (use the additional hint).

$$D \subseteq \bigcup_{i=1}^{n} O(x_i, y_i).$$

Note that for each i, F is injective in $O(x_i, y_i)$, there is at most one solution of $F(x,y) = (0,0)$ in $O(x_i, y_i)$. Therefore, there are at most n solution in $D \subseteq \bigcup_{i=1}^{n} O(x_i, y_i)$.

An alternative solution: prove by contradiction. Suppose there are infinitely many solutions, then there must be a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ such that $(x_n, y_n) \to (x_0, y_0)$ such that $F(x_n, y_n) = 0$.