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COMP90038
Algorithms and Complexity

Lecture 4: Analysis of Algorithms
(with thanks to Harald Søndergaard)

Toby Murray

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@tobycmurray

Last Time: Time Complexity

- Measure **input size** by natural number *n*
- Measure **execution time** as number of **basic operations** performed
- **Time complexity** *t(n)* for an algorithm: number of **basic operations** as a function of *n*
- How to **compare** different *t(n)* ?
  - Asymptotic growth rate
  - $O(\mathrm{g}(\mathrm{n}))$, $\Omega(g(n))$, $\Theta(g(n))$

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  bas
- Hov
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  - $O(g(n))$, $\Omega(g(n))$, $\Theta(g(n))$

Problem	Size Measure	Basic Operation
Search in a list of *n* items	*n*	Key comparison
Multiply two matrices of floats	Matrix size (rows x columns)	Float multiplication
Compute a^n	$\log n$	Float multiplication
Graph problem	Number of nodes and edges	Visiting a node

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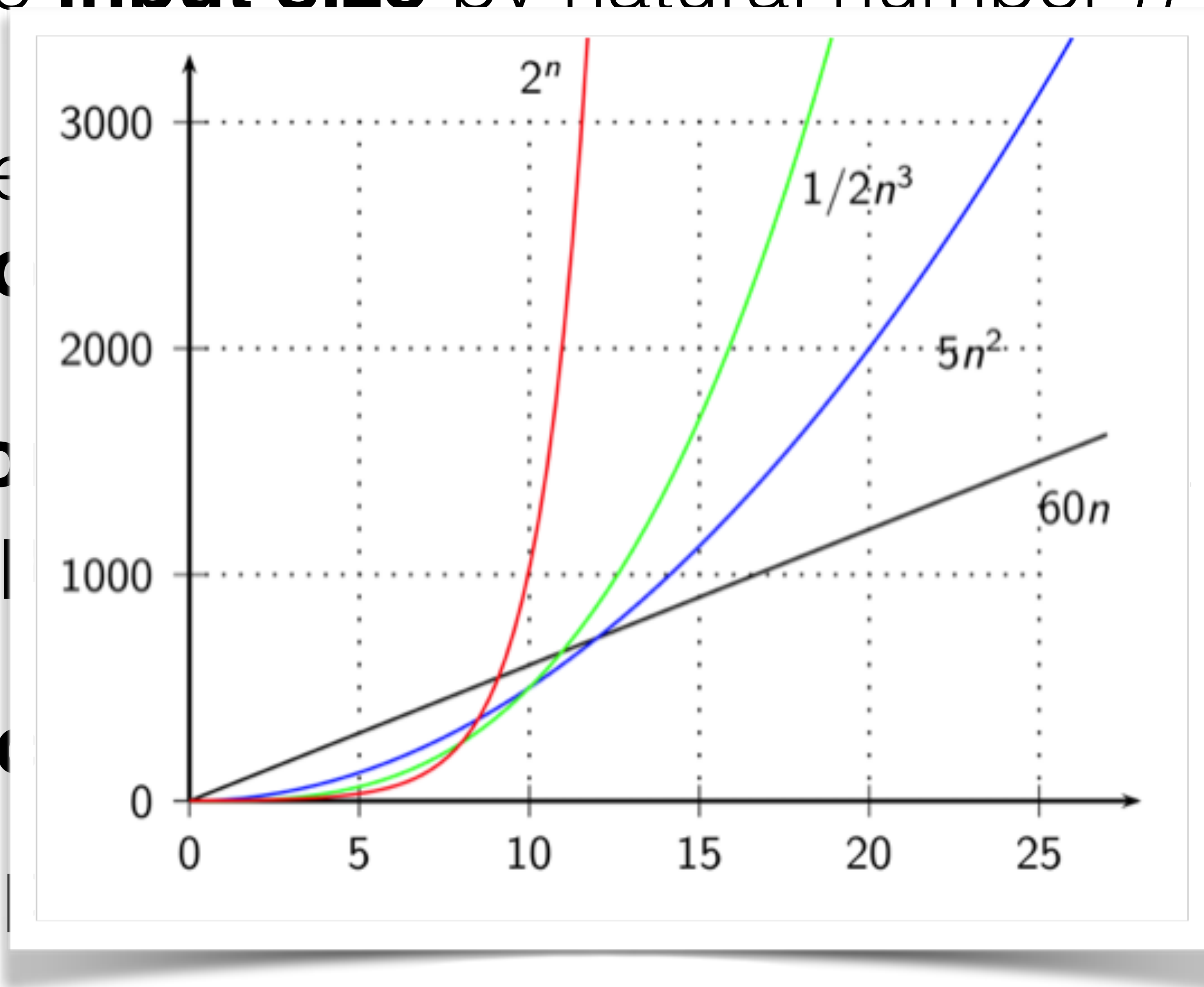

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$$f(n) < g(n) \text{ iff } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Last Time: Time Complexity

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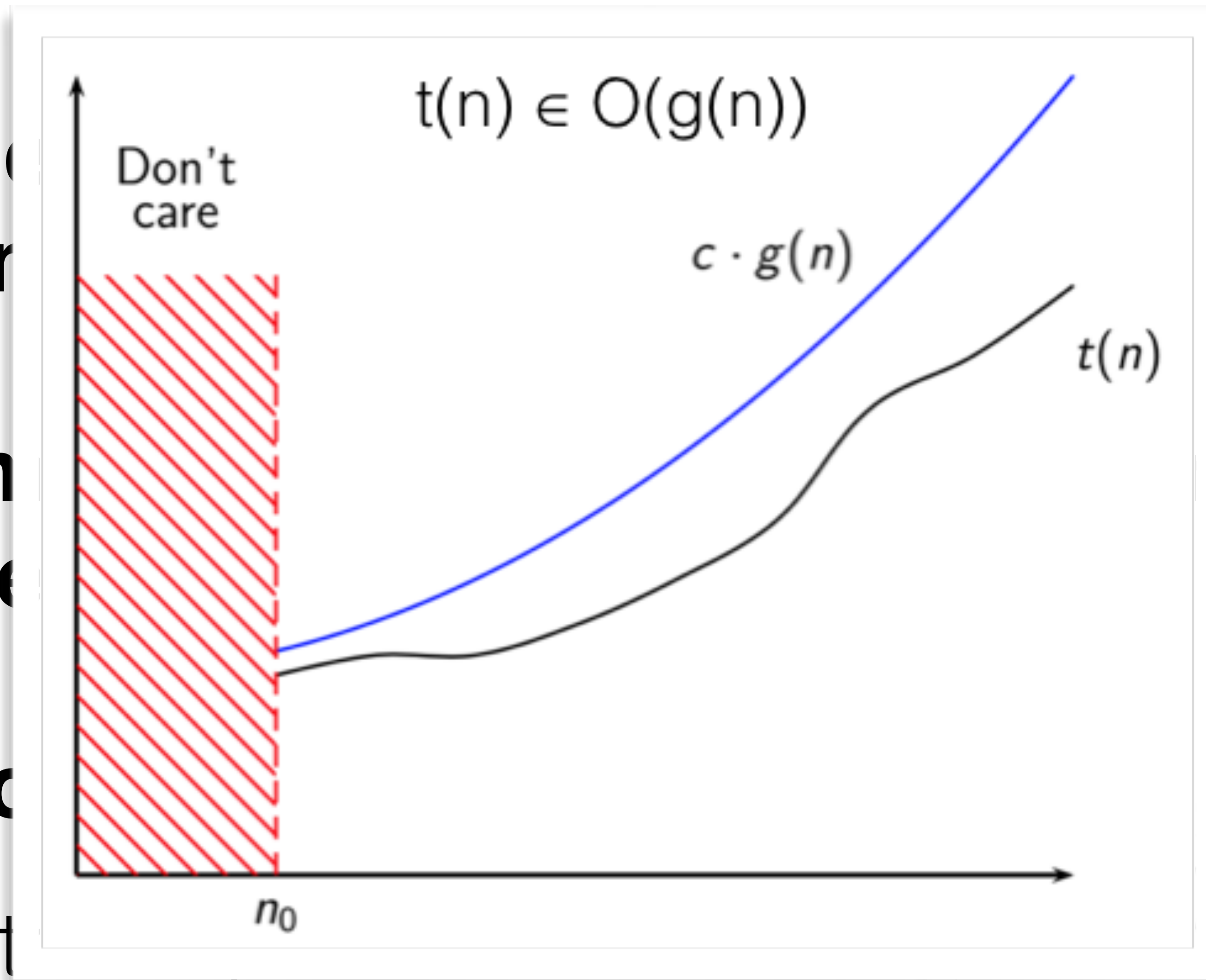

$)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

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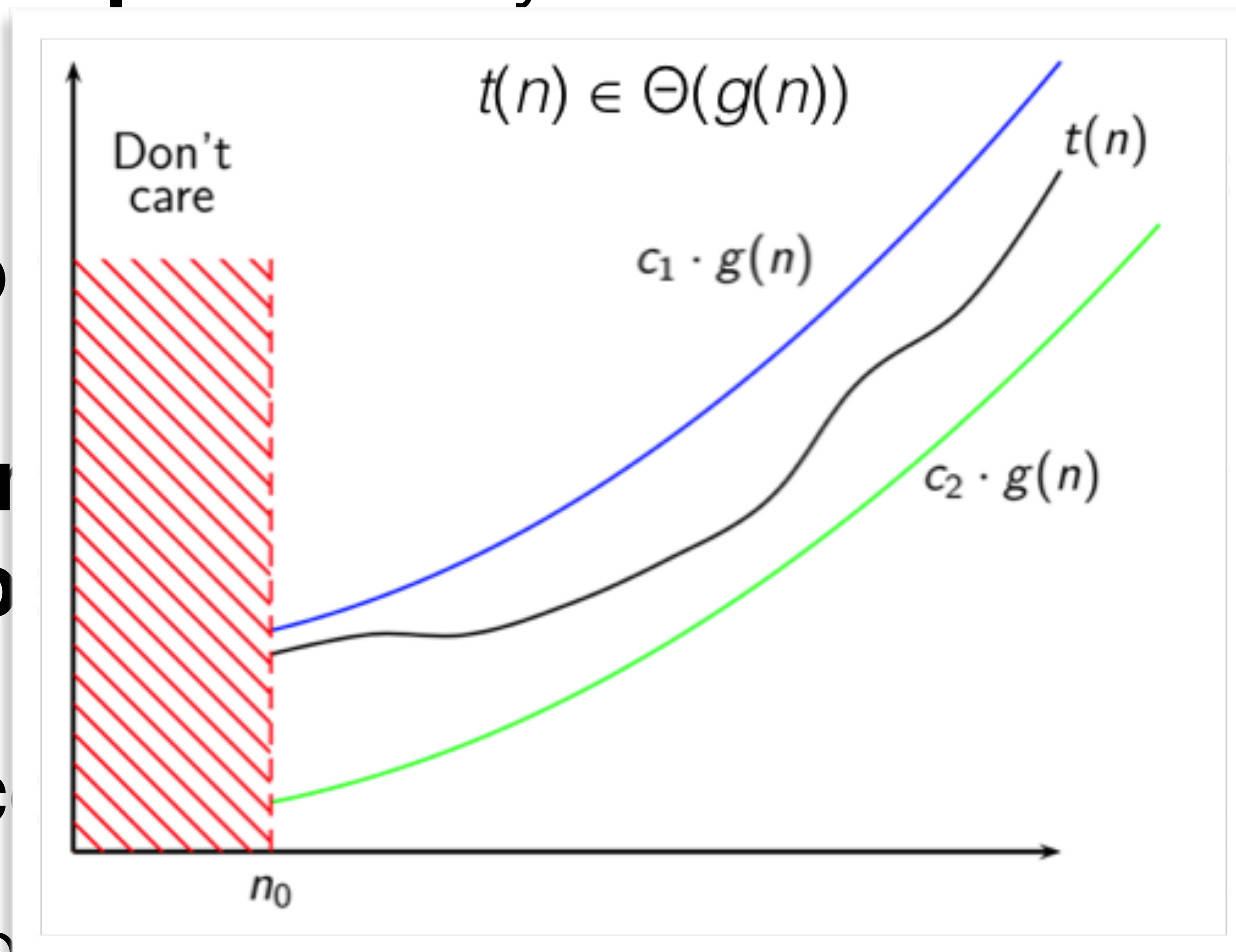

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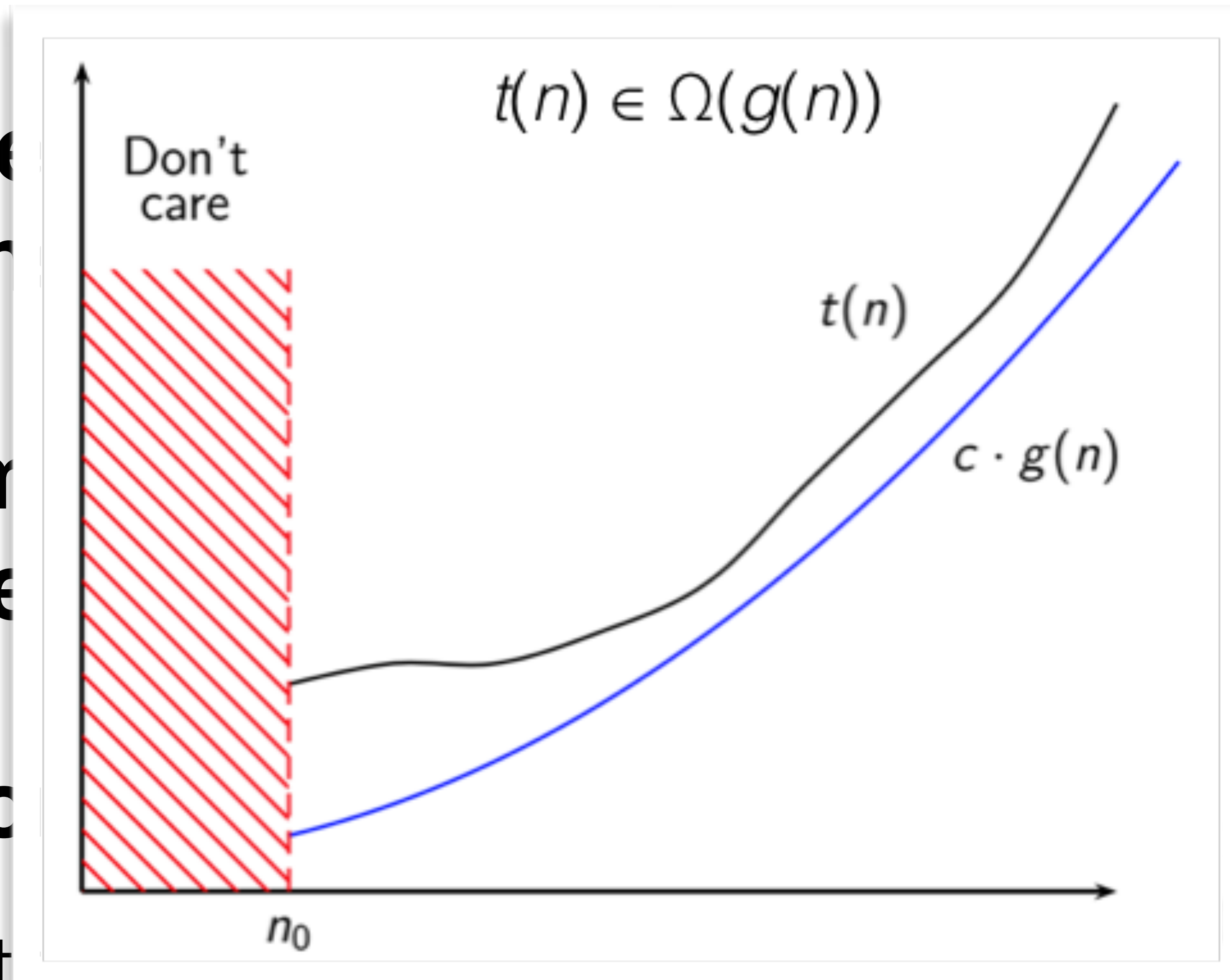

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$$f(n) < g(n) \text{ iff } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Establishing Growth Rate

- In the last lecture we proved $t(n) \in O(g(n))$ for some cases of t and g, using the definition of O directly:

  $n > n_0 \Rightarrow t(n) < c \cdot g(n)$ for some c and n_0.

- A more common approach uses

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

- Use this to show that $1000n \in O(n^2)$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000n}{n^2}$$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000\cancel{n}}{n^2}$$

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$$\lim_{n \to \infty} \frac{1000\cancel{n}}{\cancel{n^2}}$$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000\cancel{n}}{n^{\cancel{2}}}$$

$$= \lim_{n \to \infty} \frac{1000}{n}$$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000\cancel{n}}{\cancel{n^2}}$$

$$= \lim_{n \to \infty} \frac{1000}{n}$$

$$= 0$$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000\cancel{n}}{\cancel{n^2}}$$

$$= \lim_{n \to \infty} \frac{1000}{n}$$

$$= 0$$

So $1000n$ grows asymptotically slower than n^2

$1000n \in O(n^2)$

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So $1000n$ grows asymptotically slower than n^2

Thus $1000n \in O(n^2)$

O, Ω, Θ

What this tells us about how $f(n)$ relates to $O(g(n))$, $\Omega(g(n))$ and $\Theta(g(n))$

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

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L'Hôpital's Rule

$$\lim_{h \to \infty} \frac{t(n)}{g(n)} = \lim_{h \to \infty} \frac{t'(n)}{g'(n)}$$

where t' and g' are the derivatives of t and g

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- For example, show that $\log_2$ n grows slower than $\sqrt{n}$

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$$= 2\log_2 e \lim_{n\to\infty}\frac{\sqrt{n}}{n}$$

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$$= 2\log_2 e \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 2\log_2 e \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Example:
Finding Largest Element in an Array

function $\text{MAXELEMENT}(A[\cdot], n)$
  $max \leftarrow A[0]$
  for $i \leftarrow 1$ to $n - 1$ **do**
    if $A[i] > max$ **then**
      $max \leftarrow A[i]$
  return max

(where n is length of the array)

A:

23	12	42	6	69	18	3
0	1	2	3	4	5	6

Example:
Finding Largest Element in an Array

$$
\begin{array}{l}
\textbf{function } \text{MAXELEMENT}(A[\cdot], n) \\
\quad max \leftarrow A[0] \\
\quad \textbf{for } i \leftarrow 1 \text{ to } n-1 \textbf{ do} \\
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\quad \textbf{return } max
\end{array}
$$

(where n is length of the array)

A:

23	12	42	6	69	18	3
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max: 23

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A:

23	12	42	6	69	18	3
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A[i]

max: 23

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(where *n* is length of the array)

A:

23	12	42	6	69	18	3
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A[i]

max: 42

Example:
Finding Largest Element in an Array

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A:

23	12	42	6	69	18	3
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A[i]

max: 42

Example:
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23	12	42	6	69	18	3
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max: 42

A[i]

Example:
Finding Largest Element in an Array

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function MaxElement(A[·], n)
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    return max
```

(where n is length of the array)

A:

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i]

max: 69

Example:
Finding Largest Element in an Array

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(where n is length of the array)

A:

23	12	42	6	69	18	3
0	1	2	3	4	5	6

max: 69

A[i]

Example:
Finding Largest Element in an Array

(where n is length of the array)

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A[i]

max: 69

Example:
Finding Largest Element in an Array

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(where *n* is length of the array)

Size of input, *n*:
length of the array

Example:
Finding Largest Element in an Array

$$
\begin{aligned}
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&\quad\quad\quad max \leftarrow A[i] \\
&\quad \textbf{return } max
\end{aligned}
$$

(where n is length of the array)

Size of input, n: length of the array

Basic operation: comparison "A[i] > max"

Example:
Finding Largest Element in an Array

(where n is length of the array)

function $\textsc{MaxElement}(A[\cdot], n)$
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Size of input, n:
length of the array

Basic operation: comparison "A[i] > max"

Count the number of basic operations executed for an array of size n:

Example:
Finding Largest Element in an Array

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Size of input, *n*: length of the array

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Count the number of basic operations executed for an array of size *n:*

1

Example:
Finding Largest Element in an Array

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(where n is length of the array)

Size of input, n: length of the array

Basic operation: comparison "A[i] > max"

Count the number of basic operations executed for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1$$

Example: Finding Largest Element in an Array

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```
function MaxElement(A[·], n)
    max ← A[0]
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```

(where *n* is length of the array)

Size of input, *n*: length of the array

Basic operation: comparison "A[i] > max"

Count the number of basic operations executed for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1)$$

Example: Finding Largest Element in an Array

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  return max

(where n is length of the array)

Size of input, n: length of the array

Basic operation: c

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Count the number of basic

for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1)$$

Example: Finding Largest Element in an Array

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(where n is length of the array)

Size of input, n: length of the array

Basic operation: c

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Count the number of basic for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1) = n - 1$$

Example:
Finding Largest Element in an Array

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Size of input, n: length of the array

Basic operation: c

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Count the number of basic
for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1)-1+1) = n-1 \in \Theta(n)$$

Example: Selection Sort

$$
\begin{array}{l}
\textbf{function } \text{SelSort}(A[\cdot], n) \\
\quad \textbf{for } i \leftarrow 0 \text{ to } n-2 \textbf{ do} \\
\quad\quad min \leftarrow i \\
\quad\quad \textbf{for } j \leftarrow i+1 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad \textbf{if } A[j] < A[min] \textbf{ then} \\
\quad\quad\quad\quad min \leftarrow j \\
\quad\quad \text{swap } A[i] \text{ and } A[min]
\end{array}
$$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

Example: Selection Sort

function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

↑
A[i]

Example: Selection Sort

function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i]

min: 0

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 0

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

function $\text{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

function SelSort($A[\cdot]$, n)
    for $i \leftarrow 0$ to $n-2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i+1$ to $n-1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] (index 0) A[j] (index 3)

min: 1

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 3

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6
↑				↑		
A[i]				A[j]		

min: 3

Example: Selection Sort

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```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 3

Example: Selection Sort

function SelSort($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6
A[i]						A[j]

min: 3

Example: Selection Sort

function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] (index 0)   A[j] (index 6)

min: 6

Example: Selection Sort

function SELSORT($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

↑
A[i]

min: 6

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 6

↑
A[i]

Example: Selection Sort

function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

↑ A[i]

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 1

↑

A[i]

Example: Selection Sort

function $\text{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6
	A[i]	A[j]				

min: 1

Example: Selection Sort

function SelSort($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6
	↑ A[i]		↑ A[j]			

min: 1

Example: Selection Sort

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```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i] A[j]

min: 3

Example: Selection Sort

```
function SELSORT(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

3	12	42	6	69	18	23
0	1	2	3	4	5	6
	A[i]			A[j]		

min: 3

Example: Selection Sort

function SelSort($A[\cdot]$, n)
    for $i \leftarrow 0$ to $n-2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i+1$ to $n-1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i]   A[j]

min: 3

Example: Selection Sort

function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6
	A[i]					A[j]

min: 3

Example: Selection Sort

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function SELSORT($A[\cdot]$, n)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	6	42	12	69	18	23
0	1	2	3	4	5	6

A[i]

min: 3

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

3	6	42	12	69	18	23
0	1	2	3	4	5	6

A[i]

min: 3

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size n: length of the array

Basic operation: comparison A[j] < A[min]

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

C(n) =

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1$$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size n: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i)$$

Example: Selection Sort

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```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size n: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

Example: Selection Sort

function SelSort($A[\cdot]$, n)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

Input size n: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2}$$

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size n: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i) = \sum_{i=0}^{n-2} (n - 1) - \sum_{i=0}^{n-2} i$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{n(n-1)}{2} \in \Theta(n^2)$$

Example: Matrix Multiplication

function $\text{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\quad \triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

Example: Matrix Multiplication

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function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices

  for $i \leftarrow 0$ to $n-1$ **do**

    for $j \leftarrow 0$ to $n-1$ **do**

      $C[i,j] \leftarrow 0.0$

      for $k \leftarrow 0$ to $n-1$ **do**

        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$

  return C

i: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

i: 0

j: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 0 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\text{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 0 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

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function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \quad \underset{C}{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 10 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function MATRIXMULT($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

i: 0

j: 1

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 10 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

$$
\begin{array}{l}
\textbf{function } \text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n) \qquad \triangleright \text{ For } n \times n \text{ matrices} \\
\quad \textbf{for } i \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad \textbf{for } j \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad C[i,j] \leftarrow 0.0 \\
\quad\quad\quad \textbf{for } k \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad\quad C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j] \\
\quad \textbf{return } C
\end{array}
$$

i: 0

j: 1

k: 1

$$
\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}
\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}
\qquad
\begin{bmatrix} 103 & 52 \\ & \end{bmatrix}
$$

A B C

Example: Matrix Multiplication

function MATRIXMULT($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices

    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\quad\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

Basic operation: multiplication A[i,k] * B[k,j]

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) =$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \qquad 1$$

Example: Matrix Multiplication

$$
\begin{array}{l}
\textbf{function } \textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n) \qquad \triangleright \text{ For } n \times n \text{ matrices} \\
\quad \textbf{for } i \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad \textbf{for } j \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad C[i,j] \leftarrow 0.0 \\
\quad\quad\quad \textbf{for } k \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad\quad C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j] \\
\quad \textbf{return } C
\end{array}
$$

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
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                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\qquad \triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
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    return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

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$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1)$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

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$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

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$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

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$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right)$$

$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

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$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

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$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

$$= n^3$$

Example: Matrix Multiplication

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$$M(n) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1}\left(n \cdot \sum_{j=0}^{n-1} 1\right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} ca_i = c\sum_{i=l}^{u} a_i$$

$$= n^3$$

$$\in \Theta(n^3)$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

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function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

$F(5)$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$F(5) = F(4) \cdot 5$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$\begin{aligned} F(5) &= F(4) \cdot 5 \\ &= (F(3) \cdot 4) \cdot 5 \end{aligned}$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

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function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 5!
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

Analysing Recursive Algorithms

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function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) =$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$
$$M(n) =$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = 0$$
$$M(n) = \qquad\qquad 1$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$
$$M(n) = \qquad\qquad + 1$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= 0 \\
M(n) &= M(n-1) + 1
\end{aligned}
$$

Analysing Recursive Algorithms

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function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= 0 \\
M(n) &= M(n-1) + 1
\end{aligned}
$$

Need to express M(n) in **closed form** (i.e. non-recursively)

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$\begin{aligned} M(0) &= 0 \\ M(n) &= M(n-1) + 1 \end{aligned}$$

Need to express M(n) in **closed form** (i.e. non-recursively)

Try: **"telescoping"** aka **"backward substitution"**

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ? $M(n-1) = M((n-1)-1) + 1$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$M(n) =$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$
$$= M(n-2) + 2$$
$$= (M(n-3) + 1) + 2$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2)+1)+1$$
$$= M(n-2)+2$$
$$= (M(n-3)+1)+2$$
$$= M(n-3)+3$$
$$\ldots$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\dots \\ &= M(n-n) + n \end{aligned}$$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2)+1)+1$$
$$= M(n-2)+2$$
$$= (M(n-3)+1)+2$$
$$= M(n-3)+3$$
$$\ldots$$
$$= M(n-n)+n$$
$$= M(0)+n$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\dots \\ &= M(n-n) + n \\ &= M(0) + n \\ &= n \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) = {} & (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\dots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Closed form:

Telescoping
(aka Backward Substitution)

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$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$
$$= M(n-2) + 2$$
$$= (M(n-3) + 1) + 2$$
$$= M(n-3) + 3$$
$$\ldots$$
$$= M(n-n) + n$$
$$= M(0) + n$$
$$= n$$

Closed form:

$$\boxed{M(n) = n}$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Closed form:

$$\boxed{M(n) = n}$$

Complexity:

Telescoping
(aka Backward Substitution)

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$$\boxed{M(n) = M(n-1) + 1}$$

$$\boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Closed form:

$$\boxed{M(n) = n}$$

Complexity:

$$\boxed{M(n) \in \Theta(n)}$$

Example:
Binary Search in Sorted Array

function $\textsc{BinSearch}(A[\cdot], lo, hi, key)$
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lo: 0

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lo: 0

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lo: 0

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key: 41

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BinSearch(A,4,6,41)

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lo: 4

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key: 41

mid: 5

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returns 4

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$C(0) = 0$ $\qquad\qquad C(n) = \qquad\qquad + 1$

Example:
Binary Search in Sorted Array

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Basic operation: key comparison A[mid] = key

$C(0) = 0$ $\qquad$ $C(n) = C(n/2) + 1$

Telescoping

A **smoothness rule** allows us to assume that n is a power of 2

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$$C(n) \in \Theta(\log n)$$

Logarithmic Functions Have Same Rate of Growth

In O, Ω, Θ, expressions we can just write "log" for any logarithmic function no matter what the base is

Asymptotically, all logarithmic behaviour is the same, since

$$\log_a x = (\log_a b)(\log_b x)$$

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Also, since $\log n^c = c \cdot \log n$ $\quad \log n^c \in O(\log n)$

Back to Euclid

```
function EUCLID(m, n)
    while n ≠ 0 do
        r ← m mod n
        m ← n
        n ← r
    return m
```

```
function EUCLID(m, n)
    if n = 0 then
        return m
    return EUCLID(n, m mod n)
```

Back to Euclid

function $\textsc{Euclid}(m, n)$
  while $n \neq 0$ **do**
    $r \leftarrow m \ mod \ n$
    $m \leftarrow n$
    $n \leftarrow r$
  return m

function $\textsc{Euclid}(m, n)$
  if $n = 0$ **then**
    return m
  return $\textsc{Euclid}(n, m \ mod \ n)$

Running time is **linear** in size (in bits) of input, i.e.

$$O(\log(m + n))$$

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since the value of m (and n) is at least halved in every two iterations.

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since the value of *m* (and *n*) is at least halved in every two iterations.

Why? After two iterations, m becomes m mod n; also

$$1 < n < m \implies m \bmod n < m/2$$

Summarising Reasoning with Big-Oh

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(for nested loops: count number of times outer loop is executed, multiply by cost of inner loop)

Some Useful Formulas

From Stirling's formula: $n! \in O(n^{n+\frac{1}{2}})$

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$$
\begin{aligned}
\sum_{i=0}^{n} i^2 &= \tfrac{n}{3}(n+\tfrac{1}{2})(n+1) \\
\sum_{i=0}^{n}(2i+1) &= (n+1)^2 \\
\sum_{i=1}^{n} 1/i &= O(\log n)
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$$\sum_{i=0}^{n}(2i+1) = (n+1)^2$$

$$\sum_{i=1}^{n} 1/i = O(\log n)$$

See also Cormen's Appendix A or Levitin's Appendix A.

Levitin's Appendix B is a tutorial on recurrence relations.

The Road Ahead

- You'll get much more familiar with asymptotic analysis as we use it on algorithms we meet in this course.
- Next week we begin our study of algorithms by looking at **brute force** approaches

