Homework 1, Due: Friday, 9/11

This assignment is due on **Friday, September 11**, by 11:59 PM. Your assignment should be well-organized, typed (or neatly written and scanned) and saved as a .pdf for submission on Canvas. You must show all of your work to receive full credit. For problems requiring the use of MATLAB code, please remember to also submit your .m-files on Canvas as a part of your completed assignment. Your code should be appropriately commented to receive full credit.

Problems

1 Let $P_N(x)$ be the Nth order Taylor polynomial approximation of the function $f(x) = \ln(x)$ centered at $x_0 = 1$.

(a) (5 points) Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval [0.5,2].

(b) (5 points) Write a MATLAB code to find the smallest value of N required for

$$|f(x) - P_N(x)| < tol$$

when $x = 1.2$ and $tol = 10^{-5}$? Include the relevant analytic computations for this problem, along with a description of your code. Describe what your code is doing and show your code's output (which should be your answer for the N described). How does your answer change for different values of x (closer to or further from x_0) and different values of tol? Explain, and show your results for a few examples.

(c) (5 points) Plot and analyze the relative error $|f(x)-P_N(x)|$ for $N = 1, \ldots, 1000$ for different values of x. In particular, show `loglog` plots when $x = 1.2$ and for $x = 2.02$. What happens to the error as N increases? Explain your findings.

2 In calculus, the limit definition of the derivative of a function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The approximation

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

is known as a *forward difference* approximation of $f'(x)$ for a given h. In practice there is an "optimal" choice of h that minimizes round-off error along with other types of error in approximating the derivative.

Consider the problem of numerically approximating the derivative of $f(x) = x^2 \sin(x)$ at the point $x = 2.5$ using the forward difference scheme; i.e., numerically approximating $f'(2.5)$ using the formula

$$f'(2.5) \approx \frac{f(2.5+h) - f(2.5)}{h}$$

(a) (5 points) Write a MATLAB code to find the optimal h (i.e., the h for which the absolute error is smallest) by numerical experimentation. Note that you'll want to test a wide range of possible h values, with the idea (as in the definition of derivative) that $h \to 0$. Report your optimal value of h (your code should output this value).

(b) (5 points) Use MATLAB to create a plot that shows how the absolute error decreases and then starts to increase as h continues to get smaller. You may want to utilize a `loglog` plot. Make sure that your plot has a title, axis labels, a legend (if helpful), and readable font size! Include the final version of your plot as well as the code used to generate it.

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Any code that you submit should be your own. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.