An Efficient Method for Calculating Optimal Free-Space N-Impulse Trajectories

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A new technique for calculating optimum central-force transfer and rendezvous trajectories is presented. The approach employed provides a method for determining the optimum impulsive solution between two completely arbitrary states of position and velocity for a given transfer time. The method makes use of primer-vector theory to determine the time, number, and state vector of all impulses. The N-impulse trajectories obtained are optimal in the sense that the total characteristic velocity required to satisfy the specified end conditions exhibits at least a local minimum. The necessary conditions for local optimality are assured by requiring continuity of the primer vector, its derivative, and the Hamiltonian at all interior junction points. Results are presented for a typical Apollo applications program rendezvous problem where a plane change of 86° is required. Optimum rendezvous trajectories requiring up to and including four impulses were found. As shown by the problem analyzed, large savings can result if the optimum multi-impulse envelope is followed rather than using a simpler program requiring fewer impulses.

Introduction

An efficient technique for determining optimal N-impulse free-space trajectories can be applied in several problem areas. Some possible uses for such a method are 1) a trajectory scanner to obtain performance trends in interplanetary mission studies, 2) a control perturbation generator in a feedback system[1,2] used to aid convergence in continuous-thrust optimization problems, and 3) an accurate method for actual mission design in cases where thrust/weight ratios are large and burn-time/trip-time ratios are small.

For the solution of multiple-impulse trajectories, two questions are posed. When will an additional interior impulse improve the performance of the reference solution? If an interior impulse is needed, where is it located and what is its magnitude and direction such that an optimum solution is obtained? The primer-vector concept first introduced by Lawden[3] answered the first question. The extension of this theory to nonoptimal trajectories by Lion and Handelsman[4] provides a method for answering the second question.

Past attempts to obtain optimal impulsive trajectories have been concerned with highly restrictive cases.[5,6] The approach outlined in this report results in a method that is applicable to a large variety of impulsive transfer problems in which no restrictions on plane changes, orbit eccentricities, and so forth, need be imposed.

If the problem posed requires K interior impulses to be optimal, minimizing the total impulse requires determining the minimum of a function of $4K$ variables by some appropriate multivariable search technique. A number of possible multivariable search methods are outlined by Wilde.[7] The method ultimately used to obtain the results presented here is a modified descent method conceived by Fletcher and Powell.[8]

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Problem Definition

With the method presented here, the optimum N-impulse trajectory is obtained by improving, in an iterative fashion, a reference trajectory which satisfies the specified problem boundary conditions but otherwise may be chosen arbitrarily. The analysis and results presented are for fixed-time rendezvous problems that consist of transferring from one state of position, velocity, and time to another. Thus, all trajectories encountered in the iterative convergence procedure must satisfy 14 specified boundary conditions. In addition, allowable trajectories are required to be continuous in position coordinates; velocity discontinuities, of course, occur at impulse points.

For this study, the initial reference trajectory was taken to be a Lambert solution satisfying the specified boundary conditions. For simplicity of nomenclature, the problem discussed will be the determination of an optimum three-impulse solution. This approach shows how the convergence takes place when there is one interior impulse. The method is extended to multiple-interior-impulse solutions by merely increasing the dimension of the search procedure.

Criteria for an Optimum Solution

The necessary conditions for optimal impulsive trajectories were developed by Lawden[3] and are stated in terms of the "primer vector." The term primer vector was introduced by Lawden to denote a vector whose components are the three adjoint variables related to the velocity vector. Lawden's necessary conditions for optimal impulsive trajectories are 1) the primer vector and its first derivative are everywhere continuous; 2) whenever an impulse occurs, the primer vector is aligned with the impulse and has unit magnitude; 3) the primer-vector magnitude must never exceed unity on a coasting arc; and 4) the time derivative of the primer-vector magnitude must be zero at all interior junction points separating coasting arcs.

A primer-vector magnitude history for a typical Lambert problem is shown in Fig. 1a. Since the primer vector generating this history was aligned with the impulses at t_0 and t_f, this two-impulse trajectory satisfied Lawden's optimality criteria. In Fig. 1b, one type of primer-vector history characteristic of a nonoptimum two-impulse solution is shown. This paper is concerned with the use of the nonoptimal primer-vector history to determine interior impulses such that Lawden's criteria are satisfied and the total characteristic velocity required is minimized.

Criteria for Three Impulses

The development of the criteria for additional impulses is presented in Ref. 4. An outline of the procedure leading to the results of Ref. 4 is included for the proper development of the method used to obtain the results presented.

For a central force field, the equations of particle motion are

$$\dot{X} = V \qquad \dot{V} = \nabla P(X,t) \tag{1}$$

where $P(X,t)$ is the gravitational potential, ∇ is the gradient operator, and X and V are the particle position and velocity vectors, respectively. Suppose a reference trajectory T and a perturbed trajectory T' are considered for comparison. If the variation of a quantity between the perturbed and the reference trajectory is represented by the symbol δ, for example

$$\delta X(t) = X'(t) - X(t) \tag{2}$$

then it can be shown that the following relationship between perturbations in the state variables X and V and the adjoint variables λ holds between impulses:

$$\lambda^T \cdot \delta\dot{X} - \dot{\lambda}^T \cdot \delta X = \text{const} \tag{3}$$

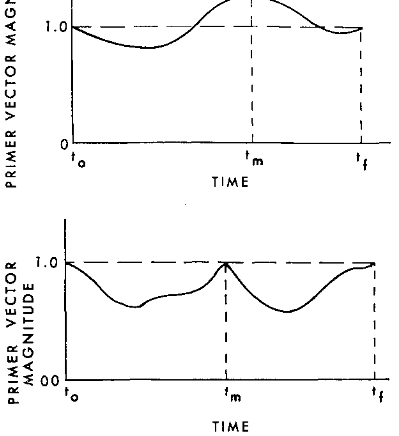
Equation (3) is of prime importance in the ensuing analysis.

The state-variable perturbations at any two times during the trajectory are related by the state-transition matrix. The relationship is

$$\begin{Bmatrix} \delta X(t) \\ \delta V(t) \end{Bmatrix} = \Phi(t,\tau) \begin{Bmatrix} \delta X(\tau) \\ \delta V(\tau) \end{Bmatrix} \tag{4}$$

or in component form

$$\left.\begin{aligned} \delta X(t) &= \phi_{11}(t,\tau)\delta X(\tau) + \phi_{12}(t,\tau)\delta V(\tau) \\ \delta V(t) &= \phi_{21}(t,\tau)\delta X(\tau) + \phi_{22}(t,\tau)\delta V(\tau) \end{aligned}\right\} \tag{5}$$



a) A typical optimum primer-vector history

b) One type of nonoptimum primer-vector history

c) Typical nonoptimum three-impulse primer-vector history

Fig. 1 Primer-vector magnitude characteristics.

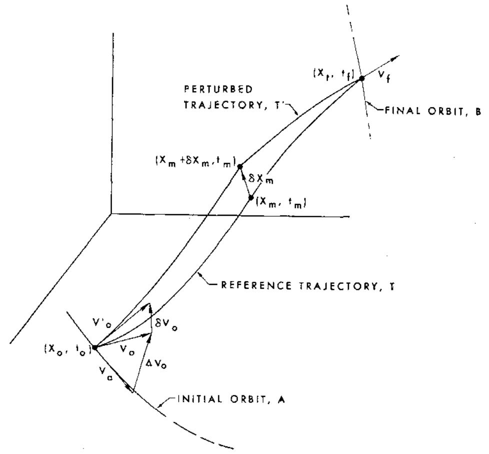


Fig. 2 Reference and perturbed trajectories.

The elements of $\phi(t,\tau)$ are influence functions that, for small perturbations, approach partial derivatives. For example

$$\phi_{11}(t,\tau) = \partial X(t)/\partial X(\tau) \tag{6}$$

Methods for computing the state transition matrix are presented in Ref. 9.

The adjoint variables λ transform in the same manner as perturbations in position and are therefore given by a relationship similar to Eq. (4)

$$\begin{Bmatrix} \lambda(t) \\ \dot{\lambda}(t) \end{Bmatrix} = \Phi(t,\tau) \begin{Bmatrix} \lambda(\tau) \\ \dot{\lambda}(\tau) \end{Bmatrix} \tag{7}$$

Over the coasting arc $t_0 \leq t \leq t_f$ the primer vector is defined by Eq. (7) together with the boundary conditions

$$\lambda(t_0) = \lambda_0 = \Delta V_0/|\Delta V_0| \qquad \lambda(t_f) = \lambda_f = \Delta V_f/|\Delta V_f| \tag{8}$$

where $|\ |$ indicates the vector magnitude. As shown in Ref. 4, the six boundary conditions of Eqs. (8) together with Eq. (7) with $t = t_f$ and $\tau = t_0$ uniquely define $\dot{\lambda}_0$, if $\phi_{12}(t_f,t_0)$ is nonsingular. Thus, the primer vector $\lambda(t)$ and its derivative $\dot{\lambda}(t)$ are uniquely defined over the entire coasting arc.

The cost of an impulsive trajectory is defined as the sum of the magnitudes of the impulsive velocities applied. For N impulses

$$J = \sum_{i=1}^{N} |\Delta V_i| \tag{9}$$

Suppose the two-impulse reference trajectory T is formed (Fig. 2) from orbit A to orbit B by applying the velocity impulses ΔV_0 at t_0 and ΔV_f at t_f. Let the perturbed trajectory T' be formed as follows:

1) At a time $t_m (t_0 < t_m < t_f)$, the position vector X_m on the reference trajectory is perturbed by an amount δX_m.

2) Two Lambert solutions are obtained between the states

$$\{X(t_0),t_0\} \rightarrow \{X(t_m) + \delta X_m,t_m\}$$

and

$$\{X(t_m) + \delta X_m,t_m\} \rightarrow \{X(t_f),t_f\}$$

3) The velocities in the two Lambert solutions of step 2 differ from the reference trajectory velocities. Let these velocity perturbations (Fig. 2) be denoted by δV_0 at time t_0, $\delta V_m{}^-$ at time $t_m{}^-$, $\delta V_m{}^+$ at time $t_m{}^+$, and δV_f at time t_f where minus indicates a quantity evaluated at t^- and plus indicates evaluation at t^+.

4) The cost increment between the perturbed and the reference trajectories is defined as

$$dJ = J' - J \tag{10}$$

where

$$\left.\begin{aligned} J &= |\Delta V_0| + |\Delta V_f| \\ J' &= |\Delta V_0 + \delta V_0| + |\delta V_m{}^+ - \delta V_m{}^-| + |\Delta V_f - \delta V_f| \end{aligned}\right\} \quad (11)$$

The preceding relationships lead to the following expression for the cost increment where only first-order terms have been retained

$$dJ = \lambda_0 \cdot \delta V_0 + |\delta V_m{}^+ - \delta V_m{}^-| - \lambda_f \cdot \delta V_f \quad (12)$$

Evaluating Eq. (3) at the end points of the two Lambert solutions gives

$$\left.\begin{aligned} \lambda_0 \cdot \delta V_0 - \dot{\lambda}_0 \cdot \delta X_0 &= \lambda_m{}^- \cdot \delta V_m{}^- - \dot{\lambda}_m{}^- \cdot \delta X_m{}^- \\ \lambda_m{}^+ \cdot \delta V_m{}^+ - \dot{\lambda}_m{}^+ \cdot \delta X_m{}^+ &= \lambda_f \cdot \delta V_f - \dot{\lambda}_f \cdot \delta X_f \end{aligned}\right\} \quad (13)$$

But, since the position at t_0 and t_f are fixed

$$\delta X_0 = \delta X_f = 0 \quad (14)$$

In the foregoing expressions, λ and $\dot{\lambda}$ are computed from the reference two-impulse trajectory and are therefore continuous, so that

$$\lambda_m{}^+ = \lambda_m{}^- = \lambda_m \quad (15)$$

and

$$\dot{\lambda}_m{}^+ = \dot{\lambda}_m{}^- = \dot{\lambda}_m \quad (16)$$

For continuity of the position vector on the perturbed trajectory, the condition

$$\delta X_m{}^+ = \delta X_m{}^- \quad (17)$$

is necessary. Using Eqs. (13–17) in Eq. (12), the following result is obtained:

$$dJ = c(1 - \lambda_m \cdot \eta) \quad (18)$$

In Eq. (18), η is a unit vector in the direction of the intermediate impulse and c is the magnitude of the intermediate impulse. The criteria for a third impulse can be established by inspection of Eq. (18). Note that if $|\lambda_m|$ is greater than unity, the Lambert reference trajectory may be improved by applying a midcourse impulse in the direction of λ_m at the time t_m. To first order, the greatest decrease in cost will be obtained if the intermediate impulse is applied when $|\lambda_m|$ is a maximum.

Calculation of the Interior Impulse

Once the desirability of a third impulse is established, the question of how it may be calculated arises. The midcourse impulse is determined by forming the perturbed trajectory T', which is a three-impulse solution satisfying the prescribed boundary conditions. Then, on each conic of T', Eq. (4) must hold, or

$$\begin{Bmatrix} \delta X_m{}^- \\ \delta V_m{}^- \end{Bmatrix} = \phi(t_m,t_0) \begin{Bmatrix} \delta X_0 \\ \delta V_0 \end{Bmatrix} \qquad \begin{Bmatrix} \delta X_m{}^+ \\ \delta V_m{}^+ \end{Bmatrix} = \phi(t_m,t_f) \begin{Bmatrix} \delta X_f \\ \delta V_f \end{Bmatrix} \quad (19)$$

Equation (14) is used in Eqs. (19), and continuity of the position vector at the intermediate impulse is required to obtain the perturbation equation,

$$\delta X_m = A^{-1}(\delta V_m{}^+ - \delta V_m{}^-) \quad (20)$$

where

$$A = \phi_{22}(t_m,t_f)\phi_{12}{}^{-1}(t_m,t_f) - \phi_{22}(t_m,t_0)\phi_{12}{}^{-1}(t_m,t_0) \quad (21)$$

It is known from Eq. (18) that the midcourse impulse should be applied along $\lambda_{\max}$. Therefore, with c equal to the midcourse impulse magnitude, Eq. (20) becomes

$$\delta X_m = cA^{-1}\lambda_m/|\lambda_m| \quad (22)$$

where λ_m is $\lambda_{\max}$ (see Fig. 1b). As long as the magnitude of the intermediate impulse c is small enough to insure that the first-order perturbation theory holds, the cost of the perturbed trajectory will be less than the reference trajectory cost.

If second-order terms are retained in the vector subtraction of Eq. (10), a quadratic expression in δV_0 and δV_f is obtained.‡ It is possible, by using this expression within the second-order limitations imposed, to obtain the value of c which provides a maximum improvement in the performance function J. Because δX_0 and δX_f are zero for the rendezvous problem, based on Eqs. (19) it may be shown that

$$\delta V_0 = \phi_{12}{}^{-1}(t_m,t_0)\delta X_m \qquad \delta V_f = \phi_{12}{}^{-1}(t_m,t_f)\delta X_m \quad (23)$$

If Eqs. (22) and (23) are used in the second-order expression for dJ, the expression becomes quadratic in the independent variable c. Setting the derivative of this expression with respect to c equal to zero and solving for c gives

$$c = \frac{(\beta\cdot\Delta V_f)/|\Delta V_f| - (\alpha\cdot\Delta V_0)/|\Delta V_0| - 1}{[\alpha\cdot\alpha - (\alpha\cdot\Delta V_0)^2/|\Delta V_0|^2]/|\Delta V_0| + [\beta\cdot\beta - (\beta\cdot\Delta V_f)^2/|\Delta V_f|^2]/|\Delta V_f|} \quad (24)$$

where

$$\left.\begin{aligned} \alpha &= \phi_{12}{}^{-1}(t_m,t_0)A^{-1}\lambda_m/|\lambda_m| \\ \beta &= \phi_{12}{}^{-1}(t_m,t_f)A^{-1}\lambda_m/|\lambda_m| \end{aligned}\right\} \quad (25)$$

as the best first guess for the magnitude of the intermediate impulse.

With the position vector perturbation δX_m established by Eq. (22), the subarcs of T' can be determined from two Lambert solutions. The midcourse impulse is given by the velocity discontinuity on T' at t_m.

Convergence to the Optimum Trajectory

The initial three-impulse trajectory is not necessarily optimum. The primer-vector history for each two-impulse segment of T' is determined in the same way as the primer-vector history for the original two-impulse trajectory. Primer boundary conditions (λ is a unit vector along each impulse) are satisfied at each impulsive time (t_0, t_m, and t_f). With these boundary conditions, λ is forced to satisfy continuity requirements at the midcourse impulse. A typical primer-vector history for this trajectory is shown in Fig. 1c; note that at time t_m a cusp may exist in the primer-vector curve. This is characteristic of a nonoptimum trajectory in which the primer-vector derivative is discontinuous. If the initial three-impulse trajectory is nonoptimum, it must be possible to find a neighboring three-impulse trajectory requiring less total cost.

The reference trajectory T is now considered to be the initial three-impulse trajectory, and T' is considered to be a perturbed three-impulse trajectory. The perturbed three-impulse trajectory consists of two Lambert trajectories between the states

$$\left.\begin{aligned} \{X_0,t_0\} &\to \{X_m + dX_m, t_m + dt_m\} \\ \{X_m + dX_m, t_m + dt_m\} &\to \{X_f, t_f\} \end{aligned}\right\} \quad (26)$$

where X_m and t_m refer to position and time on the initial three-impulse trajectory. The differential perturbations dX_m and the variations δX_m are related, as shown in Fig. 3. It is now possible, analogous to the development of Eq. (12), to

‡ The derivation is the work of D. R. Glandorf of Lockheed Electronics Co.

write

$$dJ = \dot{\lambda}_m{}^{+}\cdot \delta X_m{}^{+} - \dot{\lambda}_m{}^{-}\cdot \delta X_m{}^{-} + |\delta V_m{}^{+} - \delta V_m{}^{-}| - \lambda_m\cdot(\delta V_m{}^{+} - \delta V_m{}^{-}) \quad (27)$$

for the differential cost between the perturbed and reference three-impulse trajectories. On the reference trajectory λ_m is chosen to be a unit vector along the midcourse impulse $(\delta V_m{}^{+} - \delta V_m{}^{-})$ so that the last two terms in Eq. (27) cancel, leaving

$$dJ = \dot{\lambda}_m{}^{+}\cdot \delta X_m{}^{+} - \dot{\lambda}_m{}^{-}\cdot \delta X_m{}^{-} \quad (28)$$

Because the time at which the interior impulse is applied varies in the perturbed and reference solution and since X is required to be continuous (Fig. 3), the total change in position of the intermediate impulse satisfies the equations

$$dX_m = \delta X_m{}^{-} + \dot{X}^{-}dt_m \qquad dX_m = \delta \dot{X}_m{}^{+} + X^{+}dt_m \quad (29)$$

It can easily be shown that to first order, Eqs. (29) hold also for $\dot{X}^{-}$ and $\dot{X}^{+}$ calculated on the reference trajectory rather than the perturbed trajectory, as is indicated in Fig. 3. Therefore, using Eqs. (29) in Eq. (28) results in

$$dJ = (\dot{\lambda}_m{}^{+} - \dot{\lambda}_m{}^{-})\cdot dX_m - (\dot{\lambda}_m{}^{+}\cdot \dot{X}^{+} - \dot{\lambda}_m{}^{-}\cdot \dot{X}^{-})dt_m \quad (30)$$

where the quantities in the parentheses are determined from the reference trajectory. Equation (30) can then be written in the form

$$dJ = \nabla J \cdot dZ \quad (31)$$

where the vector Z is the combined position vector and time.

The problem of finding the optimum three-impulse solution then becomes one of minimizing $J(Z)$ by some multivariable search method. Since the gradient components can be expressed in terms of the primer-vector derivative and velocities on the reference three-impulse trajectory and thus are readily available, a minimizing technique which employs the gradient components is indicated. One such technique which exhibits quadratic convergence characteristics near the optimum is presented in Ref. 8. Note from Eqs. (30) and (31) that when the minimum of J is obtained (i.e., $\nabla J = 0$), Lawden's first optimality condition is satisfied.

The method of applying the theory developed to obtain the multi-impulse solutions shown in this study is as follows:

1) Given the initial and final rendezvous states and the transfer time, the two possible solutions to Lambert's problem are calculated. The two-impulse reference trajectory is arbitrarily chosen to be the Lambert solution which gives the smaller value of J. It is immediately apparent that the duality of solutions to Lambert's problem leads to the possibility of multiple extremal solutions for the impulsive rendezvous problem. For example, if the two solutions to a given Lambert problem exhibit the primer-vector magnitude histories shown in Figs. 1a and 1b, respectively, then clearly there are at least two solutions to the problem posed which satisfy Lawden's optimality criteria. The first is an optimum two-impulse solution; the second will require more than two impulses to be optimal. A policy decision must be made to choose the two-impulse reference trajectory from which convergence to the optimal N-impulse solution begins. The obvious decision, and the one used in this study, is to select the cheaper of the two solutions as the reference trajectory.

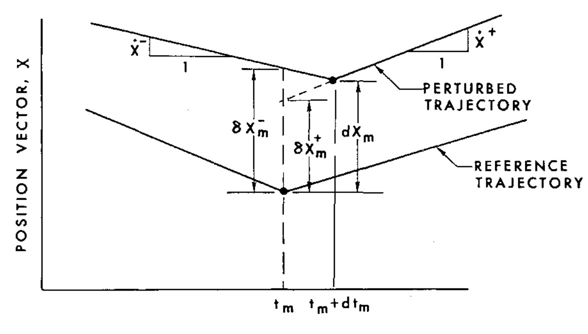


Fig. 3 Relationship between position variations and perturbations.

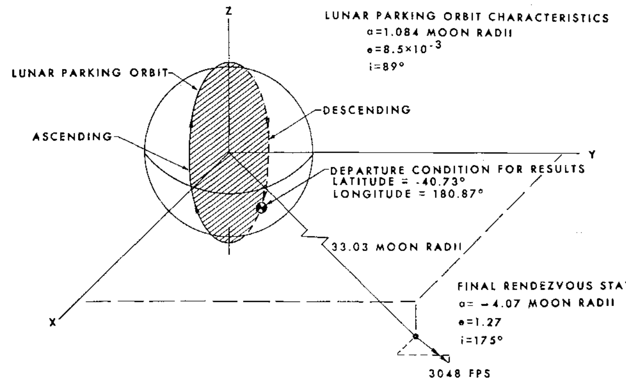


Fig. 4 Typical Apollo Applications Program transfer problem.

2) Once a reference trajectory is chosen, the primer-vector history is computed and tested according to Lawden's optimality criteria. If $|\lambda|$ exceeds unity on any subarc, an additional impulse will improve the performance. In this study, impulses are added one at a time at the point in the time history where $|\lambda|$ is maximum. If the reference trajectory were a two-impulse trajectory, the addition of an intermediate impulse would create a nonoptimum three-impulse trajectory. The modified gradient search to a relative minimum then begins, using the gradient components indicated in Eq. (30) in a four-dimensional multivariable search. (The independent variables consist of three components of the intermediate impulse location X_m and the time t_m at which the intermediate impulse is applied.)

3) In each step of the convergence process, policy decisions must again be made to determine the appropriate sequence of subarcs which comprise the total trajectory. In computing the cost function J for an N-impulse trajectory, there are N-1 decisions as to which Lambert solution to accept. These decisions could be based on the most optimal policy; that is, compute all the 2^{N-1} possible values of J and let the trajectory consist of the sequence of subarcs requiring the lowest total cost. This is an expensive approach since convergence may require the computation of several hundred nonoptimal trajectories. The calculation of 2^{N-1} possible solutions in order to select each nonoptimal trajectory would require a prohibitive amount of machine time. The method used in this study is to first establish the N-impulse reference trajectory from which convergence begins. Subsequent trajectories are found by making independent policy decisions at N-1 impulse points. Decision I selects the following subarc which requires the smallest change in angular momentum at the impulse point if the total energy change is positive; decision II selects the other solution. The preceding decisions are reversed for retrograde trajectories where total energy change is negative. This logic requires only the computation of a simple cross product and a test at each impulse point.

The modified gradient search procedure continues until convergence for the given number of impulses is achieved ($|\nabla J|$ approaches zero within a specified tolerance). If the converged trajectory satisfies Lawden's criteria, execution terminates; if not, an additional impulse is added and the convergence process is continued.

Example Problem

To exhibit the generality of the method previously outlined in this paper, a problem of current interest in the Apollo

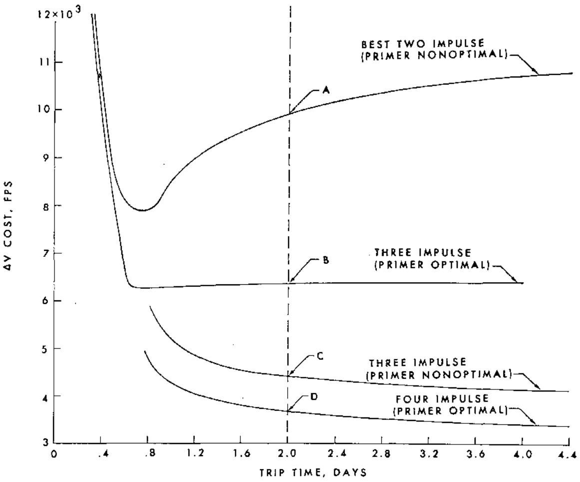
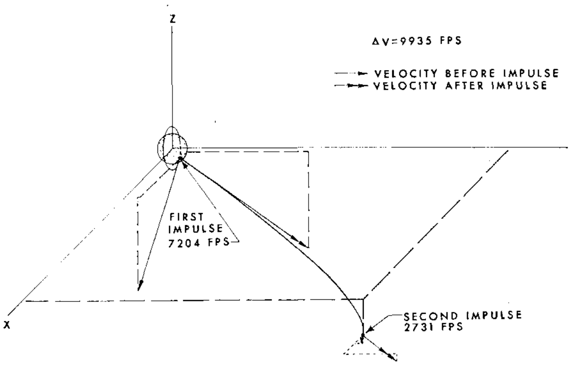


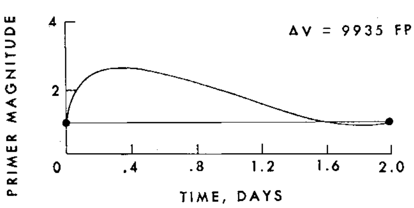
Fig. 5 Multiple-impulse performance.

Applications Program was solved. The nature of this problem is shown in Fig. 4. Note that the rendezvous requires an 86° plane change from a near-circular near-polar lunar orbit to a transearth velocity vector at the sphere of influence of the moon. For a given departure time (i.e., a given state in the lunar orbit), the rendezvous with the fixed final state was calculated for various transfer times. The cost curves for the particular departure state noted in Fig. 4 are shown in Fig. 5. A cost curve is characterized 1) by the number of impulses required to accomplish the rendezvous and 2) by whether Lawden's optimality criteria are satisfied by the solution.

With regard to the classical necessary conditions for optimality (i.e., that λ, $\dot{\lambda}$, and H are continuous), it should be noted that two different three-impulse extremals were found, both of which satisfy the classical necessary conditions for optimality. The characteristics of an entire family of extremals that were found in this study for the particular departure state noted in Fig. 4 and for the transfer time of 2 days are illustrated in Figs. 6–9. In Fig. 6, only the two-impulse solution having the lower cost of the two possible solutions is shown. The primer-magnitude time history, also illustrated in Fig. 6, indicates that an improvement in the cost may be obtained with an additional impulse. When an intermediate impulse is applied and the multivariable search procedure discussed in the text is conducted, the optimal three-impulse solution shown in Fig. 7 results. This solution is found by employing policies I and II for conic selection at the first and second impulse points, respectively. This solution satisfies Lawden's optimality criteria as shown by the primer-magnitude time history in Fig. 7. Another characteristics of this three-impulse solution is that the first of the two transfer angles is less than π. A second three-impulse solution may be found, using policy I (minimum angular momentum change) at both the first and second impulse points to select the conic arcs. This second three-impulse solution, characterized by having a first transfer angle greater than π, is illustrated in Fig. 8. The cost is noted to be lower than the previously calculated three-impulse solution, but the solu-

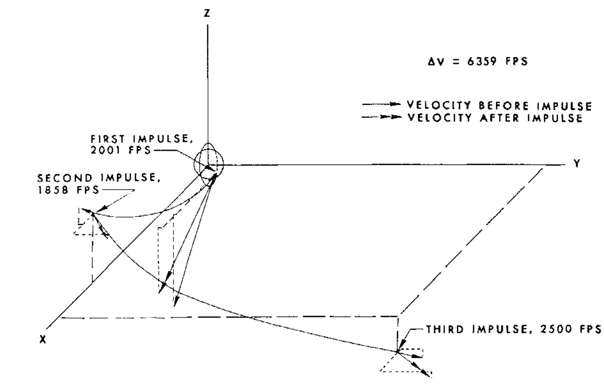


a) Case A, best two-impulse trajectory

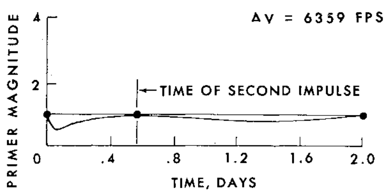


b) Case A, primer history

Fig. 6 Case A, two-impulse trajectory and primer history.

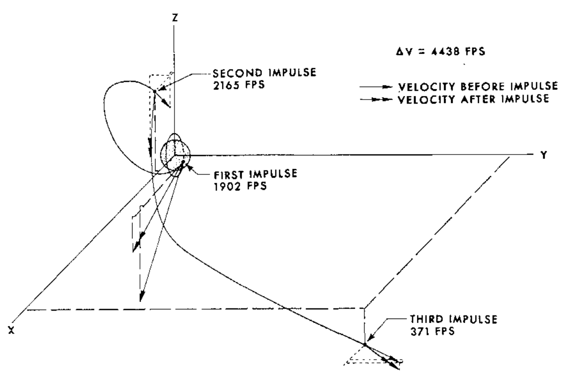


a) Case B, optimal three-impulse trajectory

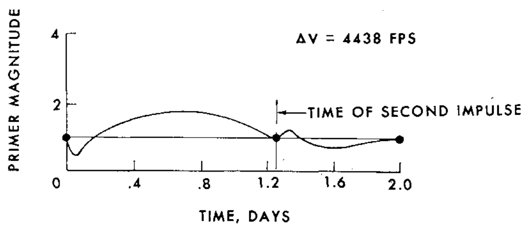


b) Case B, primer history

Fig. 7 Case B, optimal three-impulse trajectory and primer history.

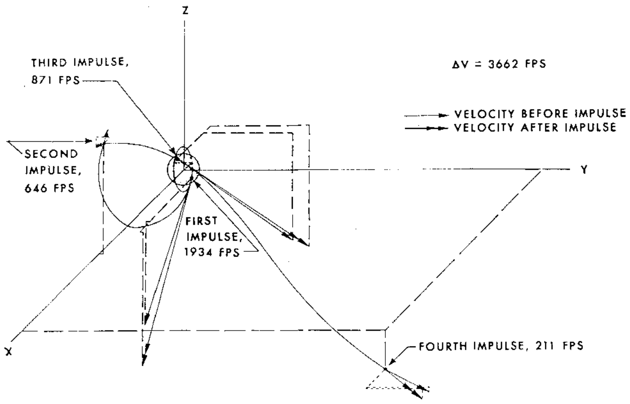


a) Case C, nonoptimum three-impulse trajectory

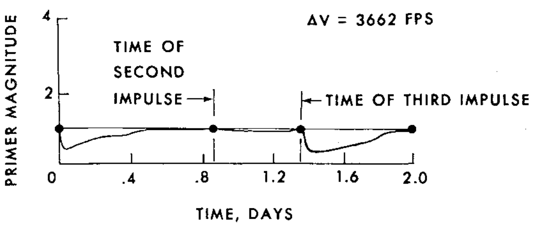


b) Case C, primer history

Fig. 8 Case C, nonoptimum three-impulse trajectory and primer history.



a) Case D, optimal four-impulse trajectory



b) Case D, primer history

Fig. 9 Case D, optimal four-impulse trajectory and primer history.

tion does not satisfy Lawden's optimality conditions as indicated by the primer-magnitude time history. Application of an additional impulse when the primer magnitude is a maximum and conducting a multivariable (eight variable) search results in the optimum four-impulse solution illustrated in Fig. 9.

Note that attempts to find extremals in addition to those shown were fruitless. When the convergence process was begun with different policy sequences than those indicted previously, the same solutions as those shown resulted, although the convergence path from the reference to the extremal solution was necessarily different. The existence of other extremals for this problem has not, however, been disproved.

Typical times for solutions on a UNIVAC 1108 computer in FORTRAN IV double precision were 2 and 6 min for the three- and four-impulse solutions, respectively. That all the solutions obtained do in fact satisfy the classical necessary optimality criteria is assured by requiring the root-mean-

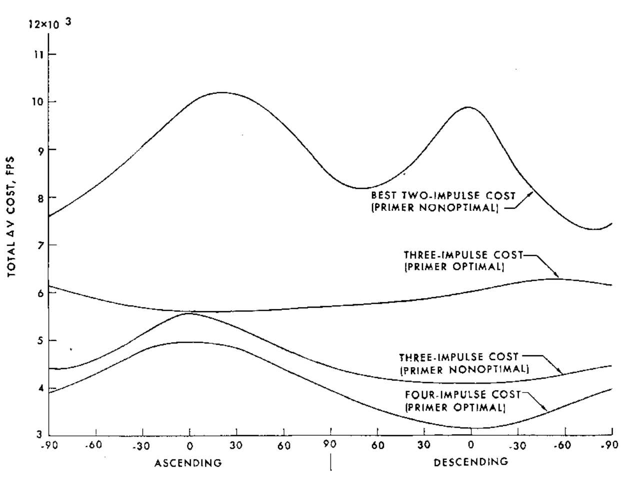


Fig. 10 Minimum-cost transfer performance.

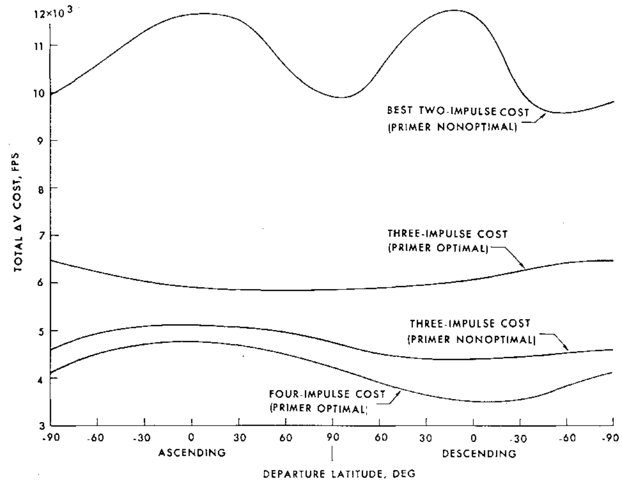


Fig. 11 Two-day transfer performance.

square magnitude of the cost-function gradient in a normalized system of units to be less than 10^{-5} for cutoff to occur. If the results of Fig. 5 are obtained for various initial states, cross plotting yields much information. For example, if the number of impulses is fixed, the best time to leave lunar orbit can be established. This time can be determined from Fig. 10. Similarly, for a fixed transfer time, the cross plot of Fig. 11 results, again indicating the best time to begin the rendezvous maneuver.

Conclusions

The method for obtaining optimal impulsive trajectories outlined by Lion and Handelsman can be used as the basis for a computer program to obtain multi-impulse solutions to general problems. The method was found to be not only powerful but efficient and accurate as well. Solutions requiring up to and including four impulses were found for the Apollo Applications Program problem investigated in this study. The results show, in general, that additional impulses improve performance significantly. Also, results indicate that for a fixed number of impulses, multiple solutions satisfying the classical necessary optimality conditions may exist. These solutions may, however, be differentiated by examining them in the light of Lawden's optimality criteria.

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