COMP90038
Algorithms and Complexity

Lecture 15: Balanced Trees

(with thanks to Harald Søndergaard & Michael Kirley)

Andres Munoz-Acosta

munoz.m@unimelb.edu.au

Peter Hall Building G.83

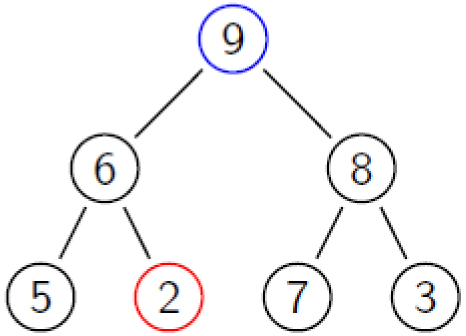
Recap

- Last week we talked about:
  - Two representations: Heaps and Binary Search Trees
  - An algorithm: Heapsort
  - An strategy: Transform-and-conquer through pre-sorting

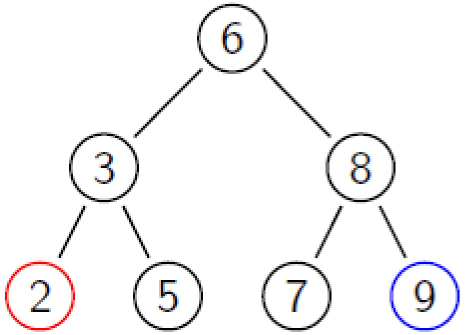
Differences between heaps and BSTs

- We have the array [2 3 5 6 7 8 9]

- As a heap:
  - **Each child has a priority (key) which is no greater than its parent's.** This guarantees that the root of the tree is a maximal element.
  - It must be a complete tree (filled top to bottom, left to right)
  - There are many valid heaps!!!



- As a BST:
  - Let the root be *r*; then each element in the **left subtree is smaller** than *r* and each element in the **right sub-tree is larger** than *r*.
  - A BST is never a heap!!!



Heapsort and Pre-sorting

- Heapsort:
  - Uses the fact that the root of a heap is always the maximal element.
  - It iterates the sequence: Build the heap – eject the root – build the heap – eject the root …
- Pre-sorting
  - Simplify the problem (through sorting the data) such that an efficient algorithm can be used.

Finding anagrams using pre-sorting

- You are given a very long list of words:

{health, revolution, foolish, garner, drive, praise, traverse, anger, ranger, … scoop, fall, praise}

- Find the anagrams in the list.

- An approach is to sort each word, sort the list of words, and then find the repeats…

Exercise: Finding Anagrams

health	aehhlt	aeerstv	1
revolution	eilnoortvu	aegnr	1
foolish	fhiloos	aegnrr	1
garner	aegnrr	aegnrr	**2** (This element is an anagram)
drive	deirv	aehhlt	1
praise	aeiprs	aeiprs	1
traverse	aeerstv	afll	1
anger	aegnr	coops	1
ranger	aegnrr	deirv	1
…	…	…	…
scoop	coops	eilnoortvu	1
fall	afll	fhiloos	1
truly	lrtuy	lrtuy	1

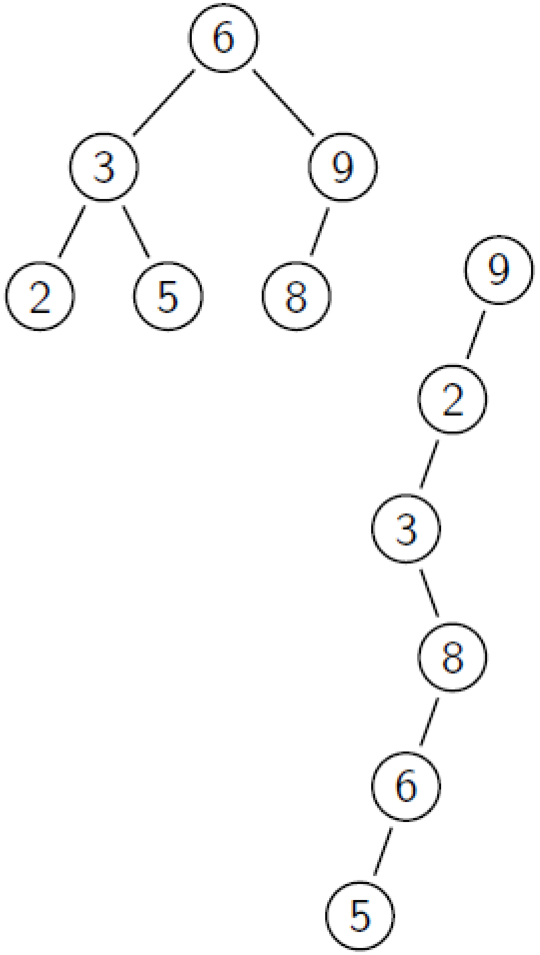
Sort each word Sort the list Find repeats

Approaches to Balanced Binary Search Trees

- If a BST is "reasonably" balanced, search involves $\Theta(\log n)$ comparisons in the worst case.

- If the BST is “unbalanced”, search could be linear.

- To optimise performance, it is important to keep trees “reasonably” balanced.



Approaches to Balanced Binary Search Trees

- Instance simplification approaches: Self-balancing trees
  - **AVL trees**
  - Red-black trees
  - Splay trees
- Representational changes:
  - **2–3 trees**
  - 2–3–4 trees
  - B-trees

AVL Trees

- Named after Adelson-Velsky and Landis.

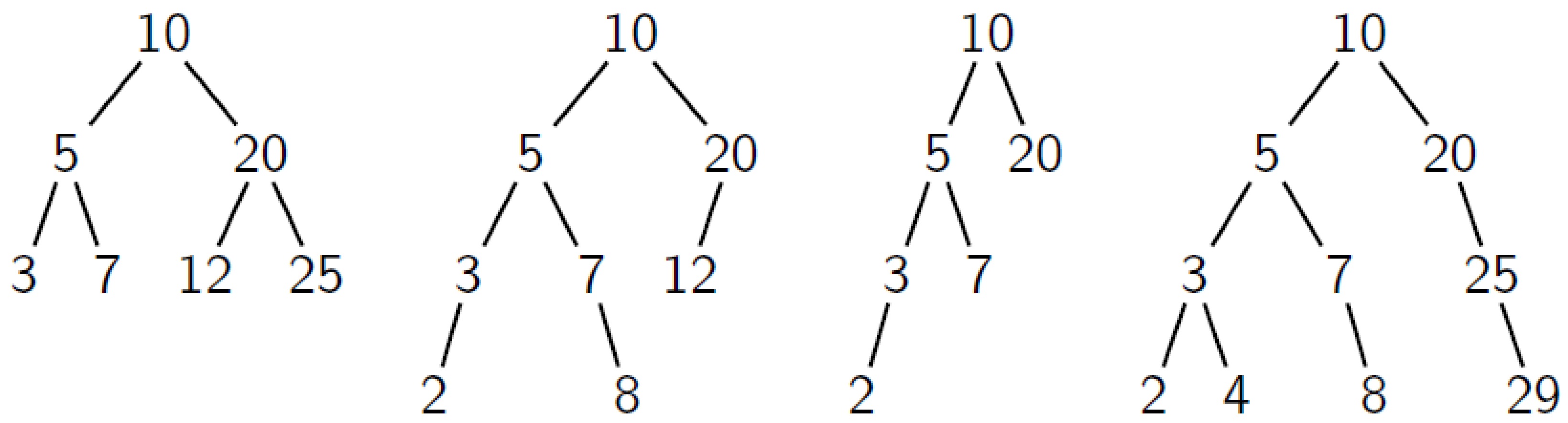
- Recall that we defined the height of the empty tree as -1.

- For a binary (sub-) tree, let the **balance factor** be the difference between the height of its left sub-tree and that of its right sub-tree.

- An **AVL tree** is a BST in which the balance factor is -1, 0, or 1, for every sub-tree.

AVL Trees: Examples and Counter-Examples

- Which of these are AVL trees?



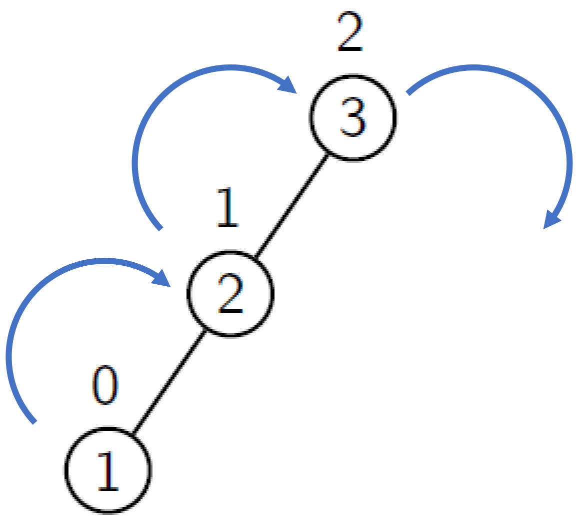
Building an AVL Tree

- As with standard BSTs, insertion of a new node always takes place at the fringe of the tree.

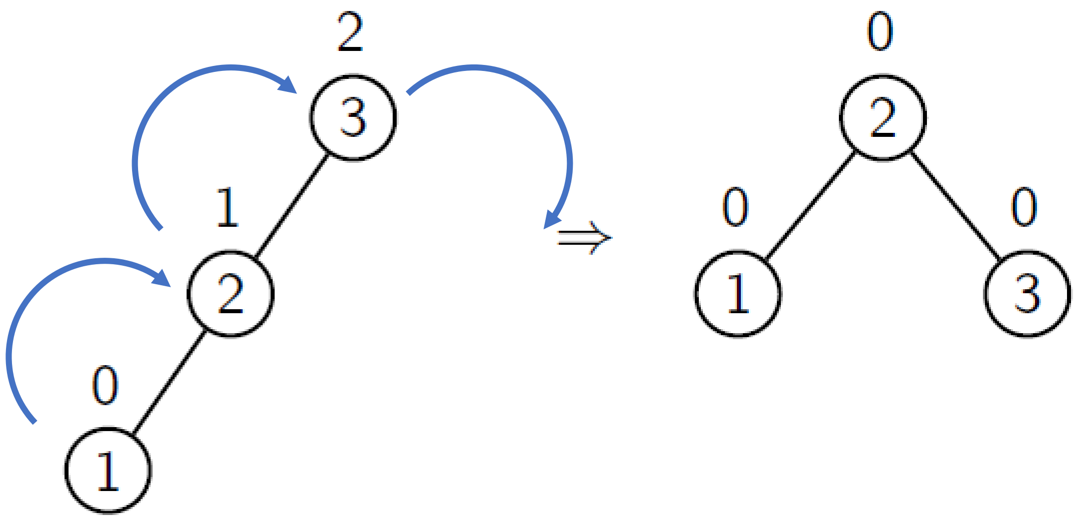
- If insertion of the new node makes the AVL tree unbalanced (some nodes get balance factors of 2 or -2), transform the tree to regain its balance.

- Regaining balance can be achieved with one or two simple, local transformations, so-called **rotations**.

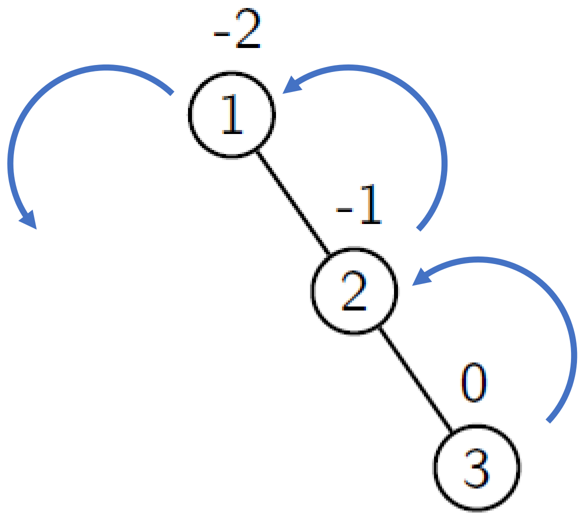
AVL Trees: R-Rotation



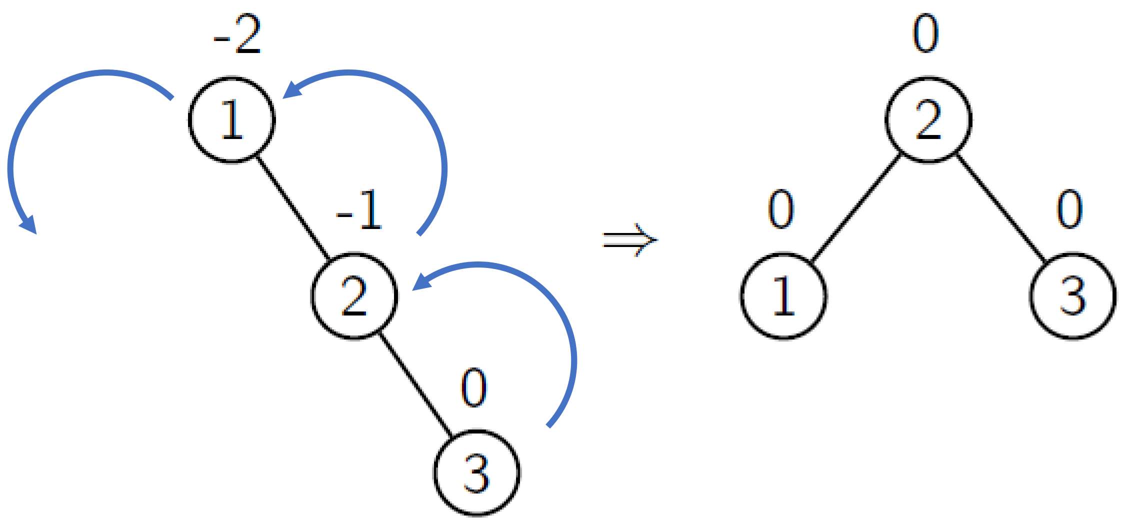
AVL Trees: R-Rotation



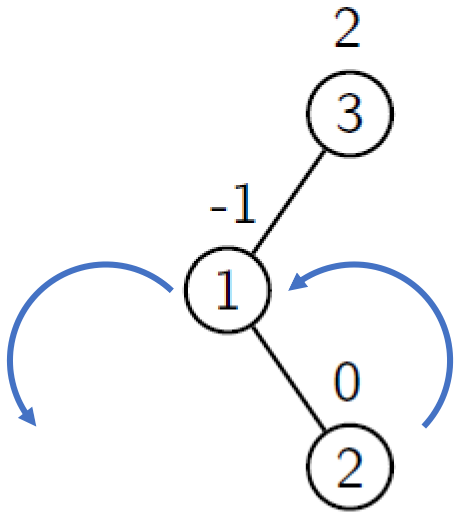
AVL Trees: L-Rotation



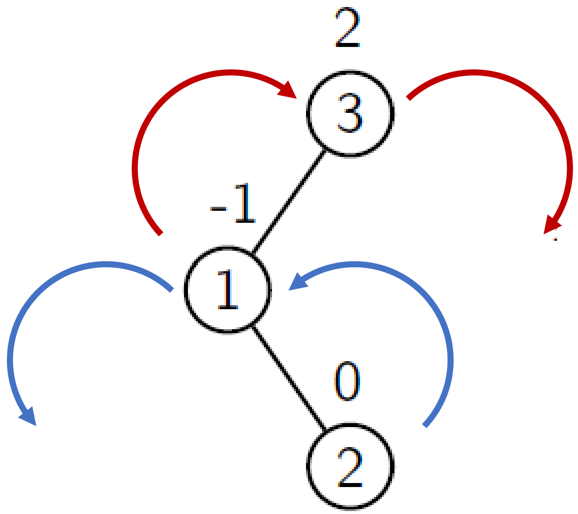
AVL Trees: L-Rotation



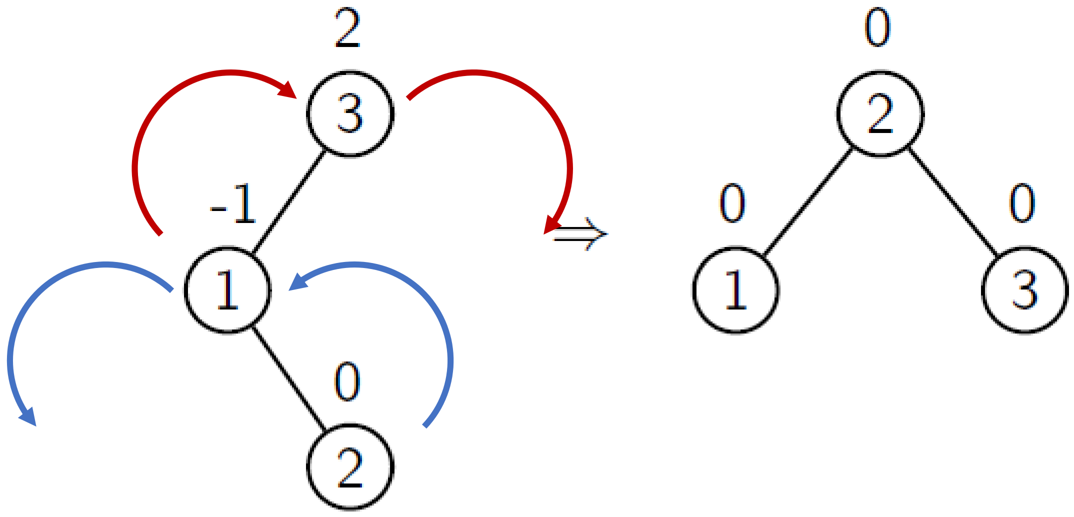
AVL Trees: LR-Rotation



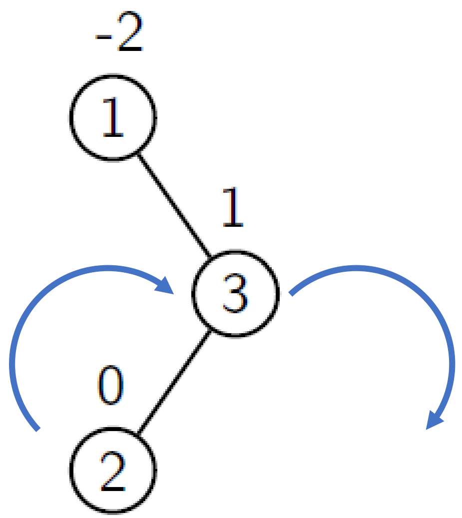
AVL Trees: LR-Rotation



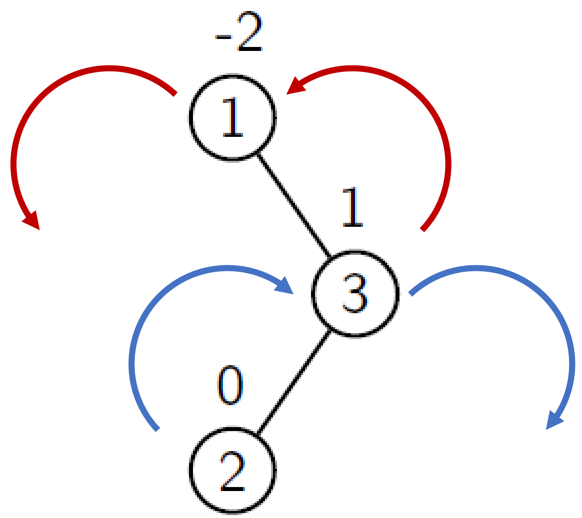
AVL Trees: LR-Rotation



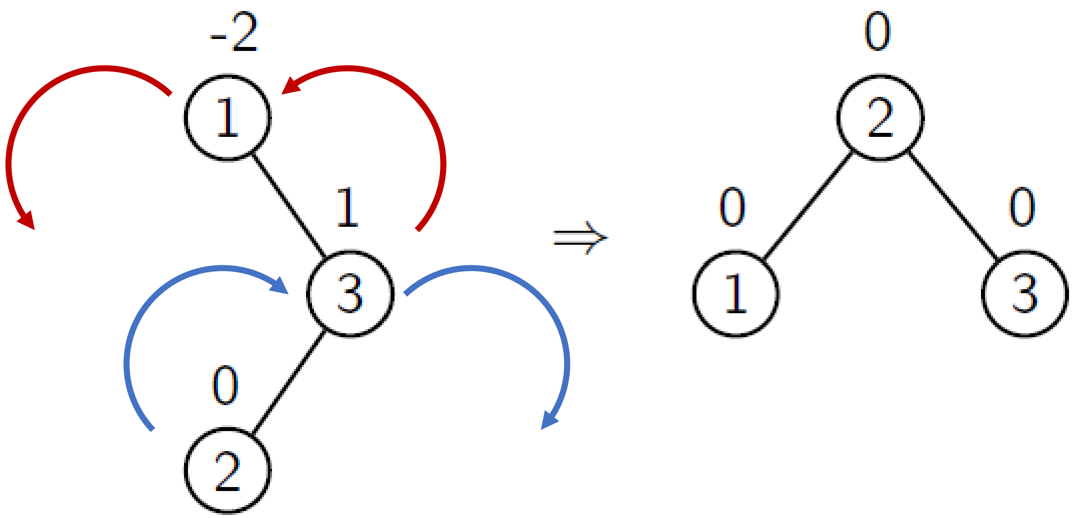
AVL Trees: RL-Rotation



AVL Trees: RL-Rotation

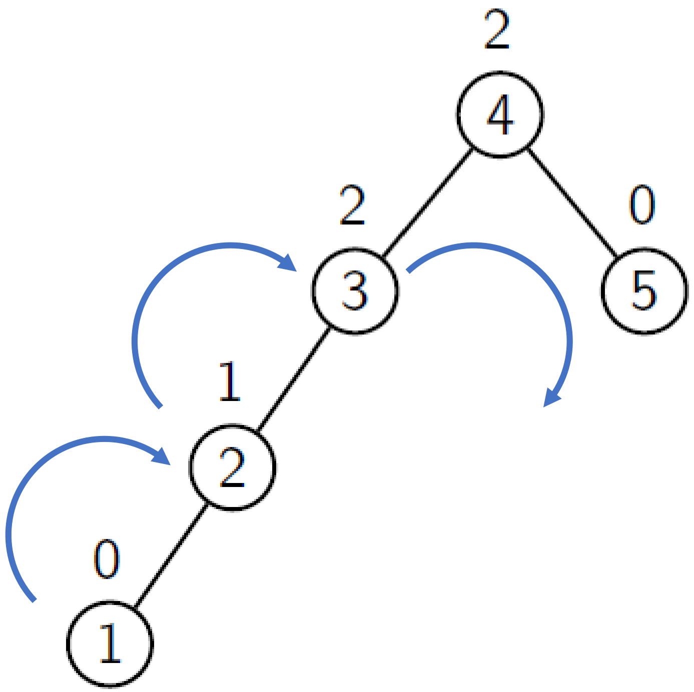


AVL Trees: RL-Rotation



AVL Trees: Where to Perform the Rotation

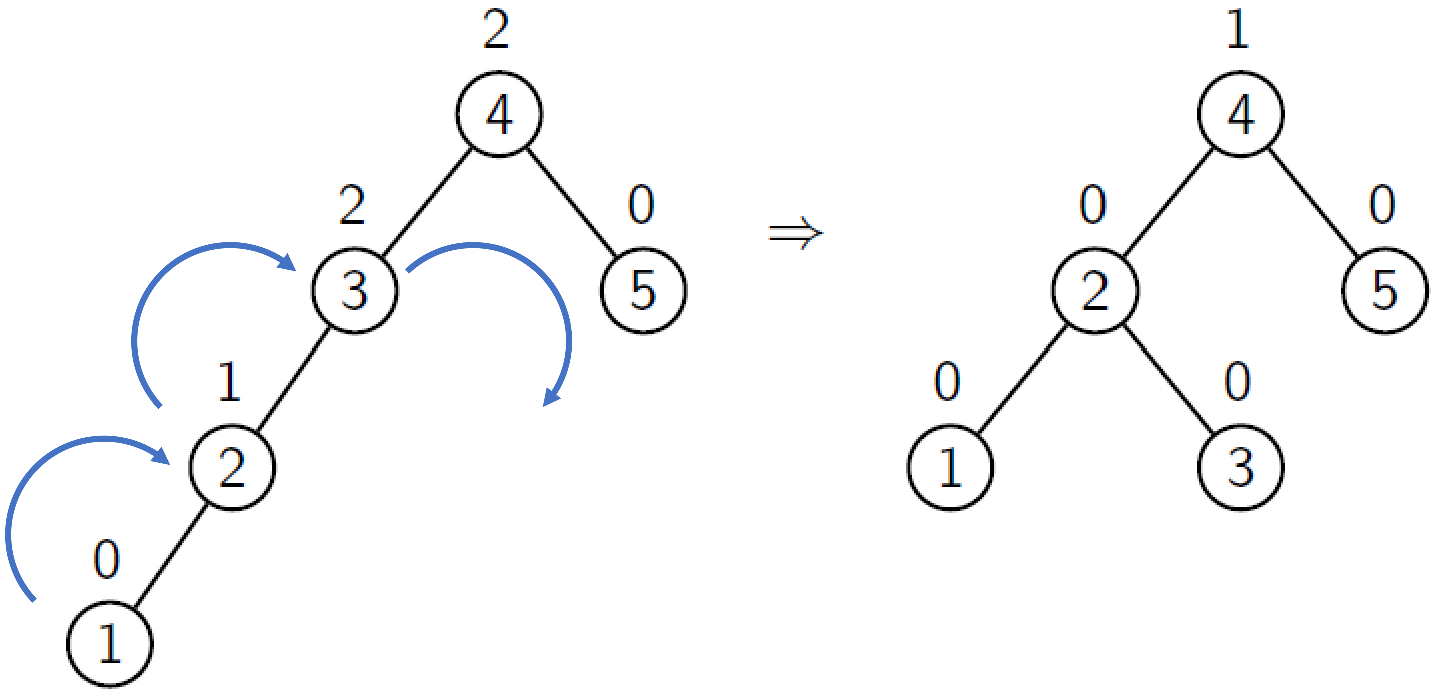
- Along an unbalanced path, we may have several nodes with balance factor 2 (or -2):



- It is always the **lowest** unbalanced subtree that is re-balanced.

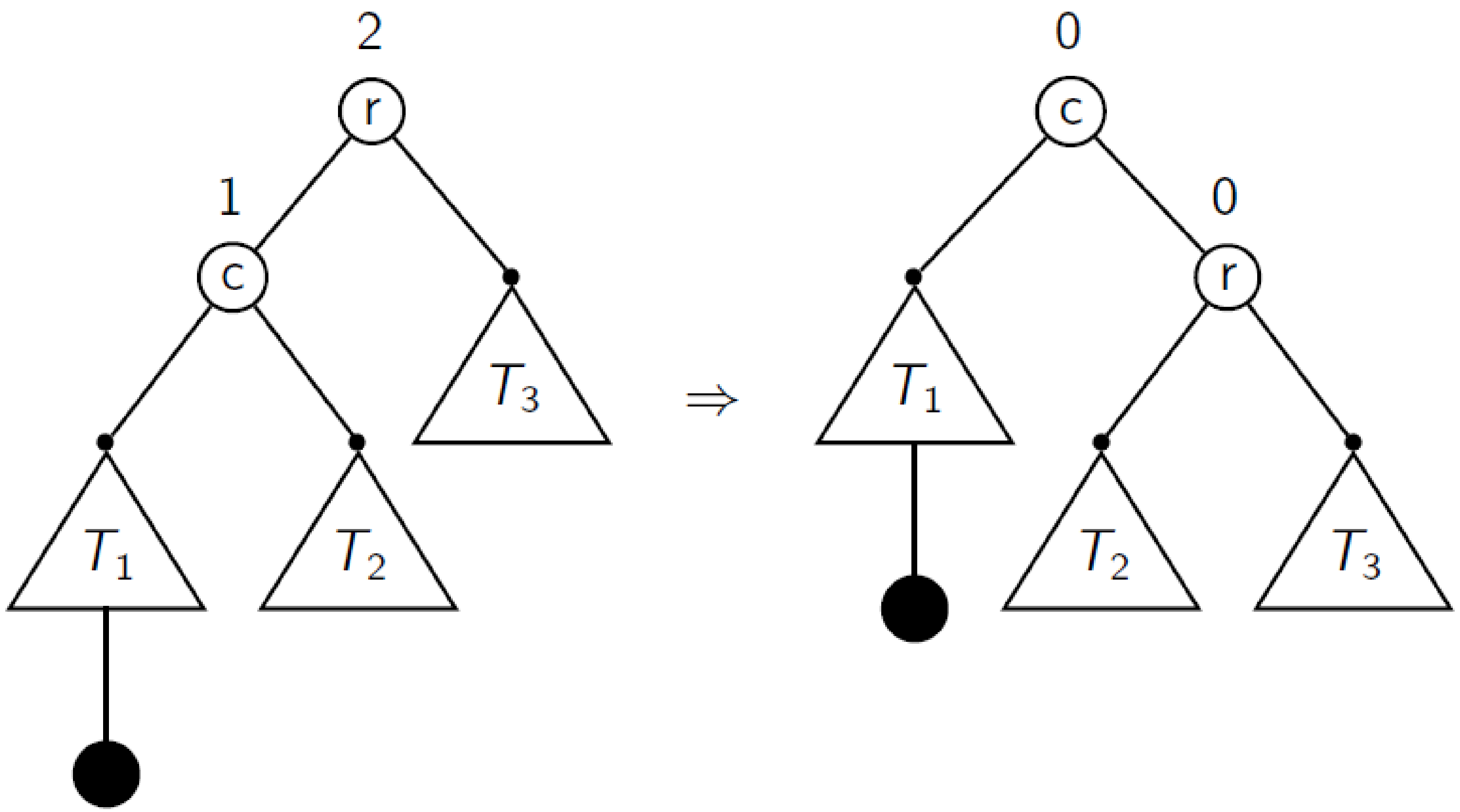
AVL Trees: Where to Perform the Rotation

- Along an unbalanced path, we may have several nodes with balance factor 2 (or -2):



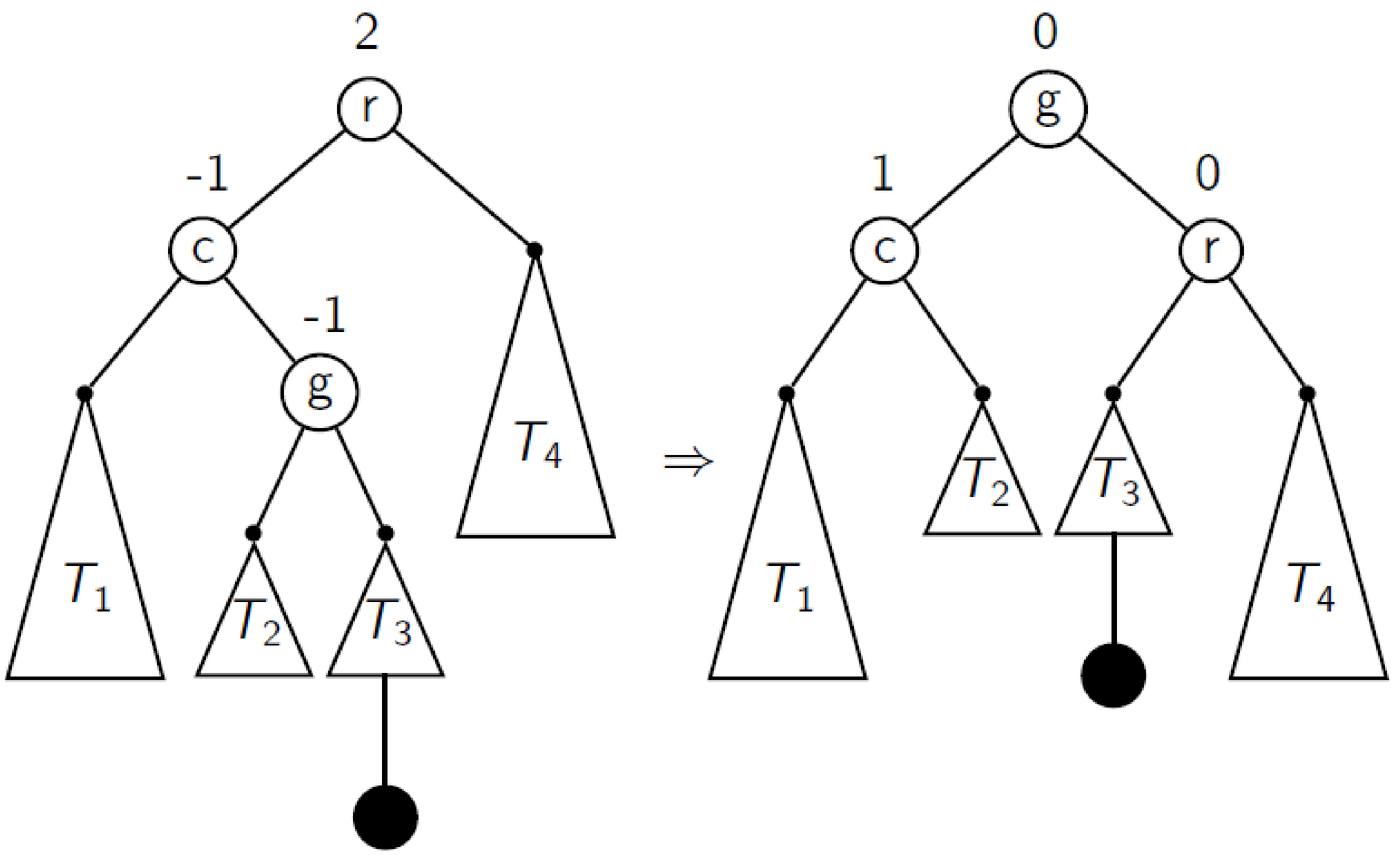
- It is always the **lowest** unbalanced subtree that is re-balanced.

AVL Trees: The Single Rotation, Generally



- This shows an **R-rotation**; an **L-rotation** is similar.

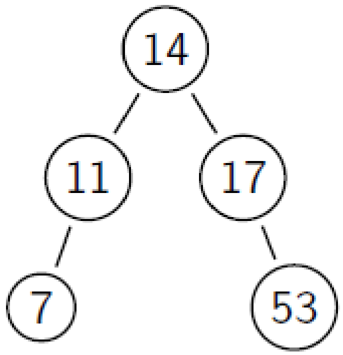
AVL Trees: The Double Rotation, Generally



- This shows an **LR-rotation**; an **RL-rotation** is similar.

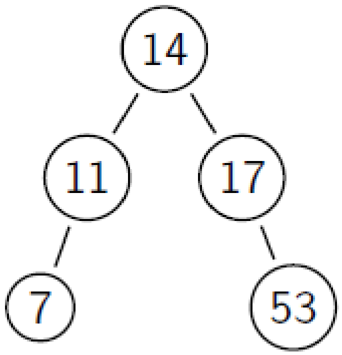
Example

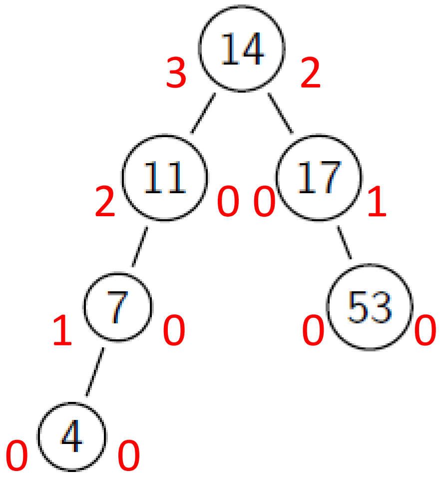
- On the tree below, insert the elements {4, 13, 12}

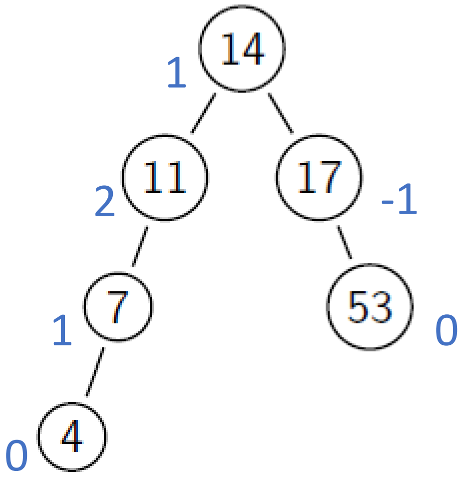


Example

- On the tree below, insert the elements {4, 13, 12}

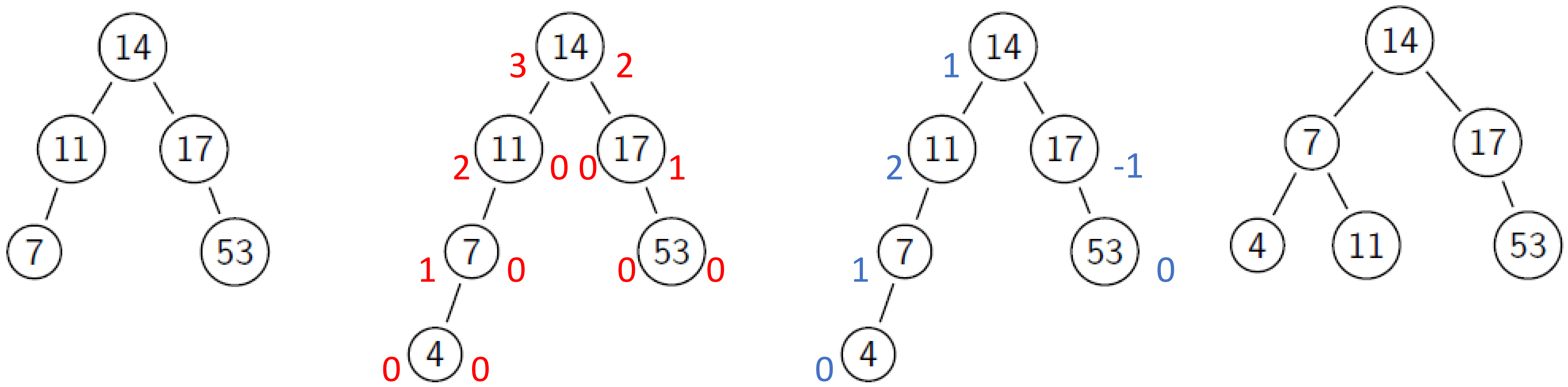






Example

- On the tree below, insert the elements {4, 13, 12}

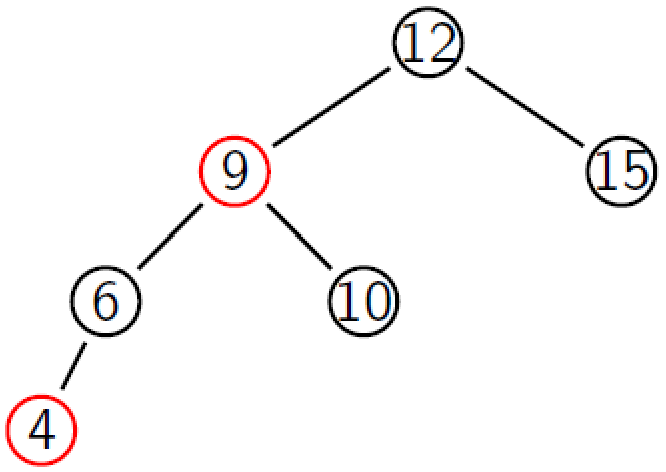


Properties of AVL Trees

- The notion of "balance" that is implied by the AVL condition is sufficient to guarantee that the depth of an AVL tree with n nodes is $\Theta(\log n)$.
- For random data, the depth is very close to $\log_2 n$, the optimum.
- In the worst case, search will need at most 45% more comparisons than with a perfectly balanced BST.
- **Deletion** is harder to implement than insertion, but also $\Theta(\log n)$.

Other Kinds of Balanced Trees

- A **red-black tree** is a BSTs with a slightly different concept of "balanced". Its nodes are coloured red or black, so that
  - No red node has a red child.
  - Every path from the root to the fringe has the same number of black nodes.



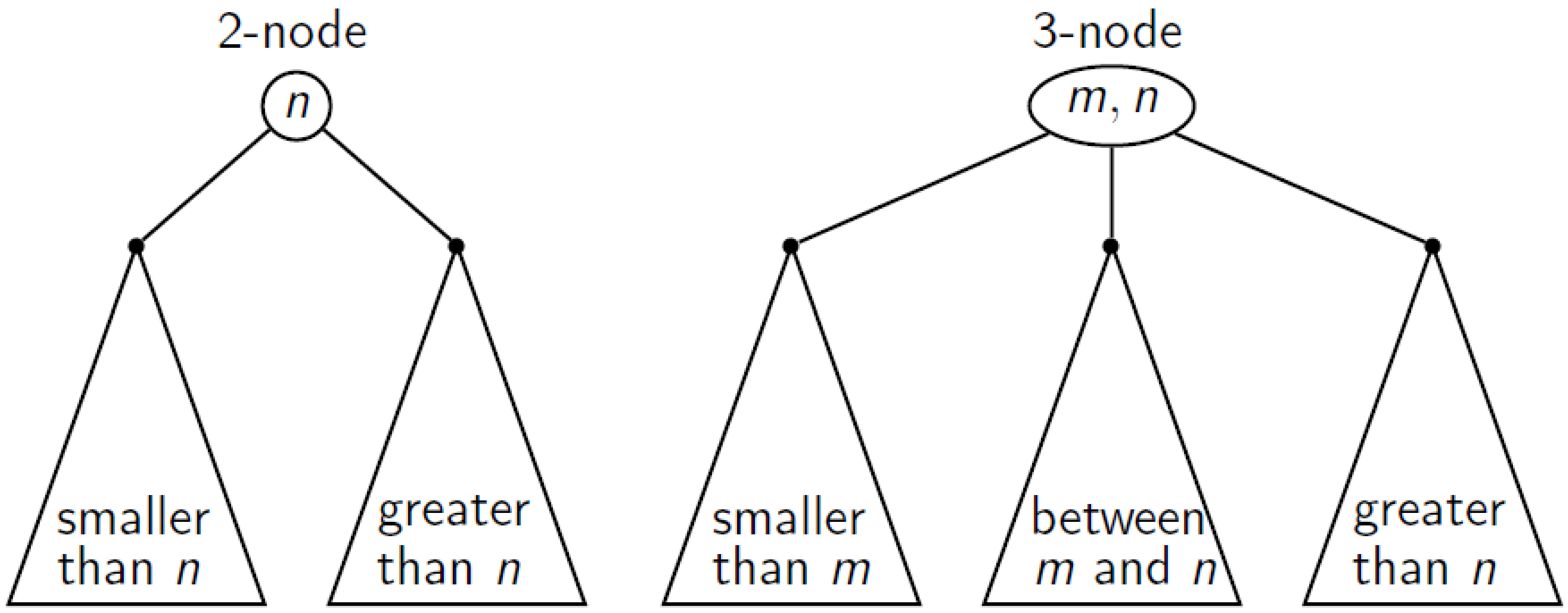
A worst-case red-black tree (the longest path is twice as long as the shortest path).

- A **splay tree** is a BST which is not only self-adjusting, but also **adaptive**. Frequently accessed items are brought closer to the root, so their access becomes cheaper.

2–3 Trees

- 2–3 trees and 2–3–4 trees are search trees that allow more than one item to be stored in a tree node.
- A node that holds a single item has two children (or none, if it is a leaf).
- A node that holds two items (a so-called **3-node**) has three children (or none, if it is a leaf).
- And for 2–3–4 trees, a node that holds three items (a **4-node**) has four children (or none, if it is a leaf).
- This allows for a simple way of keeping search trees **perfectly** balanced.

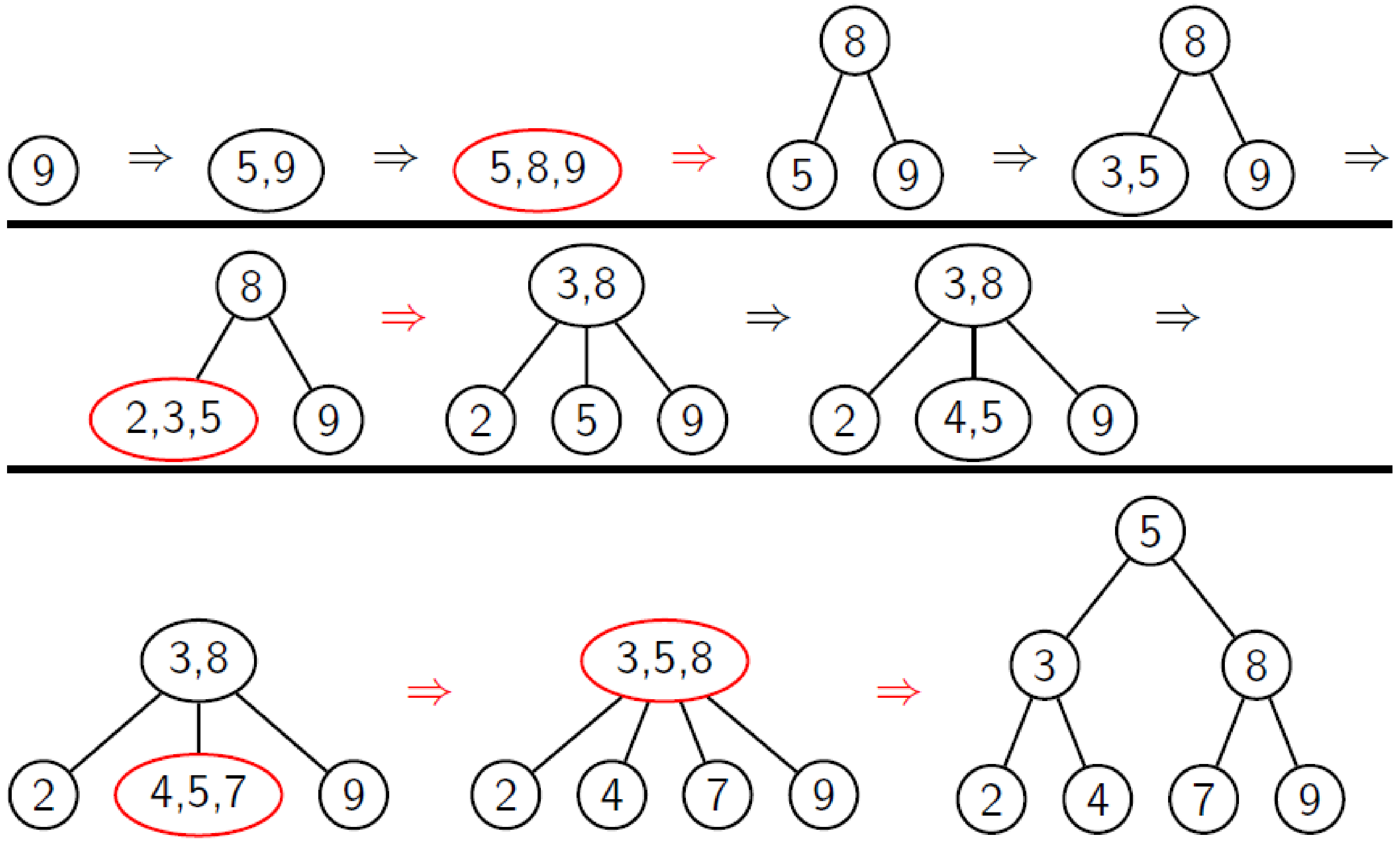
2-Nodes and 3-Nodes



Insertion in a 2–3 Tree

- To insert a key *k*, pretend that we are searching for *k*.
- This will take us to a leaf node in the tree, where *k* should now be inserted; if the node we find there is a 2-node, *k* can be inserted without further ado.
- Otherwise we had a 3-node, and the two inhabitants, together with *k*, momentarily form a node with three elements; in sorted order, call them k_1, k_2, and k_3.
- We now **split** the node, so that k_1 and k_3 form their own individual 2-nodes. The middle key, k_2 is **promoted** to the parent node.
- The promotion may cause the parent node to overflow, in which case **it** gets split the same way. The only time the tree's height changes is when the root overflows.

Example: Build a 2–3 Tree from {9, 5, 8, 3, 2, 4, 7}



Exercise: 2–3 Tree Construction

- Build the 2–3 tree that results from inserting these keys, in the given order, into an initially empty tree:

  C, O, M, P, U, T, I, N, G

2–3 Tree Analysis

- Worst case search time results when all nodes are 2-nodes. The relation between the number n of nodes and the height h is:

$$n = 1 + 2 + 4 + \ldots + 2^h = 2^{h+1} - 1$$

- That is, $\log_2(n+1) = h+1$.

- In the best case, all nodes are 3-nodes:

$$n = 2 + 2\times 3 + 2\times 3^2 + \ldots + 2\times 3^h = 3^{h+1} - 1$$

- That is, $\log_3(n+1) = h+1$.

- Hence we have $\log_3(n+1) - 1 \le h \le \log_2(n+1) - 1$.

- Useful formula: $\displaystyle\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \text{ for } a \neq 1$

Next lecture

- How to buy time, by spending a bit of space.