School of Computing and Information Systems
COMP90038 Algorithms and Complexity Tutorial Week 6

Sample Answers

The exercises

33. Write an algorithm to classify all edges of an undirected graph, so that depth-first tree edges can be distinguished from back edges.

Answer: Here is how we can classify the edges:

```
function ClassifyEdges(⟨V, E⟩)
    mark each node in V with 0
    for each v in V do
        if v is marked 0 then
            DfsExplore(v)

function DfsExplore(v)
    mark v with 1
    for each edge (v, w) do
        if w is marked with 0 then
            classify (v, w) (and thereby (w, v)) as "tree edge"
            DfsExplore(w)
        else
            if (v, w) (and thereby (w, v)) is not a "tree edge" then
                classify (v, w) (and thereby (w, v)) as "back edge"
```

34. Show how the algorithm from the previous question can be utilised to decide whether an undirected graph is cyclic.

Answer: Once we know the difference between tree and back edges, this is easy. We just change depth-first exploration slightly:

```
function IsCyclic(⟨V, E⟩)
    mark each node in V with 0
    for each v in V do
        if v is marked 0 then
            if DfsExplore(v) = True then
                return True
    return False

function DfsExplore(v)
    mark v with 1
    for each edge (v, w) do                    ▷ w is v's neighbour
        if w is marked with 0 then
            if DfsExplore(w) then
                return True
        else
            if (v, w) is a back edge then
                return True
    return False
```

35. Explain how one could also use breadth-first search to see whether an undirected graph is cyclic. Which of the two traversals, depth-first and breadth-first, will be able to find cycles faster? (If there is no clear winner, give an example where one is better, and another example where the other is better.)

Answer: A graph has a cycle iff its breadth-first forest contains a cross edge. Sometimes depth-first search finds a cycle faster, sometimes not. Below, on the left, is a case where depth-first search finds the cycle faster, before all the nodes have been visited. On the right is an example where breadth-first search finds the cycle faster.

36. Design an algorithm to check whether an undirected graph is 2-colourable, that is, whether its nodes can be coloured with just two colours in such a way that no edge connects two nodes of the same colour. Hint: Adapt one of the graph traversal algorithms.

Answer: An undirected graph can be checked for two-colourability by performing a DFS traversal.

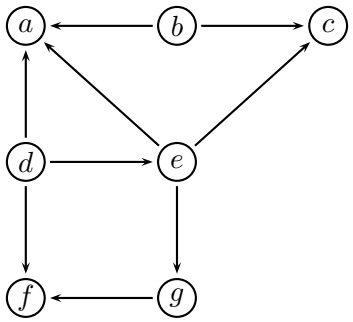
This begins by first assigning a colour of 0 (that is, no colour) to each vertex. Assume the two possible "colours" are 1 and 2.

Then traverse each vertex in the graph, colouring the vertex and then recursively colouring (via DFS) each neighbour with the opposite colour. If we encounter a vertex with the same colour as its sibling then a two-colouring is not possible.

```
function IsTwoColourable(G)
    let ⟨V, E⟩ = G
    for each v in V do
        colour[v] ← 0
    for each v in V do
        if colour[v] = 0 then
            Dfs(v, 1)
    return True

function Dfs(v, currentColour)
    colour[v] ← currentColour
    for each node u in V adjacent to v do
        if u is marked with currentColour then
            return False
        if u is marked with 0 then
            Dfs(u, 3 − currentColour)
```

37. Apply the DFS-based topsort algorithm to linearize the following graph:



Answer: Here is how the traversal stack develops:

$$\begin{array}{rrr} & & f_{7,4} \\ & & g_{6,5} \\ & c_{3,2} & e_{5,6} \\ a_{1,1} & b_{2,3} & d_{4,7} \end{array}$$

The reverse of the popping order is: d, e, g, f, b, c, a.

38. Apply insertion sort to the list S, O, R, T, X, A, M, P, L, E.

Answer: Have fun :)

39. For what kind of array is the time complexity of insertion sort linear?

Answer: Insertion sort is linear-time for already-sorted arrays, and arrays that are "almost-sorted". The definition of this is that the number of "inversions" is $O(n)$ where n is the size of the array. In next week's exercises we define inversions properly and ask for an efficient algorithm to count their number.

40. Trace how interpolation search proceeds when searching for 42 in an array containing (in index positions 0..21)

$$20, 20, 21, 23, 25, 26, 26, 27, 29, 29, 29, 30, 32, 33, 34, 36, 38, 40, 41, 43, 43, 45$$

Answer: The first probe will be at position $x = 0 + \lfloor \frac{42-20}{45-20}(21-0) \rfloor = \lfloor \frac{22}{25} \cdot 21 \rfloor = 18$. At index 18 we find 41, so the entire segment from 0 to 18 is discarded. Interpolation sort is now ready to repeat the process for 43, 43, 45. However, it is essential to first check the extreme elements. Since $43 > 42$, the process must stop straight away, reporting failure.

41. (Optional.) For evenly distributed keys, interpolation search is $O(\log \log n)$. Show that $\log \log n$ has a smaller order of growth than $\log n$. Hint: Differential calculus tells us that $(\log x)' = \Theta(\frac{1}{x})$ and the chain rule says that $(f \circ g)'(x) = f'(g(x))g'(x)$.

Answer: We want to show that $\lim_{n\to\infty} \frac{\log \log n}{\log n} = 0$. By l'Hôpital's rule, it suffices to show that $\lim_{n\to\infty} \frac{1}{n \log n} / \frac{1}{n} = 0$. But $\frac{1}{n \log n} / \frac{1}{n} = \frac{1}{\log n}$, which approaches 0 (albeit slowly) as n approaches ∞.