MVE550 - Obligatory assignment 2

Thijs Van Meurs - Pierre-Eliot Jourdan

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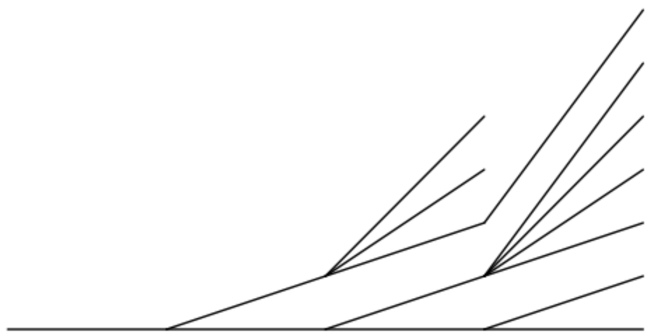

Figure 1: The 5 first steps of the Branching process for question 1.

1 Question 1

In this exercise, we consider a branching process that has been observed for $Z_0, ...Z_4$ (see fig 1). We assume that the offspring is Poisson distributed with expectation λ and that we use the improper prior $\pi(\lambda) \infty_\lambda \frac{1}{\lambda}$.

1.1 a)

In this question, we want to compute the posterior distribution for λ.

First, we notice the improper prior can be written as Gamma(0,0). Indeed, when we have a look at the theoretical expression of Gamma(r,x) we get :

$$f(\lambda; r, x) = \frac{\lambda^{r-1} * e^{-\lambda x} * x^r}{\Gamma(r)}.$$

So, by setting the parameter x and r to 0 (and as $0^0 = 1$) we get :

$$f(\lambda; 0, 0) = \frac{1}{\lambda}$$

We observe on fig 1 that the tree leads to 9 branches. We can conclude that the information in the given figure can be described as 9 stochastic variables distributed as Poisson(λ).

In order to compute the posterior distribution for λ, let's apply the formula for the Poisson Gamma conjugacy (appendix C of the Bayesian inference compendium). The first parameter in the Gamma function corresponds to the current number of individuals in the population and the second, the current number of branches in the tree.

Thus, by applying the conjugacy to generations $Z_0, ...Z_4$, we get successively : Gamma(0,0), Gamma(1,1), Gamma(3,2), Gamma(8,5) and Gamma(15,9).

Thus, according to the conjugacy rules, the posterior distribution for λ is a Gamma(15,9).

1.2 b)

In this question, we will describe the R code we made.

We used the Extinction Probability Theorem 4.2 which says that the probability of extinction is the lowest s such that $G(s) = s$, with G the generating function (of the Poisson law in our case). Thus, in our code we try to find the minimum of the function : $f\_extinct = |e^{\lambda(s-1)} - s|$ which will give us the roots of the equation $G(s) = s$ and then the probability extinction. We notice one obvious root : $|e^{\lambda(s-1)} - s| = 0$ for $s = 1$.

Here is why we use the function $f\_singleExtinct$: to minimize $f\_extinct$ on the range $[0; 1[$ which will give us the probability of extinction for 1 branch.

As we have 7 branches at the 4th generation (Z_4), we need to compute the probability that all branches go extinct. Hence the function $f\_fullExtinct = f\_singleExtinct^b$ is the function for determining the probability that the entire process goes extinct, with $b = X_n$ ($b = 7, n = 4$).

1.3 c)

In this question, we want to compute the probability of extinction for the branching process taking the uncertainty in λ into account. According to a), we know the posterior distribution for λ is $Gamma(\lambda; 15, 9)$ and according to b), the function $f\_fullExtinct(\lambda)$ is the probability that the entire process goes extinct.

We can deduce the integral representing the searched probability :

$$\int_0^\infty f\_fullExtinct(\lambda) * Gamma(\lambda; 15, 9), d\lambda = 0.0724382$$

(we computed the result using R).

1.4 d)

To get the extinction probability, we can derive it using the integral or with simulation. In question 1.3 we calculated this semi-analytically using the integral. To comply with the stochastic nature of real world scenarios, we simulate the extinction probability using 100000 variables for λ.
First, we simulate the extinction probability from generation $n = 4$, as is shown in figure 1. We compute the extinction probability in accordance to

$$\frac{1}{n} \sum_{i=0}^{n} f\_fullExtinct(\lambda_i) = 0.0727206$$

with $n = 100000$ and $[\lambda_1, \ldots, \lambda_n] \overset{i.i.d.}{\sim} Gamma(15, 9)$.
The second part is to simulate the entire branching process from the ground up. If a branching process does not go extinct, it will create offsprings indefinitely. For this reason we will have to consider an evaluation point in time. If a branching process is able to reach this generation n_{eval}, it will most likely not go extinct. Therefore n_{eval} should not be chosen too small. If a larger value is chosen, the performance (calculation speed) will diminish. We choose $n_{eval} = 100$ as emperical evaluation point: if the branching process reaches 100 generations, we count that as survival. Extinct, otherwise. With n values for λ (see begin section 1.4), and running the simulation as specified above, we yield a result with extinction probability = 0.07319.

1.5 e)

In this question, we are looking for the maximum likelihood estimate (MLE) for λ. Since the offspring is Poisson distributed with expectation λ, we can deduce the likelihood :

$$L(\lambda) = \prod_{k=1}^{9} f_\lambda(X_k)$$

with f_λ the probability distribution of the Poisson law and the independant random variables X_k representing the new individuals at each generation Z_i. Thus :

$$L(\lambda) = \prod_{k=1}^{9} \frac{e^{-\lambda} * \lambda^{X_k}}{X_k!} = e^{-9\lambda} * \frac{\lambda^{X_1+X_2+...+X_9}}{X_1! * X_2! * ... * X_9!} = \frac{e^{-9\lambda} * \lambda^{15}}{X_1! * X_2! * ... * X_9!}$$

Let's take the derivative of $MLE(\lambda)$:

$$L'(\lambda) = \frac{1}{X_1! * X_2! * ... * X_9!} * (e^{-9\lambda} * 15\lambda^{14} - 9e^{-9\lambda} * \lambda^{15}) = \frac{e^{-9\lambda}}{X_1! * X_2! * ... * X_9!} * \lambda^{14} * (15 - 9\lambda)$$

Let's find the maximum of $MLE(\lambda)$:

$$L'(\lambda) = 0 => \lambda^{14} * (15 - 9\lambda) = 0.$$

We deduce the solutions of the previous equation : $\lambda = 0$ and $\lambda = \frac{5}{3}$. As $L''(0) > 0$, this is a minimum and therefore $\frac{5}{3}$ is the desired estimate of λ.

Then, let's compute the probability of extinction for the branching process using this estimate for λ, applying the R code in b) :

$$f\_fullExtinct(\frac{5}{3}) = 0.0003766258$$

The probability of extiction of this brancing process using $\lambda = \frac{5}{3}$ is 0.038%.

2 Question 2

x	2.3	5.4	4.6	5.7	8.9	3.2	7.1
y	1.7	3.4	2.0	4.9	5.5	3.7	5.4

Table 1: Values for the speed (x) and braking distance (y) for question 2.

2.1 a)

In this question, we are going to express the likelihood for the parameter vector θ that we will use in our R code (all the $y_i|\theta$ are independent) :

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f_\theta(y_i|\theta) = \prod_{i=1}^{n} Normal(\theta_1 + x_i\theta_2, 1)$$

Hence the logarithm of the likelihood function for this θ :

$$l(\theta) = log(\mathcal{L}(\theta)) = log(\prod_{i=1}^{n} Normal(\theta_1 + x_i\theta_2, 1)) = \sum_{i=1}^{n} log(Normal(\theta_1 + x_i\theta_2, 1))$$

(see Q2.R *part 2a* for the computation)

2.2 b)

In this question, we are going to express the density of the prior at the parameter vector θ that we will use in our R code. According to the instructions :

$$\pi(\theta) = \pi(\theta_1, \theta_2) = Normal(\theta_1; 0, 10^2)Normal(\theta_2; 1, 10^2)$$

Hence the logarithm of the density of the prior for this θ :

$$log(\pi(\theta)) = log(Normal(\theta_1; 0, 10^2)Normal(\theta_2; 1, 10^2))$$
$$= log(Normal(\theta_1; 0, 10^2)) + log(Normal(\theta_2; 1, 10^2))$$

2.3 c)

In this question, we implemented a MCMC algorithm (see Q2.R, *part c*) using random walk proposal. The algorithm begins with a random parameter θ and we propose the value:

$$\theta' = \theta + (e_1, e_2) = \theta + Normal(0, 0.2^2) + Normal(0, 0.1^2)$$

In the R code, we consider an acceptance ratio α (*alpha*): ratio of the probability with θ over the probability with the proposed θ'. After that, let's compute the acceptance probability p : if $p < \delta$, we accept the proposed value θ' else we keep the old value θ. We repeat this process N = 1000 times.

Using the accepted θ', we can compute the probability of the braking distance to be less than 8 when the speed is 10. We get :

$$p(y_8 < 8 | x_8 = 10) = 0.9035233$$

The normalized distribution of the braking distance as a function of speed can be found in figure 2.

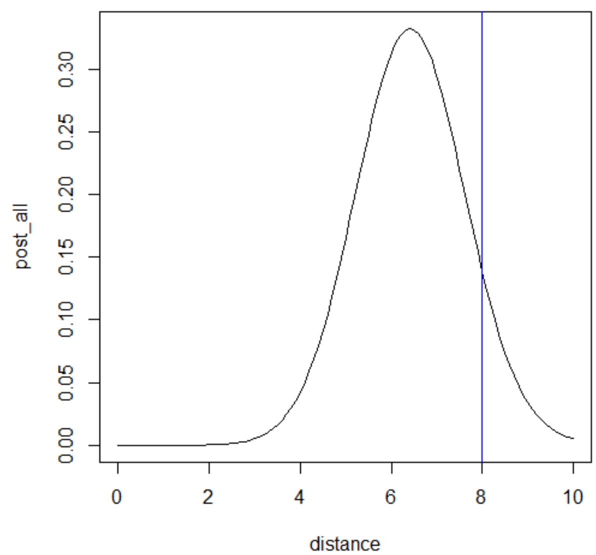

Figure 2: The normalized probability distribution of braking at distance $x_8 = 10$.