COMP90038
Algorithms and Complexity

Lecture 20: Greedy Algorithms – Prim and Dijkstra

(with thanks to Harald Søndergaard & Michael Kirley)

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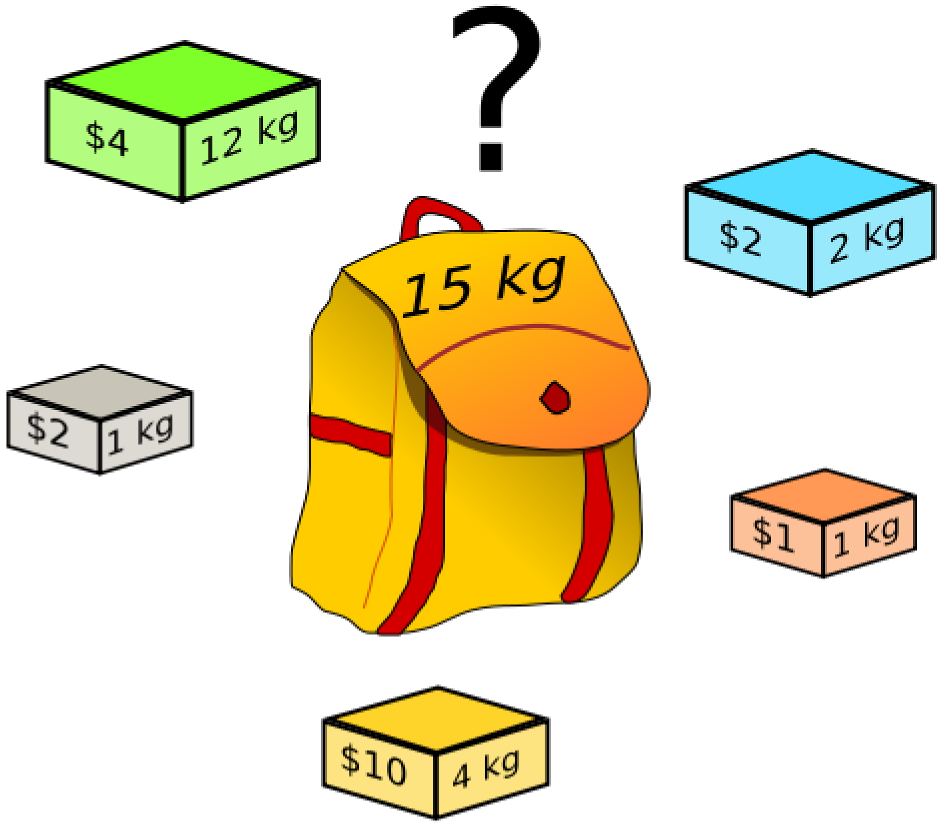
Recap

- We have talked a lot about **dynamic programming:**
  - DP is bottom-up problem solving technique.
  - Similar to divide-and-conquer; however, problems are overlapping, making tabulation a requirement.
  - Solutions often involve recursion.

- We applied this idea to two graph problems:
  - Computing the **transitive closure** of a directed graph; and
  - **Finding shortest distances** in weighted directed graphs.

A practice challenge

- Can you solve the problem in the figure?
    - W = 15
    - w = [1 1 2 4 12]
    - v = [1 2 2 10 4]
- Because it is a larger instance, **memoing** is preferable.
    - How many states do we need to evaluate?
- FYI the answer is $15 {1,2,3,4}



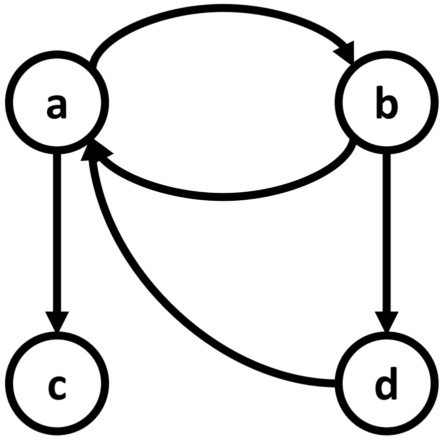
The table

		j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
w	*v*	*i*																
		0																
1	1	1		**1**	**1**	**1**	-1	-1	-1	-1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	2	2		**2**	-1	**3**	-1	-1	-1	-1	-1	**3**	-1	**3**	-1	**3**	-1	**3**
2	2	3		-1	-1	**4**	-1	-1	-1	-1	-1	-1	-1	**5**	-1	-1	-1	**5**
4	10	4		-1	-1	**4**	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	**15**
12	4	5		-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	**15**

- We know that we include all the elements up to 4 because the last column (15) is the cumulative sum of the values.

Warshall’s algorithm

- Warshall's algorithm computes the **transitive closure** of a directed graph.
  - An edge (*a*,*d*) is in the transitive closure of graph *G* iff there is a path in *G* from *a* to *d*.



- **Is there a path** from node *i* to node *j* using nodes [1 … *k*] as “stepping stones”?

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Such path will exist if and only if we can:
  - step from *i* to *j* using only nodes [1 … *k*-1], or
  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

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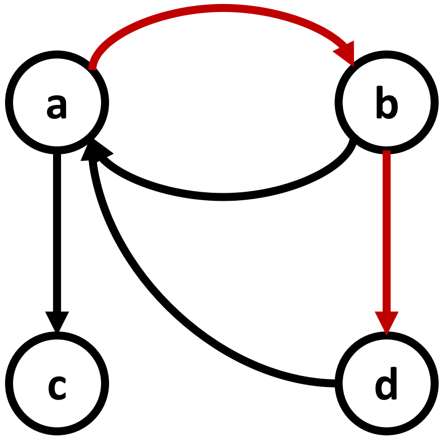


$$\begin{bmatrix} 0 & 1 & 1 & \boxed{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

- If *G*'s adjacency matrix is *A* then we can express the recurrence relation as:

$$R[i,j,0] = A[i,j]$$

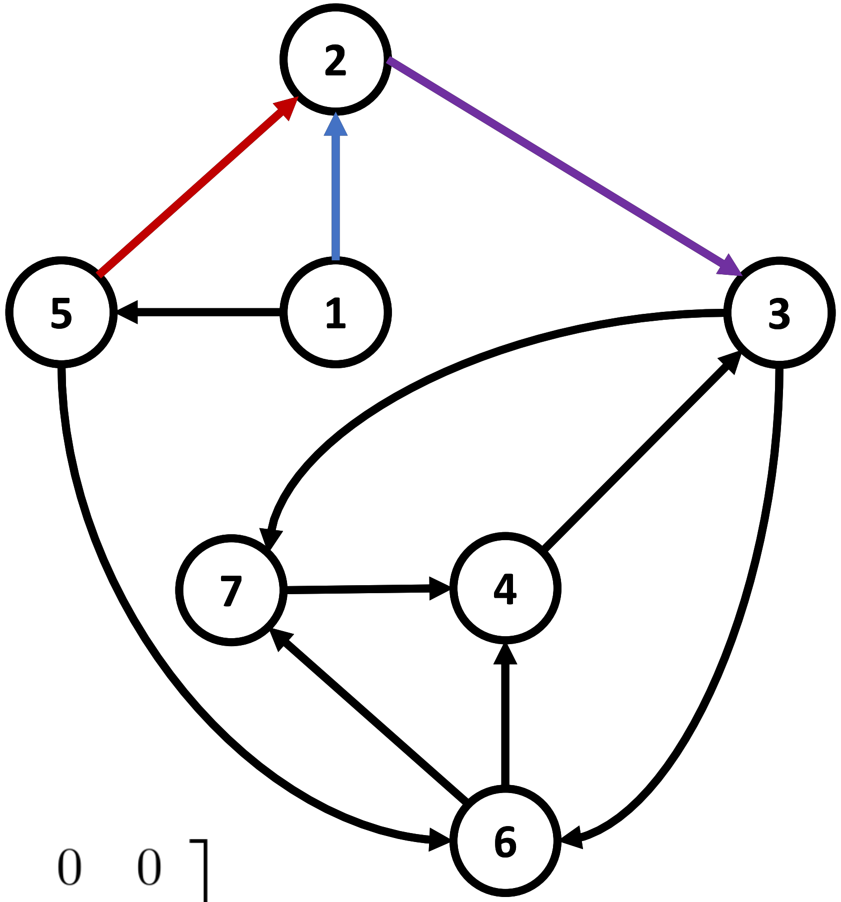
$$R[i,j,k] = R[i,j,k-1] \textbf{ or } (R[i,k,k-1] \textbf{ and } R[k,j,k-1])$$

- We examined the simplest version of the algorithm.

$$
\begin{array}{l}
\textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{if } A[i,k] \textbf{ then} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad \textbf{if } A[k,j] \textbf{ then} \\
\quad\quad\quad\quad\quad A[i,j] \leftarrow 1
\end{array}
$$

Warshall's Algorithm

- Let's visualize the steps.
- Using node 2 (*k*=2), we can reach node 3 from nodes 1 and 5.

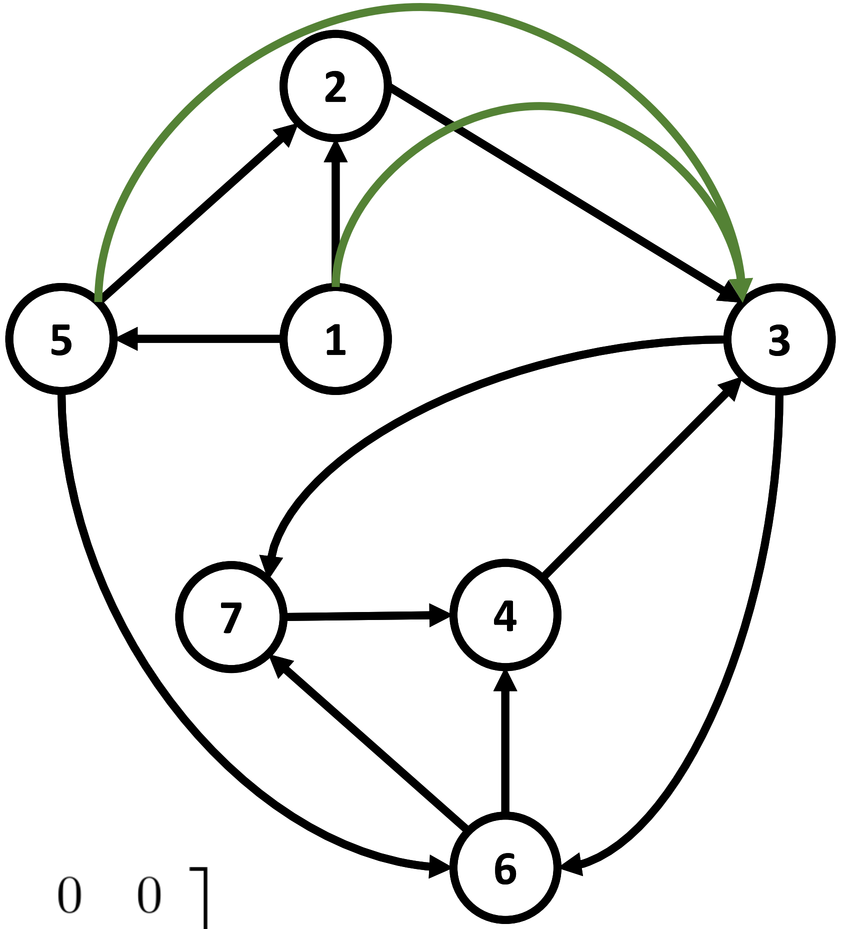


$$\begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's Algorithm

- Let's visualize the steps.

- Using node 2 (k=2), we can reach node 3 from nodes 1 and 5.



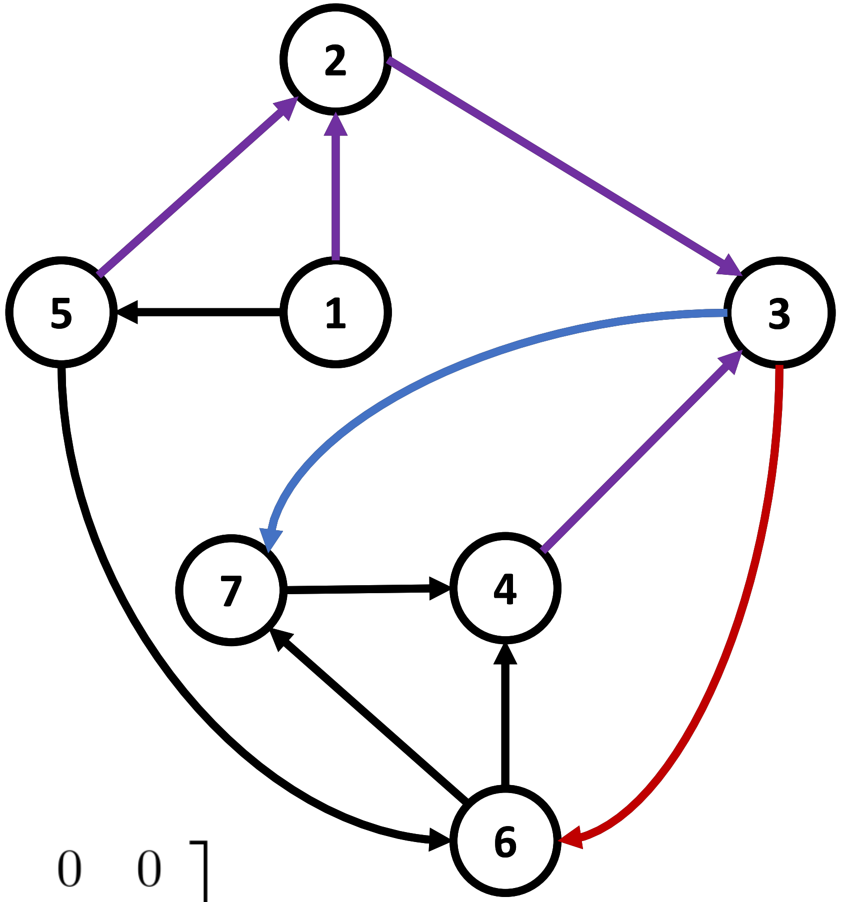
$$\begin{bmatrix} 0 & 1 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Warshall’s Algorithm

- Let’s visualize the steps.

- Using node 2 (*k*=2), we can reach node 3 from nodes 1 and 5.

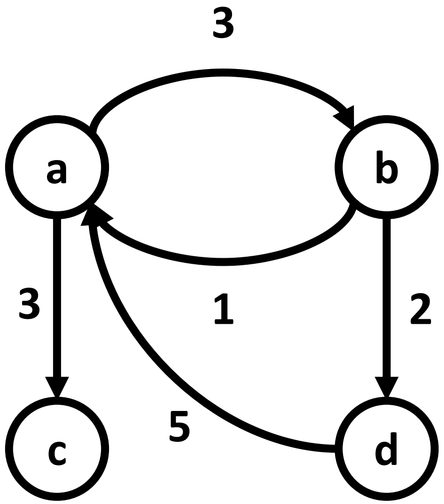
- Using node 3 (k=3) we can reach: Nodes [6 7] from nodes [1,2,5]



$$\begin{bmatrix} 0 & 1 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{1} \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 1 & \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Floyd's Algorithm

- Floyd's algorithm solves the **all-pairs shortest-path** problem for weighted graphs with **positive weights**.
  - It works for **directed** as well as **undirected** graphs.



- **What is the shortest path** from node *i* to node *j* using nodes [1 … *k*] as "stepping stones"?

- Such path will exist if and only if we can:
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  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

$$\begin{bmatrix} \infty & 3 & 3 & \infty \\ 1 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty \end{bmatrix}$$

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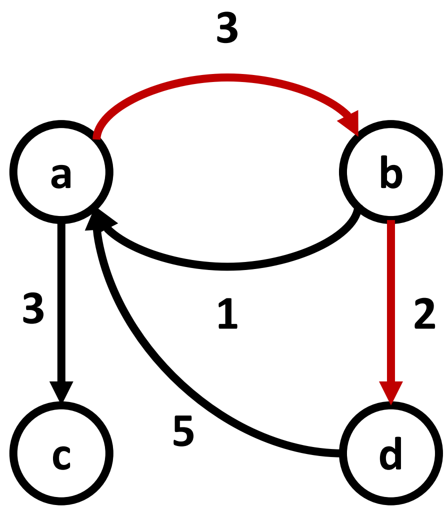


$$\begin{bmatrix} \infty & 3 & 3 & \boxed{5} \\ 1 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty \end{bmatrix}$$

Floyd's Algorithm

- If *G*'s weight matrix is *W* then we can express the recurrence relation as:

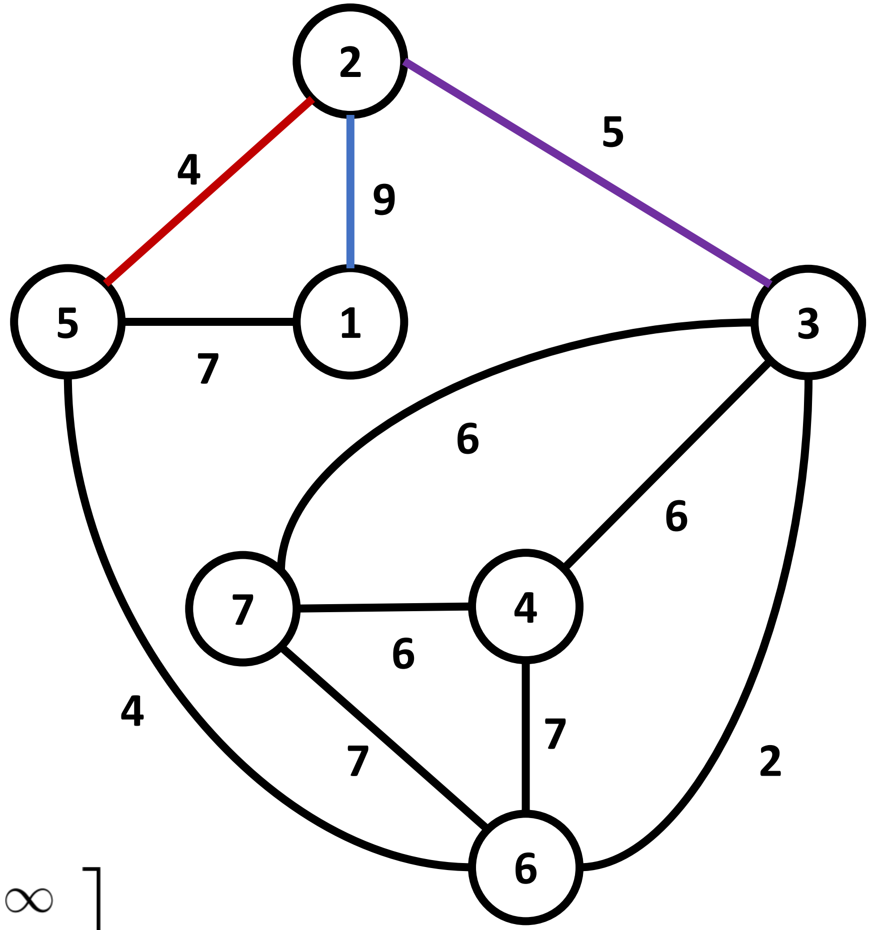
$$D[i, j, 0] = W[i, j]$$

$$D[i, j, k] = \min\left(D[i, j, k-1], D[i, k, k-1] + D[k, j, k-1]\right)$$

- A simpler version updating D:

$$
\begin{array}{l}
\textbf{function } \textsc{Floyd}(W[\cdot,\cdot], n) \\
\quad D \leftarrow W \\
\quad \textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad D[i, j] \leftarrow \min\left(D[i, j], D[i, k] + D[k, j]\right) \\
\quad \textbf{return } D
\end{array}
$$

Floyd's Algorithm

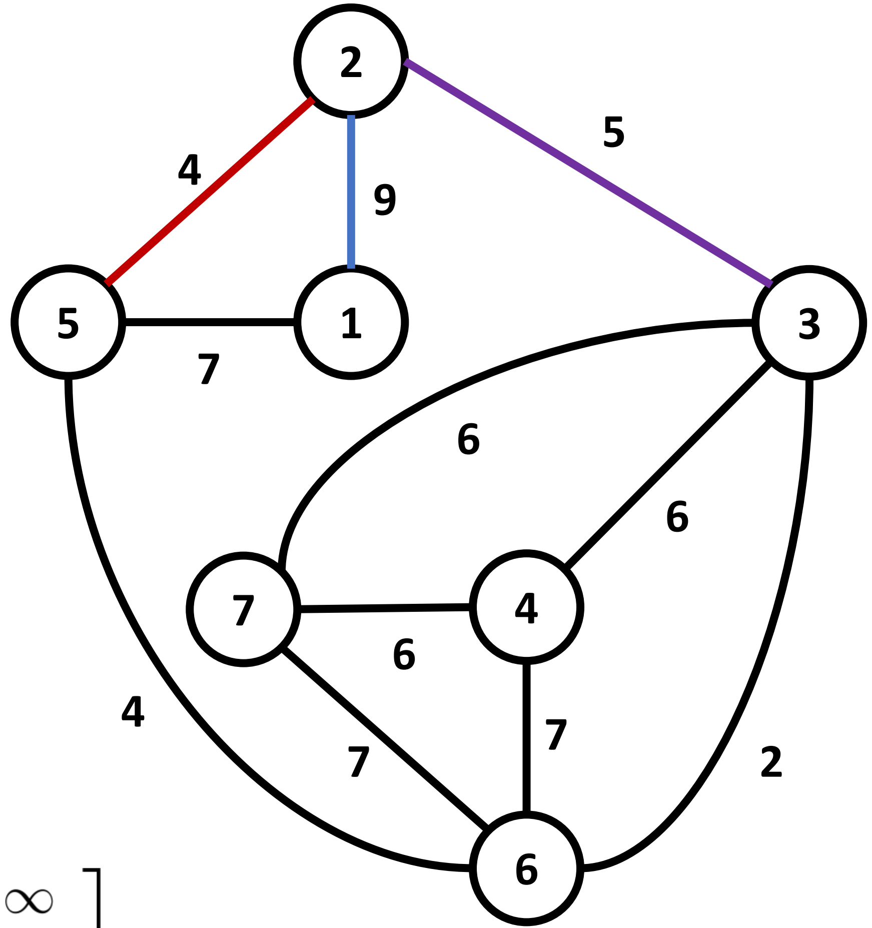


- For *k*=2
  - We can go 1 → 2 → 3, the distance 1 → 3 is 9 + 5 = 14
  - We can go 5 → 2 → 3, the distance of 5 → 3 is 4 + 5 = 9

$$\begin{bmatrix} 0 & \boxed{9} & \infty & \infty & 7 & \infty & \infty \\ 9 & 0 & \boxed{5} & \infty & 4 & \infty & \infty \\ \infty & 5 & 0 & 6 & \infty & 2 & 6 \\ \infty & \infty & 6 & 0 & \infty & 7 & 6 \\ 7 & \boxed{4} & \infty & \infty & 0 & 4 & \infty \\ \infty & \infty & 2 & 7 & 4 & 0 & 7 \\ \infty & \infty & 6 & 6 & \infty & 7 & 0 \end{bmatrix}$$

Floyd's Algorithm

- For *k*=2
  - We can go 1 → 2 → 3, the distance 1 → 3 is 9 + 5 = 14
  - We can go 5 → 2 → 3, the distance of 5 → 3 is 4 + 5 = 9
- The distance matrix gets updated to:



$$\begin{bmatrix} 0 & 9 & \boxed{14} & \infty & 7 & \infty & \infty \\ 9 & 0 & 5 & \infty & 4 & \infty & \infty \\ \boxed{14} & 5 & 0 & 6 & \boxed{9} & 2 & 6 \\ \infty & \infty & 6 & 0 & \infty & 7 & 6 \\ 7 & 4 & \boxed{9} & \infty & 0 & 4 & \infty \\ \infty & \infty & 2 & 7 & 4 & 0 & 7 \\ \infty & \infty & 6 & 6 & \infty & 7 & 0 \end{bmatrix}$$

Greedy Algorithms

- A problem solving strategy is to take the **locally best** choice among all feasible ones.
  - Once we do this, our decision is **irrevocable**.

- We want to change 30 cents using the smallest number of coins.
  - If we assume coin denominations of {25, 10, 5, 1}, we could use as many 25-cent pieces as we can, then do the same for 10-cent pieces, and so on, until we have reached 30 cents (25+5).

  - This **greedy** strategy would not work for denominations {25, 10, 1} (25+1+1+1+1+1 compared to 10+10+10).

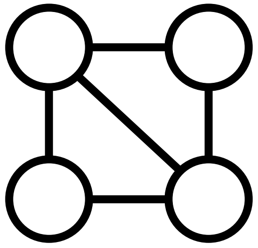
Greedy Algorithms

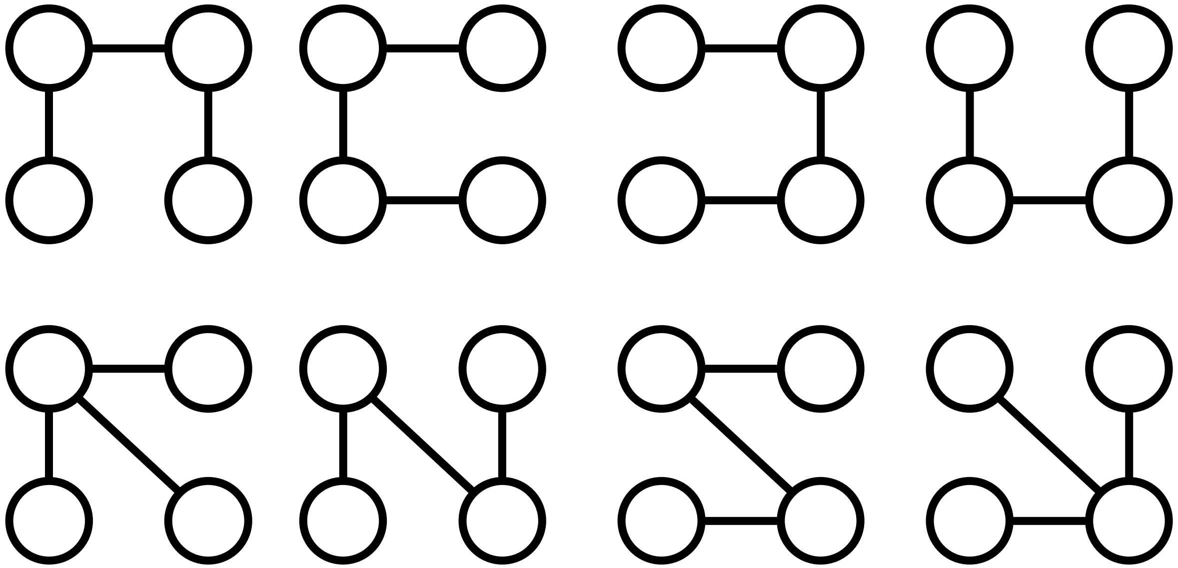
- In general, it is unusual that **locally best** choices yield **global best** results.
  - However, there are problems for which **a greedy algorithm is correct and fast**.
  - In some other problems, a greedy algorithm serve as an acceptable **approximation algorithm.**

- Here we shall look at:
  - Prim's algorithm for finding **minimum spanning trees**
  - Dijkstra's algorithm for **single-source shortest paths**

What is an Spanning Tree?

- Recall that a **tree** is a connected graph with no cycles.
- A **spanning tree** of a graph $\langle V,E \rangle$ is a tree $\langle V,E' \rangle$ where E' is a subset of E
- For example, the graph on the left has eight different spanning trees:

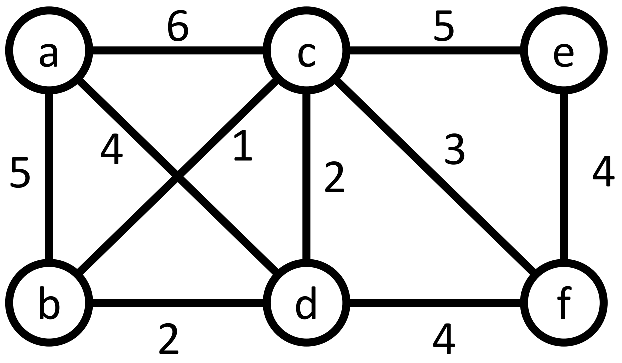




Minimum Spanning Trees of Weighted Graphs

- For a **weighted graph**, some spanning trees are more desirable than others.
  - For example, suppose we have a set of “stations” to connect in a network, and also some possible connections, each with its own **cost**.

- This is the problem of finding a spanning tree with the smallest possible cost.
  - Such tree is a **minimum spanning tree** for the graph.



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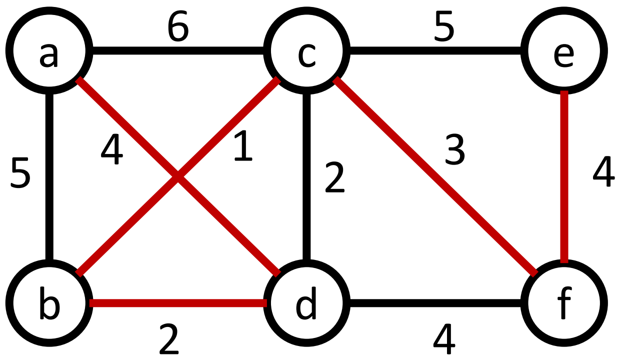


Prim’s Algorithm

- Prim's algorithm is an example of a greedy algorithm.
  - It constructs a sequence of subtrees *T,* by **adding to the latest tree the closest node not currently on it**.

- A simple version:

$$
\begin{array}{l}
\textbf{function } \textsc{Prim}(\langle V, E\rangle) \\
\quad V_T \leftarrow \{v_0\} \\
\quad E_T \leftarrow \emptyset \\
\quad \textbf{for } i \leftarrow 1 \text{ to } |V| - 1 \textbf{ do} \\
\qquad \text{find a minimum-weight edge } (v, u) \in V_T \times (V \setminus V_T) \\
\qquad V_T \leftarrow V_T \cup \{u\} \\
\qquad E_T \leftarrow E_T \cup \{(v, u)\} \\
\quad \textbf{return } E_T
\end{array}
$$

Prim's Algorithm

- But how to find the **minimum-weight edge** (*v*,*u*)?

- A standard way to do this is to organise the nodes that are not yet included in the spanning tree *T* as a **priority queue**, organised in a **min-heap** by edge **cost**.

- The information about which nodes are connected in *T* can be captured by an array *prev* of nodes, indexed by *V*. Namely, when (*v*,*u*) is included, this is captured by setting *prev*[*u*] = v.

Prim's Algorithm

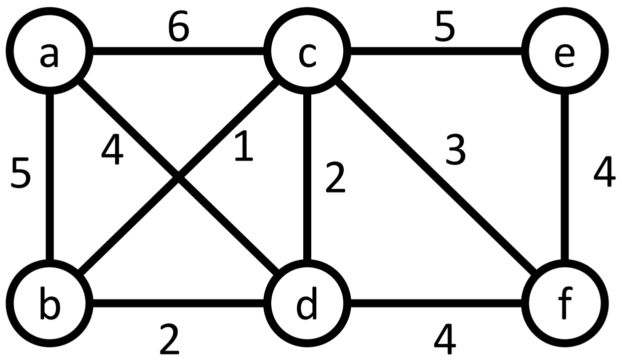
- The complete algorithm is:

```
function PRIM(⟨V, E⟩)
    for each v ∈ V do
        cost[v] ← ∞
        prev[v] ← nil
    pick initial node v_0
    cost[v_0] ← 0
    Q ← INITPRIORITYQUEUE(V)                    ▷ priorities are cost values
    while Q is non-empty do
        u ← EJECTMIN(Q)
        for each (u, w) ∈ E do
            if weight(u, w) < cost[w] then
                cost[w] ← weight(u, w)
                prev[w] ← u
                UPDATE(Q, w, cost[w])           ▷ rearranges priority queue
```

Prim's Algorithm

- On the first loop, we only create the table

function $\textsc{Prim}(\langle V, E\rangle)$
    $\boxed{\begin{array}{l}\textbf{for } \text{each } v \in V \textbf{ do}\\ \quad cost[v] \leftarrow \infty\\ \quad prev[v] \leftarrow nil\end{array}}$
    pick initial node v_0
    $cost[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $weight(u, w) < cost[w]$ **then**
                $cost[w] \leftarrow weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, cost[w])$



Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil

Prim's Algorithm

- Then we pick the first node as the initial one

function $\textsc{Prim}(\langle V, E \rangle)$
  for each $v \in V$ **do**
    $cost[v] \leftarrow \infty$
    $prev[v] \leftarrow nil$
  pick initial node v_0
  $cost[v_0] \leftarrow 0$
  $Q \leftarrow \textsc{InitPriorityQueue}(V)$
  while Q is non-empty **do**
    $u \leftarrow \textsc{EjectMin}(Q)$
    for each $(u, w) \in E$ **do**
      if $weight(u, w) < cost[w]$ **then**
        $cost[w] \leftarrow weight(u, w)$
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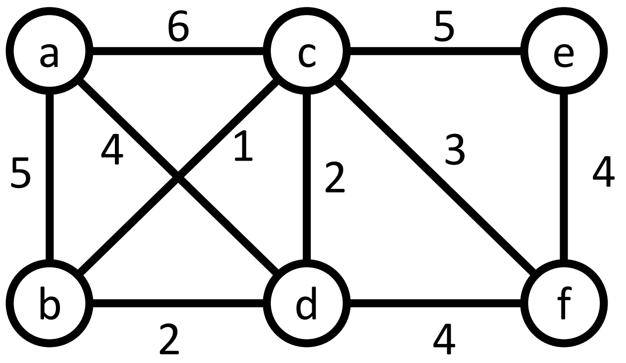
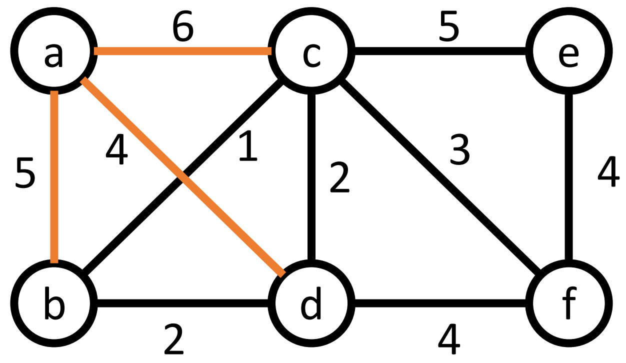
        $\textsc{Update}(Q, w, cost[w])$



Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	**0**	∞	∞	∞	∞	∞
	prev	**nil**	nil	nil	nil	nil	nil

Prim's Algorithm

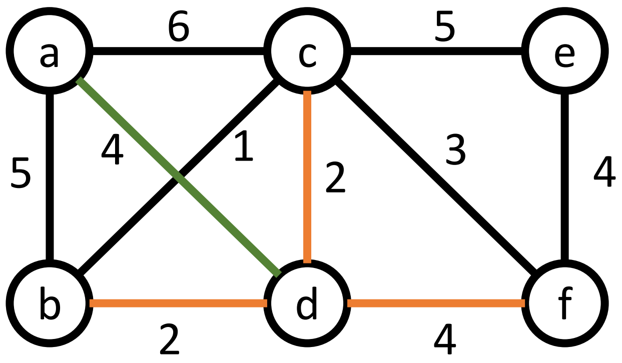


- We take the first node out of the queue and update the costs

function $\textsc{Prim}(\langle V, E\rangle)$
  for each $v \in V$ **do**
    $cost[v] \leftarrow \infty$
    $prev[v] \leftarrow nil$
  pick initial node v_0
  $cost[v_0] \leftarrow 0$
  $Q \leftarrow \textsc{InitPriorityQueue}(V)$
  while Q is non-empty **do**
    $u \leftarrow \textsc{EjectMin}(Q)$
    for each $(u, w) \in E$ **do**
      if $weight(u, w) < cost[w]$ **then**
        $cost[w] \leftarrow weight(u, w)$
        $prev[w] \leftarrow u$
        $\textsc{Update}(Q, w, cost[w])$

Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil

Prim's Algorithm

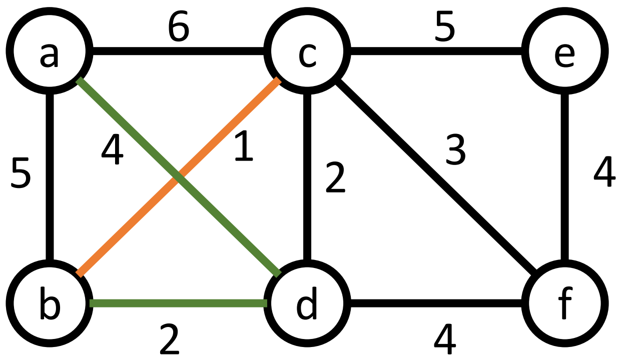


- We eject the node with the lowest cost and update the queue.

function $\textsc{Prim}(\langle V, E\rangle)$
    for each $v \in V$ **do**
        $cost[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    pick initial node v_0
    $cost[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $weight(u, w) < cost[w]$ **then**
                $cost[w] \leftarrow weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, cost[w])$

Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil
a,d	cost		2	2		∞	4
	prev		d	d		nil	d

Prim's Algorithm

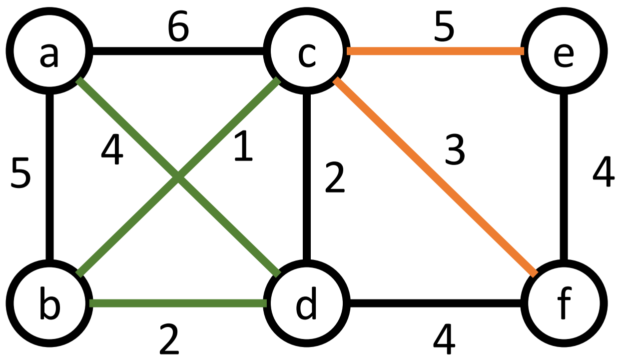


- We eject the next node based on alphabetical order.
  Why is f not updated?

function PRIM($\langle V, E\rangle$)
    for each $v \in V$ **do**
        $cost[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    pick initial node v_0
    $cost[v_0] \leftarrow 0$
    $Q \leftarrow$ INITPRIORITYQUEUE(V)
    while Q is non-empty **do**
        $u \leftarrow$ EJECTMIN(Q)
        for each $(u, w) \in E$ **do**
            if $weight(u, w) < cost[w]$ **then**
                $cost[w] \leftarrow weight(u, w)$
                $prev[w] \leftarrow u$
                UPDATE($Q, w, cost[w]$)

Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil
a,d	cost		2	2		∞	4
	prev		d	d		nil	d
a,d,b	cost			1		∞	4
	prev			b		nil	d

Prim's Algorithm



- We now update f

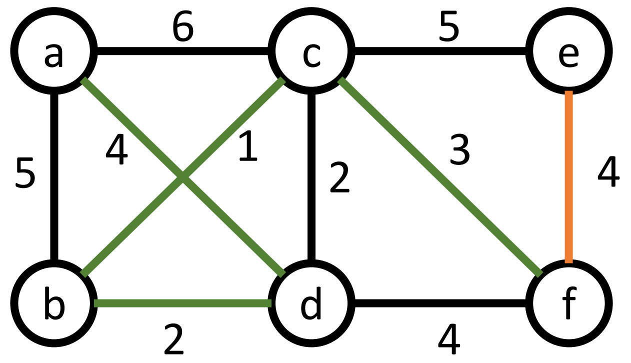
function $\textsc{Prim}(\langle V, E\rangle)$
  for each $v \in V$ **do**
    $cost[v] \leftarrow \infty$
    $prev[v] \leftarrow nil$
  pick initial node v_0
  $cost[v_0] \leftarrow 0$
  $Q \leftarrow \textsc{InitPriorityQueue}(V)$
  while Q is non-empty **do**
    $u \leftarrow \textsc{EjectMin}(Q)$
    for each $(u, w) \in E$ **do**
      if $weight(u, w) < cost[w]$ **then**
        $cost[w] \leftarrow weight(u, w)$
        $prev[w] \leftarrow u$
        $\textsc{Update}(Q, w, cost[w])$

Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil
a,d	cost		2	2		∞	4
	prev		d	d		nil	d
a,d,b	cost			1		∞	4
	prev			b		nil	d
a,d,b,c	cost					5	3
	prev					c	c

Prim's Algorithm

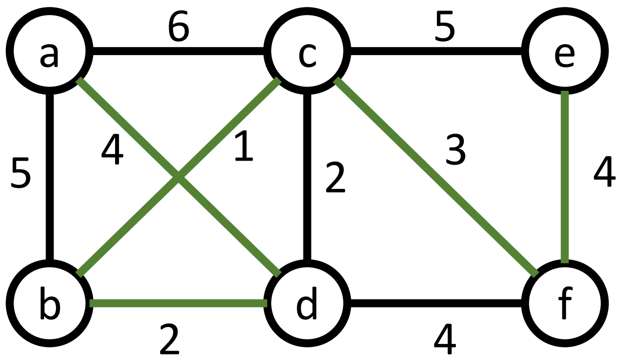
- We reach the last choice

function $\textsc{Prim}(\langle V, E\rangle)$
    for each $v \in V$ **do**
        $cost[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    pick initial node v_0
    $cost[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $weight(u, w) < cost[w]$ **then**
                $cost[w] \leftarrow weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, cost[w])$



Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil
a,d	cost		2	2		∞	4
	prev		d	d		nil	d
a,d,b	cost			1		∞	4
	prev			b		nil	d
a,d,b,c	cost					5	3
	prev					c	c
a,d,b,c,f	cost					4	
	prev					f	

Prim's Algorithm



- The resulting tree is {a,d,b,c,f,e}

function $\textsc{Prim}(\langle V, E \rangle)$
    for each $v \in V$ **do**
        $cost[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    pick initial node v_0
    $cost[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $weight(u, w) < cost[w]$ **then**
                $cost[w] \leftarrow weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, cost[w])$

Tree T		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		5	6	4	∞	∞
	prev		a	a	a	nil	nil
a,d	cost		2	2		∞	4
	prev		d	d		nil	d
a,d,b	cost			1		∞	4
	prev			b		nil	d
a,d,b,c	cost					5	3
	prev					c	c
a,d,b,c,f	cost					4	
	prev					f	
a,d,b,c,f,e	cost						
	prev						

Analysis of Prim’s Algorithm

- First, a crude analysis: For each node, we look through the edges to find those incident to the node, and pick the one with smallest cost. Thus we get $O(|V| \times |E|)$. However, we are using cleverer data structures.

- Using adjacency lists for the graph and a min-heap for the priority queue, we perform $|V|$ - 1 heap deletions (each at cost $O(\log |V|)$) and $|E|$ updates of priorities (each at cost $O(\log |V|)$).

- Altogether $(|V|-1+|E|)\ O(\log |V|)$.

- Since, in a connected graph, $|V|-1 \leq |E|$, this is $O(|E| \log |V|)$.

Dijkstra's Algorithm

- Another classical greedy weighted-graph algorithm is **Dijkstra's algorithm**, whose overall structure is the same as Prim's.

- Recall that Floyd's algorithm gave us the shortest paths, **for every pair of nodes**, in a (directed or undirected) weighted graph. It assumed an adjacency matrix representation and had complexity $O(|V|^3)$.

- **Dijkstra's algorithm** is also a shortest-path algorithm for (directed or undirected) weighted graphs. It finds all shortest paths **from a fixed start node**. Its complexity is the same as that of Prim's algorithm.

Dijkstra's Algorithm

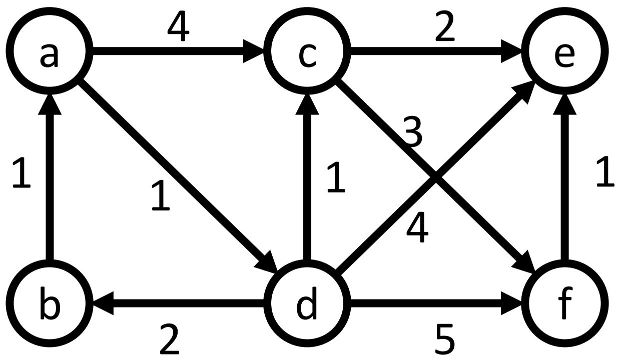
- The complete algorithm is:

```
function DIJKSTRA(⟨V, E⟩, v_0)
    for each v ∈ V do
        dist[v] ← ∞
        prev[v] ← nil
    dist[v_0] ← 0
    Q ← INITPRIORITYQUEUE(V)                         ▷ priorities are distances
    while Q is non-empty do
        u ← EJECTMIN(Q)
        for each (u, w) ∈ E do
            if dist[u] + weight(u, w) < dist[w] then
                dist[w] ← dist[u] + weight(u, w)
                prev[w] ← u
                UPDATE(Q, w, dist[w])                ▷ rearranges priority queue
```

Dijkstra's Algorithm

- On the first loop, we only create the table

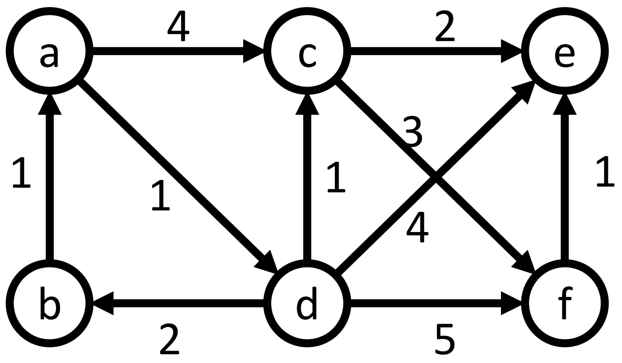
$$
\begin{array}{l}
\textbf{function } \textsc{Dijkstra}(\langle V, E\rangle, v_0) \\
\quad \boxed{\begin{array}{l} \textbf{for } \text{each } v \in V \textbf{ do} \\ \quad dist[v] \leftarrow \infty \\ \quad prev[v] \leftarrow nil \end{array}} \\
\quad dist[v_0] \leftarrow 0 \\
\quad Q \leftarrow \textsc{InitPriorityQueue}(V) \\
\quad \textbf{while } Q \text{ is non-empty } \textbf{do} \\
\qquad u \leftarrow \textsc{EjectMin}(Q) \\
\qquad \textbf{for } \text{each } (u, w) \in E \textbf{ do} \\
\qquad\quad \textbf{if } dist[u] + weight(u, w) < dist[w] \textbf{ then} \\
\qquad\qquad dist[w] \leftarrow dist[u] + weight(u, w) \\
\qquad\qquad prev[w] \leftarrow u \\
\qquad\qquad \textsc{Update}(Q, w, dist[w])
\end{array}
$$



Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil

Dijkstra's Algorithm

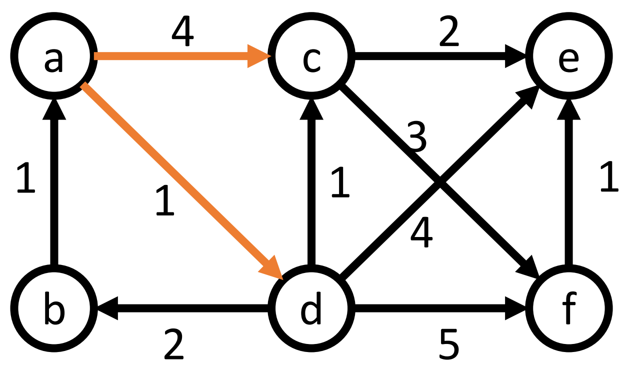
- Then we pick the first node as the initial one



function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
    for each $v \in V$ **do**
        $\mathit{dist}[v] \leftarrow \infty$
        $\mathit{prev}[v] \leftarrow \mathit{nil}$
    $\boxed{\mathit{dist}[v_0] \leftarrow 0}$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $\mathit{dist}[u] + \mathit{weight}(u, w) < \mathit{dist}[w]$ **then**
                $\mathit{dist}[w] \leftarrow \mathit{dist}[u] + \mathit{weight}(u, w)$
                $\mathit{prev}[w] \leftarrow u$
                $\textsc{Update}(Q, w, \mathit{dist}[w])$

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	**0**	∞	∞	∞	∞	∞
	prev	**nil**	nil	nil	nil	nil	nil

Dijkstra's Algorithm

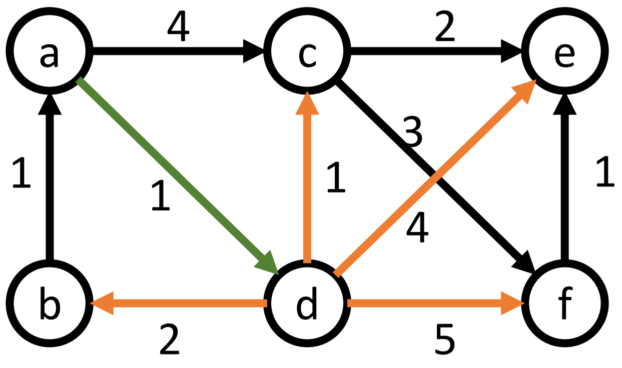


- Then we pick the first node as the initial one

```
function DIJKSTRA(⟨V, E⟩, v_0)
    for each v ∈ V do
        dist[v] ← ∞
        prev[v] ← nil
    dist[v_0] ← 0
    Q ← INITPRIORITYQUEUE(V)
    while Q is non-empty do
        u ← EJECTMIN(Q)
        for each (u, w) ∈ E do
            if dist[u] + weight(u, w) < dist[w] then
                dist[w] ← dist[u] + weight(u, w)
                prev[w] ← u
                UPDATE(Q, w, dist[w])
```

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil

Dijkstra's Algorithm

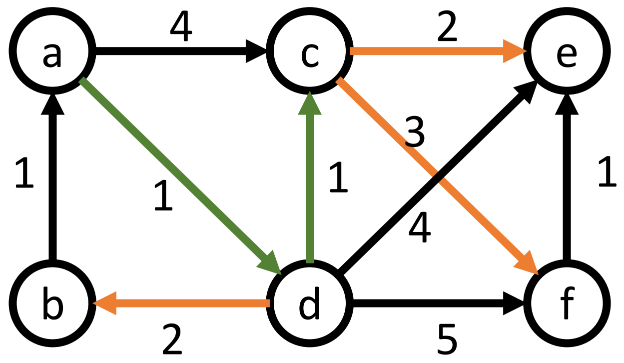


- Then eject the node with the shortest distance from the queue. Then, **we update all the paths by adding 1.**

function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
  for each $v \in V$ **do**
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow nil$
  $dist[v_0] \leftarrow 0$
  $Q \leftarrow \textsc{InitPriorityQueue}(V)$
  while Q is non-empty **do**
    $u \leftarrow \textsc{EjectMin}(Q)$
    for each $(u, w) \in E$ **do**
      if $dist[u] + weight(u, w) < dist[w]$ **then**
        $dist[w] \leftarrow dist[u] + weight(u, w)$
        $prev[w] \leftarrow u$
        $\textsc{Update}(Q, w, dist[w])$

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil
a,d	cost		3	2		5	6
	prev		d	d		d	d

Dijkstra's Algorithm



- Our next node will be the one with the shortest path in overall (b)

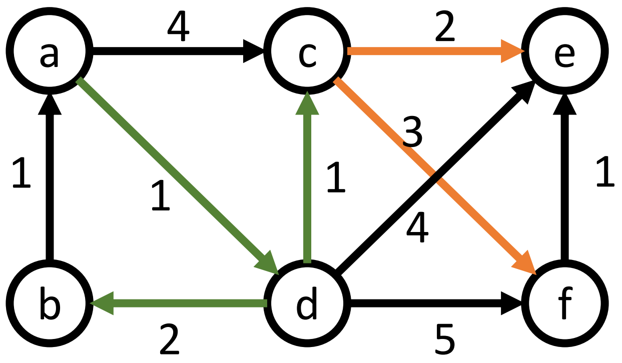
$$
\begin{array}{l}
\textbf{function } \textsc{Dijkstra}(\langle V, E\rangle, v_0) \\
\quad \textbf{for} \text{ each } v \in V \textbf{ do} \\
\qquad dist[v] \leftarrow \infty \\
\qquad prev[v] \leftarrow nil \\
\quad dist[v_0] \leftarrow 0 \\
\quad Q \leftarrow \textsc{InitPriorityQueue}(V) \\
\quad \textbf{while } Q \text{ is non-empty } \textbf{do} \\
\qquad u \leftarrow \textsc{EjectMin}(Q) \\
\qquad \textbf{for} \text{ each } (u, w) \in E \textbf{ do} \\
\qquad\quad \textbf{if } dist[u] + weight(u, w) < dist[w] \textbf{ then} \\
\qquad\qquad dist[w] \leftarrow dist[u] + weight(u, w) \\
\qquad\qquad prev[w] \leftarrow u \\
\qquad\qquad \textsc{Update}(Q, w, dist[w])
\end{array}
$$

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil
a,d	cost		3	2		5	6
	prev		d	d		d	d
a,d,c	cost		3			4	5
	prev		d			c	c

Dijkstra's Algorithm

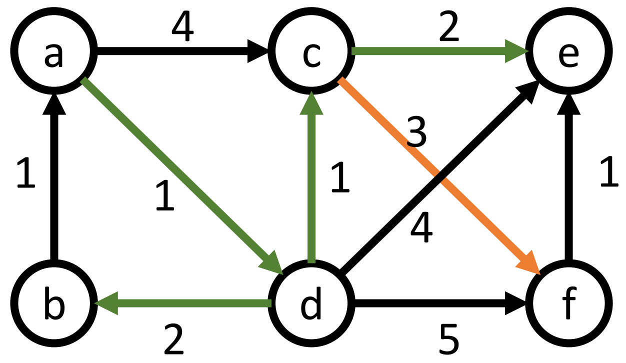
- Now, we continue evaluating from (c)

function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
    for each $v \in V$ **do**
        $dist[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    $dist[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $dist[u] + weight(u, w) < dist[w]$ **then**
                $dist[w] \leftarrow dist[u] + weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, dist[w])$



Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil
a,d	cost		3	2		5	6
	prev		d	d		d	d
a,d,c	cost		3			4	5
	prev		d			c	c
a,d,c,b	cost					4	5
	prev					c	c

Dijkstra's Algorithm

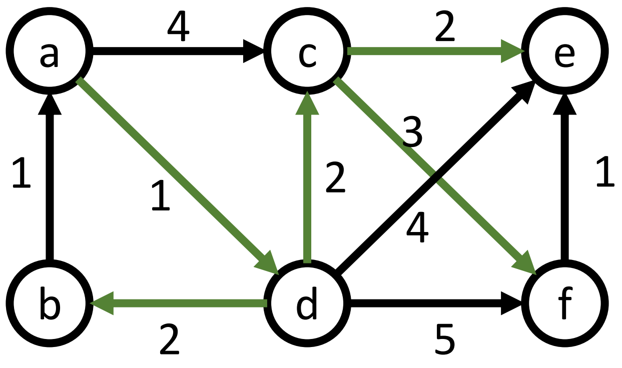


- We arrive at our last decision.

function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
    for each $v \in V$ **do**
        $dist[v] \leftarrow \infty$
        $prev[v] \leftarrow nil$
    $dist[v_0] \leftarrow 0$
    $Q \leftarrow \textsc{InitPriorityQueue}(V)$
    while Q is non-empty **do**
        $u \leftarrow \textsc{EjectMin}(Q)$
        for each $(u, w) \in E$ **do**
            if $dist[u] + weight(u, w) < dist[w]$ **then**
                $dist[w] \leftarrow dist[u] + weight(u, w)$
                $prev[w] \leftarrow u$
                $\textsc{Update}(Q, w, dist[w])$

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil
a,d	cost		3	2		5	6
	prev		d	d		d	d
a,d,c	cost		3			4	5
	prev		d			c	c
a,d,c,b	cost					4	5
	prev					c	c
a,d,c,b,e	cost						5
	prev						c

Dijkstra's Algorithm



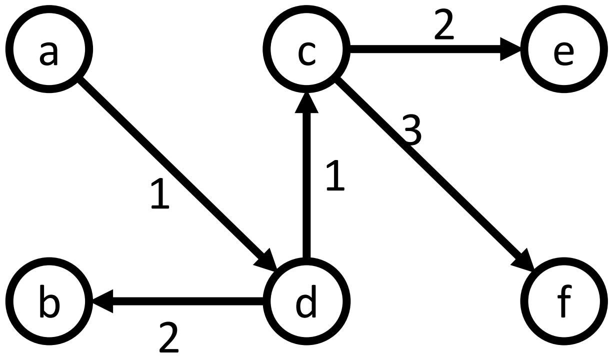
- Our complete tree is {a,d,c,b,e,f}

function $\textsc{Dijkstra}(\langle V, E\rangle, v_0)$
  for each $v \in V$ **do**
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow nil$
  $dist[v_0] \leftarrow 0$
  $Q \leftarrow \textsc{InitPriorityQueue}(V)$
  while Q is non-empty **do**
    $u \leftarrow \textsc{EjectMin}(Q)$
    for each $(u, w) \in E$ **do**
      if $dist[u] + weight(u, w) < dist[w]$ **then**
        $dist[w] \leftarrow dist[u] + weight(u, w)$
        $prev[w] \leftarrow u$
        $\textsc{Update}(Q, w, dist[w])$

Covered		a	b	c	d	e	f
	cost	∞	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
	cost	0	∞	∞	∞	∞	∞
	prev	nil	nil	nil	nil	nil	nil
a	cost		∞	4	1	∞	∞
	prev		nil	a	a	nil	nil
a,d	cost		3	2		5	6
	prev		d	d		d	d
a,d,c	cost		3			4	5
	prev		d			c	c
a,d,c,b	cost					4	5
	prev					c	c
a,d,c,b,e	cost						5
	prev						c
a,d,c,b,e,f	cost						
	prev						

Tracing paths

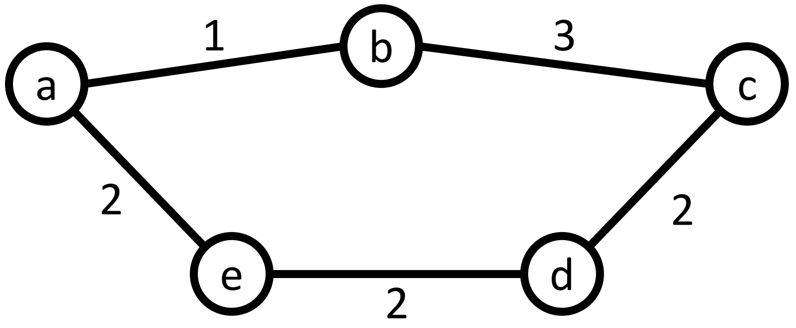
- The array `prev` is not really needed, unless we want to retrace the shortest paths from node *a*



- This tree is referred as the **shortest-path tree**

Spanning trees and Shortest-Path trees

- The shortest-path tree that results from Dijkstra’s algorithm is very similar to the minima spaning tree.



- Exercise:
  - Which edge is missing in the minimal spanning tree?
  - Which edge is missing from the shortest-path tree?
  - Assume that you always started from node a.

Next lecture

- We will have a look to Huffman encoding for data compression