Obligatory assignment 3 MVE550, autumn 2021

Petter Mostad

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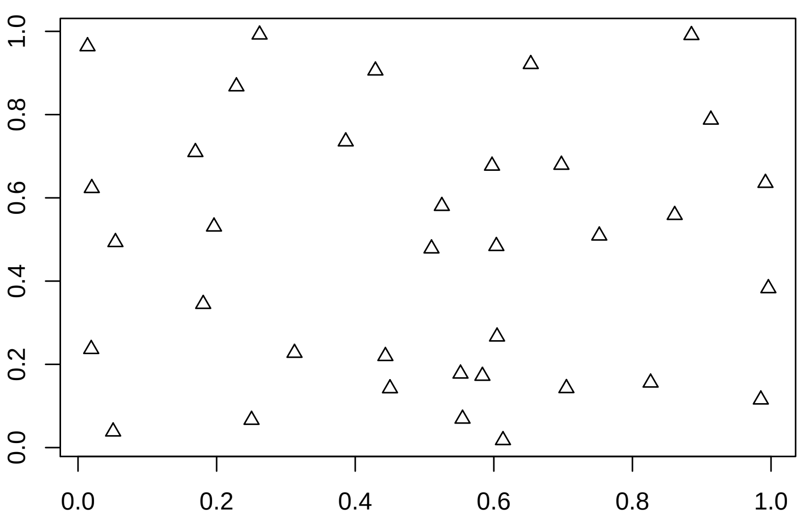

Figure 1: The data used in Question 1 below

1. We consider the positions of trees in a square area. Our data is shown in Figure 1, which shows the position of 36 trees in the square $[0, 1] \times [0, 1]$.

   (a) Assume trees are placed in the square $[0, 1] \times [0, 1]$ according to a spatial Poisson process with parameter $\lambda = 36$. Compute the probability that there are 6 trees or more in the area $[0.2, 0.6] \times [0.2, 0.6]$.

   (b) Assume like above trees are placed in the square $[0, 1] \times [0, 1]$ according to a spatial Poisson process with parameter $\lambda = 36$. Compute the probability that there are exactly 4 trees in the square $[0.2, 0.6] \times [0.2, 0.6]$ and at the same time exactly 4 trees in the square $[0.4, 0.8] \times [0.4, 0.8]$.

(c) Assume like above trees are placed in the square $[0, 1] \times [0, 1]$ according to a spatial Poisson process with parameter $\lambda = 36$. Write R code to simulate this process, so that your code can output a figure showing the placement of trees in the square $[0, 1] \times [0, 1]$. Show one such example figure.

(d) Now, assume that λ has the prior $\pi(\lambda) \propto_\lambda 1/\lambda$, and that our data are those illustrated in Figure 1, where we have observed 36 trees in a square of size 1. Derive the posterior for λ. Extend your code from (c) to a simulation which uses this posterior instead of a fixed λ.

(e) Consider the stochastic process you simulated from in (d). Let X be the random variable representing the average over all points of the distance from this point to its nearest neighbour. In other words, if $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ are the simulated points, define

$$X = \frac{1}{k} \sum_{i=1}^{k} \min_{j=1,\ldots,i-1,i+1,\ldots,k} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$

Use simulation to derive and plot a histogram of a random sample from the distribution of X.

(f) In the data shown in Figure 1, one can compute that the value of X is 0.1358. Use your results from (e) to discuss whether the Poisson model is a good model for these tree data, and if not, why not / what should be changed.

(g) OPTIONAL: Consider the stochastic process you simulated from in (d). Consider Y, the random variable representing the number of points in the circle centered at $(0.5, 0.5)$ and with radius 0.1. Determine the name and the parameters of the distribution Y has.

State	1	3	2	3	2	1	2	3	1	2
Duration	1.32	6.21	1.63	2.44	5.11	0.29	2.87	4.3	2.76	1.92

Table 1: The data used in Question 2 below. The list shows the first 10 consecutive states of the chain, and for each state, the duration of the stay in that state.

2. We study a continuous-time Markov chain with the three possible states 1, 2, and 3. It is specified with a vector $q = (q_1, q_2, q_3)$ of parameters for the Exponential distribution holding times at each of the states and a transition matrix

$$\tilde{P} = \begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{32} & 0 \end{bmatrix}$$

for the *embedded* discrete Markov chain.

(a) Assume $(q_1, q_2, q_3) = (1/5, 1, 1/2)$, $(p_{12}, p_{13}) = (0.5, 0.5)$, $(p_{21}, p_{23}) = (0.5, 0.5)$, and $(p_{31}, p_{32}) = (0.5, 0.5)$. Compute the infinitesimal generator matrix Q, and also the limiting distributon for the chain.

(b) Now, assume we would like to use the data shown in Table 1 to learn about the parameters of the chain. We use as a prior a product of independent priors $\pi(q_1) \propto_{q_1} 1/q_1$, $\pi(q_2) \propto_{q_2} 1/q_2$, and $\pi(q_3) \propto_{q_3} 1/q_3$, multiplied with independent priors for the rows of $\tilde{P}$, specified as distributions on p_{12}, p_{21}, and p_{31}:

$$\begin{aligned} p_{12} &\sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right) \\ p_{21} &\sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right) \\ p_{31} &\sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

Write down the joint posterior distribution for all the parameters, $q_1, q_2, q_3, p_{12}, p_{21}, p_{31}$.

(c) Based on this posterior, write R code to simulate from the posterior for the infinitesimal generator matrix Q for the continuous-time Markov chain, and output an example simulation of Q.