COMP90038
Algorithms and Complexity

Lecture 11: Sorting with Divide-and-Conquer
(with thanks to Harald Søndergaard)

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Divide and Conquer

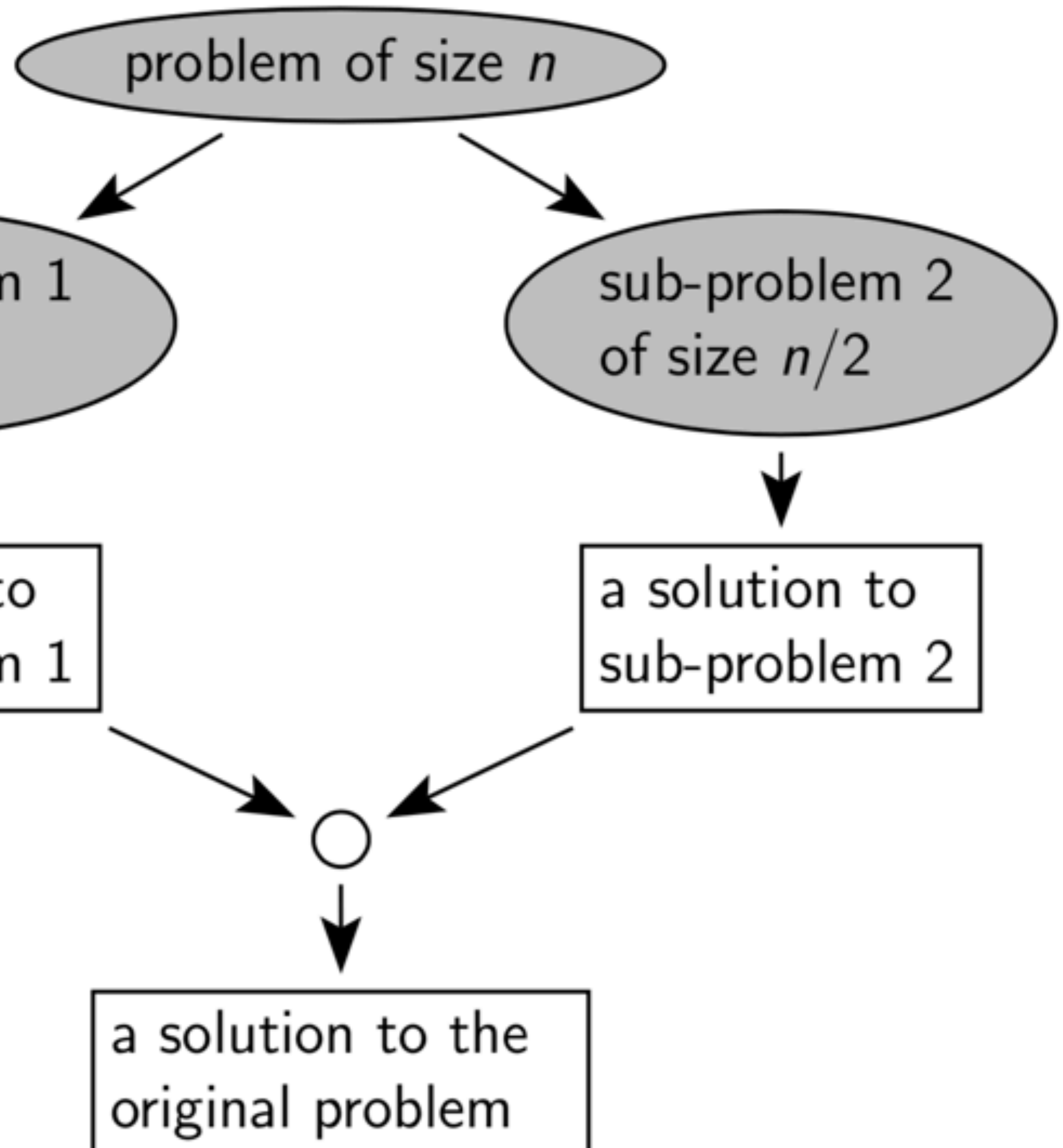
- We earlier studied recursion as a powerful problem solving technique.

- The **divide-and-conquer** strategy tries to make the most of this idea:

  1. Divide the given problem instance into smaller instances.
  2. Solve the smaller instances recursively.
  3. Combine the smaller solutions to solve the original instance.

- This works best when the smaller instances can be made to be of equal (or near-equal) size.

Split-Solve-and-Join Approach



Divide and Conquer Algorithms

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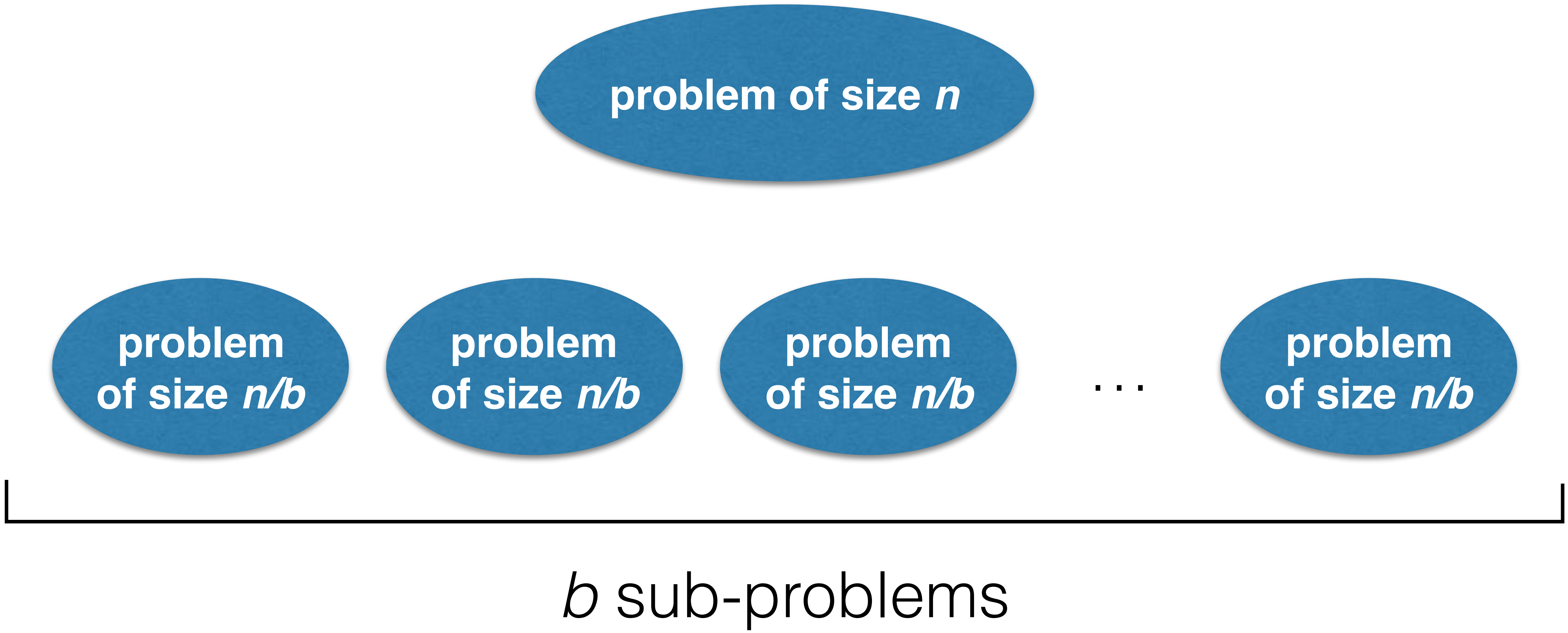
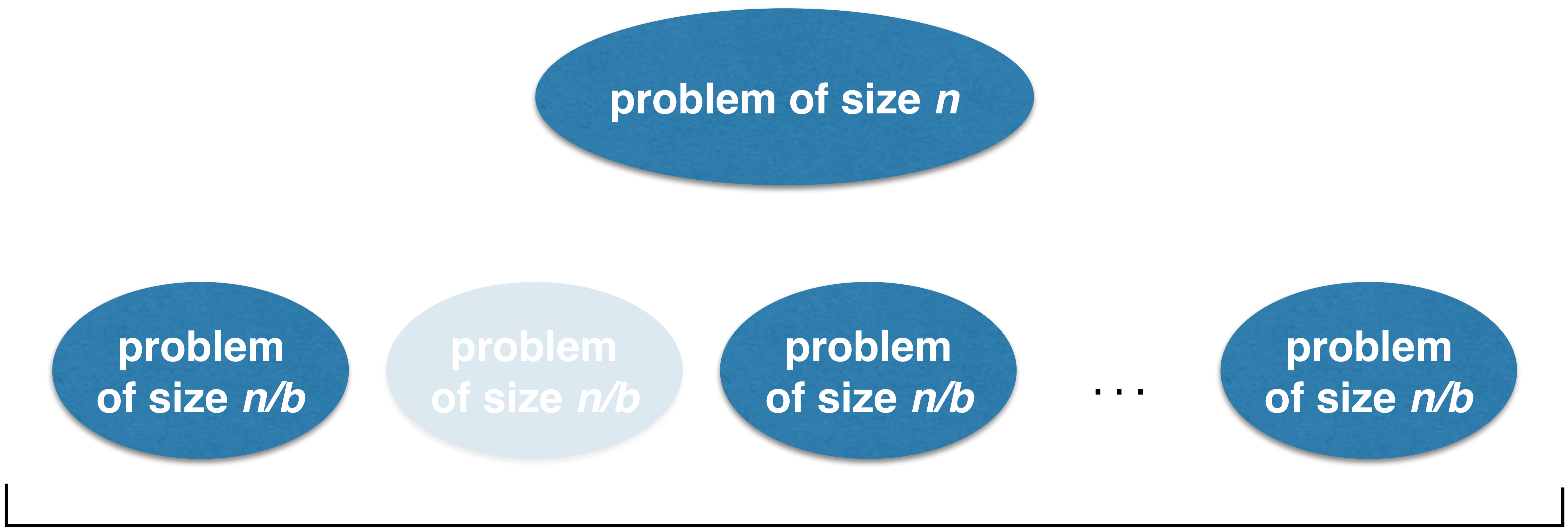
- We will discuss:
  - The Master Theorem
  - Mergesort
  - Quicksort
  - Tree traversal
  - Closest Pair revisited

Divide-and-Conqer General Case

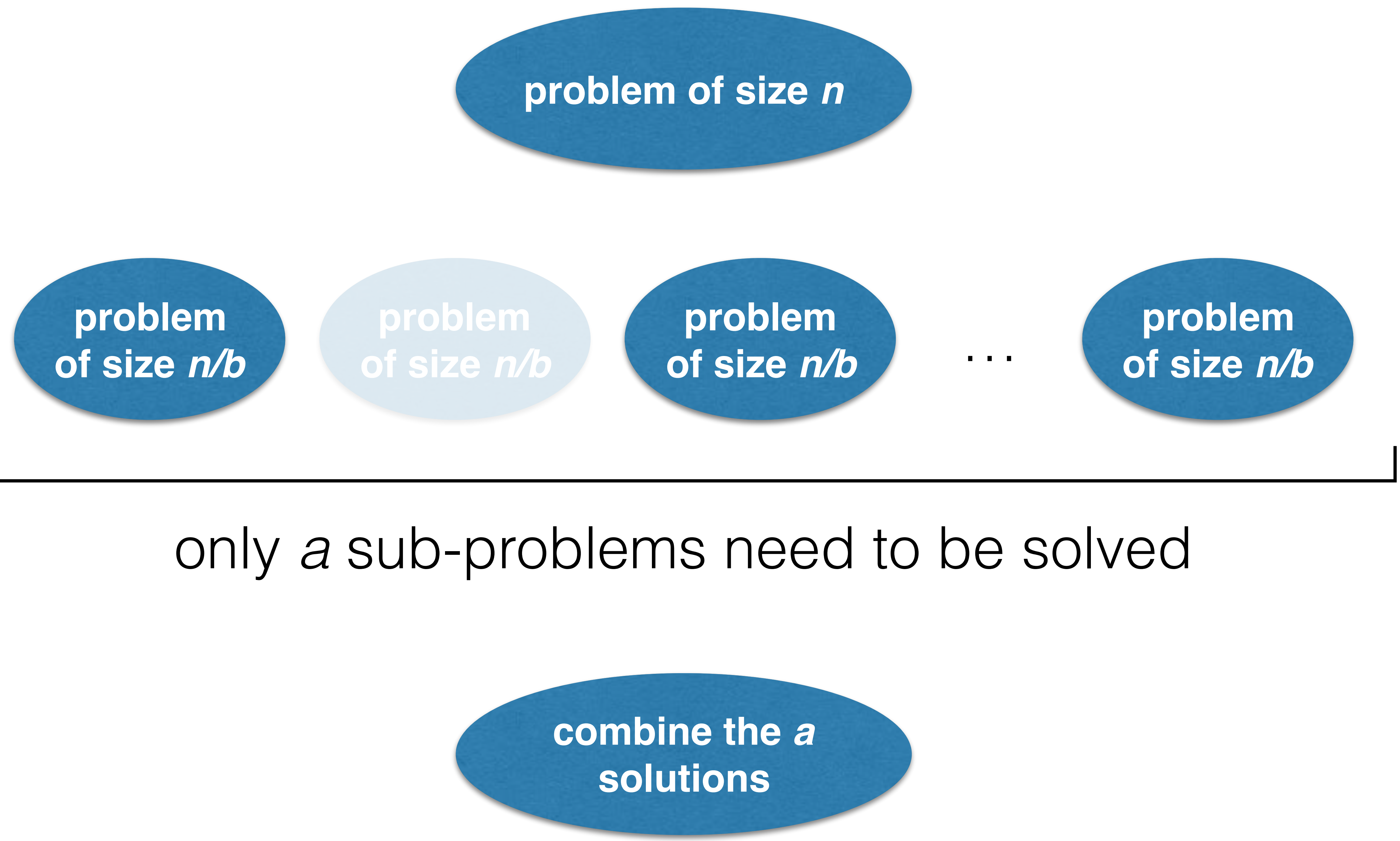


Divide-and-Conqer General Case



only *a* sub-problems need to be solved

Divide-and-Conqer General Case



Divide-and-Conquer Recurrences

- What is the time required to solve a problem of size n by divide-and-conquer?

- For the general case, assume we split the problem into b instances (each of size n/b), of which a need to be solved:

$$T(n) = aT(n/b) + f(n)$$

  where $f(n)$ expresses the time spent on dividing a problem into b sub-problems and combining the a results.

- (A very common case is $T(n) = 2T(n/2) + n$.)

- How to find closed forms for these recurrences?

The Master Theorem

- (A proof is in Levitin's Appendix B.)

- For integer constants $a \geq 1$ and $b > 1$, and function f with $f(n) \in \Theta(n^d)$, $d \geq 0$, the recurrence

$$T(n) = aT(n/b) + f(n)$$

(with T(1) = c) has solutions, and

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- Note that we also allow a to be greater than b.

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$

a = 2, b = 2, d = 1

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So, by the Master Theorem, $T(n) \in \Theta(n \log n)$

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$

a = 2, b = 2, d = 1

$1 \times n$

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

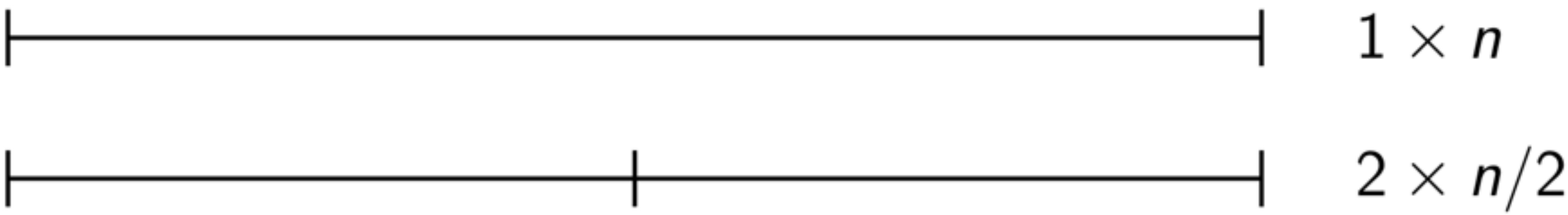
$T(n) = 2(2T(n/4) + (n/2)) + n$

$1 \times n$

Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

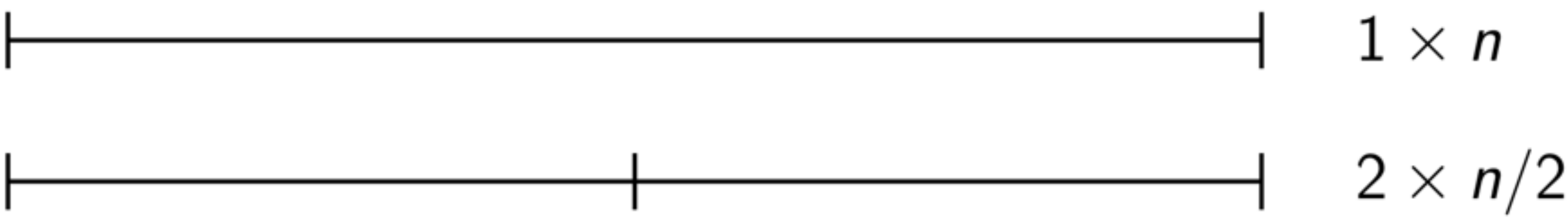
$T(n) = 4T(n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

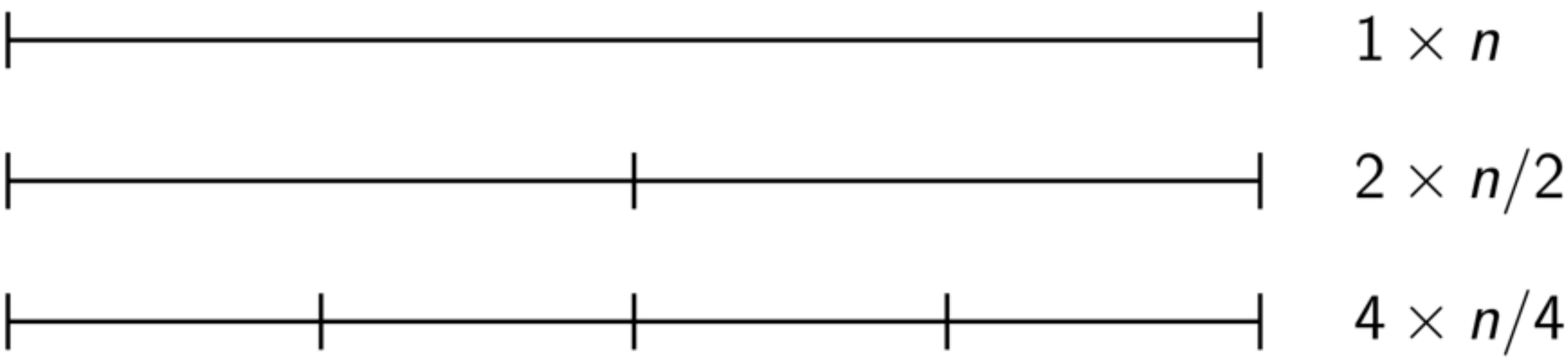
$T(n) = 4(2T(n/8) + n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

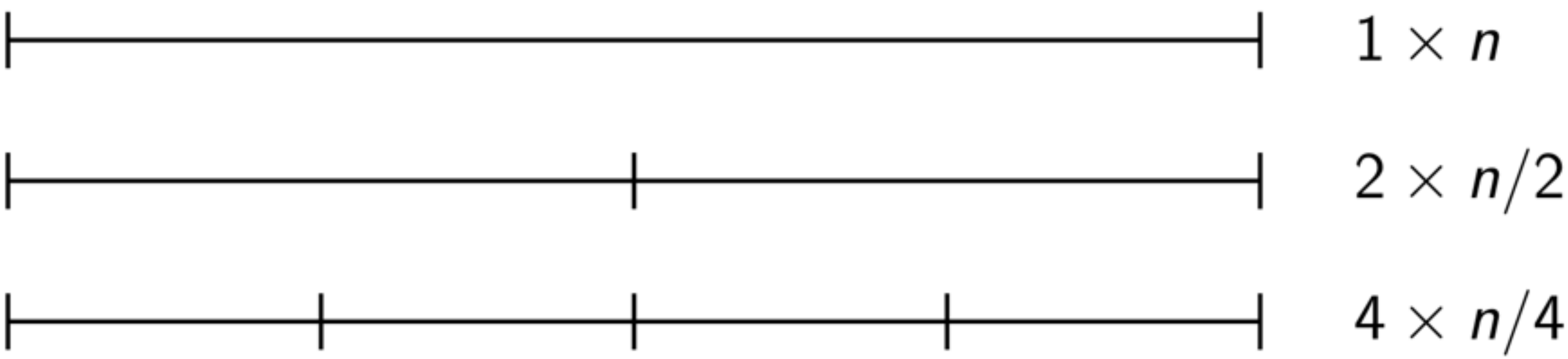
$T(n) = 8T(n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

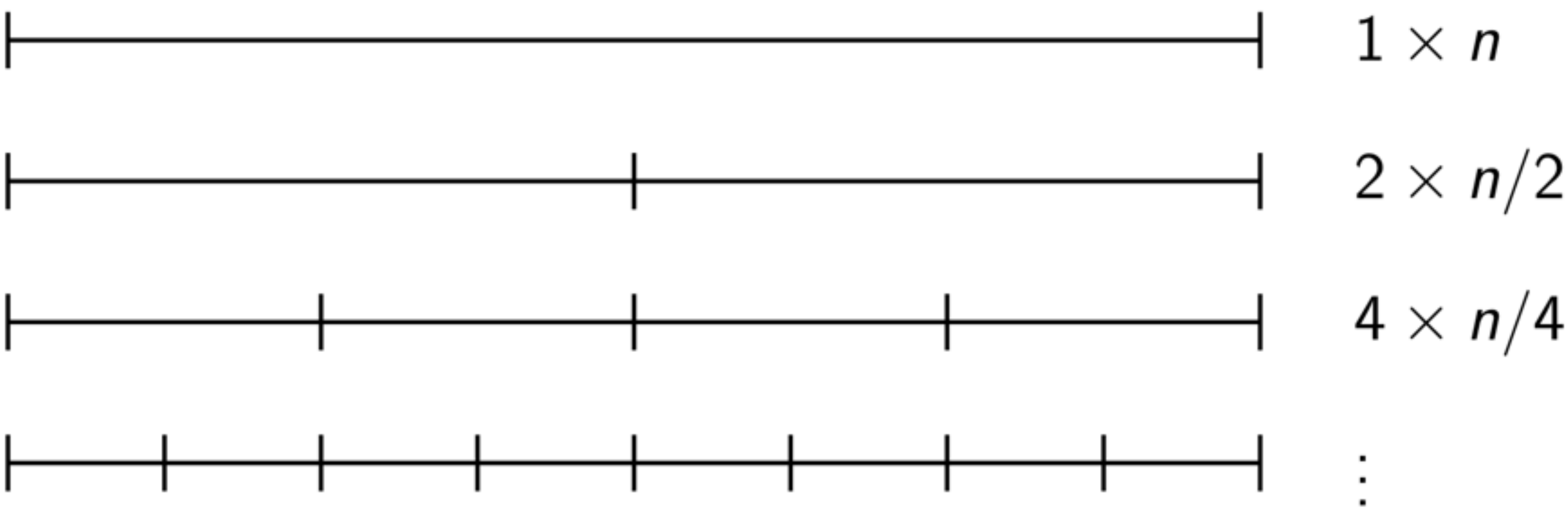
$T(n) = 8(2T(n/16) + n/8) + 4(n/4) + 2(n/2) + n$



Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ a = 2, b = 2, d = 1

$T(n) = 16T(n/16) + 8(n/8) + 4(n/4) + 2(n/2) + n$



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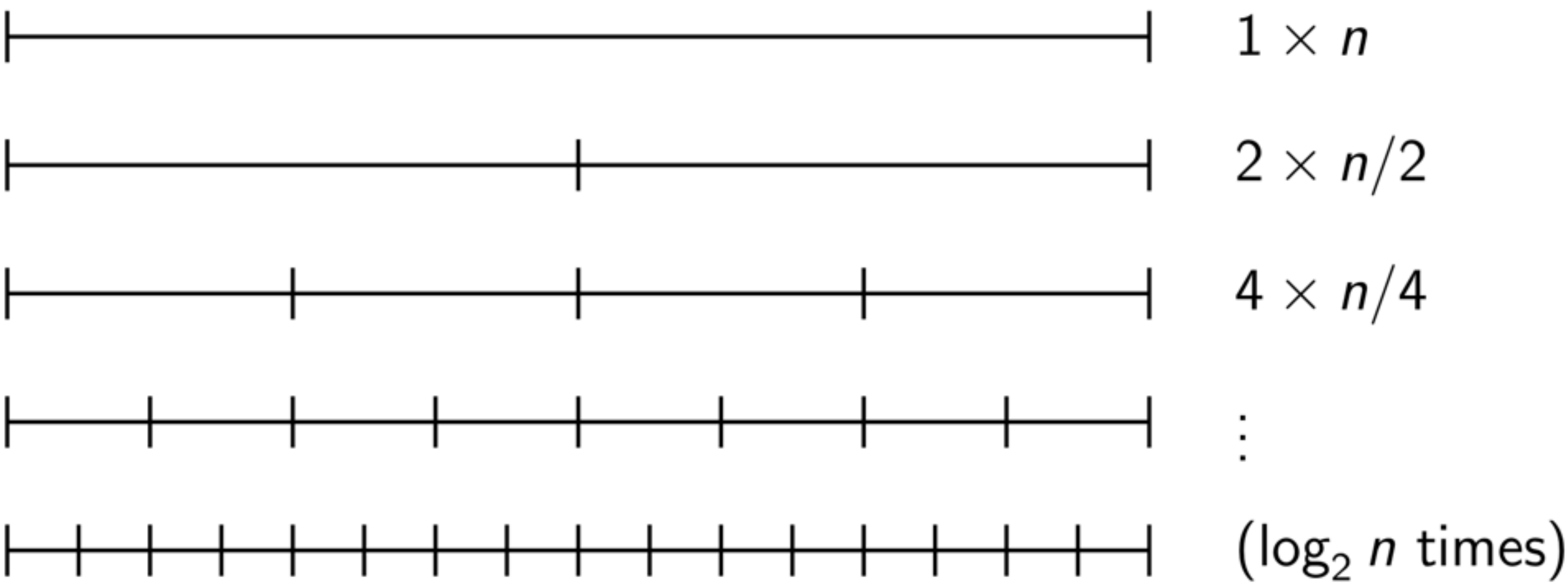
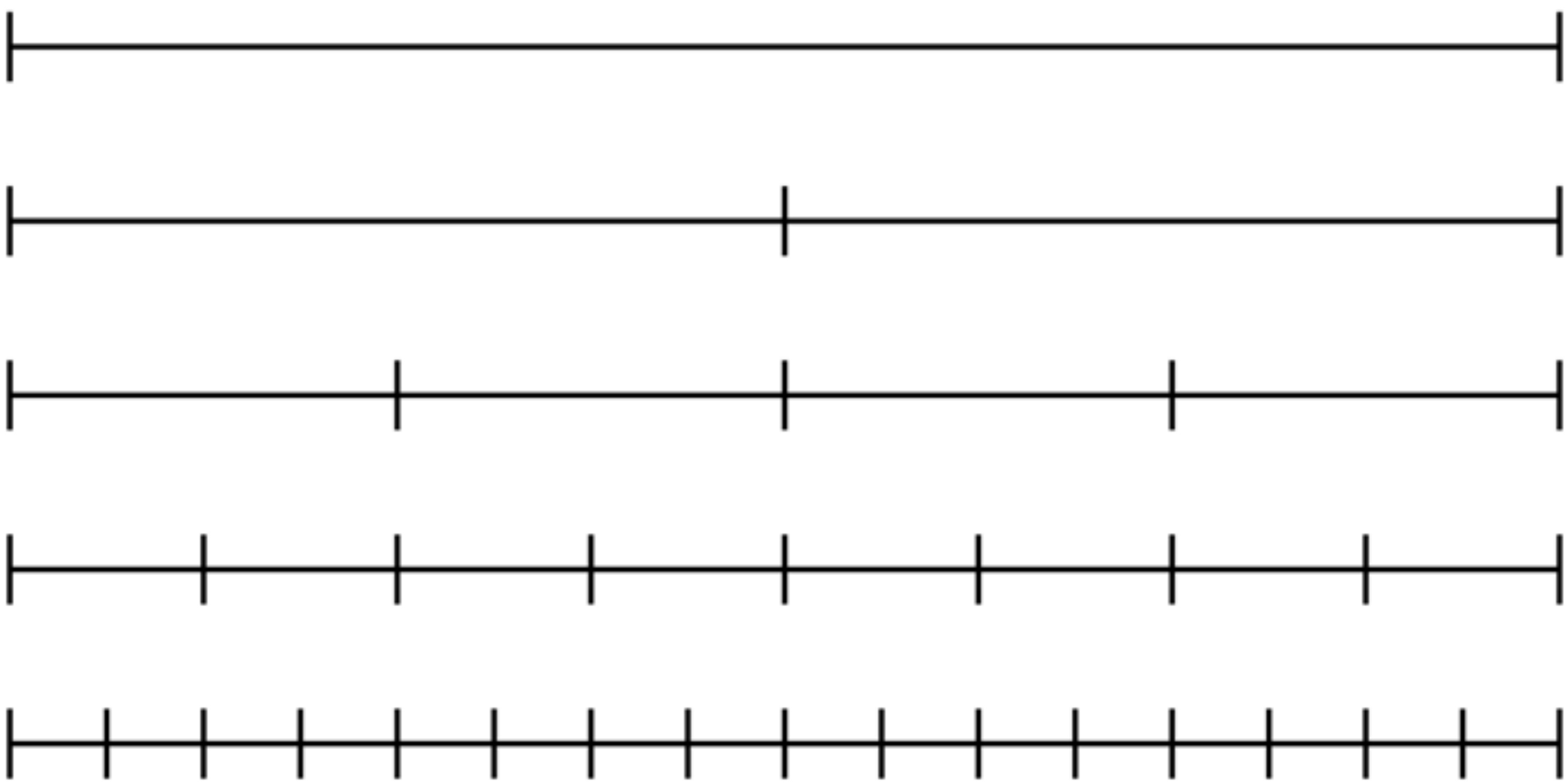


Master Theorem: Example 1

$T(n) = 2T(n/2) + n$ $\quad$ a = 2, b = 2, d = 1

$T(n) \in \Theta(n \log n)$



$1 \times n$

$2 \times n/2$

$4 \times n/4$

$\vdots$

$(\log_2 n \text{ times})$

Master Theorem: Example 2

$T(n) = 4T(n/4) + n$

a = 4, b = 4, d = 1

$a = b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ $\quad\quad$ a = 4, b = 4, d = 1

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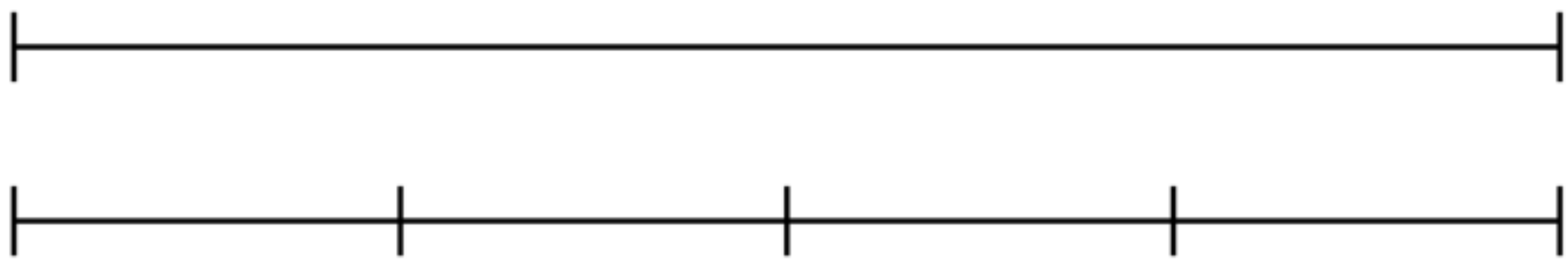

Master Theorem: Example 2

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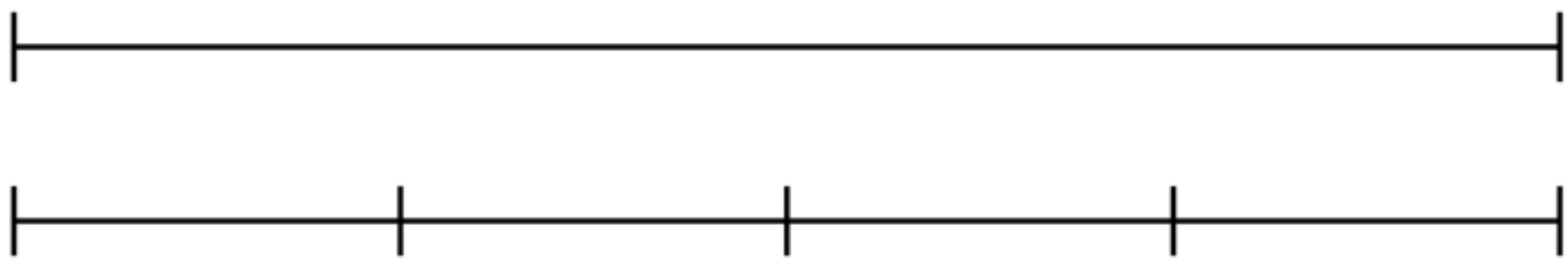
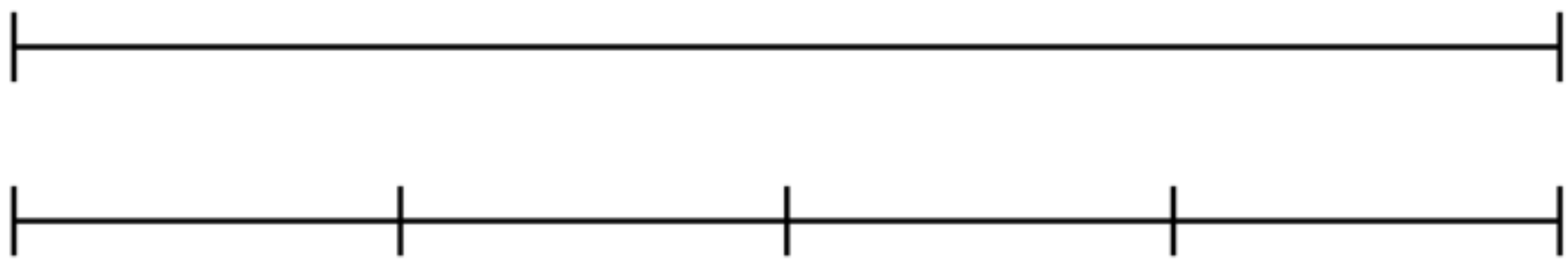


Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

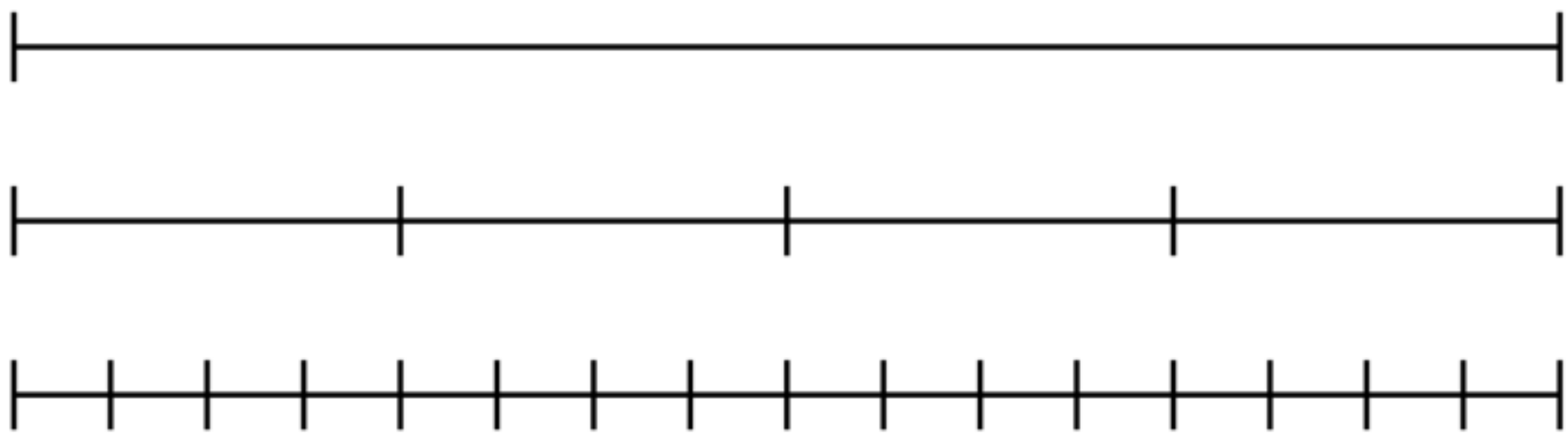
$T(n) = 16(4T(n/64) + n/16)\ 4(n/4) + n$



Master Theorem: Example 2

$T(n) = 4T(n/4) + n$ a = 4, b = 4, d = 1

$T(n) = 64T(n/64) + 16(n/16) + 4(n/4) + n$



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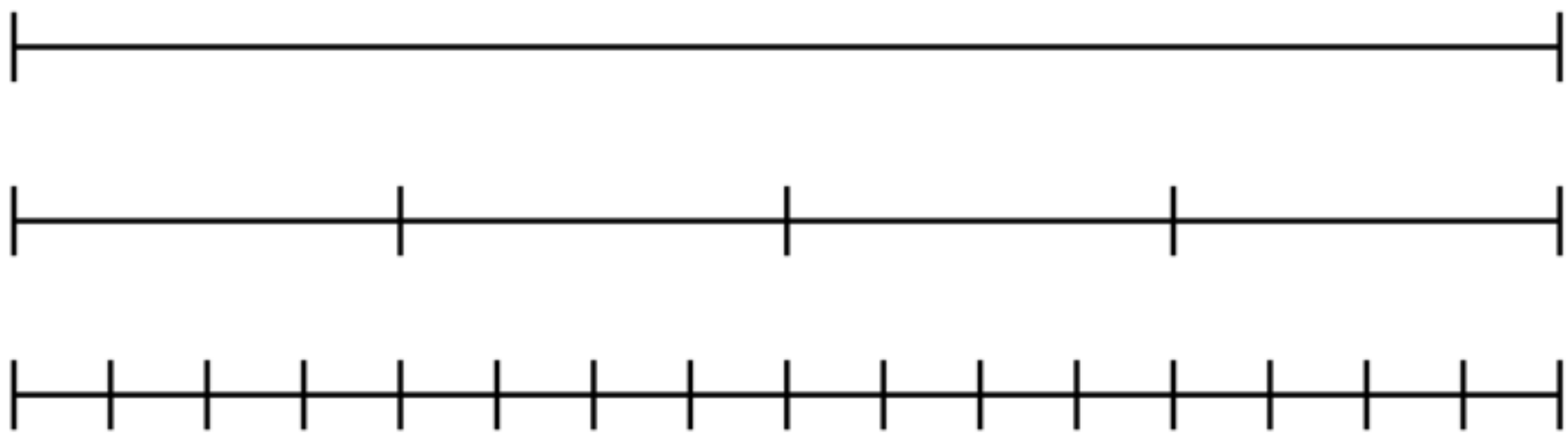

⋮

(log$_4$ n times)

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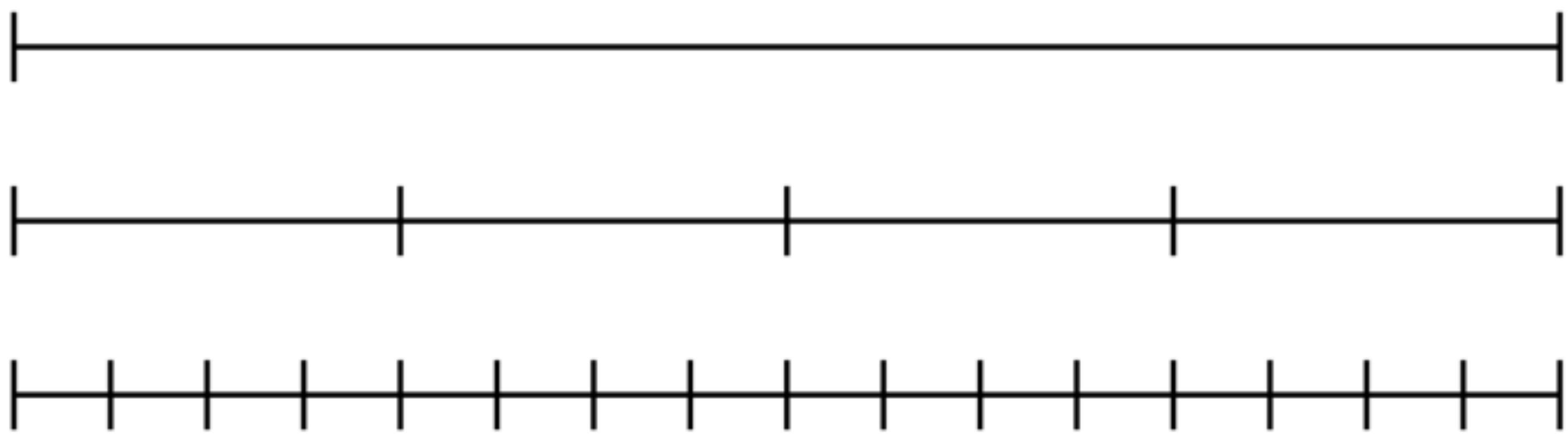

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$T(n) \in \Theta(n \log n)$

Master Theorem: Example 3

$T(n) = T(n/2) + n$ $\qquad$ a = 1, b = 2, d = 1

Master Theorem: Example 3

$T(n) = T(n/2) + n$

$a = 1, b = 2, d = 1$

$a < b^d$

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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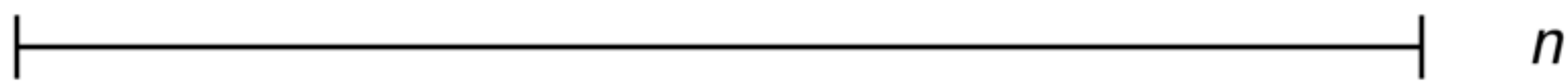

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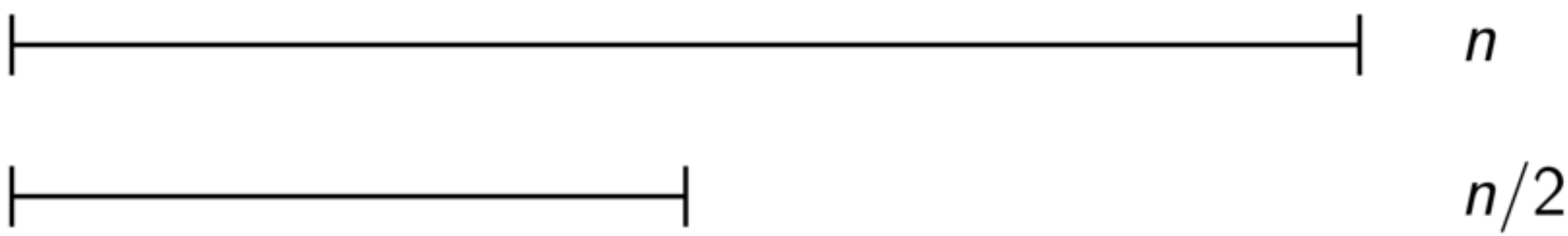

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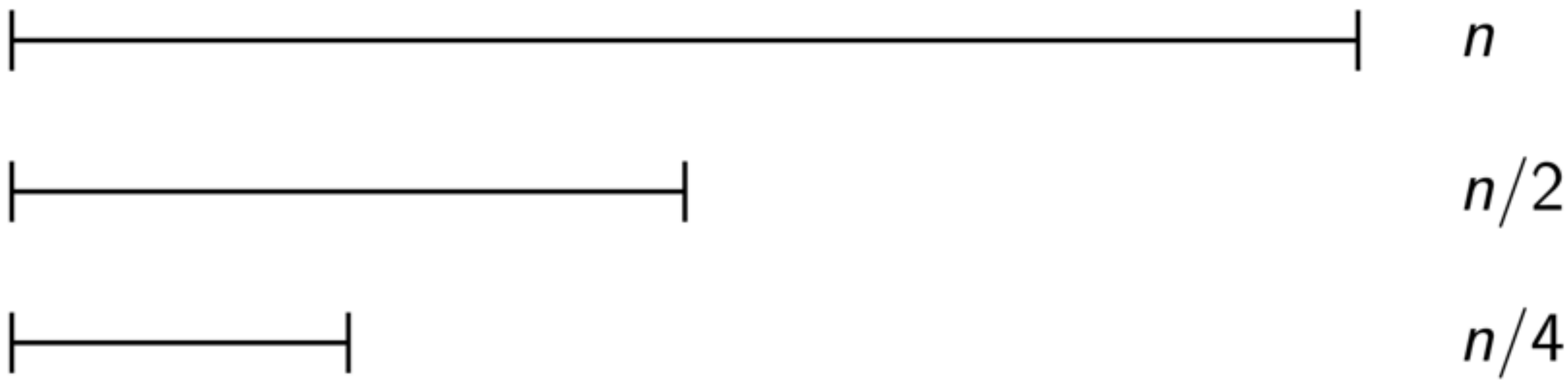
$T(n) = T(n/8) + n/4 + n/2 + n$



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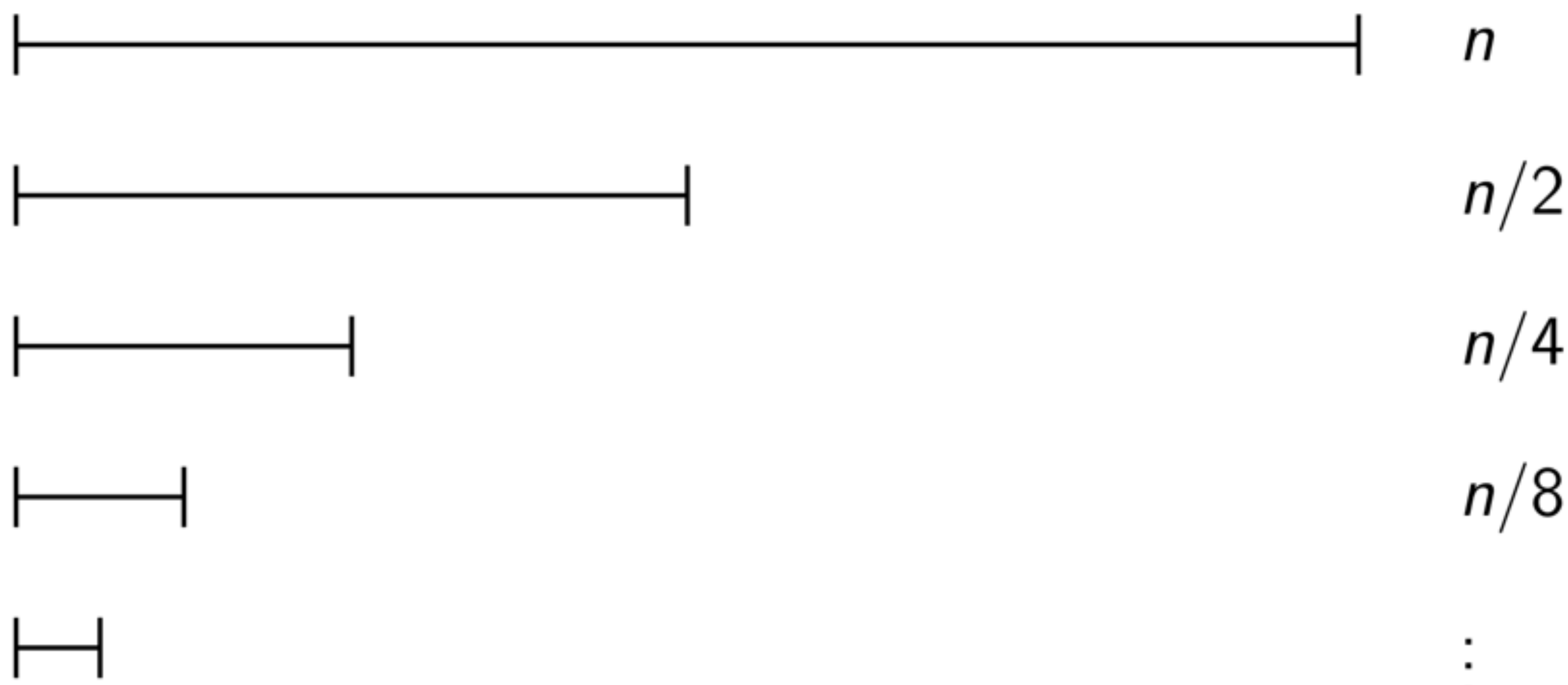

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$T(n) \in \Theta(n)$

Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$ $\qquad$ a = 2, b = 2, d = 2

Master Theorem: Example 4

$T(n) = 2T(n/2) + n^2$

a = 2, b = 2, d = 2

a < b^d

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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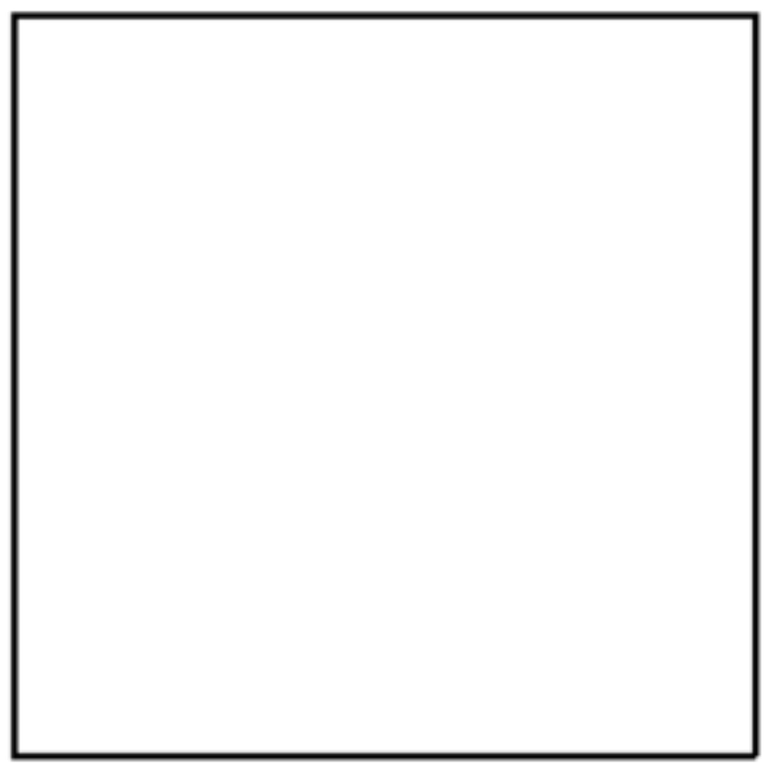

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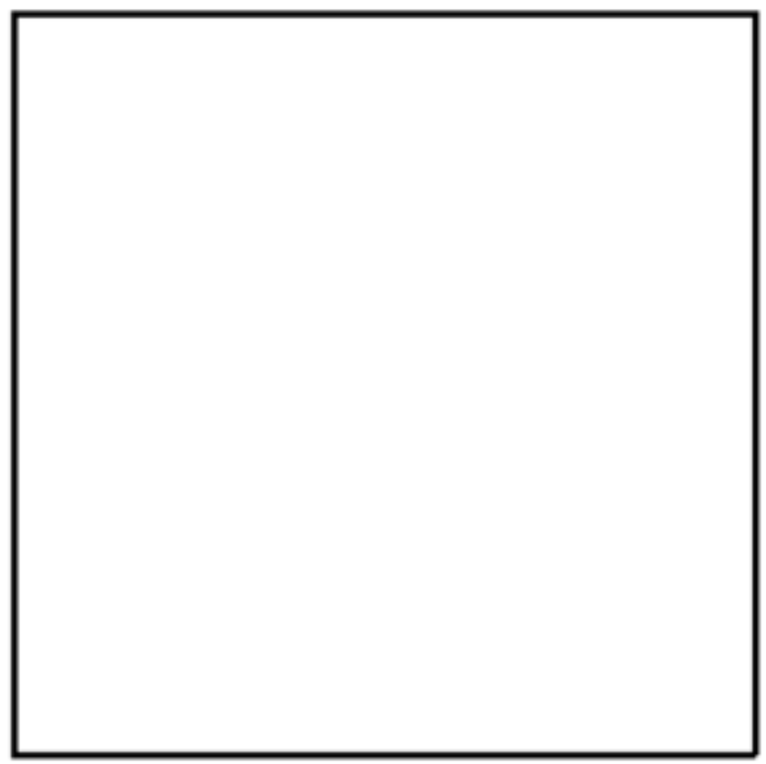

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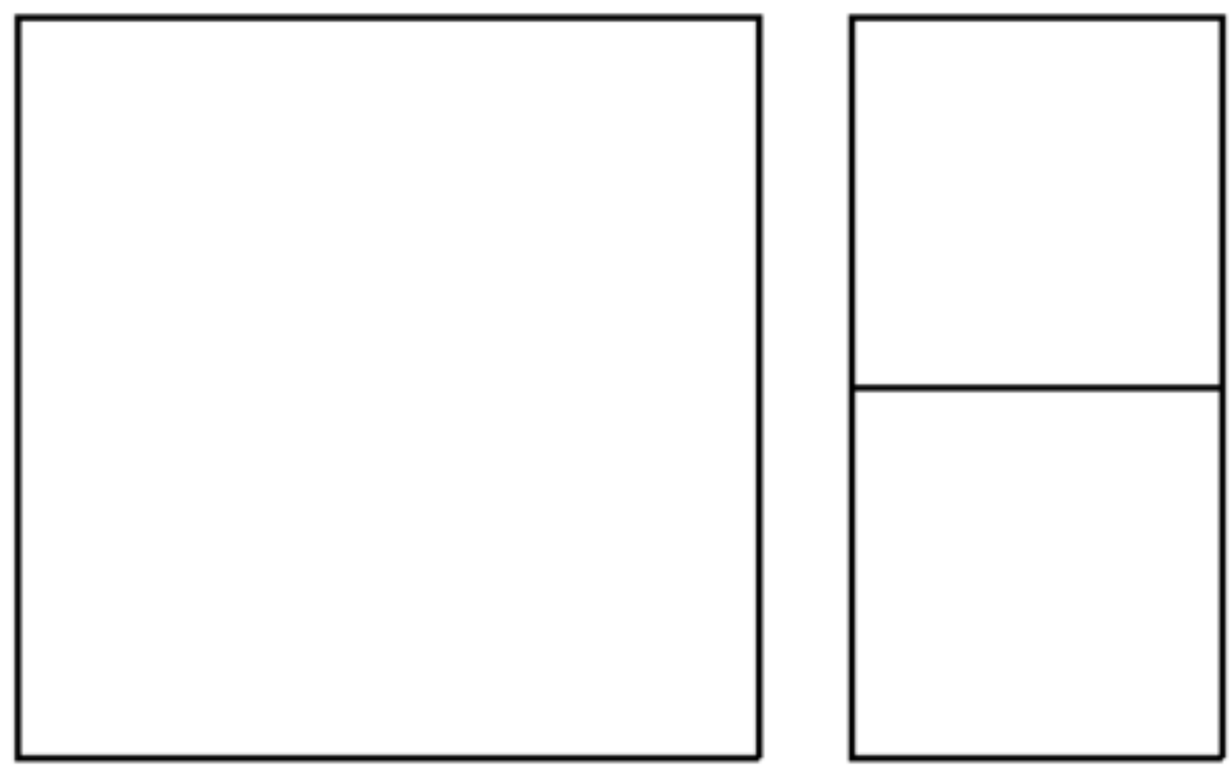

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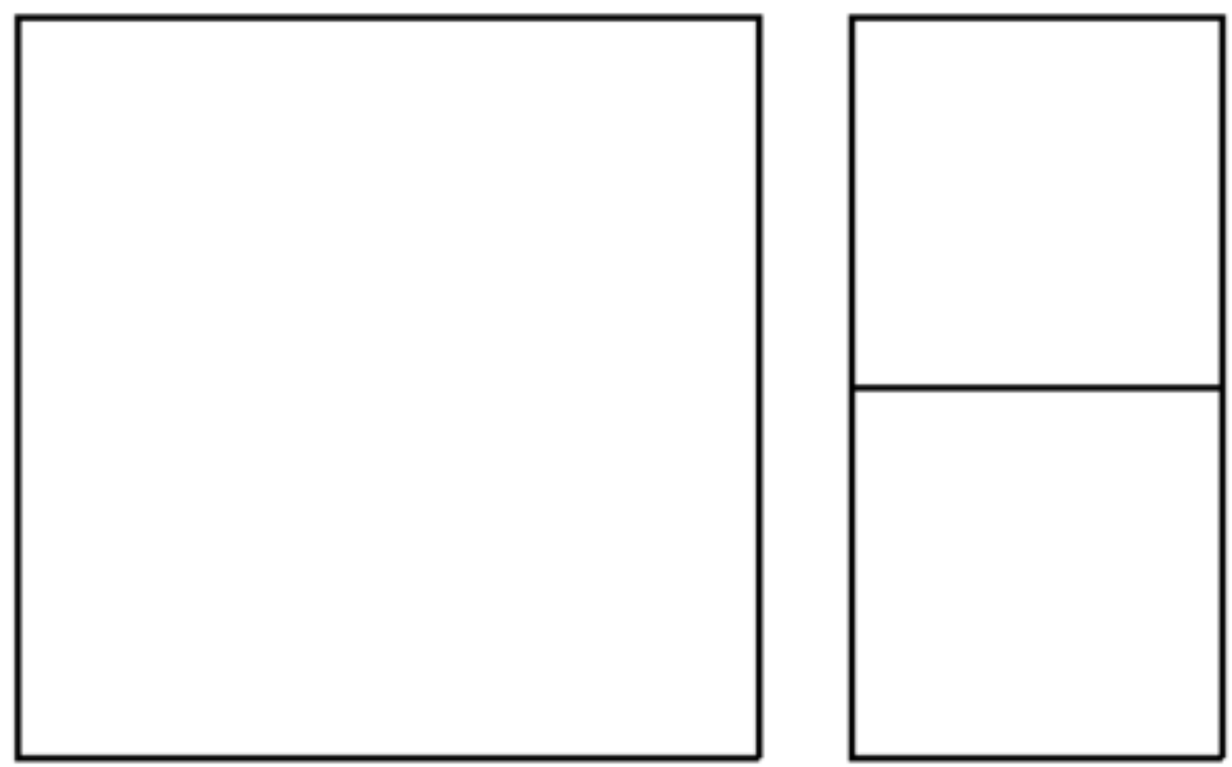

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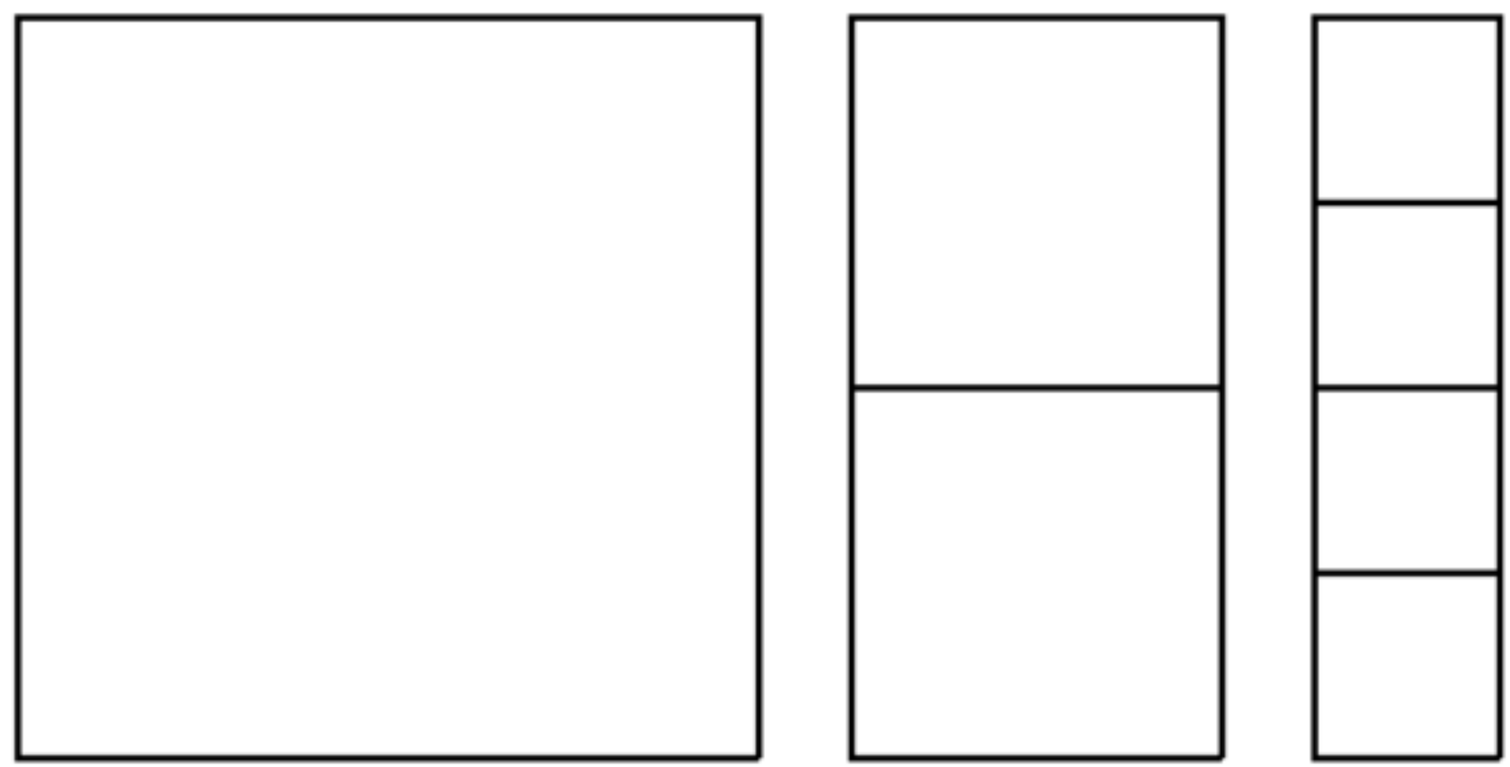

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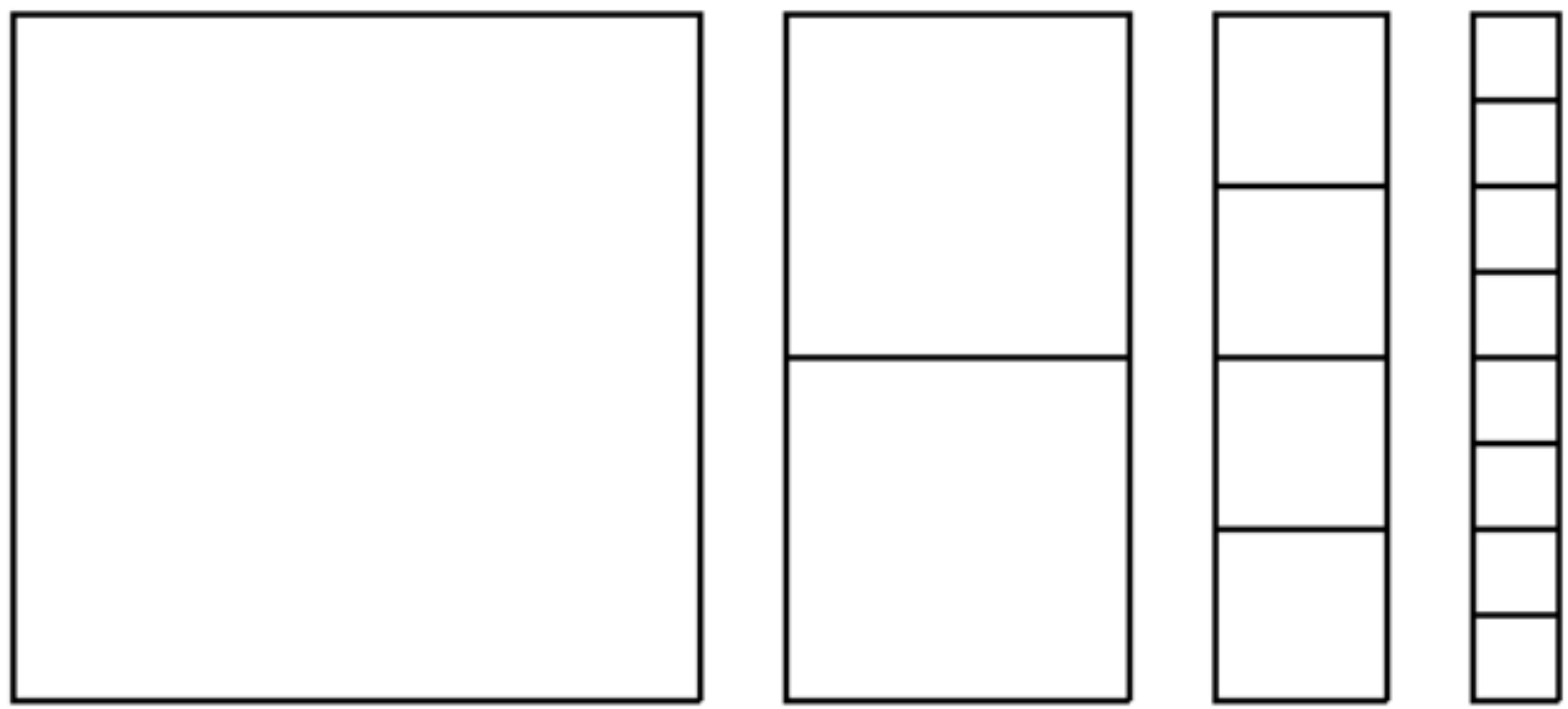

Master Theorem: Example 4

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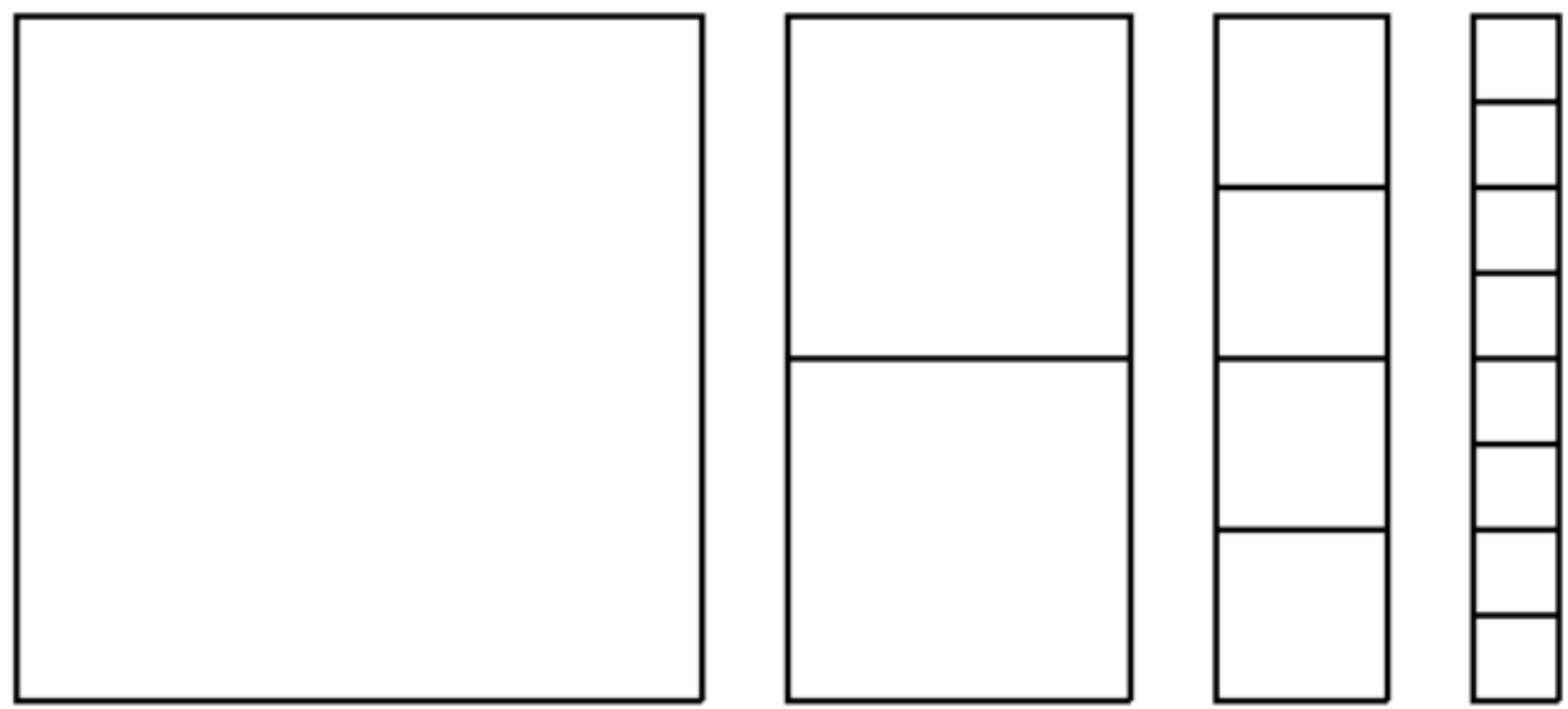
$T(n) \in \Theta(n^2)$



Mergesort

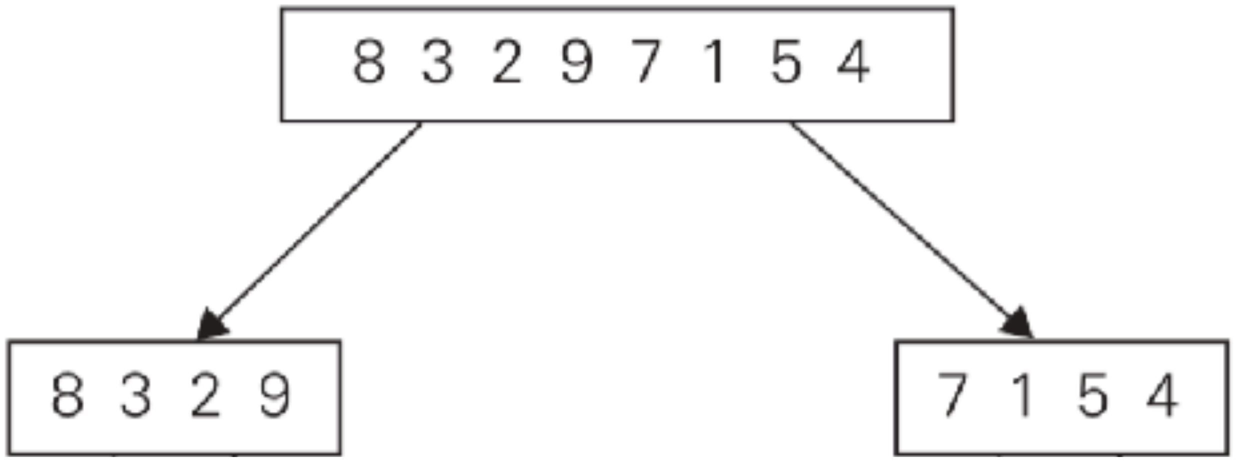
- Perhaps the most obvious application of divide-and-conquer:
- To sort an array (or a list), cut it into two halves, sort each half, and merge the two results.

```
procedure MERGESORT(A[·], n)                  ▷ Sort A[0]..A[n − 1]
    if n > 1 then
        for i ← 0 to ⌊n/2⌋ − 1 do             ▷ Copy left half of A to B
            B[i] ← A[i]
        for i ← 0 to ⌈n/2⌉ − 1 do             ▷ Copy right half of A to C
            C[i] ← A[⌊n/2⌋ + i]
        MERGESORT(B, ⌊n/2⌋)                    ▷ Sort B
        MERGESORT(C, ⌈n/2⌉)                    ▷ Sort C
        MERGE(B, ⌊n/2⌋, C, ⌈n/2⌉, A)           ▷ Merge B and C into A
```

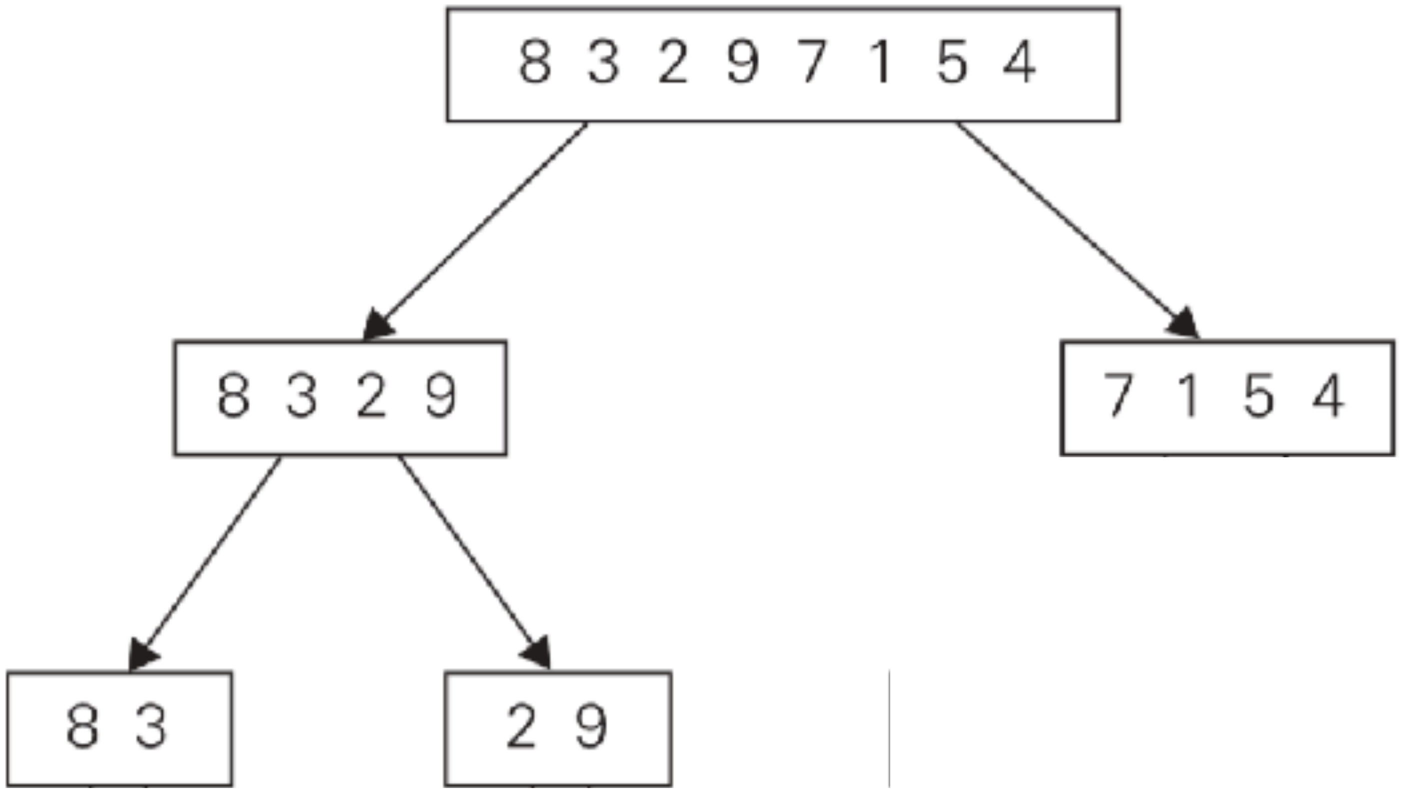
Mergesort

8	3	2	9	7	1	5	4

Mergesort



Mergesort



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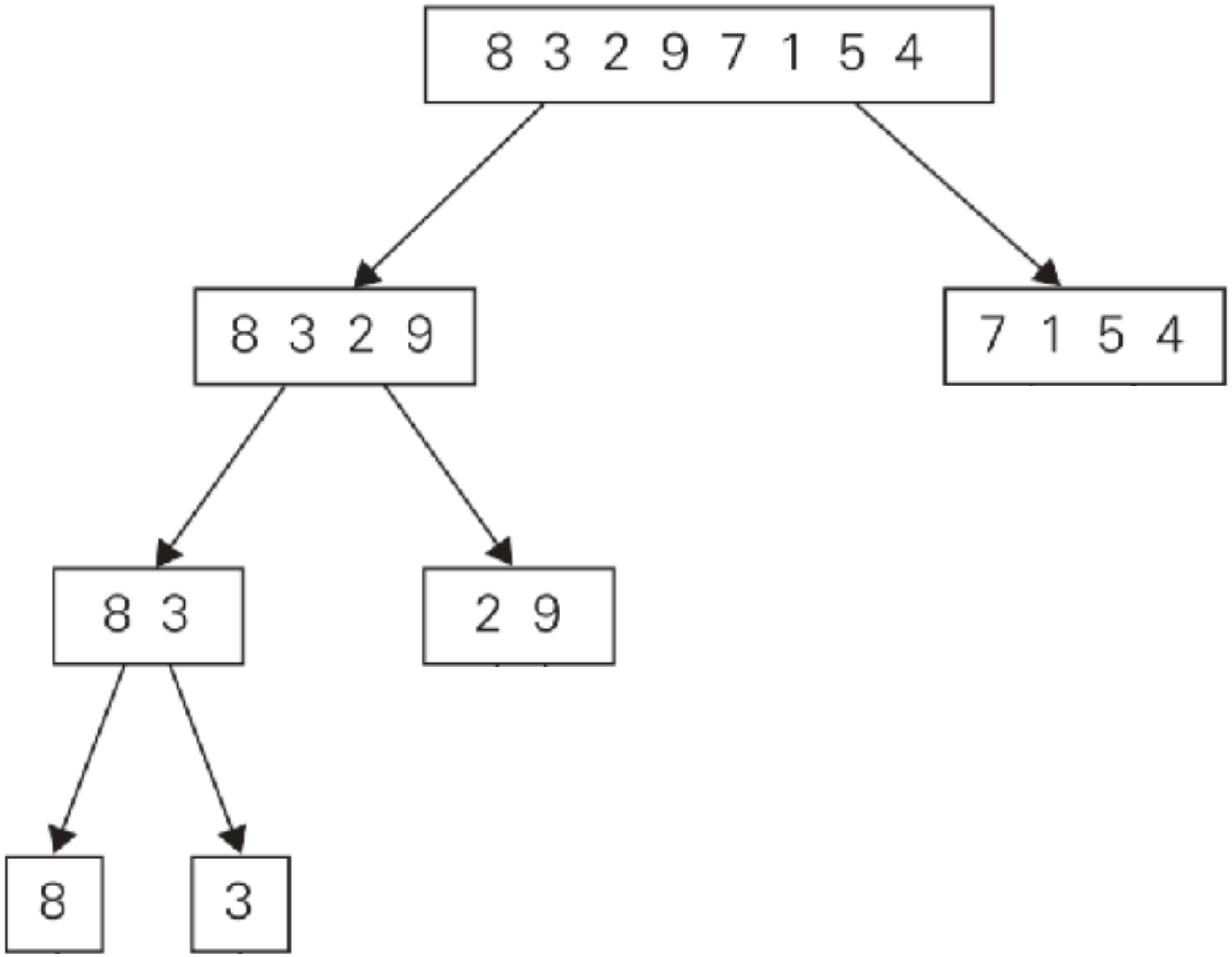

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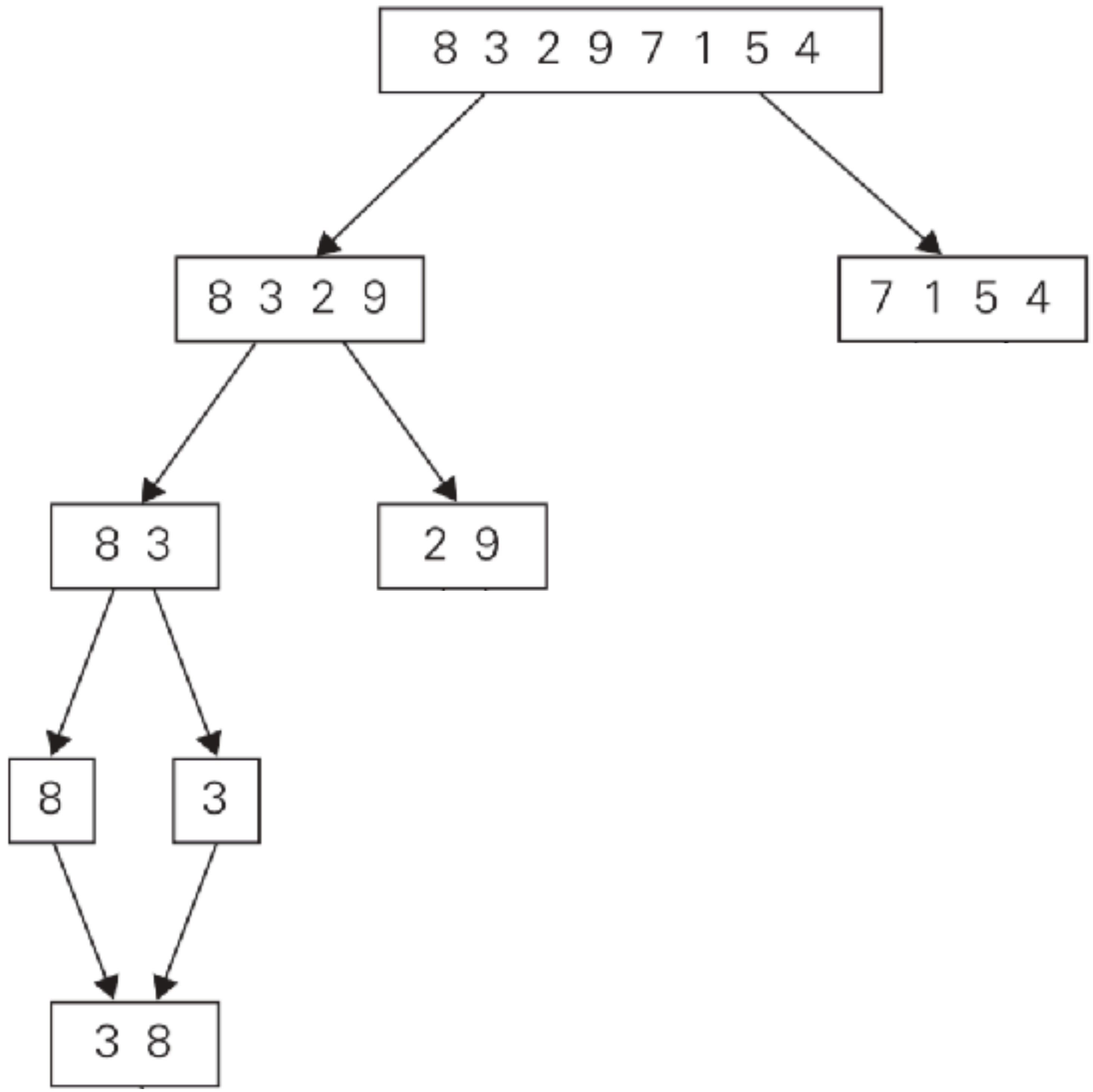

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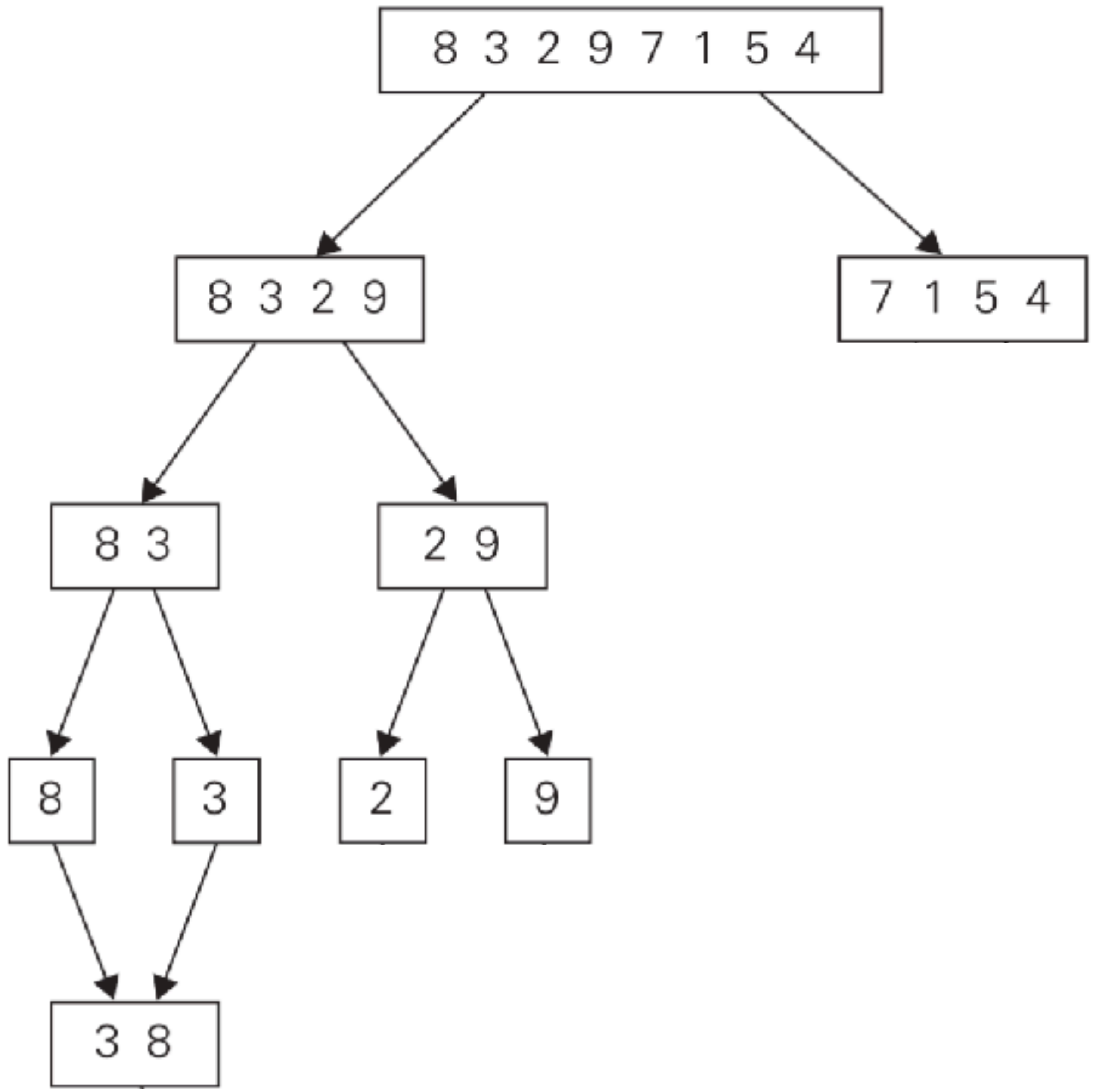
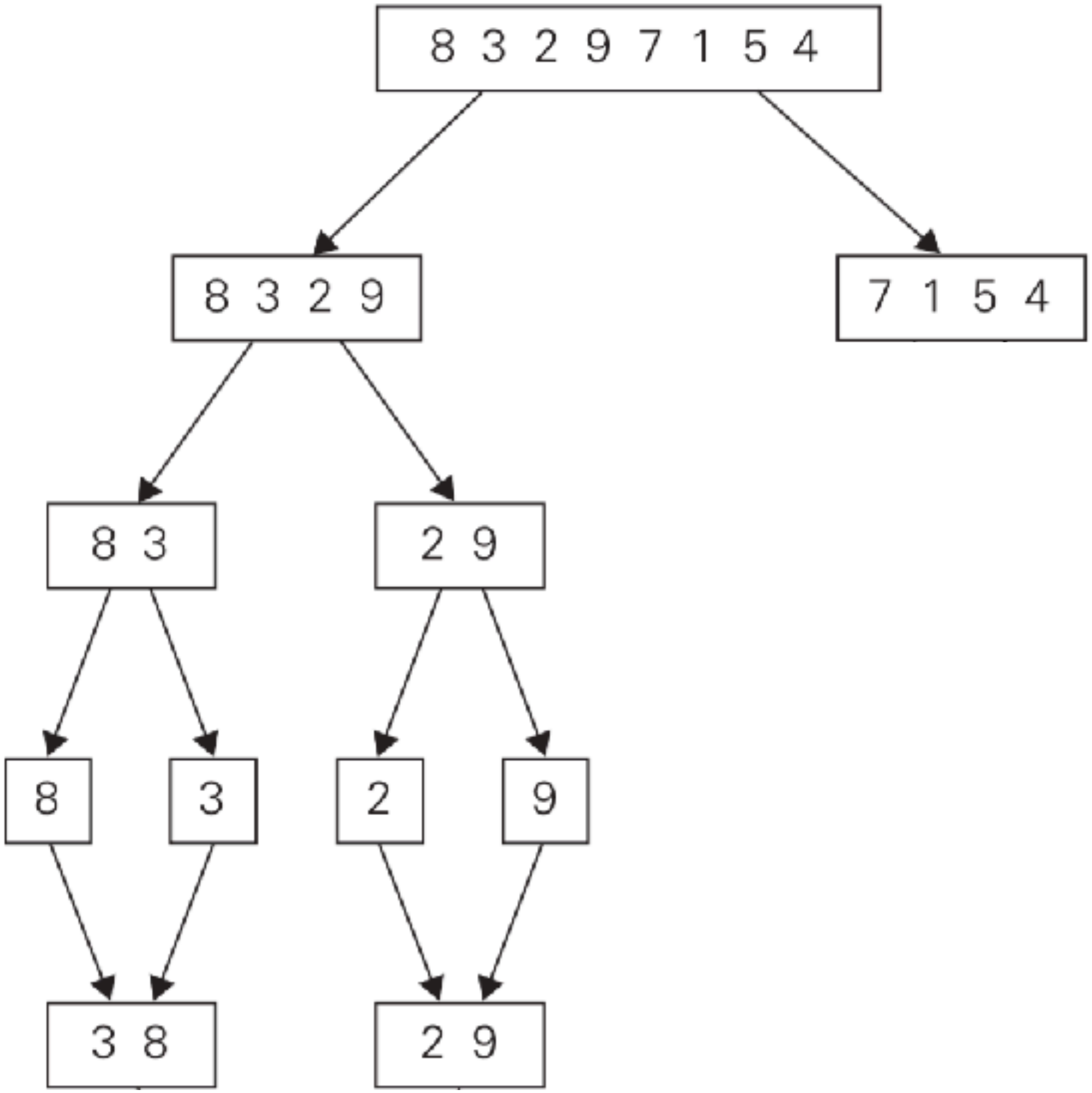
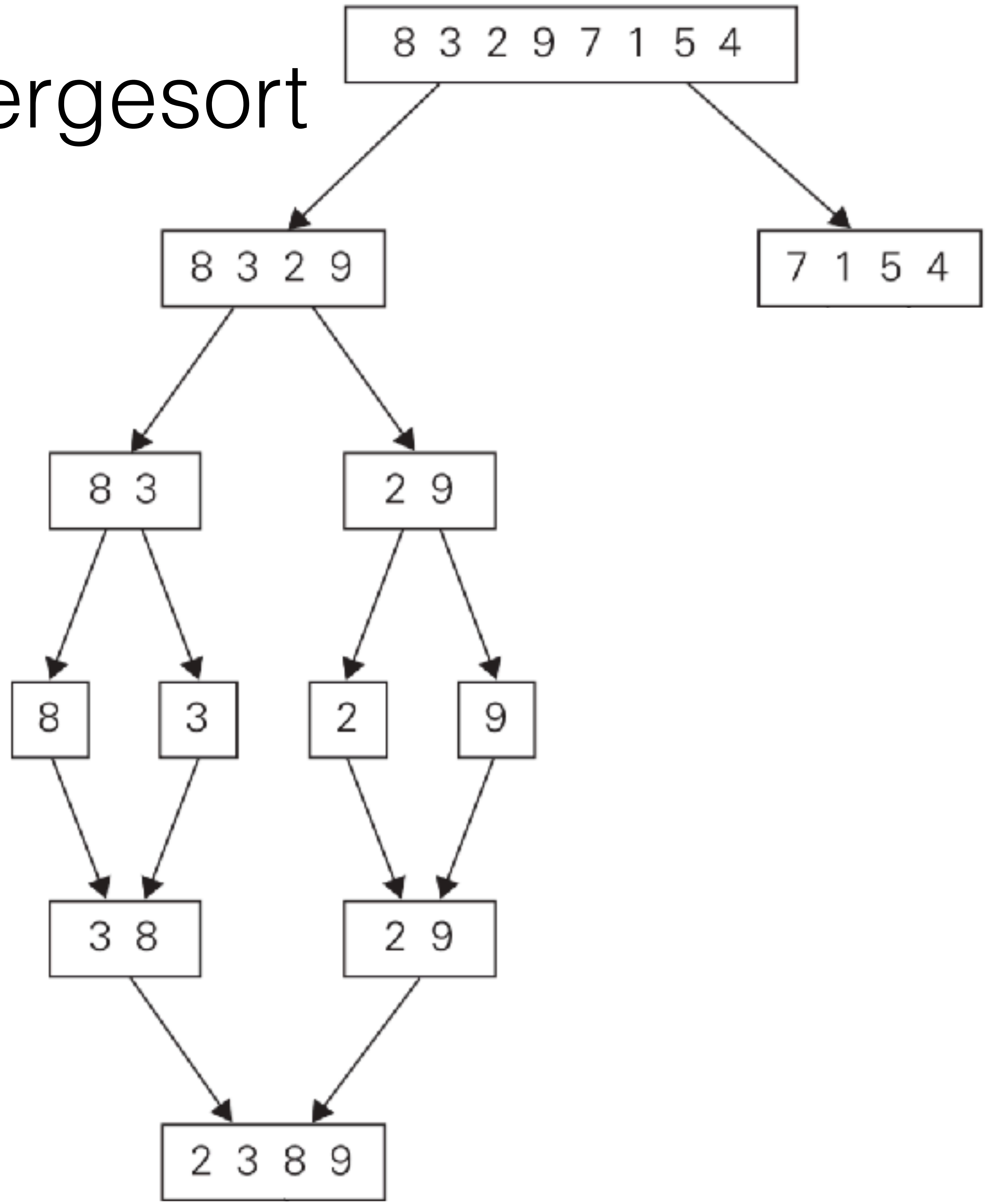


Mergesort



Mergesort



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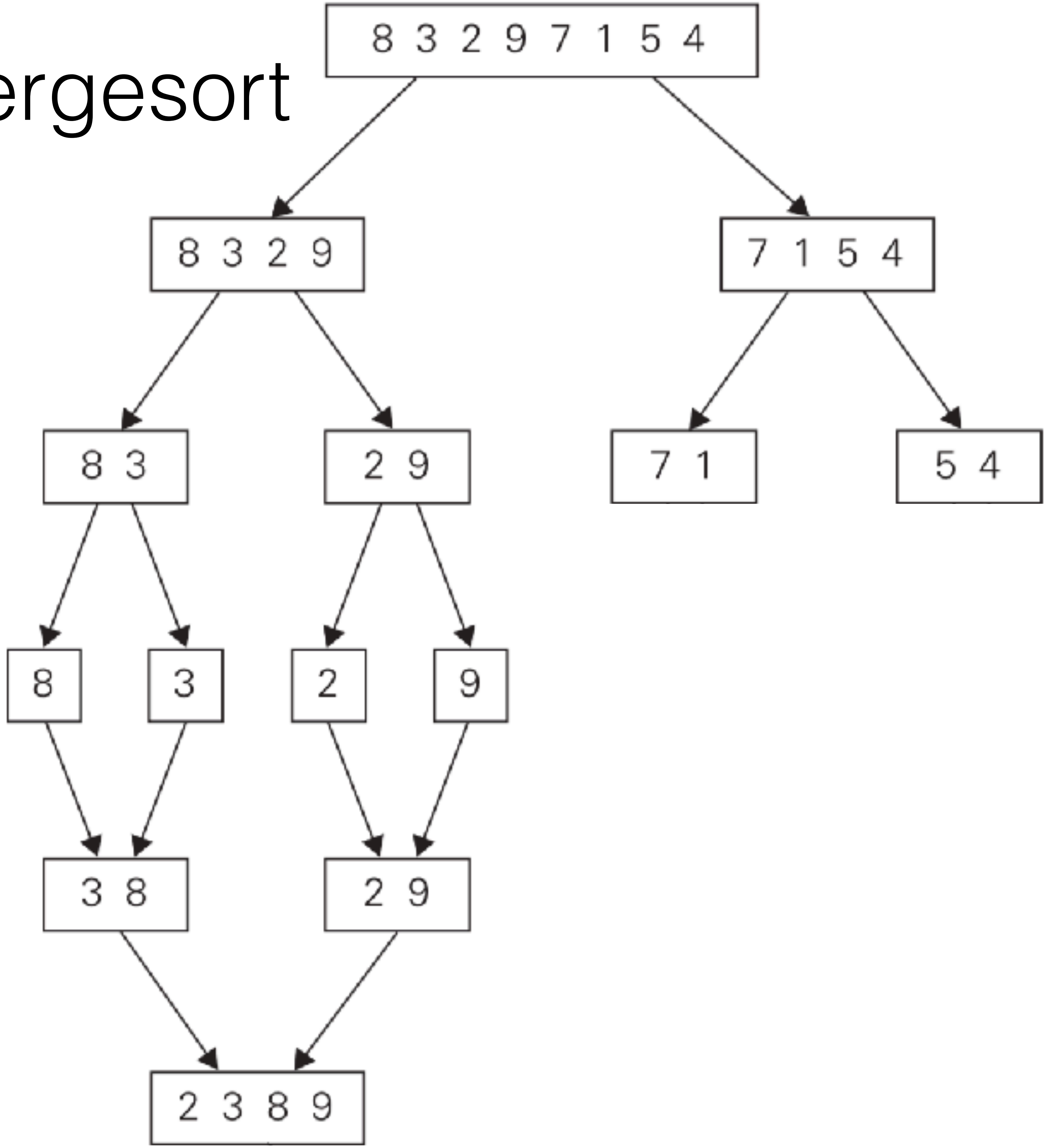
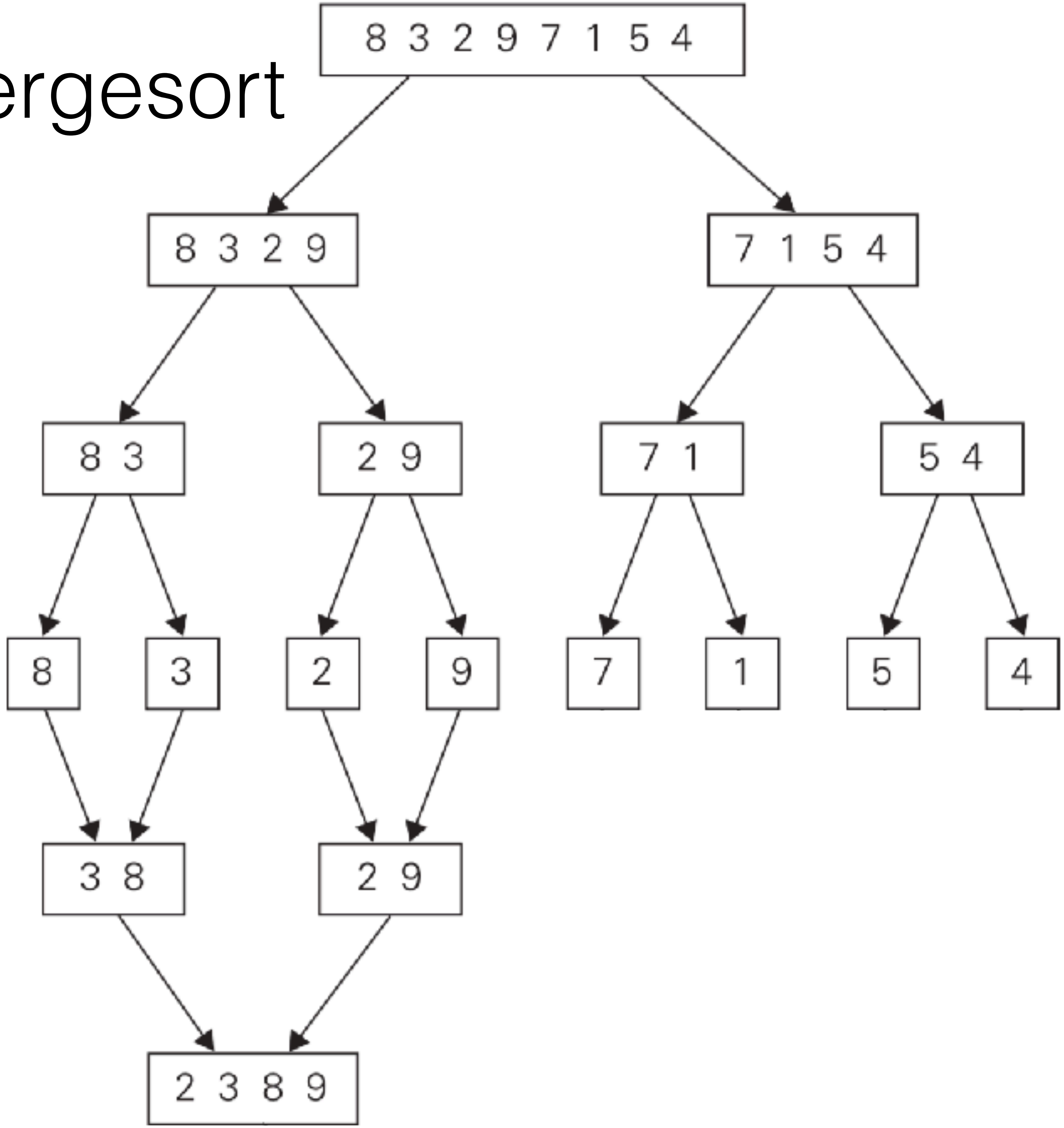
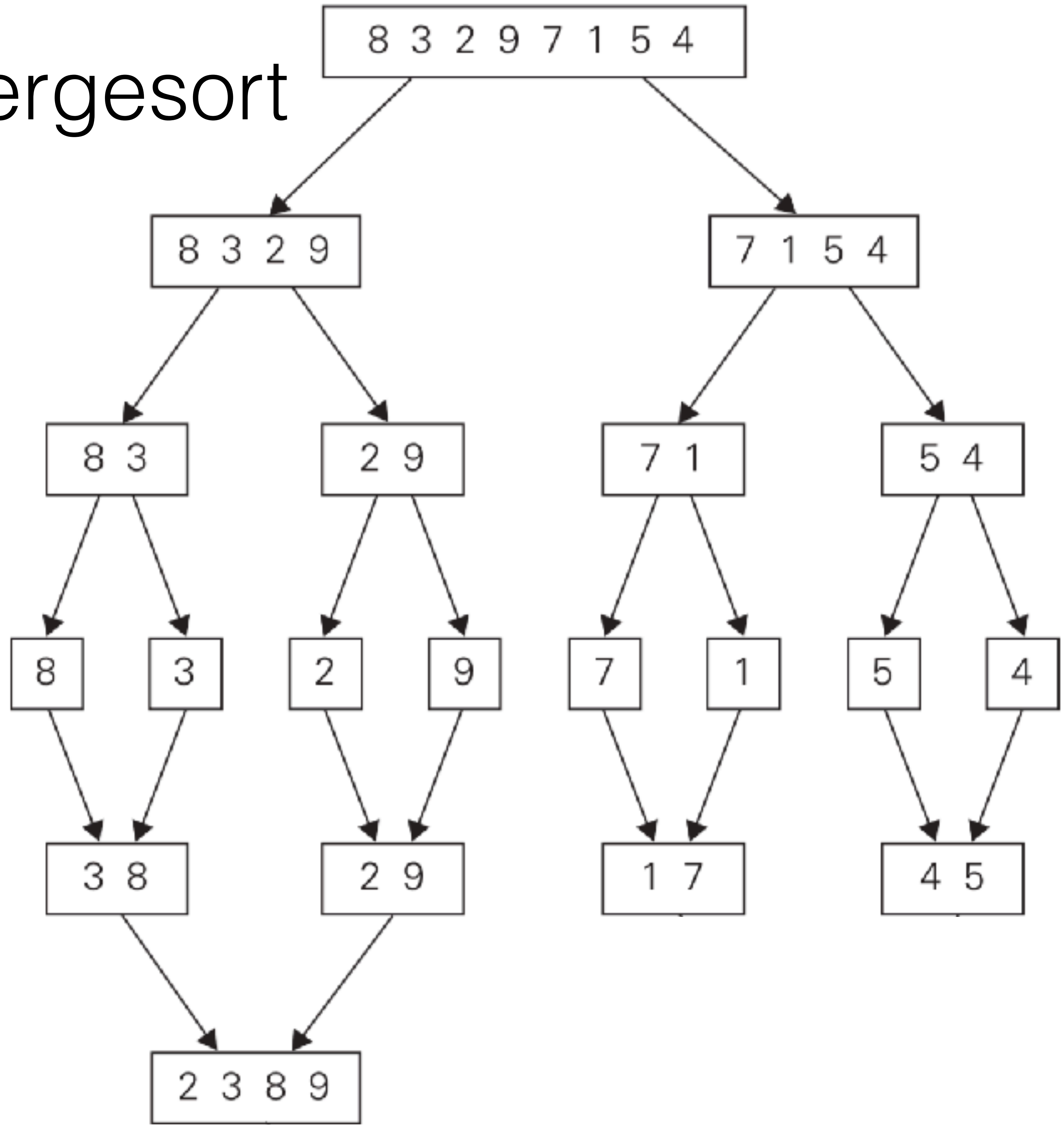


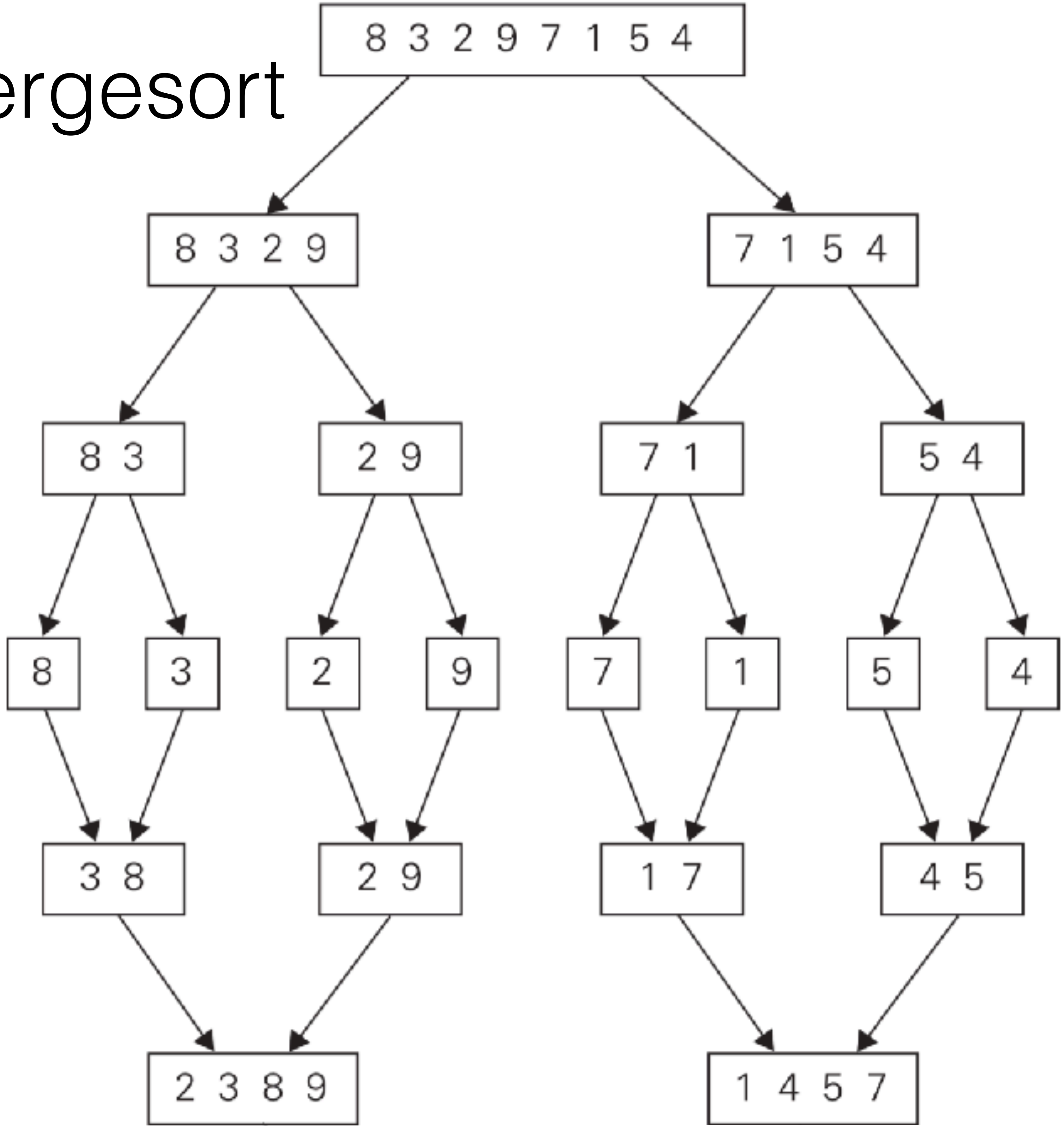
Mergesort



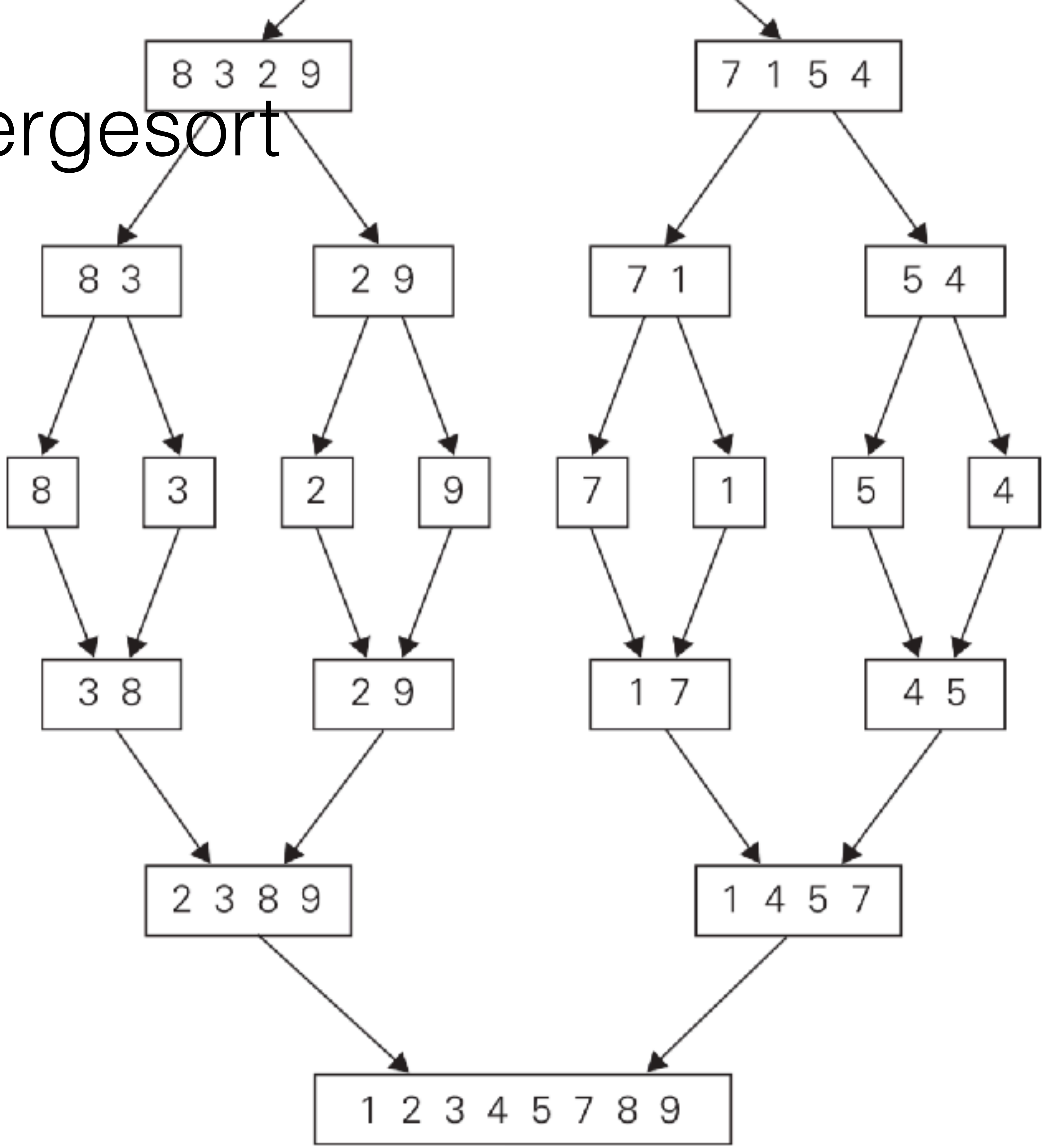
Mergesort



Mergesort



Mergesort



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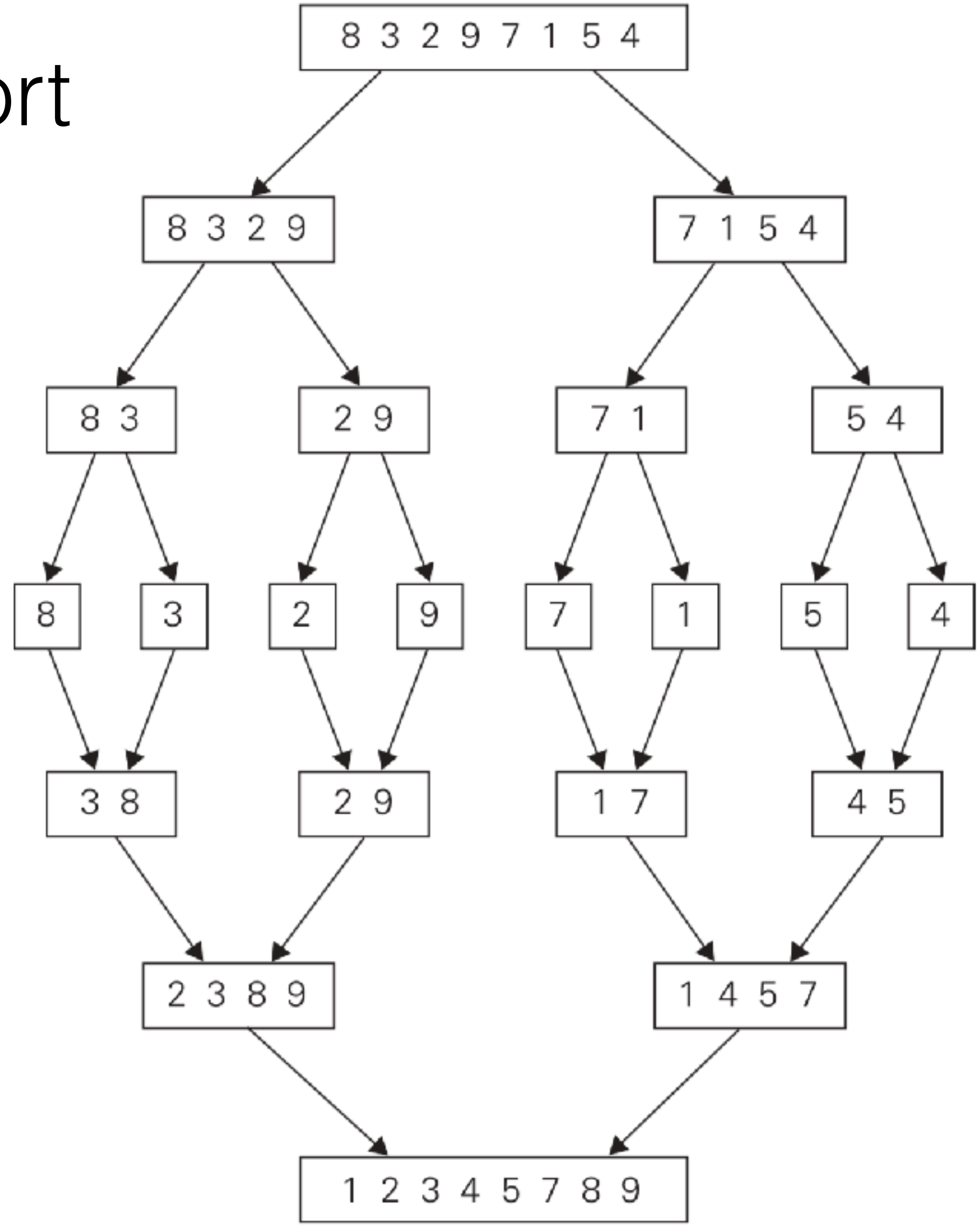

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2 9
1 7
4 5
2 3 8 9
1 4 5 7
1 2 3 4 5 7 8 9

Mergesort: Merging Arrays

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procedure $\textsc{Merge}(B[\cdot], p, C[\cdot], q, A[\cdot])$
    $i \leftarrow 0; j \leftarrow 0; k \leftarrow 0$
    while $i < p$ and $j < q$ **do**
        if $B[i] \leq C[j]$ **then**
            $A[k] \leftarrow B[i]$
            $i \leftarrow i + 1$
        else
            $A[k] \leftarrow C[j]$
            $j \leftarrow j + 1$
        $k \leftarrow k + 1$
    if $i = p$ **then**
        copy $C[j]..C[q-1]$ to $A[k]..A[p+q-1]$ $\triangleright$ (a for loop)
    else
        copy $B[i]..B[p-1]$ to $A[k]..A[p+q-1]$ $\triangleright$ (a for loop)

Mergesort: Merging Arrays

B:

2	3	8	9
0	1	2	3

i

C:

1	4	5	7
0	1	2	3

j

p: 4

q: 4

A:

0	1	2	3	4	5	6	7

k

Mergesort: Merging Arrays

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B:

0	1	2	3
2	3	8	9

i

C:

0	1	2	3
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p: 4

q: 4

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1							

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Mergesort: Merging Arrays

B:

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0	1	2	3

i (at index 1)

C:

1	4	5	7
0	1	2	3

j (at index 1)

p: 4

q: 4

A:

1	2						
0	1	2	3	4	5	6	7

k (at index 2)

Mergesort: Merging Arrays

B:

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0	1	2	3
	i		

C:

1	4	5	7
0	1	2	3
	j		

p: 4

q: 4

A:

1	2	3					
0	1	2	3	4	5	6	7
		k					

Mergesort: Merging Arrays

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Mergesort: Merging Arrays

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q: 4

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1	2	3	4				
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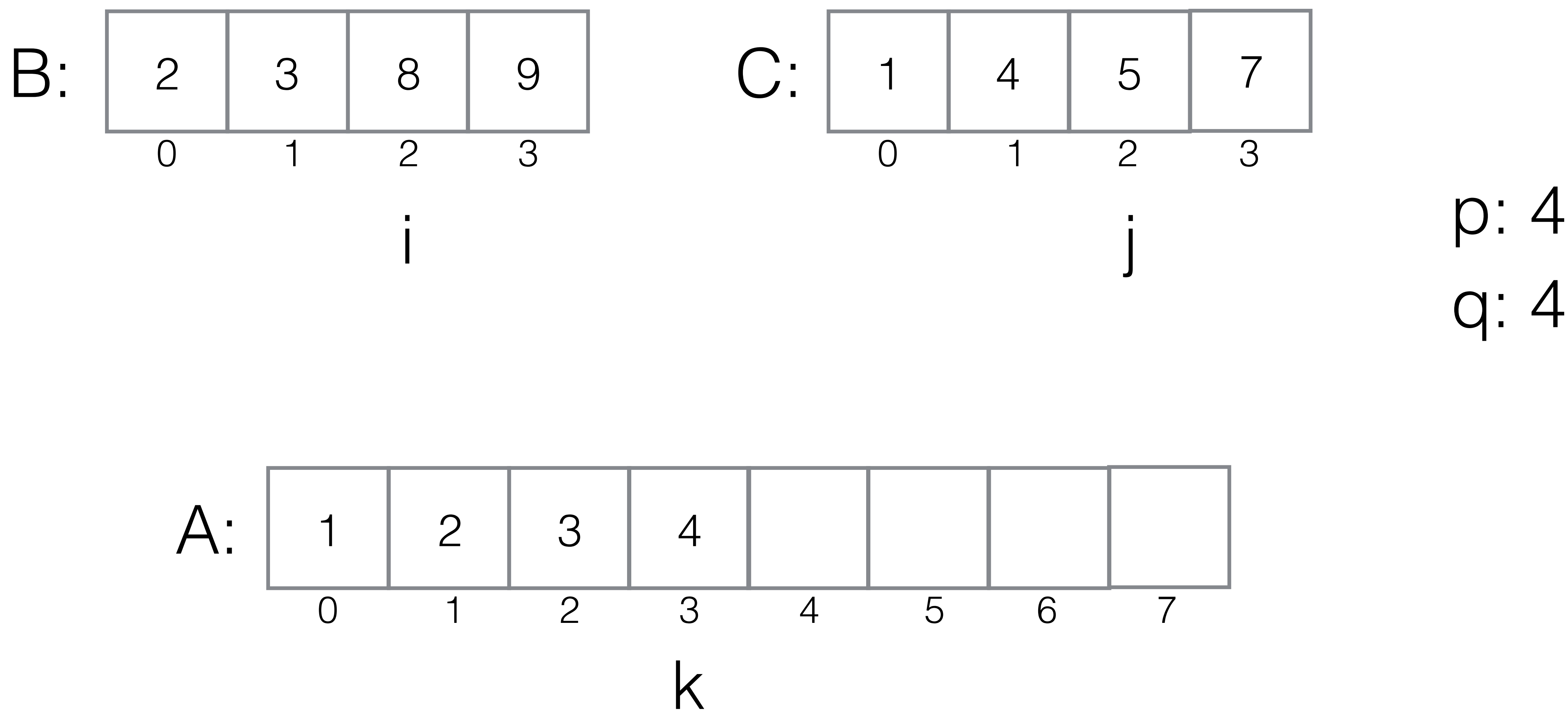
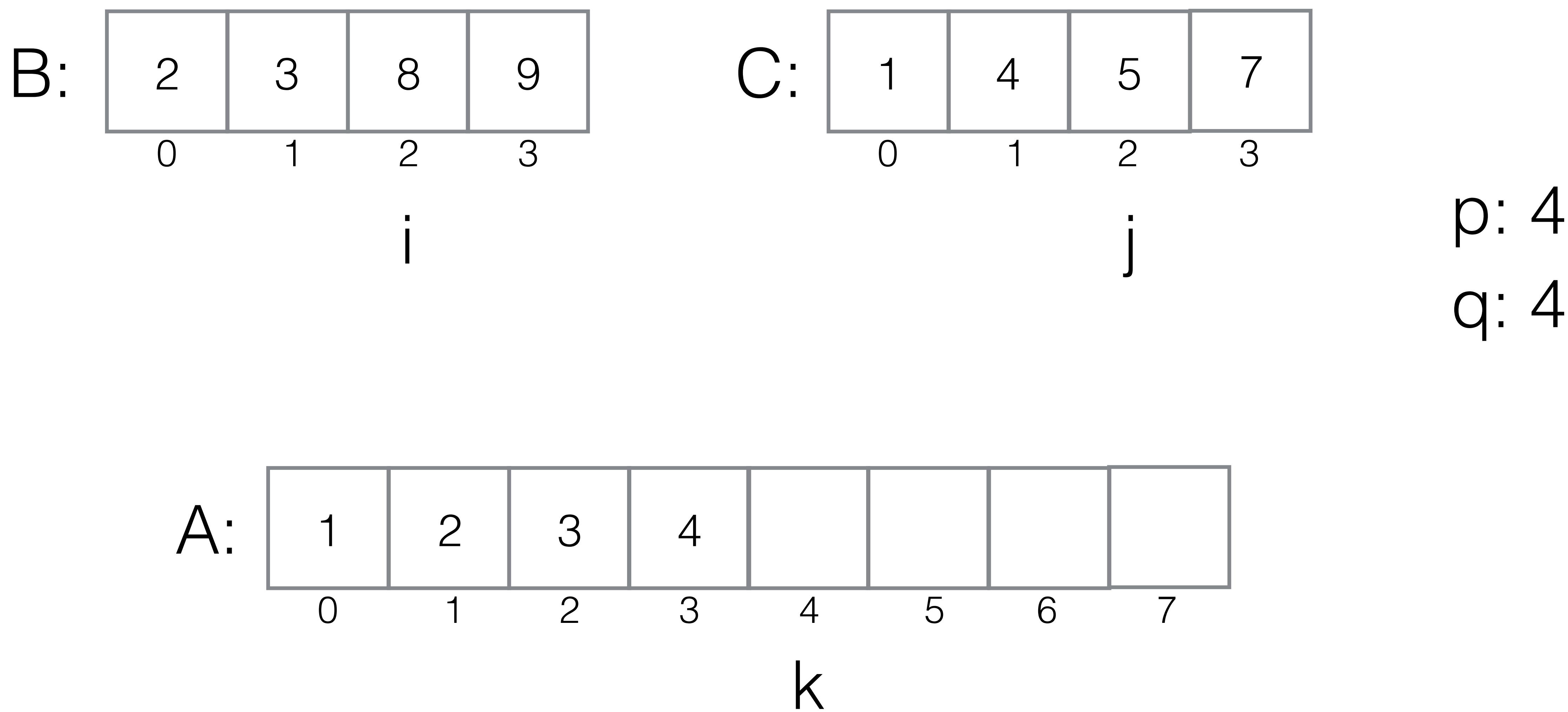
k

Mergesort: Merging Arrays



Mergesort: Merging Arrays



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Mergesort: Merging Arrays

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        if $B[i] \leq C[j]$ **then**
            $A[k] \leftarrow B[i]$
            $i \leftarrow i + 1$
        else
            $A[k] \leftarrow C[j]$
            $j \leftarrow j + 1$
        $k \leftarrow k + 1$
    if $i = p$ **then**
        copy $C[j]..C[q-1]$ to $A[k]..A[p+q-1]$ $\triangleright$ (a for loop)
    else
        copy $B[i]..B[p-1]$ to $A[k]..A[p+q-1]$ $\triangleright$ (a for loop)

Mergesort: Merging Arrays

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1	2	3	4	5	7	8	9
0	1	2	3	4	5	6	7

k

Mergesort: Analysis

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- How many comparisons will MERGE need to make in the worst case, when given arrays of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$?

- If the largest and second-largest elements are in different arrays, then n – 1 comparisons. Hence the cost equation for Mergesort is

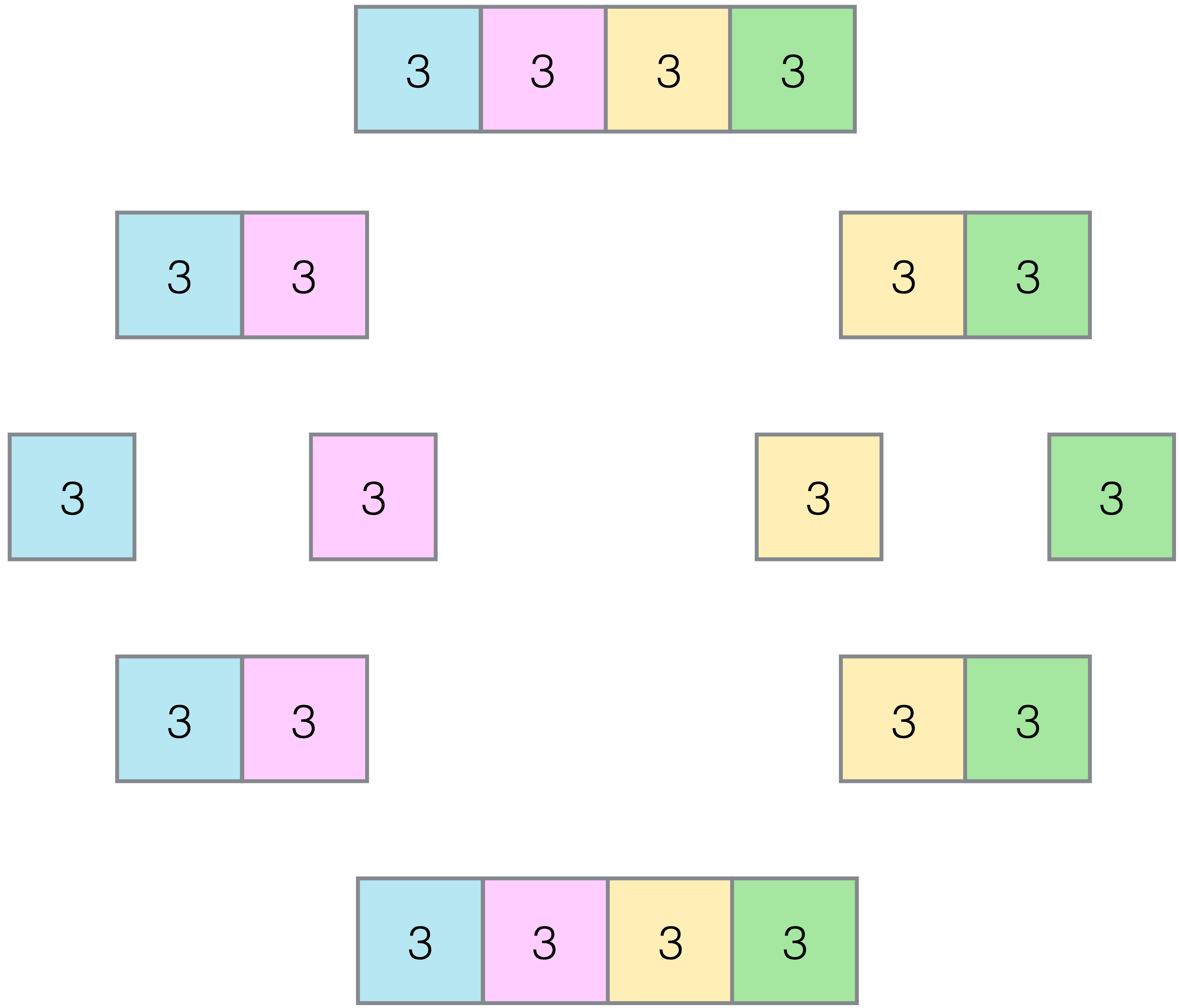
$$C(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C(n/2) + n - 1 & \text{otherwise} \end{cases}$$

- By the Master Theorem, $C(n) \in \Theta(n \log n)$.

Mergesort: Properties

- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.
- Is mergesort stable? ?
- Is mergesort in-place? no
- If comparisons are fast, mergesort ranks between quicksort and heapsort (covered next week) for time, assuming random data.
- Mergesort is the method of choice for linked lists and for very large collections of data.

Mergesort: Stability



Mergesort: Properties

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- For large n, the number of comparisons made tends to be around 75% of the worst-case scenario.

- Is mergesort stable? yes

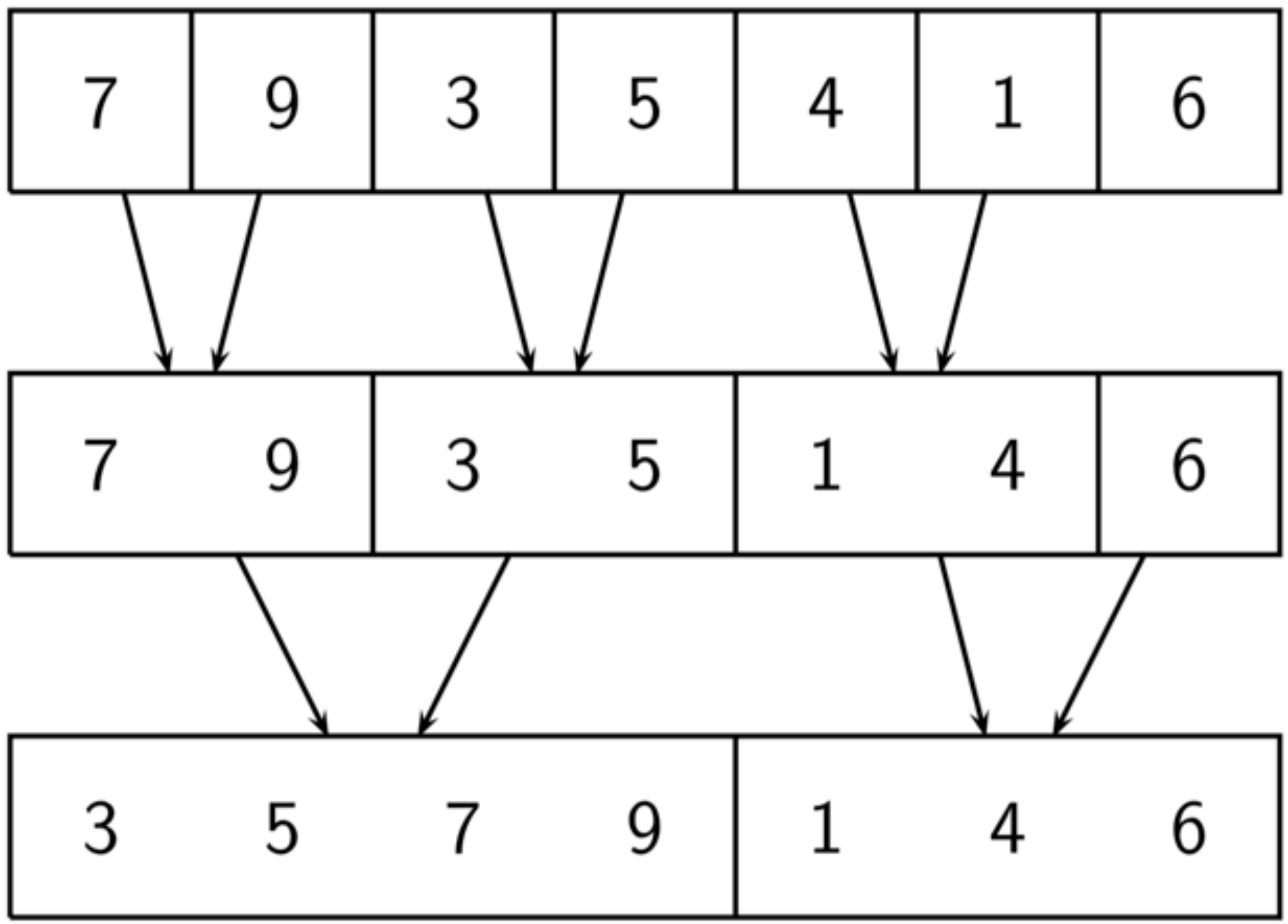
- Is mergesort in-place? no

- If comparisons are fast, mergesort ranks between quicksort and heapsort (covered next week) for time, assuming random data.

- Mergesort is the method of choice for linked lists and for very large collections of data.

Bottom-Up Mergesort

- An alternative way of doing mergesort:
- Generate **runs** of length 2, then of length 4, and so on:



Quicksort

- Quicksort takes a divide-and-conquer approach that is different to mergesort's.

- It uses the **partitioning** idea from QuickSelect, picking a pivot element, and partitioning the array around that, so as to obtain this situation:

$$\underbrace{A[0] \dots A[s-1]}_{\text{all are } \leq A[s]} \quad A[s] \quad \underbrace{A[s+1] \dots A[n-1]}_{\text{all are } \geq A[s]}$$

- The element A[s] will be in its final position (it is the (s + 1)th smallest element).

- All that then needs to be done is to sort the segment to the left, recursively, as well as the segment to the right.

Quicksort

- Very short and elegant:

$$
\begin{array}{l}
\textbf{procedure } \textsc{Quicksort}(A[\cdot], lo, hi) \\
\quad \textbf{if } lo < hi \textbf{ then} \\
\qquad s \leftarrow \textsc{Partition}(A, lo, hi) \\
\qquad \textsc{Quicksort}(A, lo, s-1) \\
\qquad \textsc{Quicksort}(A, s+1, hi)
\end{array}
$$

- Initial call: Quicksort(A, 0, n – 1).

Quicksort: Example

procedure $\text{QUICKSORT}(A[\cdot], lo, hi)$
    if $lo < hi$ **then**
        $s \leftarrow \text{PARTITION}(A, lo, hi)$
        $\text{QUICKSORT}(A, lo, s-1)$
        $\text{QUICKSORT}(A, s+1, hi)$

A:

9	23	8	41	22	3	37
0	1	2	3	4	5	6

Quicksort: Example

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procedure $\textsc{Quicksort}(A[\cdot], lo, hi)$
    if $lo < hi$ **then**
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Quicksort: Example

procedure QUICKSORT($A[\cdot], lo, hi$)
  if $lo < hi$ **then**
    $s \leftarrow$ PARTITION(A, lo, hi)
    QUICKSORT($A, lo, s - 1$)
    QUICKSORT($A, s + 1, hi$)

A:

3	8	9	41	22	23	37
0	1	2	3	4	5	6

Quicksort: Example

procedure $\textsc{Quicksort}(A[\cdot], lo, hi)$
  if $lo < hi$ **then**
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	0	1	2	3	4	5	6

Quicksort: Example

```
procedure Quicksort(A[·], lo, hi)
    if lo < hi then
        s ← Partition(A, lo, hi)
        Quicksort(A, lo, s − 1)
        Quicksort(A, s + 1, hi)
```

A:

3	8	9	22	23	37	41
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Quicksort: Example

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procedure $\textsc{Quicksort}(A[\cdot], lo, hi)$
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Hoare Partitioning

- The standard way of doing partitioning in Quicksort

function $\textsc{Partition}(A[\cdot], lo, hi)$
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
    repeat
        while $i < hi$ and $A[i] \leq p$ **do** $i \leftarrow i + 1$
        while $j \geq lo$ and $A[j] > p$ **do** $j \leftarrow j - 1$
        $swap(A[i], A[j])$
    until $i \geq j$
    $swap(A[i], A[j])$ ▷ Undo the last swap
    $swap(A[lo], A[j])$ ▷ Bring pivot to its correct position
    return j

Hoare Partitioning

function PARTITION($A[\cdot]$, lo, hi)
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
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i						j

p: 9

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0	1	2	3	4	5	6

j (at index 2) i (at index 3)

p: 9

Hoare Partitioning

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```
function PARTITION(A[·], lo, hi)
    p ← A[lo]; i ← lo; j ← hi
    repeat
        while i < hi and A[i] ≤ p do i ← i + 1
        while j ≥ lo and A[j] > p do j ← j − 1
        swap(A[i], A[j])
    until i ≥ j
    swap(A[i], A[j])
    swap(A[lo], A[j])
    return j
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p: 9

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  return j

A:

9	3	8	41	22	23	37
0	1	2	3	4	5	6
		j	i			

p: 9

Hoare Partitioning

function $\text{PARTITION}(A[\cdot], lo, hi)$
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi$
    repeat
        while $i < hi$ and $A[i] \leq p$ **do** $i \leftarrow i + 1$
        while $j \geq lo$ and $A[j] > p$ **do** $j \leftarrow j - 1$
        $swap(A[i], A[j])$
    until $i \geq j$
    $swap(A[i], A[j])$
    $swap(A[lo], A[j])$
    return j

A:

	8	3	9	41	22	23	37
	0	1	2	3	4	5	6
			j	i			

p: 9

Quicksort Analysis: Best Case Analysis

- The best case happens when the pivot is the median; that results in two sub-tasks of equal size.

$$C_{best}(n) = \begin{cases} 0 & \text{if } n < 2 \\ 2C_{best}(n/2) + n & \text{otherwise} \end{cases}$$

The 'n' is for the n key comparisons performed by Partition.

- By the Master Theorem, $C_{best}(n) \in \Theta(n \log n)$, just as for mergesort, so quicksort's best case is (asymptotically) no better than mergesort's worst case.

Quicksort Worst Case

A:

Quicksort Analysis: Worst Case Analysis

- The worst case happens if the array is already sorted.

- In that case, we don't really have divide-and-conquer, because each recursive call deals with a problem size that has only been decremented by 1:

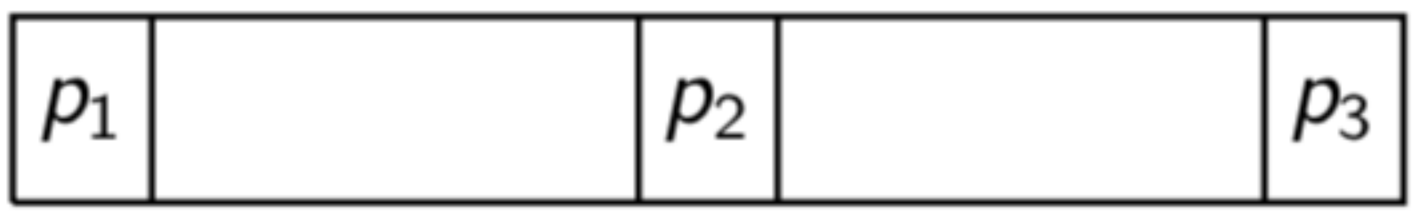
$$C_{worst}(n) = \begin{cases} 0 & \text{if } n < 2 \\ C_{worst}(n-1) + n & \text{otherwise} \end{cases}$$

- That is, $C_{worst}(n) = n + (n - 1) + \cdots + 3 + 2 \in \Theta(n^2)$.

Quicksort Improvements: Median-of-Three

- It would be better if the pivot was chosen randomly.

- A cheap and useful approximation to this is to take the median of three candidates, A[lo], A[hi], and A[$\lfloor$(lo + hi)/2$\rfloor$].



- Reorganise the three elements so that p_1 is the median, and p_3 is the largest of the three.

- Now run quicksort as before.

Quicksort Improvements: Median-of-Three

- In fact, with median-of-three, we can have a much faster version than before, simplifying tests in the innermost loops:

function PARTITION($A[\cdot], lo, hi$)
    $p \leftarrow A[lo]$; $i \leftarrow lo$; $j \leftarrow hi + 1$
    repeat
        ~~**while** $i < hi$ **and** $A[i] \leq p$ **do** $i \leftarrow i + 1$~~
        repeat $i \leftarrow i + 1$ **until** $A[i] \geq p$
        ~~**while** $j \geq lo$ **and** $A[j] > p$ **do** $j \leftarrow j - 1$~~
        repeat $j \leftarrow j - 1$ **until** $A[j] \leq p$
        $swap(A[i], A[j])$
    until $i \geq j$
    $swap(A[i], A[j])$
    $swap(A[lo], A[j])$
    return j

Quicksort Improvements: Early Cut-Off

- A second useful improvement is to stop quicksort early and switch to insertion sort. This is easily implemented:

$$
\begin{array}{l}
\textbf{procedure } \textsc{Sort}(A[\cdot], n) \\
\quad \textsc{QuickAlmostSort}(A, 0, n-1) \\
\quad \textsc{InsertionSort}(A, n)
\end{array}
$$

$$
\begin{array}{l}
\textbf{procedure } \textsc{QuickAlmostSort}(A[\cdot], lo, hi) \\
\quad \textbf{if } lo + 10 < hi \textbf{ then} \\
\quad\quad s \leftarrow \textsc{Partition}(A, lo, hi) \\
\quad\quad \textsc{QuickAlmostSort}(A, lo, s-1) \\
\quad\quad \textsc{QuickAlmostSort}(A, s+1, hi)
\end{array}
$$

Quicksort Properties

- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.
- A major reason for its speed is the very tight inner loop in PARTITION.
- Although mergesort has a better performance guarantee, quicksort is faster on average.
- In the best case, we get $\lceil \log 2\ n \rceil$ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.
- Is quicksort stable? ?
- Is it in-place? yes

Quicksort Stability

2	2	1

i   j

Quicksort Stability

2	2	1
	i	j

Quicksort Stability

2	2	1

j

i

Quicksort Stability

1	2	2
		j
		i

Quicksort Stability

1	2	2

Quicksort Stability

1	2	2

This is where we finished

2	2	1

This is where we started

Not stable

Quicksort Properties

- With these (and other) improvements, quicksort is considered the best available sorting method for arrays of random data.
- A major reason for its speed is the very tight inner loop in PARTITION.
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- In the best case, we get $\lceil \log 2\ n \rceil$ recursive levels. It can be shown that on random data, the expected number is $2 \log_e n \approx 1.38 \log_2 n$. So quicksort's average behaviour is very close to the best-case behaviour.
- Is quicksort stable? no
- Is it in-place? yes

Next up

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- Tree traversal methods, plus we apply the divide-and-conquer technique to the closest-pair problem.