COMP90038
Algorithms and Complexity

Lecture 23: Revision

(with thanks to Harald Søndergaard & Michael Kirley)

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What is examinable?

- **Data structures**: Stacks, queues, trees. Binary search trees, AVL trees, heaps (priority queues). Graphs: directed, undirected, weighted.

- **Sorting algorithms**: Selection sort, insertion sort, (shellsort), mergesort, quicksort, heapsort, distribution counting.

- **Search algorithms**: Sequential search, binary search, BSTs, AVL trees, 2–3 trees, hashing, *k*th-largest element.

- **Graph algorithms**: DFS, BFS, topsort, (cyclicity, connectedness), Warshall, Floyd, Prim, Dijkstra.

What is examinable?

- **String manipulation algorithms**: Brute-force string search, Horspool (Huffman encoding, Boyer-Moore, Knuth-Morris-Pratt and Rabin-Karp are not examinable)

- **Other algorithms**: Closest pairs, knapsack, …

- **Algorithmic techniques**: Brute force, decrease-and-conquer, divide-and-conquer, transform-and-conquer (presorting, representation change), dynamic programming, greedy methods, time/space tradeoffs.

- **Algorithm analysis**: Asymptotics, the Master Theorem for divide-and-conquer.

Priority queues and heaps

- A **priority queue** is a set of elements, each one has an associated **priority** which determines its ejection order
- A **heap** is a **partially ordered binary tree**
  - It must be a **complete tree** (filled from top to bottom, left to right)
  - Each child has a **priority** which is **less or equal** than its parent (max-heap)
  - We represent the heap as an array, starting from position 1, such that the children of node i will be nodes $2i$ and $2i + 1$

Heapsort

- It iterates the sequence: Build the heap – eject the root – build the heap – eject the root …
- An exercise: sort [8 45 11 97 1 78 82]

8	45	**11**	97	1	**78**	**82**
8	45	**82**	97	1	**78**	**11**
8	**45**	82	**97**	**1**	78	11
8	**97**	82	**45**	**1**	78	11
8	**97**	**82**	45	1	78	11
97	**8**	**82**	45	1	78	11
97	**8**	82	**45**	**1**	78	11
97	**45**	82	**8**	**1**	78	11

97	45	82	8	1	78	**11**
11	45	82	8	1	78	**97**
82	45	**11**	8	1	78	**97**
82	45	78	8	1	**11**	**97**
11	45	78	8	1	**82**	**97**
78	45	**11**	8	1	**82**	**97**
1	45	11	8	**78**	**82**	**97**

45	**1**	11	8	**78**	**82**	**97**
8	1	11	**45**	**78**	**82**	**97**
11	1	**8**	**45**	**78**	**82**	**97**
8	1	**11**	**45**	**78**	**82**	**97**
1	**8**	**11**	**45**	**78**	**82**	**97**
1	**8**	**11**	**45**	**78**	**82**	**97**

Transform and Conquer

- The idea is to make a change, such that the problem can be solved faster. Two main approaches:
  - Instance simplification, e.g., pre-sorting
  - Representational change, e.g., use a binary search tree

- A **binary search tree** is a binary tree, which must satisfy:
  - For the root *r*, each element in the **left subtree is smaller** than *r*, and each element in the **right subtree is larger** than r

- Balanced trees are preferable as in the worst case search requires $\Theta(\log n)$

AVL Trees and 2-3 Trees

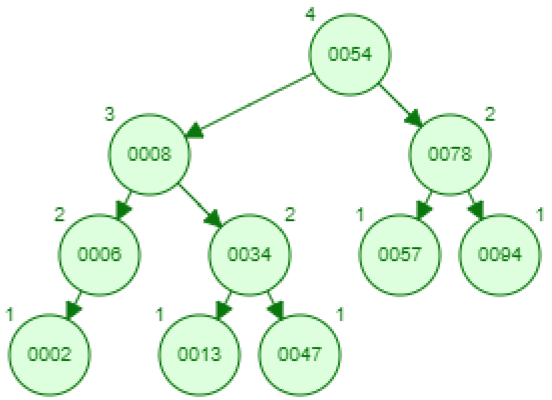
- These are approaches to keep the tree balanced

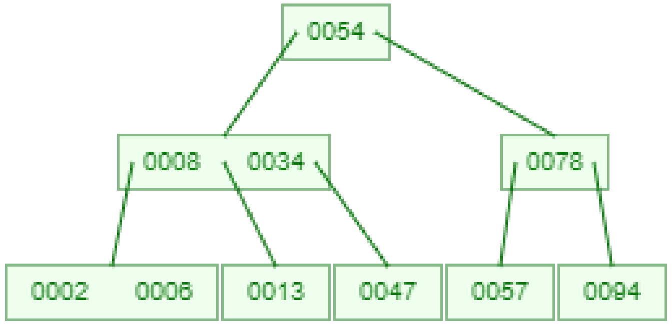
- An AVL tree rotates its nodes, such that the balance factor remains below 2 or -2

- A 2-3 tree allows a maximum of 2 keys and three children per node.

Example

- Build an AVL tree and a 2-3 tree with the data [8 6 54 78 94 13 57 47 2 34]





- https://www.cs.usfca.edu/~galles/visualization/Algorithms.html

Sorting by counting

- Lets go through this example carefully:
  - The keys are: [1 2 3 4 5]
  - The data is: [5 5 1 5 4 1 2 3 5 5 1 5 5 3 5 1 3 5 4 5]
- Lets count the appearances of each key:

Key	1	2	3	4	5
Occurrences	4	1	3	2	10

- Lets add up the occurrences

Occurrences	4	1	3	2	10
	0+4	4+1	5+3	8+2	10+10
Cumulation	4	5	8	10	20

Sorting by counting

- Lets sort the data:

Key	1	2	3	4	5
Cumulation	4	5	8	10	20
P[20]					19
P[10]				9	
P[19]					18
P[8]			7		

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Unsorted	5	5	1	5	4	1	2	3	5	5	1	5	5	3	5	1	3	5	4	5
Sorted	1	1	1	1	2	3	3	3	4	4	5	5	5	5	5	5	5	5	5	5

Horspool’s algorithm

- Lets go through this example carefully:
  - The pattern is ‘ACGT’ (A=1, T=2, G=3, C=4 → P[.] = [1 4 3 2], m = 4)
  - The string is GACCGCGTGAGATAACGTCA
- This algorithm creates the table of shifts:

$$
\begin{array}{l}
\textbf{function } \textsc{FindShifts}(P[\cdot], m) \\
\quad \textbf{for } i \leftarrow 0 \text{ to } \mathit{alphasize} - 1 \textbf{ do} \\
\quad\quad \mathit{Shift}[i] \leftarrow m \\
\quad \textbf{for } j \leftarrow 0 \text{ to } m - 2 \textbf{ do} \\
\quad\quad \mathit{Shift}[P[j]] \leftarrow m - (j + 1)
\end{array}
$$

	A	T	G	C
After first loop	4	4	4	4
j=0	**3**	4	4	4
j=1	3	4	4	**2**
j=2	3	4	**1**	2

Horspool's algorithm

- We append a **sentinel** at the end of the data to guarantee completion
  - The string is now GACCGCGTGAGATAACGTCA**ACGT**

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
STRING	G	A	C	C	G	C	G	T	G	A	G	A	T	A	A	C	G	T	C	A	A	C	G	T
T[.]	3	1	4	4	3	4	3	2	3	1	3	1	2	1	1	4	3	2	4	1	2	1	1	4
FAILED (C!=T, SHIFT BY C)	A	C	G	**T**																				
FAILED (C!=T, SHIFT BY C)			A	C	G	**T**																		
FAILED (G!=A, SHIFT BY T)					**A**	C	G	T																
FAILED (A!=T, SHIFT BY A)									A	C	G	**T**												
FAILED (A!=T, SHIFT BY A)												A	C	G	**T**									
FOUND AT 17															**A**	**C**	**G**	**T**						

- Any change in P will change the shift table.

Hashing

- **Hashing** is a standard way of implementing the abstract data type "dictionary", a collection of <attribute name, value> pairs.

- The challenges in implementing a **hash table** are:
  - Design an easy (cheap) to compute hash function that distribute the keys evenly.
  - Handling of same addresses (collisions) for different key values

- We examined three approaches
  - Separate Chaining
  - Linear probing
  - Double Hashing

An excercise

- With the hash functions $h(k) = k \bmod 11$ and $s(k) = 3 + k \bmod 7$, draw the hash table that results after inserting in the given order: [35 20 26 62 48]

Index	0	1	2	3	4	5	6	7	8	9	10
Separate Chaining			35		26			62		20	
					48						
Linear Probing			35		26	48		62		20	
Double Hashing	48		35		26			62		20	

Recap

- **Dynamic programming** is a bottom-up problem solving technique. The idea is to divide the problem into smaller, overlapping ones. The results are tabulated and used to find the complete solution.
  - Solutions often involves recursion.

- Dynamic programming is often used on **Combinatorial Optimization** problems.
  - We are trying to find the **best** possible **combination** subject to some **constraints**

- Discussed a few problems
  - Coin row problem
  - Knapsack problem
  - Transitive closure of a matrix
  - All pairs shortest paths

The coin row problem

- You are shown a group of coins of different denominations ordered in a row.

- **You can keep some of them**, as long as you **do not pick two adjacent ones**.
  - Your objective is to **maximize your profit** , i.e., you want to take the largest amount of money.

- The solution can be expressed as the recurrence:

$$S(n) = \max\left(c_n + S\left(n-2\right), S\left(n-1\right)\right) \text{ for } n > 1$$

$$S(1) = c_1$$

$$S(0) = 0$$

The coin row problem

- Try to solve the problem C = [50 10 20 20 50 50 20 5 5 10]

Step	0	50	10	20	20	50	50	20	5	5	10
0	0										
1		50									
2	0	50	50								
3		50	50	70							
4			50	70	70						
5				70	70	120					
6					70	120	120				
7						120	120	140			
8							120	140	140		
9								140	140	145	
10									140	145	150

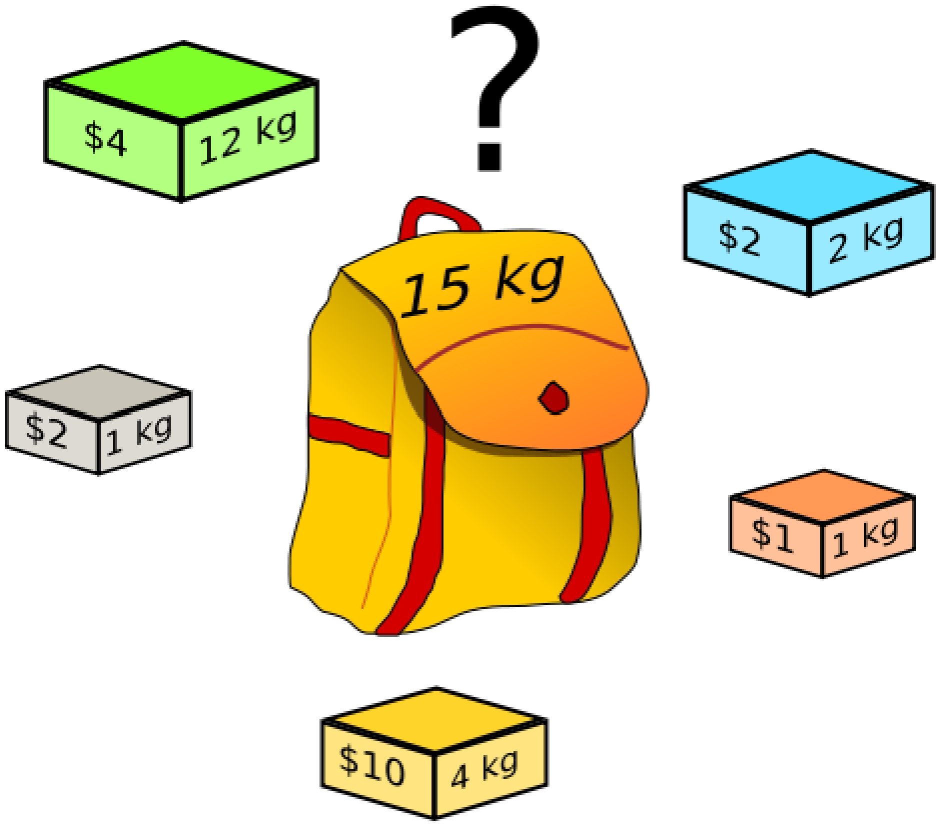
The knapsack problem

- We also talked about the **knapsack problem**:

- Given a list of *n* items with:
  - Weights $\{w_1, w_2, \ldots, w_n\}$
  - Values $\{v_1, v_2, \ldots, v_n\}$
- and a knapsack (container) of capacity W

- Find the **combination** of items with the **highest value** that would **fit into the knapsack**

- All values are positive integers



The knapsack problem

- The algorithm was based on a recursion, with a base **state**:

$$K(i, w) = 0 \text{ if } i = 0 \text{ or } w = 0$$

- And a general case:

$$K(i, w) = \begin{cases} \max(K(i-1, w), K(i-1, w-w_i) + v_i) & \text{if } w \geq w_i \\ K(i-1, w) & \text{if } w < w_i \end{cases}$$

- Another exercise:
  - The knapsack capacity W = 10
  - The values are {8, 24, 13, 19}
  - The weights are {7, 4, 6, 5}

The knapsack problem

			j	0	1	2	3	4	5	6	7	8	9	10
v	w	i												
		0		0	0	0	0	0	0	0	0	0	0	0
8	7	1		0	0	-1	-1	0	0	0	-1	-1	-1	8
24	4	2		0	-1	-1	-1	24	24	-1	-1	-1	-1	24
13	6	3		0	-1	-1	-1	-1	24	-1	-1	-1	-1	37
19	5	4		0	-1	-1	-1	-1	-1	-1	-1	-1	-1	43

function $\mathrm{MFK}\textsc{nap}(i, j)$
  if $i < 1$ or $j < 1$ **then**
    return 0
  if $F(i, j) < 0$ **then**
    if $j < w(i)$ **then**
      $value = \mathrm{MFK}\textsc{nap}(i-1, j)$
    else
      $value = \max(\mathrm{MFK}\textsc{nap}(i-1, j), v(i) + \mathrm{MFK}\textsc{nap}(i-1, j-w(i)))$
    $F(i, j) = value$
  return $F(i, j)$

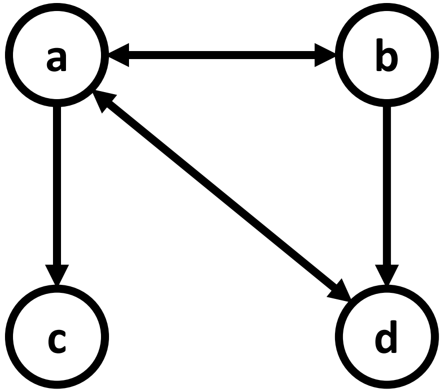
Warshall’s algorithm

- Warshall's algorithm computes the **transitive closure** of a directed graph, by building a path through the following the rules:
  - step from *i* to *j* using only nodes [1 … *k*-1], or
  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

- We examined the simplest version of the algorithm.

$$
\begin{array}{l}
\textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{if } A[i,k] \textbf{ then} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad \textbf{if } A[k,j] \textbf{ then} \\
\quad\quad\quad\quad\quad A[i,j] \leftarrow 1
\end{array}
$$

Warshall's Algorithm



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$k = 1, i = 2, j = 2$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$k = 1, i = 2, j = 3$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$k = 1, i = 4, j = 2$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$k = 1, i = 4, j = 3$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$k = 1, i = 4, j = 4$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$k = 2, i = 1, j = 1$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Floyd's Algorithm

- Floyd's algorithm solves the **all-pairs shortest-path** problem for weighted graphs with **positive weights**, by building a path following these rules:
  - step from *i* to *j* using only nodes [1 … *k*-1], or
  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

- We examined a simple version updating D:

$$
\begin{array}{l}
\textbf{function } \textsc{Floyd}(W[\cdot,\cdot], n) \\
\quad D \leftarrow W \\
\quad \textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right) \\
\quad \textbf{return } D
\end{array}
$$

Floyd's Algorithm

$$\begin{bmatrix} \infty & 5 & 1 & 2 \\ 4 & \infty & \infty & 10 \\ \infty & \infty & \infty & \infty \\ 3 & \infty & \infty & \infty \end{bmatrix}$$

$$\begin{bmatrix} \infty & 5 & 1 & 2 \\ 4 & 9 & 5 & 10 \\ \infty & \infty & \infty & \infty \\ 3 & 8 & 4 & 5 \end{bmatrix}$$

k=1

$$\begin{bmatrix} 9 & 5 & 1 & 2 \\ 4 & 9 & 5 & 6 \\ \infty & \infty & \infty & \infty \\ 3 & 8 & 4 & 5 \end{bmatrix}$$

k=2

$$\begin{bmatrix} 9 & 5 & 1 & 2 \\ 4 & 9 & 5 & 6 \\ \infty & \infty & \infty & \infty \\ 3 & 8 & 4 & 5 \end{bmatrix}$$

k=3

$$\begin{bmatrix} 5 & 5 & 1 & 2 \\ 4 & 9 & 5 & 6 \\ \infty & \infty & \infty & \infty \\ 3 & 8 & 4 & 5 \end{bmatrix}$$

k=4

Greedy algorithms

- A **greedy algorithm** takes the **locally best** choice among all feasible ones. Such choice is **irrevocable**.
  - Greedy algorithm can provide good **approximations**.

- We applied this idea to two graph problems :
  - Prim's algorithm for finding **minimum spanning trees**
  - Dijkstra's algorithm for **single-source shortest path**

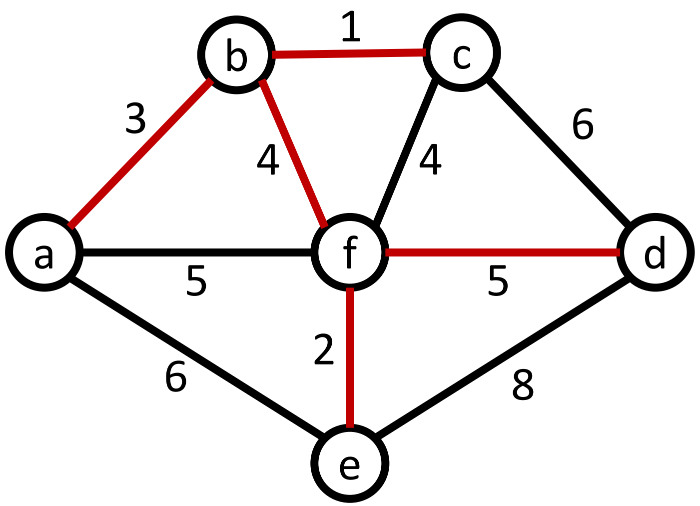
Prim's Algorithm

- We examined the complete algorithm, that uses priority queues:

$$
\begin{array}{ll}
\textbf{function } \textsc{Prim}(\langle V, E\rangle) & \\
\quad \textbf{for } \text{each } v \in V \textbf{ do} & \\
\quad\quad \mathit{cost}[v] \leftarrow \infty & \\
\quad\quad \mathit{prev}[v] \leftarrow \mathit{nil} & \\
\quad \text{pick initial node } v_0 & \\
\quad \mathit{cost}[v_0] \leftarrow 0 & \\
\quad Q \leftarrow \textsc{InitPriorityQueue}(V) & \triangleright \text{ priorities are cost values} \\
\quad \textbf{while } Q \text{ is non-empty } \textbf{do} & \\
\quad\quad u \leftarrow \textsc{EjectMin}(Q) & \\
\quad\quad \textbf{for } \text{each } (u, w) \in E \textbf{ do} & \\
\quad\quad\quad \textbf{if } \mathit{weight}(u, w) < \mathit{cost}[w] \textbf{ then} & \\
\quad\quad\quad\quad \boxed{\mathit{cost}[w] \leftarrow \mathit{weight}(u, w)} & \\
\quad\quad\quad\quad \mathit{prev}[w] \leftarrow u & \\
\quad\quad\quad\quad \textsc{Update}(Q, w, \mathit{cost}[w]) & \triangleright \text{ rearranges priority queue}
\end{array}
$$

Another example

- Let's work with the following graph, but starting from b:



Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	∞	0	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
b	cost	3		1	∞	∞	4
	prev	b		b	nil	nil	b
b,c	cost	3			6	∞	4
	prev	b			c	nil	b
b,c,a	cost				6	6	4
	prev				c	a	b
b,c,a,f	cost				5	2	
	prev				f	f	
b,c,a,f,e	cost				5		
	prev				f		
b,c,a,f,e,d	cost						
	prev						

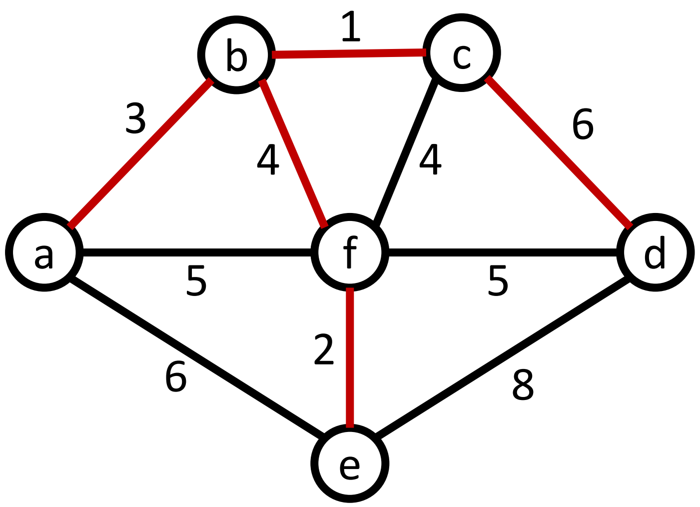
Dijkstra's Algorithm

- **Dijkstra's algorithm** finds all shortest paths **from a fixed start node**. Its complexity is the same as that of Prim's algorithm.

function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
    for each $v \in V$ **do**
        $dist[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    $dist[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$ ▷ priorities are distances
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $dist[u] + weight(u, w) < dist[w]$ **then**
                $\boxed{dist[w] \leftarrow dist[u] + weight(u, w)}$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, dist[w])$ ▷ rearranges priority queue

Another example

- Let’s work with this graph again, but starting from b:



Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
b	cost	3		1	∞	∞	4
	prev	b		b	nil	nil	b
b,c	cost	3			7	∞	4
	prev	b			c	nil	b
b,c,a	cost				7	9	4
	prev				c	a	b
b,c,a,f	cost				7	6	
	prev				c	f	
b,c,a,f,e	cost				7		
	prev				c		
b,c,a,f,e,d	cost						
	prev						

- Toby will be back for Wednesday.
- Thanks for your attention.
- Please do not forget to fill in the SES.