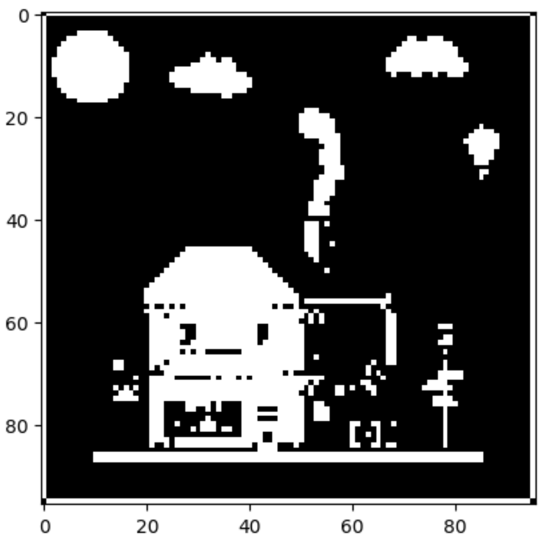
1. Image Denoising Using loopy Belief Propagation

This part explores max-product Loopy **belief propagation** (Loopy-BP) method for denoising binary images. Each element of the matrix can be either 1 or -1, with 1 representing white pixels and -1 representing black pixels.

Data preparation

```
!pip install wget

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.image as mpimg
import PIL.Image as Image
from os.path import exists
from wget import download
from tqdm import tqdm

filename, url = "trc113gqu9651.png", "https://i.redd.it/trc113gqu9651.png"

def load_img():
    if not exists(filename):
        download(url)

    with open(filename, 'rb') as fp:
        img2 = Image.open(fp).convert('L')
        img2 = img2.resize((96, 96), Image.LANCZOS)
        img2 = np.array(img2)
    return (img2 > 120) * 2.0 - 1

img_true = load_img()
plt.imshow(img_true, cmap='gray')
```

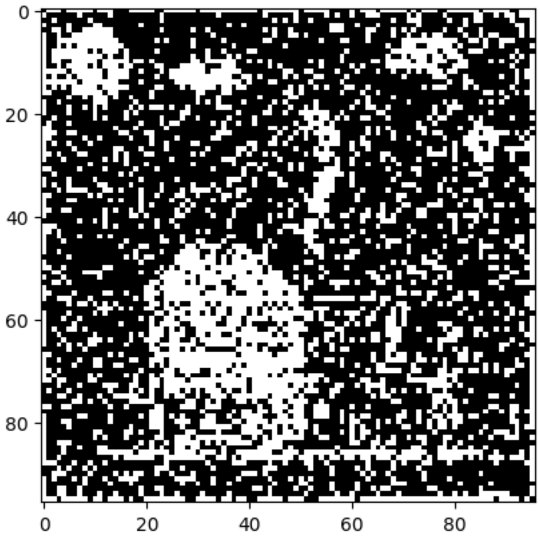
```
Collecting wget
  Downloading wget-3.2.zip (10 kB)
  Preparing metadata (setup.py) ... done
Building wheels for collected packages: wget
  Building wheel for wget (setup.py) ... done
  Created wheel for wget: filename=wget-3.2-py3-none-any.whl size=9656 sha256=110c95b3052006e953d043540eb3c8e31394fc2951ae0d121f0c1397b9
  Stored in directory: /root/.cache/pip/wheels/40/b3/0f/a40dbd1c6861731779f62cc4babcb234387e11d697df70ee97
Successfully built wget
Installing collected packages: wget
Successfully installed wget-3.2
<matplotlib.image.AxesImage at 0x7b1986790950>
```



Introducing noise into the image, for each pixel, swap its value between 1 and -1 with rate 0.2.

```
def gen_noisyimg(img, noise=.05):
        swap = np.random.binomial(1, noise, size=img.shape)
        return img * (1 - 2 * swap)

noise = 0.2
img_noisy = gen_noisyimg(img_true, noise)
plt.imshow(img_noisy, cmap='gray')
```

```
<matplotlib.image.AxesImage at 0x7b19861afe10>
```



The Loopy BP algorithm

Implementing the **max-product** BP to obtain the MAP estimate.

Initialization:

For discrete node x_j with 2 possible states, $m_{i\to j}$ can be written as a 2 dimensional real vector $\mathrm{m}_{i,j}$ with $m_{i\to j}(x_j) = \mathrm{m}_{i,j}[index(x_j)]$. We initialize them uniformly to $m_{i\to j}(x_j) = 1/2$.

For a number of iterations:

For node x_j in $\{x_s\}_{s=1}^n$:

1. Compute the product of inbound messages from neighbours of x_j:
$$\prod_{k\in N(j)\neq i} m_{k\to j}(x_j)$$
2. Compute potentials $\psi_j(x_j) = \exp(\beta x_j y_j)$ and $\psi_{ij}(x_i, x_j) = \exp(J x_i x_j)$. This expression specifically holds when $x \in \{-1, +1\}$.
3. Maximize over $x_j = \{-1, +1\}$ to get $m_{j\to i}(x_i)$:
$$m_{j\to i}(x_i) = \max_{x_j} \psi_j(x_j)\psi_{ij}(x_i, x_j) \prod_{k\in N(j)\neq i} m_{k\to j}(x_j)$$
4. Normalize messages for stability $m_{j\to i}(x_i) = m_{j\to i}(x_i)/\sum_{x_i} m_{j\to i}(x_i)$.

Compute beliefs after message passing is done.
$$b(x_i) \propto \psi_i(x_i) \prod_{j\in N(i)} m_{j\to i}(x_i).$$

Initialization

Initialize the message between neighbor pixels uniformly as $m_{j\to i}(x_i) = 1/k$. Since each pixel can only be 1 or -1, message has two values $m_{j\to i}(1)$ and $m_{j\to i}(-1)$. We also initialize hyperparameters J and β.

```
y = img_noisy.reshape([img_true.size, ])
num_nodes = len(y)
init_message = np.zeros([2, num_nodes, num_nodes]) + .5
J = 1.0
beta = 1.0
```

```
def get_neighbors_of(node):
        neighbors = []
        m = int(np.sqrt(num_nodes))
        if (node + 1) % m != 0:
```

```
            neighbors += [node + 1]
        if node % m != 0:
            neighbors += [node - 1]
        if node + m < num_nodes:
            neighbors += [node + m]
        if node - m >= 0:
            neighbors += [node - m]

        return set(neighbors)
```

1.1 Implement message passing in BP

```
    def get_message(node_from, node_to, messages):
        # product of received messages
        neighbors = get_neighbors_of(node_from)
        neighbors_clean = neighbors.difference(set[node_to])
        mess_received = np.prod(messages[:, list(neighbors_clean), node_from], axis=1)

        # exponent when x_i is 1 and -1
        i_positive = beta * y[node_from] + J
        i_negative = beta * y[node_from] - J

        # final message maximizing over x_j
        final_mess = np.vstack([np.max(np.exp(i_positive * np.array([1,-1])) * mess_received), \
        np.max(np.exp(i_negative * np.array([1,-1])) * mess_received)])
        return final_mess.reshape([2,])
        pass


    def step_bp(step, messages):
        for node_from in range(num_nodes):
            for node_to in get_neighbors_of(node_from):
                m_new = get_message(node_from, node_to, messages)
                # normalize
                m_new = m_new / np.sum(m_new)

                messages[:, node_from, node_to] = step * m_new + (1. - step) * \
                    messages[:, node_from, node_to]
        return messages
```

```
    # Run loopy BP update for 10 iterations
    num_iter = 10
    step = 0.5
    for it in range(num_iter):
        init_message = step_bp(step, init_message)
        print(it + 1,'/',num_iter)
```

```
1 / 10
2 / 10
3 / 10
4 / 10
5 / 10
6 / 10
7 / 10
8 / 10
9 / 10
10 / 10
```

1.2 Computing belief from messages

Calculating the unnormalized belief for each pixel

$$\tilde{b}(x_i) = \psi_i(x_i) \prod_{j \in N(i)} m_{j \to i}(x_i),$$

and normalizing the belief across all pixels

$$b(x_i) = \frac{\tilde{b}(x_i)}{\sum_{x_j} \tilde{b}(x_j)}.$$

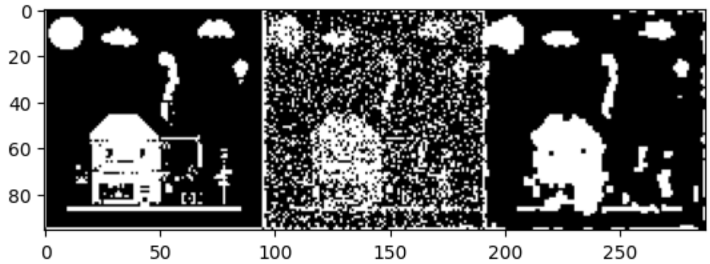
```
    def update_beliefs(messages):
        beliefs = np.zeros([2, num_nodes])
        for node in range(num_nodes):
            neighbors = get_neighbors_of(node)
            received_mess = np.prod(messages[:, list(neighbors), node], axis=1)
            each_belief = np.exp(beta * y[node] * np.array([1, -1])) * received_mess
            beliefs[:, node] += each_belief / np.sum(each_belief)
        return beliefs
```

```
# call update_beliefs() once
beliefs = update_beliefs(init_message)
```

To get the denoised image, we use 0.5 as the threshold and consider pixel with belief less than threshold as black while others as white, which is the same as choosing the pixel with maximum probability

```
pred = 2. * ((beliefs[0, :] > .5) + .0) - 1.
img_out = pred.reshape(img_true.shape)

plt.imshow(np.hstack([img_true, img_noisy, img_out]), cmap='gray')
```

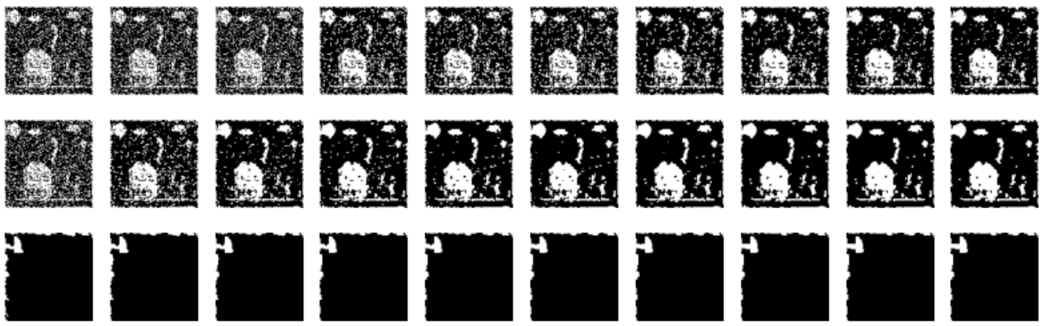
```
<matplotlib.image.AxesImage at 0x7b1986ba8090>
```



1.3 Momentum in belief propagation

```
def test_trajectory(step_size, max_step=10):
        # re-initialize each time
        messages = np.zeros([2, num_nodes, num_nodes]) + .5
        images = []

        for i in range(max_step):
            messages = step_bp(step_size, messages)
            beliefs = update_beliefs(messages)
            pred = 2. * ((beliefs[0, :] > .5) + .0) - 1.
            img_out = pred.reshape(img_true.shape)
            images.append(img_out)
        return images
```

```
def plot_series(images):
    n = len(images)
    fig, ax = plt.subplots(1, n)
    for i in range(n):
        ax[i].imshow(images[i], cmap='gray')
        ax[i].set_axis_off()
    fig.set_figwidth(10)
    fig.show()

for i in [.1, .3, 1.]:
    plot_series(test_trajectory(i,10))
```

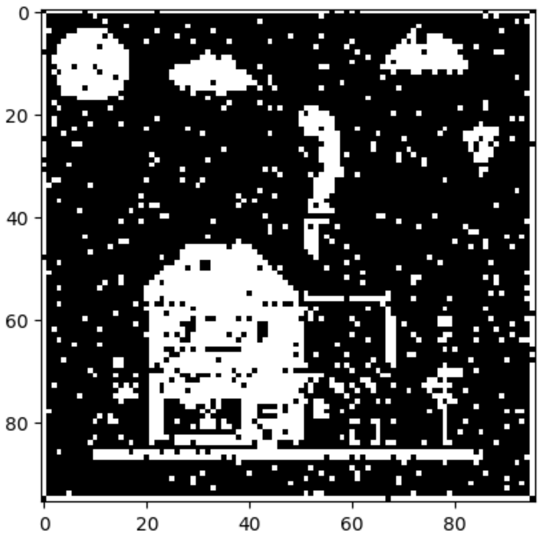


1.4 Noise level and the hyperparameter J

Exploring how the level of noise in the image influences our choice in the hyperparameter J.

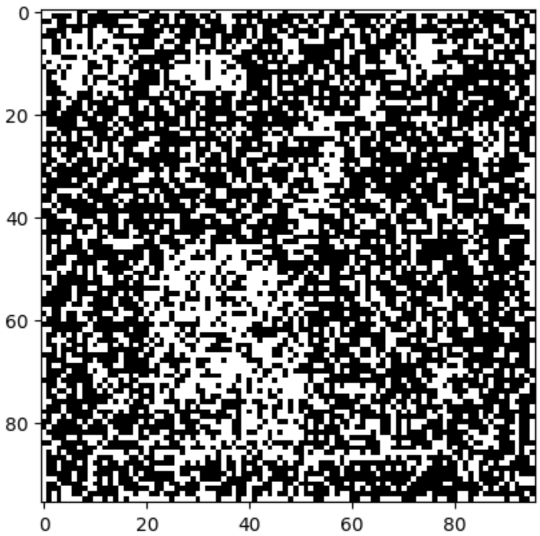
```
# image of noise 0.05
img_1 = gen_noisyimg(img_true, noise=0.05)
plt.imshow(img_1, cmap='gray')
```

```
<matplotlib.image.AxesImage at 0x7b19862cdc90>
```

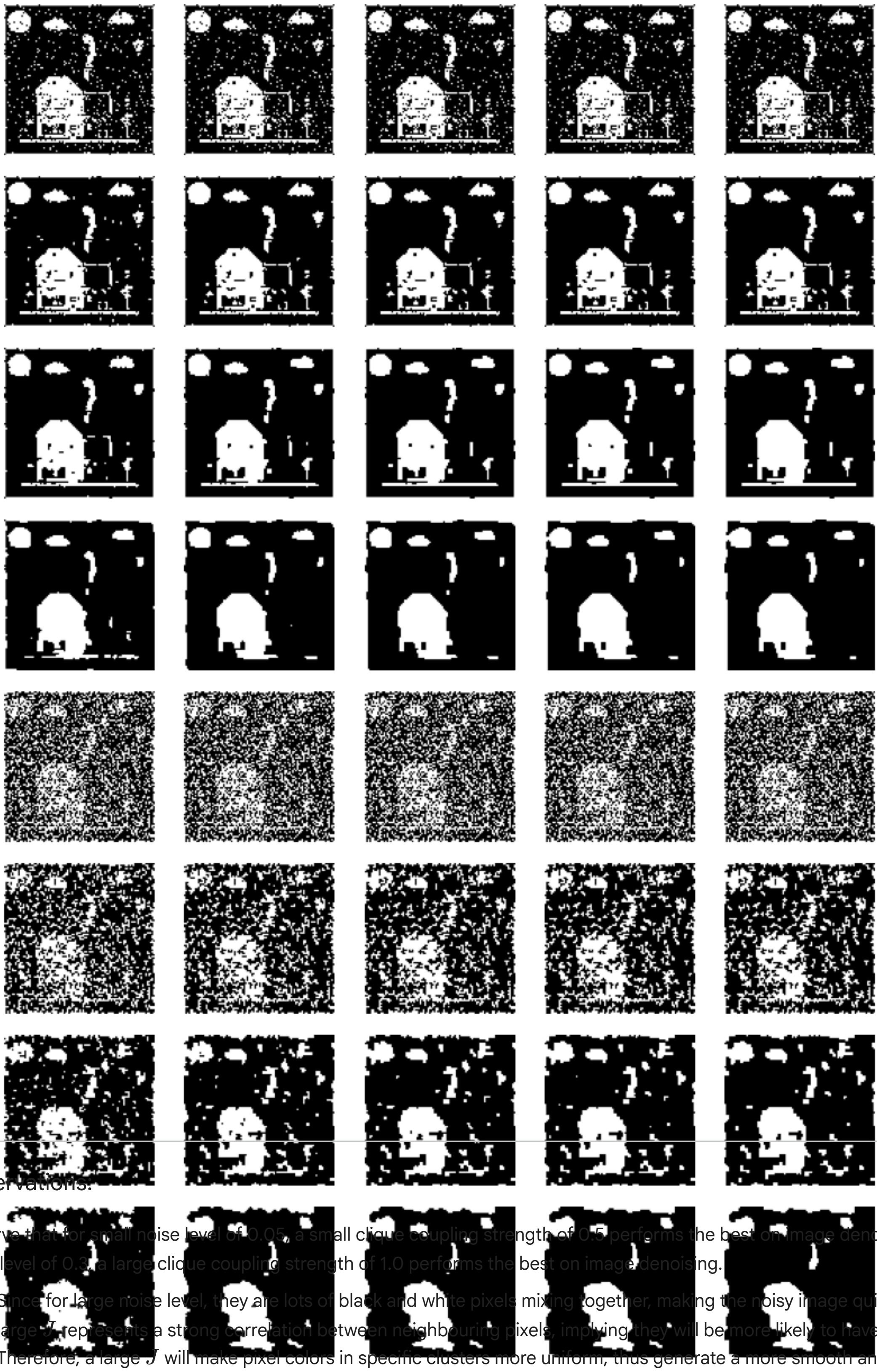


```
# image of noise 0.3
img_2 = gen_noisyimg(img_true, noise=0.3)
plt.imshow(img_2, cmap='gray')
```

```
<matplotlib.image.AxesImage at 0x7b1969f14090>
```



```
strength = np.array([0.1, 0.5, 1, 5])

# noise level = 0.05
for J in [0.1, 0.5, 1, 5]:
    y = img_1.reshape([img_true.size, ])
    plot_series(test_trajectory(0.8, 5))

# noise level = 0.3
for J in [0.1, 0.5, 1, 5]:
    y = img_2.reshape([img_true.size, ])
    plot_series(test_trajectory(0.8, 5))
```



Observations:

Observe that for small noise level of 0.05, a small clique coupling strength of 0.5 performs the best on image denoising. While for large noise level of 0.3, a large clique coupling strength of 1.0 performs the best on image denoising.

- Since for large noise level, they are lots of black and white pixels mixing together, making the noisy image quite unclear. While large J represents a strong correlation between neighbouring pixels, implying they will be more likely to have the same color. Therefore, a large J will make pixel colors in specific clusters more uniform, thus generate a more smooth and clear image.
- When noise level is small, a large J will cause oversmoothing issue. It will generate large clusters of same color, thus losing detailed information of true image. So a small J is perferable for small noise level.

2. Markov chain Monte Carlo in the TrueSkill model

We'll consider a slightly simplified version of the original trueskill model. We assume that each player has a true, but unknown skill $z_i \in \mathbb{R}$. We use N to denote the number of players.

The prior:

The prior over each player's skill is a standard normal distribution, and all player's skills are *a priori* independent.

The likelihood:

For each observed game, the probability that player i beats player j, given the player's skills z_A and z_B, is:

$$p(A \text{ beat } B | z_A, z_B) = \sigma(z_A - z_B)$$

where

$$\sigma(y) = \frac{1}{1 + \exp(-y)}$$

We chose this function simply because it's close to zero or one when the player's skills are very different, and equals one-half when the player skills are the same. This likelihood function is the only thing that gives meaning to the latent skill variables $z_1 \dots z_N$.

```
!pip install wget
import os
import os.path

import matplotlib.pyplot as plt
import wget

import pandas as pd


import numpy as np
from scipy.stats import norm
import scipy.io
import scipy.stats
import torch
import random
from torch.distributions.normal import Normal

from functools import partial

import matplotlib.pyplot as plt
```

```
Requirement already satisfied: wget in /usr/local/lib/python3.11/dist-packages (3.2)
```

2.1 Implementing the TrueSkill Model

2.1.a

Given a $K \times N$ array where each row is a setting of the skills for all N players, it returns a $K \times 1$ array, where each row contains a scalar giving the log-prior for that set of skills.

```
def log_joint_prior(zs_array):
    N = zs_array.shape[1]
    log_prior = -0.5 * N * np.log(2 * np.pi) - \
    0.5 * np.sum(zs_array ** 2, axis = 1)
    return log_prior.reshape(-1, 1)
    pass
```

2.1.

Given a pair of skills z_a and z_b, **logp_a_beats_b** evaluates the log-likelihood that player with skill z_a beat player with skill z_b under the model detailed above, and **logp_b_beats_a** is vice versa.

```
def logp_a_beats_b(z_a, z_b):
    return -torch.logaddexp(torch.tensor([0]), z_b - z_a)
    pass

def logp_b_beats_a(z_a, z_b):
    return -torch.logaddexp(torch.tensor([0]), z_a - z_b)
    pass
```

2.2 Examining the posterior for only two players and toy data

We'll first consider the case where we only have 2 players, A and B. We'll examine how the prior and likelihood interact when conditioning on different sets of games.

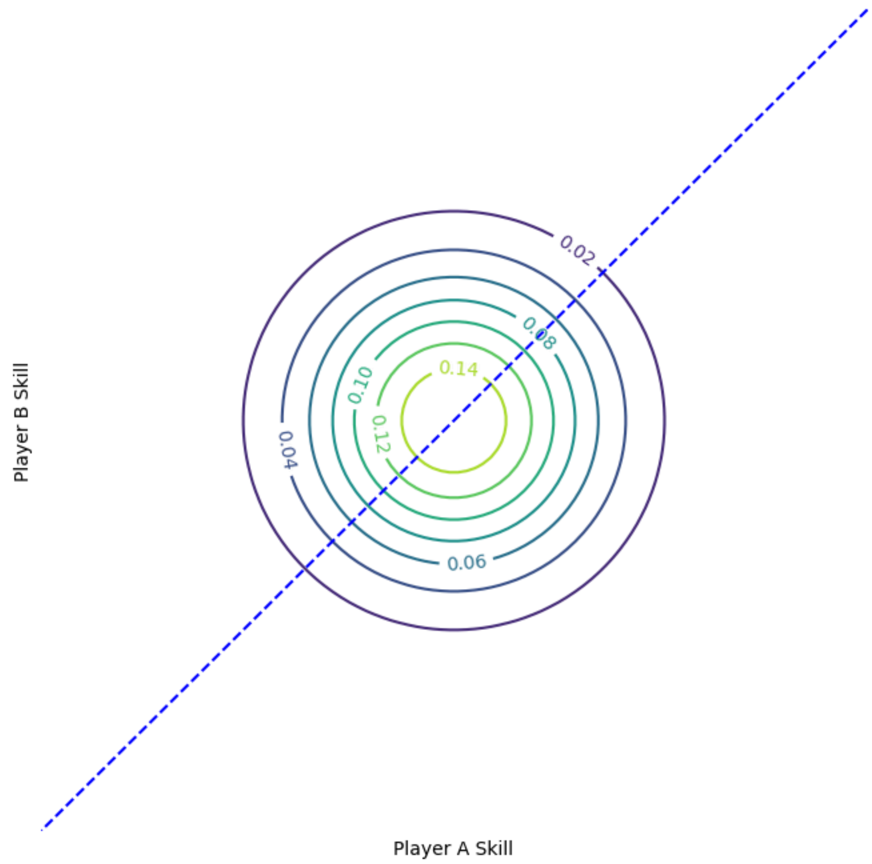
```
# Plotting helper functions
def plot_isocontours(ax, func, steps=100):
        x = torch.linspace(-4, 4, steps=steps)
        y = torch.linspace(-4, 4, steps=steps)
        X, Y = torch.meshgrid(x, y, indexing="ij")
        Z = func(X, Y)
```

```
        cs = plt.contour(X, Y, Z )
        plt.clabel(cs, inline=1, fontsize=10)
        ax.set_yticks([])
        ax.set_xticks([])

def plot_2d_fun(f, x_axis_label="", y_axis_label="", scatter_pts=None):
        fig = plt.figure(figsize=(8,8), facecolor='white')
        ax = fig.add_subplot(111, frameon=False)
        ax.set_xlabel(x_axis_label)
        ax.set_ylabel(y_axis_label)
        plot_isocontours(ax, f)
        if scatter_pts is not None:
            plt.scatter(scatter_pts[:,0], scatter_pts[:, 1])
        plt.plot([4, -4], [4, -4], 'b--')     # Line of equal skill
        plt.show(block=True)
        plt.draw()
```

2.2.

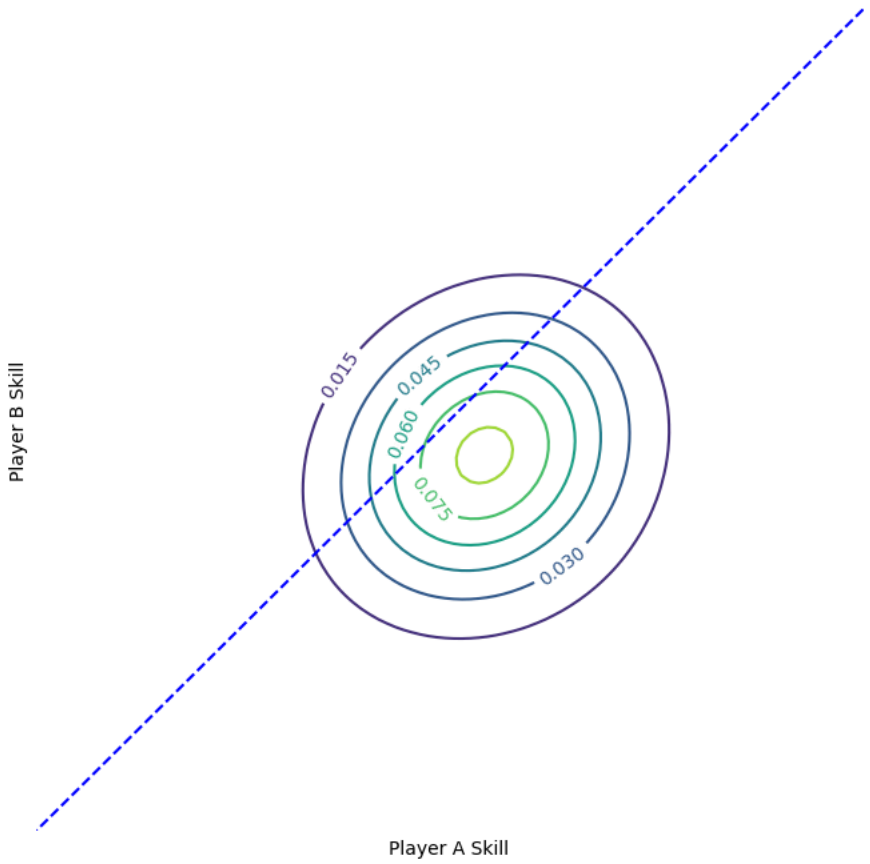
Isocontours of the joint prior over 2 players' skills.

```
def log_prior_over_2_players(z1, z2):
    return -torch.log(torch.tensor([2 * torch.pi])) - \
    0.5 * (z1 ** 2 + z2 ** 2)
    pass

def prior_over_2_players(z1, z2):
    return torch.exp(log_prior_over_2_players(z1, z2))

plot_2d_fun(prior_over_2_players, "Player A Skill", "Player B Skill")
```



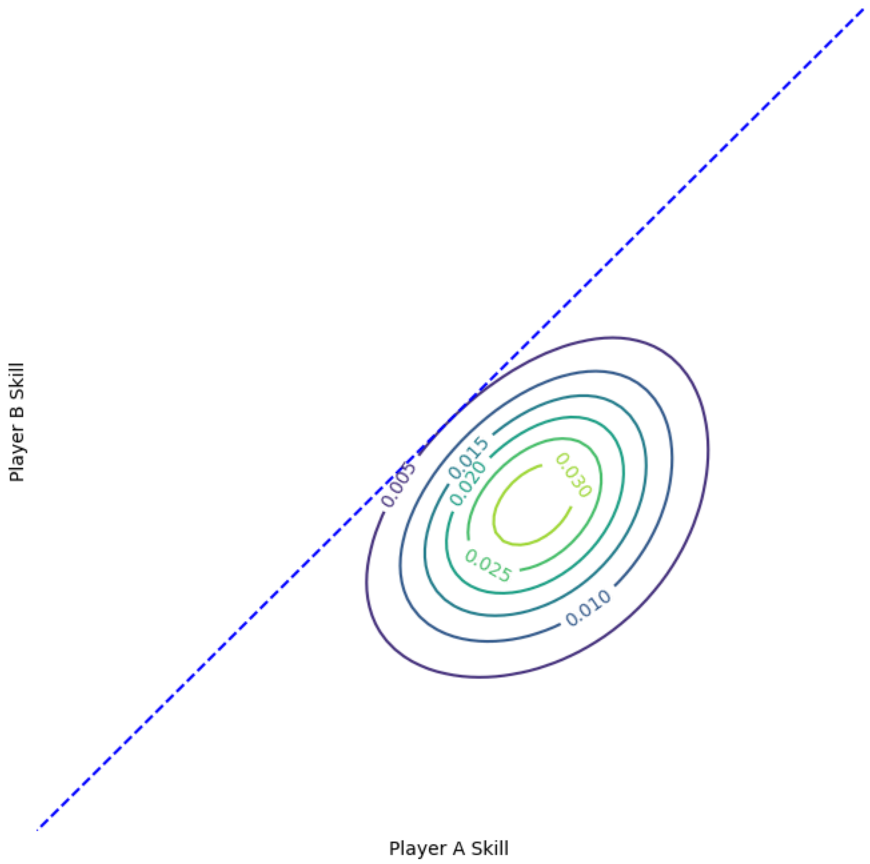

```
<Figure size 640x480 with 0 Axes>
```

```
def log_posterior_A_beat_B(z1, z2):
    return log_prior_over_2_players(z1, z2) + logp_a_beats_b(z1, z2)
    pass

def posterior_A_beat_B(z1, z2):
    return torch.exp(log_posterior_A_beat_B(z1, z2))

plot_2d_fun(posterior_A_beat_B, "Player A Skill", "Player B Skill")
# Note that the posterior probabilities shown are unnormalized
```



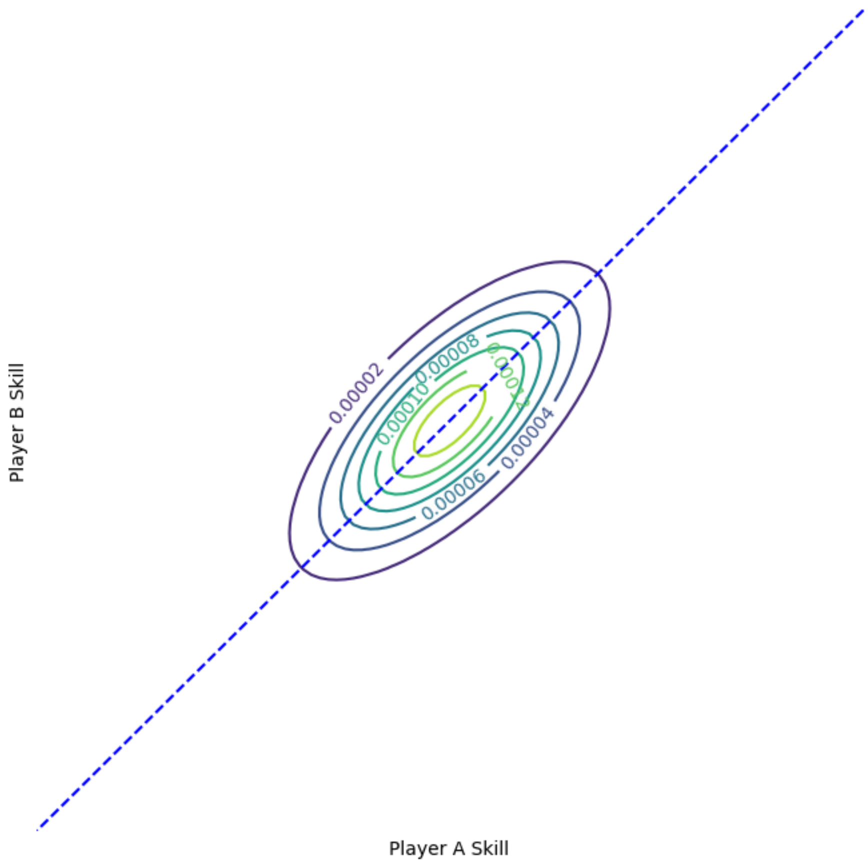
<Figure size 640x480 with 0 Axes>

```
def  log_posterior_A_beat_B_5_times(z1,  z2):
    return  log_prior_over_2_players(z1,  z2)  +  5  *  logp_a_beats_b(z1,  z2)
    pass

def  posterior_A_beat_B_5_times(z1,  z2):
    return  torch.exp(log_posterior_A_beat_B_5_times(z1,  z2))

plot_2d_fun(posterior_A_beat_B_5_times,  "Player  A  Skill",  "Player  B  Skill")
```



<Figure size 640x480 with 0 Axes>

```
def log_posterior_beat_each_other_5_times(z1, z2):
    return log_prior_over_2_players(z1, z2) + 5 * logp_a_beats_b(z1, z2) + \
    5 * logp_b_beats_a(z1, z2)
    pass

def posterior_beat_each_other_5_times(z1, z2):
    return torch.exp(log_posterior_beat_each_other_5_times(z1, z2))

plot_2d_fun(posterior_beat_each_other_5_times, "Player A Skill", "Player B Skill")
```




```
<Figure size 640x480 with 0 Axes>
```

2.3 Hamiltonian and Langevin Monte Carlo on Two Players and Toy Data

```
random.seed(42)
torch.manual_seed(42)
```

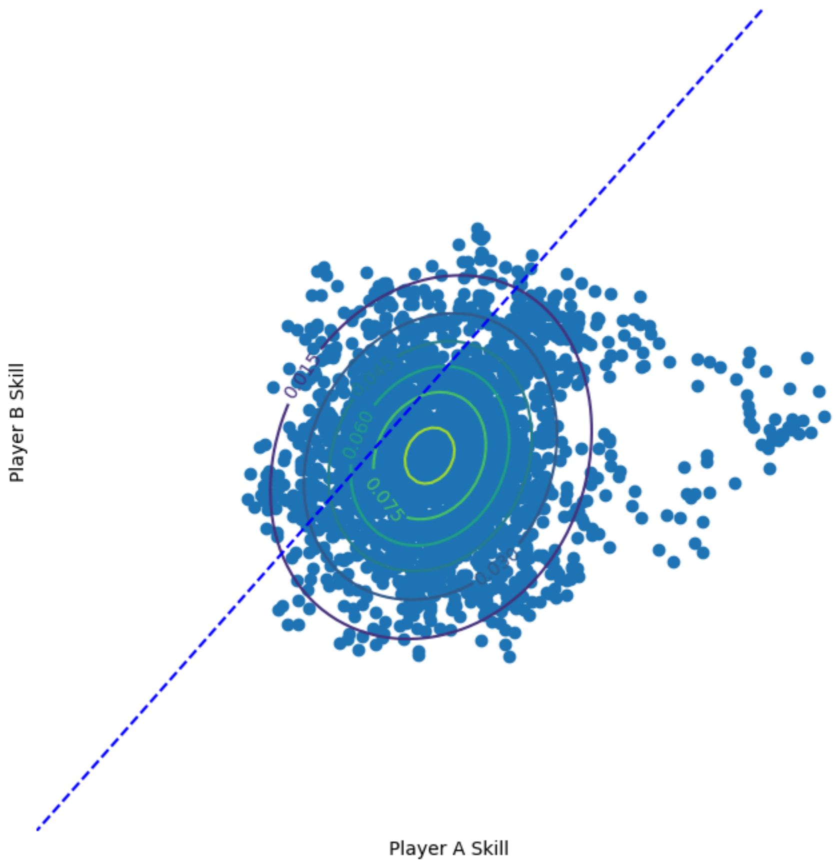
```
<torch._C.Generator at 0x7b1969d69a90>
```

```
# Hamiltonian Monte Carlo
from tqdm import trange, tqdm_notebook  # Progress meters

def leapfrog(params_t0, momentum_t0, stepsize, logprob_grad_fun):
    # Performs a reversible update of parameters and momentum
    momentum_thalf = momentum_t0     + 0.5 * stepsize * logprob_grad_fun(params_t0)
    params_t1 =         params_t0          +            stepsize * momentum_thalf
    momentum_t1 =       momentum_thalf + 0.5 * stepsize * logprob_grad_fun(params_t1)
    return params_t1, momentum_t1


def iterate_leapfrogs(theta, v, stepsize, num_leapfrog_steps, grad_fun):
    for i in range(0, num_leapfrog_steps):
        theta, v = leapfrog(theta, v, stepsize, grad_fun)
    return theta, v

def metropolis_hastings(state1, state2, log_posterior):
    accept_prob = torch.exp(log_posterior(state2) - log_posterior(state1))
    if random.random() < accept_prob:
        return state2  # Accept
    else:
        return state1  # Reject

def draw_samples_hmc(num_params, stepsize, num_leapfrog_steps, n_samples, log_posterior):
    theta = torch.zeros(num_params)
```

```
    def log_joint_density_over_params_and_momentum(state):
        params, momentum = state
        m = Normal(0., 1.)
        return m.log_prob(momentum).sum(axis=-1) + log_posterior(params)

    def grad_fun(zs):
        zs = zs.detach().clone()
        zs.requires_grad_(True)
        y = log_posterior(zs)
        y.backward()
        return zs.grad



    sampleslist = []
    for i in trange(0, n_samples):
        sampleslist.append(theta)

        momentum = torch.normal(0, 1, size = np.shape(theta))

        theta_new, momentum_new = iterate_leapfrogs(theta, momentum, stepsize, num_leapfrog_steps, grad_fun)

        theta, momentum = metropolis_hastings((theta, momentum), (theta_new, momentum_new), log_joint_density_over_params_and_mc
    return torch.stack((sampleslist))
```

```
# Hyperparameters
num_players = 2
num_leapfrog_steps = 20
n_samples = 2500
stepsize = 0.01

def log_posterior_a(zs):
    z1, z2 = zs[0], zs[1]
    return log_posterior_A_beat_B(z1, z2)

samples_a = draw_samples_hmc(num_players, stepsize, num_leapfrog_steps, n_samples, log_posterior_a)
plot_2d_fun(posterior_A_beat_B, "Player A Skill", "Player B Skill", samples_a)
```

```
100%|████████████| 2500/2500 [00:35<00:00, 70.89it/s]
```



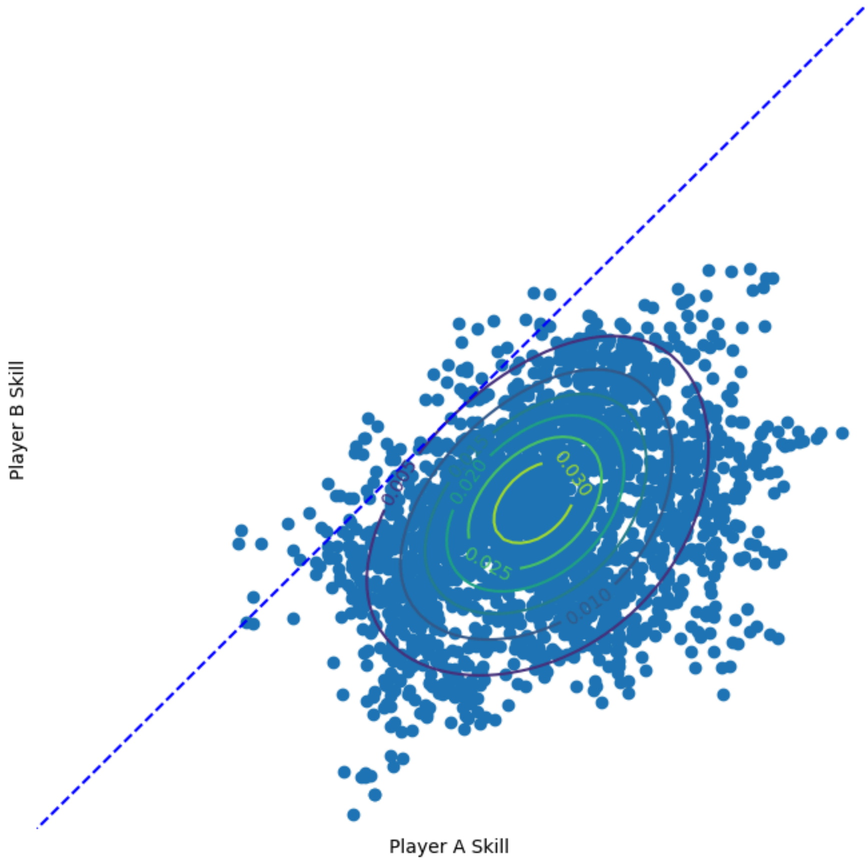

```
<Figure size 640x480 with 0 Axes>
```

2.3.a

Joint posterior where we observe player A winning 5 games against player B.

```
# Hyperparameters
num_players = 2
num_leapfrog_steps = 20
n_samples = 2500
stepsize = 0.01


def log_posterior_b(zs):
    return log_posterior_A_beat_B_5_times(zs[0], zs[1])
    pass


# Run HMC and plot the posterior contour and the samples
samples_b = draw_samples_hmc(num_players, stepsize, num_leapfrog_steps, n_samples, log_posterior_b)

# plot
plot_2d_fun(posterior_A_beat_B_5_times, "Player A Skill", "Player B Skill", samples_b)
```

```
100%|████████████| 2500/2500 [00:36<00:00, 68.41it/s]
```



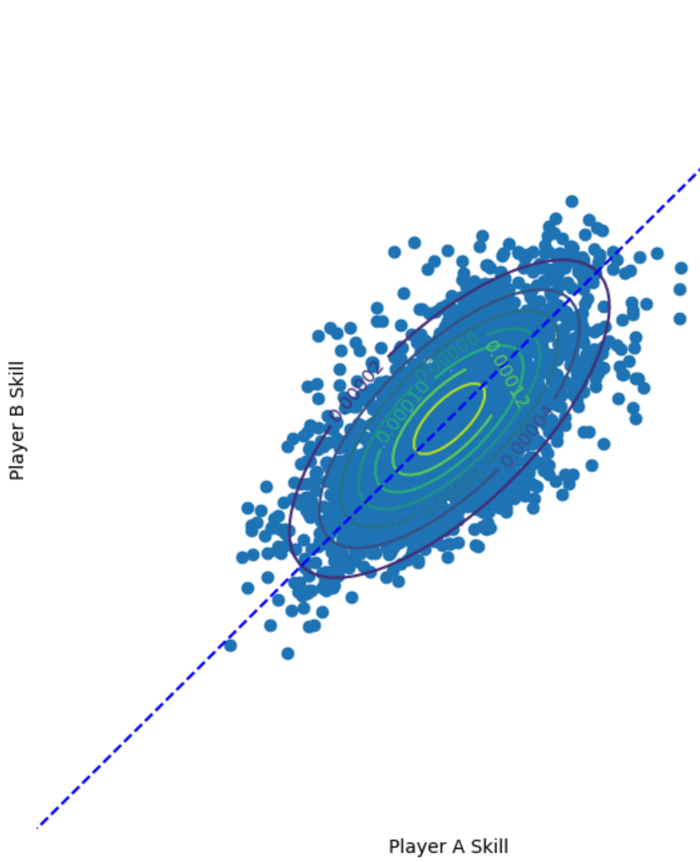

```
<Figure size 640x480 with 0 Axes>
```

2.3.b

Joint posterior where we observe player A winning 5 games and player B winning 5 games.

```
# Hyperparameters
num_players = 2
num_leapfrog_steps = 20
n_samples = 2500
stepsize = 0.01

def log_posterior_c(zs):
    return log_posterior_beat_each_other_5_times(zs[0], zs[1])
    pass

# Run HMC and plot the posterior contour and the samples
samples_c = draw_samples_hmc(num_players, stepsize, num_leapfrog_steps, n_samples, log_posterior_c)

# plot
plot_2d_fun(posterior_beat_each_other_5_times, "Player A Skill", "Player B Skill", samples_c)
```

```
100%|██████████| 2500/2500 [00:45<00:00, 54.48it/s]
```




```
<Figure size 640x480 with 0 Axes>
```

2.3.c

Langevin Monte Carlo algorithm with the Metropolis-Hastings filter. To sample from a posterior distribution $p(z|D)$ with LMC, starting from some intialization z_0, we iteratively compute the proposal

$$z'_{t+1} = z_t + \eta \nabla \log p(z|D) + \sqrt{2 * \eta} W,$$

where $W \sim N(0, I)$. Then, we accept z'_{t+1} according to the Metrapolis-Hastings Algorithm, i.e. we define

$$A = \frac{p(z'_{t+1}|D) \exp\left(-\|z_t - z'_{t+1} - \eta * \nabla \log p(z'_{t+1}|D)\|^2\right)}{p(z_t|D) \exp\left(-\|z'_{t+1} - z_t - \eta * \nabla \log p(z_t|D)\|^2\right)}.$$

We then generate $u \sim \mathrm{Unif}(0, 1)$, and accept the proposal iff $u \leq A$, in other words

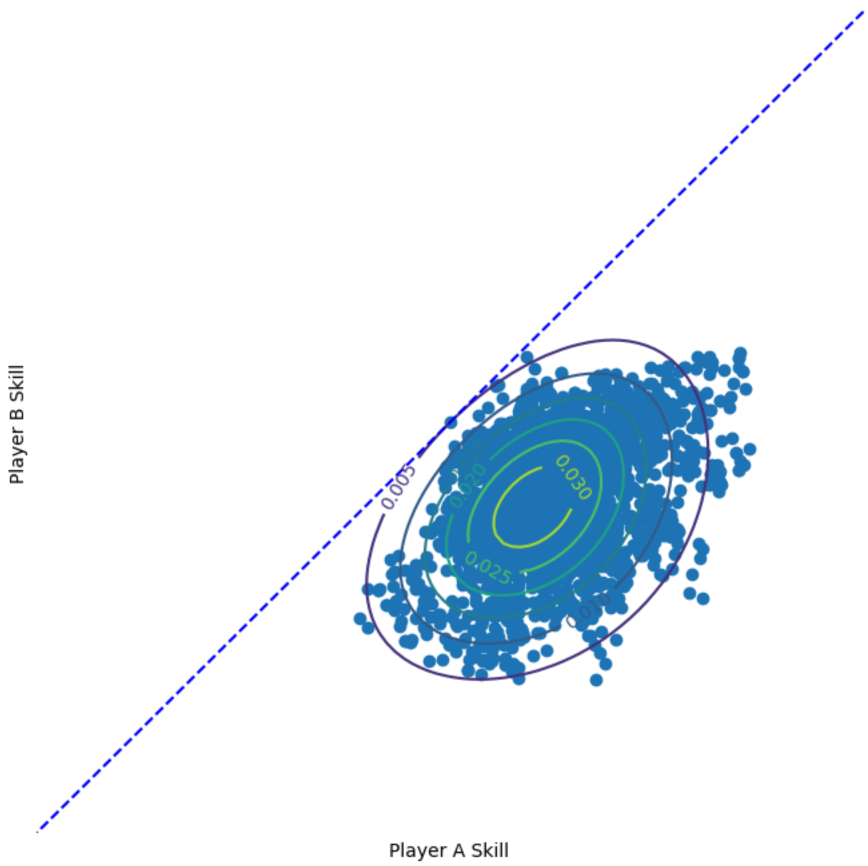
$$z_{t+1} = \begin{cases} z'_{t+1} & \text{if } u \leq A \\ z_t & \text{if } u > A \end{cases}.$$

```
def draw_samples_lmc(num_params, stepsize, n_samples, log_posterior):
    zs = torch.zeros(num_params)

    def grad_log_posterior(zs):
        zs = zs.detach().clone()
        zs.requires_grad_(True)
        y = log_posterior(zs)
        y.backward()
        return zs.grad

    sampleslist = []
    for i in trange(0, n_samples):
        sampleslist.append(zs)

        # compute z'_(t+1)
        W = torch.normal(mean=0, std=1, size=zs.shape)
        grad = grad_log_posterior(zs)
        proposal = zs + stepsize * grad + W * torch.sqrt(torch.tensor([2 * stepsize]))

        # compute A
        grad_new = grad_log_posterior(proposal)
        log_A = (log_posterior(proposal) - torch.norm(zs - proposal - \
        stepsize * grad_new) ** 2) - (log_posterior(zs) - \
        torch.norm(proposal - zs - stepsize * grad) ** 2)

        A = torch.exp(log_A)
```

```
        # rejection rule construction
        u = torch.rand(1)
        if u <= A:
            zs = proposal
    return torch.stack((sampleslist))
```

```
num_players = 2
n_samples = 2500
stepsize = 0.01
key=42

samples_b_lmc = draw_samples_lmc(num_players, stepsize, n_samples, log_posterior_b)

ax = plot_2d_fun(posterior_A_beat_B_5_times, "Player A Skill", "Player B Skill", samples_b_lmc)
```

```
100%|██████████| 2500/2500 [00:02<00:00, 899.39it/s]
```




```
<Figure size 640x480 with 0 Axes>
```

Q 2.4 Approximate inference conditioned on real data [26 points]

The dataset contains data on 2500 games amongst 33 Premier League teams:

- names is a 33 by 1 matrix, whose i'th entry is the name of player i.
- games is a 2500 by 2 matrix of game outcomes, one row per game.

It is based on the following kaggle dataset: https://www.kaggle.com/datasets/evangower/premier-league-matches-19922022

```
# Download the dataset
!curl -L -o premier-league-matches-19922022.zip\
  https://www.kaggle.com/api/v1/datasets/download/evangower/premier-league-matches-19922022
!unzip premier-league-matches-19922022.zip
```

```
  % Total    % Received % Xferd  Average Speed   Time    Time     Time  Current
                                 Dload  Upload   Total   Spent    Left  Speed
  0     0    0     0    0     0      0      0 --:--:-- --:--:-- --:--:--     0
100 81859  100 81859    0     0   244k      0 --:--:-- --:--:-- --:--:--  244k
Archive:  premier-league-matches-19922022.zip
  inflating: premier-league-matches.csv
```

```
from sklearn.preprocessing import LabelEncoder

def load_games():
        dataset = pd.read_csv("premier-league-matches.csv")
        mini_ds = dataset[dataset['FTR'] != 'D'][-2500:]
        all_teams = pd.concat((mini_ds['Home'], mini_ds['Away'])).unique()
```

```
    encoder = LabelEncoder()
    encoder.fit(all_teams)
    mini_ds['HomeId'] = encoder.transform(mini_ds['Home'])
    mini_ds['AwayId'] = encoder.transform(mini_ds['Away'])

    winner_ids = np.where(mini_ds['FTR'] == 'H', mini_ds['HomeId'], mini_ds['AwayId'])
    loser_ids = np.where(mini_ds['FTR'] == 'H', mini_ds['AwayId'], mini_ds['HomeId'])
    games = np.column_stack((winner_ids, loser_ids))
    names = encoder.classes_

    return games, names
```

```
def log_games_likelihood(zs, games):
    # games is an array of size (num_games x 2)
    # zs is an array of size (num_players)

    winning_player_ixs = games[:,0]
    losing_player_ixs = games[:,1]

    winning_player_skills = zs[winning_player_ixs]

    losing_player_skills = zs[losing_player_ixs]

    log_likelihoods = logp_a_beats_b(winning_player_skills, losing_player_skills)

    return torch.sum(log_likelihoods)
```

```
def log_joint_probability(zs, games):
    N = zs.shape[0]
    log_prior = -0.5 * N * torch.log(torch.tensor([2 * np.pi])) - \
    0.5 * torch.sum(zs ** 2)
    return log_prior + log_games_likelihood(zs, games)
    pass
```

Run Langevin Monte Carlo on the posterior over all skills conditioned on all the chess games from the dataset.

```
# Hyperparameters
num_players = len(names)
n_samples = 10000
stepsize = 0.01

def log_posterior(zs):
    return log_joint_probability(zs, games)
    pass

all_games_samples = draw_samples_lmc(num_players, stepsize, n_samples, log_posterior)
```

```
100%|████████████| 10000/10000 [00:13<00:00, 722.07it/s]
```

```
unsort_mean_skills = torch.mean(all_games_samples, axis = 0)
mean_skills, indices = torch.sort(unsort_mean_skills, descending=True)
var_skills = torch.var(all_games_samples, axis = 0)[indices]

plt.xlabel("Player Rank")
plt.ylabel("Player Skill")
plt.errorbar(range(num_players), mean_skills, var_skills)
```

```
<ErrorbarContainer object of 3 artists>
```