Regular language model:

$$S(q,d) \;=\; \prod_{t \in q} \left(\frac{\mid d \mid}{\mid d \mid + \mu} \cdot \frac{f_{d,t}}{\mid d \mid} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$$=\; \prod_{t \in q} \left(\frac{\cancel{\mid d \mid}}{\mid d \mid + \mu} \cdot \frac{f_{d,t}}{\cancel{\mid d \mid}} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$$=\; \prod_{t \in q} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

Note now that $\log(AB) = \log(A) + \log(B) = \sum_{x \in \{A,B\}} \log(x)$,
and that the log transformation is monotonic (i.e. if $0 < a < b$, then $\log(a) < \log(b)$), so we get:

$$S(q,d) \;=\; \prod_{t \in q} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$$\stackrel{\text{rank}}{=} \sum_{t \in q} \log \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

For the term-oriented language model, we need to first observe that terms in the query but not in the document have a $f_{d,t} = 0$, so we can omit the $\frac{f_{d,t}}{\mid d \mid + \mu}$, so we only consider this expression for terms in both the query and the document:

$$S(q,d) \;=\; \prod_{t \in q} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$$=\; \prod_{t \in q \cap d} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right) \prod_{t \in q - d} \left(\frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$\frac{F_t}{F}$ is going to be the same for every document and we only care about the document ranking. Let's take it out from the second product (of terms in the query, but not in the document):

$$S(q,d) \;=\; \prod_{t \in q \cap d} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right) \prod_{t \in q - d} \left(\frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right)$$

$$\stackrel{\text{rank}}{=} \prod_{t \in q \cap d} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right) \prod_{t \in q - d} \left(\frac{\mu}{\mid d \mid + \mu} \right)$$

As stated in the lecture notes, the following is trivially true:

$$\frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \times \frac{\mid d \mid + \mu}{\mu} \cdot \frac{F}{F_t} = 1$$

I'm going to multiply this expression for every term in both the query and the document (which I can do, because I'm multiplying by a bunch of ones):

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t \in q \cap d} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right) \prod_{t \in q - d} \left(\frac{\mu}{\mid d \mid + \mu} \right)$$

$$\stackrel{\text{rank}}{=} \prod_{t \in q \cap d} \left(\frac{f_{d,t}}{\mid d \mid + \mu} + \frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \right) \prod_{t \in q \cap d} \left(\frac{\mu}{\mid d \mid + \mu} \cdot \frac{F_t}{F} \times \frac{\mid d \mid + \mu}{\mu} \cdot \frac{F}{F_t} \right) \prod_{t \in q - d} \left(\frac{\mu}{\mid d \mid + \mu} \right)$$

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I'm going to take the second of those two expressions I just added, rearrange the leftmost product (they have the same domain, namely the terms in both the query and the document) to include it (but not the one with μ in the numerator), and then simplify over the plus:

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mid d\mid +\mu}+\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\times\frac{\mid d\mid +\mu}{\mu}\cdot\frac{F}{F_t}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mid d\mid +\mu}+\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\left(\frac{\mid d\mid +\mu}{\mu}\cdot\frac{F}{F_t}\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mid d\mid +\mu}\cdot\frac{\mid d\mid +\mu}{\mu}\cdot\frac{F}{F_t}+\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\cdot\frac{\mid d\mid +\mu}{\mu}\cdot\frac{F}{F_t}\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\cancel{\mid d\mid +\mu}}\cdot\frac{\cancel{\mid d\mid +\mu}}{\mu}\cdot\frac{F}{F_t}+\frac{\cancel{\mu}}{\cancel{\mid d\mid +\mu}}\cdot\frac{\cancel{F_t}}{\cancel{F}}\cdot\frac{\cancel{\mid d\mid +\mu}}{\cancel{\mu}}\cdot\frac{\cancel{F}}{\cancel{F_t}}\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

I still don't care about multiplying through by a bunch of $\frac{F_t}{F}$:

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\cdot\frac{F_t}{F}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

I now note that the last two products are the same expression, one over the terms in both the query and the document, the other in the query but not the document — this is clearly all of the terms in the query.

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q\cap d}\left(\frac{\mu}{\mid d\mid +\mu}\right)\prod_{t\in q-d}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

The second product there doesn't actually depend on the term — which means I'm just multiplying by $\frac{\mu}{\mid d\mid +\mu}$ for every term in the query. I've just described exponentiation!

$$S(q,d) \stackrel{\text{rank}}{=} \prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)\prod_{t\in q}\left(\frac{\mu}{\mid d\mid +\mu}\right)$$

$$\stackrel{\text{rank}}{=} \left(\frac{\mu}{\mid d\mid +\mu}\right)^{|q|}\times\prod_{t\in q\cap d}\left(\frac{f_{d,t}}{\mu}\cdot\frac{F}{F_t}+1\right)$$

Then I log-transform to get my final formulation:

$$S(q, d) \stackrel{\text{rank}}{=} (\frac{\mu}{\mid d \mid + \mu})^{\mid q \mid} \times \prod_{t \in q \cap d} (\frac{f_{d,t}}{\mu} \cdot \frac{F}{F_t} + 1)$$

$$\stackrel{\text{rank}}{=} \log((\frac{\mu}{\mid d \mid + \mu})^{\mid q \mid} \times \prod_{t \in q \cap d} (\frac{f_{d,t}}{\mu} \cdot \frac{F}{F_t} + 1))$$

$$\stackrel{\text{rank}}{=} \log((\frac{\mu}{\mid d \mid + \mu})^{\mid q \mid}) + \log(\prod_{t \in q \cap d} (\frac{f_{d,t}}{\mu} \cdot \frac{F}{F_t} + 1))$$

$$\stackrel{\text{rank}}{=} \mid q \mid \log(\frac{\mu}{\mid d \mid + \mu}) + \sum_{t \in q \cap d} \log(\frac{f_{d,t}}{\mu} \cdot \frac{F}{F_t} + 1)$$