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LECTURES ON ASTRODYNAMICS

(Sixth edition, corrected and updated)

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ABSTRACT

These class notes contain an essence of the author's lectures giving basic knowledge in different areas of Astrodynamics. The notes begin with necessary mathematical information (chapter 1). Then a detailed analysis of the two-body problem is given (chapters 2–4). The Kepler's laws, the Newton's law of gravity, and first integrals of the two-body problem are given in chapter 2. Then the Keplerian motion is analyzed using both the classical approach (chapter 3) and a universal approach (chapter 4). The classical approach means an individual consideration of the motion for different orbit types. The universal approach gives an effective method of the motion calculation unified for all orbit types. The perturbed motion is considered in chapter 5. This chapter gives a brief introduction into the theory of the perturbed motion and describes influence of the main natural perturbations, both gravitational and non-gravitational ones, on the motion. Chapters 2–5 are applicable to the both natural and artificial celestial bodies.

The spacecraft impulsive maneuvers changing different orbital parameters and performing inter-orbital transfers are analyzed in chapter 6. A simplified approach to the maneuver optimization and analysis is considered in the chapter. Interplanetary transfers including the gravity assist maneuvers are analyzed in chapter 8. This analysis uses the patched-conic approach, which is described in the chapter. The Lambert problem solution (i.e. determination of the transfer orbit between two given positions in a given time) also is necessary for the interplanetary transfer calculation. This problem is analyzed in details and solved in the notes (chapter 7). An advanced approach is used giving a universal solution to the problem, i.e. solution unified with respect to the orbit types and number of complete revolutions. Maneuvering of a spacecraft in the sphere of influence of a planet is considered in chapter 9. The goal of this maneuvering is transfer of the spacecraft approaching the planet from the incoming hyperbola to an operational orbit. An introduction into the space navigation, i.e. orbit tracking and determination and orbital correction maneuvers, is given in chapter 10. Autonomous navigation also is briefly considered there. The optimization of the orbital maneuvers based on the Pontryagin's maximum principle and Lawden's primer vector is considered in chapter 12. The basic elements of the optimization theory and its application to the spacecraft maneuvering are given in the chapter.

The electric propulsion (low thrust) and some aspects of optimization of the electrically propelled transfers are considered in chapter 13. Chapter 14 gives an approach to the optimization of both impulsive and low thrust if the thrust direction is under a constraint. A method of the optimization of low-thrust transfer between two given positions is given in chapter 15. This method is based on a linearization of the transfer near a close Keplerian orbit. A way of providing any desired accuracy of the optimization is suggested. The optimization method is also applicable to the case of constrained thrust direction considered in chapter 14. Chapter 16 considers a spiral low-thrust transfer between given orbits; this type of transfer takes place near a planet. Planar transfers with transversal thrust between two circular orbits or between circular and parabolic orbits are considered. For this case simple formulas for the calculation of the transfer parameters are obtained in this chapter. A general case of the spiral low thrust transfer between given orbits in an arbitrary gravity field is considered in chapter 17. A simple and effective method of the transfer optimization based on the linearization of motion near reference orbits is described. The method is applicable to different transfer types and also in the cases of partly given final orbit or constrained thrust direction. It is shown that the method gives a locally optimal solution.

The state transition matrix, which is widely used in different areas of the Astrodynamics, is considered in chapter 11. In particular, calculation of this matrix is necessary for the spacecraft navigation and correction maneuvers and for the transfer optimization considered in chapters 10, 12, 13, 15, 17. An effective method of the state transition matrix calculation and inversion is described in chapter 11. This method is based on the matrix decomposition simplifying the matrix calculation and unifying the matrix for different orbit types.

Chapters 6, 7, 9, 11, 14 – 17 are based on the methods developed by the author.

PREFACE

This is a corrected and updated version of the *Lectures on Astrodynamics* notes. These notes originally were composed of the viewgraphs of the lectures given by author to students and personnel of INPE. The goal of the lectures was to give basic knowledge in different areas of the space flight dynamics, from the basis (like Kepler's laws and Newton's law of gravity) to very special topics (such as state transition matrix, optimization of low thrust transfers, constrained thrust direction etc.). The desire to cover as many areas of the flight dynamics as possible in the short lecture course has inevitably led to the necessity of skipping some details. Just a brief impression is given about some of the topics (such as perturbations and navigation in chapters 5 and 10 respectively). A simplified approach is used for the impulsive inter-orbital transfer consideration and just a few important transfers are considered (see chapter 6). Although the reader of the notes can consider other transfers independently using the suggested approach. Other topics (two-body problem, Lambert problem solution) are given in details in chapters 2–4 and 7. A detailed description of an effective method of the state transition matrix calculation is given as well (chapter 11) because this matrix plays an important role in different areas of the astrodynamics.

More than 400 corrections, changes and additions were made to the notes after publication of their fifth edition in 2007. A new chapter "Low-Thrust Transfers between Given Orbits" (chapter 17) was added; this chapter gives a method of spiral inter-orbital transfer optimization. A subsection about Hohmann transfer between neighboring orbits also was added to chapter 6. A few explanations and elucidation were added to chapters 6, 13, 14. Values of the planetary gravitational parameters in Table 2.1 were renewed; some figures were replaced by clearer ones. The annexes to chapters 6, 14, 16 were moved to the end of the class notes. Many changes and additions were made making the explanation clearer and understanding easier. Numerous mistakes, misprints and inaccuracies also were corrected. Other corrections were mostly editorial and stylistic.

Necessary notation and equations are repeated for convenience at the beginning of most of the chapters. Important equations and statements are emphasized by framing them. The style of the notes is deliberately short, simple, and rigorous, sometimes schematic, because the notes imply an additional explanation during the lectures. Nevertheless, the notes can be used by students and specialists as a manual or a handbook.

Contents

1. Mathematical Background

Overhead dots

$$x = x(t)$$

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2} = \frac{d\dot{x}}{dt}$$

Vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \{x_1, x_2, ..., x_n\} \qquad = \text{column } n\text{-vector}$$

$$\vec{x} = \vec{x}(t): \quad \dot{\vec{x}} = \{\dot{x}_1, \dot{x}_2, ..., \dot{x}_n\}$$

<u>Transposition</u>: $\vec{x}^T = [x_1 \;\; x_2 \;\; ... \;\; x_n]$ = row

<u>Dot product</u>: $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + ... + x_n y_n = \sum_{i=1}^{n} x_i y_i$

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

Vectors $\vec{x}_1, ..., \vec{x}_m$ $(m \le n)$ are <u>linearly independent</u> if the equality $\sum_{i=1}^{m} c_i \vec{x}_i = \vec{0}$ is fulfilled only for $c_1 = ... = c_m = 0$

Matrices

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \qquad = m\times n\text{-matrix}$$

If $m = n$ then A is n-order matrix or n-matrix

$$A = A(t): \quad \dot{A} = \begin{bmatrix} \dot{A}_{11} & \dot{A}_{12} & \dots & \dot{A}_{1n} \\ \dot{A}_{21} & \dot{A}_{22} & \dots & \dot{A}_{2n} \\ \dots & \dots & \dots & \dots \\ \dot{A}_{m1} & \dot{A}_{m2} & \dots & \dot{A}_{mn} \end{bmatrix}$$

Scalar factor: $$cA = \begin{bmatrix} cA_{11} & cA_{21} & \dots & cA_{m1} \\ cA_{12} & cA_{22} & \dots & cA_{m2} \\ \dots & \dots & \dots & \dots \\ cA_{1n} & cA_{2n} & \dots & cA_{mn} \end{bmatrix}$$

A = $l\times m$-matrix:

$\vec{x}$ = m-vector: $\vec{y} = A\vec{x}$ = l-vector, $y_i = \sum_{j=1}^{n} A_{ij} x_j$

B = $m\times n$-matrix: $C = AB$ = $l\times n$-matrix, $C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$

Transposition: $$A^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ A_{12} & A_{22} & \dots & A_{m2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{mn} \end{bmatrix}$$ = transposed $n\times m$-matrix

Inverse matrix:

A = n-matrix, $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ = unit matrix

A^{-1} = inverse matrix: $AA^{-1} = A^{-1}A = I$

Important properties of the transposed and inverse matrices:

$$(AB)^T = B^T A^T, \quad (AB)^{-1} = B^{-1} A^{-1}$$

$$\frac{dA^{-1}}{dt} = -A^{-1}\dot{A}A^{-1}$$

Rank of matrix:

Rank of an $m\times n$-matrix A is equal to rank (A) = r ($r \le \min(m,n)$) if there are r linearly independent columns or rows of the matrix A and any $r + 1$ columns or rows are linearly dependent.

n-matrix A can be inverted if and only if rank (A) = n .

Derivatives with respect to a vector

$y = y(\vec{x}), \quad \vec{x} = \{x_1, x_2, ..., x_n\}$:

$$\frac{\partial y}{\partial \vec{x}} = \mathrm{grad}_{\vec{x}}\, y = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix} \qquad = \text{gradient (row)}$$

$\vec{y} = \vec{y}(\vec{x}), \quad \vec{x} = \{x_1, ..., x_n\}, \quad \vec{y} = \{y_1, ..., y_m\}$

$$Y = \frac{\partial \vec{y}}{\partial \vec{x}} \quad \text{is } m\times n\text{-matrix}, \quad Y_{ij} = \frac{\partial y_i}{\partial x_j}, \quad i = 1,...,m, \quad j = 1,...,n$$

Extremum of a function

Extremum = minimum or maximum of a function.

<u>Scalar argument:</u>

$x = x(t)$: $\quad \dot{x} = 0 \quad$ is necessary condition of $\min x$ or $\max x$

<u>Vector argument:</u>

$y = y(\vec{x})$: $\quad \dfrac{\partial y}{\partial \vec{x}} = \vec{0}^T \quad$ is necessary condition of $\min y$ or $\max y$

Newton–Raphson method of solving algebraic equations

<u>Scalar equation:</u>

$$y(x) = a, \quad y' = \frac{dy}{dx}$$

x_0 – first guess; nth iteration:

$$\boxed{x_n = x_{n-1} - \frac{y(x_{n-1}) - a}{y'(x_{n-1})}, \quad n = 0, 1, ...}$$

<u>Vector equation:</u>

$$\vec{y}(\vec{x}) = \vec{a}, \quad \vec{x}, \vec{y}, \vec{a} \quad \text{are } n\text{-vectors}, \qquad Y(\vec{x}) = \frac{\partial \vec{y}(\vec{x})}{\partial \vec{x}}$$

$\vec{x}_0$ = first guess; in nth iteration

$$\boxed{\vec{x}_n = \vec{x}_{n-1} - Y^{-1}(\vec{x}_{n-1})\left[\vec{y}(\vec{x}_{n-1}) - \vec{a}\right], \quad n = 0, 1, ...}$$

Vectors in 3-dimentional space

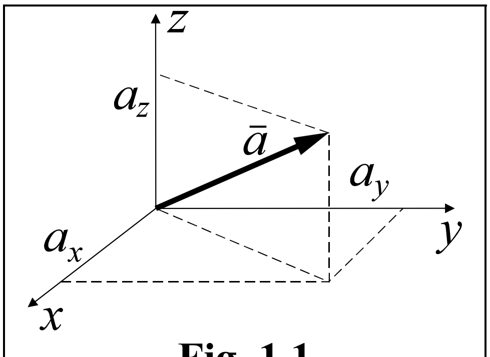

Fig. 1.1

$\vec{a} = \{a_x, a_y, a_z\}$ is 3-vector

$$a = |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

= <u>vector magnitude (modulus)</u>

$\vec{a}^0 = \dfrac{\vec{a}}{a}$ is <u>unit vector</u>

$$|\vec{a}^0| = 1$$

$$\frac{\partial a}{\partial \vec{a}} = \begin{bmatrix} \dfrac{\partial a}{\partial a_x} & \dfrac{\partial a}{\partial a_y} & \dfrac{\partial a}{\partial a_z} \end{bmatrix} = \begin{bmatrix} \dfrac{a_x}{a} & \dfrac{a_y}{a} & \dfrac{a_z}{a} \end{bmatrix} = \vec{a}^{0T}$$

<u>Dot product:</u> $\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b} = a_x b_x + a_y b_y + a_z b_z$

<u>Cross product:</u> $\vec{a}\times\vec{b} = \{a_y b_z - a_z b_y\,,\ a_z b_x - a_x b_z\,,\ a_x b_y - a_y b_x\}$

Let φ be angle between $\vec{a}$ and $\vec{b}$

$$\vec{a}\cdot\vec{b} = ab\cos\varphi, \quad |\vec{a}\times\vec{b}| = ab\sin\varphi$$

Hamiltonian system

$$H = H(\vec{x}), \quad \vec{x} = \begin{bmatrix} \vec{x}' \\ \vec{x}'' \end{bmatrix}, \quad \vec{x}' = \{x_1', \ldots, x_n'\}, \quad \vec{x}'' = \{x_1'', \ldots, x_n''\}$$

$$\boxed{\dot{\vec{x}}' = \left(\frac{\partial H}{\partial \vec{x}''}\right)^T, \quad \dot{\vec{x}}'' = -\left(\frac{\partial H}{\partial \vec{x}'}\right)^T} \tag{1.1}$$

H = <u>Hamiltonian</u>;
$\vec{x}', \vec{x}''$ = <u>Hamiltonian variables</u>, <u>canonical variables</u>;
Eq. (1.1) = <u>Hamiltonian equation</u>, <u>canonical form</u> of the differential equation.

Due to (1.1) $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial \vec{x}'}\dot{\vec{x}}' + \dfrac{\partial H}{\partial \vec{x}''}\dot{\vec{x}}'' = 0$

$\Rightarrow$ $\boxed{H = \text{const}}$ = first integral.

Hypersurface

$\vec{x} = \{x_1, ..., x_n\}$ = vector in an n-dimensional space

$f(\vec{x}) = 0$ = equation of a <u>hypersurface</u> S

If $f(\vec{x}) = A\vec{x} + \vec{b}$ then S is <u>hyperplane</u>

Let $f(\vec{x}_0) = f(\vec{x}) = 0 \quad \Rightarrow \quad \vec{x}_0, \vec{x} \in S$

$\vec{x} \to \vec{x}_0 \quad \Rightarrow \quad d\vec{x} = \vec{x} - \vec{x}_0 \in P$ = <u>tangent hyperplane</u>

$$f(\vec{x}) = f(\vec{x}_0) + \left.\frac{\partial f(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}_0} d\vec{x} \quad \Rightarrow \quad \left.\frac{\partial f(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}_0} d\vec{x} = 0$$

$\Rightarrow \quad \dfrac{\partial f(\vec{x})}{\partial \vec{x}} = \mathrm{grad}_{\vec{x}} f(\vec{x})$ is orthogonal to P and S.

Manifold

Equations

$$f_1(\vec{x}) = 0, ..., f_m(\vec{x}) = 0 \tag{1.2}$$

define m hypersurfaces $S_1, ..., S_m$.

Intersection M of the hypersurfaces $S_1, ..., S_m$ (i.e. $\vec{x} \in M$ if $\vec{x}$ satisfies all Eqs. (1.2)) is <u>manifold</u> if the vectors

$$\frac{\partial f_1}{\partial \vec{x}}, \; ..., \; \frac{\partial f_m}{\partial \vec{x}}$$

are linearly independent.

Intersection of the tangent hyperplanes to the hypersurfaces $S_1, ..., S_m$ at $\vec{x} \in M$ is a <u>tangent plane</u> to the manifold M at $\vec{x}$.

Any linear combination

$$\sum_{i=1}^{m} c_i \frac{\partial f_i(\vec{x})}{\partial \vec{x}}$$

is orthogonal to the tangent plane and the manifold.

2. Two Body Problem

2.1. Equations of motion

Kepler's laws based upon observations (see Fig. 2.1):

1. A planet moves in an ellipse with the Sun in one of two foci (F_1 or F_2).

2. A planet sweeps out equal areas in equal times:

$$\boxed{t_2 - t_1 = t_4 - t_3 \quad \Rightarrow \quad S_1 = S_2}$$

3. Squares of the periods of the planets are proportional to the cubes of their average distances from the Sun:

$$\boxed{\frac{P_1^2}{P_2^2} = \frac{a_1^3}{a_2^3}}$$

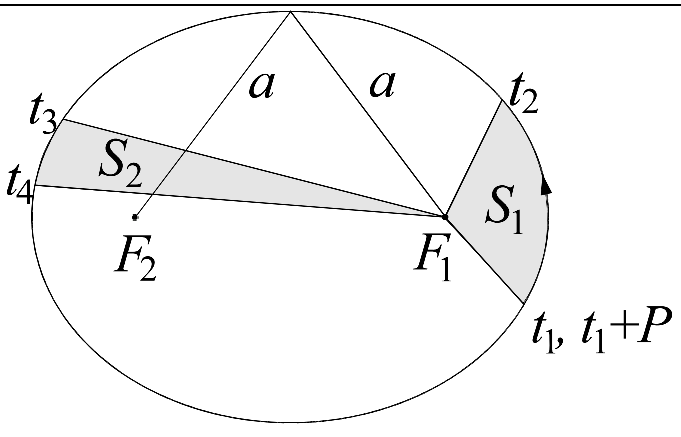

Fig. 2.1

Newton's universal law of gravity (law of the inverse squares) derived from the Kepler's laws:

$$F = \frac{GMm}{r^2} \quad = \text{ attraction force, gravitational force}$$

$G = 6.672 \cdot 10^{-20}$ km^3 kg^{-1} s^{-2} is universal constant of gravitation;
M, m = attracting masses;
r = distance between M and m.
Vector form of the law:

$$\vec{F} = -\frac{GMm}{r^2}\vec{r}^{\,0},$$

$$\boxed{\vec{r}^{\,0} = \frac{\vec{r}}{r}, \quad r = |\vec{r}|},$$

$\vec{r} = \{x, y, z\}$ is position vector, radius-vector.

Acceleration (see Fig.2.2)

$\vec{r} = \vec{\rho} - \vec{\rho}_0$

1. Inertial frame:

$$\ddot{\vec{\rho}} = -\frac{GM}{r^3}\vec{r}, \qquad \ddot{\vec{\rho}}_0 = \frac{Gm}{r^3}\vec{r} \qquad (2.1)$$

$$\Rightarrow \quad \ddot{\vec{r}} = -\frac{G(M+m)}{r^3}\vec{r}$$

2. Barycentric frame: $\vec{\rho}_0 = -\dfrac{m}{M}\vec{\rho}$

   and (2.1) is used with

$$\vec{r} = \frac{M+m}{M}\vec{\rho} = -\frac{M+m}{m}\vec{\rho}_0$$

3. M-centric frame:

$$\ddot{\vec{r}} = \ddot{\vec{\rho}} - \ddot{\vec{\rho}}_0 = -\frac{G(M+m)}{r^3}\vec{r} \qquad (2.2)$$

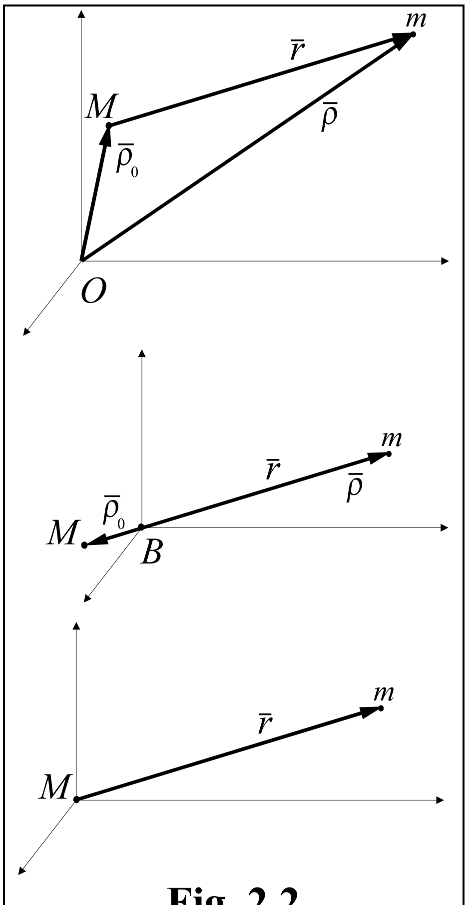

Fig. 2.2

$\mu_1 = GM$, $\mu_2 = Gm$ = <u>gravitational parameters, gravitational (gravity) constants</u>

$\mu = \mu_1 + \mu_2 = G(M + m)$

If $m << M \quad \Rightarrow \quad \mu \approx GM$

<u>Newton's law of gravity</u>:

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}} \tag{2.3}$$

<u>Another forms</u>:

1. $U = \dfrac{\mu}{r}$ is <u>gravitational potential</u>;

$$\boxed{\ddot{\vec{r}} = \left(\frac{\partial U}{\partial \vec{r}}\right)^T} \tag{2.4}$$

$\Rightarrow$ the force is <u>conservative</u>.

2. $\vec{v} = \dot{\vec{r}}, \quad \vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}, \quad \vec{f}(\vec{x}) = \begin{bmatrix} \vec{v} \\ -\dfrac{\mu}{r^3}\vec{r} \end{bmatrix}$

$$\boxed{\dot{\vec{x}} = \vec{f}(\vec{x})} \tag{2.5}$$

$\vec{v} = \{v_x, v_y, v_z\}$ is <u>velocity vector</u>,
$\vec{x}$ is <u>state vector</u>.

> Equations (2.1 – 2.5) describe <u>two body problem</u>, <u>Keplerian motion</u>, unperturbed motion

Table 2.1. Gravitational parameters and mean distances of the Sun, planets, and Moon

Celestial body	μ, km^3/s^2	Mean distance from the Sun	
		astron. units	10^6 km
Sun	$132712.440 \cdot 10^6$	–	–
Mercury	22032.080	0.387	57.909
Venus	324858.599	0.723	108.209
Earth	398600.433	1.000	149.598
Mars	42828.314	1.524	227.941
Jupiter	126712767.858	5.203	778.293
Saturn	37940626.061	9.555	1429.371
Uranus	5794549.007	19.218	2874.995
Neptune	6836534.064	30.110	4504.346
Pluto	981.601	39.518	5911.775
Moon*	4902.801	0.00257	0.3844

* Mean distance from Earth

2.2. Gravity field of a body of finite size

Just two massive dots were considered above.

Consider a body of a finite size.

Designate (see Fig. 2.3):

dm, dV = elementary mass and volume

$\sigma = \sigma\left(\vec{R}\right)$ = mass density

$$dm = \sigma dV$$

$$\int_V dm = M$$

$$U = G\int_V \frac{\sigma dV}{r} = G\int_V \frac{\sigma dV}{\left|\vec{\rho} - \vec{R}\right|}$$

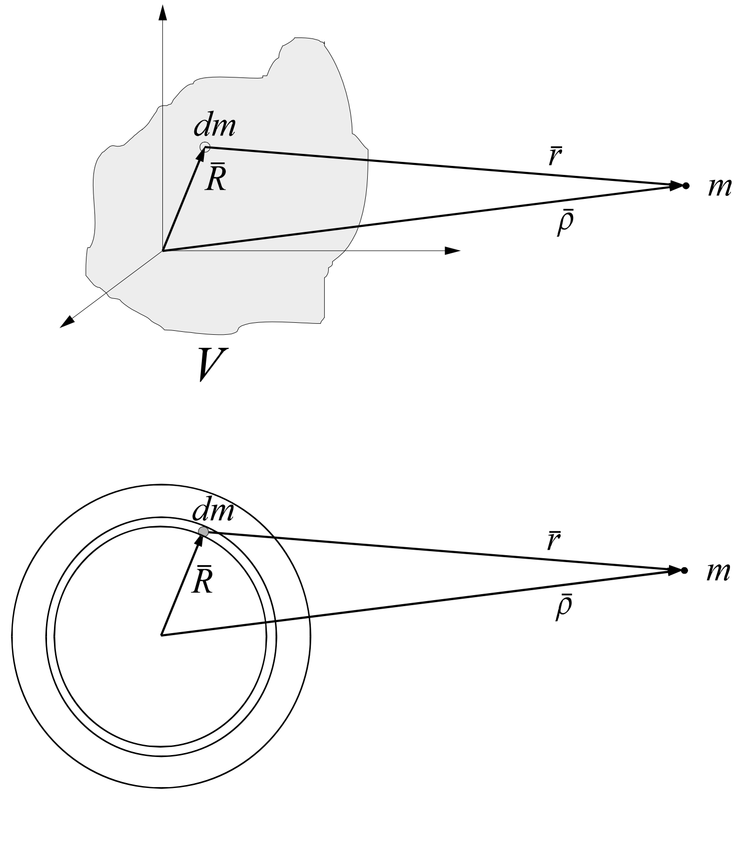

Fig. 2.3

$R = \left|\vec{R}\right|$:

$$\boxed{\text{If} \quad \sigma = \sigma\left(R\right) \quad \text{then} \quad U = \frac{GM}{\rho} = \frac{\mu}{\rho}}$$

A body with spherically distributed density attracts as a dot of the same mass located in the body mass center,
i.e. the body has <u>central gravity field</u>.

2.3. First integrals

Multiplying (2.3) by $\vec{v} = \dot{\vec{r}}$ gives:

Left hand: $\dot{\vec{r}} \cdot \ddot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\dot{\vec{r}} \cdot \dot{\vec{r}}) = \frac{d}{dt}\frac{v^2}{2}$

Right hand: $\dot{\vec{r}} \cdot \left(-\frac{\mu}{r^3}\vec{r} \right) = \frac{d}{dt}\frac{\mu}{r}$

$$\Rightarrow \quad \boxed{h = v^2 - \frac{2\mu}{r}} \tag{2.6}$$

= <u>integral of energy</u>
h = const is <u>energy constant</u> (doubled energy)

Multiplying (2.3) by $\vec{r}$ gives:

Left hand: $\vec{r} \times \ddot{\vec{r}} = \frac{d}{dt}\vec{r} \times \vec{v} - \frac{d}{dt}\vec{v} \times \vec{v} = \frac{d}{dt}\vec{r} \times \vec{v}$

(since $\vec{v} \times \vec{v} = \vec{0}$)

Right hand: $\vec{r} \times \left(-\frac{\mu}{r^3}\vec{r} \right) = -\frac{\mu}{r^3}\vec{r} \times \vec{r} = \vec{0}$

$$\Rightarrow \quad \boxed{\vec{c} = \vec{r} \times \vec{v}} \tag{2.7}$$

= <u>angular momentum</u>, <u>integral of areas</u>
$\vec{c}$ = const is <u>constant of areas</u>, constant vector of angular momentum

$$\boxed{\vec{r} \cdot \vec{c} = 0 \quad \Rightarrow \quad \text{the motion is planar}}$$

$$\boxed{\text{vector } \vec{c} \text{ is orthogonal to the orbit plane}}$$

Multiplying (2.3) by $\vec{c}$ gives:

Left hand: $\ddot{\vec{r}} \times \vec{c} = \frac{d}{dt}(\vec{v} \times \vec{c})$

Right hand: $-\frac{\mu}{r^3}\vec{r} \times \vec{c} = -\frac{\mu}{r^3}\vec{r} \times (\vec{r} \times \vec{v}) = -\frac{\mu}{r^3}(\vec{r}r\dot{r} - \vec{v}r^2)$

$$= \mu\left(-\frac{\vec{r}}{r^2}\dot{r} + \frac{\dot{\vec{r}}}{r}\right) = \frac{d}{dt}\left(\mu\frac{\vec{r}}{r}\right)$$

(since $\vec{r} \cdot \vec{v} = r\dot{r}$)

$$\Rightarrow \quad \boxed{\vec{l} = -\mu\frac{\vec{r}}{r} + \vec{v} \times \vec{c}} \tag{2.8}$$

= <u>Laplace integral</u>

$\vec{l} = \text{const}$ is <u>Laplace vector</u> located in the orbit plane

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Relations:

$$\boxed{\vec{c} \cdot \vec{l} = 0} \tag{2.9}$$

$$\boxed{l^2 = \mu^2 + c^2 h} \tag{2.10}$$

where $l = |\vec{l}|, c = |\vec{c}|$

$\Rightarrow$ integrals $h, \vec{c}, \vec{l}$ give 5 independent constants.

Designate

$$H = \frac{h}{2}$$

Due to (2.5, 2.6)

$$\boxed{\dot{\vec{r}} = \frac{\partial H}{\partial \vec{v}}, \quad \dot{\vec{v}} = -\frac{\partial H}{\partial \vec{r}}} \tag{2.11}$$

$\Rightarrow$ $\frac{h}{2}$ is the Hamiltonian and (2.11) is the canonical form of the equations of motion

2.4. Orbit types

$\vec{l} \cdot \vec{r} = lr\cos\vartheta$ (ϑ is angle between $\vec{l}$ and $\vec{r}$)

From (2.8) obtain:

$$\vec{l} \cdot \vec{r} = -\mu r + c^2 \quad \Rightarrow \quad r = \frac{c^2}{\mu + l\cos\vartheta}$$

Define parameters taking into account (2.10):

$$p = \frac{c^2}{\mu} \qquad = \underline{\text{semilatus rectum}}\text{, focal parameter} \tag{2.12}$$

$$e = \frac{l}{\mu} = \sqrt{1 + \frac{c^2}{\mu^2}h} \qquad = \underline{\text{eccentricity}} \tag{2.13}$$

ϑ = <u>true anomaly</u>

$$\Rightarrow \quad \boxed{r = \frac{p}{1 + e\cos\vartheta}} \tag{2.14}$$

Eq. (2.14) describes a conic section in polar coordinates and gives a generalized 1st Kepler's law.

$$\boxed{\begin{array}{ll} e < 1 & = \text{elliptic orbits} \\ e = 1 & = \text{parabolic orbits} \\ e > 1 & = \text{hyperbolic orbits} \end{array}} \tag{2.15}$$

$l \geq 0 \quad \Rightarrow \quad e \geq 0$

$\vartheta = 0$:

$$\boxed{r = r_\pi = \frac{p}{1+e}} \tag{2.16}$$

= <u>periapsis</u> (pericenter) <u>radius</u>

Vector $\vec{l}$ is always directed to the minimal orbital distance (periapsis) along the <u>apsid line</u> (see Fig. 2.4)

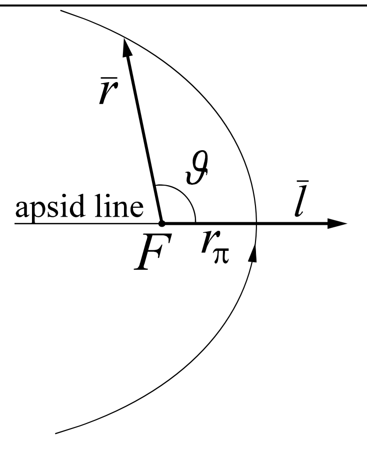


Fig. 2.4

From (2.7)

$$c = rv\sin\varphi = rv_n = r^2\dot{\vartheta} \tag{2.17}$$

(see Fig. 2.5);

$$dS = \frac{r \cdot r d\vartheta}{2} = \frac{cdt}{2}$$

$$\Rightarrow \quad \boxed{S = \frac{c}{2}\left(t_2 - t_1\right)} \tag{2.18}$$

Eq. (2.18) gives the 2nd Kepler's law.

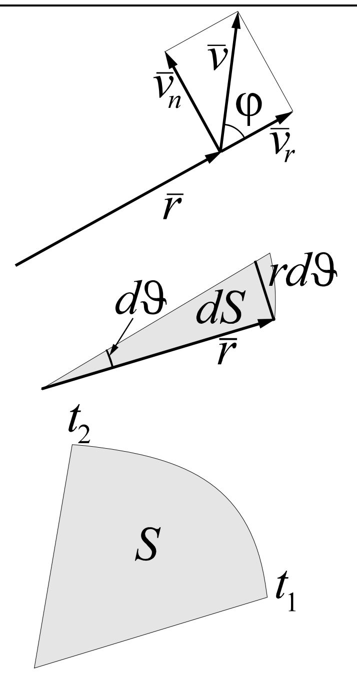


Fig. 2.5

2.5. Energetic orbit classification

Using (2.11, 2.13, 2.16) obtain (see Fig. 2.6):

$$v_r = \dot{r} = \frac{pe\sin\vartheta}{(1+e\cos\vartheta)^2}\dot{\vartheta} = \sqrt{\frac{\mu}{p}}e\sin\vartheta \qquad (2.19)$$

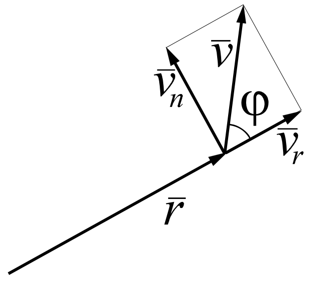


Fig. 2.6

= radial velocity.

$$c = rv\sin\varphi$$

$$\Rightarrow \quad v_n = \frac{c}{r} = \sqrt{\frac{\mu}{p}}(1+e\cos\vartheta) \qquad (2.20)$$

= transversal velocity,

$$v^2 = v_r^2 + v_n^2 = \frac{\mu}{p}\left(1+e^2+2e\cos\vartheta\right) \qquad (2.21)$$

$\Rightarrow$ Eq. (2.6) using (2.14, 2.21) gives

$$h = v^2 - \frac{2\mu}{r} = \frac{\mu}{p}\left[1+e^2+2e\cos\vartheta - 2(1+e\cos\vartheta)\right] = -\frac{\mu}{p}\left(1-e^2\right) \qquad (2.22)$$

$$\Rightarrow \quad \boxed{\begin{array}{lll} h<0 & \left(\text{i.e. } v^2 < \dfrac{2\mu}{r}\right) & = \text{elliptic orbits} \\ h=0 & \left(\text{i.e. } v^2 = \dfrac{2\mu}{r}\right) & = \text{parabolic orbits} \\ h>0 & \left(\text{i.e. } v^2 > \dfrac{2\mu}{r}\right) & = \text{hyperbolic orbits} \end{array}} \qquad (2.23)$$

3. Orbit Types and Orbital Elements

3.1. Necessary formulas

$$
\begin{array}{ll}
h = v^2 - \dfrac{2\mu}{r} = -\dfrac{\mu}{p}\left(1-e^2\right) & = \text{integral of energy} \\
\vec{c} = \vec{r} \times \vec{v} & = \text{angular momentum} \\
c = \left|\vec{c}\right| = \sqrt{\mu p} = r v_n = r^2 \dot{\vartheta} & \\
l = \left|\vec{l}\right| = \mu e & = \text{Laplace integral} \\
p = \dfrac{c^2}{\mu} & = \text{semilatus rectum} \\
e = \sqrt{1 + \dfrac{c^2}{\mu^2} h} & = \text{eccentricity} \\
r = \dfrac{p}{1 + e\cos\vartheta} & = \text{orbit radius} \\
r_\pi = \dfrac{p}{1+e} & = \text{periapsis radius} \\
v_r = \dot{r} = \sqrt{\dfrac{\mu}{p}} e \sin\vartheta & = \text{radial velocity} \\
v_n = \dfrac{c}{r} = \sqrt{\dfrac{\mu}{p}}\left(1 + e\cos\vartheta\right) & = \text{transversal velocity}
\end{array}
\qquad (3.1)
$$

3.2. Elliptic orbits

$$\boxed{e < 1, \quad h < 0, \quad v^2 < \frac{2\mu}{r}}$$

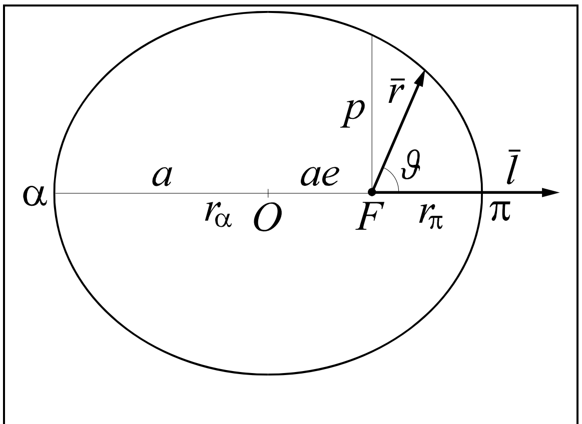


Fig. 3.1

$\vartheta = 0: \quad r_\pi = F\pi = \dfrac{p}{1+e}$ = periapsis (pericenter) radius,

$\vartheta = \pi: \quad r_\alpha = F\alpha = \dfrac{p}{1-e}$ = <u>apoapsis</u> (apocenter) radius

(see Fig. 3.1);

$$r_\pi \le r \le r_\alpha$$

$$\vartheta = \frac{\pi}{2}: \quad r = p$$

$a = O\pi = O\alpha$ = <u>semimajor axis</u>, mean radius

$$\boxed{a = \frac{r_\pi + r_\alpha}{2} = \frac{p}{1-e^2}} \tag{3.2}$$

$$\Rightarrow \quad \boxed{h = -\frac{\mu}{a}, \quad v^2 = \frac{2\mu}{r} - \frac{\mu}{a}} \tag{3.3}$$

<u>Circular orbit</u>:

$$e = 0\,, \quad r = a = p$$

$v = \sqrt{\dfrac{\mu}{a}}$ is <u>circular velocity</u>

Flight time

E = eccentric anomaly (see Fig. 3.2)

$$r\sin\vartheta = a\sin E\sqrt{1-e^2}$$
$$a\cos E - r\cos\vartheta = ae$$

$$\Rightarrow\ r = \sqrt{a^2\sin^2 E\left(1-e^2\right) + (a\cos E - ae)^2}$$

$$\Rightarrow\quad \boxed{r = a\left(1-e\cos E\right)} \qquad (3.4)$$

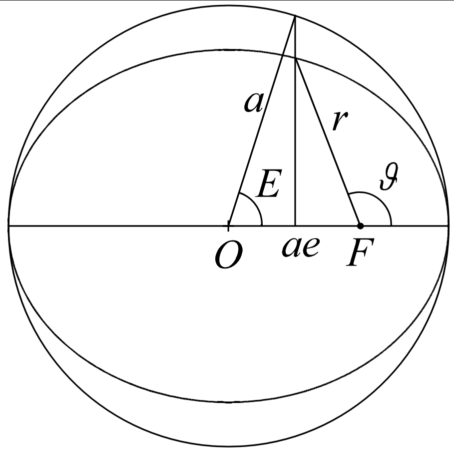


Fig. 3.2

Comparing r in (3.1) and (3.4) and taking into account (3.2) obtain:

$$\cos\vartheta = \frac{\cos E - e}{1-e\cos E} \qquad (3.5)$$

Using equality $\cos\alpha = \dfrac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}$ for any α obtain from (3.5):

$$\boxed{\tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}} \qquad (3.6)$$

Taking into account (3.5) and the equality $2\cos^2\alpha/2 = 1+\cos\alpha$ obtain from (3.6):

$$\dot{\vartheta} = \frac{\sqrt{1-e^2}}{1-e\cos E}\dot{E} \qquad (3.7)$$

On the other hand it can be found from (3.1, 3.4):

$$a^2\left(1-e\cos E\right)^2\dot{\vartheta} = \sqrt{\mu p}$$

$\Rightarrow$ obtain using (3.2):

$$\frac{a^{3/2}}{\sqrt{\mu}}\left(1-e\cos E\right)\dot{E} = 1$$

$$\Rightarrow\quad \boxed{t - t_\pi = \frac{a^{3/2}}{\sqrt{\mu}}\left(E - e\sin E\right)} \qquad (3.8)$$

= Kepler's equation,

t_π is time of the periapsis passage, periapsis time.

Orbital period corresponds to $E = 2\pi \quad \Rightarrow$ from (3.8)

$$\boxed{P = 2\pi \frac{a^{3/2}}{\sqrt{\mu}} \quad = \text{ orbital period}} \tag{3.9}$$

This gives the 3rd Kepler's law:

$$\frac{P_1^2}{P_2^2} = \frac{a_1^3}{a_2^3}$$

$$n = \frac{\sqrt{\mu}}{a^{3/2}} \qquad = \underline{\text{mean motion}}$$

$$M = n(t - t_\pi) \qquad = \underline{\text{mean anomaly}}$$

$$\Rightarrow \quad \boxed{M = E - e\sin E \quad = \text{ Kepler's equation}} \tag{3.10}$$

Assume that $\tau = t - t_\pi$ is given $\Rightarrow M$ is also given. To find E from (3.10) the Newton–Raphson method can be used (see Chapter 1).

$$\frac{d}{dE}(E - e\sin E) = 1 - e\cos E$$

$\Rightarrow$ at nth iteration

$$E_n = E_{n-1} - \frac{E_{n-1} - e\sin E_{n-1} - M}{1 - e\cos E_{n-1}}, \quad n = 0, 1, ... \tag{3.11}$$

First guess for the iterative procedure (3.11) can be taken as

$E_0 = 0$ or $E_0 = M$

3.3. Parabolic orbits

$$\boxed{e = 1, \quad h = 0, \quad v^2 = \frac{2\mu}{r}}$$

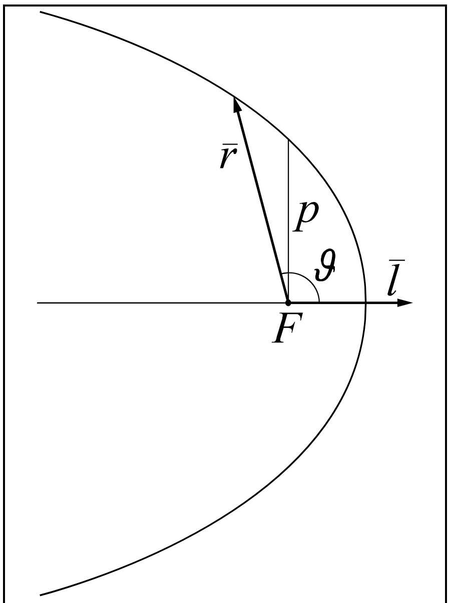


Fig. 3.3

In (2.14):

$\vartheta \to \pi \quad \text{or} \quad \vartheta \to -\pi \quad \Rightarrow \quad r \to \infty$

$r \to \infty \quad \Rightarrow \quad v \to 0$

$$r_\pi = \frac{p}{2}$$

$$\boxed{a = \frac{p}{1 - e^2} = \infty}$$

$$\boxed{v_p = \sqrt{\frac{2\mu}{r}} \quad = \text{ parabolic velocity}} \tag{3.12}$$

Flight time

From (3.1):

$$\frac{p^2\dot{\vartheta}}{(1+\cos\vartheta)^2} = \sqrt{\mu p} \quad \Rightarrow \quad \frac{p^{3/2}}{\sqrt{\mu}} \frac{d\vartheta}{(1+\cos\vartheta)^2} = dt$$

$$\Rightarrow \quad \boxed{t - t_\pi = \frac{p^{3/2}}{2\sqrt{\mu}}\left(\tan\frac{\vartheta}{2} + \frac{1}{3}\tan^3\frac{\vartheta}{2}\right)} \tag{3.13}$$

Assume that $\tau = t - t_\pi$ is given. Variable $x = \tan\dfrac{\vartheta}{2}$ can be found from the cubic equation (3.13) analytically or using the Newton–Raphson method (see Chapter 1): at nth iteration of the method

$$x_n = x_{n-1} - \frac{\dfrac{x^3}{3} + x - \dfrac{2\sqrt{\mu}}{p^{3/2}}(t - t_\pi)}{x^2 + 1}, \quad n = 0, 1, \ldots \tag{3.14}$$

First guess for the iterative procedure (3.14) can be taken as follows:

$$x_0 = 0 \quad \text{or} \quad x_0 = \frac{2\sqrt{\mu}}{p^{3/2}}(t - t_\pi) \quad \text{or} \quad x_0 = \left[\frac{6\sqrt{\mu}}{p^{3/2}}(t - t_\pi)\right]^{\frac{1}{3}}$$

After x is found $\vartheta = 2\arctan x$.

3.4. Hyperbolic orbits

$$\boxed{e > 1, \quad h > 0, \quad v^2 > \frac{2\mu}{r}}$$

$$\cos\vartheta_* = -\frac{1}{e}, \quad -\vartheta_* < \vartheta < \vartheta_*$$

$$\vartheta \to \vartheta_* \text{ or } \vartheta \to -\vartheta_* \quad \Rightarrow \quad r \to \infty$$

$$r \to \infty \quad \Rightarrow \quad v \to v_\infty$$

v_∞ = <u>v-infinity, excess velocity, asymptotic velocity</u>

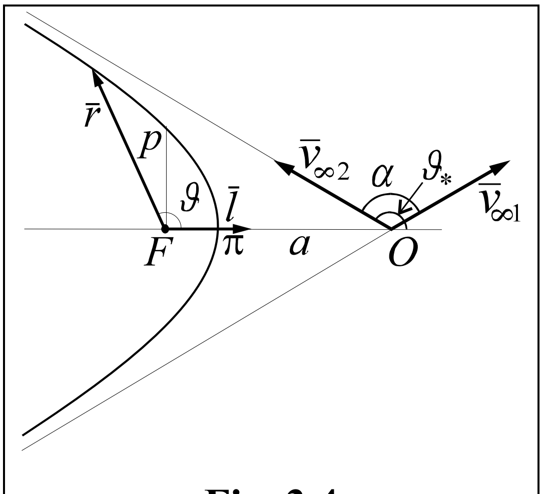


Fig. 3.4

$$\Rightarrow \quad \boxed{v = \sqrt{\frac{2\mu}{r} + v_\infty^2} = \sqrt{v_p^2 + v_\infty^2}}$$

v_p = <u>parabolic velocity</u>

$a = O\pi$ (see Fig. 3.4)

$$\boxed{a = \frac{p}{e^2 - 1}} \tag{3.15}$$

$$\Rightarrow \quad \boxed{h = \frac{\mu}{a} = v_\infty^2, \quad v^2 = \frac{2\mu}{r} + \frac{\mu}{a}} \tag{3.16}$$

α = angle between the incoming ($\vec{v}_{\infty 1}$) and outgoing ($\vec{v}_{\infty 2}$) excess velocities (<u>turn angle</u>) (see Fig. 3.4), $|\vec{v}_{\infty 1}| = |\vec{v}_{\infty 2}| = v_\infty$.

$$\frac{\alpha}{2} = \vartheta_* - \frac{\pi}{2}, \quad \cos\vartheta_* = -\frac{1}{e} \quad \Rightarrow \quad \sin\frac{\alpha}{2} = \frac{1}{e}$$

$$c^2 = r_\pi^2 v_\pi^2 = r_\pi^2\left(\frac{2\mu}{r_\pi} + v_\infty^2\right), \quad h = v_\infty^2$$

$$\Rightarrow \quad e = \sqrt{1 + \frac{c^2}{\mu^2} h} = 1 + \frac{r_\pi v_\infty^2}{\mu}$$

$$\Rightarrow \quad \boxed{\sin\frac{\alpha}{2} = \frac{1}{1 + \dfrac{r_\pi v_\infty^2}{\mu}}} \tag{3.17}$$

Flight time

$e > 1 \quad \Rightarrow \quad$ in (3.7)

$$\dot{\vartheta} = \frac{\sqrt{e^2 - 1}}{1 - e\cos E} i\dot{E} \;, \qquad \boxed{i = \sqrt{-1}} \tag{3.18}$$

Define new variable H:

$$E = iH\,, \quad H = -iE$$

and take into account equalities $\cos iH = \cosh H\,, \quad \sin iH = -i\sinh H$

$\Rightarrow$ obtain from (3.6, 3.18):

$$\boxed{\tan\frac{\vartheta}{2} = \sqrt{\frac{e+1}{e-1}}\tanh\frac{H}{2}} \tag{3.19}$$

$$\dot{\vartheta} = \frac{\sqrt{e^2 - 1}}{e\cosh H - 1}\dot{H}$$

According to (3.2) and (3.15) replace a with $-a$ in (3.4, 3.8):

$$\boxed{r = a\left(e\cosh H - 1\right)} \tag{3.20}$$

$$\boxed{t - t_\pi = \frac{a^{3/2}}{\sqrt{\mu}}\left(e\sinh H - H\right)} \tag{3.21}$$

= Kepler's equation for hyperbolic orbits.

Assume that $\tau = t - t_\pi$ is given. Like it has been done for elliptic orbits (see (3.11)) obtain:

$$H_n = H_{n-1} - \frac{e\sinh H_{n-1} - H_{n-1} - \dfrac{\sqrt{\mu}}{a^{3/2}}\tau}{e\cosh H_{n-1} - 1}\,, \quad n = 0, 1, \ldots \tag{3.22}$$

3.5. Calculation of position and velocity (orbit propagation)

Formulation of the propagation problem:

Initial state vector $\vec{x}_0 = \vec{x}(t_0) = \begin{bmatrix} \vec{r}_0 \\ \vec{v}_0 \end{bmatrix}$ is given.

The problem is to find the state vector $\vec{x} = \vec{x}(t) = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}$

at a given time t.

Designate (see Fig. 3.5):

☊ = ascending node
☋ = descending node
Ω = longitude of the ascending node
ω = argument of periapsis
$u = \omega + \vartheta$ = argument of latitude
i = inclination

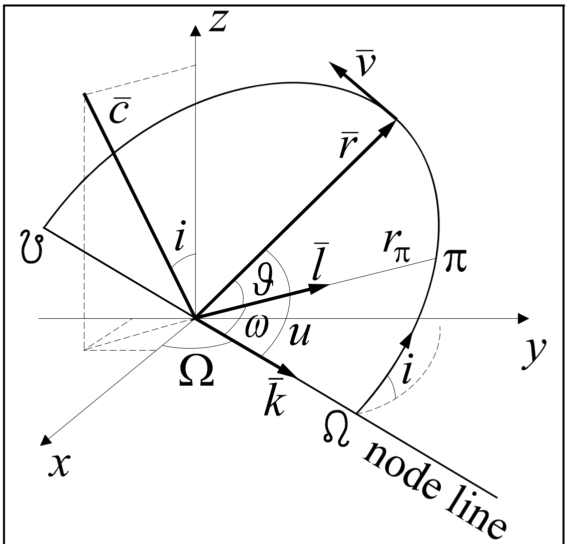


Fig. 3.5

Ω, ω, i are constants and can be calculated from $\vec{r}_0, \vec{v}_0$ as follows (see Fig. 3.5):

$$\left.\begin{aligned}
&\cos i = \frac{c_z}{c} \\
&\sin\Omega = \frac{c_x}{\sqrt{c_x^2 + c_y^2}}, \quad \cos\Omega = -\frac{c_y}{\sqrt{c_x^2 + c_y^2}} \\
&\sin\omega = \frac{\left|\vec{l} \times \vec{k}\right|}{\mu e}, \quad \cos\omega = \frac{\vec{l} \cdot \vec{k}}{\mu e}
\end{aligned}\right\} \tag{3.23}$$

where $\vec{k} = \{\cos\Omega, \sin\Omega, 0\}$ and $\vec{c}, \vec{l}$ are given by (3.1) with $\vec{r}, \vec{v}$ replaced by $\vec{r}_0, \vec{v}_0$.

ϑ can be found from (3.6, 3.8) for elliptic orbits, from (3.13) for parabolic orbits, and from (3.19, 3.21) for hyperbolic orbits.

$$\boxed{\begin{aligned} \vec{r}^{\,0} &= \begin{bmatrix} \cos\Omega\,\cos u - \sin\Omega\,\sin u\,\cos i \\ \sin\Omega\,\cos u + \cos\Omega\,\sin u\,\cos i \\ \sin u\,\sin i \end{bmatrix} \\ \vec{n}^{\,0} &= \begin{bmatrix} -\cos\Omega\,\sin u - \sin\Omega\,\cos u\,\cos i \\ -\sin\Omega\,\sin u + \cos\Omega\,\cos u\,\cos i \\ \cos u\,\sin i \end{bmatrix} \end{aligned}} \tag{3.24}$$

= unit vectors of the radial and transversal directions,

$$\boxed{\begin{aligned} \vec{r} &= r\,\vec{r}^{\,0} \\ \vec{v} &= v_r\,\vec{r}^{\,0} + v_n\,\vec{n}^{\,0} \end{aligned}} \tag{3.25}$$

where r, v_r, v_n are given by (3.1).

3.6. Orbital elements

$a,\ e,\ i,\ \Omega,\ \omega,\ t_\pi$

= set of the <u>orbital elements</u> completely defining the orbit.
Another options:

instead of a: r_π, p, P, n;
instead of e, ω: $e\cos\omega$, $e\sin\omega$;
instead of t_π: $t - t_\pi$, M (for elliptic orbits), ϑ, u.

$$a = \frac{\mu}{|h|} = \frac{1}{\left|\frac{2}{r} - \frac{v^2}{\mu}\right|}, \quad e = \sqrt{1 + \frac{c^2}{\mu^2}h} = \sqrt{1 + \frac{r^2v^2 - (\vec{r}\cdot\vec{v})^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right)},$$

$$\cos i = \frac{c_z}{c} = \frac{xv_y - yv_x}{\sqrt{r^2v^2 - (\vec{r}\cdot\vec{v})^2}}, \quad \tan\Omega = -\frac{c_x}{c_y} = \frac{zv_y - yv_z}{zv_x - xv_z},$$

$$\cos\omega = \frac{\vec{l}\cdot\vec{k}}{\mu e}, \quad \vec{l} = -\mu\frac{\vec{r}}{r} + \vec{v}\times(\vec{r}\times\vec{v}), \quad \vec{k} = \begin{bmatrix}\cos\Omega\\ \sin\Omega\\ 0\end{bmatrix},$$

$$t - t_\pi = \begin{cases} \dfrac{a^{3/2}}{\sqrt{\mu}}(E - e\sin E) & \text{for elliptic orbits,}\\[2ex] \dfrac{p^{3/2}}{2\sqrt{\mu}}\left(\tan\dfrac{\vartheta}{2} + \dfrac{1}{3}\tan^3\dfrac{\vartheta}{2}\right) & \text{for parabolic orbits,}\\[2ex] \dfrac{a^{3/2}}{\sqrt{\mu}}(e\sinh H - H) & \text{for hyperbolic orbits,}\end{cases}$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\vartheta}{2}, \quad \tanh\frac{H}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\vartheta}{2},$$

$$p = \frac{c^2}{\mu} = \frac{r^2v^2 - (\vec{r}\cdot\vec{v})^2}{\mu}, \quad r_\pi = \frac{p}{1+e}, \quad P = 2\pi\frac{a^{3/2}}{\sqrt{\mu}}, \quad n = \frac{\sqrt{\mu}}{a^{3/2}},$$

$$M = \frac{\sqrt{\mu}}{a^{3/2}}(t - t_\pi), \quad \cos\vartheta = \frac{1}{e}\left(\frac{p}{r} - 1\right), \quad u = \omega + \vartheta.$$

4. Universal Formulas for Keplerian Motion

4.1. Necessary formulas and prefatory remarks

$$
\boxed{
\begin{aligned}
& h = v^2 - \frac{2\mu}{r} \\
& \vec{c} = \vec{r} \times \vec{v} \\
& c^2 = \vec{c} \cdot \vec{c} = r^2 v^2 - (\vec{r} \cdot \vec{v})^2 \\
& \vec{l} = -\mu \frac{\vec{r}}{r} + \vec{v} \times \vec{c} \\
& \text{Elliptic orbits:} \\
& t - t_\pi = \frac{a^{3/2}}{\sqrt{\mu}} (E - e \sin E) \\
& r = a(1 - e \cos E) \\
& \text{Hyperbolic orbits:} \\
& t - t_\pi = \frac{a^{3/2}}{\sqrt{\mu}} (e\,\mathrm{sh} H - H) \\
& r = a(e\,\mathrm{ch} H - 1)
\end{aligned}
}
\tag{4.1}
$$

Disadvantages of the approach to the Keplerian motion calculation considered in Chapter 3 are:

- different formulas for different orbit types;
- the formulas do not work directly with t_0, t, it is necessary to calculate also t_π and corresponding parameters;
- the formulas do not work for near-parabolic orbits, i.e. if a value is very big.

4.2. Stumpf functions and their properties

Stumpf functions:

$$\boxed{c_n = c_n(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{(2m+n)!}, \quad n = 0, 1, 2, \ldots} \tag{4.2}$$

From (4.2):

$$c_n = \frac{1}{n!} + \sum_{m=1}^{\infty} \frac{(-x)^m}{(2m+n)!} = \frac{1}{n!} - x \sum_{k=0}^{\infty} \frac{(-x)^k}{(2k+n+2)!} \quad (k = m-1)$$

$$\Rightarrow \quad \boxed{c_n = \frac{1}{n!} - x\, c_{n+2}} \tag{4.3}$$

$\Rightarrow$ if one needs c_n for $n_1 \le n \le n_2$ it is sufficient to find c_{n_2-1}, c_{n_2} (or c_{n_1}, c_{n_1+1} if $x \ne 0$).

From (4.2):

$$\frac{dc_n}{dx} = -\sum_{m=1}^{\infty} \frac{m(-x)^{m-1}}{(2m+n)!} = -\frac{1}{2}\sum_{m=1}^{\infty} \frac{(2m+n)(-x)^{m-1}}{(2m+n)!} + \frac{n}{2}\sum_{m=1}^{\infty} \frac{(-x)^{m-1}}{(2m+n)!}$$

$$= -\frac{1}{2}\sum_{k=0}^{\infty} \frac{(-x)^k}{(2k+n+1)!} + \frac{n}{2}\sum_{k=1}^{\infty} \frac{(-x)^k}{(2k+n+2)!} \quad (k = m-1)$$

$$\Rightarrow \quad \boxed{\frac{dc_n}{dx} = \frac{nc_{n+2} - c_{n+1}}{2}} \tag{4.4}$$

Using (4.2) and equalities

$$\cos y = \sum_{m=0}^{\infty} \frac{(-y)^{2m}}{(2m)!}, \quad \sin y = \sum_{m=0}^{\infty} \frac{(-y)^{2m+1}}{(2m+1)!},$$

$$\cosh y = \sum_{m=0}^{\infty} \frac{y^{2m}}{(2m)!}, \quad \sinh y = \sum_{m=0}^{\infty} \frac{y^{2m+1}}{(2m+1)!}$$

obtain from (4.2) the finite expressions for some of the Stumpf functions given in Table 4.1.

Table 4.1. Finite expressions for Stumpff functions

	$x > 0$	$x < 0$
c_0	$\cos\sqrt{x}$	$\cosh\sqrt{-x}$
c_1	$\dfrac{\sin\sqrt{x}}{\sqrt{x}}$	$\dfrac{\sinh\sqrt{-x}}{\sqrt{-x}}$
c_2	$\dfrac{1-\cos\sqrt{x}}{x}$	$\dfrac{\cosh\sqrt{-x}-1}{-x}$
c_3	$\dfrac{\sqrt{x}-\sin\sqrt{x}}{x\sqrt{x}}$	$\dfrac{\sinh\sqrt{-x}-\sqrt{-x}}{-x\sqrt{-x}}$

Other relations between Stumpf functions.

It can be obtained from Table 4.1 that

$$c_1^2 - c_0 c_2 = c_2 \tag{4.5}$$

Eqs. (4.3, 4.5) give

$$\begin{aligned} c_2^2 - c_1 c_3 &= c_3 - 2c_4 \\ c_3^2 - c_2 c_4 &= \frac{c_4}{2} - 2c_5 + 2c_6 \end{aligned} \tag{4.6}$$

4.3. Universal formula for the flight time

Define a universal variable s (fictitious time, generalized eccentric anomaly) by equation

$$\boxed{\dot{s} = \frac{1}{r}, \quad s(t_0) = 0} \tag{4.7}$$

Note that

$$\boxed{s > 0 \quad \text{if} \quad t > t_0} \tag{4.8}$$

Taking into account integral of energy (see (4.1)) and equality $v^2 = \vec{v} \cdot \vec{v}$ obtain from (4.7):

$$s = \int_{t_0}^{t} \frac{dt}{r} = \frac{1}{2\mu} \int_{\vec{r}_0}^{\vec{r}} \vec{v} \cdot d\vec{r} - \frac{h(t - t_0)}{2\mu},$$

$$\int_{\vec{r}_0}^{\vec{r}} \vec{v} \cdot d\vec{r} = \vec{r} \cdot \vec{v} - \vec{r}_0 \cdot \vec{v}_0 - \int_{\vec{v}_0}^{\vec{v}} \vec{r} \cdot d\vec{v} = \vec{r} \cdot \vec{v} - \vec{r}_0 \cdot \vec{v}_0 + \mu \int_{t_0}^{t} \frac{dt}{r}$$

$$\Rightarrow \quad \boxed{s = \frac{\vec{r} \cdot \vec{v} - \vec{r}_0 \cdot \vec{v}_0 - h\tau}{\mu}} \tag{4.9}$$

where

$$\boxed{\tau = t - t_0} \tag{4.10}$$

From (4.1):

$$1 = \frac{a^{3/2}}{\sqrt{\mu}} (1 - e\cos E) \dot{E} = \sqrt{\frac{a}{\mu}} r\dot{E}$$

$$1 = \frac{a^{3/2}}{\sqrt{\mu}} (e\cosh H - 1) \dot{H} = \sqrt{\frac{a}{\mu}} r\dot{H}$$

$$\Rightarrow \quad \dot{E} = \dot{H} = \sqrt{\frac{\mu}{a}} \frac{1}{r} = \sqrt{\frac{\mu}{a}} \dot{s}, \quad E(t_0) = E_0, \quad H(t_0) = H_0 \tag{4.11}$$

$$\Rightarrow \quad \boxed{s = \begin{cases} \sqrt{\dfrac{a}{\mu}}\left(E - E_0\right) & \text{for elliptic orbits,} \\ \sqrt{\dfrac{a}{\mu}}\left(H - H_0\right) & \text{for hyperbolic orbits} \end{cases}} \tag{4.12}$$

Define variable

$$\boxed{x = -hs^2} \tag{4.13}$$

$$\Rightarrow \quad \boxed{E - E_0 = \sqrt{x}\,, \quad H - H_0 = \sqrt{-x}} \tag{4.14}$$

Consider elliptic orbits; Eqs. (4.1, 4.11) give

$$\dot{r} = ae\sin E\,\dot{E} = \frac{\sqrt{\mu a}\,e\sin E}{r}$$

$$\Rightarrow \quad \vec{r}\cdot\vec{v} = r\dot{r} = \sqrt{\mu a}\,e\sin E \tag{4.15}$$

From (4.14):

$$\sin E = \sin\left(E_0 + \sqrt{x}\right) = \sin E_0 \cos\sqrt{x} + \cos E_0 \sin\sqrt{x} \tag{4.16}$$

$\Rightarrow$ using (4.14, 4.15) and equality $e\cos E_0 = 1 - \dfrac{r_0}{a}$ (see (4.1))

Eq. (4.9) give

$$\tau = \frac{\vec{r}\cdot\vec{v} - \vec{r}_0\cdot\vec{v}_0 - \mu s}{h} = \frac{\vec{r}_0\cdot\vec{v}_0\left(1 - \cos\sqrt{x}\right) - \sqrt{\mu a}\left(1 - \dfrac{r_0}{a}\right)\sin\sqrt{x} + \mu s}{-hs^2}\,s^2 \tag{4.17}$$

Eqs. (4.17, 4.3, 4.13) and Table 4.1 give:

$$\boxed{\tau = r_0 s c_1 + \vec{r}_0\cdot\vec{v}_0\, s^2 c_2 + \mu s^3 c_3} \tag{4.18}$$

= <u>universal Kepler's equation</u>.

Using expression $\dfrac{dt}{ds} = \dfrac{1}{\dot{s}}$ obtain from (4.1, 4.7, 4.10):

$$r = \frac{d\tau}{ds} \tag{4.19}$$

It follows from (4.4) and the equality $\dfrac{dx}{ds} = 2\dfrac{x}{s}$ that

$$\begin{aligned}\frac{d}{ds}\left(s^n c_n\right) &= ns^{n-1}c_n + s^n \frac{nc_{n+2} - c_{n+1}}{2} \cdot \frac{dx}{ds} \\ &= ns^{n-1}c_n + s^{n-1}\left[n\left(\frac{1}{n!} - c_n\right) - \frac{1}{(n-1)!} + c_{n-1}\right] = s^{n-1}c_{n-1}\end{aligned} \tag{4.20}$$

$\Rightarrow$ obtain from (4.18–4.20):

$$\boxed{r = r_0 c_0 + \vec{r}_0 \cdot \vec{v}_0\, s c_1 + \mu s^2 c_2} \tag{4.21}$$

4.4. Solving the universal Kepler's equation

Assume that τ is given and use the Newton–Raphson method (see Chapter 1) for finding s from (4.18). Using (4.18, 4.19, 4.21) obtain at *n*th iteration:

$$s_n = s_{n-1} - \frac{r_0 s c_1 + \vec{r}_0 \cdot \vec{v}_0 s^2 c_2 + \mu s^3 c_3 - \tau}{r_0 c_0 + \vec{r}_0 \cdot \vec{v}_0 s c_1 + \mu s^2 c_2}, \quad n = 0, 1, \ldots \tag{4.22}$$

Taking into account (4.8, 4.9) the first guess can be taken as follows:

$$s_0 = 0 \quad \text{or} \quad s_0 = -\frac{h\tau}{\mu} \quad (\text{if } h < 0)$$

4.5. Calculation of position and velocity (orbit propagation)

Represent position $\vec{r} = \vec{r}(t)$ and velocity $\vec{v} = \vec{v}(t) = \dot{\vec{r}}$ in the form:

$$\boxed{\begin{aligned} \vec{r} &= f\vec{r}_0 + g\vec{v}_0 \\ \vec{v} &= \dot{f}\vec{r}_0 + \dot{g}\vec{v}_0 \end{aligned}} \tag{4.23}$$

To find f, g apply dot and cross production of the first equation in (4.23) by the Laplace vector

$$\vec{l} = -\mu\frac{\vec{r}}{r} + \vec{v}\times\vec{c} = -\mu\frac{\vec{r}_0}{r_0} + \vec{v}_0\times\vec{c}$$

using equations

$$\begin{aligned} &\vec{l}\cdot\vec{r} = c^2 - \mu r, \quad \vec{l}\cdot\vec{v} = -\mu\frac{\vec{r}\cdot\vec{v}}{r} = -\mu\dot{r}, \\ &\vec{l}\times\vec{r} = (\vec{v}\times\vec{c})\times\vec{r} = (\vec{r}\cdot\vec{v})\vec{c}, \\ &\vec{l}\times\vec{v} = -\frac{\mu}{r}\vec{r}\times\vec{v} + (\vec{v}\times\vec{c})\times\vec{v} = \left(v^2 - \frac{\mu}{r}\right)\vec{c} \end{aligned}$$

$$\Rightarrow \quad \left.\begin{aligned} &c^2 - \mu r = f\left(c^2 - \mu r_0\right) - g\mu\dot{r}_0 \\ &\vec{r}\cdot\vec{v} = f\vec{r}_0\cdot\vec{v}_0 + g\left(v^2 - \frac{\mu}{r_0}\right) \end{aligned}\right\} \tag{4.24}$$

It can be derived from (4.24) using (4.18, 4.21) that

$$\boxed{f = 1 - \frac{\mu s^2 c_2}{r_0}, \quad g = \tau - \mu s^3 c_3} \tag{4.25}$$

Eqs. (4.20, 4.7) give:

$$\frac{d}{dt}\left(s^n c_n\right) = \frac{d}{ds}\left(s^n c_n\right)\dot{s} = \frac{s^{n-1}c_{n-1}}{r}$$

$\Rightarrow$ obtain from (4.25):

$$\boxed{\dot{f} = -\frac{\mu s c_1}{r_0 r}, \quad \dot{g} = 1 - \frac{\mu s^2 c_2}{r}} \tag{4.26}$$

5. Perturbations

5.1. Necessary formulas

$$\boxed{\begin{aligned}
&\vec{c} = \vec{r} \times \vec{v}, \quad c = |\vec{c}| = r v_n = \sqrt{\mu p} \\
&p = \frac{c^2}{\mu} = \frac{r^2 v_n^2}{\mu} \\
&e^2 = 1 + \frac{c^2}{\mu^2} h = 1 + \frac{r^2 v_n^2}{\mu^2}\left(v^2 - \frac{2\mu}{r}\right) \\
&r = \frac{p}{1 + e\cos\vartheta} \\
&v^2 = v_r^2 + v_n^2 \\
&v_r = \sqrt{\frac{\mu}{p}} e \sin\vartheta, \quad v_n = \frac{c}{r} = \sqrt{\frac{\mu}{p}}\left(1 + e\cos\vartheta\right) \\
&r^2 \dot{\vartheta} = c
\end{aligned}} \tag{5.1}$$

5.2. General information about perturbations

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}} \tag{5.2}$$

= unperturbed motion. Equation of perturbed motion is

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \vec{\varphi}} \tag{5.3}$$

where $\vec{\varphi} = \vec{\varphi}(\vec{r}, \vec{v}, t)$ is the perturbing acceleration or perturbation. Usually

$$\boxed{\varphi = |\vec{\varphi}| << \left|-\frac{\mu}{r^3}\vec{r}\right| = \frac{\mu}{r^2}} \tag{5.4}$$

Types of perturbations:

1. Gravitational perturbations:

- nonspherical shape of the planets;
- attraction of other celestial bodies.

Gravitational perturbations are conservative.

2. Nongravitational perturbations:

- atmospheric drag;
- solar radiation pressure;
- jet perturbations due to a gas leakage from the spacecraft or ice evaporation from the comet.

Another form (via state vector):

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \vec{g}(\vec{x}, t) \tag{5.5}$$

$$\boxed{\vec{x} = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}, \quad \vec{f}(\vec{x}) = \begin{bmatrix} \vec{v} \\ -\frac{\mu}{r^3}\vec{r} \end{bmatrix}, \quad \vec{g} = \vec{g}(\vec{x}, t) = \begin{bmatrix} \vec{0} \\ \vec{\varphi} \end{bmatrix}} \tag{5.6}$$

Simplest way of solving equation (5.5) is a numerical integration.

5.3. Osculating elements

$$\boxed{\vec{q} = \{a,\ e,\ i,\ \Omega,\ \omega,\ t_\pi\}} \tag{5.7}$$

= vector of orbital elements.

$$\vec{q} = \vec{q}(\vec{x}) \tag{5.8}$$

where state vector $\vec{x}$ satisfies (5.5). Using (5.8, 5.5) obtain:

$$\dot{\vec{q}} = \frac{\partial \vec{q}}{\partial \vec{x}} \dot{\vec{x}} = \frac{\partial \vec{q}}{\partial \vec{x}} \left[\vec{f}\left(\vec{x}\right) + \vec{g}\left(\vec{x}, t\right) \right]$$

If $\vec{g}\left(\vec{x}, t\right) = \vec{0}$ (i.e. the motion is Keplerian, unperturbed) then $\vec{q} = \text{const}$

$$\Rightarrow \quad \frac{\partial \vec{q}}{\partial \vec{x}} \vec{f}\left(\vec{x}\right) = \vec{0}$$

$$\Rightarrow \quad \dot{\vec{q}} = \frac{\partial \vec{q}}{\partial \vec{x}} \vec{g}\left(\vec{x}, t\right) \tag{5.9}$$

and in general

$$\Rightarrow \quad a = a(t),\ \ e = e(t),\ \ i = i(t),\ \ \Omega = \Omega(t),\ \ \omega = \omega(t),\ \ t_\pi = t_\pi(t) \tag{5.10}$$

= osculating (instantaneous) elements; corresponding instantaneous orbit is osculating orbit (see Fig. 5.1).

Obtain from (5.9, 5.6):

$$\boxed{\dot{\vec{q}} = \frac{\partial \vec{q}}{\partial \vec{v}} \vec{\varphi}} \tag{5.11}$$

= differential equations for the osculating elements.

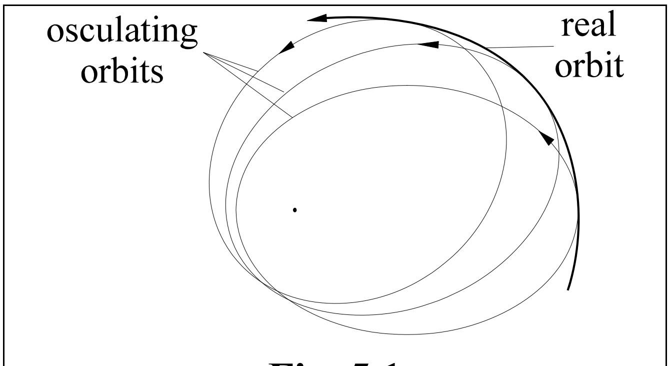


Fig. 5.1

Define orbital frame $\xi\eta\varsigma$ (see Fig. 5.2):

– origin is in the body mass center;
– ξ axis is directed along $\vec{r}$;
– η axis lies in the same plane as $\vec{r}, \vec{v}$ and has acute angle with $\vec{v}$;
– ζ axis completes the right frame.

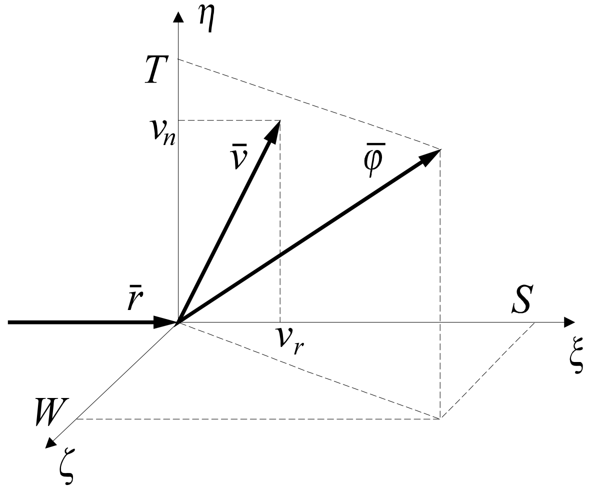


Fig. 5.2

Designate in the frame:

$$\vec{\varphi} = \{S, T, W\}, \quad \vec{v} = \{v_r, v_n, v_\zeta\} \tag{5.12}$$

where $v_\zeta = 0,\ \dot{v}_\zeta = W$ (see Fig. 5.2 and Eq. (5.1))

$\Rightarrow$ Eq. (5.11) gives:

$$\boxed{\dot{\vec{q}} = \frac{\partial \vec{q}}{\partial v_r} S + \frac{\partial \vec{q}}{\partial v_n} T + \frac{\partial \vec{q}}{\partial v_\zeta} W} \tag{5.13}$$

It follows from (5.1) that

$$\boxed{\begin{aligned} \dot{a} &= \sqrt{\frac{p}{\mu}} \frac{2a}{1-e^2} \left(e \sin\vartheta\, S + \frac{p}{r} T \right) \\ \dot{e} &= \sqrt{\frac{p}{\mu}} \left\{ \sin\vartheta\, S + \left[\left(1 + \frac{r}{p}\right) \cos\vartheta + \frac{r}{p} e \right] T \right\} \end{aligned}} \tag{5.14}$$

The cross production of (5.3) by $\vec{r}$ is

$$\vec{r} \times \ddot{\vec{r}} = \vec{r} \times \vec{\varphi} \tag{5.15}$$

(since $\vec{r} \times \vec{r} = \vec{0}$).

$$\dot{\vec{c}} = \frac{d}{dt} \vec{r} \times \vec{v} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \ddot{\vec{r}} \tag{5.16}$$

$\Rightarrow$ Eqs. (5.15, 5.16) give

$$\boxed{\dot{\vec{c}} = \vec{r} \times \vec{\varphi}} \tag{5.17}$$

One can obtain from (5.17) taking into account (3.23) (see Chapter 3):

$$\boxed{\begin{aligned} \dot{i} &= \sqrt{\frac{p}{\mu}}\frac{r}{p}\cos u\, W \\ \dot{\Omega} &= \sqrt{\frac{p}{\mu}}\frac{r}{p}\frac{\sin u}{\sin i}W \\ \dot{\omega} &= \sqrt{\frac{p}{\mu}}\left[-\frac{\cos\vartheta}{e}S+\frac{1}{e}\left(1+\frac{r}{p}\right)\sin\vartheta\, T-\frac{r}{p}\frac{\sin u}{\tan i}W\right] \end{aligned}} \tag{5.18}$$

(here $u=\omega+\vartheta$ is argument of latitude, see Chapter 3).

5.4. Secular and long-periodic perturbations

Consider elliptic orbits. Assume that the functions

$$S = S(\vec{q}, \vartheta), \; T = T(\vec{q}, \vartheta), \; W = W(\vec{q}, \vartheta) \tag{5.19}$$

are given. Then due to (5.7, 5.14, 5.18, 5.19)

$$\dot{\vec{q}} = \vec{\psi}(\vec{q}, \vartheta)$$

is a given function. First and last equations of Eqs. (5.1) give

$$\frac{dt}{d\vartheta} = \frac{1}{\dot{\vartheta}} = \frac{r^2}{\sqrt{\mu p}}$$

$$\Rightarrow \quad \frac{d\vec{q}}{d\vartheta} = \dot{\vec{q}}\frac{dt}{d\vartheta} = \vec{\psi}(\vec{q}, \vartheta)\frac{r^2}{\sqrt{\mu p}} = \vec{\chi}(\vec{q}, \vartheta) \tag{5.20}$$

is a given function. Assume that the inequality (5.4) is fulfilled
$\Rightarrow$ osculating elements $\vec{q}$ are changing slowly with time
$\Rightarrow$ it can be assumed that
$\vec{q} = \text{const}$ in $\vec{\chi}(\vec{q}, \vartheta)$ within one revolution, i.e. for $0 \le \vartheta \le 2\pi$

$$\Rightarrow \quad \boxed{\Delta\vec{q} \approx \int_0^{2\pi} \vec{\chi}(\vec{q}, \vartheta)\, d\vartheta} \tag{5.21}$$

is a change of the elements in one revolution. The procedure (5.21) averages short-periodic variations of the elements and gives secular and long-periodic variations:

- secular variation is growing with time;
- long-periodic variation is a periodic (or nearly periodic) variation with period higher than the orbital one.

5.5. Gravitational influence of other celestial bodies

Let μ_0, μ' be the gravitational parameters of the central (main) body and of the considered trial one respectively,

$$\mu = \mu_0 + \mu' \qquad (5.22)$$

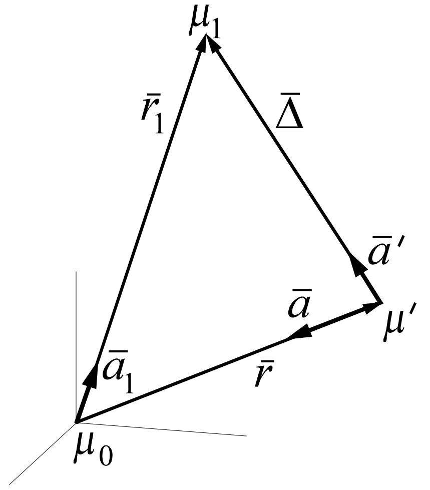


Fig. 5.3

Consider a third massive body with position vector $\vec{r}_1$ and gravitational parameter μ_1 and introduce (see Fig. 5.3):

$\vec{a} = -\dfrac{\mu}{r^3}\vec{r}$ = main (central) acceleration;

$\vec{a}' = \dfrac{\mu_1}{\Delta^3}\vec{\Delta}$ = acceleration from the third body, where

$\vec{\Delta} = \vec{r}_1 - \vec{r}, \quad \Delta = \left|\vec{\Delta}\right|;$

$\vec{a}_1 = \dfrac{\mu_1}{r_1^3}\vec{r}_1$ = acceleration of the frame origin caused by the third body $\Rightarrow$ $-\vec{a}_1$ is the acceleration of inertia

$$\Rightarrow \quad \boxed{\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} + \mu_1\left(\frac{\vec{\Delta}}{\Delta^3} - \frac{\vec{r}_1}{r_1^3}\right)} \qquad (5.23)$$

Inequality (5.4) is fulfilled when:

1. $\mu_1 << \mu$;
2. $r << r_1, \; r << \Delta$.

Examples

1. μ_0 = Sun, μ' = Earth, μ_1 = another planet.

   $\mu_1 << \mu_0 < \mu, \quad r_1 \sim r \sim \Delta$

   $\Rightarrow$ (5.4) is fulfilled.

2. μ_0 – Earth, $\mu' = 0$ – an Earth satellite, μ_1 – Sun.

   $\mu_1 >> \mu_0 = \mu$, but $r << r_1, \; r << \Delta$

   $\Rightarrow$ (5.4) is fulfilled.

Variations

$$\Delta a = 0$$

Variations of the other elements depend on the position of the perturbing body relatively to the orbit plane and apsid line of the perturbed body.

Variation of the eccentricity (see Fig. 5.4):

(a) if the projection of the perturbing body on the orbit plane is in quadrant I or III then

$$\Delta e < 0 \;\Rightarrow\; \Delta r_\pi > 0,\; \Delta r_\alpha < 0;$$

(b) if the projection of the perturbing body on the orbit plane is in quadrant II or IV then

$$\Delta e > 0 \;\Rightarrow\; \Delta r_\pi < 0,\; \Delta r_\alpha > 0.$$

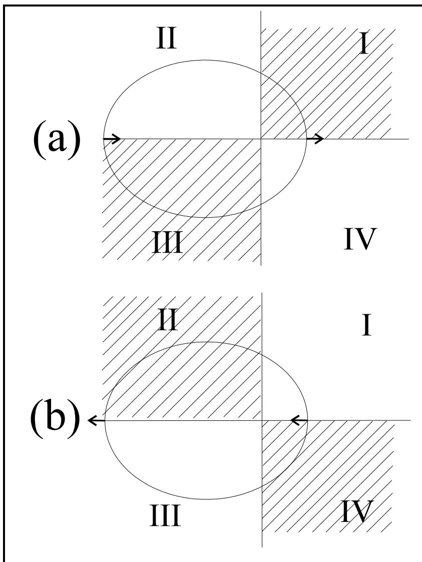


Fig. 5.4

5.6. Sphere of influence

Designate (see Fig. 5.5):

μ_0, μ_1 = gravitational parameters of the Sun and the planet;

$\vec{r}_0, \vec{r}_1$ = heliocentric and planet-centric position vectors of the body (μ in Fig. 5.5);

$\vec{\rho} = \vec{r}_0 - \vec{r}_1$ = position of the planet relatively to the Sun.

Comparing Figs. 5.3 and 5.5, obtain from (5.23) equations of motion of the body in the heliocentric frame (1) and planet-centric frame (2) as follows:

$$(1)\quad \ddot{\vec{r}}_0 = \vec{f}_0 + \vec{\varphi}_0, \quad \vec{f}_0 = -\frac{\mu_0}{r_0^3}\vec{r}_0, \quad \vec{\varphi}_0 = -\mu_1\left(\frac{\vec{r}_1}{r_1^3} + \frac{\vec{\rho}}{\rho^3}\right),$$

$$(2)\quad \ddot{\vec{r}}_1 = \vec{f}_1 + \vec{\varphi}_1, \quad \vec{f}_1 = -\frac{\mu_1}{r_1^3}\vec{r}_1, \quad \vec{\varphi}_1 = -\mu_0\left(\frac{\vec{r}_0}{r_0^3} - \frac{\vec{\rho}}{\rho^3}\right)$$

Sphere of influence is a surface surrounding the planet in which

$$\left|\vec{\varphi}_0\right| \Big/ \left|\vec{f}_0\right| = \left|\vec{\varphi}_1\right| \Big/ \left|\vec{f}_1\right|$$

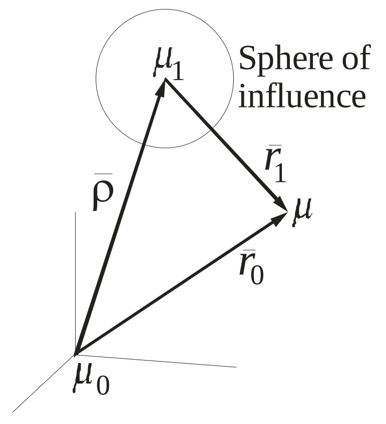


Fig. 5.5

Radius R_S of the sphere of influence is

$$\boxed{R_S \approx \rho\left(\frac{\mu_1}{\mu_0}\right)^{2/5}} \tag{5.24}$$

Table 5.1. Radii of the planet's spheres of influence

Planet	R_S, 10^6 km
Mercury	0.11
Venus	0.62
Earth	0.93
Mars	0.58
Jupiter	48.2
Saturn	54.6
Uranus	51.8
Neptune	87.0
Pluto	3.36

5.7. Influence of the planet oblateness

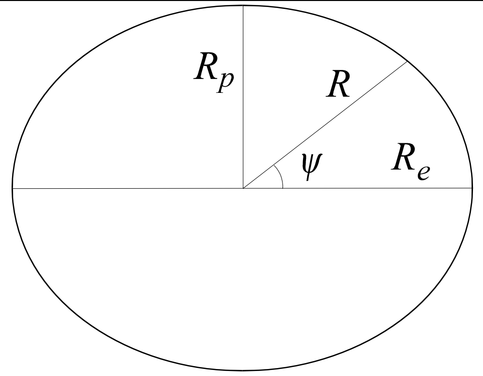


Fig. 5.6

Designate (see Fig. 5.6):

R_e, R_p = equatorial and polar radii of the planet;

$\alpha = \dfrac{R_e - R_p}{R_e}$ = oblateness coefficient ($\alpha \ll 1 \;\Rightarrow\; \alpha^2 \approx 0$);

$R_0 = R_e\left(1 - \dfrac{\alpha}{3}\right)$ = mean integral radius of the planet;

ψ = the planet-centric latitude.

Reference ellipsoid:

$$\boxed{R = R_e\left(1 - \alpha \sin^2 \psi\right) \approx R_0 + \alpha R_0 \left(\frac{1}{3} - \sin^2 \psi\right)} \tag{5.25}$$

= the ellipsoid radius in the latitude ψ .

$$\boxed{U = \frac{\mu}{r} + U_{pert}, \quad U_{pert} = \frac{3}{2} J_2 \frac{\mu}{r}\left(\frac{R_e}{r}\right)^2 \left(\frac{1}{3} - \sin^2 \psi\right)} \tag{5.26}$$

U is gravitational potential, U_{pert} is perturbing function where

$$J_2 = \text{const}$$

is a coefficient of the 2nd zonal harmonic.

The perturbing acceleration

$$\vec{\varphi} = \left(\frac{\partial U_{pert}}{\partial \vec{r}}\right)^T$$

can be obtained from (5.26) taking into account the equation

$$\sin \psi = \frac{z}{r}$$

Values of J_2 for the planets are given in Table 5.2.

Table 5.2. Values of J_2

Planet	J_2
Mercury	?
Venus	~ 0
Earth	$1.08263 \cdot 10^{-3}$
Mars	$1.96 \cdot 10^{-3}$
Jupiter	$1.471 \cdot 10^{-2}$
Saturn	$1.667 \cdot 10^{-2}$
Uranus	$1.3 \cdot 10^{-2}$
Neptune	$5.0 \cdot 10^{-2}$
Pluto	?

Secular perturbations are

$$\boxed{\begin{aligned} &\dot{a} = \dot{e} = \dot{i} = 0 \\ &\dot{\Omega} = -\frac{3}{2} J_2 n \left(\frac{R_e}{p}\right)^2 \cos i \\ &\dot{\omega} = \frac{3}{4} J_2 n \left(\frac{R_e}{p}\right)^2 \left(5\cos^2 i - 1\right) \end{aligned}} \tag{5.27}$$

where p is semilatus rectum and $n = \dfrac{\sqrt{\mu}}{a^{3/2}}$ is mean motion of the orbit.

Variations of the elements in one revolution are

$$\boxed{\begin{aligned} &\Delta a = \Delta e = \Delta i = 0 \\ &\Delta\Omega = -3\pi J_2 \left(\frac{R_e}{p}\right)^2 \cos i \\ &\Delta\omega = \frac{3}{2} \pi J_2 \left(\frac{R_e}{p}\right)^2 \left(5\cos^2 i - 1\right) \end{aligned}} \tag{5.28}$$

Solution of the equation $5\cos^2 i - 1 = 0$ is $i = 63.4°$

$$\Rightarrow \quad \boxed{\begin{aligned} &i < 63.4^{\circ} \quad \Rightarrow \quad \Delta\omega > 0 \\ &i > 63.4^{\circ} \quad \Rightarrow \quad \Delta\omega < 0 \end{aligned}}$$

5.8. Influence of the atmosphere

The atmospheric drag force is

$$F = c_x S_m \frac{\rho v^2}{2}$$

where

c_x = a dimensionless drag coefficient (usually $2 \le c_x \le 2.2$),
S_m = the spacecraft middle section,
ρ = the atmospheric density,
v = the spacecraft velocity.

The perturbing acceleration is

$$\boxed{\vec{\varphi} = -\frac{F}{m}\frac{\vec{v}}{v} = -c_x \frac{S_m}{m}\frac{\rho v}{2}\vec{v}} \tag{5.29}$$

High elliptic orbit: the atmospheric drag affects like a braking impulse in the periapsis (see Fig. 5.7)

$\Rightarrow$ $r_\pi \approx \text{const}$, r_α is gradually lowering down.

Circular orbit: the altitude is gradually lowering down (i.e. the spacecraft is descending in a spiral trajectory).

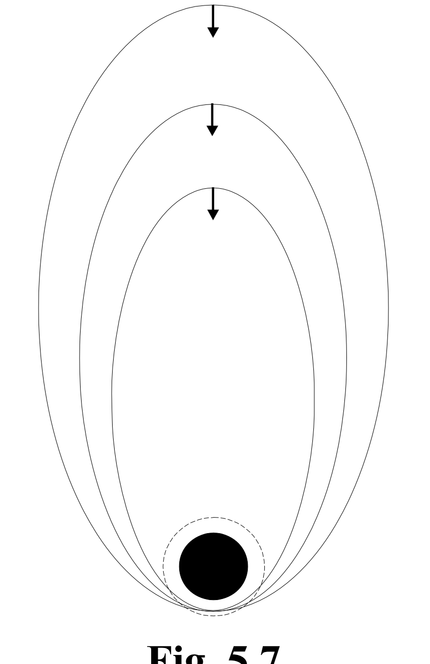


Fig. 5.7

Secular variations of the elements:

$$\Delta a < 0, \quad \Delta e < 0$$
$$\Delta i = \Delta\Omega = \Delta\omega = 0$$

(rotation of the atmosphere together with the planet is not taken into account).

6. Orbital Maneuvers

6.1. Useful formulas

Consider elliptic orbits:

$$\left.\begin{aligned} &v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \\ &a = \frac{r_\pi + r_\alpha}{2}, \quad r_\pi = a(1-e), \quad r_\alpha = a(1+e) \end{aligned}\right\} \tag{6.1}$$

Define parameter

$$\xi = \frac{r_\pi}{r_\alpha} = \frac{1-e}{1+e} \tag{6.2}$$

1. $r = r_\pi$

$$v_\pi^2 = \frac{\mu}{a}\left(\frac{2}{1-e} - 1\right) = \frac{\mu}{a}\,\frac{1+e}{1-e}$$

$$\Rightarrow \quad \boxed{v_\pi = \sqrt{\frac{\mu}{a}\frac{r_\alpha}{r_\pi}} = \sqrt{\frac{\mu}{a\xi}}} \tag{6.3}$$

2. $r = r_\alpha$

$$v_\alpha^2 = \frac{\mu}{a}\left(\frac{2}{1+e} - 1\right) = \frac{\mu}{a}\frac{1-e}{1+e}$$

$$\Rightarrow \quad v_\alpha = \sqrt{\frac{\mu}{a}\frac{r_\pi}{r_\alpha}} = \sqrt{\frac{\mu}{a}\xi} = \xi v_\pi \tag{6.4}$$

Another way for obtaining (6.4):

$$\vec{c} = \vec{r} \times \vec{v},$$

$$c = |\vec{c}| = r_\pi v_\pi = r_\alpha v_\alpha \quad \Rightarrow \quad v_\alpha = \xi v_\pi$$

6.2. Jet propulsion

Designate:

t_0, t = time before the jet thrust and current time ($t \geq t_0$);

$m_0 = m(t_0)$, $m = m(t)$ = initial and current spacecraft mass;

$m_p = m_p(t)$ = consumed propellant;

dm_p = propellant consumption in an infinitively small time dt;

$dm = -dm_p$ = decrement of the spacecraft mass in the time dt;

$\dot{m}_p = \dfrac{dm_p}{dt}$ = mass flow rate;

$\dot{m} = \dfrac{dm}{dt} = -\dot{m}_p \quad (\dot{m} \leq 0)$;

u = exhaust (or eject) velocity (assume u = const);

dv = increment of the spacecraft velocity in the time dt.

The conservation of momentum law gives:

$$m\,dv = u\,dm_p \tag{6.5}$$

$$\Rightarrow \quad \frac{dv}{u} = -\frac{dm}{m}$$

$$\Rightarrow \quad \boxed{\Delta v = v - v_0 = u \ln \frac{m_0}{m}} \tag{6.6}$$

= increment of the spacecraft velocity caused by the jet thrust.
From Eq. (6.6) obtain the propellant consumption:

$$m_p = m_0 - m = m_0 \left[1 - \exp\left(-\frac{\Delta v}{u} \right) \right] \tag{6.7}$$

The thrust force is

$$F_T = m \frac{dv}{dt}$$

$\Rightarrow$ from Eq. (6.5) obtain:

$$\boxed{F_T = -\dot{m} u} \tag{6.8}$$

The jet acceleration is

$$\boxed{\alpha = \frac{F_T}{m} = -\frac{\dot{m} u}{m}} \tag{6.9}$$

6.3. Impulsive maneuver

Impulsive maneuver is an <u>instant change of the orbital velocity</u> of a spacecraft by means of the jet thrust.

$$\vec{v}_- = \vec{v}(t-0), \quad \vec{v}_+ = \vec{v}(t+0)$$
$$\vec{v}_+ = \vec{v}_- + \Delta\vec{v}$$

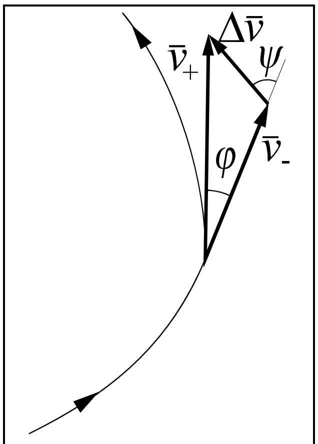


Fig. 6.1

$\Delta\vec{v}$ is the impulse (see Fig. 6.1)

$$\Delta v = |\Delta\vec{v}| = |\vec{v}_+ - \vec{v}_-| = \sqrt{v_-^2 + v_+^2 - 2v_-v_+\cos\varphi}$$

= <u>maneuver delta-V</u>, <u>maneuver cost</u>
(φ is the angle between $\vec{v}_-$ and $\vec{v}_+$, see Fig. 6.1).

The problem is to minimize m_p
$\Rightarrow$ due to Eq. (6.7) to minimize Δv .

6.4. Optimal impulsive change in some orbital parameters

Below the spacecraft velocity $\vec{v}_-$ before the maneuver will be designated as $\vec{v}$.

1. Changing orbital energy

$$h = v^2 - \frac{2\mu}{r} \tag{6.10}$$

= doubled orbital energy.

r is fixed during the maneuver $\Rightarrow$ it follows from (6.10) that

$$\Delta h \approx \frac{\partial h}{\partial \vec{v}} \Delta \vec{v} = 2\vec{v} \cdot \Delta \vec{v} = 2v\Delta v \cos\psi \tag{6.11}$$

where ψ is the angle between $\vec{v}$ and $\Delta\vec{v}$ (see Fig. 6.1).

$$\begin{aligned} &\Delta v \text{ is given} \Rightarrow \max|\Delta h| \text{ corresponds to } \max v \text{ and } \psi = 0 \text{ or } \psi = \pi; \\ &\Delta h \text{ is given} \Rightarrow \min \Delta v \text{ corresponds to } \max v \text{ and } \psi = 0 \text{ or } \psi = \pi \end{aligned} \tag{6.12}$$

It follows from (6.1) that

$$\max v = v_\pi \quad = \text{periapsis velocity.}$$

Elliptic orbit:

$$\Delta h \approx \frac{\mu}{a^2} \Delta a$$

$\Rightarrow$ conclusion (6.12) is also correct for Δa.

Conclusion:

> An impulse in periapsis collinear to the spacecraft velocity is optimal for changing orbital energy or semimajor axis.
> h and a increase if $\Delta\vec{v} \uparrow\uparrow \vec{v}_\pi$ and decrease if $\Delta\vec{v} \uparrow\downarrow \vec{v}_\pi$.

2. Changing apoapsis radius

r is fixed during the maneuver $\Rightarrow$ 1st equation of (6.1) gives

$$2\vec{v}\cdot\Delta\vec{v} \approx \frac{\mu}{a^2}\Delta a = \frac{\mu}{2a^2}\left(\Delta r_\pi + \Delta r_\alpha\right)$$

Δr_α is given $\Rightarrow$ $\min \Delta v$ is reached when $\Delta\vec{v} \uparrow\uparrow \vec{v}$ (or $\Delta\vec{v} \uparrow\downarrow \vec{v}$)
and $v = \max$

In periapsis $v = v_\pi = \max$ and $\Delta r_\pi = 0$.

Conclusion:

> An impulse in periapsis collinear to the spacecraft velocity is optimal for changing the apoapsis radius.
> r_α increases if $\Delta\vec{v} \uparrow\uparrow \vec{v}_\pi$ and decreases if $\Delta\vec{v} \uparrow\downarrow \vec{v}_\pi$.

3. Changing periapsis radius

$$\Delta r_\pi \approx \frac{\partial r_\pi}{\partial \vec{v}}\Delta\vec{v} = \pm\left|\frac{\partial r_\pi}{\partial \vec{v}}\right|\left|\Delta\vec{v}\right|$$

$$\frac{\partial r_\pi}{\partial \vec{v}} = \frac{\left(r^2\vec{v}_n - r_\pi^2\vec{v}\right)^T}{\mu e} \qquad (\vec{v} = \vec{v}_r + \vec{v}_n\text{, see also Section 13.6})$$

$\vec{c} = \vec{r}\times\vec{v}$ = angular momentum,

$c = |\vec{c}| = r v_n = \sqrt{\mu p}$=const

$$\left|r^2\vec{v}_n - r_\pi^2\vec{v}\right|^2 = r^4 v_n^2 - 2r^2 r_\pi^2 v_n^2 + r_\pi^4 v^2 = \left(r^2 - 2r_\pi^2\right)c^2 + r_\pi^4\left(\frac{2\mu}{r} - \frac{\mu}{a}\right)$$

$$\frac{d}{dr}\left|r^2\vec{v}_n - r_\pi^2\vec{v}\right|^2 = 2c^2 r - \frac{2\mu r_\pi^4}{r^2} = 2\mu\left(pr - \frac{r_\pi^4}{r^2}\right) > 0$$

since $r > r_\pi$, $p > r_\pi$

$\Rightarrow$ optimal impulse is in apoapsis

Conclusion:

> An impulse in apoapsis collinear to the spacecraft velocity is optimal for changing the periapsis radius.
> r_π increases if $\Delta\vec{v} \uparrow\uparrow \vec{v}_\alpha$ and decreases if $\Delta\vec{v} \uparrow\downarrow \vec{v}_\alpha$.

4. Turning orbit plane

Consider changing the spacecraft orbit plane in a given angle ε without any change in the orbit size and shape, i.e.

$$|\vec{v}_-| = |\vec{v}_+| = v \quad \text{(see Fig. 6.2)}$$

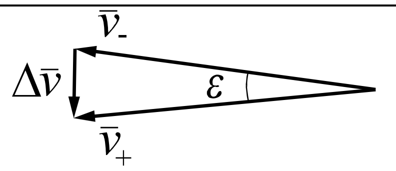


Fig. 6.2

$$\Rightarrow \quad \Delta v = 2v \sin\frac{\varepsilon}{2}$$

$\Rightarrow$ min Δv corresponds to $\min v = v_\alpha$ = apoapsis velocity.

Conclusion:

> An impulse in apoapsis is optimal for changing the spacecraft orbit plane.

6.5. One-impulse transfer

1. Transfer from a circular orbit to an elliptic one

Designate (see Fig. 6.3):

r_0 = radius of the initial orbit and periapsis radius of the final orbit;

r_1 = apoapsis radius of the final orbit;

$\xi = \dfrac{r_0}{r_1}$;

$v_0 = \sqrt{\dfrac{\mu}{r_0}}$ = velocity in the initial orbit;

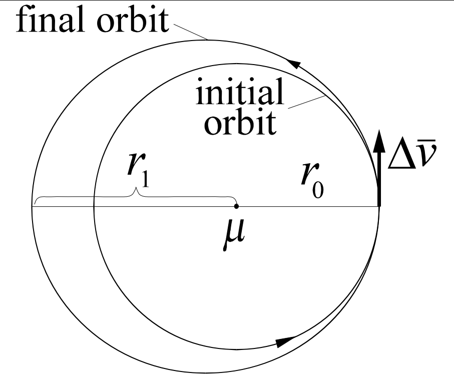


Fig. 6.3

$v_1 = \sqrt{\dfrac{2\mu}{r_0+r_1}\dfrac{r_1}{r_0}}$ = velocity in periapsis of the final orbit.

$$v_1 = \sqrt{\frac{\mu}{r_0}\frac{2}{1+\xi}} = v_0\sqrt{\frac{2}{1+\xi}}$$

$$\Rightarrow \quad \boxed{\Delta v = v_1 - v_0 = v_0\left(\sqrt{\frac{2}{1+\xi}} - 1\right)} \qquad (6.13)$$

Consider a small maneuver, i.e. $\Delta r = r_1 - r_0 << r_0$.
The impulse is tangent (see Section 6.4) $\Rightarrow$ Eq. (6.1) gives

$$2v\,\Delta v \approx \frac{\mu}{a^2}\Delta a$$

In the considered case $a = r_0$, $\Delta a = \dfrac{r_0+r_1}{2} - r_0 = \dfrac{\Delta r}{2}$, $v = v_0 = \sqrt{\dfrac{\mu}{r_0}}$

$$\Rightarrow \quad \boxed{\Delta v \approx \frac{\sqrt{\mu}}{4r_0^{3/2}}\Delta r = \frac{v_0}{4r_0}\Delta r = \frac{n_0}{4}\Delta r = \frac{\pi}{2P_0}\Delta r} \qquad (6.14)$$

where n_0, P_0 are angular velocity (mean motion) and period of the initial orbit.

2. Transfer from a circular orbit to a parabolic one

In this case $v_1 = \sqrt{\dfrac{2\mu}{r_0}} = \sqrt{2}v_0$

$$\Rightarrow \quad \boxed{\Delta v = v_0\left(\sqrt{2}-1\right)} \tag{6.15}$$

3. Transfer from a circular orbit to a hyperbolic one

In this case $v_1 = \sqrt{\dfrac{2\mu}{r_0} + v_\infty^2} = v_0\sqrt{2+\left(\dfrac{v_\infty}{v_0}\right)^2}$

$$\Rightarrow \quad \boxed{\Delta v = \sqrt{\frac{2\mu}{r_0} + v_\infty^2} - \sqrt{\frac{\mu}{r_0}} = v_0\left[\sqrt{2+\left(\frac{v_\infty}{v_0}\right)^2} - 1\right]} \tag{6.16}$$

Note. The transfers considered are reversible, i.e. Δv of the transfer from elliptic, parabolic, or hyperbolic orbit to a circular one is given by (6.13, 6.15, 6.16) respectively.

6.6. Two-impulse transfer

Consider orbital transfer between a circular orbit and a coplanar elliptic orbit (see Fig. 6.4).

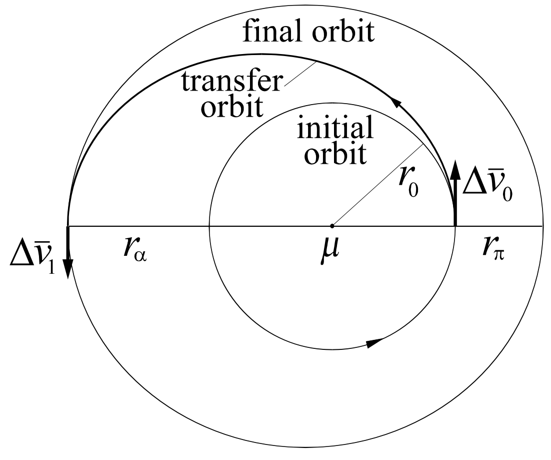


Fig. 6.4

Designate:

r_0 = radius of the initial circular orbit;

r_π, r_α = periapsis and apoapsis radii of the final elliptic orbit;

$$\xi_0 = \frac{r_0}{r_\alpha},\ \ \xi_1 = \frac{r_\pi}{r_\alpha} \quad (0 < \xi_1 \le 1)$$

v_0, v_1 = the spacecraft velocities in the initial orbit and at the arrival point of the final orbit

v_b, v_e = the spacecraft velocities at the beginning and end of the transfer orbit

$\Delta v_0, \Delta v_1$ = first and second impulses

Δv = total impulse

$$\Delta v_0 = |v_b - v_0|, \quad \Delta v_1 = |v_e - v_1|, \quad \Delta v = \Delta v_0 + \Delta v_1 \tag{6.17}$$

<u>Transfer to apoapsis of the final orbit</u> (see Fig. 6.4)

$$\left.\begin{aligned} v_0 &= \sqrt{\frac{\mu}{r_0}},\ \ v_1 = \sqrt{\frac{2\mu}{r_\pi + r_\alpha}\frac{r_\pi}{r_\alpha}} = v_0\sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} \\ v_b &= \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_\alpha}{r_0}} = v_0\sqrt{\frac{2}{1+\xi_0}},\ \ v_e = \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_0}{r_\alpha}} = v_0\xi_0\sqrt{\frac{2}{1+\xi_0}} \end{aligned}\right\} \tag{6.18}$$

Designate total impulse as Δv_α in this case.

Transfer to periapsis of the final orbit (see Fig. 6.5)

$$\left.\begin{aligned} v_0 &= \sqrt{\frac{\mu}{r_0}},\ v_1 = \sqrt{\frac{2\mu}{r_\pi + r_\alpha}\frac{r_\alpha}{r_\pi}} = v_0\sqrt{\frac{2\xi_0}{\xi_1(1+\xi_1)}} \\ v_b &= \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_\pi}{r_0}} = v_0\sqrt{\frac{2\xi_1}{\xi_0+\xi_1}},\ v_e = \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_0}{r_\pi}} = v_0\xi_0\sqrt{\frac{2}{\xi_1(\xi_0+\xi_1)}} \end{aligned}\right\} \tag{6.19}$$

Designate total impulse as Δv_π in this case.

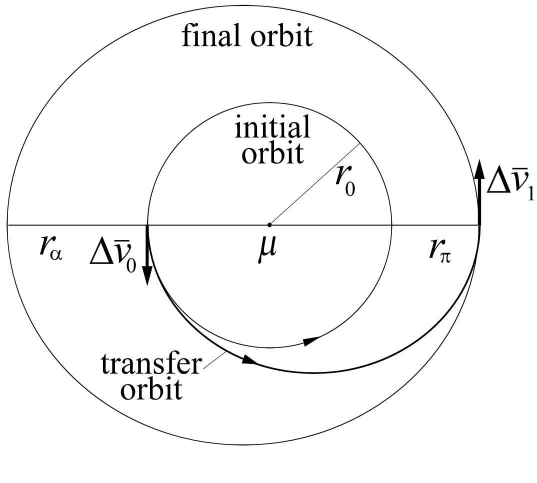


Fig. 6.5

Final orbit is outside the initial one (as is shown in Figs. 6.4, 6.5), i.e. $r_0 \le r_\pi \le r_\alpha$

$$\Rightarrow \quad 0 < \xi_0 \le \xi_1 \le 1 \tag{6.20}$$

$$\Delta v_0 = v_b - v_0, \quad \Delta v_1 = v_1 - v_e \tag{6.21}$$

$\Rightarrow$ due to (6.17–6.19, 6.21)

$$\begin{aligned} \Delta v_\alpha &= v_0\left[\sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} + (1-\xi_0)\sqrt{\frac{2}{1+\xi_0}} - 1\right] \\ \Delta v_\pi &= v_0\left[\sqrt{\frac{2\xi_0}{\xi_1(1+\xi_1)}} + (\xi_1-\xi_0)\sqrt{\frac{2}{\xi_1(\xi_0+\xi_1)}} - 1\right] \end{aligned} \tag{6.22}$$

As is shown in Annex A, in this case $\Delta v_\alpha \le \Delta v_\pi$ (see (A.4))
$\Rightarrow$ optimal is transfer to apoapsis and

$$\boxed{\Delta v = v_0\left[\sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} + (1-\xi_0)\sqrt{\frac{2}{1+\xi_0}} - 1\right]} \tag{6.23}$$

Final orbit is inside the initial one, i.e. $r_\pi \le r_\alpha \le r_0$

$$\Rightarrow \quad 0 \le \xi_1 \le 1 \le \xi_0 \tag{6.24}$$

$$\Delta v_0 = v_0 - v_b, \quad \Delta v_1 = v_e - v_1 \tag{6.25}$$

$\Rightarrow$ due to (6.17–6.19, 6.25)

$$\begin{aligned} \Delta v_\alpha &= v_0 \left[1 - \sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} - (1-\xi_0)\sqrt{\frac{2}{1+\xi_0}} \right] \\ \Delta v_\pi &= v_0 \left[1 - \sqrt{\frac{2\xi_0}{\xi_1(1+\xi_1)}} - (\xi_1 - \xi_0)\sqrt{\frac{2}{\xi_1(\xi_0+\xi_1)}} \right] \end{aligned} \tag{6.26}$$

As is shown in Annex A, in this case $\Delta v_\pi \le \Delta v_\alpha$ (see (A.5))
$\Rightarrow$ optimal is transfer to periapsis and

$$\boxed{\Delta v = v_0 \left[1 - \sqrt{\frac{2\xi_0}{\xi_1(1+\xi_1)}} - (\xi_1 - \xi_0)\sqrt{\frac{2}{\xi_1(\xi_0+\xi_1)}} \right]} \tag{6.27}$$

Initial and final orbits intersect, i.e. $r_\pi \le r_0 \le r_\alpha$

$$\Rightarrow \quad 0 < \xi_1 \le \xi_0 \le 1 \tag{6.28}$$

$\Rightarrow$ due to (6.17–6.19) $\Delta v_0 = v_b - v_0,\ \Delta v_1 = v_e - v_1,$

$$\Delta v_\alpha = v_0 \left[\sqrt{2(1+\xi_0)} - \sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} - 1 \right] \tag{6.29}$$

for transfer to apoapsis and $\Delta v_0 = v_0 - v_b,\ \Delta v_1 = v_1 - v_e,$

$$\Delta v_\pi = v_0 \left[\sqrt{\frac{2\xi_0}{\xi_1(1+\xi_1)}} - \sqrt{\frac{2(\xi_0+\xi_1)}{\xi_1}} + 1 \right] \tag{6.30}$$

for transfer to periapsis.

As is shown in Annex A, in this case $\Delta v_\alpha \le \Delta v_\pi$ (see (A.8))
$\Rightarrow$ optimal is transfer to apoapsis and

$$\boxed{\Delta v = v_0 \left[\sqrt{2(1+\xi_0)} - \sqrt{\frac{2\xi_0\xi_1}{1+\xi_1}} - 1 \right]} \tag{6.31}$$

Note. The transfers considered are reversible, i.e. Δv of the transfer from elliptic orbit to a circular one is given by (6.23, 6.27, 6.31).

Hohmann transfer

Consider a special case of the circular-to-elliptic orbit transfer: optimal transfer between two circular coplanar orbits (Hohmann transfer). Designate:

r_1 = radius of the final orbit

$$\Rightarrow \quad r_\pi = r_\alpha = r_1 \quad \Rightarrow \quad \xi_0 = \frac{r_0}{r_1}, \quad \xi_1 = 1$$

and

$\xi_0 < 1$ if the final orbit is outside the initial one,
$\xi_0 > 1$ if the final orbit is inside the initial one.

Eqs. (6.23, 6.27) give

$$\boxed{\Delta v = v_0 \left| \sqrt{\frac{2}{1+\xi_0}}(1-\xi_0) - \left(1-\sqrt{\xi_0}\right) \right|} \tag{6.32}$$

Fig. 6.6 shows the value $\Delta v / v_0$ versus ξ_0 for the Hohmann transfer outside the initial orbit ($\xi_0 < 1$). The value $\xi_0^{(1)} = \arg\max \Delta v$ can be found from the equation

$$\frac{d}{d\xi_0}\left(\frac{\Delta v}{v_0}\right) = 0$$

$$\Rightarrow \quad \xi_0^3 + 9\xi_0^2 + 15\xi_0 - 1 = 0 \tag{6.33}$$

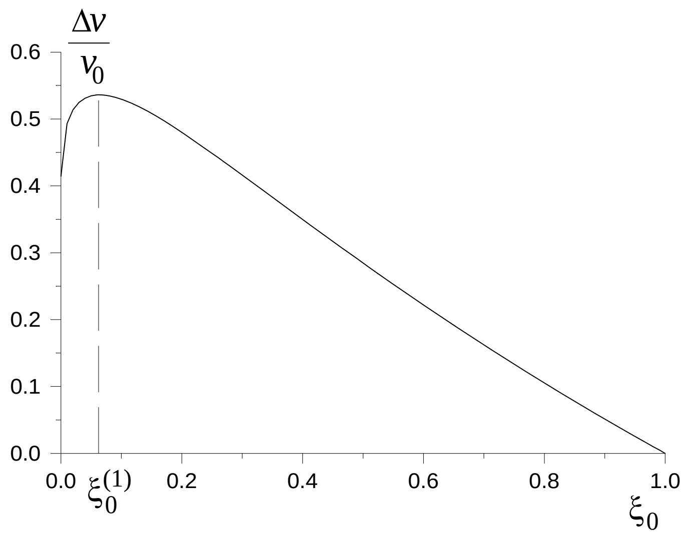


Fig. 6.6

The only solution to (6.33) between 0 and 1 is

$$\boxed{\xi_0^{(1)} = 0.064178 \quad \Rightarrow \quad r_1^{(1)} = 15.58172\ r_0} \tag{6.34}$$

Note. This result is also applicable for the transfer inside the initial orbit (i.e. for $\xi_0 > 1$) by means of replacing ξ_0 by $1/\xi_0$ in Fig. 6.6 and Eq. (6.34). In this case

$$r_1^{(1)} = 0.064178\ r_0$$

Hohmann transfer between neighboring orbits

Assume that

$$r_0 = r_1(1-\varepsilon),\ |\varepsilon| \ll 1 \quad \Rightarrow \quad \xi_0 = 1-\varepsilon$$

$\Rightarrow$ Eq. (6.32) becomes

$$\Delta v = v_0 \left| \sqrt{\frac{2}{2-\varepsilon}}\varepsilon - 1 + \sqrt{1-\varepsilon} \right| \approx v_0 \left| \sqrt{1+\frac{\varepsilon}{2}}\varepsilon - 1 + 1 - \frac{\varepsilon}{2} \right|$$

$$\Rightarrow \quad \boxed{\Delta v = \frac{v_0 |\varepsilon|}{2}} \tag{6.35}$$

6.7. Three-impulse transfer

<u>Transfer between two circular coplanar orbits</u> (bi-elliptic and bi-parabolic transfers).

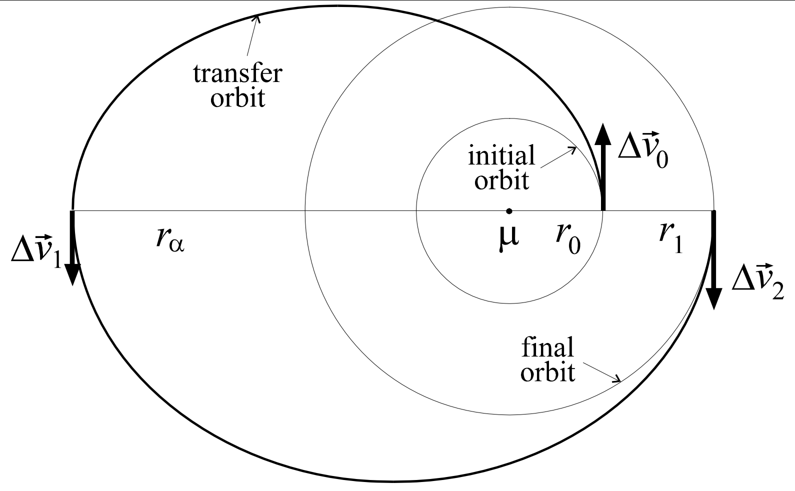


Fig. 6.7

Designate (see Fig. 6.7):

r_0 = radius of the initial circular orbit;
r_1 = radius of the final circular orbit;
r_α = apoapsis radius of the transfer orbit;

$$\xi_0 = \frac{r_0}{r_1}, \quad \xi = \frac{r_0}{r_\alpha}, \quad \xi_1 = \frac{r_1}{r_\alpha} = \frac{\xi}{\xi_0} \quad (0 < \xi_0 < 1,\ 0 < \xi_1 \le 1);$$

$0 \le \xi \le \xi_0$:

$\xi = \xi_0$ = two-impulse transfer ($r_\alpha = r_1$);
$0 < \xi < \xi_0$ = bi-elliptic transfer ($r_1 < r_\alpha < \infty$);
$\xi = 0$ = bi-parabolic transfer ($r_\alpha = \infty$);

$$\Delta v_0 = \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_\alpha}{r_0}} - \sqrt{\frac{\mu}{r_0}} = v_0\left(\sqrt{\frac{2}{1+\xi}} - 1\right);$$

$$\Delta v_1 = \sqrt{\frac{2\mu}{r_1 + r_\alpha}\frac{r_1}{r_\alpha}} - \sqrt{\frac{2\mu}{r_0 + r_\alpha}\frac{r_0}{r_\alpha}} = v_0\left(\sqrt{\frac{2\xi\xi_1}{1+\xi_1}} - \xi\sqrt{\frac{2}{1+\xi}}\right);$$

$$\Delta v_2 = \sqrt{\frac{2\mu}{r_1 + r_\alpha}\frac{r_\alpha}{r_1}} - \sqrt{\frac{\mu}{r_1}} = v_0\left(\sqrt{\frac{2\xi_0}{1+\xi_1}} - \sqrt{\xi_0}\right);$$

$$\boxed{\Delta v = \Delta v_0 + \Delta v_1 + \Delta v_2 = v_0\left[\sqrt{\frac{2}{1+\xi}}(1-\xi) + \sqrt{2(\xi_0+\xi)} - 1 - \sqrt{\xi_0}\right]} \tag{6.36}$$

Find value $\xi_0^{(2)}$ of ξ_0 starting from which three-impulse bi-elliptic transfer becomes preferable.

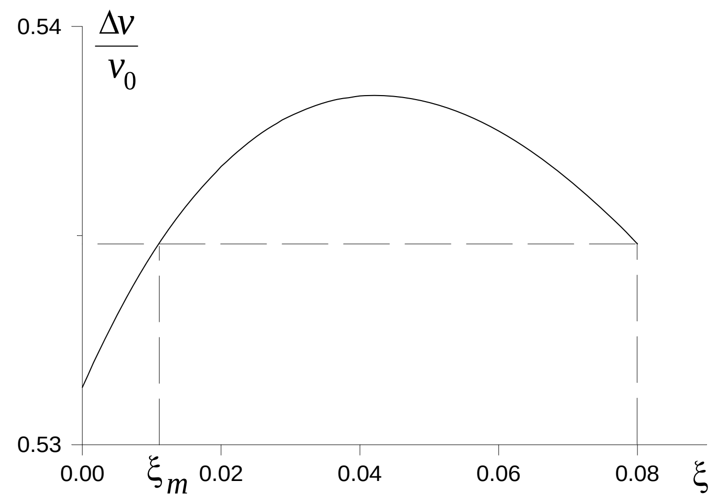


Fig. 6.8

Fig. 6.8 shows the value $\Delta v/v_0$ versus ξ for $\xi_0 = 0.08$.

If $\xi < \xi_m$ then bi-elliptic three-impulse transfer is better than the two-impulse (Hohmann) one; here ξ_m is a solution to the equation

$$\Delta v(\xi = \xi_m) = \Delta v(\xi = \xi_0) \tag{6.37}$$

The value $\xi_0^{(2)}$ can be found from the equation

$$\Delta v(\xi = 0) = \Delta v(\xi = \xi_0).$$

From (6.36) obtain:

$$\sqrt{2} + \sqrt{2\xi_0} - 1 - \sqrt{\xi_0} = \sqrt{\frac{2}{1+\xi_0}}(1-\xi_0) + 2\sqrt{\xi_0} - 1 - \sqrt{\xi_0}$$

$$\Rightarrow \quad \sqrt{\frac{2}{1+\xi_0}}(1-\xi_0) + \left(2-\sqrt{2}\right)\sqrt{\xi_0} - \sqrt{2} = 0$$

Solution to the equation is:

$$\boxed{\xi_0^{(2)} = 0.083761 \quad \Rightarrow \quad r_1^{(2)} = 11.93877\ r_0} \tag{6.38}$$

Find the value $\xi_0^{(3)}$ of ξ_0 for which any three-impulse bi-elliptic transfer is preferable, i.e.

$$\arg\max_{\xi} \Delta v = \xi_0^{(3)}$$

(see Fig. 6.9).

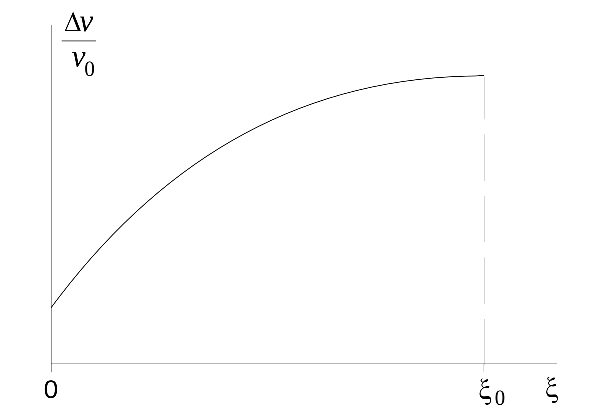


Fig. 6.9

From (6.36) obtain:

$$\frac{d}{d\xi}\frac{\Delta v}{v_0} = -\sqrt{\frac{2}{1+\xi}} - \frac{1}{2}\sqrt{\frac{2}{1+\xi}}\frac{1-\xi}{1+\xi} + \frac{1}{2}\sqrt{\frac{2}{\xi_0+\xi}} = 0$$

$$\Rightarrow \quad (3+\xi_0)\xi^2 + 6(1+\xi_0)\xi - (1-9\xi_0) = 0$$

$$\Rightarrow \quad \xi = \frac{-3(1+\xi_0)+2\sqrt{3-2\xi_0}}{3+\xi_0} = \xi_0$$

$$\Rightarrow \quad \xi_0^4 + 12\xi_0^3 + 42\xi_0^2 + 44\xi_0 - 3 = 0$$

$$\Rightarrow \quad (\xi_0+3)\left(\xi_0^3 + 9\xi_0^2 + 15\xi_0 - 1\right) = 0 \tag{6.39}$$

Comparing (6.39) and (6.33) and taking into account (6.34) obtain that the only solution to (6.39) between 0 and 1 is

$$\boxed{\xi_0^{(3)} = \xi_0^{(1)} = 0.064178 \quad \Rightarrow \quad r_1^{(3)} = 15.58172\ r_0} \tag{6.40}$$

<u>Conclusion</u>. The comparison of different transfer types gives the optimal transfers in terms of $\min \Delta v$:

- $r_0 < r_1 < r_1^{(2)}$ = two-impulse (Hohmann) transfer is optimal;
- $r_1 = r_1^{(2)}$ = two-impulse transfer and three-impulse bi-parabolic one are equally optimal; any bi-elliptic transfer is worse;
- $r_1 > r_1^{(2)}$ = three-impulse bi-parabolic transfer is optimal and:
- $r_1^{(2)} < r_1 < r_1^{(3)}$ = three-impulse bi-elliptic transfer with $r_\alpha > r_0/\xi_m$ is better than the Hohmann transfer (ξ_m is a solution to (6.37), see also Fig. 6.8);
- $r_1 \geq r_1^{(3)}$ = any three-impulse bi-elliptic transfer is better than the Hohmann one.

<u>Note</u>. The results are applicable if the final orbit is inside the initial one. In this case

$$r_1^{(2)} = 0.083761\ r_0, \quad r_1^{(3)} = 0.064178\ r_0$$

and the conclusions should be respectively changed.

2. Turning circular orbit plane

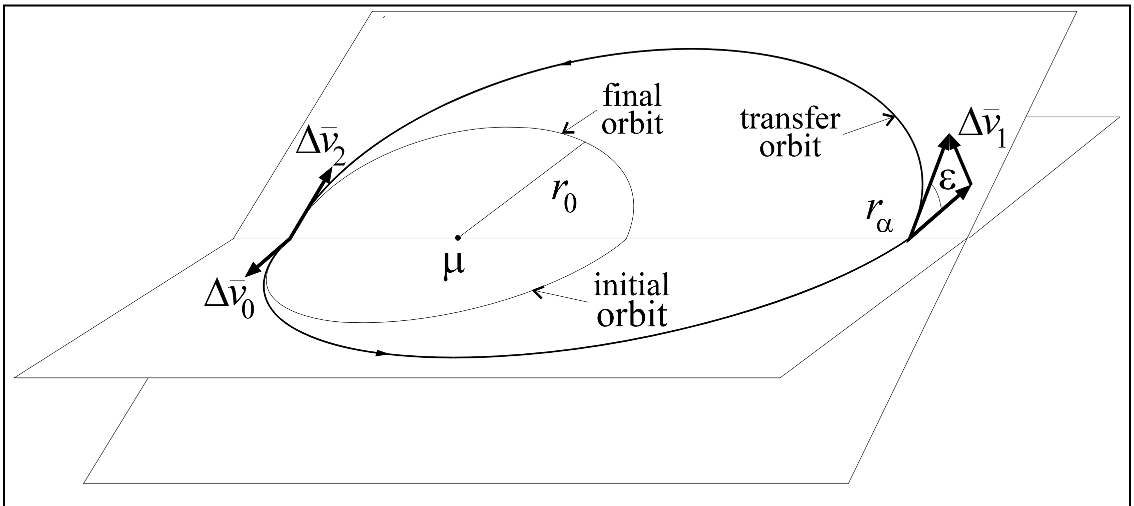


Fig. 6.10

Designate (see Fig. 6.10):

r_0 = radius of the initial and final circular orbits;

r_α = apoapsis radius of the transfer orbit;

ε = angle between the initial and final circular orbits;

$\xi = \dfrac{r_0}{r_\alpha}, \quad 0 \le \xi \le 1,$

$\xi = 1$ = one-impulse transfer ($r_\alpha = r_0$);

$0 < \xi < 1$ = three-impulse bi-elliptic transfer ($r_0 < r_\alpha < \infty$);

$\xi = 0$ = three-impulse bi-parabolic transfer ($r_\alpha = \infty$);

$v_0 = \sqrt{\dfrac{\mu}{r_0}}$ = velocity in the initial and final circular orbits;

$v_\pi = \sqrt{\dfrac{2\mu}{r_0 + r_\alpha}\dfrac{r_\alpha}{r_0}} = v_0\sqrt{\dfrac{2}{1+\xi}}$ = velocity in the periapsis of the transfer orbit;

$v_\alpha = \sqrt{\dfrac{2\mu}{r_0 + r_\alpha}\dfrac{r_0}{r_\alpha}} = v_0\xi\sqrt{\dfrac{2}{1+\xi}}$ = velocity in the apoapsis of the transfer orbit;

$$\Delta v_0 = v_\pi - v_0 = v_0\left(\sqrt{\frac{2}{1+\xi}} - 1\right);$$

$$\Delta v_1 = 2v_\alpha \sin\frac{\varepsilon}{2} = 2v_0\sqrt{\frac{2}{1+\xi}}\sin\frac{\varepsilon}{2};$$

$$\Delta v_2 = \Delta v_0;$$

$$\boxed{\Delta v = \Delta v_0 + \Delta v_1 + \Delta v_2 = 2v_0\left[\sqrt{\frac{2}{1+\xi}}\left(1+\xi\sin\frac{\varepsilon}{2}\right) - 1\right]} \qquad (6.41)$$

Fig. 6.11 shows the value $\Delta v/v_0$ versus ξ for $\varepsilon = 45°$.

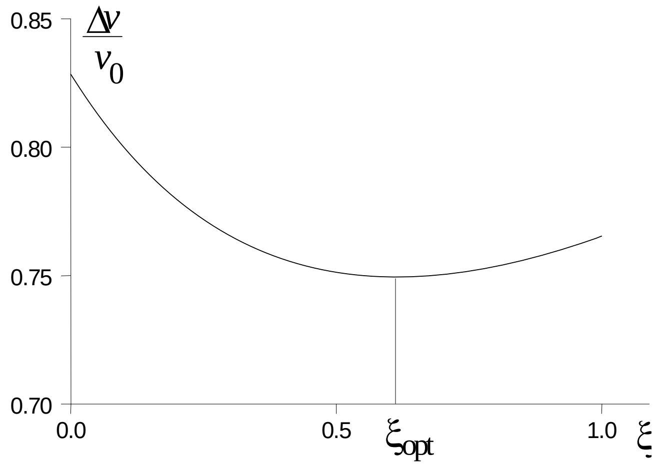


Fig. 6.11

Find the optimal value

$$\xi_{opt} = \arg\min_{\xi} \Delta v$$

(see Fig. 6.11):

$$\frac{d}{d\xi}\frac{\Delta v}{v_0} = -\sqrt{\frac{2}{1+\xi}}\,\frac{1+\xi\sin\frac{\varepsilon}{2}}{1+\xi} + 2\sqrt{\frac{2}{1+\xi}}\sin\frac{\varepsilon}{2} = 0$$

1) $\xi = 1 \;\Rightarrow\; \varepsilon = \varepsilon^{(1)} = 38.94°$;
2) $\xi = 0 \;\Rightarrow\; \varepsilon = \varepsilon^{(2)} = 60°$;
3) $0 < \xi < 1,\;\; \varepsilon^{(1)} < \varepsilon < \varepsilon^{(2)}$

$$\Rightarrow \quad \boxed{\xi_{opt} = \frac{1}{\sin\frac{\varepsilon}{2}} - 2 \quad\Rightarrow\quad r_{\alpha\,opt} = \frac{\sin\frac{\varepsilon}{2}}{1-2\sin\frac{\varepsilon}{2}}r_0} \qquad (6.42)$$

Conclusion:

$\varepsilon \le \varepsilon^{(1)}$	= one-impulse transfer is optimal;
$\varepsilon^{(1)} < \varepsilon < \varepsilon^{(2)}$	= three-impulse bi-elliptic transfer with r_α given by (6.42) is optimal;
$\varepsilon \ge \varepsilon^{(2)}$	= three-impulse bi-parabolic transfer is optimal.

7. Lambert Problem

7.1. Statement of the problem

Designate (see Fig. 7.1):

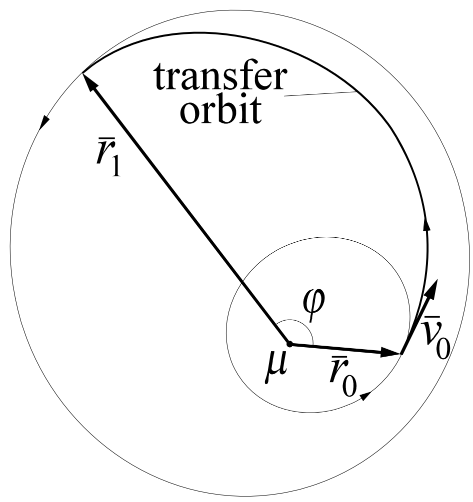


Fig. 7.1

t_0, t_1 = times of the departure and arrival respectively;

$\vec{r}_0, \vec{r}_1$ = position vectors of the departure and arrival points respectively;

$\vec{v}_0$ = spacecraft velocity at the beginning of the <u>transfer orbit</u> (i.e. the spacecraft orbit from $\vec{r}_0$ to $\vec{r}_1$);

$\tau = t_1 - t_0$ = <u>time of flight</u>;

φ = <u>transfer angle</u>.

The problem is to find the transfer orbit for given $\vec{r}_0, \vec{r}_1, \tau$ (<u>Lambert problem</u>).

7.2. Necessary formulas

Designate (see Chapter 4):

s = universal variable,

$$\dot{s} = \frac{1}{r}, \quad s(t_0) = 0, \tag{7.1}$$

$$x = -hs^2 \tag{7.2}$$

where h is the constant of energy;

$c_n = c_n(x) \quad (n = 0,1,...)$ is Stumpf functions.

The following expressions obtained in Chapter 4 will be used below (see (4.2, 4.4 – 4.6)):

$$c_n(0) = \frac{1}{n!} \tag{7.3}$$

$$\frac{dc_n}{dx} = \frac{nc_{n+2} - c_{n+1}}{2} \tag{7.4}$$

$$c_1^2 - c_0 c_2 = c_2 \tag{7.5}$$

$$c_2^2 - c_1 c_3 = c_3 - 2c_4, \quad c_3^2 - c_2 c_4 = \frac{c_4}{2} - 2c_5 + 2c_6 \tag{7.6}$$

Let $s = s(t_1)$; then

$$\tau = r_0 s c_1 + \vec{r}_0 \cdot \vec{v}_0 s^2 c_2 + \mu s^3 c_3 \tag{7.7}$$

= universal Kepler's equation (see (4.18));

$$r_1 = \frac{d\tau}{ds} = r_0 c_0 + \vec{r}_0 \cdot \vec{v}_0 s c_1 + \mu s^2 c_2; \tag{7.8}$$

$$\vec{r}_1 = f\vec{r}_0 + g\vec{v}_0, \tag{7.9}$$

where

$$f = 1 - \frac{\mu s^2 c_2}{r_0}, \quad g = \tau - \mu s^3 c_3 \tag{7.10}$$

(see (4.21, 4.23, 4.25)).

7.3. Equation for the Lambert problem

$\vec{r}_0, \vec{v}_0$ completely define the transfer orbit and $\vec{r}_0$ is given $\Rightarrow$

$\boxed{\text{to solve the Lambert problem it is necessary to determine } \vec{v}_0}$

Eq. (7.9) gives

$$\boxed{\vec{v}_0 = \frac{1}{g}\left(\vec{r}_1 - f\,\vec{r}_0\right)} \tag{7.11}$$

$\Rightarrow$ due to Eqs. (7.11, 7.10):

$\boxed{\text{to solve the Lambert problem it is sufficient to find } x, s}$

Assume that $\varphi \neq 2\pi$; it can be shown that in this case $c_2 \neq 0$

$\Rightarrow$ Eqs. (7.7, 7.10) give

$$\vec{r}_0 \cdot \vec{v}_0 = \frac{\tau - \mu s^3 c_3}{s^2 c_2} - \frac{r_0 s c_1}{s^2 c_2} = \frac{g}{s^2 c_2} - \frac{r_0 c_1}{s c_2} \tag{7.12}$$

Substitute (7.12) into (7.8):

$$r_1 = r_0 c_0 + \left(\frac{g}{s^2 c_2} - \frac{r_0 c_1}{s c_2}\right) s c_1 + \mu s^2 c_2 = r_0\left(c_0 - \frac{c_1^2}{c_2}\right) + \frac{g c_1}{s c_2} + \mu s^2 c_2 \tag{7.13}$$

Taking into account (7.5) obtain from (7.13):

$$\frac{g c_1}{s c_2} = r_0 + r_1 - \mu s^2 c_2 \tag{7.14}$$

Multiply (7.9) by $\vec{r}_0$ using (7.10, 7.12, 7.14):

$$\vec{r}_0 \cdot \vec{r}_1 = \left(1 - \frac{\mu s^2 c_2}{r_0}\right) r_0^2 + g\,\vec{r}_0 \cdot \vec{v}_0 = r_0^2 - \mu s^2 c_2 r_0 + \frac{g^2}{s^2 c_2} - \frac{g c_1}{s c_2}$$

$$= r_0^2 - \mu s^2 c_2 r_0 + \frac{g^2}{s^2 c_2} - r_0^2 - r_0 r_1 + \mu s^2 c_2 r_0 \tag{7.15}$$

Using the equality

$$r_0 r_1 + \vec{r}_0 \cdot \vec{r}_1 = r_0 r_1 (1 + \cos\varphi) = 2 r_0 r_1 \cos^2 \frac{\varphi}{2}$$

obtain from (7.15):

$$g^2 = 2 r_0 r_1 \cos^2 \frac{\varphi}{2} s^2 c_2 \tag{7.16}$$

Designate $\vec{c} = \vec{r}_0 \times \vec{v}_0$, $c = |\vec{c}|$, and multiply (7.9) by $\vec{r}_0$:

$$\vec{r}_0 \times \vec{r}_1 = g\vec{c} \tag{7.17}$$

On the other hand

$$\vec{r}_0 \times \vec{r}_1 = r_0 r_1 \sin\varphi \frac{\vec{c}}{c} \tag{7.18}$$

$\operatorname{sgn}\sin\varphi = \operatorname{sgn}\cos\frac{\varphi}{2}$ $\Rightarrow$ Eqs. (7.17, 7.18) give: $\operatorname{sgn} g = \operatorname{sgn}\cos\frac{\varphi}{2}$
$\Rightarrow$ from (7.16) obtain:

$$g = \sqrt{2 r_0 r_1} \cos\frac{\varphi}{2} s\sqrt{c_2} \tag{7.19}$$

Define:

$$\boxed{\rho = \frac{\sqrt{2 r_0 r_1}}{r_0 + r_1} \cos\frac{\varphi}{2}} \tag{7.20}$$

$$\boxed{u = \sqrt{1 - \rho \frac{c_1}{\sqrt{c_2}}}} \tag{7.21}$$

and substitute g from (7.19) into (7.14):

$$\rho \frac{c_1}{\sqrt{c_2}} = 1 - \frac{\mu s^2 c_2}{r_0 + r_1}$$

$$\Rightarrow \quad \boxed{s = \sqrt{\frac{r_0 + r_1}{\mu c_2}} u} \tag{7.22}$$

Substitute g from (7.10) and s from (7.22) into (7.19) using (7.20):

$$\tau - \mu\left(\frac{r_0 + r_1}{\mu c_2}\right)^{\frac{3}{2}} u^3 c_3 = (r_0 + r_1)\rho\sqrt{\frac{r_0 + r_1}{\mu c_2}} u\sqrt{c_2} \tag{7.23}$$

Dividing (7.23) by $(r_0 + r_1)^{3/2}$ and defining a dimensionless time

$$\boxed{\sigma = \frac{\sqrt{\mu}}{(r_0 + r_1)^{3/2}}\tau} \tag{7.24}$$

transform (7.23) into the equation:

$$\boxed{F(x,\rho) = \frac{c_3}{c_2^{3/2}} u^3 + \rho u = \sigma} \tag{7.25}$$

Eq. (7.25) is the equation for x. After x is found from (7.25) we can obtain s from (7.22) and then calculate f, g from (7.10) and $\vec{v}_0$ from (7.11).

Fig. 7.2 shows function $F(x,\rho)$ versus $y = \sqrt{|x|}\,\text{sgn}(x)$ for different values of ρ.

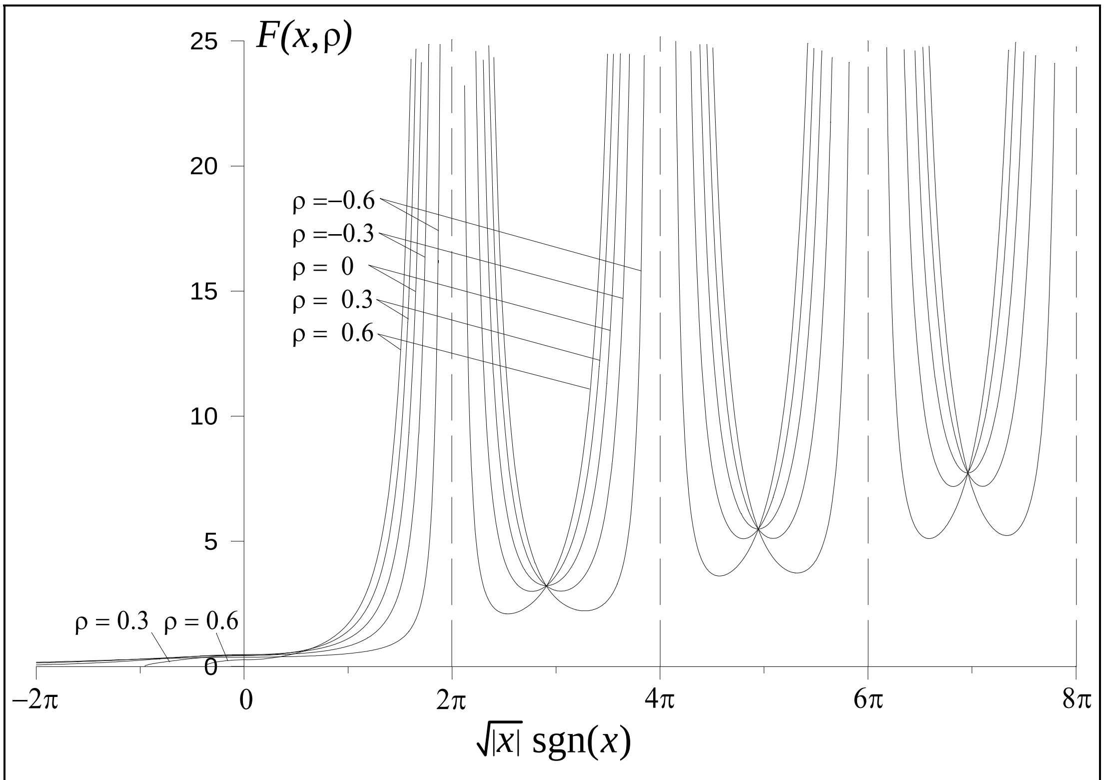


Fig. 7.2

7.4. Analysis of equation (7.25)

Domain of the function $F(x, \rho)$

As is seen in (7.20, 7.24)

$$-\frac{1}{\sqrt{2}} \le \rho \le \frac{1}{\sqrt{2}}, \qquad \sigma > 0 \tag{7.26}$$

Due to (7.21) it must be

$$\rho \frac{c_1}{\sqrt{c_2}} \le 1 \tag{7.27}$$

As follows from Table 4.1

$$\frac{c_1(x)}{\sqrt{c_2(x)}} = \sqrt{2} c_0\left(\frac{x}{4}\right) = \begin{cases} \sqrt{2}\cos\dfrac{\sqrt{x}}{2} & \text{if} \quad x \ge 0, \\ \sqrt{2}\cosh\dfrac{\sqrt{-x}}{2} & \text{if} \quad x < 0 \end{cases} \tag{7.28}$$

$\Rightarrow$ due to (7.26, 7.28) the inequality (7.27) can be broken only if

$$\rho > 0, \quad x < 0$$

$\Rightarrow$ Eqs. (7.26, 7.28) give

$$x \ge -4\ln^2\left(\frac{1}{\sqrt{2}\rho} + \sqrt{\frac{1}{2\rho^2} - 1}\right) \tag{7.29}$$

For $x > 0$

$$c_2(x) = \frac{1 - \cos\sqrt{x}}{x} \tag{7.30}$$

$\Rightarrow$ Eqs. (7.25, 7.30) give

$$x \ne 2\pi k \quad (k = 1, 2, \ldots) \tag{7.31}$$

Type of the transfer orbit

Obtain from the equation $x = -hs^2$:

$$\begin{aligned} x &< 0 \quad = \quad \text{hyperbolic orbit;} \\ x &= 0 \quad = \quad \text{parabolic orbit;} \\ x &> 0 \quad = \quad \text{elliptic orbit.} \end{aligned} \tag{7.32}$$

From (7.25) using (7.6) obtain derivative of the function $F(x,\rho)$:

$$F_x = \frac{dF(x,\rho)}{dx} = \frac{c_3^2 - c_5 + 4c_6}{4c_2^{3/2}} u^3 + \left(3\frac{c_3}{c_2^{3/2}} u^2 + \rho \right) \frac{\rho\sqrt{c_2}}{8u} \tag{7.33}$$

It can be shown that

$$F_x > 0 \quad \text{for} \quad x < (2\pi)^2 \tag{7.34}$$

(see Fig. 7.2). From (7.3, 7.25) obtain for the parabolic orbit:

$$F(0,\rho) = \frac{1}{3}\left(\sqrt{2} + \rho\right)\sqrt{1 - \sqrt{2}\rho} = \sigma_{par} \tag{7.35}$$

where σ_{par} is the dimensionless time of the parabolic transfer

$\Rightarrow$ due to (7.25, 7.34) obtain:

$$\boxed{\begin{aligned} \sigma &< \sigma_{par} \quad = \text{ hyperbolic orbit} \\ \sigma &= \sigma_{par} \quad = \text{ parabolic orbit} \\ \sigma &> \sigma_{par} \quad = \text{ elliptic orbit} \end{aligned}} \tag{7.36}$$

Elliptic transfer orbit

Consider x within the limits

$$(2\pi k)^2 < x < [2\pi(k+1)]^2 \quad (k = 0, 1, ...) \tag{7.37}$$

and define variables:

$$z = \frac{\sqrt{x}}{2} - \pi k, \tag{7.38}$$

$$\xi = \xi(z) = \frac{\pi k + z - \sin z \cos z}{\sin^3 z}, \tag{7.39}$$

$$\eta = \eta(z) = \sqrt{2}\rho - 2\cos z + \sqrt{2}\rho \cos^2 z \tag{7.40}$$

Using Table 4.1 (see Chapter 4) transform Eqs. (7.21, 7.25, 7.33) for the elliptic transfer orbit as follows:

$$u = \sqrt{1 - \sqrt{2}\rho \cos z}, \tag{7.41}$$

$$F(x,\rho) = \Phi(z,k,\rho) = \left(\frac{\xi u^2}{\sqrt{2}} + \rho\right) u = \sigma, \tag{7.42}$$

$$4\sqrt{x} F_x(x,\rho) = \Phi_z(z,k,\rho) = \frac{u^2\left(4u^2 + 3\xi\eta\right) + 2\rho^2 \sin^2 z}{2\sqrt{2} u \sin z} \tag{7.43}$$

Number of the solutions

Fig. 7.3 illustrates solution to (7.25):

> If $x<(2\pi)^2$ then there is the only solution (y_0 in Fig. 7.3) with $\varphi < 2\pi$

Consider $x>(2\pi)^2$; if

$$(2\pi k)^2 < x < \left[2\pi(k+1)\right]^2 \quad (k=1,2,\ldots) \tag{7.44}$$

then the transfer orbit has k complete orbits (orbital revolutions), i.e.

$$2\pi k < \varphi < 2\pi(k+1)$$

Let $F_k = F(x_k,\rho) = \min\limits_x F(x,\rho)$ for x satisfying (7.44) (see Fig. 7.3),

$$x_k = \arg\min_x F(x,\rho)$$

> $\sigma > F_k$: two solutions $x_k^{(1)}, x_k^{(2)}$ (y_1 and y_2 for k = 1 in Fig. 7.3)
> $\sigma = F_k$: one solution x_k (y_3 for k = 2 in Fig. 7.3)
> $\sigma < F_k$: no solutions (k = 3 in Fig. 7.3)

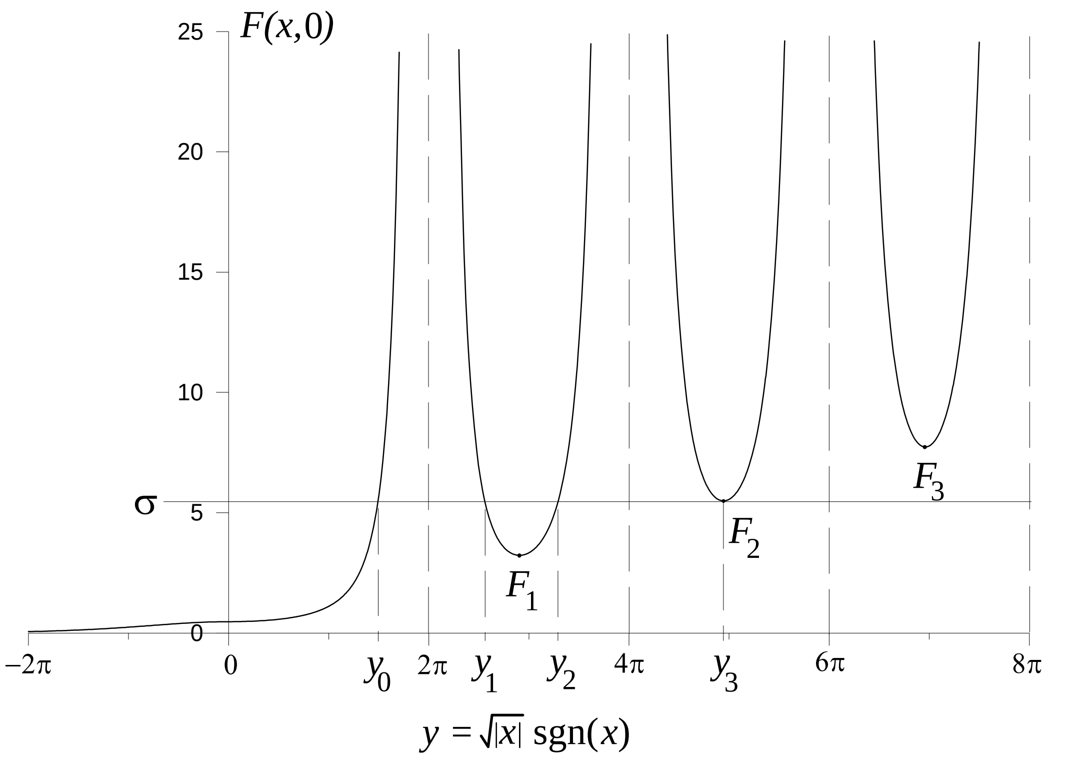


Fig. 7.3

Consider two solutions $x_k^{(1)}, x_k^{(2)}$ for $k \geq 1$:

$$(2\pi k)^2 < x_k^{(1)} < x_k < x_k^{(2)} < \left[2\pi(k+1)\right]^2 \quad (k = 1, 2, ...) \tag{7.45}$$

where x_k is a solution to the equation

$$F_x(x, \rho) = 0 \tag{7.46}$$

Using (7.38 – 7.40, 7.43) reduce equation (7.46) to the form:

$$u^2\left(4u^2 + 3\xi\eta\right) + 2\rho^2 \sin^2 z = 0 \tag{7.47}$$

After z_k is found from (7.47) x_k is given by

$$x_k = 4\left(z_k + \pi k\right)^2 \tag{7.48}$$

7.5. Solution of equation (7.25)

To solve Eq. (7.25) use the Newton–Raphson method:

$$x_n = x_{n-1} - \frac{F(x_{n-1}, \rho) - \sigma}{F_x(x_{n-1}, \rho)} \quad (n = 0, 1, ...) \tag{7.49}$$

where $F_x(x, \rho)$ is given by (7.33).

<u>First guess for the procedure (7.49)</u>
First guess x_0 for which

$$F(x_0, \rho) > \sigma$$

provides convergence of the procedure (7.49) (see Fig. 7.4). Therefore the first guess can be taken as follows:

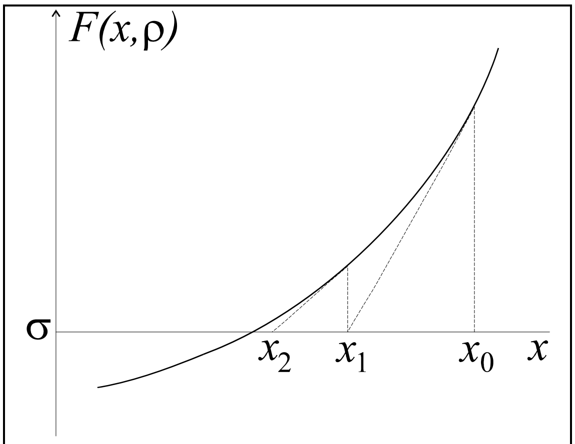


Fig. 7.4

1. <u>Hyperbolic and parabolic transfer orbits</u>

   It can be taken

   $x_0 = 0$

   <u>Note.</u> If $\rho > 0$ then the condition (7.29) must be checked in the procedure (7.49).

2. <u>Elliptic transfer orbits</u>

   Consider (7.41 – 7.43) and seek the first guess for (7.38), i.e.

$$z_0 = \frac{\sqrt{x_0}}{2} - \pi k \quad (0 < z_0 < \pi) \tag{7.50}$$

   Eq. (7.41) gives

$$u > u_m \equiv \sqrt{1 - \sqrt{2}\,|\rho|} > 0 \tag{7.51}$$

   a) Zero complete revolutions ($k = 0$).

   Let us seek the first guess in the form

$$z_0 = \pi - \varepsilon_0 \tag{7.52}$$

From (7.39, 7.52) for $k = 0$ obtain:

$$\xi\left(z_0\right) = \frac{\pi - \varepsilon_0 + \frac{1}{2}\sin 2\varepsilon_0}{\sin^3 \varepsilon_0} > \frac{\pi - \varepsilon_0 + \varepsilon_0 - \frac{8}{12}\varepsilon_0^3}{\varepsilon_0^3} = \frac{\pi - \frac{2}{3}\varepsilon_0^3}{\varepsilon_0^3} \quad (7.53)$$

$\Rightarrow$ Eqs. (7.42, 7.51, 7.53) give

$$\Phi\left(z_0, 0, \rho\right) > \left(\frac{\pi - \frac{2}{3}\varepsilon_0^3}{\varepsilon_0^3} u_m^2 + \rho\right) u_m = \sigma$$

$$\Rightarrow \boxed{\varepsilon_0 = \left(\frac{\pi}{\frac{2}{3}u_m^3 + \sigma - \rho u_m}\right)^{\frac{1}{3}} u_m} \quad (7.54)$$

b) k complete revolutions ($k > 0$).
Let us seek the first guess in the forms

$$z_0^{(1)} = \varepsilon_1, \quad z_0^{(2)} = \pi - \varepsilon_2$$

for the solutions $z^{(1)}, z^{(2)}$ respectively. Then from (7.39) obtain:

$$\xi\left(z_0^{(1)}\right) = \frac{\pi k + \varepsilon_1 - \frac{1}{2}\sin 2\varepsilon_1}{\sin^3 \varepsilon_1} > \frac{\pi k}{\varepsilon_1^3}$$

$$\Rightarrow \Phi\left(z_0^{(1)}, k, \rho\right) > \left(\frac{\pi k}{\sqrt{2}\varepsilon_1^3} u_m^2 + \rho\right) u_m = \sigma$$

$$\Rightarrow \boxed{\varepsilon_1 = \left[\frac{\pi k}{\sqrt{2}\left(\sigma - \rho u_m\right)}\right]^{\frac{1}{3}} u_m} \quad (7.55)$$

Similarly to (7.54) obtain:

$$\boxed{\varepsilon_2 = \left[\frac{\pi\left(k+1\right)}{\frac{2}{3}u_m^3 + \sigma - \rho u_m}\right]^{\frac{1}{3}} u_m} \quad (7.56)$$

Conclusion. The first guess can be taken as follows:

$\sigma \le \sigma_{par}$: $x_0 = 0$ (if $\rho > 0$ then (7.29) must be checked);

$\sigma > \sigma_{par}$, $k = 0$: $x_0 = 4(\pi - \varepsilon_0)^2$;

$\sigma > \sigma_{par}$, $k > 0$: $\begin{cases} x_0 = 4(\pi k + \varepsilon_1)^2 & \text{(1st solution);} \\ x_0 = 4[\pi(k+1) - \varepsilon_2]^2 & \text{(2nd solution)} \end{cases}$

(it must be $\sigma \ge F_k$ in the $k > 0$ case);

here $\sigma_{par}, \varepsilon_0, \varepsilon_1, \varepsilon_2$ are given by (7.35, 7.54 – 7.56) using (7.51).

7.6. Case of collinear vectors $\vec{r}_0, \vec{r}_1$

If $\vec{r}_1 \uparrow\downarrow \vec{r}_2$ the transfer orbit plane is uncertain, however radial v_r and transversal v_n components of $\vec{v}_0$ can be found.

From (7.12) taking into account equations

$$v_r = \frac{\vec{r}_0 \cdot \vec{v}_0}{r_0}, \quad g = 0$$

obtain:

$$\boxed{v_r = -\frac{r_0 c_1}{s c_2}} \tag{7.57}$$

Eqs. (7.57) and

$$v_0^2 = v_r^2 + v_n^2, \quad h = v_0^2 - \frac{2\mu}{r_0}$$

give

$$\boxed{v_n = \sqrt{h - v_r^2 + \frac{2\mu}{r_0}}} \tag{7.58}$$

where using (7.2, 7.22) h can be found as follows:

$$h = -\frac{\mu c_2 x}{(r_0 + r_1) u^2},$$

x is a solution to (7.25).

8. Interplanetary Flights

8.1. Method of patched conics

Divide the spacecraft interplanetary trajectory into the parts (see Fig. 8.1):

I = flight in the sphere of influence of the departure planet;

II = heliocentric transfer;

III = flight in the sphere of influence of the destination planet.

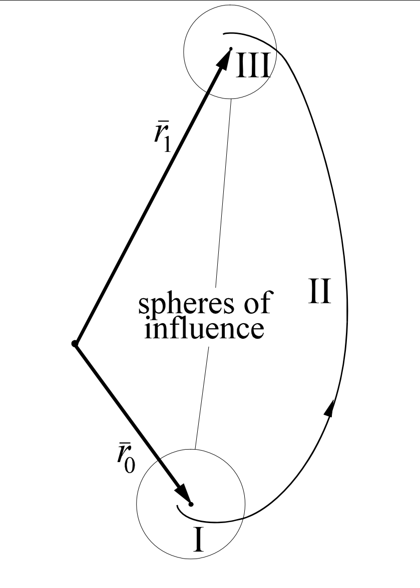


Fig. 8.1

<u>Flight in the sphere of influence</u> (see Fig. 8.2)

Neglect all the perturbations within the sphere of influence

$\Rightarrow$ the trajectory is hyperbolic.

Designate (see Fig. 8.2):

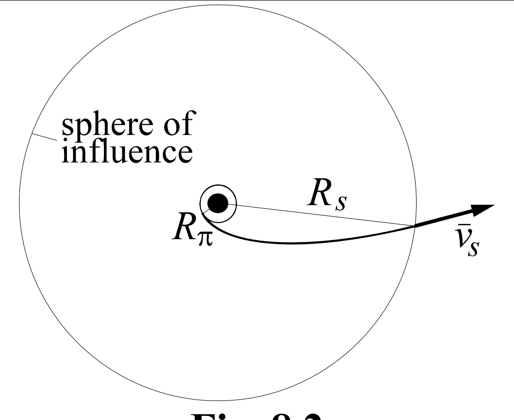


Fig. 8.2

R_π = periapsis radius (i.e. launch point at the departure planet or closest approach to the destination planet);

R_S = radius of sphere of influence;

$\vec{v}_S$ = spacecraft velocity at the sphere of influence border.

$$\boxed{R_\pi << R_S \quad \Rightarrow \quad \vec{v}_S \approx \vec{v}_\infty} \tag{8.1}$$

Heliocentric flight

Let r be the heliocentric distance of the planet.

$$\boxed{R_S << r \quad \Rightarrow \quad \text{assume } R_S = 0}$$

Designate:

τ_0, τ_1 = times of flight in spheres of influence of the departure and destination planets respectively,

τ = time of the heliocentric flight.

$$\boxed{\tau_0, \tau_1 << \tau}$$

$\Rightarrow$ considering the heliocentric part of the spacecraft orbit assume

$$\boxed{\tau_0 = \tau_1 = 0}$$

The considered way of the 'gluing' the orbit parts is the patched conic approach (method).

8.2. Launch Δv and flyby velocity for the interplanetary transfer

Assume that the spacecraft is launched from a low circular orbit around the departure planet (LDPO). Designate:

μ, μ_0, μ_1 = gravitational parameters of the Sun, departure planet, and destination planet respectively;

R_0, R_1 = radius of the launch LDPO and closest approach distance to the destination planet respectively;

t_0, t_1 = times of launch and arrival;

$\vec{r}_0, \vec{r}_1$ = heliocentric positions of the departure and destination planets at t_0 and t_1 respectively;

$\vec{v}_0, \vec{v}_1$ = spacecraft velocities in the heliocentric transfer orbit at t_0 and t_1 respectively;

$\vec{u}_0, \vec{u}_1$ = velocities of the departure and destination planets at t_0 and t_1 respectively.

$\vec{r}_0, \vec{r}_1, \tau = t_1 - t_0$ are given

$\Rightarrow$ $\vec{v}_0$ is the solution of the Lambert problem,

$\vec{v}_1$ is given by

$$\vec{v}_1 = \dot{f}\vec{r}_0 + \dot{g}\vec{v}_0$$

where

$$\dot{f} = -\frac{\mu s c_1}{r_0 r_1}, \quad \dot{g} = 1 - \frac{\mu s^2 c_2}{r_1}$$

(see (4.23, 4.26) in Chapter 4).

Define

$\boxed{\vec{v}_{\infty 0} = \vec{v}_0 - \vec{u}_0}$ = departure-planet-centric spacecraft velocity on the border of the planetary sphere of influence $\approx$ outgoing excess velocity;

$\boxed{\vec{v}_{\infty 1} = \vec{v}_1 - \vec{u}_1}$ = destination-planet-centric spacecraft velocity on the border of the planetary sphere of influence $\approx$ incoming excess velocity.

As was shown in Section 6.4

$$\boxed{\Delta v = \sqrt{\frac{2\mu_0}{R_0} + v_{\infty 0}^2} - \sqrt{\frac{\mu_0}{R_0}}} \tag{8.2}$$

= launch delta-V in LDPO;

$$\boxed{V_1 = \sqrt{\frac{2\mu_1}{R_1} + v_{\infty 1}^2}} \tag{8.3}$$

= the spacecraft flyby velocity at the closest approach to the destination planet.

$$\vec{r}_0 = \vec{r}_0\left(t_0\right), \quad \vec{r}_1 = \vec{r}_1\left(t_1\right)$$

$\Rightarrow$ the solution of the Lambert problem depends only on t_0, t_1

$$\Rightarrow \quad \boxed{\Delta v = \Delta v\left(t_0, t_1\right), \quad V_1 = V_1\left(t_0, t_1\right)} \tag{8.4}$$

<u>C_3 parameter</u>

$$\boxed{v_\infty^2 = C_3}$$

$v_{\infty 0}^2$ = launch C_3, $v_{\infty 1}^2$ = arrival C_3.

C_3 is a doubled orbital energy.

8.3. On optimal transfer

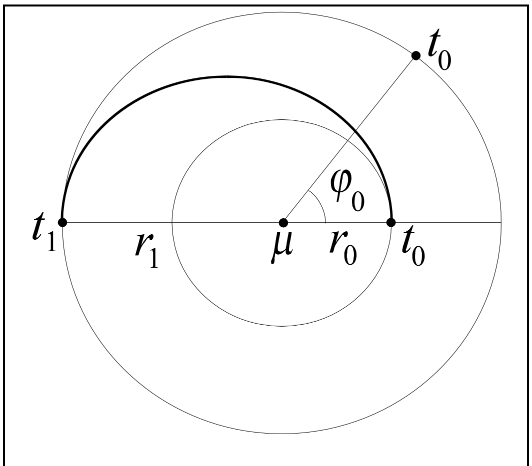


Fig. 8.3

Assume that the departure and destination planet orbits are circular and coplanar

$\Rightarrow$ the transfer angle value $\varphi = \pi$ is optimal (Hohmann transfer, see Chapter 6 and Fig. 8.3). The flight time is:

$$t_1 - t_0 = \pi \frac{a^{3/2}}{\sqrt{\mu}}$$

where

$$a = \frac{r_0 + r_1}{2}$$

is semimajor axis of the transfer orbit. Introduce:

$$\xi = \frac{r_0}{r_1};$$

$n_j = \dfrac{\sqrt{\mu}}{r_j^{3/2}}$ = mean motions of the departure (j = 0) and destination (j = 1) planets;

$$P_j = \frac{2\pi}{n_j} \tag{8.5}$$

= orbital periods of the departure (j = 0) and destination (j = 1) planets;

φ_0 = heliocentric angle between the planets at t_0 (see Fig. 8.3).

$$\boxed{\varphi_0 = \pi - n_1(t_1 - t_0) = \pi\left[1 - \left(\frac{1+\xi}{2}\right)^{3/2}\right]} \tag{8.6}$$

Eq. (8.6) defines initial geometry of the planets necessary for the optimal transfer.

Angular velocity of the destination planet relatively to the departure one is

$$n = |n_1 - n_0|$$

Time necessary to repeat the same mutual position of the planets is

$$P = \frac{2\pi}{|n_1 - n_0|} \tag{8.7}$$

⇒ Eqs. (8.5, 8.7) give

$$\boxed{P = \frac{P_0 P_1}{|P_1 - P_0|}} \tag{8.8}$$

Eq. (8.8) gives the time between two optimal launch dates for the considered case.

If the departure planet is Earth:

$$\boxed{P = \frac{P_1}{|P_1 - 1|}} \tag{8.9}$$

= launch date period given in years.

Table 8.1. Periods of the optimal launches from Earth to the planets

Destination planet	P, year
Mercury	0.32
Venus	1.60
Mars	2.14
Jupiter	1.09

Eqs. (8.8, 8.9) and Table 8.1 give an <u>average</u> periodicity of the optimal launch dates.

Real planetary orbits are elliptic and mutually inclined

$\Rightarrow$ the value $\varphi = \pi$ can lead to a very high Δv (see Fig. 8.4).

φ_1 is the optimal transfer angle for the 1st semi-orbit, $0 < \varphi_1 < \pi$;
φ_2 is the optimal transfer angle for the 2nd semi-orbit, $\pi < \varphi_2 < 2\pi$.

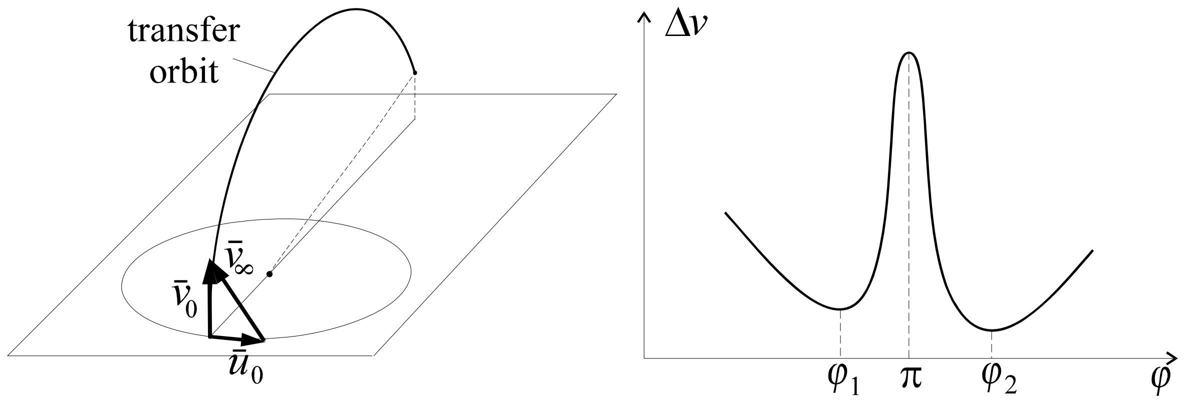


Fig. 8.4

For each launch date t_0 an optimal arrival date t_1 (i.e. the date minimizing launch Δv) can be found. Define

$$\Delta v_m = \Delta v_m(t_0) = \min_{t_1} \Delta v(t_0, t_1)$$

$$\Delta v_M = \min_{t_0, t_1} \Delta v(t_0, t_1) = \min_{t_0} \Delta v_m(t_0)$$

Fig. 8.5 illustrates functions $\Delta v(t_0, t_1)$ (level curves, Fig. 8.5a) and $\Delta v_m(t_0)$ (Fig. 8.5b).
The level curves on the left plot satisfy the inequalities

$$\Delta v_1 < \Delta v_2 < \Delta v_3 < \Delta v_4$$

If a maximum acceptable value of the launch delta-V is given it defines the launch window (see Fig. 8.5b).

If the launch window is given it defines the launch delta-V value (see Fig. 8.5b).

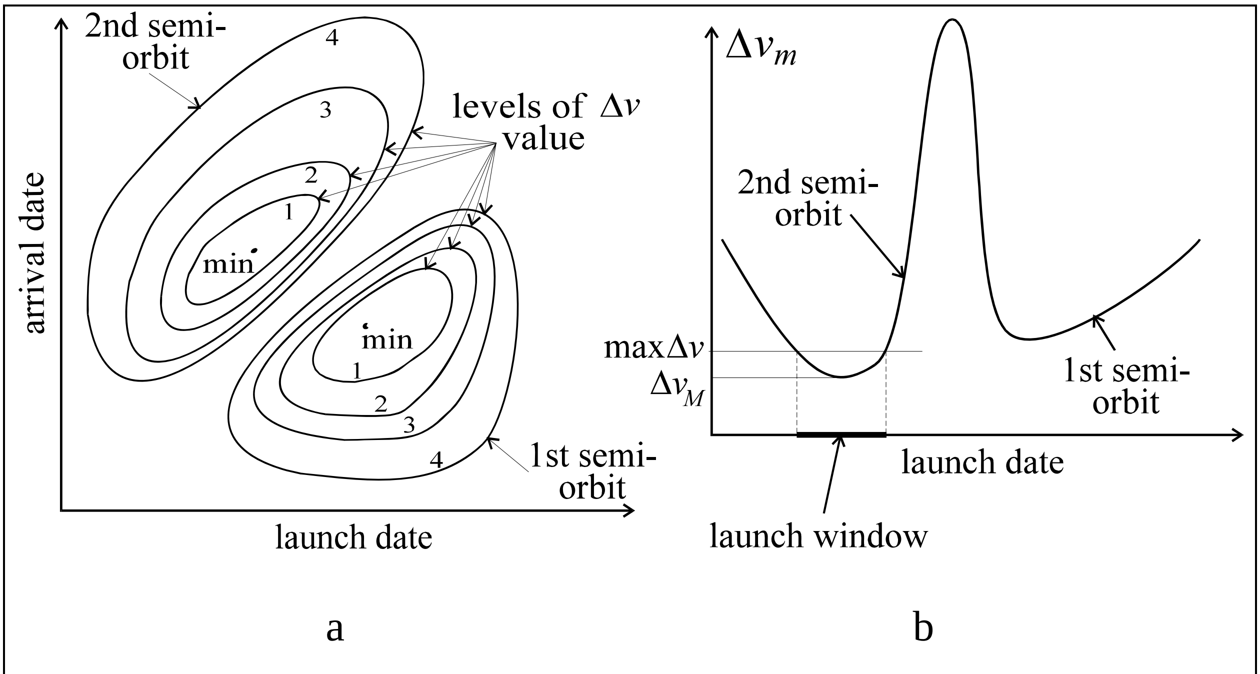


Fig. 8.5

8.4. Gravity assist maneuvers

Consider a spacecraft flight through the sphere of influence of a planet. Designate (see Fig. 8.6):

$\vec{v}_\infty$, $\vec{v}'_\infty$ = incoming and outgoing excess velocities of the spacecraft;

$$\vec{v} = \vec{u} + \vec{v}_\infty, \;\; \vec{v}' = \vec{u} + \vec{v}'_\infty \qquad (8.10)$$

= the spacecraft heliocentric velocities before entry into and after exit from the sphere of influence respectively

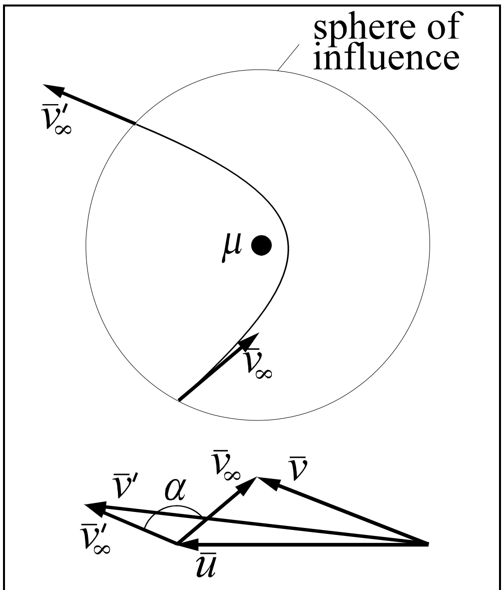


Fig. 8.6

(the flight time in the sphere of influence is assumed to be small relatively to the planet orbital period, therefore in (8.10) the planet velocity $\vec{u}$ is the same for the entry and exit).

$\vec{v}_1 \neq \vec{v}'_1 \;\Rightarrow$ the planet flyby changes the spacecraft velocity.

Due to (8.10)

$$\boxed{\Delta\vec{v} = \vec{v}' - \vec{v} = \vec{v}'_\infty - \vec{v}_\infty}$$

= an ‘impulse’ produced by the gravity assist maneuver.

Gravity assist = swingby

8.5. Changing the orbital energy by means of swingby

Consider a <u>planar free swingby</u>, i.e. $|\vec{v}_\infty| = |\vec{v}'_\infty| = v_\infty$.

Designate (see Fig. 8.7):

- μ = gravitational parameter of the planet;
- $\vec{r}, \vec{u}$ = heliocentric position and velocity of the planet during the swingby;
- $\vec{R}_\pi, \vec{v}_\pi$ = periapsis position and velocity of the planet flyby;
- α = turn angle;
- δ = angle between $\vec{r}$ and $\vec{R}_\pi$

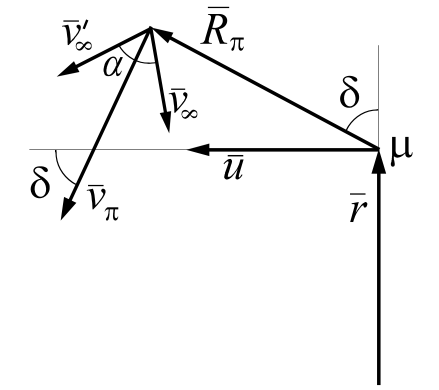


Fig. 8.7

Assuming the planet orbit circular, δ is also the angle between $\vec{u}$ and $\vec{v}_\pi$ (see Fig. 8.7). The spacecraft heliocentric velocities before and after the swingby can be found from (8.10) as follows:

$$\left.\begin{aligned} v^2 &= u^2 + v_\infty^2 + 2uv_\infty \cos\left(\delta + \frac{\alpha}{2}\right) \\ v'^2 &= u^2 + v'^2_\infty + 2uv'_\infty \cos\left(\delta - \frac{\alpha}{2}\right) \end{aligned}\right\} \tag{8.11}$$

The orbital energy is (see (2.6))

$$E = \frac{h}{2} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\Rightarrow \quad \Delta E = \frac{v'^2}{2} - \frac{v^2}{2} = uv_\infty\left[\cos\left(\delta - \frac{\alpha}{2}\right) - \cos\left(\delta + \frac{\alpha}{2}\right)\right] = 2uv_\infty \sin\delta \sin\frac{\alpha}{2} \tag{8.12}$$

If $\vec{v}'_\infty, \vec{v}_\infty$ in Fig. 8.7 are the incoming and outgoing excess velocities respectively then v', v in (8.12) are the spacecraft heliocentric velocities before and after the swingby respectively and

$$\Delta E = -2uv_\infty \sin\delta \sin\frac{\alpha}{2} \tag{8.13}$$

$\Rightarrow$ due to (8.12, 8.13, 3.17) the orbital energy change is

$$\boxed{\Delta E = \pm 2uv_\infty \sin\delta \sin\frac{\alpha}{2} = \pm\frac{2uv_\infty \sin\delta}{1 + \frac{R_\pi v_\infty^2}{\mu}}} \tag{8.14}$$

8.6. Flight in sphere of influence of a planet

Consider a flight between three planets with the gravitational parameters μ_0, μ_1, μ_2 respectively (see Fig. 8.8): from the departure planet ($\vec{r}_0$ at t_0) via the gravity assist planet ($\vec{r}_1$ at t_1) to the destination planet ($\vec{r}_2$ at t_2).

Two parts of the spacecraft trajectory ($\vec{r}_0$ to $\vec{r}_1$ in time $t_1 - t_0$ and $\vec{r}_1$ to $\vec{r}_2$ in time $t_2 - t_1$) can be found from two solutions of the Lambert problem. These solutions give $\vec{v}_\infty$ and $\vec{v}'_\infty$.

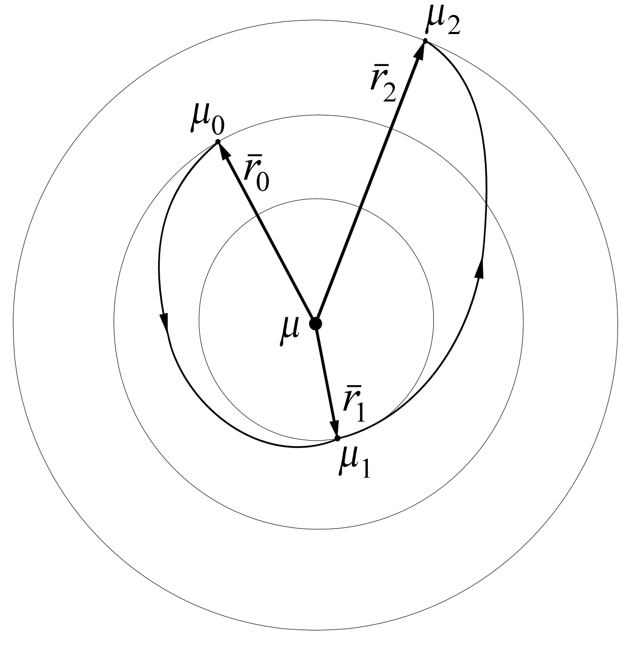


Fig. 8.8

The problem is to find the spacecraft trajectory in the sphere of influence of the gravity assist planet joining the two parts of the heliocentric orbit (i.e. to find the trajectory having the incoming and outgoing excess velocities $\vec{v}_\infty$ and $\vec{v}'_\infty$ respectively).

Thus, $\vec{v}_\infty$ and $\vec{v}'_\infty$ are given; designate $v_\infty = |\vec{v}_\infty|, \quad v'_\infty = |\vec{v}'_\infty|$;

the angle α between $\vec{v}_\infty$ and $\vec{v}'_\infty$ is the turn angle and

$$\cos\alpha = \frac{\vec{v}_\infty \cdot \vec{v}'_\infty}{v_\infty v'_\infty} \tag{8.15}$$

Since the radius of sphere of influence is much smaller than the heliocentric distances assume that the spacecraft can entry and exit the sphere of influence in any points of its surface without changing the heliocentric trajectory, i.e. both excess velocity vectors can be moved along the surface in parallel to themselves (see Fig. 8.9).

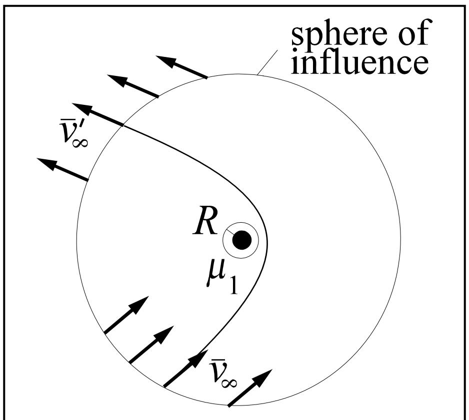


Fig. 8.9

Free planet swingby

Let $v_\infty = v'_\infty$; in this case the spacecraft trajectory is hyperbolic in the plane containing $\vec{v}_\infty$ and $\vec{v}'_\infty$ and the planet mass center.

The turn angle is (see Eq. (3.17)):

$$\sin\frac{\alpha}{2} = \frac{1}{1+\dfrac{R_\pi v_\infty^2}{\mu_1}} \tag{8.16}$$

$$\Rightarrow \quad \boxed{R_\pi = \frac{\mu_1}{v_\infty^2}\left(\frac{1}{\sin\dfrac{\alpha}{2}} - 1\right)} \tag{8.17}$$

= periapsis distance of the trajectory.

Let R be a minimum distance the spacecraft can approach the planet to (R = planet radius + atmosphere altitude + safety distance), i.e.

$$R_\pi \geq R$$

Free swingby is only possible when $v_\infty = v'_\infty$ and $\sin\dfrac{\alpha}{2} \leq \dfrac{1}{1+\dfrac{R v_\infty^2}{\mu_1}}$

Powered swingby

Let $v_\infty \neq v'_\infty$; assume that $v_\infty < v'_\infty$. In this case an active maneuver in the sphere of influence is necessary. In a simplest case optimal is a one-impulse maneuver.

Define angles:

$$\left.\begin{aligned} \beta &= \frac{\pi - \alpha}{4} \\ \lambda &= \arctan\left(\frac{v'_\infty - v_\infty}{v'_\infty + v_\infty}\tan\beta\right) \\ \psi &= \beta - \lambda \end{aligned}\right\} \tag{8.18}$$

Optimal distance R_i of the impulse application and the impulse value Δv are:

$$\boxed{\begin{aligned} R_i &= \frac{2\mu\sin^2\psi}{v_\infty^2\cos^2\lambda\left(2\cos^2\beta - \cos^2\lambda\right)}, \\ \Delta v &= \left(v_\infty + v'_\infty\right)\sin\lambda \end{aligned}} \tag{8.19}$$

(see Fig. 8.10; here $\vec{v}_-$ and $\vec{v}_+$ are the spacecraft velocities just before and after the impulse).

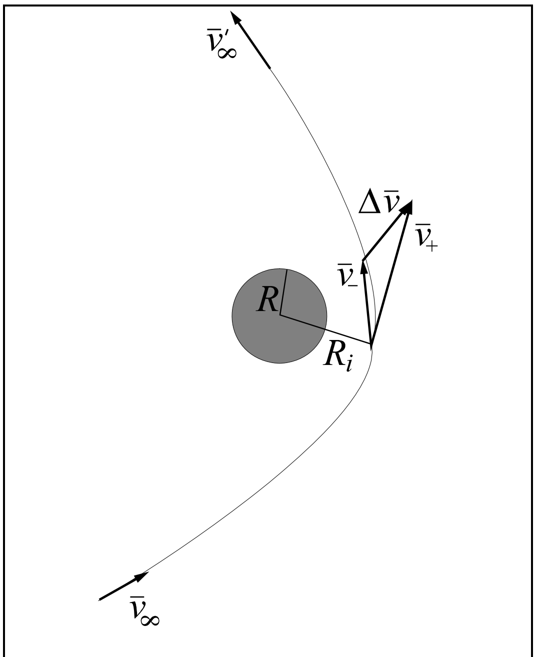


Fig. 8.10

Delta-V as a function of times

Due to (8.15, 8.18, 8.19) delta-V can be represented as

$$\Delta v = f\left(\vec{v}_\infty, \vec{v}'_\infty\right) \tag{8.20}$$

Assume that the times t_0, t_1, t_2 of the departure, swingby, and arrival are given and

$$\vec{r}_0 = \vec{r}_0\left(t_0\right),\ \vec{r}_1 = \vec{r}_1\left(t_1\right),\ \vec{r}_2 = \vec{r}_2\left(t_2\right)$$

(see Fig. 8.8). Vectors $\vec{v}_\infty$ and $\vec{v}'_\infty$ are given by the solutions of the Lambert problem for the transfers between $\vec{r}_0, \vec{r}_1$ and $\vec{r}_1, \vec{r}_2$ respectively

$$\Rightarrow \quad \vec{v}_\infty = \vec{v}_\infty\left(t_0, t_1\right), \quad \vec{v}'_\infty = \vec{v}'_\infty\left(t_1, t_2\right)$$

$\Rightarrow$ due to (8.20)

$$\boxed{\Delta v = \Delta v\left(t_0, t_1, t_2\right)} \tag{8.21}$$

8.7. Formulation of the optimization problem for multi-planet transfer

Consider a transfer to n planets; designate:

$t_0, \vec{r}_0$ = launch time and the departure planet position at t_0;

$t_j, \vec{r}_j$ = jth swingby time and the jth planet position at t_j ($j = 1, \ldots, n-1$);

$t_n, \vec{r}_n$ = time of arrival to the final planet and the planet position at t_n

After the spacecraft trajectory is found by means of solving the Lambert problem n times, the total delta-V can be found as follows:

$$\Delta v = \sum_{j=0}^{n} \Delta v_j \tag{8.22}$$

where

Δv_0 = launch delta-V;

Δv_j = delta-V in the sphere of influence of the jth planet, $j = 1, \ldots, n-1$ ($\Delta v_j = 0$ if the swingby is free);

Δv_n = braking delta-V near the final planet ($\Delta v_n = 0$ if no braking is necessary).

Due to (8.4, 8.21)

$$\Delta v_0 = \Delta v_0(t_0, t_1), \quad \Delta v_n = \Delta v_n(t_{n-1}, t_n),$$
$$\Delta v_j = \Delta v_j(t_{j-1}, t_j, t_{j+1}), \quad j = 1, \ldots, n-1$$

$\Rightarrow$ in (8.22)

$$\Delta v = \Delta v(\vec{t}) \tag{8.23}$$

where

$$\vec{t} = \{t_0, t_1, \ldots, t_n\}. \tag{8.24}$$

The optimization problem is to find

$$\Delta v_m = \min_{\vec{t}} \Delta v \tag{8.25}$$

<u>Note</u>. In practice deep space active maneuvers also can be used (not considered here).

8.8. VEGA and ΔVEGA maneuvers

Venus and Earth Gravity Assists (VEGA) maneuver

Consider an Earth–Venus–Earth–further transfer (see Fig. 8.11).

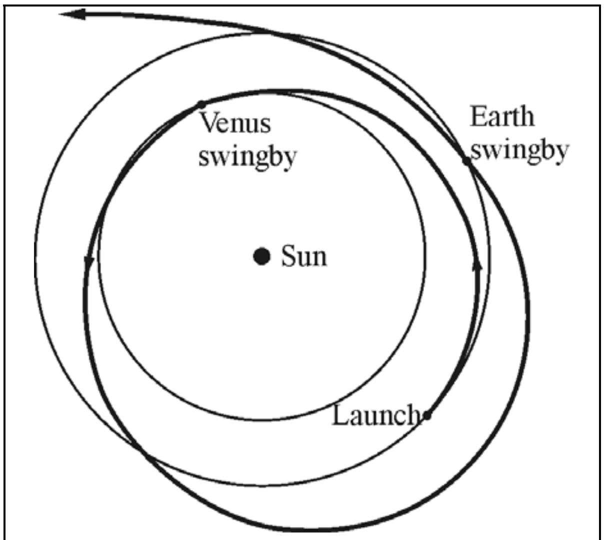


Fig. 8.11

Designate:

t_0, t_1, t_2 = times of launch, Venus and Earth swingbys respectively;

$\vec{u}_0, \vec{u}_1, \vec{u}_2$ = velocities of Earth, Venus, and Earth at t_0, t_1, t_2 respectively;

$\vec{v}_0$ = spacecraft heliocentric velocity after launch;

$\vec{v}_j, \vec{v}'_j$ = spacecraft heliocentric velocities respectively before and after the Venus (j = 1) and Earth (j =2) swingbys;

$\vec{v}_{\infty 0}$ = launch excess velocity;

$\vec{v}_{\infty j}, \vec{v}'_{\infty j}$ = incoming and outgoing excess velocities during the Venus (j = 1) and Earth (j =2) swingbys.

(a) Launch:

$\vec{v}_{\infty 0} \uparrow\downarrow \vec{u}_0 \quad \Rightarrow \quad v_0 = u_0 - v_{\infty 0}$

(b) Venus swingby (free $\Rightarrow v'_{\infty 1} = v_{\infty 1}$):

$\vec{v}_{\infty 1}$ is inclined to $\vec{u}_1$,

$\vec{v}'_{\infty 1} \uparrow\uparrow \vec{u}_1 \quad \Rightarrow \quad v'_1 = u_1 + v_{\infty 1} > v_1$

(c) Earth swingby (free $\Rightarrow v'_{\infty 2} = v_{\infty 2}$):

$\vec{v}_{\infty 2}$ is inclined to $\vec{u}_2$,

$\vec{v}'_{\infty 2} \uparrow\uparrow \vec{u}_2 \quad \Rightarrow \quad v'_2 = u_2 + v_{\infty 2} > v_2$

(see Fig. 8.12).

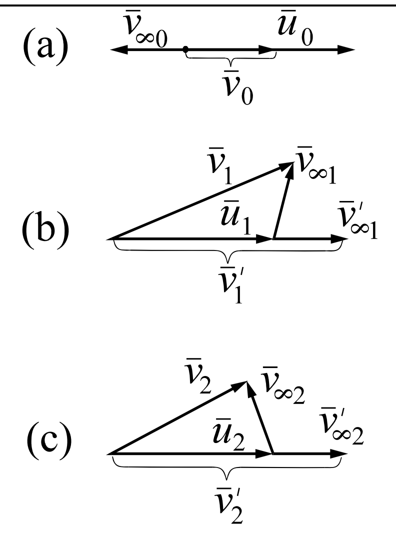


Fig. 8.12

> Duration of the VEGA maneuver is 13–17 months
>
> Gain in delta-V is up to 2.5 km/s

Delta-V and Earth Gravity Assist (ΔVEGA) maneuver

The idea is same as for VEGA: return to Earth with a higher excess velocity (see Fig. 8.13).

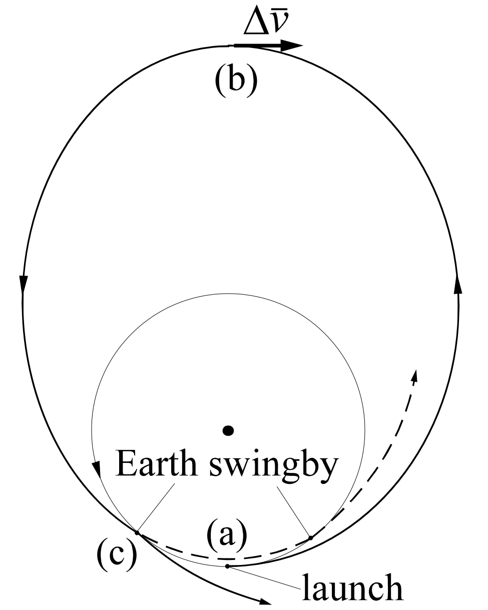


Fig. 8.13

(a) Launch to a heliocentric orbit with perihelion at Earth orbit and *n*-year period (n = 2, 3, ...):

$$\vec{v}_{\infty 0} \uparrow\uparrow \vec{u}_0$$

(b) Braking impulse near the orbit aphelion:

$$\Delta\vec{v} \uparrow\downarrow \vec{v}_\alpha$$

(c) Earth swingby:

$\vec{v}_{\infty 2}$ is inclined to $\vec{u}_2$ (see Fig. 8.12c),

$$\vec{v}'_{\infty 2} \uparrow\uparrow \vec{u}_2 \;\Rightarrow\; v'_2 = u_2 + v_{\infty 2} > v_2$$

The maneuver of duration of 2, 3, ... years is called:

2^-, 3^-, ... ΔVEGA maneuver if the Earth swingby occurs before perihelion (solid trajectory in Fig. 8.13);

2^+, 3^+, ... ΔVEGA maneuver if the Earth swingby occurs after perihelion (dashed trajectory in Fig. 8.13).

Gain in delta-V is up to 1.1 km/s

Disadvantages of the VEGA and ΔVEGA maneuvers

- VEGA maneuver requires a special mutual position of Earth and Venus and is possible every 1.6 years; the maneuver duration is more than one year.
- ΔVEGA maneuver has longer duration and smaller gain than VEGA.

9. Maneuvering in the Sphere of Influence of a Planet

9.1. Introduction and notation

The spacecraft entry into the sphere of influence of the destination planet and maneuvering near the planet in order to fulfill the mission goal are considered in this chapter. The mission goal may be the planet and its moon(s) exploration.

Designate:

μ = gravity constant of the planet;
R = minimum possible distance of the planet approach;
a = semimajor axis;
e = eccentricity;
r_π, r_α = periapsis and apoapsis radii;
i = orbit inclination, $0 \le i \le \pi$;
α = turn angle of the spacecraft excess velocity, $0 < \alpha < \pi$

9.2. Necessary formulas

$$a = \frac{r_\pi + r_\alpha}{2}$$ = semimajor axis

$$r_\pi = \frac{p}{1+e}, \quad r_\alpha = \frac{p}{1-e}$$ = periapsis and apoapsis radii

$$\Rightarrow \quad p = \frac{2r_\pi r_\alpha}{r_\pi + r_\alpha} \tag{9.1}$$

= semilatus rectum

$$c = rv_n = \sqrt{\mu p} \tag{9.2}$$

= magnitude of the angular momentum

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \frac{2\mu}{r}\left(1 - \frac{r}{r_\pi + r_\alpha}\right) \tag{9.3}$$

= integral of energy

$$v_\alpha = \sqrt{\frac{2\mu}{r_\pi + r_\alpha}\frac{r_\pi}{r_\alpha}}, \quad v_\pi = \frac{r_\alpha}{r_\pi} v_\alpha \tag{9.4}$$

= apoapsis and periapsis velocities

$$\sin\frac{\alpha}{2} = \frac{1}{1 + \dfrac{r_\pi v_\infty^2}{\mu}} \tag{9.5}$$

= turn angle

9.3. Spacecraft incoming orbit

Assume the periapsis radius r_π of the spacecraft incoming hyperbolic orbit to be given (any value $r_\pi \geq R$ may be selected, see Section 8.6). Define the orbit plane by unit basis vectors

$$\vec{e}_1 = \frac{\vec{v}_\infty}{v_\infty}, \quad \vec{e}_2 \perp \vec{e}_1 \tag{9.6}$$

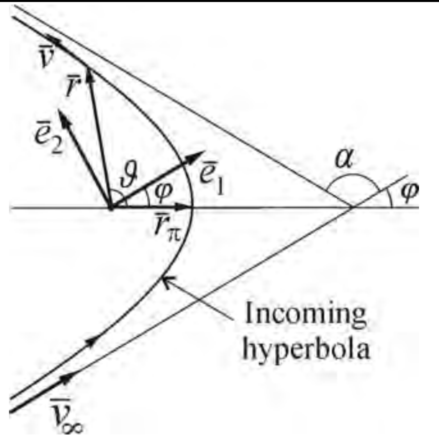


Fig. 9.1

(see Fig. 9.1). The planet-centric orbit plane may arbitrarily rotate around vector $\vec{v}_\infty$ (see Section 8.6) $\Rightarrow$ $\vec{e}_2$ may be selected arbitrarily. Consider a fixed basis vector $\vec{e}_{20} \perp \vec{e}_1$ lying in the frame basal plane
$\Rightarrow$ $\vec{e}_{20} = \{e_{20x}, e_{20y}, 0\}$, and let

$$\vec{e}_{30} = \vec{e}_1 \times \vec{e}_{20} = \{e_{30x}, e_{30y}, e_{30z}\} \tag{9.7}$$

$$\Rightarrow \quad \vec{e}_2 = \vec{e}_{20} \cos\psi + \vec{e}_{30} \sin\psi \tag{9.8}$$

where angle ψ may be selected arbitrarily (see Fig. 9.2). Vector $\vec{e}_{20}$ may be found from equations

$$e_{20x}^2 + e_{20y}^2 = 1$$

$$e_{1x}e_{20x} + e_{1y}e_{20y} = 0$$

To resolve the ambiguity assume $e_{30z} > 0$ in (9.7).

Define angle $\varphi = \dfrac{\pi - \alpha}{2}$

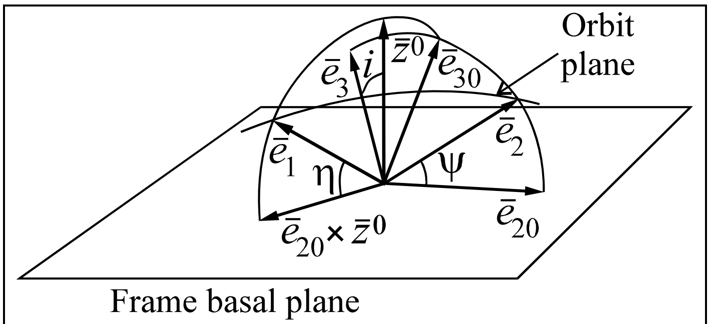


Fig. 9.2

$\left(0 < \varphi < \dfrac{\pi}{2}\right.$, see Fig. 9.1) $\Rightarrow$ due to (9.5)

$$\cos\varphi = \frac{1}{1 + \dfrac{r_\pi v_\infty^2}{\mu}},$$

$$\vec{r}_\pi = r_\pi(\vec{e}_1 \cos\varphi - \vec{e}_2 \sin\varphi) \tag{9.9}$$

$$\vec{r} = r(\vec{e}_1 \cos(\vartheta - \varphi) + \vec{e}_2 \sin(\vartheta - \varphi)) \tag{9.10}$$

<u>Orbital elements</u>

$$\boxed{\begin{aligned} a &= \frac{\mu}{v_\infty^2} \\ e &= 1 + \frac{r_\pi v_\infty^2}{\mu} \end{aligned}}$$

(see Section 3.4). Consider unit basis vector

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_2 = \{e_{3x}, e_{3x}, e_{3x}\} \tag{9.11}$$

Eqs. (9.7, 9.8, 9.11) give

$$\vec{e}_3 = \vec{e}_1 \times \vec{e}_{20} \cos\psi + \vec{e}_1 \times \vec{e}_{30} \sin\psi = \vec{e}_1 \times \vec{e}_{20} \cos\psi - \vec{e}_{20} \sin\psi \tag{9.12}$$

$\vec{e}_3$ is orthogonal to the orbit plane

$$\Rightarrow \quad \boxed{\begin{aligned} &\cos i = e_{3z} \\ &\tan\Omega = -\frac{e_{3x}}{e_{3y}} \\ &\cos\omega = \frac{\vec{r}_\pi \cdot \vec{k}}{r_\pi}, \ \vec{k} = \begin{bmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{bmatrix} \end{aligned}} \tag{9.13}$$

(see Section 3.6). Let $\vec{z}^{\,0} = \{0,0,1\}$ be a normal to the frame basal plane. Then

$$\cos i = \vec{z}^{\,0} \cdot \vec{e}_3 = \vec{z}^{\,0} \cdot \vec{e}_1 \times \vec{e}_{20} \cos\psi = \vec{e}_1 \cdot \vec{e}_{20} \times \vec{z}^{\,0} \cos\psi$$

Let η be angle between $\vec{v}_\infty$ and the frame basal plane. Unit vector $\vec{e}_{20} \times \vec{z}^{\,0}$ is directed along projection of $\vec{e}_1$ onto the basal plane

$\Rightarrow$ due to (9.6) $\vec{e}_1 \cdot \vec{e}_{20} \times \vec{z}^{\,0} = \cos\eta$

$$\Rightarrow \quad \boxed{\cos i = \cos\eta \cos\psi} \tag{9.14}$$

$$\Rightarrow \quad \boxed{\eta \le i \le \pi - \eta} \tag{9.15}$$

9.4. One-impulse capturing maneuver

The spacecraft capturing maneuver at the periapsis of the incoming hyperbolic orbit is considered. Minimum value of the braking Δv_0 is reached at the minimum distance R.

If r_α of the planet orbit is given then

$$\boxed{\Delta v_0 = \sqrt{\frac{2\mu}{R} + v_\infty^2} - \sqrt{\frac{2\mu}{R} - \frac{\mu}{a}}} \tag{9.16}$$

where $a = \dfrac{R + r_\alpha}{2}$.

If the orbital period P is given then in (9.16)

$$a = \sqrt[3]{\frac{P^2 \mu}{4\pi^2}}$$

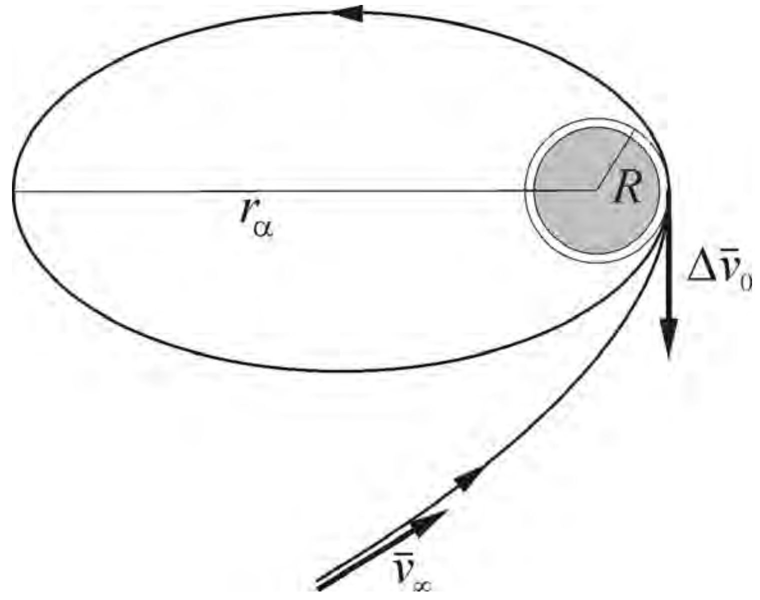


Fig. 9.3

9.5. Dropping a probe on the planet

Assume that a probe is to be detached from the spacecraft to enter the planet atmosphere. For this the probe velocity should be changed in a value Δv_p in order to lower periapsis radius of its orbit
$\Rightarrow$ optimal (in terms of $\min \Delta v_p$) is the probe separation maneuver at apoapsis of the spacecraft orbit (see Section 6.4).

Let R_a be radius of the planet atmosphere $(R_a \le R)$ and θ be angle of the probe entry into the atmosphere (see Fig. 9.4),

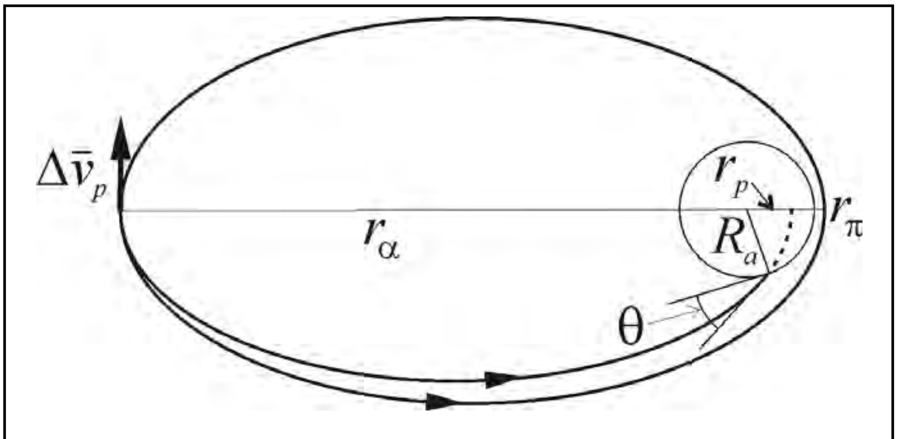


Fig. 9.4

$$\cos\theta = \frac{v_n}{v} \tag{9.17}$$

where velocities v, v_n are calculated at R_a (see Fig. 9.5).
Eqs. (9.1 – 9.3) give

$$v^2 = \frac{2\mu}{R_a}\left(1 - \frac{R_a}{r_\pi + r_\alpha}\right) \tag{9.18}$$

$$v_n^2 = \frac{\mu p}{R_a^2} = \frac{2\mu}{R_a}\frac{r_\pi r_\alpha}{r_\pi + r_\alpha} \tag{9.19}$$

$\Rightarrow$ from Eqs. (9.17 – 9.19) obtain:

$$\boxed{r_p = \frac{(r_\alpha - R_a)R_a\cos^2\theta}{r_\alpha - R_a\cos^2\theta}} \tag{9.20}$$

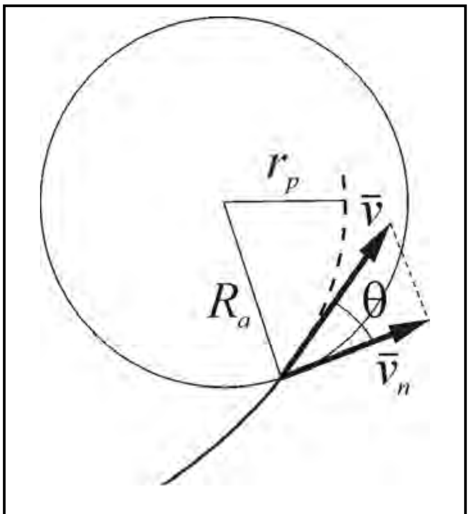


Fig 9.5

Let v_α, $v_{\alpha p}$ be apoapsis velocities of the spacecraft and probe after the separation $\Rightarrow$ $\Delta v_p = v_\alpha - v_{\alpha p}$

$$\Rightarrow \quad \boxed{\Delta v_p = \sqrt{\frac{2\mu}{r_\pi + r_\alpha}\frac{r_\pi}{r_\alpha}} - \sqrt{\frac{2\mu}{r_p + r_\alpha}\frac{r_p}{r_\alpha}}} \tag{9.21}$$

If $\Delta r_\pi = r_\pi - r_p << R_a$ then

$$\boxed{\Delta v_p \approx \frac{v_\pi}{4a}\Delta r_\pi} \tag{9.22}$$

9.6. Insertion into a given circular planet orbit

Consider insertion of the spacecraft into a circular planet orbit given by radius r_0 and unit vector $\vec{c}^{\,0}$ of the orbit angular momentum ($\vec{c}^{\,0}$ is normal to the orbit plane). The following three-impulse maneuver is used for this (see Fig. 9.6):

– impulse Δv_0 at distance $r_\pi = |\vec{r}_\pi|$: capturing maneuver (see Section 9.4);
– impulse Δv_1 at distance r_α: raising periapsis to r_0 and turning the orbit plane (if necessary);
– impulse Δv_2 at distance r_0: insertion into the given circular orbit.

Designate

$$\gamma_1 = \vec{e}_1 \cdot \vec{c}^{\,0}, \;\; \gamma_2 = \vec{e}_{20} \cdot \vec{c}^{\,0}, \;\; \gamma_3 = \vec{e}_{30} \cdot \vec{c}^{\,0} \tag{9.23}$$

= direction cosines of $\vec{c}^{\,0}$ in a $\vec{e}_1, \vec{e}_{20}, \vec{e}_{30}$ frame (see Fig. 9.2)

$$\Rightarrow \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{9.24}$$

<u>Parameters of the incoming orbit</u>

In order to perform this maneuver periapsis of the spacecraft incoming hyperbola should lie in the plane of the final circular orbit (see Fig. 9.6) $\Rightarrow$ $\vec{r}_\pi \cdot \vec{c}^{\,0} = 0$ $\Rightarrow$ due to (9.9, 9.8, 9.23)

$$\gamma_1 \cos\varphi - \gamma_2 \sin\varphi \cos\psi - \gamma_3 \sin\varphi \sin\psi = 0 \tag{9.25}$$

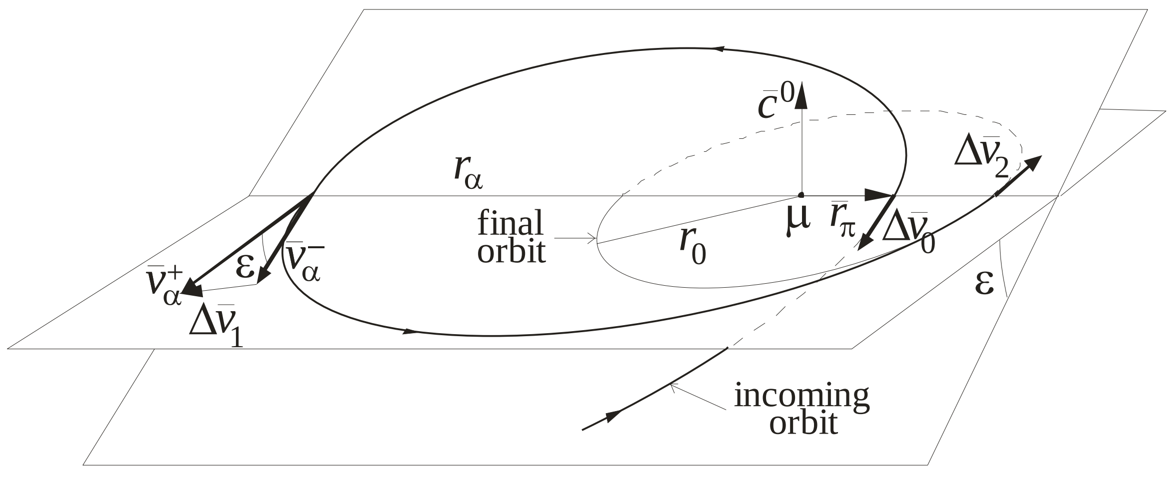


Fig. 9.6

$\Rightarrow$ taking into account (9.24) ψ angle can be found from (9.25) as follows:

$$\tan\frac{\psi}{2} = \frac{\gamma_3 \sin\varphi \pm \sqrt{\sin^2\varphi - \gamma_1^2}}{\gamma_1 \cos\varphi + \gamma_2 \sin\varphi} \tag{9.26}$$

$$\cos\psi = \frac{\gamma_1\gamma_2 \sin\varphi \pm \gamma_3\sqrt{\sin^2\varphi - \gamma_1^2}}{\left(1-\gamma_1^2\right)\sin\varphi} \tag{9.27}$$

$\Rightarrow$ the considered three-impulse maneuver is possible only if $\sin\varphi \geq |\gamma_1|$

Let ε be angle between the spacecraft initial hyperbolic orbit and final circular orbit (see Fig. 9.6) $\Rightarrow$ due to (9.7, 9.12, 9.23)

$$\cos\varepsilon = \vec{e}_3 \cdot \vec{c}^{\,0} = -\gamma_2 \sin\psi + \gamma_3 \cos\psi \tag{9.28}$$

Note that there are two solutions of Eq. (9.25) for ψ. That one is selected which provides $\min\varepsilon$.

Angles of the incoming hyperbola orientation are given by (9.13) using (9.6 – 9.9, 9.12).

<u>Special case</u>

Consider a case when inclination of the final circular orbit = 0, i.e.

$$\varepsilon = i, \quad \vec{c}^{\,0} = \pm\vec{z}^{\,0} = \{0,\, 0,\, \pm 1\}$$

$\Rightarrow$ due to (9.23) $\gamma_1 = \sin\eta,\ \gamma_2 = 0,\ \gamma_3 = \cos\eta$

$$\Rightarrow \quad \cos\psi = \pm\frac{\sqrt{\sin^2\varphi - \sin^2\eta}}{\sin\varphi\cos\eta} \tag{9.29}$$

$\Rightarrow$ the considered three-impulse maneuver is possible only if $\varphi \geq \eta$.

Eqs. (9.14, 9.29) give:

$$\cos\varepsilon = \cos i = \pm\sqrt{1 - \frac{\sin^2\eta}{\sin^2\varphi}} \tag{9.30}$$

$$\Rightarrow \quad \sin\varepsilon = \sin i = \left|\frac{\sin\eta}{\sin\varphi}\right|, \quad 0 \leq \varepsilon, i \leq \frac{\pi}{2} \tag{9.31}$$

Values of the impulses

Δv_0 is given by (9.16).

Let v_α^-, v_α^+ be the spacecraft apoapsis velocities before and after Δv_1 respectively

$$\Rightarrow \quad v_\alpha^- = \sqrt{\frac{2\mu}{R+r_\alpha}\frac{r_\pi}{r_\alpha}}, \; v_\alpha^+ = \sqrt{\frac{2\mu}{r_0+r_\alpha}\frac{r_0}{r_\alpha}}$$

(see Chapter 6),

$$\boxed{\Delta v_1 = \sqrt{{v_\alpha^-}^2 + {v_\alpha^+}^2 - 2v_\alpha^- v_\alpha^+ \cos\varepsilon}} \tag{9.32}$$

(see Fig. 9.6) where $\cos\varepsilon$ is given by (9.28) or (9.30).

Velocities in the final circular orbit before and after Δv_2 are

$$v_0^- = \sqrt{\frac{2\mu}{r_0+r_\alpha}\frac{r_\alpha}{r_0}}, \quad v_0^+ = \sqrt{\frac{\mu}{r_0}}$$

$$\Rightarrow \quad \boxed{\Delta v_2 = \sqrt{\frac{2\mu}{r_0+r_\alpha}\frac{r_\alpha}{r_0}} - \sqrt{\frac{\mu}{r_0}}} \tag{9.33}$$

Total maneuver cost is

$$\boxed{\Delta v = \Delta v_0 + \Delta v_1 + \Delta v_2} \tag{9.34}$$

10. Orbit Determination and Correction Maneuvers

10.1. Launch errors

Designate:

$\vec{x}_0, \vec{x}$ = the spacecraft state vectors at the initial time t_0 and at a given time t respectively,

$$\vec{x} = \vec{x}\left(\vec{x}_0, t\right) \tag{10.1}$$

Let $\vec{x}_{0\text{nom}}$ be a value of $\vec{x}_0$ for a nominal spacecraft orbit hitting the target. True launch parameters are slightly different from the required (nominal) ones due to the launch errors. The value of the state vector in the true orbit is

$$\vec{x}_{0\text{true}} = \vec{x}_{0\text{nom}} + \Delta\vec{x}_0, \tag{10.2}$$

$\Delta\vec{x}_0$ is the launch errors vector.

Let $\Delta\vec{x}$ be the errors of the state vector $\vec{x}$ at the given time t.

$$\Delta\vec{x} \approx \frac{\partial\vec{x}}{\partial\vec{x}_0}\Delta\vec{x}_0 \tag{10.3}$$

$$\Phi = \Phi\left(t, t_0\right) = \frac{\partial\vec{x}}{\partial\vec{x}_0} \tag{10.4}$$

= state transition matrix.

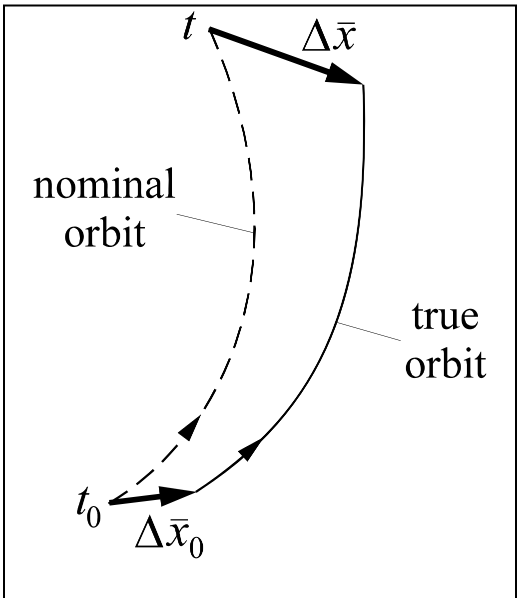


Fig. 10.1

10.2. Orbit tracking

The tracking is a measurement of orbital parameters from a ground station.

Let $\vec{r}$ and $\vec{r}_s$ be the positions of the spacecraft and ground station respectively,

$$\vec{\rho} = \vec{r} - \vec{r}_s = \left\{\rho_x, \rho_y, \rho_z\right\}$$

= the spacecraft position relatively to the ground station (see Fig. 10.2).

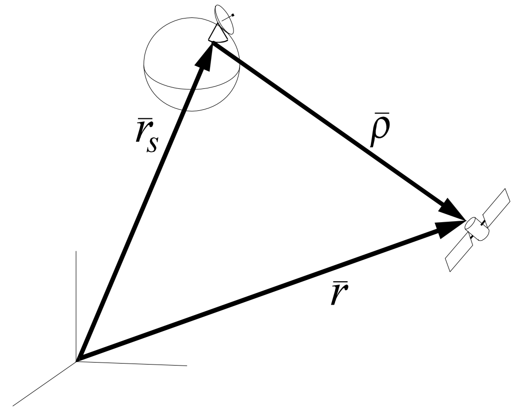


Fig. 10.2

Types of the tracking data:

$$\left.\begin{aligned}
&\rho = |\vec{\rho}| = |\vec{r} - \vec{r}_s| \quad \text{– range observations;}\\
&\dot{\rho} = \frac{\vec{\rho}\cdot\dot{\vec{\rho}}}{\rho} = \frac{(\vec{r} - \vec{r}_s)\cdot(\dot{\vec{r}} - \dot{\vec{r}}_s)}{|\vec{r} - \vec{r}_s|} \quad \text{– Doppler observations;}\\
&\vec{\rho}^{\,0} = \frac{\vec{\rho}}{\rho} = \frac{\vec{r} - \vec{r}_s}{|\vec{r} - \vec{r}_s|} \quad \text{– angular observations.}
\end{aligned}\right\} \qquad (10.5)$$

Unit vector $\vec{\rho}^{\,0}$ defines two angles (see Fig. 10.3):

$$\left.\begin{aligned}
&\sin\delta = \rho_z^0;\\
&\cos\alpha = \frac{\rho_x^0}{\cos\delta}, \quad \sin\alpha = \frac{\rho_y^0}{\cos\delta}.
\end{aligned}\right\} \qquad (10.6)$$

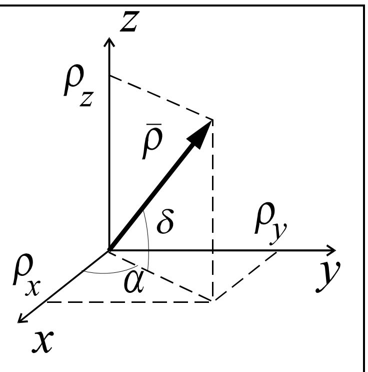


Fig. 10.3

The angles are:

- Optical astrometric observations:
  right ascension α (counted in the Earth equatorial plane) and declination δ (angle with the Earth equator plane) (see Fig. 10.3);
- Radiometric observations:
  azimuth a (counted in the local horizontal plane from the direction to the North in the clock-wise direction) and elevation e (angle with the local horizon).

10.3. Orbit determination. Least Square Method

Tracking data contain observation errors. Designate:

$\tilde{\psi},\ \psi_{\text{true}}$ = observed and true values of the measured parameter respectively;

$\varepsilon = \tilde{\psi} - \psi_{\text{true}}$ = observation error, a random value.

Mathematical expectation:

$E(\varepsilon) = m$ = mean value;

$E\left[(\varepsilon - m)^2\right] = \sigma^2, \quad \sigma$ = standard deviation.

It is assumed that $m = 0$ and σ is known.

Formulation of the problem:

There are n observations (i.e. measured values of the parameters)

$$\tilde{\psi}_1, ..., \tilde{\psi}_n \tag{10.7}$$

obtained at times $t_1, ..., t_n$ respectively. The problem is to determine the orbit using the observations (see Fig. 10.4)

$\Rightarrow$ it is necessary to obtain $\vec{x}_0 = \vec{x}(t_0)$ (sought-for parameters).

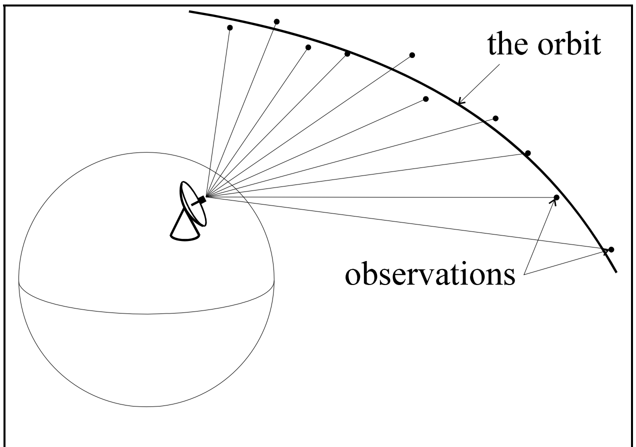


Fig. 10.4

Let

$$\psi_j = \psi\left(\vec{x}_j\right) \quad (j = 1, ..., n) \tag{10.8}$$

be values of the parameters calculated for

$$\vec{x}_j = \vec{x}\left(\vec{x}_0, t_j\right) \quad (j = 1, ..., n) \tag{10.9}$$

using (10.5, 10.6),

$\sigma_j \ (j = 1, ..., n)$ be standard deviations of the observations (10.7);

$$\boxed{L = \sum_{j=1}^{n} \left(\frac{\tilde{\psi}_j - \psi_j}{\sigma_j} \right)^2} \tag{10.10}$$

is the function of the Least Square Method (Least Squares),

$\dfrac{1}{\sigma_j^2}$ = weight of jth observation.

Vector-matrix form of the function is

$$\boxed{L = \left(\tilde{\vec{\psi}} - \vec{\psi}\right)^T W \left(\tilde{\vec{\psi}} - \vec{\psi}\right)} \tag{10.11}$$

where

$$\tilde{\vec{\psi}} = \{\tilde{\psi}_1, ..., \tilde{\psi}_n\}, \quad \vec{\psi} = \{\psi_1, ..., \psi_n\}$$

are vector of the observations and vector of the calculated parameters respectively,

$$W = \begin{bmatrix} \dfrac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \dfrac{1}{\sigma_n^2} \end{bmatrix} \tag{10.12}$$

is a diagonal weight matrix.

Special case: $\sigma_1 = \sigma_2 = ... = \sigma_n = 1$ (unweighed observations)

$\Rightarrow \ W = I$ = unit matrix

$$\Rightarrow \quad L = \sum_{j=1}^{n} (\tilde{\psi}_j - \psi_j)^2 = \left(\tilde{\vec{\psi}} - \vec{\psi}\right) \cdot \left(\tilde{\vec{\psi}} - \vec{\psi}\right)$$

Least Square Method gives a value $\vec{x}_0$ minimizing function L.

Necessary condition of minimum:

$$\frac{\partial L}{\partial \vec{x}_0} = -2\left(\tilde{\vec{\psi}} - \vec{\psi}\right)^T W \frac{\partial \vec{\psi}}{\partial \vec{x}_0} = \vec{0}^T \tag{10.13}$$

Define matrix

$$\Psi = \frac{\partial \vec{\psi}}{\partial \vec{x}_0}$$

$$\Rightarrow \qquad \vec{y} \equiv \Psi^T W \left(\tilde{\vec{\psi}} - \vec{\psi}\right) = \vec{0} \tag{10.14}$$

A jth row of Ψ is

$$\frac{\partial \psi_j}{\partial \vec{x}_0} = \frac{\partial \psi_j}{\partial \vec{x}_j} \frac{\partial \vec{x}_j}{\partial \vec{x}_0} = \frac{\partial \psi_j}{\partial \vec{x}_j} \Phi\left(t_j, t_0\right)$$

where $\dfrac{\partial \psi_j}{\partial \vec{x}_j}$ can be found from (10.5, 10.6), $\Phi\left(t_j, t_0\right)$ is the state transition matrix (see (10.4)). Eq. (10.14) can be written in the form

$$\vec{y}\left(\vec{x}_0\right) = \vec{0} \tag{10.15}$$

To solve Eq. (10.15) apply the Newton–Raphson procedure.

$$\frac{\partial \vec{y}}{\partial \vec{x}_0} \approx -\Psi^T W \frac{\partial \vec{\psi}}{\partial \vec{x}_0} = -\Psi^T W \Psi$$

(neglecting derivative of Ψ with respect to $\vec{x}_0$ in (10.14))
$\Rightarrow$ in an ith iteration of the procedure

$$\boxed{\vec{x}_{0,i+1} = \vec{x}_{0i} + \left(\Psi_i^T W \Psi_i\right)^{-1} \Psi_i^T W \left(\tilde{\vec{\psi}} - \vec{\psi}_i\right) \quad (i = 0, 1, \ldots)} \tag{10.16}$$

where $\Psi_i, \vec{\psi}_i$ are values of $\Psi, \vec{\psi}$ calculated in $\vec{x}_{0i}$.
The nominal value $\vec{x}_{0\text{nom}}$ is usually taken as a first guess for the procedure (10.16).

Let $\vec{x}_0$ be a solution to (10.15), $\vec{x} = \vec{x}\left(\vec{x}_0\right)$ and $\vec{\psi} = \vec{\psi}\left(\vec{x}\right)$. Then

$\tilde{\vec{\psi}} - \vec{\psi}$ is the vector of <u>residuals</u>, or O–C (observed minus calculated).

10.4. Orbit correction maneuvers

After the orbit is determined by the Least Squares it is necessary to correct the launch errors in order to hit the target.

Let $\vec{r}, \vec{r}'$ be the target and spacecraft positions respectively at the nominal arrival time t_a,

$$\Delta\vec{r} = \vec{r}' - \vec{r}$$

is the miss. Represent the state transition matrix in the form

$$\Phi\left(t_a, t_0\right) = \begin{bmatrix} R_{r0} & R_{v0} \\ V_{r0} & V_{v0} \end{bmatrix}$$

where

$$R_{r0} = \frac{\partial\vec{r}}{\partial\vec{r}_0}, \quad R_{v0} = \frac{\partial\vec{r}}{\partial\vec{v}_0}, \quad V_{r0} = \frac{\partial\vec{v}}{\partial\vec{r}_0}, \quad V_{v0} = \frac{\partial\vec{v}}{\partial\vec{v}_0}$$

are 3×3-matrices. Then

$$\Delta\vec{r} \approx \frac{\partial\vec{r}}{\partial\vec{x}_0}\Delta\vec{x}_0 = R_{r0}\Delta\vec{r}_0 + R_{v0}\Delta\vec{v}_0$$

where $\Delta\vec{x}_0 = \begin{bmatrix} \Delta\vec{r}_0 \\ \Delta\vec{v}_0 \end{bmatrix}$ is the vector of the launch errors.

A correcting impulse $\Delta\vec{v}$ is applied at a time t_c:

$$-\Delta\vec{r} \approx R_{vc}\Delta\vec{v}$$

where $R_{vc} = \dfrac{\partial\vec{r}}{\partial\vec{v}\left(t_c\right)}$ is a 3×3-matrix

(see Fig. 10.5)

$$\Rightarrow \quad \boxed{\Delta\vec{v} \approx -R_{vc}^{-1}\Delta\vec{r}} \qquad (10.17)$$

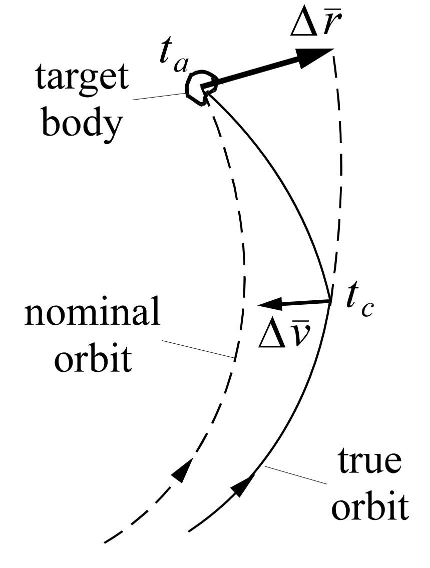


Fig. 10.5

Selection of the time t_c of the orbit correction maneuver:

- the orbit should be determined accurately enough
  $\Rightarrow\ t_c > t_d$ where $t_d - t_0$ is a time interval necessary for the orbit determination;
- the value of Δv should be minimal
  $\Rightarrow\ t_c = \arg \min\limits_{t_d < t_c < t_a} \Delta v$

10.5. Autonomous navigation

Autonomous tracking is onboard imaging camera observations; it provides much more accurate determination of the spacecraft motion relatively to the target celestial body than the ground-based tracking. Images of the body among the stars give accurate angular observations. After the relative motion is determined a targeting correction maneuver is carried out.

Assume the spacecraft trajectory relatively to the target body is a straight line and the relative velocity is constant. Let δ be the measured angular discrepancy of the body

$$\Rightarrow \qquad \Delta v = 2v\sin\frac{\delta}{2} \approx v\delta$$

(see Fig. 10.6)

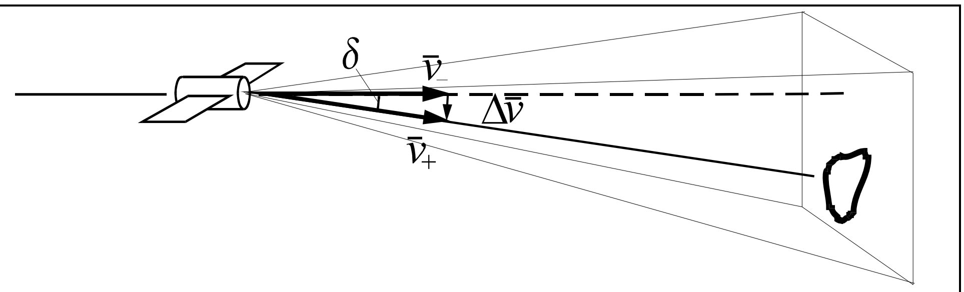


Fig. 10.6

Two ways of the autonomous navigation:

- ground-based processing of the observations and the control of the spacecraft;
- automated on-board processing and control (if the time is critically short).

11. State Transition Matrix

11.1. Necessary formulas

$$
\left.\begin{aligned}
&\dot{s} = \frac{1}{r}, \quad s(t_0) = 0 \\
&s = \frac{\vec{r}\cdot\vec{v} - \vec{r}_0\cdot\vec{v}_0 - h(t - t_0)}{\mu}
\end{aligned}\right\} = \text{universal variable}
$$

$$
\begin{aligned}
&x = -hs^2 \\
&c_n = c_n(x) \quad (n = 0, 1, \ldots) && = \text{Stumpff functions} \\
&h = v^2 - \frac{2\mu}{r} = -\frac{\mu}{a} && = \text{integral of energy} \\
&\vec{c} = \vec{r}\times\vec{v} && = \text{angular momentum} \\
&\vec{l} = -\frac{\mu}{r}\vec{r} + \vec{v}\times\vec{c} && = \text{Laplace integral} \\
&c = |\vec{c}| = \sqrt{r^2v^2 - (\vec{r}\cdot\vec{v})^2}, \quad l = |\vec{l}| \\
&p = \frac{c^2}{\mu} && = \text{semilatus rectum} \\
&r = r_0c_0 + \vec{r}_0\cdot\vec{v}_0\, s c_1 + \mu s^2 c_2
\end{aligned}
\tag{11.1}
$$

11.2. Variational and adjoint (costate) variational equations

Consider a vector $\vec{x} = \vec{x}(t) = \{x_1, ..., x_n\}$ satisfying the equation

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \tag{11.2}$$

and a vector $\vec{y} = \vec{y}(\vec{x}) = \{y_1, ..., y_n\}$.
Designate

$$X = \frac{\partial \vec{x}}{\partial \vec{y}}, \quad Y = \frac{\partial \vec{y}}{\partial \vec{x}} \tag{11.3}$$

and assume that the matrix X is non-singular, i.e.

$$X^{-1} = Y \tag{11.4}$$

Eqs. (11.2, 11.3) give

$$\frac{d}{dt}\frac{\partial \vec{x}}{\partial \vec{y}} = \frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial \vec{y}}$$

$$\Rightarrow \quad \boxed{\dot{X} = FX} \tag{11.5}$$

where

$$F = \frac{\partial \vec{f}(\vec{x})}{\partial \vec{x}} \tag{11.6}$$

Eq. (11.5) is <u>variational equation</u>.

$$\frac{\partial \vec{y}}{\partial \vec{x}}\frac{\partial \vec{x}}{\partial \vec{y}} = YX = I$$

where I is the unit n×n-matrix

$$\Rightarrow \quad \frac{d}{dt}(YX) = \dot{Y}X + Y\dot{X} = \dot{Y}X + YFX = 0$$

$\Rightarrow$ since matrix X is non-singular then

$$\boxed{\dot{Y} = -YF} \tag{11.7}$$

= <u>adjoint (costate) variational equation</u>.

Assume matrices (11.3) non-singular and let $X_i = \dfrac{\partial \vec{x}}{\partial y_i}$ and $Y_i = \dfrac{\partial y_i}{\partial \vec{x}}$ be columns of matrix X and rows of matrix Y respectively ($i = 1, \ldots, n$). All X_i satisfy equation (11.5) and all Y_i satisfy equation (11.7)

$\Rightarrow$ for any vector $\vec{x} = \vec{x}(t) = \{x_1, ..., x_n\}$ satisfying Eq. (11.2) and a shortened vector $\vec{y}' = \vec{y}'(\vec{x}) = \{y_1, ..., y_m\}$ $(m < n)$ matrix $\dfrac{\partial \vec{x}}{\partial \vec{y}'}$ satisfies variational equation (11.5) and matrix $\dfrac{\partial \vec{y}'}{\partial \vec{x}}$ satisfies adjoint variational equation (11.7).

11.3. Definition of the state transition matrix

Consider motion in the 3-dimentional space;

$$\vec{x}_0 = \vec{x}(t_0) = \begin{bmatrix} \vec{r}_0 \\ \vec{v}_0 \end{bmatrix}, \quad \vec{x} = \vec{x}(t) = \begin{bmatrix} \vec{r} \\ \vec{v} \end{bmatrix}$$

are state 6-vectors at the initial and current times,

$$\boxed{\Phi = \Phi(t, t_0) = \frac{\partial \vec{x}}{\partial \vec{x}_0}} \tag{11.8}$$

is the <u>state transition matrix</u>,

$$\Phi_{ij} = \frac{\partial x_i}{\partial x_{0j}} \quad (i, j = 1, \dots, 6)$$

Matrix Φ satisfies variational equation (11.5):

$$\boxed{\dot{\Phi} = F\Phi, \quad \Phi(t_0, t_0) = I} \tag{11.9}$$

where I is the unit 6×6-matrix.

The state vector satisfies Eq. (11.2). For the two body problem

$$\vec{f}(\vec{x}) = \begin{bmatrix} \vec{v} \\ -\frac{\mu}{r^3}\vec{r} \end{bmatrix}$$

in (11.2) (see Chapter 2) $\Rightarrow$ Eq. (11.6) gives

$$\boxed{F = \begin{bmatrix} 0 & I_3 \\ G & 0 \end{bmatrix}} \tag{11.10}$$

where I_3 is the unit 3×3-matrix,

$$\boxed{G = \frac{\mu}{r^3}\left(3\frac{\vec{r}\vec{r}^T}{r^2} - I_3\right)} \tag{11.11}$$

is a symmetric 3×3-matrix.

<u>Note</u>. The state transition matrix will be calculated in this Chapter <u>only for the two body problem</u>.

11.4. Inverse state transition matrix

$$\Phi^{-1} = \Phi(t_0, t) = \frac{\partial \vec{x}_0}{\partial \vec{x}} \tag{11.12}$$

Due to (11.4, 11.5, 11.7) matrix Φ^{-1} satisfies adjoint (costate) variational equation:

$$\frac{d}{dt}\Phi^{-1} = -\Phi^{-1}F, \quad \Phi^{-1}(t_0, t_0) = I \tag{11.13}$$

Define 6×6-matrix

$$J = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix} \tag{11.14}$$

It is easy to obtain:

$$J^{-1} = J^T = -J \quad \Rightarrow \quad JJ = -I \tag{11.15}$$

<u>Theorem 11.1</u>. The state transition matrix is <u>symplectic</u>, i.e.

$$\boxed{\Phi^T J \Phi = J \quad \Rightarrow \quad \Phi^{-1} = -J\Phi^T J} \tag{11.16}$$

<u>Proof</u>. Eqs. (11.10, 11.14) and the equality $G^T = G$ give

$$F^T J = \begin{bmatrix} -G & 0 \\ 0 & I_3 \end{bmatrix} = -JF \tag{11.17}$$

Due to (11.9, 11.17)

$$\frac{d}{dt} J\Phi^T J = J\dot{\Phi}^T J = J\Phi^T F^T J = -J\Phi^T JF, \tag{11.18}$$

i.e. matrix $-J\Phi^{-1}J$ satisfies Eq. (11.13).

Initial conditions in (11.9) and Eq. (11.15) give

$$J\Phi^T(t_0, t_0) J = JJ = -I \tag{11.19}$$

Eqs. (11.18, 11.19) prove the theorem.

Representing matrix Φ in the form

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \tag{11.20}$$

Eq. (11.16) can be written as

$$\boxed{\Phi^{-1} = \begin{bmatrix} \Phi_{22}^T & -\Phi_{12}^T \\ -\Phi_{21}^T & \Phi_{11}^T \end{bmatrix}} \tag{11.21}$$

11.5. Approach to the state transition matrix calculation

Let $\vec{q} = \vec{q}(\vec{x})$ be a 6-vector of independent first integrals

$\Rightarrow \quad \vec{q}(\vec{x}) = \vec{q}(\vec{x}_0)$

$$\Rightarrow \quad \frac{\partial \vec{q}}{\partial \vec{x}} \frac{\partial \vec{x}}{\partial \vec{x}_0} = \frac{\partial \vec{q}}{\partial \vec{x}_0} \tag{11.22}$$

Introduce matrix

$$A = A(t) = \frac{\partial \vec{q}}{\partial \vec{x}}, \quad A_0 = A(t_0) = \frac{\partial \vec{q}}{\partial \vec{x}_0} \tag{11.23}$$

$\Rightarrow$ Eqs. (11.8, 11.22, 11.23) give

$$\boxed{\Phi = A^{-1} A_0} \tag{11.24}$$

Due to (11.23) matrix A satisfies adjoint (costate) variational equation (11.7):

$$\boxed{\dot{A} = -AF} \tag{11.25}$$

Represent matrix A in the form

$$A = \begin{bmatrix} P & Q \end{bmatrix} \tag{11.26}$$

where P, Q are 6×3-matrices $\Rightarrow$ it follows from (11.26, 11.10) that

$$\boxed{\dot{P} = -QG, \quad \dot{Q} = -P} \tag{11.27}$$

Eqs. (11.15, 11.16, 11.24) give

$$A_0^{-1} A = -J A_0^T A^{T^{-1}} J \;\Rightarrow\; A_0^{-1} A J = J A_0^T A^{T^{-1}} \;\Rightarrow\; A J A^T = A_0 J A_0^T$$

$$\Rightarrow \quad \boxed{A^{-1} = J A^T K^{-1}} \tag{11.28}$$

where

$$K = A_0 J A_0^T \tag{11.29}$$

is a constant matrix.

11.6. Calculation of five rows of matrix *A*

$\vec{c}, \vec{l}, h$ are seven first integrals defining five independent ones. Introduce constant vectors $\vec{p}_1, \vec{p}_2$ such that

$$q_1 = \vec{p}_1 \cdot \vec{c},\ q_2 = \vec{p}_2 \cdot \vec{c},\ q_3 = \vec{p}_1 \cdot \vec{l},\ q_4 = \vec{p}_2 \cdot \vec{l},\ q_5 = h \tag{11.30}$$

are independent. Represent matrix A in the form

$$A = \begin{bmatrix} \vec{a}_1^T & \vec{b}_1^T \\ \cdots & \cdots \\ \vec{a}_6^T & \vec{b}_6^T \end{bmatrix} \tag{11.31}$$

where

$$\vec{a}_j^T = \frac{\partial q_j}{\partial \vec{r}}, \quad \vec{b}_j^T = \frac{\partial q_j}{\partial \vec{v}} \quad (j = 1, \ldots, 5) \tag{11.32}$$

are rows of the matrices P, Q respectively (see (11.26)).
As follows from (11.27)

$$\dot{\vec{a}}_j = -G\vec{b}_j, \quad \dot{\vec{b}}_j = -\vec{a}_j \quad (j = 1, \ldots, 6)$$

Eqs. (11.30, 11.32) give

$$\left.\begin{array}{l} \left.\begin{array}{l} \vec{a}_i = \vec{v} \times \vec{p}_i, \quad \vec{b}_i = \vec{p}_i \times \vec{r}, \\ \vec{a}_{i+2} = \dfrac{\mu}{r^3}\vec{r} \times (\vec{p}_i \times \vec{r}) + \vec{v} \times (\vec{v} \times \vec{p}_i), \\ \vec{b}_{i+2} = \vec{p}_i \times \vec{c} - \vec{r} \times (\vec{v} \times \vec{p}_i) \end{array}\right\} i = 1, 2 \\ \vec{a}_5 = \dfrac{\mu}{r^3}\vec{r}, \quad \vec{b}_5 = \vec{v} \end{array}\right\} \tag{11.33}$$

11.7. Calculation of sixth row of matrix *A*

Theorem 11.2. Consider a $2n$-dimensional Hamiltonian system:

$$\dot{\vec{x}} = \vec{f}\,(\vec{x},t), \quad \vec{x} = \begin{bmatrix} \vec{x}' \\ \vec{x}'' \end{bmatrix}, \quad \vec{f} = \vec{f}\,(\vec{x},t) = \begin{bmatrix} \partial H/\partial \vec{x}'' \\ -\partial H/\partial \vec{x}' \end{bmatrix},$$

$$\vec{x}' = \{x_1', ..., x_n'\}, \quad \vec{x}'' = \{x_1'', ..., x_n''\}$$

Assume that independent first integrals $q_1, ..., q_{2n-1}$ are known and

$$A_i = \mathrm{grad}_{\vec{x}}\, q_i = \frac{\partial q_i}{\partial \vec{x}} \quad (i = 1, ..., 2n-1)$$

Consider matrix A consisting of the rows $A_1, ..., A_{2n-1}$ and

$$A_{2n} = \left(\vec{\gamma} - \vec{f}\int_{t_0}^{t} \lambda\, dt\right)^T J \tag{11.34}$$

where

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

$2n$-vector $\vec{\gamma} = \vec{\gamma}(t)$ and function $\lambda = \lambda(t)$ satisfy the equations

$$\begin{aligned} &A_i \vec{\gamma} = d_i \quad (i = 1, ..., 2n-1), \\ &\dot{\vec{\gamma}} - F\vec{\gamma} = \lambda\, \vec{f}, \end{aligned} \tag{11.35}$$

d_i are any constants, $F = \dfrac{\partial \vec{f}}{\partial \vec{x}}$.

Then matrix A satisfies the adjoint (costate) variational equation

$$\dot{A} = -AF$$

and A is non-singular if and only if

$$\sum_{i=1}^{2n-1} d_i \lambda_i \neq 0 \tag{11.36}$$

where parameters λ_i are given by

$$\sum_{i=1}^{2n-1} \lambda_i A_i = \mathrm{grad}_{\vec{x}}\, H = \frac{\partial H}{\partial \vec{x}}$$

Due to (11.31) row A_6 is

$$A_6 = \begin{bmatrix} \vec{a}_6^T & \vec{b}_6^T \end{bmatrix}$$

It can be found using (11.34, 11.35) that

$$\vec{a}_6 = d_1\vec{a}_6' + d_2\vec{a}_6'', \quad \vec{b}_6 = d_1\vec{b}_6' + d_2\vec{b}'' \tag{11.37}$$

where d_1, d_2 are any constants satisfying the inequality

$$d_1 h + d_2 l^2 \neq 0 \tag{11.38}$$

((11.38) follows from (11.36)). Using theorem 11.2 one can obtain:

$$\vec{a}_6' = -3\frac{\mu}{r^3}\tau\vec{r} + \vec{v}, \quad \vec{b}_6' = 2\vec{r} - 3\tau\vec{v}, \tag{11.39}$$

$$\vec{a}_6'' = \left(\mu\dot{r} - 2\frac{\mu}{r^3}S\right)\vec{r} + \dot{S}\vec{v}, \quad \vec{b}_6'' = \dot{S}\vec{r} - 2S\vec{v} \tag{11.40}$$

where

$$\tau = t - t_0, \quad S = c^2\tau - \mu\int_{t_0}^{t} r\,dt, \quad \dot{S} = c^2 - \mu r \tag{11.41}$$

Conclusion

Due to (11.38) the constants d_1, d_2 can be taken as follows:

For a given small value $\varepsilon > 0$:

1. If $|h| \geq \varepsilon\frac{\mu}{r_0}$ (i.e. all orbits except parabolic and near-parabolic ones) then $d_1 = 1,\ d_2 = 0$
   (i.e. 6th row of matrix A is given by (11.39)).

2. If $|h| < \varepsilon\frac{\mu}{r_0}$ (i.e. parabolic and near-parabolic orbits) then $d_1 = 0,\ d_2 = 1$
   (i.e. 6th row of matrix A is given by (11.40)).

11.8. Calculation of integral in (11.41)

<u>Orbits different from parabolic and near-parabolic ones</u> $\left(|h| \geq \varepsilon \dfrac{\mu}{r_0}\right)$

From (11.1) obtain:

$$r' = \frac{dr}{ds} = \frac{\dot{r}}{\dot{s}} = \vec{r} \cdot \vec{v}, \quad r'' = \frac{d^2 r}{ds^2} = \frac{v^2 - \dfrac{\mu}{r}}{\dot{s}} = hr + \mu \tag{11.42}$$

To calculate integral in (11.41) the expression for semilatus rectum (see (11.1)) and (11.42) can be used:

$$p = \frac{r^2 v^2 - (\vec{r} \cdot \vec{v})^2}{\mu} = 2r + \frac{hr^2}{\mu} - \frac{r'^2}{\mu}$$

$$\Rightarrow \quad r^2 = \frac{\mu p}{h} - 2\frac{\mu r}{h} + \frac{r'^2}{h} \tag{11.43}$$

Using (11.43) and expression $dt = r\,ds$ obtain:

$$\int_{t_0}^{t} r\,dt = \int_0^s r^2 ds = \frac{\mu}{h} ps - 2\frac{\mu}{h}\int_0^s r\,ds + \frac{1}{h}\int_0^s r'^2 ds \tag{11.44}$$

where

$$\int_0^s r'^2 ds = \int_{r_0}^{r} r' dr = rr' - r_0 r_0' - \int_0^s rr'' ds$$
$$= rr' - r_0 r_0' - h\int_0^s r^2 ds - \mu\int_0^s r\,ds,$$

$$\int_0^s r\,ds = \int_{t_0}^{t} dt = \tau$$

$\Rightarrow$ Eq. (11.44) using expression for s (see (11.1)) gives

$$\boxed{\int_{t_0}^{t} r\,dt = \frac{(r+p)r' - (r_0 + p)r_0' - (3\mu + ph)\tau}{2h}} \tag{11.45}$$

Parabolic orbits ($h = 0$)

$x = 0$ in (11.1)

$\Rightarrow$ the Stumpff functions are $c_0 = c_1 = 1,\ c_2 = \dfrac{1}{2}$

$$\Rightarrow\ r = r_0 + r_0' s + \frac{\mu s^2}{2}$$

$$\Rightarrow\ r^2 = r_0^2 + 2r_0 r_0' s + \left(r_0'^2 + \mu r_0\right) s^2 + \mu r_0' s^3 + \frac{\mu^2}{4} s^4$$

$$\Rightarrow\ \boxed{\int_{t_0}^{t} r\,dt = \int_0^s r^2 ds = r_0^2 s + r_0 r_0' s^2 + \frac{r_0'^2 + \mu r_0}{3} s^3 + \frac{\mu r_0'}{4} s^4 + \frac{\mu^2}{20} s^5} \tag{11.46}$$

where $s = \dfrac{r' - r_0'}{\mu}$.

Near-parabolic orbits $\left(|h| < \varepsilon \dfrac{\mu}{r_0}\right)$

Consider function

$$R(s) = \int_0^s r^2 ds = \int_{t_0}^{t} r\,dt \tag{11.47}$$

Integral (11.47) can be calculated by expansion of the value $R(0)$ near $R(s)$ in Taylor series using the expressions

$$\frac{d^{2n-1}r}{ds^{2n-1}} = h^{n-1} r', \quad \frac{d^{2n}r}{ds^{2n}} = h^{n-1} r'' \tag{11.48}$$

Finally it can be found that

$$\boxed{\begin{aligned} \int_{t_0}^{t} r\,dt = r^2 s - r r' s^2 + \frac{r'^2 + r r''}{3} s^3 - 2\mu s^4 \left[r'' s c_5(x) - r' c_4(x)\right] \\ - 8 s^4 \left[\left(h r'^2 + r''^2\right) s c_5(4x) - r' r'' c_4(4x)\right] \end{aligned}} \tag{11.49}$$

11.9. Selection of vectors $\vec{p}_1, \vec{p}_2$ and inversion of matrix A

Using (11.31, 11.33, 11.37, 11.39, 11.40) calculate matrix (11.29):

$$K = \begin{bmatrix} 0 & k_1 & 0 & -k_2 & 0 & d_1 m_1 \\ -k_1 & 0 & k_2 & 0 & 0 & d_1 m_2 \\ 0 & -k_2 & 0 & -hk_1 & 0 & d_2 n_1 \\ k_2 & 0 & hk_1 & 0 & 0 & d_2 n_2 \\ 0 & 0 & 0 & 0 & 0 & -d_1 h - d_2 l^2 \\ -d_1 m_1 & -d_1 m_2 & -d_2 n_1 & -d_2 n_2 & d_1 h + d_2 l^2 & 0 \end{bmatrix} \tag{11.50}$$

where

$$\begin{aligned} &k_1 = \vec{c} \cdot \vec{p}_1 \times \vec{p}_2, \quad k_2 = \vec{l} \cdot \vec{p}_1 \times \vec{p}_2, \\ &m_i = \vec{c} \cdot \vec{p}_i, \quad n_i = c^2 \vec{l} \cdot \vec{p}_i \quad (i = 1, 2) \end{aligned} \tag{11.51}$$

Due to (11.28) matrix A is singular if and only if matrix K is singular

$\Rightarrow$ the problem is to select vectors $\vec{p}_1, \vec{p}_2$ providing non-singularity of matrix (11.50).

<u>Elliptic and hyperbolic orbits</u> $\left(l \neq 0,\ h \geq \varepsilon \dfrac{\mu}{r_0} \right)$

In this case $d_1 = 1,\ d_2 = 0$ (see Sect. 11.7). Taking

$$\boxed{\vec{p}_1 = \frac{\vec{l}}{l}, \quad \vec{p}_2 = \frac{\vec{l} \times \vec{c}}{l\,c}} \tag{11.52}$$

$\Rightarrow$ $k_1 = -c,\ k_2 = m_1 = m_2 = 0$ (n_1, n_2 are not used)

$$\Rightarrow \quad K^{-1} = \frac{1}{hc} \begin{bmatrix} 0 & h & 0 & 0 & 0 & 0 \\ -h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & -c & 0 \end{bmatrix}$$

and (11.28, 11.31) give

$$\boxed{A^{-1} = \frac{1}{hc} \begin{bmatrix} -h\vec{b}_2 & h\vec{b}_1 & \vec{b}_4 & -\vec{b}_3 & -c\vec{b}_6 & c\vec{b}_5 \\ h\vec{a}_2 & -h\vec{a}_1 & -\vec{a}_4 & \vec{a}_3 & c\vec{a}_6 & -c\vec{a}_5 \end{bmatrix}} \tag{11.53}$$

Circular, elliptic, and hyperbolic orbits $\left(h \geq \varepsilon \frac{\mu}{r_0} \right)$

In this case $d_1 = 1,\ d_2 = 0$ (see Sect. 11.7). Taking

$$\boxed{\vec{p}_1 = \frac{\vec{r}_0}{c}, \quad \vec{p}_2 = \frac{\vec{v}_0}{c}} \tag{11.54}$$

$\Rightarrow \quad k_1 = 1,\ k_2 = m_1 = m_2 = 0 \quad (n_1, n_2$ are not used)

$$\Rightarrow \quad K^{-1} = \frac{1}{h} \begin{bmatrix} 0 & -h & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and (11.28, 11.31) give

$$\boxed{A^{-1} = \frac{1}{h} \begin{bmatrix} h\vec{b}_2 & -h\vec{b}_1 & -\vec{b}_4 & \vec{b}_3 & -\vec{b}_6 & \vec{b}_5 \\ -h\vec{a}_2 & h\vec{a}_1 & \vec{a}_4 & -\vec{a}_3 & \vec{a}_6 & -\vec{a}_5 \end{bmatrix}} \tag{11.55}$$

Parabolic and near-parabolic orbits $\left(|h| < \varepsilon \frac{\mu}{r_0} \right)$

In this case $d_1 = 0,\ d_2 = 1$ (see Sect. 11.7). Taking

$$\boxed{\vec{p}_1 = \frac{\vec{c}}{c}, \quad \vec{p}_2 = \frac{\vec{l} \times \vec{c}}{l^2 c}} \tag{11.56}$$

$\Rightarrow \quad k_1 = 0,\ k_2 = 1,\ n_1 = n_2 = 0 \quad (m_1, m_2$ are not used)

$$\Rightarrow \quad K^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and (11.28, 11.31) give

$$\boxed{A^{-1} = \begin{bmatrix} -\vec{b}_4 & \vec{b}_3 & -\vec{b}_2 & \vec{b}_1 & -\vec{b}_6 & \vec{b}_5 \\ \vec{a}_4 & -\vec{a}_3 & \vec{a}_2 & -\vec{a}_1 & \vec{a}_6 & -\vec{a}_5 \end{bmatrix}} \tag{11.57}$$

12. Optimization of the Orbital Maneuvers

12.1. Statement of general optimization problem

Equation of motion is

$$\dot{\vec{x}} = \vec{f}\left(\vec{x},\vec{u}\right) \tag{12.1}$$

where

$$\vec{x} = \left\{x^0, x^1, ..., x^n\right\} \in X \quad = \text{state vector,} \tag{12.2}$$

$$\vec{u} = \left\{u^1, ..., u^m\right\} \in U \quad = \text{control,} \tag{12.3}$$

$$\vec{f} = \vec{f}\left(\vec{x},\vec{u}\right) = \left\{f^0,\ f^1, ...,\ f^n\right\} \tag{12.4}$$

Assume that

$\vec{f},\ \dfrac{\partial \vec{f}}{\partial \vec{x}}$ are continuous functions in $X \times U$,

$\vec{u} = \vec{u}\left(t\right)$ is a piecewise continuous function.

Let t_0, t_1 be initial and final times and assume that the vector

$$\vec{x}\left(t_0\right) = \vec{x}_0 = \left\{x_0^0, x_0^1, ..., x_0^n\right\} \tag{12.5}$$

is given.

Designate

$$\vec{x}\left(t_1\right) = \vec{x}_1 = \left\{x_1^0, x_1^1, ..., x_1^n\right\} \tag{12.6}$$

($\vec{x}_1$ is not necessarily given).

The following cases will be considered:

1) t_1 is free (i.e. the t_1 value also to be optimized);
2) t_1 is fixed.

Assume that the functional (cost function, payoff function, performance index)

$$J = \int_{t_0}^{t_1} f^0\left(\vec{x},\vec{u}\right) dt \tag{12.7}$$

is to be minimized. From (12.1, 12.2, 12.4–12.6) obtain:

$$J = x_1^0 - x_0^0 \tag{12.8}$$

The optimization problem is:

> To find control $\vec{u} = \vec{u}\left(t\right)$ $\left(t_0 \le t \le t_1\right)$ providing transfer of the object from $\vec{x}_0$ to $\vec{x}_1$ and minimizing the functional (12.7).

12.2. Pontryagin's maximum principle

Consider a vector variable

$$\vec{p} = \{ p_0,\, p_1, \ldots,\, p_n \} \tag{12.9}$$

satisfying the adjoint (costate) variational equation

$$\dot{\vec{p}}^T = -\vec{p}^T \frac{\partial \vec{f}\,(\vec{x},\vec{u})}{\partial \vec{x}} \tag{12.10}$$

($\vec{p}$ is adjoint, or costate, variable).

If a control $\vec{u} = \vec{u}(t)$ is chosen then there is a solution $\vec{x} = \vec{x}(t)$ of equation (12.1) and matrix $\partial \vec{f} / \partial \vec{x}$ in (12.10) is a known function of time $\Rightarrow$ there is a solution of equation (12.10)

$$\vec{p} = \vec{p}(t)$$

Define the <u>Hamiltonian</u> as

$$\boxed{H = H(\vec{x}, \vec{p}, \vec{u}) = \vec{p}^T \vec{f} = \sum_{i=0}^{n} p_i f^i} \tag{12.11}$$

$\Rightarrow$ due to (12.1, 12.10)

$$\boxed{\dot{\vec{x}} = \left(\frac{\partial H}{\partial \vec{p}} \right)^T , \quad \dot{\vec{p}} = -\left(\frac{\partial H}{\partial \vec{x}} \right)^T} \tag{12.12}$$

<u>Theorem 12.1 (Pontryagin's maximum principle)</u>. Let $\vec{u}_{\text{opt}} = \vec{u}_{\text{opt}}(t)$ be an optimal control transferring the object from $\vec{x}_0$ to $\vec{x}_1$ in time $t_1 - t_0$. Then:

1°. $H_{\text{opt}} = H\left(\vec{x}(t), \vec{p}(t), \vec{u}_{\text{opt}}(t)\right) = \max\limits_{\vec{u} \in U} H(\vec{x}, \vec{p}, \vec{u})$;

2°. $H_{\text{opt}} = \text{const}$ for $t_0 \le t \le t_1$;

3°. If t_1 is free then $H_{\text{opt}} = 0$ for $t_0 \le t \le t_1$ and $f_0(t_1) = 0$;

4°. $p_0(t_1) < 0$.

Proof

1° (only for fixed t_1 and free $\vec{x}_1$). Consider an infinitely small time interval $\tau-\varepsilon \le t \le \tau$ ($t_0 < \tau < t_1$, $\varepsilon > 0$ is infinitely small) and a variation $\delta\vec{u}(t)$ of the optimal control at this interval:

$$\vec{u}(t) = \begin{cases} \vec{u}_{\text{opt}}(t) \text{ if } t_0 \le t < \tau-\varepsilon \text{ or } \tau < t \le t_1 \\ \vec{u}_{\text{opt}}(t) + \delta\vec{u}(t) \text{ if } \tau-\varepsilon \le t \le \tau \end{cases} \tag{12.13}$$

($\delta\vec{u}(t)$ can be of a finite value). Designate variation of $\vec{x}(t)$ caused by $\delta\vec{u}(t)$ as

$$\delta\vec{x}(t) = \vec{x}(t) - \vec{x}_{\text{opt}}(t)$$

($\delta\vec{x}(t) = \vec{0}$ for $t < \tau-\varepsilon$).

$$\delta\vec{x}(\tau) = \left[\dot{\vec{x}}(\tau) - \dot{\vec{x}}_{\text{opt}}(\tau)\right]\varepsilon + o(\varepsilon) \tag{12.14}$$

Due to (12.1)

$$\delta\vec{x}(\tau) = \left[\vec{f}\left(\vec{x}(\tau), \vec{u}(\tau)\right) - \vec{f}\left(\vec{x}_{\text{opt}}(\tau), \vec{u}_{\text{opt}}(\tau)\right)\right]\varepsilon + o(\varepsilon) \tag{12.15}$$

Designate

$$\vec{f}_{\text{opt}} = \vec{f}\left(\vec{x}_{\text{opt}}(\tau), \vec{u}_{\text{opt}}(\tau)\right)$$

and consider a time $t > \tau$:

$$\dot{\vec{x}}(t) = \dot{\vec{x}}_{\text{opt}}(t) + \delta\dot{\vec{x}}(t) = \vec{f}_{\text{opt}} + \left.\frac{\partial\vec{f}}{\partial\vec{x}}\right|_{\text{opt}} \delta\vec{x}(t)$$

$$\Rightarrow \quad \delta\dot{\vec{x}}(t) = \left.\frac{\partial\vec{f}}{\partial\vec{x}}\right|_{\text{opt}} \delta\vec{x}(t) \tag{12.16}$$

(variational equation) $\Rightarrow$ due to (12.10, 12.16)

$$\frac{d}{dt}\vec{p}^{\,T}\delta\vec{x} = -\vec{p}^{\,T}\frac{\partial\vec{f}}{\partial\vec{x}}\delta\vec{x} + \vec{p}^{\,T}\frac{\partial\vec{f}}{\partial\vec{x}}\delta\vec{x} = \vec{0}$$

$$\Rightarrow \quad \vec{p}^{\,T}\delta\vec{x} = \text{const} \quad \Rightarrow \quad \vec{p}^{\,T}(t_1)\delta\vec{x}(t_1) = \vec{p}^{\,T}(\tau)\delta\vec{x}(\tau) \tag{12.17}$$

Put

$$p_0(t_1) = -1, \quad p_1(t_1) = \ldots = p_n(t_1) = 0 \qquad (12.18)$$

$\Rightarrow$ taking into account (12.8) obtain:

$$\vec{p}^T(t_1)\delta\vec{x}(t_1) = -\delta x^0(t_1) = -\delta J\,; \qquad (12.19)$$

$\delta J = J - J_{\text{opt}} \ge 0 \quad \Rightarrow \quad \vec{p}^T(t_1)\delta\vec{x}(t_1) \le 0 \quad \Rightarrow$ due to (12.17)

$$\vec{p}^T(\tau)\delta\vec{x}(\tau) \le 0 \qquad (12.20)$$

Eqs. (12.15, 12.20) give (since $\varepsilon > 0$)

$$\vec{p}^T(\tau)\,\vec{f}\left(x(\tau),\vec{u}(\tau)\right) \le \vec{p}^T(\tau)\,\vec{f}\left(x_{\text{opt}}(\tau),\vec{u}_{\text{opt}}(\tau)\right)$$

$\Rightarrow$ due to (12.11)

$$H\left(\vec{x}(\tau),\vec{p}(\tau),\vec{u}(\tau)\right) \le H\left(\vec{x}_{\text{opt}}(\tau),\vec{p}(\tau),\vec{u}_{\text{opt}}(\tau)\right)$$

τ is an arbitrary time $\Rightarrow$ for any t

$$H_{\text{opt}} = H\left(\vec{x}_{\text{opt}}(t),\vec{p}(t),\vec{u}_{\text{opt}}(t)\right) = \max_{\vec{u}\in U} H\left(\vec{x},\vec{p},\vec{u}\right)$$

$$\Rightarrow \boxed{\left.\frac{\partial H}{\partial \vec{u}}\right|_{\vec{u}=\vec{u}_{\text{opt}}} = \vec{0}} \qquad (12.21)$$

2°. Eqs. (12.12, 12.21) give

$$\begin{aligned}\dot{H}_{\text{opt}} &= \frac{\partial H_{\text{opt}}}{\partial \vec{x}}\dot{\vec{x}} + \frac{\partial H_{\text{opt}}}{\partial \vec{p}}\dot{\vec{p}} + \frac{\partial H_{\text{opt}}}{\partial \vec{u}}\dot{\vec{u}} \\ &= \frac{\partial H_{\text{opt}}}{\partial \vec{x}}\left(\frac{\partial H_{\text{opt}}}{\partial \vec{p}}\right)^T - \frac{\partial H_{\text{opt}}}{\partial \vec{p}}\left(\frac{\partial H_{\text{opt}}}{\partial \vec{x}}\right)^T = 0\end{aligned}$$

$$\Rightarrow \boxed{H_{\text{opt}} = \text{const}} \qquad (12.22)$$

3°. Consider free t_1 and assume that the value

$$t_1 = t_{\text{opt}} \tag{12.23}$$

provides min J

$$\Rightarrow \quad \left.\frac{dJ}{dt}\right|_{t=t_{\text{opt}}} = f_0\left(t_{\text{opt}}\right) = 0 \tag{12.24}$$

$\Rightarrow$ due to (12.23, 12.24)

$$\boxed{f_0\left(t_1\right) = 0}$$

Now the same problem can be considered as a fixed t_1 one

$\Rightarrow$ the results obtained above are applicable $\Rightarrow$ due to (12.18)

$$H_{\text{opt}}\left(t_{\text{opt}}\right) = -f_0\left(t_{\text{opt}}\right) = 0$$

$\Rightarrow$ due to (12.22)

$$\boxed{H_{\text{opt}}\left(t\right) = 0}$$

4°. $p_0\left(t_1\right) < 0$ due to (12.18).

Note. Multiplication of $\vec{p}\left(t\right)$ (and hence of H) by any constant $c > 0$ does not change the conclusions of theorem 12.1
$\Rightarrow$ it will be assumed below that

$$\boxed{p_0\left(t_1\right) = -1} \tag{12.25}$$

Special cases

1. $\vec{f}\left(\vec{x}, \vec{u}\right)$ does not depend on x^0

$\Rightarrow$ due to (12.11, 12.12) $\dot{p}_0\left(t\right) = 0 \Rightarrow p_0\left(t\right) \equiv -1 \quad \left(t_0 \le t \le t_1\right)$

$$\Rightarrow \boxed{H = -f^0 + \sum_{i=1}^{n} p_i f^i} \tag{12.26}$$

2. The problem of minimum time:

$$J = t_1 - t_0 \quad \Rightarrow \quad f^0\left(\vec{x}, \vec{u}\right) \equiv 1$$

$$\Rightarrow \boxed{H = -1 + \sum_{i=1}^{n} p_i f^i} \tag{12.27}$$

12.3. Non-autonomous system

Assume that the equation for $\vec{x}$ is

$$\dot{\vec{x}} = \vec{f}\left(\vec{x}, \vec{u}, t\right)$$

(non-autonomous system).
Introduce new variable

$$\dot{x}^{n+1} = 1, \quad x^{n+1}\left(t_0\right) = t_0 \quad \Rightarrow \quad x^{n+1} = t$$

Extend vectors $\vec{x}$, $\vec{f}$:

$$\tilde{\vec{x}} = \left\{x^0, ..., x^n, x^{n+1}\right\}, \quad \tilde{\vec{f}} = \left\{f^0, ..., f^n, 1\right\}$$

$$\Rightarrow \quad \dot{\tilde{\vec{x}}} = \tilde{\vec{f}}\left(\tilde{\vec{x}}, \vec{u}\right)$$

is autonomous system $\Rightarrow$ the problem is reduced to the solved one.

12.4. Transversality conditions

Consider shortened vectors

$$\hat{\vec{x}} = \left\{ x^1, ..., x^n \right\}, \quad \hat{\vec{p}} = \left\{ p_1, ..., p_n \right\}$$

and generalized boundary conditions

$$\hat{\vec{x}}_0 = \hat{\vec{x}}(t_0) \in X_0, \quad \hat{\vec{x}}_1 = \hat{\vec{x}}(t_1) \in X_1; \tag{12.28}$$

here X_0, X_1 are manifolds defined by the equations

$$X_0 : \ \vec{g}_0\left(\hat{\vec{x}}_0\right) = \vec{0}, \qquad X_1 : \ \vec{g}_1\left(\hat{\vec{x}}_1\right) = \vec{0} \tag{12.29}$$

where

$$\vec{g}_0 = \vec{g}_0\left(\hat{\vec{x}}_0\right) = \left\{ g_0^1, ..., g_0^{m_0} \right\}, \quad \vec{g}_1 = \vec{g}_1\left(\hat{\vec{x}}_1\right) = \left\{ g_1^1, ..., g_1^{m_1} \right\}$$

(Due to (12.8) x_0^0 must be given and x_1^0 is determined from $\min J$.)

<u>Theorem 12.2 (transversality conditions)</u>. If $\vec{u}(t)$ is an optimal control transferring the object from $\hat{\vec{x}}_0$ to $\hat{\vec{x}}_1$ subject to (12.28) then $\hat{\vec{p}}(t_0), \hat{\vec{p}}(t_1)$ are orthogonal to X_0, X_1 respectively.

In other words

$$\begin{aligned} \hat{\vec{p}}(t_0) &= \sum_{i=1}^{m_0} c_0^i \left(\left. \frac{\partial g_0^i}{\partial \hat{\vec{x}}} \right|_{t_0} \right)^T = \left(\left. \frac{\partial \vec{g}_0}{\partial \hat{\vec{x}}} \right|_{t_0} \right)^T \vec{c}_0 \\ \hat{\vec{p}}(t_1) &= \sum_{i=1}^{m_1} c_1^i \left(\left. \frac{\partial g_1^i}{\partial \hat{\vec{x}}} \right|_{t_1} \right)^T = \left(\left. \frac{\partial \vec{g}_1}{\partial \hat{\vec{x}}} \right|_{t_1} \right)^T \vec{c}_1 \end{aligned} \tag{12.30}$$

where

$$\vec{c}_0 = \left\{ c_0^1, ..., c_0^{m_0} \right\}, \quad \vec{c}_1 = \left\{ c_1^1, ..., c_1^{m_1} \right\}$$

are any constant vectors

$$\Rightarrow \quad p_i(t_0) = \left(\left. \frac{\partial \vec{g}_0}{\partial x_i} \right|_{t_0} \right)^T \vec{c}_0, \quad p_i(t_1) = \left(\left. \frac{\partial \vec{g}_1}{\partial x_i} \right|_{t_1} \right)^T \vec{c}_1 \quad (i = 1, ..., n) \tag{12.31}$$

Proof

Due to (12.17, 12.19, 12.20)

$$\vec{p}^T \delta\vec{x} \le 0$$

for any $t \in [t_0, t_1]$, also

$$\left.\frac{\partial J}{\partial \vec{x}}\right|_{J = J_{\text{opt}}} = \vec{0}$$

$\Rightarrow$ $\vec{p}^T d\vec{x} = 0$ for an infinitely small variation $d\vec{x} = \vec{x}' - \vec{x}$

$$\Rightarrow \vec{p}^T(t_0)\, d\vec{x}(t_0) = \vec{p}^T(t_1)\, d\vec{x}(t_1) = 0 \tag{12.32}$$

$dx_0^0 = 0$ (since x_0^0 is given), $dx_1^0 = dJ = 0$ $\Rightarrow$ (12.32) is fulfilled also for the shortened vectors $\hat{\vec{p}}, \hat{\vec{x}}$, i.e.

$$\hat{\vec{p}}^T(t_0)\, d\hat{\vec{x}}(t_0) = \hat{\vec{p}}^T(t_1)\, d\hat{\vec{x}}(t_1) = 0 \tag{12.33}$$

Vectors $\left(\dfrac{\partial g_j^1}{\partial \hat{\vec{x}}}\right)^T, \ldots, \left(\dfrac{\partial g_j^n}{\partial \hat{\vec{x}}}\right)^T$ are normal to X_j at $\hat{\vec{x}}$ (j = 0, 1);

$d\hat{\vec{x}} \in T_j$ where T_j is the tangent plane for X_j at $\hat{\vec{x}}$ (j = 0, 1)

$$\left(\vec{g}_j\left(\hat{\vec{x}}'\right) - \vec{g}_j\left(\hat{\vec{x}}\right) = \frac{\partial \vec{g}_j}{\partial \hat{\vec{x}}}\, d\hat{\vec{x}} = 0\right);$$

Eqs. (12.33) are fulfilled for any $d\hat{\vec{x}} \in T_j$ (j = 0, 1)

$$\Rightarrow \quad \hat{\vec{p}}(t_j) = \sum_{i=1}^{m_0} c_j^i \left(\frac{\partial g_j^i}{\partial \hat{\vec{x}}}\right)^T = \left(\frac{\partial \vec{g}_j}{\partial \hat{\vec{x}}}\right)^T \vec{c}_j \quad (j = 0, 1)$$

Special cases

Define index $j = 0$ or 1.

1. $m_j = 1$, i.e. the jth boundary condition is

$$g_j\left(\hat{\vec{x}}_0\right) = 0$$

(the manifold X_j is a hypersurface)

$$\Rightarrow \quad \hat{\vec{p}}\left(t_j\right) = c_j \left(\left. \frac{\partial g_j}{\partial \hat{\vec{x}}} \right|_{t_j} \right)^T$$

2. $m_j = n$ and

$$\vec{g}_j\left(\hat{\vec{x}}_j\right) = A\hat{\vec{x}}_j - \vec{b} = \vec{0}$$

where A is a non-singular n-matrix, $\vec{b}$ is an n-vector

$$\Rightarrow \quad \hat{\vec{x}}\left(t_j\right) = A^{-1}\vec{b} = \hat{\vec{x}}_j$$

i.e. the manifold X_j is a dot (see (12.5, 12.6))

$\Rightarrow \quad \vec{p}\left(t_j\right) = A^T\vec{c}$ is any vector.

3. $m_j = m, \quad 1 \le m < n$ and m coordinates are given:

$$g_j^i = x^i\left(t_j\right) - x_j^i = 0 \quad \left(i = 1, ..., m\right)$$

$$\Rightarrow \quad \left. \frac{\partial \vec{g}_j}{\partial \hat{\vec{x}}} \right|_{t_j} = \begin{bmatrix} 1 & & 0 & \cdots & 0 \\ & \ddots & & & \\ 0 & & 1 & \cdots & 0 \end{bmatrix}$$

$\Rightarrow$ due to (12.31)

$$p_i\left(t_j\right) = c_i \left(i = 1, ..., m\right), \; p_i\left(t_j\right) = 0 \left(i = m+1, ..., n\right)$$

$\Rightarrow \quad p_1\left(t_j\right), ..., p_m\left(t_j\right)$ can take any values and

$$\boxed{p_{m+1}\left(t_j\right) = \; ... \; = p_n\left(t_j\right) = 0} \tag{12.34}$$

$m = 0$:

$$\boxed{\text{if } \hat{\vec{x}}_j \text{ is free then } \hat{\vec{p}}\left(t_j\right) = \vec{0}}$$

$\Rightarrow$ if $\hat{\vec{x}}_1$ is free then (12.18) is a necessary condition of optimality.

12.5. Maximum principle for jet propulsion

Designate:

$m = m(t)$	= current spacecraft mass $(t_0 \le t \le t_1)$;
$m_0 = m(t_0), \quad m_1 = m(t_1)$;	
$m_p = m_p(t) = m_0 - m$	= propellant mass consumed by t;
$\dot{m} = \dfrac{dm}{dt} \le 0$;	
$\dot{m}_p = \dfrac{dm_p}{dt} = -\dot{m} \ge 0$	= mass flow rate;
u = const	= exhaust velocity;
$\vec{\alpha} = \vec{\alpha}(t)$	= vector of the jet acceleration;

$$\alpha = |\vec{\alpha}| = \frac{\dot{m}_p u}{m} = \frac{\dot{m}_p u}{m_0 - m_p} \tag{12.35}$$

(see Eq. (6.9)).

$\vec{\alpha}$ is the control, i.e. $\vec{u} \equiv \vec{\alpha}$.
Assume that

$$0 \le \dot{m}_p \le \gamma \tag{12.36}$$

$$\Rightarrow \qquad 0 \le \alpha \le \frac{\gamma u}{m_0 - m_p} \tag{12.37}$$

The equations of motion are

$$\boxed{\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{f}_v + \vec{\alpha} \end{aligned}} \tag{12.38}$$

where $\vec{f}_v = \vec{f}_v(\vec{r})$ is an acceleration caused by external forces (the case of $\vec{f}_v = \vec{f}_v(\vec{r}, \vec{v})$ is not considered).

For the two body problem

$$\vec{f}_v = -\frac{\mu}{r^3}\vec{r}$$

Take the cost function as follows:

$$J = m_p(t_1) = \int_{t_0}^{t_1} \dot{m}_p \, dt \tag{12.39}$$

$$\Rightarrow \quad \vec{x}(t) = \left\{x^0, x^1, ..., x^6\right\} = \left\{m_p, \vec{r}, \vec{v}\right\}; \tag{12.40}$$

$$x^0(t_0) = m_p(t_0) = 0, \quad x^0(t_1) = m_p(t_1) = J$$

The Hamiltonian is

$$H = p_0 \dot{m}_p + \vec{p}_r^{\,T} \vec{v} + \vec{p}_v^{\,T} \vec{f}_v + \vec{p}_v^{\,T} \vec{\alpha}, \tag{12.41}$$

$$\vec{p} = \left\{ p_0, \vec{p}_r, \vec{p}_v \right\}$$

<u>Optimal thrust direction</u>

Due to (12.41) $\max\limits_{\vec{\alpha}} H$ is reached if $\vec{\alpha} \uparrow\uparrow \vec{p}_v$

$\Rightarrow$ due to (12.35)

$$\vec{\alpha} = \frac{\dot{m}_p u}{m} \frac{\vec{p}_v}{p_v} \tag{12.42}$$

where $p_v = |\vec{p}_v|$.

$\vec{p}_v$ is the <u>Lawden's primer vector</u>.

Optimal thrust is always directed along the primer vector

$$\Rightarrow \quad \vec{p}_v^{\,T} \vec{\alpha} = p_v \alpha$$

$$\Rightarrow \quad H = p_0 \dot{m}_p + \vec{p}_r^{\,T} \vec{v} + \vec{p}_v^{\,T} \vec{f}_v + p_v \alpha \tag{12.43}$$

Eqs. (12.12, 12.41) give

$$\begin{aligned} \dot{\vec{p}}_r &= -\left(\frac{\partial H}{\partial \vec{r}}\right)^T = -\left(\frac{\partial \vec{f}_v}{\partial \vec{r}}\right)^T \vec{p}_v \\ \dot{\vec{p}}_v &= -\left(\frac{\partial H}{\partial \vec{v}}\right)^T = -\vec{p}_r \end{aligned} \tag{12.44}$$

$$\Rightarrow \quad \boxed{\ddot{\vec{p}}_v = \left(\frac{\partial \vec{f}_v}{\partial \vec{r}}\right)^T \vec{p}_v} \tag{12.45}$$

$\vec{r} = \vec{r}(t)$ is a continuous function

$\Rightarrow \quad \vec{f}_v(\vec{r}), \dfrac{\partial \vec{f}_v(\vec{r})}{\partial \vec{r}}$ are continuous functions of time

$\Rightarrow$ due to (12.45) $\boxed{\vec{p}_v,\ \dot{\vec{p}}_v,\ \ddot{\vec{p}}_v \text{ are continuous functions of time}}$

For the two body problem

$$\frac{\partial \vec{f}_v}{\partial \vec{r}} = \left(\frac{\partial \vec{f}_v}{\partial \vec{r}}\right)^T = \frac{\mu}{r^3}\left(3\frac{\vec{r}\vec{r}^T}{r^2} - I_3\right) = G \tag{12.46}$$

(see (11.10))

$$\Rightarrow \quad \boxed{\ddot{\vec{p}}_v = G\vec{p}_v} \tag{12.47}$$

<u>Optimal thrust value</u>

Eqs. (12.35, 12.43) give

$$H = \kappa \dot{m}_p + \vec{p}_r^{\,T}\vec{v} + \vec{p}_v^{\,T}\vec{f}_v \tag{12.48}$$

where it is designated

$$\kappa = \kappa(t) = p_0 + \frac{p_v u}{m} \tag{12.49}$$

Due to (12.36, 12.48) $\max\limits_{\dot{m}_p} H$ is reached if:

$$\boxed{\begin{array}{llll} \kappa > 0 & \Rightarrow & \dot{m}_p = \gamma & \text{(maximum thrust)} \\ \kappa < 0 & \Rightarrow & \dot{m}_p = 0 & \text{(zero thrust)} \\ \kappa = 0 & \Rightarrow & 0 \le \dot{m}_p \le \gamma & \text{(intermediate thrust)} \end{array}} \tag{12.50}$$

(see Fig. 12.1).

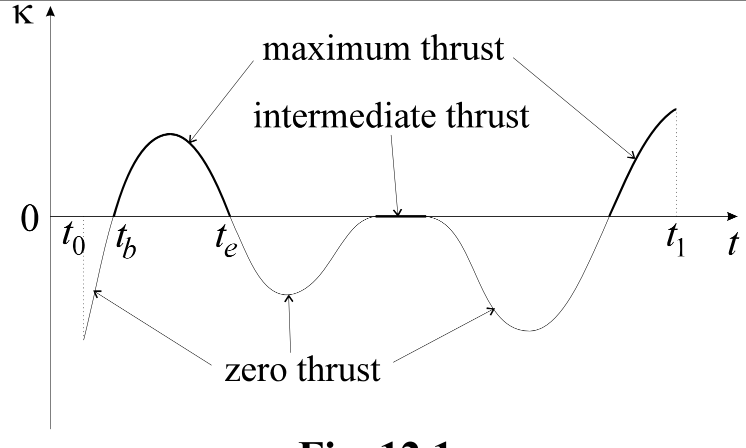


Fig. 12.1

From (12.12, 12.48) taking into account equations

$$m = m_0 - m_p,\ \ \dot{m} = -\dot{m}_p$$

obtain:

$$\dot{p}_0 = -\frac{\partial H}{\partial m_p} = \frac{p_v u\,\dot{m}_p}{\left(m_0 - m_p\right)^2} = -p_v u\,\frac{\dot{m}}{m^2} = -p_v u\,\frac{d}{dt}\left(\frac{1}{m}\right)$$

$$= \frac{\dot{p}_v u}{m} - \frac{d}{dt}\left(\frac{p_v u}{m}\right) \tag{12.51}$$

$\Rightarrow$ (12.49) gives

$$\boxed{\dot{\kappa} = \frac{\dot{p}_v u}{m}} \tag{12.52}$$

$\kappa = \kappa(t)$ is a <u>switching function</u>.

<u>First integral of the problem</u>

$H_{\text{opt}} = C$ where C is a constant (see theorem 12.1)

$\Rightarrow$ from (12.44, 12.48) obtain:

$$\boxed{\kappa\dot{m}_p + \vec{p}_v^{\,T}\vec{f}_v - \dot{\vec{p}}_v^{\,T}\vec{v} = C} \tag{12.53}$$

= first integral of the problem.

If t_1 is free then $C = 0$ (see theorem 12.1)

$$\Rightarrow\quad \boxed{\kappa\dot{m}_p + \vec{p}_v^{\,T}\vec{f}_v - \dot{\vec{p}}_v^{\,T}\vec{v} = 0} \tag{12.54}$$

= first integral for free t_1.

12.6. Maximum and zero thrust

Maximum thrust: $\dot{m}_p = -\dot{m} = \gamma$; assume that $\gamma = \text{const}$.

Designate times of the beginning and end of the maximum thrust as t_b, t_e (see Fig. 12.1),

$$\Delta t = t_e - t_b$$

is time interval of the maximum thrust

$$\Rightarrow \quad m_e = m_b - \gamma \Delta t \tag{12.55}$$

where $m_b = m(t_b)$, $m_e = m(t_e)$.

Change of the spacecraft velocity due to (6.6, 12.55) is

$$\boxed{\Delta v = u \ln \frac{m_b}{m_b - \gamma \Delta t}} \tag{12.56}$$

Zero thrust: $\dot{m}_p = \dot{m} = 0 \quad \Rightarrow \quad m = \text{const}$

$\Rightarrow$ from (12.52) and the inequality $\kappa < 0$ (see (12.50)) obtain:

$$\boxed{\kappa = \frac{p_v u}{m} - c, \quad c = \text{const} > 0} \tag{12.57}$$

Consider the two body problem; due to (12.10) vector

$$\hat{\vec{p}} = \{\vec{p}_r, \vec{p}_v\}$$

satisfies the adjoint (costate) variational equation for the Keplerian motion:

$$\dot{\hat{\vec{p}}}^T = -\hat{\vec{p}}^T F$$

where $F = \begin{bmatrix} 0 & I_3 \\ G & 0 \end{bmatrix}$ (see (11.6, 11.9, 12.46)). Matrix $A = A(t)$ defined in Section 11.5 and calculated in Sections 11.6–11.8 is a fundamental solution of the adjoint (costate) variational equation for the Keplerian motion

$$\Rightarrow \quad \hat{\vec{p}} = A^T \vec{\beta}$$

where $\vec{\beta}$ is a constant 6-vector

$$\Rightarrow \quad \boxed{\vec{p}_r = P^T \vec{\beta}, \quad \vec{p}_v = Q^T \vec{\beta}} \tag{12.58}$$

where P, Q are 6×3-matrices, $A = [P \quad Q]$ (see Section 11.5).

12.7. Impulsive thrust

Assume that $\gamma \to \infty$ $\Rightarrow$ due to (12.56) $\Delta t \to 0$ for a given Δv = <u>impulsive thrust</u> (see Fig. 12.2).

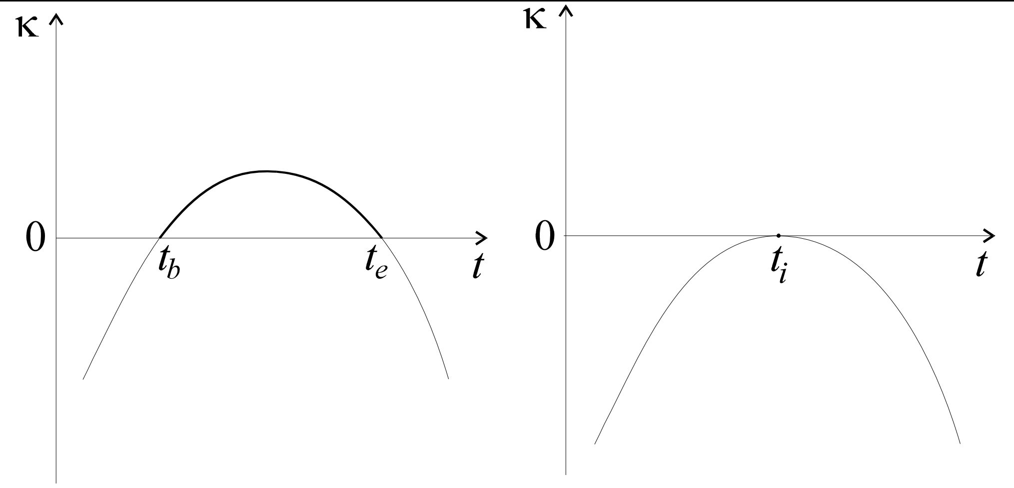


Fig. 12.2

The point of the impulsive thrust application t_i (see Fig. 12.2) is called
<u>point of junction</u>.

$$\boxed{\kappa = 0} \quad \text{for the impulsive thrust.} \tag{12.59}$$

The point of junction corresponds to $\max\limits_{t} \kappa$

$$\Rightarrow \quad \boxed{\kappa = \dot{\kappa} = 0 \quad \text{if} \quad t_0 < t_i < t_1} \tag{12.60}$$

$\Rightarrow$ due to (12.52)

$$\boxed{\dot{p}_v = 0 \quad \Rightarrow \quad \vec{p}_v \cdot \dot{\vec{p}}_v = 0 \quad \text{if} \quad t_0 < t_i < t_1} \tag{12.61}$$

If $t_i = t_0$ or $t_i = t_1$ then it may be

$$\dot{\kappa} \neq 0$$

(see Fig. 12.3) and (12.61) is not fulfilled.

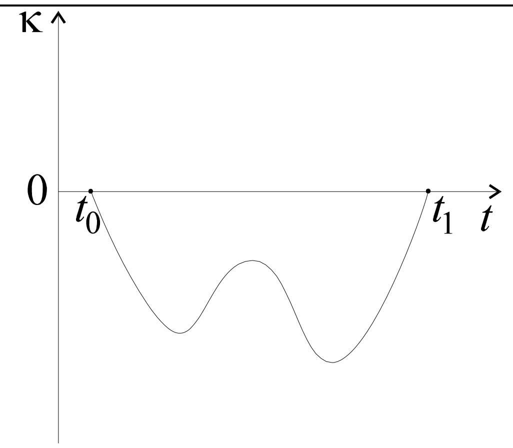


Fig. 12.3

Assume that the trajectory includes zero and impulsive thrust only (see Fig. 12.4) and in the first point of junction

$$p_v = p_m \tag{12.62}$$

Then due to (12.57, 12.59)

$$\boxed{\kappa = \frac{u}{m}\left(p_v - p_m\right)} \tag{12.63}$$

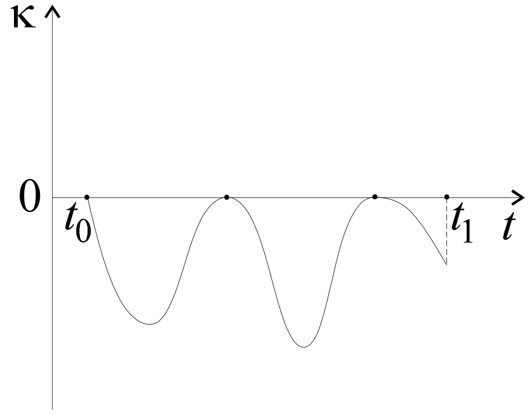


Fig. 12.4

for the zero thrust

$\Rightarrow$ (12.62) is fulfilled in all points of junction and

$$p_m = \max_{t_0 \le t \le t_1} p_v$$

$\Rightarrow$ $\kappa = \kappa(t)$ is given by (12.63) for whole time interval $t_0 \le t \le t_1$.

13. Electric Propulsion

13.1. Notation and necessary equations

Designate:

t_0, t_1 = initial and final times;

$m = m(t)$ = current spacecraft mass $(t_0 \le t \le t_1)$;

$m_0 = m(t_0)$, $m_1 = m(t_1)$;

$m_p = m_p(t) = m_0 - m$ = propellant mass consumed by t;

$\dot{m} = \dfrac{dm}{dt} \le 0$;

$\dot{m}_p = \dfrac{dm_p}{dt} = -\dot{m} \ge 0$ = mass flow rate;

u = exhaust velocity;

$F_T = \dot{m}_p u$ = jet thrust;

$\vec{\alpha} = \vec{\alpha}(t)$ = jet acceleration vector;

$\alpha = |\vec{\alpha}| = \dfrac{F_T}{m} = \dfrac{\dot{m}_p u}{m} = \dfrac{\dot{m}_p u}{m_0 - m_p}$;

W = electric power;

η = power efficiency;

$W_e = \eta W$ = effective power.

Equations of motion are

$$\dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = \vec{f}_v + \vec{\alpha} \tag{13.1}$$

where $\vec{f}_v = \vec{f}_v(\vec{r})$ is acceleration caused by external forces.
The cost function (i.e. the functional to be minimized) is

$$J = x^0(t_1) = \int_{t_0}^{t_1} f^0(\vec{r}, \vec{v}, \vec{\alpha})\,dt \to \min \tag{13.2}$$

The Hamiltonian is

$$H = p_0 f^0 + \vec{p}_r^{\,T} \vec{v} + \vec{p}_v^{\,T} \vec{f}_v + \vec{p}_v^{\,T} \vec{\alpha} \tag{13.3}$$

If $\vec{f} = \left\{ f^0, \vec{v}, \vec{f}_v + \vec{\alpha} \right\}$ does not depend on $x^0 = x^0(t)$ then

$$H = -f^0 + \vec{p}_r^{\,T} \vec{v} + \vec{p}_v^{\,T} \vec{f}_v + \vec{p}_v^{\,T} \vec{\alpha} \tag{13.4}$$

13.2. General information about electric propulsion

Electric propulsion (EP), Low thrust:

The propellant is ionized and accelerated in an electrostatic or electromagnetic field.

Table 13.1. Typical parameters of the space propulsion systems

	Chemical propulsion	Electric propulsion
Exhaust velocity, km/s	~ 3	15 to 70
Jet acceleration / g_e (g_e = 9.8066 m/s^2)	~ 0.1	10^{-5} to 10^{-4}

Chemical propulsion: impulsive thrust (runs for several minutes).
Electric propulsion: continuous thrust (can run for several months).

The effective power is

$$W_e = \eta W = \frac{\dot{m}_p u^2}{2} \tag{13.5}$$

Usually W_e, u are given

$$\Rightarrow \quad \dot{m}_p = -\dot{m} = \frac{2W_e}{u^2}, \tag{13.6}$$

$$F_T = \dot{m}_p u = \frac{2W_e}{u}, \tag{13.7}$$

$$\alpha = \frac{F_T}{m} = \frac{2W_e}{mu} \tag{13.8}$$

Typical example:

$$\left.\begin{array}{l} W = 2000 \text{ W} \\ \eta = 0.5 \\ m_0 = 250 \text{ kg} \\ u = 20 \text{ km/s} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} W_e = 1000 \text{ W} \\ \dot{m} = 5 \text{ mg/s} \\ F_T = 0.1 \text{ N} \\ \alpha = 4\cdot 10^{-4} \text{ m/s}^2 \approx 4\cdot 10^{-5}\, g_e \end{array}\right.$$

13.3. Types of the low thrust

Controllability of the thrust

1. *Limited power (LP problem).*

   Only an upper limit of the power is given, i.e.

   $$0 \le W_e \le W_{em}$$

   Exhaust velocity can be varied arbitrarily, i.e.

   $$0 \le u < \infty$$

   $\Rightarrow$ due to (13.8) the acceleration can be arbitrarily varied:

   $$0 \le \alpha < \infty$$

   From (13.5–13.8) obtain:

   $$W_e = \frac{\dot{m}_p u^2}{2} = \left(\frac{\dot{m}_p u}{m}\right)^2 \frac{m^2}{2\dot{m}_p} = \alpha^2 \frac{m^2}{2\dot{m}_p}$$

   $$\Rightarrow \dot{m}_p = \frac{m^2 \alpha^2}{2W_e} \ge \frac{m^2 \alpha^2}{2W_{em}} \qquad (13.9)$$

   $\Rightarrow$ $\boxed{\text{maximum power provides a minimum propellant consumption}}$

2. *Constant exhaust velocity (CEV problem).*

   An upper limit W_{em} of the effective power is given and

   $$u = \text{const}$$

   $\Rightarrow$ due to (13.8) the acceleration can be varied within the limits:

   $$0 \le \alpha \le \frac{2W_{em}}{mu},$$

   $$0 \le \dot{m}_p = |\dot{m}| \le \gamma \equiv \frac{2W_{em}}{u^2} \qquad (13.10)$$

Power source

1. *Solar power*.

   Solar arrays (solar electric propulsion = SEP):

   $$W_e \approx W_{e0}\left(\frac{r_0}{r}\right)^2 \tag{13.11}$$

   where r is a spacecraft heliocentric distance, r_0 is an initial value of r, $W_{e0} = W_e\left(r_0\right)$.

2. *Constant power*.

   SEP within the Earth sphere of influence,
   nuclear power (nuclear electric propulsion = NEP):

   $$W_e \approx \text{const} \tag{13.12}$$

13.4. Optimization of the low thrust

<u>LP problem</u> $(0 \le u < \infty)$

Due to (13.9) and the equality $\dot{m} = -\dot{m}_p$

$$\frac{\alpha^2}{2W_e} = -\frac{\dot{m}}{m^2} = \frac{d}{dt}\left(\frac{1}{m}\right) \quad \Rightarrow \quad \frac{1}{2}\int_{t_0}^{t_1} \frac{\alpha^2}{W_e}\,dt = \frac{1}{m_1} - \frac{1}{m_0} \qquad (13.13)$$

$\Rightarrow$ the performance index can be taken as follows:

$$\boxed{J = \frac{1}{2}\int_{t_0}^{t_1} \frac{\alpha^2}{W_e}\,dt \to \min} \qquad (13.14)$$

Due to (13.3, 13.14) the Hamiltonian is

$$H = -\frac{\alpha^2}{2W_e} + \vec{p}_r^{\,T}\vec{v} + \vec{p}_v^{\,T}\vec{f}_v + \vec{p}_v^{\,T}\vec{\alpha}\,;$$

$$\frac{\partial H}{\partial \vec{\alpha}} = -\frac{\vec{\alpha}^T}{W_e} + \vec{p}_v^{\,T} = \vec{0}^T$$

$\Rightarrow$ optimal thrust is

$$\boxed{\vec{\alpha} = W_e\,\vec{p}_v} \qquad (13.15)$$

1. *Constant power*: it can be taken

$$\boxed{J = \frac{1}{2}\int_{t_0}^{t_1} \alpha^2\,dt \to \min} \qquad (13.16)$$

$$\Rightarrow \boxed{\vec{\alpha} = \vec{p}_v} \qquad (13.17)$$

2. *Solar power*: it can be taken

$$\boxed{J = \frac{1}{2}\int_{t_0}^{t_1} (r\,\alpha)^2\,dt \to \min} \qquad (13.18)$$

$$\Rightarrow \boxed{\vec{\alpha} = \frac{\vec{p}_v}{r^2}} \qquad (13.19)$$

<u>CEV problem</u> (u = const)

Results of Sections 12.5, 12.6 can be used:

$$\boxed{J = \int_{t_0}^{t_1} \dot{m}_p \, dt \to \min} \tag{13.20}$$

The Hamiltonian is

$$H = \kappa \dot{m}_p + \vec{p}_r^{\,T} \vec{v} + \vec{p}_v^{\,T} \vec{f}_v + \vec{p}_v^{\,T} \vec{\alpha} \tag{13.21}$$

where $\kappa = \kappa(t)$ is the switching function:

$$\boxed{\begin{array}{llll} \kappa > 0 & \Rightarrow & \dot{m}_p = \gamma & \text{(maximum thrust)} \\ \kappa < 0 & \Rightarrow & \dot{m}_p = 0 & \text{(zero thrust)} \\ \kappa = 0 & \Rightarrow & 0 \le \dot{m}_p \le \gamma & \text{(intermediate thrust)} \end{array}} \tag{13.22}$$

γ is given by (13.10). Optimal thrust is

$$\boxed{\vec{\alpha} = \frac{\dot{m}_p u}{m} \frac{\vec{p}_v}{p_v}} \tag{13.23}$$

where $\vec{p}_v$ is Lawden's primer vector, $p_v = |\vec{p}_v|$.

Constant power. From (13.10, 13.12) obtain:

$$\gamma = \text{const} \tag{13.24}$$

Solar power. From (13.10, 13.11) obtain:

$$\gamma = \frac{2W_{e0}}{u^2} \left(\frac{r_0}{r} \right)^2 \tag{13.25}$$

All other results of Sections 12.5, 12.6 also can be directly applied in this case.

13.5. Locally optimal thrust

Consider an orbital parameter

$$q = q(\vec{r}, \vec{v}) \tag{13.26}$$

to be changed by low thrust and designate

$$\vec{p}^T = \frac{\partial q}{\partial \vec{v}} \tag{13.27}$$

Change of the spacecraft velocity due to the low thrust in an infinitely small time dt is

$$d\vec{v} = \vec{\alpha}\, dt \tag{13.28}$$

Hence variation of q in the time dt is

$$dq = \vec{p}^T \vec{\alpha}\, dt \tag{13.29}$$

$$\Rightarrow \begin{cases} \text{the increment of } q \text{ is maximum if } \vec{\alpha} \uparrow\uparrow \vec{p}, \\ \text{the decrement of } q \text{ is maximum if } \vec{\alpha} \uparrow\downarrow \vec{p} \end{cases}$$

(locally optimal thrust; see also Section 6.4).

Locally optimal low thrust acceleration $\vec{\alpha}$ is directed along the partials (13.27) in the same direction as $\vec{p}$ for increase of the parameter q and in the opposite direction for decrease of q .

$\frac{\partial q}{\partial \vec{x}}$ satisfies the adjoint variational equation (see Section 11.2)

$$\Rightarrow \quad \boxed{\varepsilon \vec{p} = \vec{p}_v} \tag{13.30}$$

where ε = const,

$\varepsilon > 0$ for increase of the parameter q
$\varepsilon < 0$ for decrease of the parameter q

$\vec{p}_v$ is the Lawden's primer vector providing local maximum of the Hamiltonian (see Chapter 12).

Consider motion in a planet sphere of influence or a nuclear power source $\Rightarrow$ power is constant.

LP problem. Due to (13.17, 13.30)

$$\vec{\alpha} = \varepsilon \vec{p} \tag{13.31}$$

CEV problem. Due to (13.23, 13.30)

$$\vec{\alpha} = \pm \frac{\dot{m}_p u}{m} \frac{\vec{p}}{p} \tag{13.32}$$

where '+' and '–' correspond to increase and decrease of q respectively, $\dot{m}_p$ is defined by (13.22) for $\gamma = \text{const}$.

13.6. Locally optimal thrust for the orbital elements

Designate:

$\vec{r} = \{x, y, z\}$ = position vector

$\vec{v} = \{v_x, v_y, v_z\}$ = velocity vector

$h = v^2 - \dfrac{2\mu}{r}$ = integral of energy

$\vec{c} = \vec{r} \times \vec{v} = \{c_x, c_y, c_z\}$ = angular momentum

$c = |\vec{c}|$

$\vec{c}^{\,0} = \dfrac{\vec{c}}{c}$ = unit vector of the normal to the orbit plane

$p = \dfrac{c^2}{\mu}$ = semilatus rectum

$\vec{v}_r, \vec{v}_n$ = velocity radial and transversal components

$\vec{v} = \vec{v}_r + \vec{v}_n$

u = argument of latitude

Ω = longitude of the ascending node

$\vec{k} = \{\cos\Omega, \sin\Omega, 0\}$ = unit vector of the ascending node direction

Semimajor axis: $a = \dfrac{1}{\dfrac{2}{r} - \dfrac{v^2}{\mu}}$

$$\Rightarrow \quad \boxed{\vec{p} = \left(\frac{\partial a}{\partial \vec{v}}\right)^T = \frac{2a^2}{\mu}\vec{v}} \tag{13.33}$$

Orbital period: $P = 2\pi \dfrac{a^{3/2}}{\sqrt{\mu}}$

$$\Rightarrow \quad \boxed{\vec{p} = \left(\frac{\partial P}{\partial \vec{v}}\right)^T = 3\frac{aP}{\mu}\vec{v}} \tag{13.34}$$

Eccentricity: $e=\sqrt{1+\frac{c^2}{\mu^2}h}$

$$\Rightarrow \quad \boxed{\vec{p}=\left(\frac{\partial e}{\partial \vec{v}}\right)^T=\frac{1}{\mu e}\left(p\vec{v}-\frac{r^2}{a}\vec{v}_n\right)} \tag{13.35}$$

Periapsis radius: $r_\pi=a(1-e)$

$$\Rightarrow \quad \boxed{\vec{p}=\left(\frac{\partial r_\pi}{\partial \vec{v}}\right)^T=\frac{r^2\vec{v}_n-r_\pi^2\vec{v}}{\mu e}} \tag{13.36}$$

Apoapsis radius: $r_\alpha=a(1+e)$

$$\Rightarrow \quad \boxed{\vec{p}=\left(\frac{\partial r_\alpha}{\partial \vec{v}}\right)^T=\frac{r_\alpha^2\vec{v}-r^2\vec{v}_n}{\mu e}} \tag{13.37}$$

Inclination: $\cos i=\frac{c_z}{c}$,

$$c_z=xv_y-yv_x, \quad c^2=\vec{r}\times\vec{v}\cdot\vec{c}=\vec{c}\times\vec{r}\cdot\vec{v}$$

$$\Rightarrow \quad -\sin i\frac{\partial i}{\partial \vec{v}}=\frac{1}{c}\{-y,\,x,\,0\}-\frac{c_z}{c^3}(\vec{c}\times\vec{r})^T \tag{13.38}$$

Multiply (13.38) by $\vec{r},\vec{v}$:

$$\sin i\frac{\partial i}{\partial \vec{v}}\vec{r}=\sin i\frac{\partial i}{\partial \vec{v}}\vec{v}=0 \quad \Rightarrow \quad \frac{\partial i}{\partial \vec{v}} \text{ is directed along } \vec{c}$$

$$\Rightarrow \quad -\sin i\frac{\partial i}{\partial \vec{v}}=\xi\vec{c}^{\,0}$$

To find ξ multiply (13.38) by $\vec{c}^{\,0}$ using (3.23) (see Chapter 3):

$$\xi=\frac{1}{c}\left(-yc_x^0+xc_y^0\right)=\frac{r}{c}\left[-(\sin\Omega\,\cos u+\cos\Omega\,\sin u\,\cos i)\sin\Omega\,\sin i\right.$$
$$\left.-(\cos i\cos\Omega\,\cos u-\sin\Omega\,\sin u\,\cos i)\cos\Omega\,\sin i\right]=-\frac{r\cos u\sin i}{c}$$

$$\Rightarrow \quad \boxed{\vec{p}=\left(\frac{\partial i}{\partial \vec{v}}\right)^T=\frac{r\cos u}{c}\vec{c}^{\,0}} \tag{13.39}$$

<u>Longitude of the ascending node</u>: $\tan\Omega = -\dfrac{c_x}{c_y}$,

$$c_x = yv_z - zv_y = c\sin\Omega\,\sin i,\quad c_y = zv_x - xv_z = -\cos\Omega\,\sin i \quad (13.40)$$

$$\Rightarrow \quad \frac{1}{\cos^2\Omega}\frac{\partial\Omega}{\partial\vec{v}} = -\frac{1}{c_y}\{0,-z,y\} + \frac{c_x}{c_y^2}\{z,0,-x\}$$

$$= \frac{1}{c\cos\Omega\sin i}\left(\{0,-z,y\} - \tan\Omega\{z,0,-x\}\right)$$

$$\Rightarrow \quad \frac{\partial\Omega}{\partial\vec{v}} = \frac{1}{c\sin i}\{-z\sin\Omega,-z\cos\Omega,x\sin\Omega+y\cos\Omega\} \qquad (13.41)$$

Multiply (13.41) by $\vec{r},\vec{v}$ using (13.40):

$$\frac{\partial\Omega}{\partial\vec{v}}\vec{r} = \frac{1}{c\sin i}\left(-xz\sin\Omega - yz\cos\Omega + xz\sin\Omega + yz\cos\Omega\right) = 0,$$

$$\frac{\partial\Omega}{\partial\vec{v}}\vec{v} = \frac{1}{c\sin i}\left(-zv_x\sin\Omega - zv_y\cos\Omega + xv_z\sin\Omega + yv_z\cos\Omega\right)$$

$$= \frac{1}{c\sin i}\left(-c_y\sin\Omega + c_x\cos\Omega\right) = 0$$

$$\Rightarrow \quad \frac{\partial\Omega}{\partial\vec{v}} \text{ is directed along } \vec{c} \quad \Rightarrow \quad \frac{\partial\Omega}{\partial\vec{v}} = \eta\,\vec{c}^{\,0}$$

To find η multiply (13.41) by $\vec{c}^{\,0}$ using (3.23) (see Chapter 3):

$$\eta = \frac{1}{c\sin i}\left[-zc_x^0\sin\Omega - zc_y^0\cos\Omega + \left(x\sin\Omega + y\cos\Omega\right)c_z^0\right] = \frac{\vec{c}^{\,0}\cdot\vec{k}\times\vec{r}}{c\sin i}$$

$$\Rightarrow \quad \boxed{\vec{p} = \left(\frac{\partial\Omega}{\partial\vec{v}}\right)^T = \frac{\vec{c}^{\,0}\cdot\vec{k}\times\vec{r}}{c\sin i}\vec{c}^{\,0}} \qquad (13.42)$$

14. Constrained Thrust Direction

14.1. Introduction

Optimal control of the electric propulsion (low thrust) in general needs a continuous complicated control of the spacecraft attitude in the 3-axis stabilization mode. At the same time the solar panels should be directed to the Sun during whole time of the thrust acting. This leads to a complicated spacecraft structure and attitude control system.

Simplification of the spacecraft attitude control system and stabilization mode during the flight can lower the mission cost, although puts a constraint on the thrust direction. Optimization of the low thrust transfers with the thrust direction subject to constraints is considered in this chapter.

14.2. Notation and necessary equations

Designate:

$m = m(t)$, $m_0 = m(t_0)$ = current and initial spacecraft mass;

$m_p = m_p(t) = m_0 - m$ = propellant mass consumed by t;

$\dot{m} = \dfrac{dm}{dt} \le 0$;

$\dot{m}_p = \dfrac{dm_p}{dt} = -\dot{m} \ge 0$ = mass flow rate;

u = exhaust velocity;

W_e = effective power;

$\vec{\alpha} = \vec{\alpha}(t)$ = jet acceleration vector,

$$\alpha = |\vec{\alpha}| = \frac{\dot{m}_p u}{m}, \tag{14.1}$$

$$\vec{\alpha}^0 = \begin{cases} \vec{\alpha}/\alpha & = \text{unit vector of the thrust direction for non-zero thrust;} \\ \vec{0} & = \text{for zero thrust} \end{cases} \tag{14.2}$$

$\vec{\alpha}_m, \vec{\alpha}_m^0, \alpha_m$ = optimal values of $\vec{\alpha}, \vec{\alpha}^0, \alpha$

$\vec{p}_v$ = Lawden's primer vector.

Equations of motion are

$$\dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = \vec{f}_v + \vec{\alpha} \tag{14.3}$$

where $\vec{f}_v = \vec{f}_v(\vec{r}, \vec{v})$ is acceleration caused by external forces. The performance index (i.e. the functional to be minimized) is

$$J = \int_{t_0}^{t_1} f^0 \, dt \tag{14.4}$$

Constraint on the thrust value may be written as

$$\vec{\alpha}^T \vec{\alpha} = \alpha^2 \tag{14.5}$$

where α is given by (14.1).

14.3. General equality constraint

Let us consider constraint on the thrust direction given by

$$\vec{g} = \vec{g}\left(\vec{r}, \vec{v}, t, \vec{\alpha}^0\right) = \vec{0} \tag{14.6}$$

If $\vec{g}$ explicitly depends on time it is expedient to add equation

$$\dot{t} = 1 \tag{14.7}$$

making the system autonomous (see Chapter 12)
$\Rightarrow$ due to (14.3 – 14.7) the Hamiltonian is

$$H = p_0 f_0 + \vec{p}_r^T \vec{v} + \vec{p}_v^T \vec{f}_v + \vec{p}_v^T \vec{\alpha} + \frac{\lambda_\alpha}{2}\left(\vec{\alpha}^T \vec{\alpha} - \alpha^2\right) + \vec{\lambda}_g^T \vec{g} + p_t \tag{14.8}$$

where $\lambda_\alpha, \vec{\lambda}_g, p_t$ are adjoint (costate) variables corresponding to equations (14.5, 14.6, 14.7) respectively. Adjoint variables $p_t, \vec{p}_r, \vec{p}_v$ in (14.8) satisfy equations

$$\dot{p}_t = -\frac{\partial H}{\partial t} = -\lambda_g^T \dot{\vec{g}} \tag{14.9}$$

$$\dot{\vec{p}}_r^T = -\frac{\partial H}{\partial \vec{r}} = -\vec{p}_v^T \frac{\partial \vec{f}_v}{\partial \vec{r}} - \vec{\psi}_r^T, \quad \dot{\vec{p}}_v^T = -\frac{\partial H}{\partial \vec{v}} = -\vec{p}_r^T - \vec{p}_v^T \frac{\partial \vec{f}_v}{\partial \vec{v}} - \vec{\psi}_v^T, \tag{14.10}$$

where

$$\vec{\psi}_r = \left(\frac{\partial \vec{g}}{\partial \vec{r}}\right)^T \vec{\lambda}_g, \quad \vec{\psi}_v = \left(\frac{\partial \vec{g}}{\partial \vec{v}}\right)^T \vec{\lambda}_g \tag{14.11}$$

Value $\vec{\alpha}_m^0$ of vector (14.2) providing maximum to (14.8) is

$$\vec{\alpha}_m^0 = \begin{cases} \arg\max\limits_{\vec{g}=\vec{0}} \vec{p}_v^T \vec{\alpha}^0, & \max\limits_{\vec{g}=\vec{0}} \vec{p}_v^T \vec{\alpha}^0 > 0 \\ \vec{0}, & \max\limits_{\vec{g}=\vec{0}} \vec{p}_v^T \vec{\alpha}^0 \le 0 \end{cases} \tag{14.12}$$

$$\Rightarrow \quad \vec{\alpha}_m^0 = \frac{\vec{p}_g}{p_g} = \vec{p}_g^0 \tag{14.13}$$

where $\vec{p}_g$ is a projection of $\vec{p}_v$ onto the set given by (14.6) (see Annex B), $p_g = \left|\vec{p}_g\right|$; if $\vec{p}_g = \vec{0}$ then $\vec{\alpha}_m^0 = \vec{0}$.
Assume that $\vec{\alpha}_m^0$ is found somehow and consider matrix

$$P = \vec{\alpha}_m^0 \vec{\alpha}_m^{0T} \tag{14.14}$$

According to the definition given in Annex B matrix (14.14) projects vector $\vec{p}_v$ onto the set given by (14.6), i.e.

$$\vec{p}_g = P\vec{p}_v \tag{14.15}$$

Assume that f_0 is such that

$$\frac{\partial f_0}{\partial \vec{\alpha}} = h\vec{\alpha}^T, \quad h = h(\vec{r}, \vec{v}, t, \vec{\alpha}) \tag{14.16}$$

$$\Rightarrow \quad \left(\frac{\partial H}{\partial \vec{\alpha}}\right)^T = p_0 h\vec{\alpha} + \vec{p}_v + \lambda_\alpha \vec{\alpha} + \frac{1}{\alpha}\left(I - \vec{\alpha}^0 \vec{\alpha}^{0T}\right) G^T \vec{\lambda}_g = \vec{0} \tag{14.17}$$

where $G = \dfrac{\partial \vec{g}}{\partial \vec{\alpha}^0}$, I is unit matrix. Multiplication of (14.17) from the left by the matrix (14.14) gives

$$\vec{\alpha}_m = -\frac{\vec{p}_g}{\lambda_\alpha + p_0 h} \tag{14.18}$$

$\Rightarrow$ due to (14.13) $\lambda_\alpha + p_0 h < 0$. Substitution of (14.14, 14.15, 14.18) into (14.17) gives

$$\left(I - \vec{\alpha}_m^0 \vec{\alpha}_m^{0T}\right)\left(\vec{p}_v + \frac{1}{\alpha} G^T \vec{\lambda}_g\right) = \vec{0} \tag{14.19}$$

Matrix $I - \vec{\alpha}_m^0 \vec{\alpha}_m^{0T}$ projects any vector onto the plane orthogonal to $\vec{\alpha}_m^0$
$\Rightarrow$ (14.19) is only possible if

$$\vec{p}_v + \frac{1}{\alpha} G^T \vec{\lambda}_g = x\vec{\alpha}_m^0 \tag{14.20}$$

where x is a scalar multiplier.

Limited Power (LP) case (see (13.13, 13.14)):

$$p_0 = -1, \quad f_0 = \frac{\alpha^2}{2W_e} \tag{14.21}$$

$\Rightarrow \; h = \dfrac{1}{W_e}$ in (14.16 – 14.18).

In the LP case thrust value is not limited $\Rightarrow$ (14.5) is not used
$\Rightarrow$ $\lambda_\alpha = 0$ in (14.16 – 14.18) $\Rightarrow$ due to (14.18)

$$\boxed{\vec{\alpha}_m = W_e \vec{p}_g} \tag{14.22}$$

Eqs. (14.14, 14.15, 14.22) give the optimal thrust magnitude:

$$\boxed{\alpha_m = W_e \vec{\alpha}_m^{0T} \vec{p}_g} \tag{14.23}$$

Impulsive thrust or constant exhaust velocity (CEV) case
(see (12.36, 12.39) and (13.19, 13.22)):

$$f_0 = \dot{m}_p, \tag{14.24}$$

$$0 \le \dot{m}_p \le \gamma, \quad u = \text{const} \tag{14.25}$$

$\Rightarrow$ due to (14.24) Eq. (14.16) is also fulfilled in this case with $h = 0$.
Eq. (14.13) gives the optimal thrust vector:

$$\boxed{\vec{\alpha}_m = \alpha_m \frac{\vec{p}_g}{p_g}} \tag{14.26}$$

where α_m is given by (14.1) under the conditions (14.25).
Eqs. (14.13 – 14.15) give $\vec{p}_g^T \vec{\alpha} = p_g \alpha$ $\Rightarrow$ the Hamiltonian is

$$H = \kappa' \dot{m}_p + \vec{p}_r^T \vec{v} + \vec{p}_v^T \vec{f}_v + \frac{\lambda_\alpha}{2}\left(\vec{\alpha}_m^T \vec{\alpha}_m - \alpha^2\right) + \vec{\lambda}_g^T \vec{g} + p_t \tag{14.27}$$

where

$$\kappa' = \kappa'(t) = p_0 + \frac{p_g u}{m} \tag{14.28}$$

It can be shown like it has been done in Section 12.5 for κ (see (12.52)) that

$$\dot{\kappa}' = \frac{\dot{p}_g u}{m} \tag{14.29}$$

Due to (14.25, 14.27) $\max_{\dot{m}_p} H$ is reached if

$$\boxed{\begin{array}{llll} \kappa' > 0 & \Rightarrow & \dot{m}_p = \gamma & \text{(maximum thrust)} \\ \kappa' < 0 & \Rightarrow & \dot{m}_p = 0 & \text{(zero thrust)} \\ \kappa' = 0 & \Rightarrow & 0 \le \dot{m}_p \le \gamma & \text{(intermediate thrust)} \end{array}} \tag{14.30}$$

$\Rightarrow$ κ' is the switching function for the case of constrained thrust direction (see Section 12.5).

<u>Conclusion</u>: Comparing (14.22) with (13.14) and (14.26, 14.28 – 14.30) with (12.42, 12.49, 12.52, 12.50) shows that $\vec{p}_v$ is replaced by $\vec{p}_g$ in the optimal solution.

14.4. General inequality constraint

Let us consider constraint on the thrust direction given by

$$\vec{g} = \vec{g}\left(\vec{r}, \vec{v}, t, \vec{\alpha}^0\right) \geq 0 \tag{14.31}$$

If there are two-sided inequalities they also can be reduced to (14.31) by means of decomposition into two inequality each.

Assume that dimension of vector $\vec{g}$ is n i.e.

$$\vec{g} = \{g_1, ..., g_n\}$$

$\Rightarrow$ the set given by (14.31) is intersection of the sets given by

$$g_i = g_i\left(\vec{r}, \vec{v}, t, \vec{\alpha}^0\right) \geq 0, \quad i = 1, ..., n \tag{14.32}$$

Let us introduce new control

$$\vec{\zeta} = \{\zeta_1, ..., \zeta_n\}$$

and consider vector

$$\vec{\theta} = \left\{\zeta_1^2, ..., \zeta_n^2\right\} \tag{14.33}$$

Then the inequality (14.31) may be replaced by the equality

$$\vec{g} - \vec{\theta} = \vec{0} \tag{14.34}$$

$\Rightarrow$ the Hamiltonian becomes

$$H = \kappa' \dot{m}_p + \vec{p}_r^T \vec{v} + \vec{p}_v^T \vec{f}_v + \frac{\lambda_\alpha}{2}\left(\vec{\alpha}_m^T \vec{\alpha}_m - \alpha^2\right) + \vec{\lambda}_g^T\left(\vec{g} - \vec{\theta}\right) + p_t \tag{14.35}$$

The additional necessary condition of maximum of (14.35) is

$$\frac{\partial H}{\partial \vec{\zeta}} = \vec{0}^T \tag{14.36}$$

$\Rightarrow$ Eqs. (14.35, 14.36) give

$$\lambda_{gi}\zeta_i = 0, \quad i = 1, ..., n \tag{14.37}$$

where λ_{gi} are components of vector $\vec{\lambda}_g$.

Assume that for g_i $(1 \le i \le n)$ strict inequality $g_i > 0$ is fulfilled
$\Rightarrow$ due to (14.33, 14.34) $\zeta_i > 0$ $\Rightarrow$ $\lambda_{gi} = 0$ in (14.37).

According to lemma B.1 of Annex B, in order to find $\vec{\alpha}_m^0$ the inequality

$$\vec{g}\left(\vec{r}, \vec{v}, t, \vec{p}_v^0\right) \ge \vec{0} \tag{14.38}$$

should be checked where

$$\vec{p}_v^0 = \frac{\vec{p}_v}{|\vec{p}_v|} \tag{14.39}$$

If (14.38) is fulfilled then

$$\vec{\alpha}_m^0 = \vec{p}_v^0 \tag{14.40}$$

If (14.38) is not fulfilled then $\vec{\alpha}_m^0$ is given by (14.13) and belongs to the boundary of the set specified by (14.31). This boundary is defined as follows:

k components of the vector $\vec{g}$ $= 0$ $(1 \le k \le n)$;
other $n - k$ components of the vector $\vec{g}$ > 0

$\Rightarrow$ in the boundary the constraint (14.31) can be written as

$$\vec{g}' = \vec{g}'\left(\vec{r}, \vec{v}, t, \vec{\alpha}^0\right) = \vec{0} \tag{14.41}$$

$$\vec{g}'' = \vec{g}''\left(\vec{r}, \vec{v}, t, \vec{\alpha}^0\right) > \vec{0} \tag{14.42}$$

where $\vec{g}', \vec{g}''$ are some k- and $(n - k)$-dimensional subvectors of $\vec{g}$.

As was shown above the components of vector $\vec{\lambda}_g$ corresponding to the constraints (14.42) $= 0$

$\Rightarrow$ it is sufficient to consider equality constraint (14.41) for which the problem has been solved in Section 14.3.

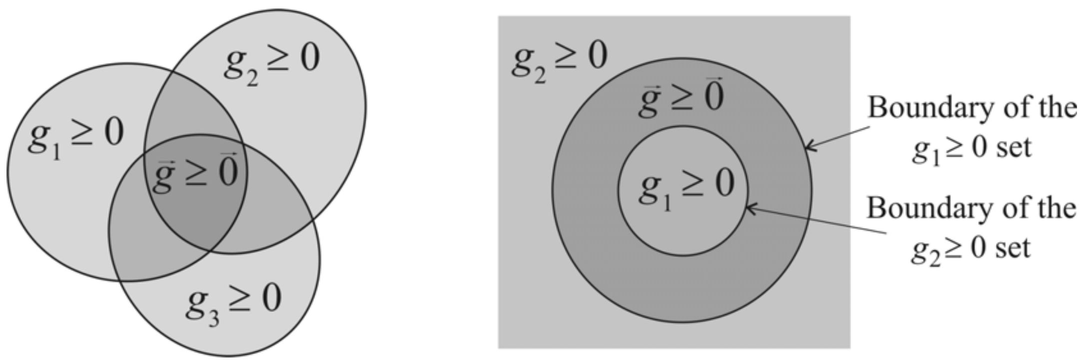
Thus, in case of the inequality constraint the optimal thrust direction is also given by (14.13) because if (14.38) is fulfilled then $\vec{p}_g = \vec{p}_v$.

14.5. An algorithm of finding optimal thrust direction for the inequality constraint

In practice finding vector $\vec{g}'$ for a given $\vec{p}_v$ may face difficulties. Assume that for any pair g_i, g_j simultaneous fulfillment of equalities

$$g_i = 0, \quad g_j = 0 \tag{14.43}$$

is possible either for finite number of $\vec{\alpha}^0$ values or impossible (this assumption is quite realistic because $\vec{\alpha}^0$ is defined by two independent scalar variables). This means that the boundaries of any pair of the sets specified by (14.32) either intersect in a finite number of points (see Fig. 14.1a) or does not intersect (see Fig. 14.1b).



a b

Fig. 14.1

According to lemmas B.1 and B.2 of Annex B vector $\vec{\alpha}_m^0$ is either given by (14.40) (if (14.38) is fulfilled) or is a unit vector of the projection of $\vec{p}_v$ onto one of the sets given by (14.32) (if $k = 1$) or is a solution of equations (14.43) (if $k \geq 2$). Thus, $\vec{\alpha}_m^0$ can be found by means of the following steps:

1° Inequality (14.38) is checked and if it is fulfilled then $\vec{\alpha}_m^0 = \vec{p}_v^0$.

2° If (14.38) is not fulfilled then all projections $\vec{p}_{gi}^{(1)}, \vec{p}_{gi}^{(2)}, \ldots$ of $\vec{p}_v$ onto the sets $g_i = 0$ ($i = 1, \ldots, n$) are determined and for all $\vec{p}_{gi}^{(\sigma)} \neq \vec{0}$ $(\sigma = 1, 2, \ldots)$ satisfying the inequality

$$\vec{g}\left(\vec{r}, \vec{v}, t, \vec{p}_{gi}^{(\sigma)0}\right) \geq \vec{0} \tag{14.44}$$

optimal thrust direction is selected as follows:

$$\vec{\alpha}_m^0 = \arg\max_{i,\sigma} \vec{p}_v^T \vec{p}_{gi}^{(\sigma)0} \tag{14.45}$$

3° If no one of the vectors $\vec{p}_{gi}^{(\sigma)} \neq \vec{0}$ satisfies (14.44) then for all pairs $i, j = 1, \ldots, n,\ i \neq j$, solutions $\vec{\alpha}_\sigma^0$ $(\sigma = 1, \ldots, M)$ of equations (14.43) are found and

$$\vec{\alpha}_m^0 = \arg\max_{\sigma} \vec{p}_v^T \vec{\alpha}_\sigma^0$$

4° If all $\vec{p}_{gi}^{(\sigma)} = \vec{0}$ $(i = 1, \ldots, n,\ \sigma = 1, 2, \ldots)$ then $\vec{\alpha}_m^0 = \vec{0}$.

Due to results of Annex B the suggested procedure of finding optimal thrust direction may be simplified in the following cases:

1. $n = 1$, i.e. $\vec{g} = g_1$ is a scalar. In this case $\vec{\alpha}_m^0 = \vec{p}_{g1}^0$ where $\vec{p}_{g1}$ is the <u>absolute</u> projection of $\vec{p}_v$ onto the set $g_1 = 0$.
2. A projection $\vec{p}_{gi}$ of $\vec{p}_v$ onto the set $g_i = 0$ is <u>absolute</u> and satisfies (14.44). Then $\vec{\alpha}_m^0 = \vec{p}_{gi}^0$ (see lemma B.3 of Annex B).
3. There is only one (i.e. <u>absolute</u>) projection $\vec{p}_{gj}$ of $\vec{p}_v$ onto the each of the sets $g_j = 0$ ($j = 1, \ldots, n$). In this case either exists a unique vector $\vec{p}_{gi}^0$ satisfying (14.44) and $\vec{\alpha}_m^0 = \vec{p}_{g1}^0$ or such a vector does not exist and $\vec{\alpha}_m^0$ is a solution of equations (14.43) (see lemma B.4 of Annex B).

14.6. Linear equality constraint

$$B\vec{\alpha}^0 = \vec{c} \tag{14.46}$$

where $B = B(\vec{r}, \vec{v}, t)$ is $n\times 3$-matrix, $\vec{c} = \vec{c}(\vec{r}, \vec{v}, t)$ is n-dimensional vector. Eq. (14.46) can be written as

$$\vec{b}_i^T \vec{\alpha}^0 = c_i \quad (i = 1, \ldots, n) \tag{14.47}$$

where $\vec{b}_i^T, c_i$ are rows of matrix B and components of vector $\vec{c}$.
Since $\vec{\alpha}^0$ is unit vector, (14.47) is only possible if $|c_i| \le |\vec{b}_i|$.

Assume that rank $B = n \le 3$ and consider different values of n:
1) $n = 1$: Eq. (14.46) gives a circle in a unit sphere;
2) $n = 2$: Eq. (14.46) gives two points a unit sphere which are intersections of two circles;
3) $n = 3$: it is only possible if $B^{-1}\vec{c}$ is a unit vector; in this case (14.46) defines a point in a unit sphere.

Consider $n \le 2$; Eq. (14.20) becomes

$$\vec{p}_v + \frac{1}{\alpha} B^T \vec{\lambda}_g = x\vec{\alpha}_m^0 \tag{14.48}$$

Multiplying (14.48) by matrix B taking into account (14.46) gives

$$\vec{\lambda}_g = -\alpha \left(BB^T\right)^{-1} \left(B\vec{p}_v - x\vec{c}\right) \tag{14.49}$$

Substituting (14.49) into (14.48) gives

$$P_0 \vec{p}_v + xB^T \left(BB^T\right)^{-1} \vec{c} = x\vec{\alpha}_m^0 \tag{14.50}$$

where

$$P_0 = I - B^T \left(BB^T\right)^{-1} B \tag{14.51}$$

Matrix (14.51) projects any vector onto the set orthogonal to B (i.e. is projective matrix, projector). It is easy to check that

$$P_0^T = P_0^2 = P_0, \quad BP_0 = P_0 B^T = 0 \tag{14.52}$$

Squaring (14.50) and using (14.52) and equality $|\vec{\alpha}^0| = 1$ give

$$x = \pm\sqrt{\frac{\vec{p}_v^T P_0 \vec{p}_v}{1-\vec{c}^T \left(BB^T\right)^{-1} \vec{c}}} \tag{14.53}$$

Designate

$$\vec{p} = P_0 \vec{p}_v, \quad p = |\vec{p}|, \quad \vec{p}^0 = \frac{\vec{p}}{p} \tag{14.54}$$

Due to (14.52, 14.54)

$$p = \sqrt{\vec{p}_v^T P_0 \vec{p}_v} \tag{14.55}$$

$\Rightarrow$ Eqs. (14.50, 14.53 – 14.55) give

$$\vec{\alpha}_m^0 = \pm\sqrt{1-\vec{c}^T \left(BB^T\right)^{-1} \vec{c}}\, \vec{p}^0 + B^T \left(BB^T\right)^{-1} \vec{c} \tag{14.56}$$

Sign in (14.56) is selected in order to provide $\max \vec{p}_v^T \vec{\alpha}^0$. Due to (14.54, 14.55) $\vec{p}_v^T \vec{p}^0 = p > 0$ $\Rightarrow$ taking into account (14.12) obtain:

$$\boxed{\vec{\alpha}_m^0 = \sqrt{1-\vec{c}^T \left(BB^T\right)^{-1} \vec{c}}\, \vec{p}^0 + B^T \left(BB^T\right)^{-1} \vec{c}} \tag{14.57}$$

if $\vec{p}_v^T \vec{\alpha}_m^0 > 0$ and $\vec{\alpha}_m^0 = \vec{0}$ if $\vec{p}_v^T \vec{\alpha}_m^0 \le 0$.

Optimal thrust magnitude is given by (14.23) for the LP case and by (14.1) taking into account (14.30) for the CEV case.

Consider case $n = 1$ (i.e. B is a row $\vec{b}_1^T$ and $\vec{c}$ is a scalar c_1). In this case $\vec{\alpha}$ belongs to the surface of a circular cone with axis directed along the vector $\vec{b}_1 \operatorname{sgn} c_1$. Designate $\vec{b}^0 = \vec{b}_1 / |\vec{b}_1|$ $\Rightarrow$ due to (14.51, 14.54)

$$\vec{p} = \left(I - \vec{b}^0 \vec{b}^{0T}\right) \vec{p}_v = \vec{b}^0 \times \left(\vec{p}_v \times \vec{b}^0\right) \tag{14.58}$$

Eq. (14.57) takes form

$$\vec{\alpha}_m^0 = \vec{p}^0 \sin\varphi + \vec{b}^0 \cos\varphi \tag{14.59}$$

where $\cos\varphi = |c_1| / |\vec{b}_1|$ (see Fig. 14.2).

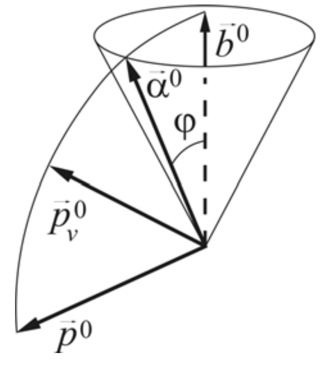


Fig. 14.2

14.7. Linear inequality constraint

$$B\vec{\alpha}^0 \geq \vec{c} \tag{14.60}$$

where $B = B(\vec{r}, \vec{v}, t)$ is $n\times3$-matrix, $\vec{c} = \vec{c}(\vec{r}, \vec{v}, t)$ is n-dimensional vector. Inequality (14.60) can be written as

$$\vec{b}_i^T \vec{\alpha}^0 \geq c_i \quad (i = 1, ..., n) \tag{14.61}$$

where $\vec{b}_i^T, c_i$ are rows of matrix B and components of vector $\vec{c}$. Each of the inequalities (14.61) specifies a segment of a unit sphere and (14.60) defines intersection of these segments (see Fig. 14.3).

Define

$$\vec{b}_i^0 = \frac{\vec{b}_i}{|\vec{b}_i|}, \quad B_{ij} = \begin{bmatrix} \vec{b}_i \\ \vec{b}_j \end{bmatrix}, \quad c_{ij} = \begin{bmatrix} c_i \\ c_j \end{bmatrix}, \tag{14.62}$$

$$\vec{p}_i = \vec{b}_i^0 \times \left(\vec{p}_v \times \vec{b}_i^0 \right), \quad \vec{p}_i^0 = \frac{\vec{p}_i}{|\vec{p}_i|} \tag{14.63}$$

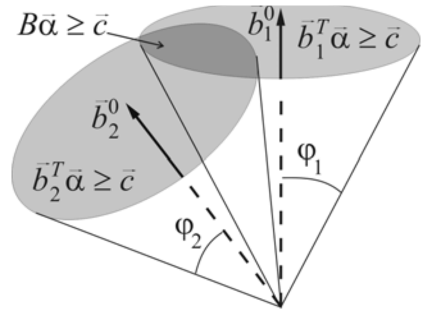


Fig. 14.3

$$P_{ij} = I - B_{ij}^T \left(B_{ij} B_{ij}^T \right)^{-1} B_{ij}, \quad \vec{p}_{ij} = P_{ij}\vec{p}_v, \quad \vec{p}_{ij}^0 = \frac{\vec{p}_{ij}}{|\vec{p}_{ij}|} \tag{14.64}$$

where $i, j = 1, \ldots, n$. If vector $\vec{p}_v$ is not collinear to any of $\vec{b}_i$ then $\vec{\alpha}_m^0$ can be found by the algorithm described in Section 14.5 where $\vec{p}_{gi}$ is reached in the vector

$$\vec{\alpha}_i^0 = \vec{p}_i^0 \sin\varphi_i + \vec{b}_i^0 \cos\varphi_i, \quad \cos\varphi_i = \frac{|c_i|}{|\vec{b}_i|} \tag{14.65}$$

and the solution of (14.43) is given by

$$\vec{\alpha}_{ij}^0 = \sqrt{1 - \vec{c}_{ij}^T \left(B_{ij} B_{ij}^T \right)^{-1} \vec{c}_{ij}}\, \vec{p}_{ij}^0 + B_{ij}^T \left(B_{ij} B_{ij}^T \right)^{-1} \vec{c}_{ij} \tag{14.66}$$

where $i, j = 1, \ldots, n$.

14.8. Linear homogeneous equality constraint

$$B\vec{\alpha} = \vec{0} \tag{14.67}$$

where $B = B(\vec{r}, \vec{v}, t)$ is $n\times 3$-matrix. Unit vector $\vec{\alpha}^0$ is replaced in (14.67) by $\vec{\alpha}$ for convenience. Two cases are possible:

1) $\operatorname{rank} B = 1 \quad \Rightarrow$ Eq. (14.67) defines a plane;

2) $\operatorname{rank} B = 2 \quad \Rightarrow$ Eq. (14.67) defines a straight line, i.e. the thrust direction is given and optimal thrust magnitude is to be found.

In the considered case $\vec{c} = \vec{0}$ in (14.46) $\Rightarrow$ due to (14.57)

$$\vec{\alpha}_m^0 = \vec{p}^0$$

$\Rightarrow$ due to (14.13) $\vec{p} = \vec{p}_g$ $\Rightarrow$ due to (14.15, 14.54) $P = P_0$, i.e. due to (14.51) matrix P in (14.15) is given by

$$\boxed{P = I - B^T\left(BB^T\right)^{-1}B} \tag{14.68}$$

Matrix (14.68) projects <u>any</u> vector onto the set given by (14.67), i.e. is a <u>projective matrix</u>. In a general case matrix (14.68) cannot be represented in the form (14.14).

14.9. Linear homogeneous inequality constraint

$$B\vec{\alpha} \geq \vec{0} \tag{14.69}$$

where $B = B(\vec{r}, \vec{v}, t)$ is $n\times3$-matrix. Inequality (14.69) defines intersection of semispaces each of which is defined by inequality

$$\vec{b}_i^T \vec{\alpha} \geq 0 \quad (i = 1, \ldots, n) \tag{14.70}$$

where $\vec{b}_i^T$ are rows of matrix B (see Fig. 14.4). According to the algorithm described in Section 14.5 the optimal thrust direction $\vec{\alpha}_m^0$ is given by one of the vectors $\vec{p}_i^0$ or $\vec{p}_{ij}^0$ given by (14.63, 14.64) for $i, j = 1, \ldots, n$ where $\vec{p}_{ij}^0$ is a solution to equations (14.43). There is unique vector $\vec{p}_i^0$ for any $i = 1, \ldots, n$ $\Rightarrow$ according to lemma B.4 of Annex B if $\exists$ i: $B\vec{p}_i^0 \geq \vec{0}$ then $\vec{\alpha}_m^0 = \vec{p}_i^0$.

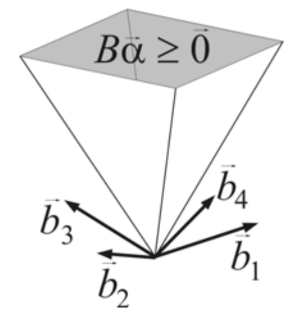


Fig. 14.4

<u>Conclusion</u>: In the case of constraint (14.69) optimal thrust is directed along vector $\vec{p}_v$ (if $B\vec{p}_v \geq \vec{0}$) or orthogonal to a vector $\vec{b}_i$ or to two vectors $\vec{b}_i, \vec{b}_j$ or equal to zero (if $\vec{p}_v^T \vec{\alpha} \leq 0$ for any $\vec{\alpha}$ satisfying (14.69)).

14.10. Matrices B, P_0 and P for linear equality constraints

Matrix BB^T in (14.51, 14.68) is singular if R = rank $B < n$. Consider matrices B, P_0, P for two possible values of R:

1. $R = 1 \Rightarrow$ for any n matrix B may be replaced by the matrix

$$B = \vec{b}^T \tag{14.71}$$

where $\vec{b}^T$ is any non-zero row of the initial matrix B

$\Rightarrow \left(BB^T\right)^{-1} = 1/b^2$, $b = |\vec{b}| \Rightarrow$ Eqs. (14.51, 14.68) give

$$P_0 = P = I - \frac{\vec{b}\vec{b}^T}{b^2} \tag{14.72}$$

2. $R = 2$; if $n = 2$ then matrix B is

$$B = \begin{bmatrix} \vec{b}_1^T \\ \vec{b}_2^T \end{bmatrix} \tag{14.73}$$

where $\vec{b}_1, \vec{b}_2$ are linearly independent vectors. If $n > 2$ then matrix B can be reduced to the view (14.73) and matrix BB^T is not singular and P_0, P are given by (14.51, 14.68).

However, matrix P can be simplified in the following way: if (14.67, 14.73) are fulfilled then $\vec{\alpha}_m^0 = \vec{b}/b$ where $\vec{b} = \vec{b}_1 \times \vec{b}_2$, $b = |\vec{b}| \Rightarrow$ due to (14.14)

$$P = \frac{\vec{b}\vec{b}^T}{b^2} \tag{14.74}$$

Note that matrix (14.73) may be replaced by the matrix

$$B = I - \frac{\vec{b}\vec{b}^T}{b^2} \tag{14.75}$$

Thus, all linear homogeneous equality constraints on the thrust direction are reduced to two cases:

1) thrust is orthogonal to a given vector $\vec{b} = \vec{b}(\vec{r}, \vec{v}, t)$, and in this case B and P are given by (14.71, 14.72);
2) thrust is directed along a given vector $\vec{b} = \vec{b}(\vec{r}, \vec{v}, t)$, and in this case B and P are given by (14.73 or 14.75, 14.74).

14.11. Unions of sets and mixed constraints

Unions of sets

All considered constraints (14.6, 14.31, 14.46, 14.60, 14.67, 14.69) give intersection of sets. Although the results can be generalized to the constraints given as a union of sets $A_1 \cup A_2 \cup \ldots \cup A_n$.

Consider an example: constraint given by

$$\vec{b}^T \vec{\alpha}^0 \geq c \quad \text{or} \quad -\vec{b}^T \vec{\alpha}^0 \geq c \tag{14.76}$$

where $c > 0$ (constraint (14.76) corresponds, for instance, to the case when there are two oppositely directed thrusters onboard the spacecraft). In this case the thrust vector lies inside the circular cone shown in Fig. 5a; Fig. 5b shows this case schematically.

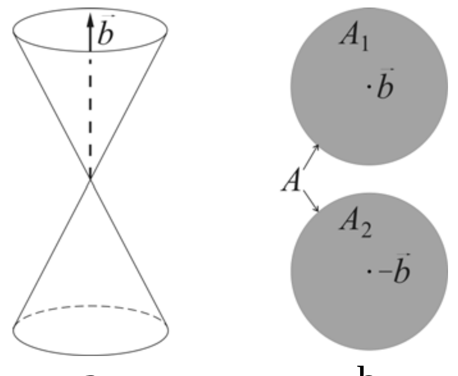


Fig. 14.5

Due to (14.59)

$$\vec{\alpha}_m^0 = \vec{p}^0 \sin\varphi \pm \vec{b}^0 \cos\varphi \tag{14.77}$$

For the optimality sign in (14.77) should be equal to the sign of $\vec{p}_v^T \vec{b}$.

Mixed constraints

The obtained results also can be generalized to the case of mixed equality and inequality constraints.

Consider an example: constraint given by

$$\vec{b}_1^T \vec{\alpha}^0 = c_1 \quad \text{and} \quad \vec{b}_2^T \vec{\alpha}^0 \geq c_2 \tag{14.78}$$

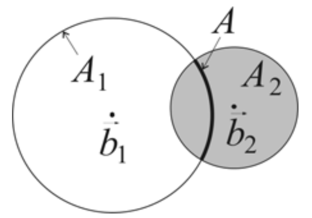


Fig. 14.6

(see Fig. 14.6). In this case the optimal thrust direction is either a projection onto the set $\vec{b}_1^T \vec{\alpha}^0 = c_1$ (i.e. is given by (14.65) with $i = 1$) or one of the intersection points of the sets $\vec{b}_1^T \vec{\alpha}^0 = c_1$, $\vec{b}_2^T \vec{\alpha}^0 = c_2$ (i.e. is given by (14.66) with $i = 1$, $j = 2$).

15. Optimization of Low Thrust Transfers

15.1. Statement of the problem and notation

A spacecraft transfer with the electric propulsion between two given positions in a given time is considered. Two-body problem and power-limited thrust are assumed.

The problem is to find the thrust vector as a function of time minimizing the propellant consumption.

Designate:

μ = gravitational parameter of the attracting center;

t = current time;

t_0, t_1 = instants of the transfer beginning and end;

$T = t - t_0$, $T_1 = t_1 - t_0$;

$\vec{r}, \vec{v}$ = position and velocity vectors,

$\vec{r}_0 = \vec{r}(t_0)$, $\vec{v}_0 = \vec{v}(t_0)$,

$r = |\vec{r}|$, $v = |\vec{v}|$,

$r_0 = r(t_0)$, $v_0 = v(t_0)$;

$\vec{x} = \{\vec{r}, \vec{v}\}$ = state vector;

$\dot{r} = \dfrac{\vec{r} \cdot \vec{v}}{r}$ = radial velocity,

$\dot{r}_0 = \dot{r}(t_0)$;

$\varphi = \varphi(t)$ = transfer angle including complete orbits,

$\cos\varphi = \dfrac{\vec{r}_0 \cdot \vec{r}_1}{r_0 r_1}$;

a = semimajor vector;

e = eccentricity;

$p = a(1 - e^2)$ = semilatus rectum;

$h = v^2 - \dfrac{2\mu}{r}$ = integral of energy;

$\vec{c} = \vec{r} \times \vec{v}$ = angular momentum (integral of areas);

$\vec{l} = -\mu \dfrac{\vec{r}}{r} + \vec{v} \times \vec{c}$ = Laplace integral;

$c = |\vec{c}| = \sqrt{\mu p}$, $l = |\vec{l}| = \mu e$;

$\vec{\alpha} = \vec{\alpha}(t)$ = thrust vector,

$\vec{\alpha}_0 = \vec{\alpha}(t_0)$,

$\alpha = |\vec{\alpha}|$;

W_e = effective power of the thrust (see Section 13.1);

$m = m(t)$ = spacecraft mass,

$m_0 = m(t_0)$, $m_1 = m(t_1)$;

$m_p = m_0 - m_1$ = propellant mass;

I = unit matrix of 3rd order.

15.2. Formalization of the problem

Equation of the spacecraft motion is

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \vec{g}\,, \tag{15.1}$$

$$\vec{f}(\vec{x}) = \left\{\vec{v},\, -\frac{\mu}{r^3}\vec{r}\right\},\ \vec{g} = \left\{\vec{0},\, \vec{\alpha}\right\} \tag{15.2}$$

Assume the boundary values to be given as follows:

$$\vec{x}(t_0) = \vec{x}_0, \quad \vec{x}(t_1) = \vec{x}_1 \tag{15.3}$$

(cases of partly given boundary values are considered in Section 15.7).

Consider unperturbed Keplerian motion described by equation

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \tag{15.4}$$

and let $\vec{y} = \vec{y}(t)$ be a solution of Eq. (15.4) with given boundary values

$$\vec{y}(t_0) = \vec{y}_0, \quad \vec{y}(t_1) = \vec{y}_1 \tag{15.5}$$

Note that if the positions in state vectors $\vec{y}_0, \vec{y}_1$ are given then velocities can be found by means of the Lambert problem solution (see Chapter 7).

Represent solution of Eq. (15.1) with boundary values (15.3) in the form

$$\vec{x} = \vec{y} + \vec{\xi} \tag{15.6}$$

and assume that values (15.5) provide

$$\left\|\vec{\xi}\right\| << \left\|\vec{y}\right\| \tag{15.7}$$

((15.7) may be possible because the thrust is low).

Linearizing Eq. (15.1) for $\vec{\xi}$ obtain:

$$\dot{\vec{\xi}} = F\vec{\xi} + \vec{g}\,, \tag{15.8}$$

$$F = \frac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix},\ G = \frac{\mu}{r^3}\left(3\frac{\vec{r}\vec{r}^T}{r^2} - I\right) \tag{15.9}$$

Boundary values for $\vec{\xi}$ are

$$\vec{\xi}(t_0) = \vec{x}_0 - \vec{y}_0 = \vec{\xi}_0, \quad \vec{\xi}(t_1) = \vec{x}_1 - \vec{y}_1 = \vec{\xi}_1 \tag{15.10}$$

The Keplerian orbit given by $\vec{y} = \vec{y}(t)$ is called reference orbit or transporting trajectory. Matrices (15.9) are calculated in this orbit.

The performance index for the limited power (LP) thrust is

$$J = \frac{1}{2}\int_{t_0}^{t_1} \frac{\alpha^2}{W_e} dt \tag{15.11}$$

(see Section 13.4) $\Rightarrow$ using (13.13) obtain:

$$m_1 = \frac{\tilde{W}_{e0}}{J + \tilde{W}_{e0}} m_0, \quad m_p = \frac{J}{J + \tilde{W}_{e0}} m_0, \tag{15.12}$$

$$\tilde{W}_{e0} = \frac{W_{e0}}{m_0}, \quad W_{e0} = W_e(t_0)$$

Eq. (15.8) is non autonomous since matrix F is a function of time. Hence Hamiltonian of the linearized problem is

$$H = -\frac{\alpha^2}{2W_e} + \vec{p}^T F \vec{\xi} + \vec{p}_v^T \vec{\alpha} + p_t \tag{15.13}$$

where

$$\vec{p} = \{\vec{p}_r, \vec{p}_v\} \tag{15.14}$$

is adjoint (costate) variable, $\vec{p}_v$ is Lawden's primer vector, p_t is an adjoint variable for the additional equation $\dot{t} = 1$ making the system autonomous (see Chapters 12, 13).

15.3. Solution of the linearized problem

Solution to Eq. (15.8) is given by the Cauchy formula

$$\vec{\xi}(t) = \Phi(t, t_0)\vec{\xi}_0 + \int_{t_0}^{t} \Phi(t, \tau)\vec{g}\,d\tau \tag{15.15}$$

where Φ is state transition matrix which can be represented as

$$\Phi(t, t_0) = A^{-1}A_0 \tag{15.16}$$

and matrix $A = A(t)$ is a general solution of the adjoint (costate) variational equation

$$\dot{A} = -AF\,, \quad A(t_0) = A_0 \tag{15.17}$$

(see Chapter 11). Matrix A is given by (11.31, 11.33) where $\vec{r}, \vec{v}$ are components of vector $\vec{y}$ (i.e. are calculated in the reference orbit), matrix A^{-1} is calculated in Section 11.9.
Like in Section 11.5, divide A into two 6×3-sub-matrices:

$$A = \begin{bmatrix} P & Q \end{bmatrix} \tag{15.18}$$

Due to (15.2, 15.16, 15.18) Eq. (15.15) becomes

$$\vec{\xi} = A^{-1}A_0\vec{\xi}_0 + A^{-1}\int_{t_0}^{t} Q\vec{\alpha}\,d\tau \tag{15.19}$$

Adjoint variable in (15.13) satisfies the equation

$$\dot{\vec{p}}^T = -\frac{\partial H}{\partial \vec{x}} = -\vec{p}^T F \tag{15.20}$$

$\Rightarrow$ since A is a general solution of Eq. (15.20),

$$\vec{p} = A^T\vec{\beta}$$

where $\vec{\beta}$ is a constant vector $\Rightarrow$ due to (15.14, 15.18)

$$\vec{p}_v = Q^T\vec{\beta} \tag{15.21}$$

Optimal thrust provides maximum to function (15.13) $\Rightarrow$ $\dfrac{\partial H}{\partial \vec{\alpha}} = \vec{0}^T$

$$\Rightarrow \quad \boxed{\vec{\alpha} = W_e \vec{p}_v = W_e Q^T \vec{\beta}} \tag{15.22}$$

(see also Chapter 13). Define

$$\vec{\Delta} = A_1 \vec{\xi}_1 - A_0 \vec{\xi}_0, \tag{15.23}$$

$$S = S(t_0, t) = \int_{t_0}^{t} W_e Q Q^T d\tau, \quad S_1 = S(t_0, t_1) \tag{15.24}$$

where $A_1 = A(t_1)$ $\Rightarrow$ due to (15.22 – 15.24) Eq. (15.19) for $t = t_1$ takes form

$$\vec{\Delta} = S_1 \vec{\beta}$$

$$\Rightarrow \quad \boxed{\vec{\beta} = S_1^{-1} \vec{\Delta}} \tag{15.25}$$

$\Rightarrow$ due to (15.22)

$$\boxed{\vec{\alpha} = W_e Q^T S_1^{-1} \vec{\Delta}} \tag{15.26}$$

Substituting (15.26) for $\alpha^2 = \vec{\alpha}^T \vec{\alpha}$ into (15.11) and taking into account (15.24) obtain:

$$\boxed{J = \frac{1}{2} \vec{\Delta}^T S_1^{-1} \vec{\Delta}} \tag{15.27}$$

Eqs. (15.19, 15.24, 15.26) give state vector $\vec{\xi} = \vec{\xi}(t)$:

$$\boxed{\vec{\xi} = A^{-1}\left(A_0 \vec{\xi}_0 + S S_1^{-1} \vec{\Delta}\right)} \tag{15.28}$$

Then state vector $\vec{x} = \vec{x}(t)$ may be found from (15.6).

15.4. Calculation of matrix S

Designate

$$B = QQ^T \tag{15.29}$$

$\Rightarrow$ due to (11.31) (see Chapter 11)

$$B_{ij} = \vec{b}_i^T \vec{b}_j \quad (i, j = 1,...,6)$$

where vectors $\vec{b}_i$ are given by (11.33). Matrix B has a view

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{33} & B_{34} & B_{35} & B_{36} \\ 0 & 0 & B_{43} & B_{44} & B_{45} & B_{46} \\ 0 & 0 & B_{53} & B_{54} & B_{55} & B_{56} \\ 0 & 0 & B_{63} & B_{64} & B_{65} & B_{66} \end{bmatrix} \tag{15.30}$$

Vectors $\vec{p}_1, \vec{p}_2$ in (11.33) given by (11.52) will be used. Then non-zero components of matrix B are

$$B_{11} = -\frac{p}{e^2}\left(\frac{r^2}{a} - 2r + p\right), \quad B_{12} = B_{21} = -\frac{1}{e^2}\sqrt{\frac{p}{\mu}}(r-p)r\dot{r}, \quad B_{22} = \frac{(r-p)^2}{e^2},$$

$$B_{33} = \frac{c^2}{e^2}\left(\frac{r^2}{a^2} + 2\frac{p}{r} - 3\frac{p}{a}\right), \quad B_{34} = B_{43} = \frac{c}{e^2}\left(\frac{r+p}{a}r - 2p\right)\dot{r},$$

$$B_{35} = B_{53} = -2\frac{c^2}{e}\left(\frac{1}{r} - \frac{1}{a}\right), \quad B_{36} = B_{63} = -2\frac{p}{e}\left[r\dot{r} - 3\mu\left(\frac{1}{r} - \frac{1}{a}\right)T\right],$$

$$B_{44} = -\frac{\mu}{e^2}\left[\frac{r^2}{a} - 2e^2 r + 2\frac{p^2}{r} - \left(3 + e^2\right)p\right], \quad B_{45} = B_{54} = 2\frac{c}{e}\dot{r},$$

$$B_{46} = B_{64} = 2\frac{c}{e}\left(r - p - 3\dot{r}T\right), \quad B_{55} = v^2, \quad B_{56} = B_{65} = 2r\dot{r} - 3v^2 T,$$

$$B_{66} = 4r^2 - 12r\dot{r}T + 9v^2T^2 \tag{15.31}$$

Consider integrals

$$R_n = \int_{t_0}^{t} r^n dt$$

(see also Section 11.8). It can be found that

$$R_{-3} = \frac{1}{p}\left(R_{-2} + \frac{\dot{r} - \dot{r}_0}{\mu}\right), \quad R_{-2} = \frac{\varphi}{c}, \quad R_{-1} = \frac{T}{a} + \frac{r\dot{r} - r_0\dot{r}_0}{\mu}, \quad R_0 = T,$$

$$R_n = \frac{a}{n+1}\left[(2n+1)R_{n-1} - npR_{n-2} - \frac{r^{n+1}\dot{r} - r_0^{n+1}\dot{r}_0}{\mu}\right], \quad n = 1,2,... \tag{15.32}$$

Also consider integrals

$$I_n = \int_{t_0}^{t} r^n \dot{r} dt = \begin{cases} \dfrac{r^{n+1} - r_0^{n+1}}{n+1}, & n = 0, 1, \pm 2, \pm 3, ... \\ \ln\dfrac{r}{r_0}, & n = -1 \end{cases} \tag{15.33}$$

$$K_1 = \int_{t_0}^{t} \varphi dt, \quad K_2 = \int_{t_0}^{t} \varphi T dt, \quad K_3 = \int_{t_0}^{t} \ln\frac{r}{r_0} dt \tag{15.34}$$

and define parameters

$$k_1 = \frac{I_0 - \dot{r}_0 T}{p}, \quad k_2 = \mu\left(2R_{-3} - \frac{R_{-2}}{a}\right), \quad k_3 = \frac{\sqrt{\mu}}{p^{3/2}}\left(1 + e^2\right) \tag{15.35}$$

Define matrix

$$\tilde{S} = \frac{S}{W_{e0}} \tag{15.36}$$

Constant power

Consider constant power, i.e.

$$W_e \equiv W_{e0} \tag{15.37}$$

Non-zero components of matrix (15.36) can be found from (15.24, 15.29, 15.31, 15.37) as follows:

$$\tilde{S}_{11} = -\frac{p}{e^2}\left(\frac{R_2}{a} - 2R_1 + pT\right),\ \ \tilde{S}_{12} = \tilde{S}_{21} = -\frac{1}{e^2}\sqrt{\frac{p}{\mu}}\left(I_2 - pI_1\right),$$

$$\tilde{S}_{22} = \frac{p}{e^2}\left(\frac{R_2}{p} - 2R_1 + pT\right),\ \ \tilde{S}_{33} = \frac{c^2}{e^2}\left(\frac{R_2}{a^2} + 2pR_{-1} - 3\frac{p}{a}T\right),$$

$$\tilde{S}_{34} = \tilde{S}_{43} = \frac{c}{e^2}\left(\frac{I_2 + pI_1}{a} - 2pI_0\right),\ \ \tilde{S}_{35} = \tilde{S}_{53} = -2\frac{c^2}{e}\left(R_{-1} - \frac{T}{a}\right),$$

$$\tilde{S}_{36} = \tilde{S}_{63} = -2\frac{p}{e}\left(4I_1 - 3r\dot{r}T\right),$$

$$\tilde{S}_{44} = -\frac{\mu}{e^2}\left[\frac{R_2}{a} - 2e^2R_1 + 2p^2R_{-1} - \left(3 + e^2\right)pT\right],$$

$$\tilde{S}_{45} = \tilde{S}_{54} = 2\frac{c}{e}I_0,\ \ \tilde{S}_{46} = \tilde{S}_{64} = 2\frac{c}{e}\left[4R_1 - (3r + p)T\right],\ \ \tilde{S}_{55} = 2\mu R_{-1} - \frac{\mu}{a}T,$$

$$\tilde{S}_{56} = \tilde{S}_{65} = 8I_1 - \frac{3}{2}\frac{\mu T^2}{a} - 6r\dot{r}T,\ \ \tilde{S}_{66} = 28R_2 + 3\frac{\mu T^3}{a} + 18r\dot{r}T^2 - 24r^2T \tag{15.38}$$

Solar power

Consider solar power supply with

$$W_e = W_{e0}\left(\frac{r_0}{r}\right)^2 \tag{15.39}$$

(see also Section 13.3). Non-zero components of matrix (15.36) can be found from (15.24, 15.29, 15.31, 15.39) as follows:

$$\tilde{S}_{11} = -\frac{p}{e^2}\left(\frac{T}{a} - 2R_{-1} + pR_{-2}\right)r_0^2, \quad \tilde{S}_{12} = \tilde{S}_{21} = -\frac{1}{e^2}\sqrt{\frac{p}{\mu}}\left(I_0 - pI_{-1}\right)r_0^2$$

$$\tilde{S}_{22} = \frac{p}{e^2}\left(\frac{T}{p} - 2R_{-1} + pR_{-2}\right)r_0^2, \quad \tilde{S}_{33} = \frac{c^2}{e^2}\left(\frac{T}{a^2} + 2pR_{-3} - 3\frac{p}{a}R_{-2}\right)r_0^2,$$

$$\tilde{S}_{34} = \tilde{S}_{43} = \frac{c}{e^2}\left(\frac{I_0 + pI_{-1}}{a} - 2pI_{-2}\right)r_0^2, \quad \tilde{S}_{35} = \tilde{S}_{53} = -2\frac{c^2}{e}\left(R_{-3} - \frac{R_{-2}}{a}\right)r_0^2,$$

$$\tilde{S}_{36} = \tilde{S}_{63} = -6\frac{p}{e}\left[k_1 + \frac{I_{-1}}{3} - \left(R_{-3} - \frac{R_{-2}}{a}\right)\mu T + \frac{\sqrt{\mu}}{p^{3/2}}e^2 K_1\right]r_0^2,$$

$$\tilde{S}_{44} = -\frac{\mu}{e^2}\left[\frac{T}{a} + 2p^2 R_{-3} - \left(3 + e^2\right)pR_{-2} - 2e^2 R_{-1}\right]r_0^2,$$

$$\tilde{S}_{45} = \tilde{S}_{54} = 2\frac{c}{e}I_{-2}\, r_0^2, \quad \tilde{S}_{46} = \tilde{S}_{64} = 2\frac{c}{e}\left(3\frac{T}{r} - pR_{-2} - 2R_{-1}\right)r_0^2, \quad \tilde{S}_{55} = k_2\, r_0^2,$$

$$\tilde{S}_{56} = \tilde{S}_{65} = \left[6k_1 + 2I_{-1} - 3k_2 T + 3k_3 K_1\right]r_0^2,$$

$$\tilde{S}_{66} = \left[4\left(1 - 9\frac{r}{p} - 3I_{-1}\right)T + 9\left(2\frac{\dot{r}_0}{p} + k_2\right)T^2 + 36\frac{R_1}{p} - 18k_3K_2 + 12K_3\right]r_0^2 \tag{15.40}$$

15.5. Properties of the solution

Theorem 15.1. Matrix (15.29) is singular, although matrix $S = S(t, t + \Delta t)$ is non-singular positive definite for any values of t and $\Delta t > 0$.

Proof

Q is 6×3-matrix $\Rightarrow$ rank $QQ^T = 3$ $\Rightarrow$ matrix (15.29) is singular.
Assume matrix S to be singular $\Rightarrow$ $\exists$ vector $\vec{s}: S\vec{s} = 0$

$$\Rightarrow \quad \vec{s}^{\,T} S\vec{s} = \int_t^{t+\Delta t} W_e \, \vec{s}^{\,T} Q Q^T \vec{s} \, dt = 0$$

On the other hand $W_e > 0$ and $\vec{s}^{\,T} QQ^T \vec{s}$ is non-negative continious function of time $\Rightarrow$ $Q^T \vec{s} \equiv \vec{0}$ in the interval Δt

$$\Rightarrow \frac{d}{dt}\left(Q^T \vec{s}\right) = \dot{Q}^T \vec{s} \equiv \vec{0}.$$

Eqs. (15.17, 15.9, 15.18) give $\dot{Q} = -P$ $\Rightarrow$ $P^T \vec{s} = \vec{0}$ $\Rightarrow$ $A^T \vec{s} = \vec{0}$ what is impossible since A is a non-singular matrix.

This contradiction proves that matrix S is non-singular and for $\forall \vec{s}: \ \vec{s}^{\,T} S\vec{s} > 0 \ \Rightarrow \ S$ is positive definite.

Thus, inversion of S in (15.25 – 15.28) is always possible.

Theorem 15.2. 1°. Optimal thrust may be zero only in isolated points and in these points the thrust changes its direction to the opposite one, i.e. the points are switching points.

2°. If $r \geq r_{\min} > 0$ then number of the switching points is finite.

Proof (only for 1°)

Assume that $\vec{\alpha} = \vec{0}$ in interval $\Delta t > 0$ $\Rightarrow$ due to (15.21) $\vec{p}_v = \vec{0}$ in Δt $\Rightarrow$ $\dot{\vec{p}}_v = \vec{0}$ in Δt $\Rightarrow$ Eqs. (15.22, 15.17, 15.9, 15.18) give $Q^T \vec{\beta} = \vec{0}$, $\dot{Q}^T \vec{\beta} = -P\vec{\beta} = \vec{0}$ $\Rightarrow$ $A^T \vec{\beta} = \vec{0}$ in Δt what is impossible since A is a non-singular matrix $\Rightarrow$ thrust may be zero only in isolated points.

Since $\vec{p}_v = \dot{\vec{p}}_v = \vec{0}$ is impossible, $\dot{\vec{p}}_v \neq \vec{0}$ when $\vec{p}_v = \vec{0}$
$\Rightarrow$ points of zero thrust are switching points.

15.6. A comment about non-zero boundary offsets

Represent boundary vectors in the form

$$\vec{\xi}_0=\left\{\vec{\rho}_0,\vec{\eta}_0\right\},\ \vec{\xi}_1=\left\{\vec{\rho}_1,\vec{\eta}_1\right\} \tag{15.41}$$

where subvectors $\vec{\rho},\vec{\eta}$ correspond to position and velocity respectively. The boundary positions in the reference orbit (transporting trajectory) (i.e. positions in (15.5)) may be taken equal to ones in the transfer trajectory (i.e. in (15.3)) $\Rightarrow$ in this case

$$\vec{\rho}_0=\vec{\rho}_1=\vec{0} \tag{15.42}$$

(see Fig. 15.1) $\Rightarrow$ boundary vector (15.23) will become

$$\vec{\Delta}=Q_1\vec{\eta}_1-Q_0\vec{\eta}_0 \tag{15.43}$$

However, it may be reasonable to select non-zero boundary offsets $\vec{\rho}_0,\vec{\rho}_1$ minimizing $\max\limits_{t_0\le t\le t_1}\left\|\vec{\xi}\right\|$ ($\vec{\xi}=\vec{\xi}(t)$ is given by (15.28)) in order to raise the linearization accuracy (see Fig. 15.2).

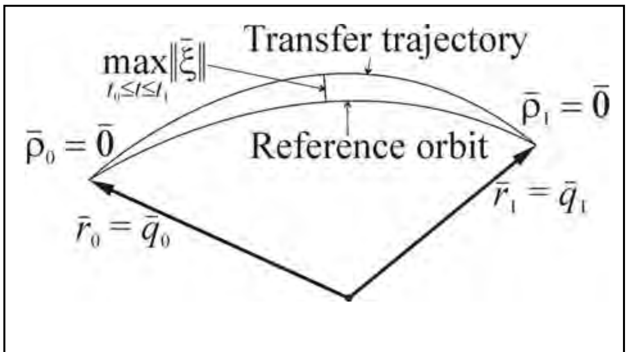


Fig. 15.1

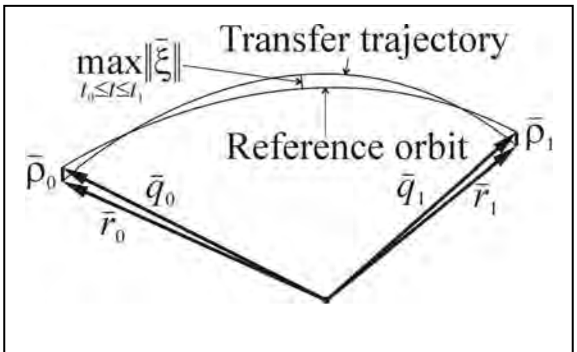


Fig. 15.2

15.7. Partly given boundary values

Consider two cases of partly given boundary values:

1. <u>Goal of the transfer is a celestial body encounter with any flyby velocity</u>

   $\Rightarrow$ $\vec{\rho}_0, \vec{\eta}_0, \vec{\rho}_1$ in (15.41) are given and $\vec{\eta}_1$ can take any value

   $\Rightarrow$ transversality condition is

   $$\vec{\alpha}_1 = \vec{\alpha}(t_1) = \vec{0} \tag{15.44}$$

   (see Section 12.4) $\Rightarrow$ Eqs. (15.26, 15.23, 15.44) give

   $$\vec{\eta}_1 = \left(Q_1^T S_1^{-1} Q_1\right)^{-1} Q_1^T S_1^{-1} \left(A_0 \vec{\xi}_0 - P_1 \vec{\rho}_1\right) \tag{15.45}$$

   If (15.42) is filfilled then (15.45) becomes

   $$\vec{\eta}_1 = \left(Q_1^T S_1^{-1} Q_1\right)^{-1} Q_1^T S_1^{-1} Q_0 \vec{\eta}_0 \tag{15.46}$$

2. <u>Launch energy is given and launch direction may be selected arbitrarily</u>

   Assume launch from Earth with given launch energy at infinity $C_3 = v_\infty^2$ (see Section 8.2) and arbitrary direction of the vector $\vec{v}_\infty$. Let $\vec{r}_0, \vec{V}_0$ be Earth position and velocity at t_0 and

   $$\vec{y}_0 = \{\vec{q}_0, \vec{u}_0\}, \qquad \vec{\xi}'_0 = \{\vec{\rho}_0, \vec{\eta}'_0\} = \{\vec{r}_0 - \vec{q}_0, \vec{V}_0 - \vec{u}_0\} \tag{15.47}$$

   $$\Rightarrow \vec{\xi}_0 = \vec{\xi}'_0 + \{\vec{0}, \vec{v}_\infty\}, \tag{15.48}$$

   $$\vec{v}_\infty \cdot \vec{v}_\infty = C_3 \tag{15.49}$$

   Transversality condition is

   $$\vec{\alpha}_0 = \lambda \vec{v}_\infty \tag{15.50}$$

   where λ is an undetermined multiplier $\Rightarrow$ Eqs. (15.26, 15.23, 15.50) give

   $$\vec{v}_\infty = \left(\lambda I + Q_0^T S_1^{-1} Q_0\right)^{-1} Q_0^T S_1^{-1} \left(A_1 \vec{\xi}_1 - P_0 \vec{\rho}_0 - Q_0 \vec{\eta}'_0\right) \tag{15.51}$$

   λ may be found from (15.49) after substituting (15.51) into there.

15.8. Thrust vector in a moving frame

Represent the primer vector in a given frame as $\vec{p}_v = \{p_1, p_2, p_3\}$.

Orbital frame: origin in the spacecraft and axes are directed along the radius-vector, transversal and normal to the orbit plane

$\Rightarrow$ unit basis vectors are

$$\vec{e}_1 = \frac{\vec{r}}{r},\ \ \vec{e}_2 = \frac{\vec{c}\times\vec{r}}{cr},\ \ \vec{e}_3 = \frac{\vec{c}}{c} \tag{15.52}$$

Applying dot product by (15.52) to (15.21) and taking into account (11.33, 11.52) obtain:

$$p_1 = \beta_3 \frac{p\dot{r}}{e} + \beta_4 \frac{c}{e}\left(\frac{p}{r} - 1\right) + \beta_5 \dot{r} + \beta_6 (2r - 3\dot{r}T),$$

$$p_2 = \beta_3 \frac{c}{e}\left(\frac{r}{a} - \frac{p}{r}\right) + \beta_4 \frac{r+p}{e}\dot{r} + \beta_5 \frac{c}{r} - 3\beta_6 \frac{c}{r}T,$$

$$p_3 = \beta_1 \frac{cr\dot{r}}{\mu e} + \beta_2 \frac{p-r}{e} \tag{15.53}$$

Tangential frame: origin in the spacecraft and axes are directed along the velocity vector, normal to it in the orbit plane and normal to the orbit plane

$\Rightarrow$ unit basis vectors are

$$\vec{e}_1 = \frac{\vec{v}}{v},\ \ \vec{e}_2 = \frac{\vec{v}\times\vec{c}}{vc},\ \ \vec{e}_3 = \frac{\vec{c}}{c} \tag{15.54}$$

Applying dot product by (15.54) to (15.21) and taking into account (11.33, 11.54) obtain:

$$p_1 = -2\beta_4 \frac{c^2}{\mu e v}\left(v^2 - \frac{\mu}{r}\right) - 2\beta_4 \frac{c\dot{r}}{ev} + \beta_5 v + \beta_6\left(2\frac{r\dot{r}}{v} - 3vT\right),$$

$$p_2 = -\beta_4 \frac{cr\dot{r}}{aev} + \beta_4 \frac{c^2 - (r\dot{r})^2}{erv} + 2\beta_6 \frac{c}{v} \tag{15.55}$$

Third component of the primer vector is given by (15.53).

15.9. Providing desired accuracy

The suggested method of low-thrust transfer optimization is approximate due to the linearization. In order to provide high accuracy of the method the time interval of linearization should be short.

Divide the time of flight T into n subintervals with boundaries $t_0, t_1, \ldots, t_{n-1}, t_n$ (with t_n replacing former t_1) and solve the problem for each subinterval separately. Subscript i and superscripts "–" or "+" will mark values of parameters at time t_i in the ith or (i +1)th subinterval respectively ($i = 1, \ldots, n-1$).

Values (15.11, 15.27, 15.23, 15.24) for ith subinterval are

$$J_i = \frac{1}{2}\int_{t_{i-1}}^{t_i} \frac{\alpha^2}{W_e} dt = \frac{1}{2}\vec{\Delta}_i^T S_i^{-1}\vec{\Delta}_i \tag{15.56}$$

$$\vec{\Delta}_i = A_i^- \vec{\xi}_i^- - A_{i-1}^+ \vec{\xi}_{i-1}^+ \tag{15.57}$$

$$S_i = S(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} W_e QQ^T d\tau \tag{15.58}$$

Function W_e in (15.56, 15.58) has initial value $W_e(r_0, t_0) = W_{e0}$ for all subintervals, matrix Q in (15.58) is calculated in ith Keplerian reference orbit (transporting trajectory) ($i = 1, \ldots, n$).

Performance index of the whole problem is

$$J = \sum_{i=1}^{n} J_i \tag{15.59}$$

ith Keplerian reference orbit ($i = 1, \ldots, n$) is given by

$$\vec{y}_i = \vec{y}_i(t) = \{\vec{q}_i, \vec{u}_i\},\ t_{i-1} \le t \le t_i,$$

$$\vec{y}_i^- = \vec{y}_i(t_i) = \{\vec{q}_i^-, \vec{u}_i^-\},\ \vec{y}_{i+1}^+ = \vec{y}_{i+1}(t_i) = \{\vec{q}_{i+1}^+, \vec{u}_{i+1}^+\}$$

Assume the reference orbits to form a continuous curve $\Rightarrow\ q_i^- = q_{i+1}^+$

$$\Rightarrow \quad \Delta\vec{y}_i = \vec{y}_{i+1}^+ - \vec{y}_i^- = \{\vec{0},\ \vec{u}_{i+1}^+ - \vec{u}_i^-\} = \{\vec{0}, \Delta\vec{u}_i\} \tag{15.60}$$

Designate

$$\vec{\xi}_i = \vec{\xi}_i^{\,+},$$

$$\vec{\Xi} = \left\{ \vec{\xi}_1, ..., \vec{\xi}_{n-1} \right\} \tag{15.61}$$

$\Rightarrow \quad \vec{\xi}_i^{\,-} = \vec{\xi}_i + \Delta \vec{y}_i$

$\Rightarrow$ Eq. (15.57) becomes

$$\begin{aligned} &\vec{\Delta}_i = A_i^- \vec{\xi}_i - A_{i-1}^+ \vec{\xi}_{i-1} + A_i^- \Delta \vec{y}_i \quad (i = 1, ..., n-1), \\ &\vec{\Delta}_n = A_n \vec{\xi}_n - A_{n-1}^+ \vec{\xi}_{n-1} \end{aligned} \tag{15.62}$$

Vector (15.61) can be found as $\vec{\Xi} = \arg\min J$

$$\Rightarrow \quad \frac{\partial J}{\partial \vec{\Xi}} = \vec{0}^T \tag{15.63}$$

Designate

$$\begin{aligned} &C_i = A_i^{-T} S_i^{-1} A_i^-, \quad D_i = C_i + A_i^{+T} S_{i+1}^{-1} A_i^+ \quad (i = 1,...,n-1), \\ &E_i = A_{i-1}^{+T} S_i^{-1} A_i^- \quad (i = 2,...,n-1) \end{aligned} \tag{15.64}$$

= matrices of 6th order,

$$D = \begin{bmatrix} D_1 & -E_2 & 0 & . & . & . & 0 \\ -E_2^T & D_2 & -E_3 & . & . & . & . \\ 0 & -E_3^T & D_3 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & D_{n-3} & -E_{n-2} & 0 \\ . & . & . & . & -E_{n-2}^T & D_{n-2} & -E_{n-1} \\ 0 & . & . & . & 0 & -E_{n-1}^T & D_{n-1} \end{bmatrix} \tag{15.65}$$

= matrix of $(6n-6)$th order,

$$\vec{d}_1 = E_1^T \vec{\xi}_0 + E_2 \Delta\vec{y}_2 - C_1 \Delta\vec{y}_1,$$
$$\vec{d}_i = E_{i+1} \Delta\vec{y}_{i+1} - C_i \Delta\vec{y}_i \quad (i = 2,...,n-2), \tag{15.66}$$
$$\vec{d}_{n-1} = E_n \vec{\xi}_n - C_{n-1} \Delta\vec{y}_{n-1}$$

= 6-dimensional vectors,

$$\vec{d} = \{\vec{d}_1, ..., \vec{d}_{n-1}\} \tag{15.67}$$

= (6n – 6)-dimensional vector.
Eq. (15.63) taking into account (15.61, 15.62, 15.64 – 15.67) gives

$$\boxed{\vec{\Xi} = D^{-1}\vec{d}} \tag{15.68}$$

15.10. Calculation procedure providing desired accuracy

The calculation procedure using the approach described in Section 15.9 is the following:

1. A reference orbit

$$\vec{y} = \vec{y}(t) = \{\vec{q}, \vec{u}\} \tag{15.69}$$

   is calculated by means of solving Lambert problem for given boundary positions $\vec{q}_0, \vec{q}_n$ and given transfer time T.

2. Time T is divided into n subintervals and instants $t_1, \ldots, t_{n-1}$ are specified and boundary values

$$\vec{y}_i = \vec{y}(t_i) = \{\vec{q}_i, \vec{u}_i\} \tag{15.70}$$

   are determined.

3. Matrices A^+_{i-1}, A^-_i, S_i are calculated for $i = 1, \ldots, n$ and matrix (15.65) and vector (15.67) are built.

4. Vector (15.61) is obtained from (15.68) and new positions

$$\vec{q}\,'_i = \vec{q}_i + \vec{\rho}_i \quad (i = 1, \ldots, n-1)$$

   are calculated. Simultaneously for a given factor $0 < \sigma < 1$ new values

$$\vec{q}\,'_0 = \sigma\vec{q}_0, \quad \vec{q}\,'_n = \sigma\vec{q}_n$$

   are calculated in order to provide $\vec{\rho}_0 \to \vec{0}$, $\vec{\rho}_n \to \vec{0}$ during the procedure.

5. New reference orbits (15.70) between each pair $\vec{q}\,'_{i-1}, \vec{q}\,'_i \ (i = 1, \ldots, n)$ are calculated by means of solving Lambert problem and vectors (15.60) are calculated.

Steps 3 to 5 are repeated until vector (15.61) is small enough. Then the values $\vec{\rho}_0 = \vec{0}$, $\vec{\rho}_n = \vec{0}$ are put.

15.11. Partly given boundary values for multiple time subintervals

Consider three cases of partly given boundary values for the time of flight divided into n subintervals.

1. <u>Launch velocity at the initial point can take any value</u>
   $\Rightarrow$ vectors $\vec{\rho}_0, \vec{\rho}_n, \vec{\eta}_n$ are given and vector $\vec{\eta}_0$ is free.
   Include $\vec{\eta}_0$ into the vector (15.61) $\Rightarrow$ extended vector (15.61) is

$$\hat{\vec{\Xi}} = \left\{ \vec{\eta}_0, \vec{\Xi} \right\} \tag{15.71}$$

Define

$$D_0 = Q_0^T S_1^{-1} Q_0, \quad E_0 = Q_0^T S_1^{-1} A_1^- \tag{15.72}$$

= 3×3 and 3×6 matrices,

$$\vec{d}_0 = -Q_0^T S_1^{-1} P_0 \vec{\rho}_0 + E_0 \Delta \vec{y}_1 \tag{15.73}$$

= 3-dimensional vector,

$$\hat{D} = \begin{bmatrix} D_0 & -E_0 & 0 & \cdots & 0 \\ -E_0^T & & & & \\ 0 & & D & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}, \tag{15.74}$$

= matrix of (6n – 3)th order,

$$\hat{\vec{d}} = \left\{ \vec{d}_0, \vec{d} \right\} \tag{15.75}$$

= (6n – 3)-dimensional vector where $\vec{\Xi}$, D, $\vec{d}$ are given by (15.61, 15.65, 15.67) $\Rightarrow$ equation

$$\frac{\partial J}{\partial \hat{\vec{\Xi}}} = \vec{0}^T \tag{15.76}$$

taking into account (15.56, 15.59, 15.62) gives

$$\boxed{\hat{\vec{\Xi}} = \hat{D}^{-1} \hat{\vec{d}}} \tag{15.77}$$

2. <u>Arrival velocity at the final point can take any value</u>
   $\Rightarrow$ vectors $\vec{\rho}_0, \vec{\eta}_0, \vec{\rho}_n$ are given and vector $\vec{\eta}_n$ is free.
   Include $\vec{\eta}_n$ into the vector (15.61) $\Rightarrow$ extended vector (15.61) is

$$\hat{\vec{\Xi}} = \left\{ \vec{\Xi}, \vec{\eta}_n \right\} \tag{15.78}$$

Define

$$D_n = Q_n^T S_n^{-1} Q_n, \quad E_n = A_{n-1}^{+T} S_n^{-1} P_n, \tag{15.79}$$

= 3×3 and 6×3 matrices,

$$\vec{d}_n = -Q_n^T S_n^{-1} P_n \vec{\rho}_n \tag{15.80}$$

= 3-dimensional vector,

$$\hat{D} = \begin{bmatrix} & & & & 0 \\ & & & & \vdots \\ & & D & & 0 \\ & & & & -E_n \\ 0 & \cdots & 0 & -E_n^T & D_n \end{bmatrix}, \tag{15.81}$$

= matrix of (6*n* – 3)th order,

$$\hat{\vec{d}} = \left\{ \vec{d}, \vec{d}_n \right\} \tag{15.82}$$

= (6*n* – 3)-dimensional vector where $\vec{\Xi}$, D, $\vec{d}$ are given by (15.61, 15.65, 15.67) $\Rightarrow$ due to equation (15.76) and taking into account (15.56, 15.59, 15.62) the extended vector (15.78) is given by (15.77).

<u>Note</u>. If both launch and arrival velocities can take any value then optimal is ballistic transfer (i.e. without any thrust run).

3. Values of the launch and/or arrival velocities are given and directions can take any values (if the initial and/or final celestial bodies are planets then the asymptotic velocities are meant)

Let $j = 0$ or $j = n$ and designate:

$\vec{r}_j, \vec{V}_j$ = position and velocity of initial ($j = 0$) or final ($j = n$) celestial body,

$\vec{w}_j$ = launch ($j = 0$) or arrival ($j = n$) velocity,

$w_j = \left|\vec{w}_j\right|$ is given,

$$\vec{\xi}'_j = \left\{\vec{r}_j - \vec{q}_j,\ \vec{V}_j - \vec{u}_j\right\} \tag{15.83}$$

(see Section 15.7)

$$\Rightarrow \quad \vec{\xi}_j = \vec{\xi}'_j + \left\{\vec{0}, \vec{w}_j\right\}, \tag{15.84}$$

$$\vec{w}_j \cdot \vec{w}_j = w_j^2 \tag{15.85}$$

Transversality conditions are

$$\vec{\alpha}_j = \lambda_j \vec{w}_j \quad (j = 0,1) \tag{15.86}$$

where λ_j are undetermined miltipliers. Define matrix

$$\hat{D} = \begin{bmatrix} \lambda_0 I + D_0 & -E_0 & 0 & \cdots & 0 \\ -E_0^T & & & & \vdots \\ 0 & & D & & 0 \\ \vdots & & & & -E_n \\ 0 & \cdots & 0 & -E_n^T & \lambda_n I + D_n \end{bmatrix}, \tag{15.87}$$

and vectors

$$\hat{\vec{\Xi}} = \left\{\vec{w}_0, \vec{\Xi}, \vec{w}_n\right\}, \tag{15.88}$$

$$\hat{\vec{d}} = \left\{ -E_0\left(\vec{\xi}'_0 - \Delta\vec{y}_1\right),\quad \vec{d}',\quad -Q_n^T S_n^{-1} A_n \vec{\xi}'_n \right\} \tag{15.89}$$

where vector $\vec{\Xi}$ and matrices D, D_0, E_0, D_n, E_n are given by (15.61, 15.65, 15.72, 15.79), $\vec{d}'$ in (15.89) is vector (15.67) with $\vec{\xi}_0, \vec{\xi}_n$ replaced by $\vec{\xi}'_0, \vec{\xi}'_n$ given by (15.83).

Due to the equation (15.76) and taking into account (15.56, 15.59, 15.62) the extended vector (15.88) is given by (15.77).

Values of the multipliers λ_0, λ_n in (15.87) are determined in the way the conditions (15.85) to be satisfied.

Note that if one of the vectors $\vec{w}_0, \vec{w}_n$ is given then the corresponding terms in (15.87 – 15.89) are absent. Order of matrix (15.87) and dimension of vectors (15.88, 15.89) are $6n$ if direction of both of vectors $\vec{w}_0, \vec{w}_n$ is not given and $6n - 3$ if one of the vectors is given.

15.12. Constrained thrust direction

Consider linear homogeneous equality constraint

$$B\vec{\alpha} = \vec{0} \tag{15.90}$$

(see Chapter 14). Due to (14.15, 14.22)

$$\boxed{\vec{\alpha} = W_e P \vec{p}_v} \tag{15.91}$$

where

$$P = I - B^T \left(BB^T\right)^{-1} B \tag{15.92}$$

is projective matrix (see Section 14.8). Vector $\vec{p}_v$ satisfies equations (14.10) where due to (14.11, 14.49) for the linear constraints

$$\left|\vec{\psi}_r\right| \sim \alpha, \quad \left|\vec{\psi}_v\right| \sim \alpha$$

$\Rightarrow$ since the considered method is approximate vectors $\vec{\psi}_r, \vec{\psi}_v$ may be neglected in Eq. (14.10) $\Rightarrow$ vector (15.14) satisfies equation (15.20) $\Rightarrow$ $\vec{p}_v$ is given by (15.21) $\Rightarrow$ due to (15.91)

$$\boxed{\vec{\alpha} = W_e P Q^T \vec{\beta}} \tag{15.93}$$

Substituting (15.93) into (15.19) shows that in order to find the optimal solution the equations (15.25 – 15.28) can be used with

$$S = S\left(t_0, t\right) = \int_{t_0}^{t} W_e Q P Q^T d\tau, \quad S_1 = S\left(t_0, t_1\right) \tag{15.94}$$

16. Simple Calculations of the Spiral Motion

16.1. Statement of the problem and basic assumptions

Consider changing the spacecraft orbital energy by means of electric propulsion for a minimum time.

> The problem is to find parameters of the spacecraft motion during the changing.

The following assumptions are taken:

- Mass flow rate and exhaust velocity are constant
- The external forces include only the planet gravitational attraction taking into account oblateness
- Initial and/or final orbit is circular
- The osculating orbit remains circular with the radius changing due to the low thrust

The last assumption is close to the reality if

jet acceleration << gravitational acceleration

(i.e. motion in a strong gravity field is considered)

16.2. Time of flight and orbit radius

Designate:

t_0, t	= initial and current time $(t \ge t_0)$;
$\tau = t - t_0$	= time of flight;
$r_0 = r(t_0),\ r = r(t)$	= initial and current orbit radius;
$v_0 = v(t_0),\ v = v(t)$	= initial and current spacecraft velocities;
u	= exhaust (or eject) velocity;
$\vec{\alpha}$	= vector of the jet acceleration;
$m_0 = m(t_0),\ m = m(t)$	= spacecraft initial and current mass;
$m_p = m_p(t)$	= consumed propellant mass;
$\dot{m}_p = \dfrac{dm_p}{dt}$	= mass flow rate;

$$\dot{m} = \frac{dm}{dt} = -\dot{m}_p \quad (\dot{m} \le 0) \tag{16.1}$$

Due to the assumptions taken in Section 16.1

$$\boxed{\dot{m}_p = |\dot{m}| = \text{const}, \quad u = \text{const}}$$

$\Rightarrow$ $\quad m_p = \dot{m}_p\,\tau, \quad m = m_0 - \dot{m}_p\,\tau,$

$$\alpha = |\vec{\alpha}| = \frac{\dot{m}_p u}{m} = \frac{\dot{m}_p u}{m_0 - \dot{m}_p\,\tau}, \tag{16.2}$$

$$\Delta v = u \ln\frac{m_0}{m} = u \ln\frac{m_0}{m_0 - \dot{m}_p \tau} \tag{16.3}$$

Integral of energy:

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

$\Rightarrow$ due to the thrust acceleration:

$$2\vec{v}\cdot\vec{\alpha} = \frac{\mu}{a^2}\dot{a} \tag{16.4}$$

$$\Rightarrow \quad |\dot{a}| = \max \ \text{ if } \begin{cases} \vec{\alpha} \uparrow\uparrow \vec{v} & \text{for orbit ascent,} \\ \vec{\alpha} \uparrow\downarrow \vec{v} & \text{for orbit descent} \end{cases}$$

$$\Rightarrow \quad \vec{v}\cdot\vec{\alpha} = \pm v\alpha, \tag{16.5}$$

"+" for ascent and "–" for descent in (16.5).

Since the orbit is circular then

$$a = r\,, \tag{16.6}$$

$$v_0 = \sqrt{\frac{\mu}{r_0}}, \quad v = \sqrt{\frac{\mu}{r}} \tag{16.7}$$

$\Rightarrow$ from (16.2, 16.4 – 16.7) obtain the following equation:

$$\pm 2\sqrt{\frac{\mu}{r}}\,\frac{\dot{m}\,u}{m_0 - \dot{m}_p\,\tau} = \frac{\mu}{r^2}\,\frac{dr}{dt} \tag{16.8}$$

Integrating (16.8) taking into account (16.7) gives

$$\boxed{\tau = \frac{m_0}{\dot{m}_p}\left[1 - \exp\left(-\frac{|v - v_0|}{u}\right)\right]} \tag{16.9}$$

$$\boxed{m_p = \dot{m}_p\,\tau = m_0\left[1 - \exp\left(-\frac{|v - v_0|}{u}\right)\right]} \tag{16.10}$$

$$\boxed{m = m_0 - m_p = m_0\,\exp\left(-\frac{|v - v_0|}{u}\right)} \tag{16.11}$$

Comparing (16.3) and (16.10) obtain:

$$\boxed{\Delta v = |v - v_0|} \tag{16.12}$$

Eqs. (16.9, 16.7) give

$$\boxed{r = \frac{r_0}{\left[1 \pm \frac{u}{v_0}\ln\left(1 - \frac{\dot{m}_p\tau}{m_0}\right)\right]^2}} \tag{16.13}$$

where "+" is for ascent and "–" is for descent.

16.3. Number of orbits and transfer angle

Designate:

N = number of the spacecraft orbits during the time τ (fractional);
φ = full transfer angle during the time τ;
$\phi = \operatorname{mod}_{2\pi} \varphi$ = phase angle $(0 \le \phi < 2\pi)$;

$$n = \frac{\sqrt{\mu}}{r^{3/2}} \quad \text{= the spacecraft mean motion;} \tag{16.14}$$

$$\eta_0 = \pm \frac{v_0}{u}, \quad \eta = \pm \frac{v}{u} \tag{16.15}$$

where "+" is for ascent and "–" is for descent
$\Rightarrow$ since $v_0 > v$ for ascent and $v_0 < v$ for descent (see (16.7)) and due to (16.12)

$$\eta - \eta_0 = -\frac{|v - v_0|}{u} = -\frac{\Delta v}{u} \tag{16.16}$$

Since the osculating orbit is assumed to be always circular then

$$\varphi = 2\pi N = \int_{t_0}^{t} n\, dt \tag{16.17}$$

From (16.17, 16.14)

$$N = \frac{1}{2\pi} \int_{t_0}^{t} n dt = \frac{\sqrt{\mu}}{2\pi} \int_{t_0}^{t} \frac{dt}{r^{3/2}} \tag{16.18}$$

Due to (C.11) (see Annex C)

$$\boxed{N = \frac{u^3}{2\pi\mu \dot{m}_p} |L_3|} \tag{16.19}$$

The transfer and phase angles are

$$\boxed{\varphi = 2\pi N, \quad \phi = 2\pi\left(N - \operatorname{int} N\right)} \tag{16.20}$$

16.4. Longitude of the ascending node

Designate:

$\Omega = \Omega(t)$ = longitude of the ascending node;

$\dot{\Omega} = \dfrac{d\Omega}{dt}$ = precession of the ascending node, i.e. secular perturbation caused by the oblateness of the planet;

$\Omega_0 = \Omega(t_0),\ \dot{\Omega}_0 = \dot{\Omega}(t_0);$

J_2 = coefficient of the second zonal harmonic;

R_e = Earth equatorial radius;

i = inclination of the orbit.

Precession of the ascending node for the circular orbit is (see Chapter 5, Eq. (5.27)):

$$\dot{\Omega} = -\frac{3}{2} J_2 n \left(\frac{R_e}{r}\right)^2 \cos i \tag{16.21}$$

$\Rightarrow$ taking into account (16.14) obtain:

$$\dot{\Omega} = \dot{\Omega}_0 \left(\frac{r_0}{r}\right)^{7/2} \tag{16.22}$$

$$\Rightarrow \quad \Delta\Omega = \Omega - \Omega_0 = \dot{\Omega}_0 r_0^{7/2} \int_{t_0}^{t} \frac{dt}{r^{7/2}} \tag{16.23}$$

Eqs. (16.7, 16.23) and (C.11) (see Annex C) give

$$\boxed{\Delta\Omega = \frac{\dot{\Omega}_0 u^7}{\dot{m}_p v_0^7} \left|L_7\right|} \tag{16.24}$$

16.5. Comparison with numerical integration

The objective is to estimate errors of the approximate formulas obtained in Sections 16.2 – 16.4, i.e. to calculate the approximate values minus accurate ones.

Consider the spacecraft ascent from a low Earth orbit and accept the following values:

$u = 15$ km/s;

$\alpha_0 = 5{\cdot}10^{-5}\, g_e$ $\quad$ ($g_e = 9{,}8066$ m/s^2)

$r_0 = 6771$ km $\quad$ (i.e. $h_0 = 400$ km)

Figs. 16.1, 16.2 show the errors and relative errors versus time; accurate value of the orbit radius r also is shown there.

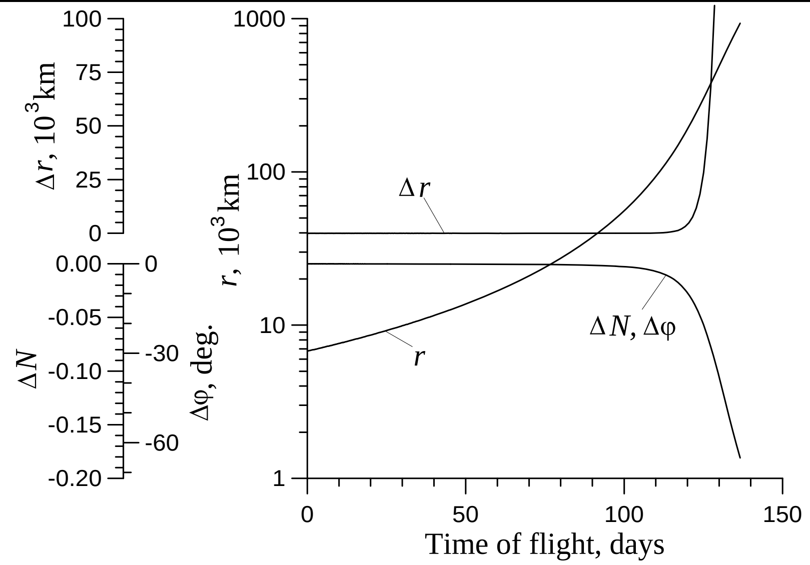


Fig. 16.1

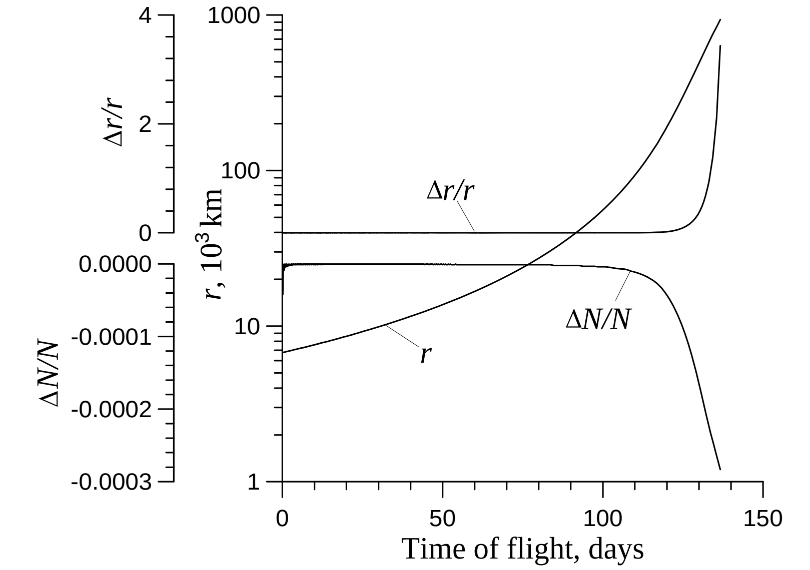


Fig. 16.2

Designate

$$g = \frac{\mu}{r^2} \quad = \text{local gravitational acceleration.}$$

Introduce dimensionless thrust acceleration

$$\boxed{\tilde{\alpha} = \frac{\alpha}{g} = \frac{\alpha r^2}{\mu}, \quad \tilde{\alpha}_0 = \frac{\alpha_0}{g_0} = \frac{\alpha_0 r_0^2}{\mu}} \tag{16.25}$$

Fig. 16.3 gives the $\tilde{\alpha}$ value versus time.

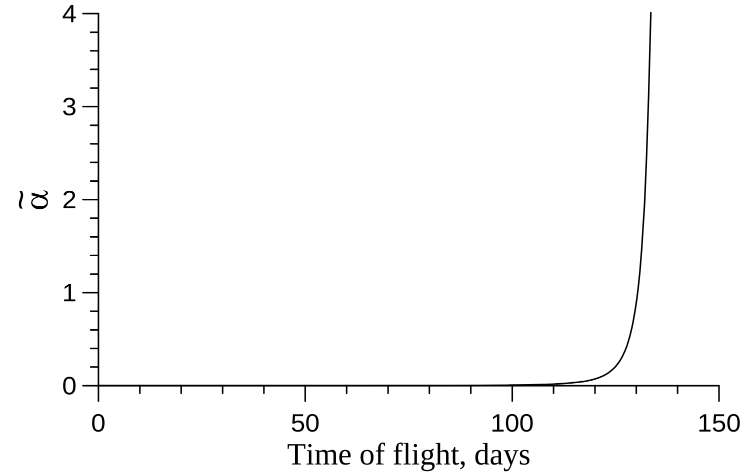


Fig. 16.3

As it is seen in Figs. 16.1 – 16.3 the errors are roughly proportional to $\tilde{\alpha}$.

Figs 16.4, 16.5 show the errors and relative errors versus radius; time of flight τ also is shown there.

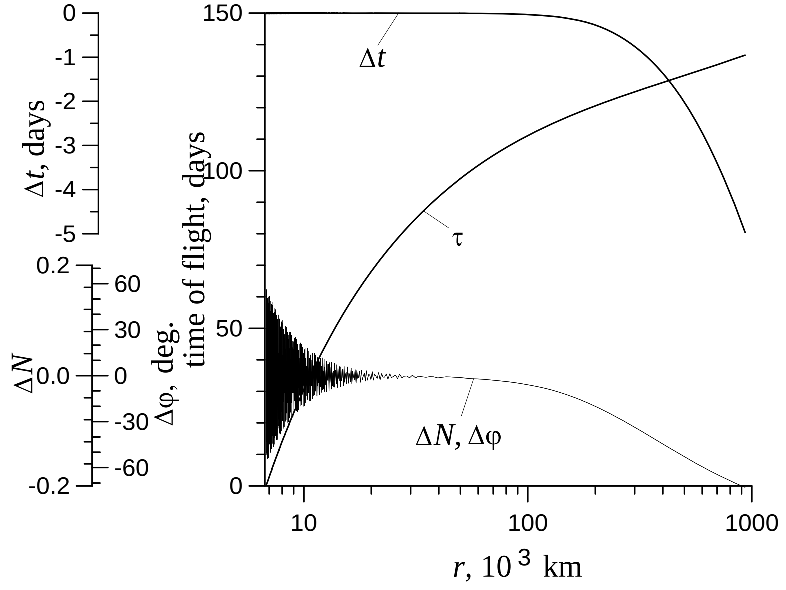


Fig. 16.4

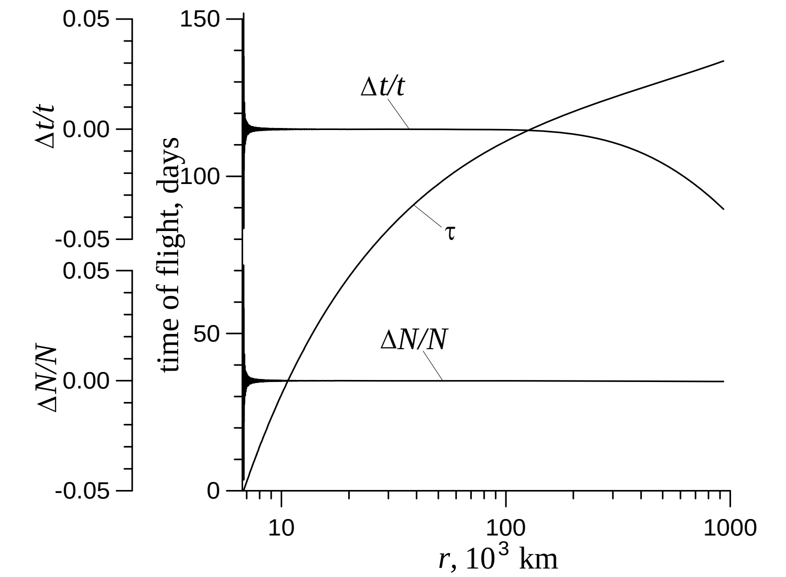


Fig. 16.5

Figures 16.6, 16.7 show the errors and relative errors versus semimajor axis; orbit radius r also is shown there.

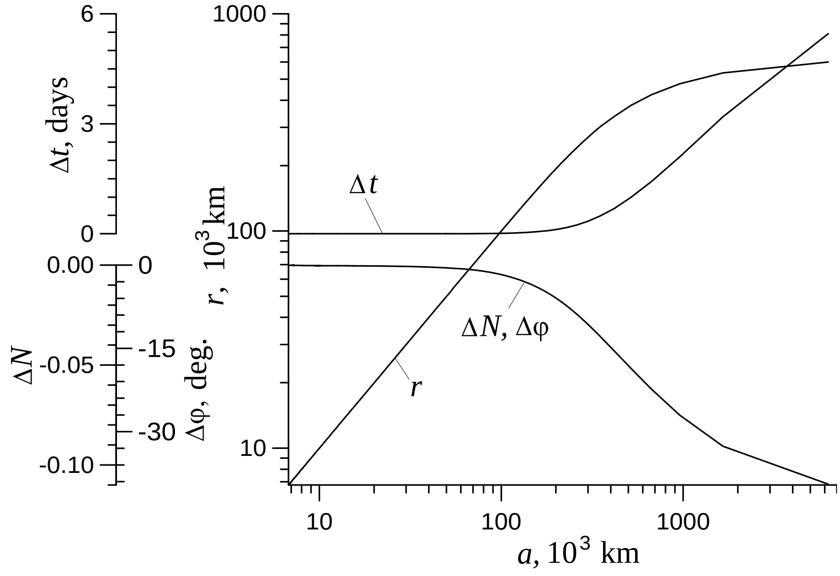


Fig. 16.6

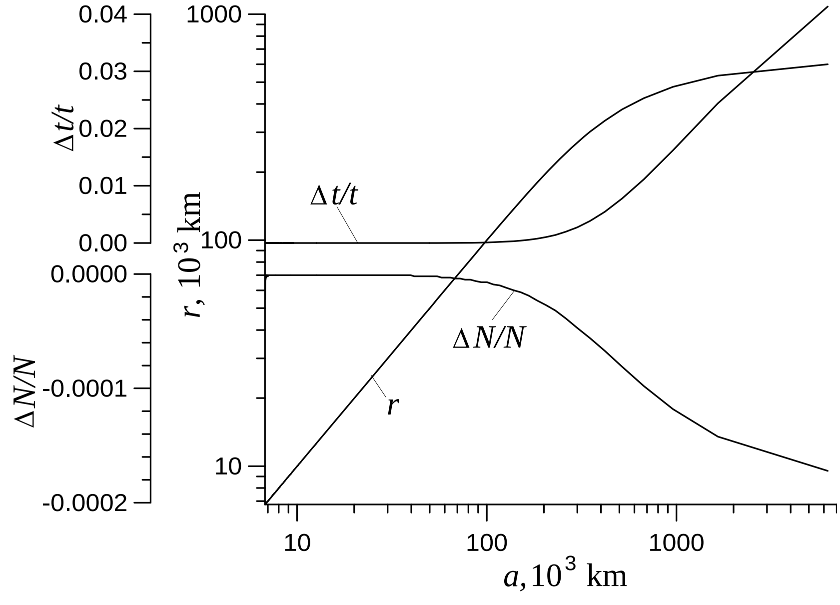


Fig. 16.7

16.6. Values of the parameters at infinity

Consider the spacecraft ascent from a circular orbit of radius r_0 to outside of sphere of influence. Usually radius of sphere of influence $R_S >> r_0$

$\Rightarrow$ we can assume $r = \infty \quad \Rightarrow \quad v = \sqrt{\frac{\mu}{r}} = 0$

$\Rightarrow$ from (16.9 – 16.11) obtain:

$$\tau_\infty = \frac{m_0}{\dot{m}_p}\left[1 - \exp\left(-\frac{v_0}{u}\right)\right] \tag{16.26}$$

$$m_\infty = m_0 \exp\left(-\frac{v_0}{u}\right) \tag{16.27}$$

$$m_{p\infty} = m_0\left[1 - \exp\left(-\frac{v_0}{u}\right)\right] \tag{16.28}$$

Designate:

$P_0 = 2\pi \frac{r_0^{3/2}}{\sqrt{\mu}}$ = period of the initial circular orbit;

Assume that $v_0 << u \quad \Rightarrow \quad \exp\left(-\frac{v_0}{u}\right) \approx 1 - \frac{v_0}{u}$

$\Rightarrow$ from (16.26 – 16.28) taking into account (16.7, 16.25) obtain:

$$\boxed{\tau_\infty \approx \frac{P_0}{2\pi\tilde{\alpha}_0}} \tag{16.29}$$

$$\boxed{m_\infty \approx m_0 - \frac{\dot{m}_p P_0}{2\pi\tilde{\alpha}_0}} \tag{16.30}$$

$$\boxed{m_{p\infty} \approx \frac{\dot{m}_p P_0}{2\pi\tilde{\alpha}_0}} \tag{16.31}$$

where $\tilde{\alpha}_0$ is given by (16.25).

Eq. (16.12) gives

$$\boxed{\Delta v_\infty = v_0} \tag{16.32}$$

(For the impulsive thrust $\Delta v_{imp\,\infty} = \left(\sqrt{2} - 1\right) v_0$

The excess $\delta v = \Delta v_\infty - \Delta v_{imp\,\infty} \approx 0.586 v_0$ is gravity loss)

Since $v = 0$ at infinity then due to (16.15) $\eta = 0$

$\Rightarrow$ Eqs. (16.9, C.6) give:

$$N_\infty = \frac{m_0 u^3}{2\pi\mu\dot{m}_p}\left(6\frac{m_\infty}{m_0} + \eta_0^3 - 3\eta_0^2 + 6\eta_0 - 6\right) \tag{16.33}$$

From (16.27) obtain:

$$\frac{m_\infty}{m_0} = e^{-\frac{v_0}{u}} = e^{-\eta_0} \approx 1 - \eta_0 + \frac{\eta_0^2}{2} - \frac{\eta_0^3}{6} + \frac{\eta_0^4}{24} \tag{16.34}$$

$\Rightarrow$ Eqs. (16.33, 16.34) using (16.2) give

$$N_\infty \approx \frac{m_0 u^3}{2\pi\mu\dot{m}_p}\frac{\eta_0^4}{4} = \frac{m_0 v_0^4}{8\pi\mu\dot{m}_p u} = \frac{\dfrac{\mu}{r_0^2}}{8\pi\mu\dfrac{\dot{m}_p u}{m_0}}$$

$\Rightarrow$ due to (16.25)

$$\boxed{N_\infty \approx \frac{1}{8\pi\tilde{\alpha}_0}} \tag{16.35}$$

Similarly from (16.24, C.6) obtain:

$$\begin{aligned}\Delta\Omega_\infty = \frac{\dot{\Omega}_0 m_0 u^7}{\dot{m}_p v_0^7}\Big(&5040\frac{m_\infty}{m_0} + \eta_0^7 - 7\eta_0^6 + 42\eta_0^5 - 210\eta_0^4 \\ &+ 840\eta_0^3 - 2520\eta_0^2 + 5040\eta_0 - 5040\Big),\end{aligned} \tag{16.36}$$

$$\begin{aligned}\frac{m_\infty}{m_0} = e^{-\frac{v_0}{u}} = e^{-\eta_0} \approx 1 - \eta_0 + \frac{\eta_0^2}{2} - \frac{\eta_0^3}{6} + \frac{\eta_0^4}{24} - \frac{\eta_0^5}{120} + \frac{\eta_0^6}{720} \\ - \frac{\eta_0^7}{5040} + \frac{\eta_0^8}{40320}\end{aligned} \tag{16.37}$$

$$\Rightarrow \quad \boxed{\Delta\Omega_\infty \approx \frac{\dot{\Omega}_0 P_0}{16\pi\tilde{\alpha}_0}} \tag{16.38}$$

Note. The obtained formulas can be used also for descent from infinity to the circular orbit of radius r_0. In this case:

m_0 is the spacecraft mass at infinity;

$v_0, \eta_0, P_0, \dot{\Omega}_0, \tilde{\alpha}_0$ are the values in the final orbit.

16.7. Values of the parameters in parabolic orbit

Mark values of all parameters at parabolic velocity by subscript *par*.

Due to Fig. 16.3 we can assume that $\tilde{\alpha}_{par} \sim 1$

$$\left(\begin{array}{l} \text{Accurate calculations show that} \\ \quad \left. \begin{array}{l} 10^{-5} g_e \le \alpha_0 \le 10^{-2} g_e \\ 10 \text{ km/s} \le u \le 50 \text{ km/s} \end{array} \right\} \quad \Rightarrow \quad 0.77 \le \tilde{\alpha}_{par} \le 0.8 \end{array} \right)$$

Accept for simplicity

$$\tilde{\alpha}_{par} = 1 \tag{16.39}$$

Note that the parabolic velocity is

$$v_{par} = \sqrt{2} v$$

where v is the circular velocity at τ_{par}. Eqs. (16.2, 16.3, 16.12, 16.25, 16.39) give

$$\tilde{\alpha}_{par} = \frac{\dot{m}_p u}{m_{par}} = \frac{\dot{m}_p u}{m_0} \exp\left(-\frac{|v - v_0|}{u} \right) = \frac{\mu}{r_{par}^2} \tag{16.40}$$

$$\Rightarrow \quad v = \sqrt{\frac{\mu}{r_{par}}} = \sqrt[4]{\frac{\mu \dot{m}_p u}{m_0}} \exp\left(-\frac{|v - v_0|}{4u} \right) \tag{16.41}$$

Assume that $|v - v_0| << 4u$

$$\Rightarrow \quad \exp\left(-\frac{|v - v_0|}{4u} \right) \approx 1 \tag{16.42}$$

On the other hand

$$\frac{\mu \dot{m}_p u}{m_0} = \left(\frac{\mu}{r_0} \right)^2 \frac{\dot{m}_p u}{m_0} \frac{r_0^2}{\mu} = v_0^4 \tilde{\alpha}_0 \tag{16.43}$$

$\Rightarrow$ Eqs. (16.41–16.43) give:

$$v \approx v_0 \sqrt[4]{\tilde{\alpha}_0}, \qquad \boxed{v_{par} \approx v_0 \sqrt[4]{4\tilde{\alpha}_0}} \tag{16.44}$$

$\Rightarrow$ substituting v in (16.9) by (16.44) obtain:

$$\boxed{\tau_{par} \approx \frac{m_0}{\dot{m}_p} \left\{ 1 - \exp\left[-\frac{v_0}{u} \left(1 - \sqrt[4]{\tilde{\alpha}_0} \right) \right] \right\}} \tag{16.45}$$

Fig. 16.8 gives inaccuracy of (16.45) (the approximate value minus accurate one) for $10^{-5} g_e \le \alpha_0 \le 10^{-2} g_e$ and 10 km/s $\le u \le 50$ km/s.

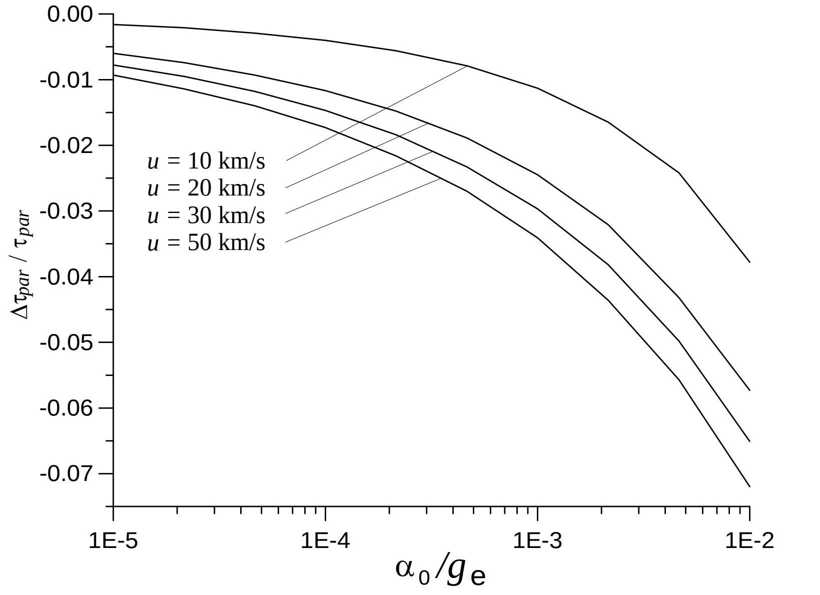


Fig. 16.8

Insert an empirical correction into (16.44) and (16.45):

$$\boxed{v_{par} \approx 0.86 v_0 \sqrt[4]{4\tilde{\alpha}_0}} \tag{16.46}$$

$$\boxed{\tau_{par} \approx \frac{m_0}{\dot{m}_p} \left\{ 1 - \exp\left[-\frac{v_0}{u} \left(1 - 0.86 \sqrt[4]{\tilde{\alpha}_0} \right) \right] \right\}} \tag{16.47}$$

Fig. 16.9 shows that the correction lowers the inaccuracy.

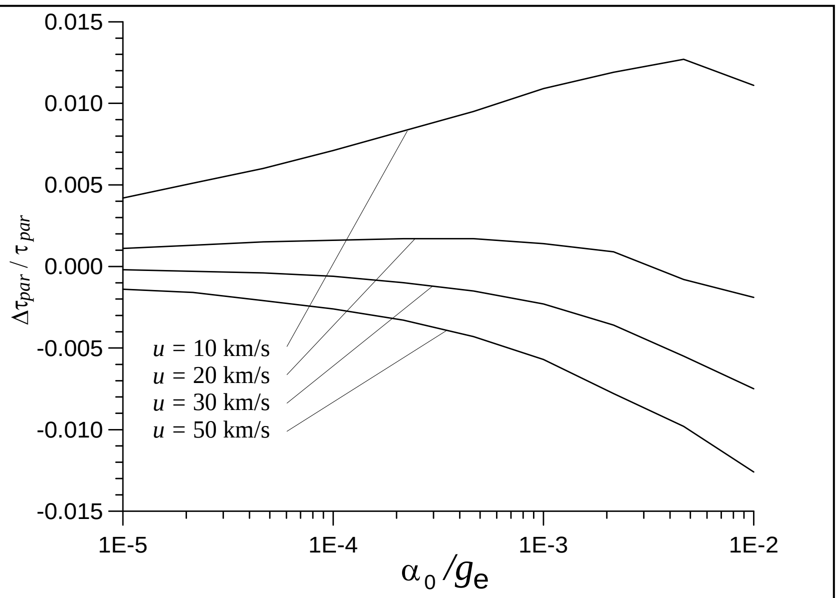


Fig. 16.9

Eqs. (16.12, 16.13) give other parameters for (16.45) and (16.47):

$$\boxed{\Delta v_{par} = v_0\left(1 - c_1\sqrt[4]{\tilde{\alpha}_0}\right)} \tag{16.48}$$

where $c_1 = 1$ for (16.45) and $c_1 = 0.86$ for (16.47),

$$\boxed{r_{par} = c_2\,\frac{r_0}{\sqrt{\tilde{\alpha}_0}}} \tag{16.49}$$

where $c_2 = 1$ for (16.45) and $c_2 = 1.35$ for (16.47).

If $v_0 << u$ then

$$\exp\left[-\frac{v_0}{u}\left(1-\sqrt[4]{\tilde{\alpha}_0}\right)\right] \approx 1 - \frac{v_0}{u}\left(1-\sqrt[4]{\tilde{\alpha}_0}\right)$$

and formulas (16.45, 16.47) take the form similar to (16.29):

$$\boxed{\tau_{par} \approx \frac{P_0}{2\pi\tilde{\alpha}_0}\left(1 - c_1\sqrt[4]{\tilde{\alpha}_0}\right)} \tag{16.50}$$

Note. The obtained formulas can be used both for ascent from a circular orbit to the parabolic one and for descent from the parabolic orbit to a circular one. In case of braking:

m_0 is the spacecraft mass in the parabolic orbit;

$v_0, P_0, \tilde{\alpha}_0$ are the values in the final circular orbit.

16.8. Conclusions

- The obtained formulas allow calculation of all parameters of the spacecraft spatial motion using low thrust for the following cases:
  - changing the circular orbit radius;
  - launch from the circular orbit to the escape trajectory;
  - braking from the trajectory entering the sphere of influence to a circular orbit
- The obtained formulas provide high accuracy if $\tilde{\alpha} << 1$
- The accuracy can get insufficient when $\tilde{\alpha} \sim 1$
- Eq. (16.13) gives rather semimajor axis than the real spacecraft position

17. Low-Thrust Transfers between Given Orbits

17.1. Basic assumptions and notation

Basic assumptions:

- an arbitrary force field is considered
- propulsion of the limited power (LP) is assumed (see Chapter 13)

Designate:

$t = 0$	= initial time of the transfer;
T	= transfer duration;
$\vec{r}, \vec{v}$	= position and velocity vectors;
$\vec{x} = \{\vec{r}, \vec{v}\}$	= state vector of the transfer trajectory,
$\vec{x}_0 = \vec{x}(0),\ \vec{x}_T = \vec{x}(T)$;	
$\vec{q} = \vec{q}(t)$	= 5-dimensional vector of osculating (instantaneous) orbital elements defining orbit;
$\vec{q}_i, \vec{q}_f$	= orbital elements of the initial and final orbits;
$\vec{y}_i, \vec{y}_f$	= state vectors of the initial and final orbits,
$\vec{y}_{i0} = \vec{y}_i(0),\ \vec{y}_{iT} = \vec{y}_i(T),\ \vec{y}_{fT} = \vec{y}_f(T)$;	
$\vec{\alpha} = \vec{\alpha}(t)$	= jet acceleration (thrust vector),
$\alpha = \lvert\vec{\alpha}\rvert$;	
$\vec{g} = \{\vec{0}, \vec{\alpha}\}$	= 6-dimensional vector;
$W_e = W_e(\vec{r}, t)$	= effective electric power of the propulsion system;
I	= unit matrix.

17.2. Necessary equations

Equation of motion in an arbitrary force field without a jet propulsion is

$$\dot{\vec{y}} = \vec{f}\left(\vec{y},t\right) \tag{17.1}$$

Variational and adjoint (or costate) variational equations are

$$\dot{\Phi} = F\Phi, \quad \dot{\Psi} = -\Psi F \tag{17.2}$$

where

$$F = \frac{\partial \vec{f}}{\partial \vec{y}} \tag{17.3}$$

Let matrices:

$$\Phi = \Phi\left(t,0\right), \quad \Psi = \Psi\left(t,0\right) \tag{17.4}$$

with initial values

$$\Phi\left(0,0\right) = I, \quad \Psi\left(0,0\right) = I \tag{17.5}$$

be general solutions to the Eqs. (17.2). Matrices (17.4) are state and costate transition matrices satisfying the following equations:

$$\Phi\left(t,0\right) = \frac{\partial \vec{y}\left(t\right)}{\partial \vec{y}\left(0\right)}, \quad \Psi\left(t,0\right) = \frac{\partial \vec{y}\left(0\right)}{\partial \vec{y}\left(t\right)}, \tag{17.6}$$

$$\Phi = \Psi^{-1} \tag{17.7}$$

For more details see Chapter 11.

17.3. Statement of the problem

Equation of motion with low thrust is

$$\boxed{\dot{\vec{x}} = \vec{f}\left(\vec{x}, t\right) + \vec{g}} \tag{17.8}$$

Performance index for the case of the LP propulsion is

$$\boxed{J = \frac{1}{2}\int_0^T \frac{\alpha^2}{W_e} dt} \tag{17.9}$$

(see Chapters 13, 15).

The problem is to find the thrust vector transferring the spacecraft between two orbits given by elements $\vec{q}_i, \vec{q}_f$ in time T and minimizing the performance index (i.e. the propellant consumption)

(see Fig. 17.1).

Boundary values:

$$\boxed{\vec{x}_0 = \vec{y}_{i0}, \quad \vec{x}_T = \vec{y}_{fT}} \tag{17.10}$$

Assume vector $\vec{y}_0$ given and vector $\vec{y}_T$ not given.

Note. The difference of the problem considered here from one of Chapter 15 is that a two-point boundary value problem (TPBVP) with given or partly given initial and final state vectors is solved in Chapter 15, while in this chapter one or both of the boundary state vectors are not given.

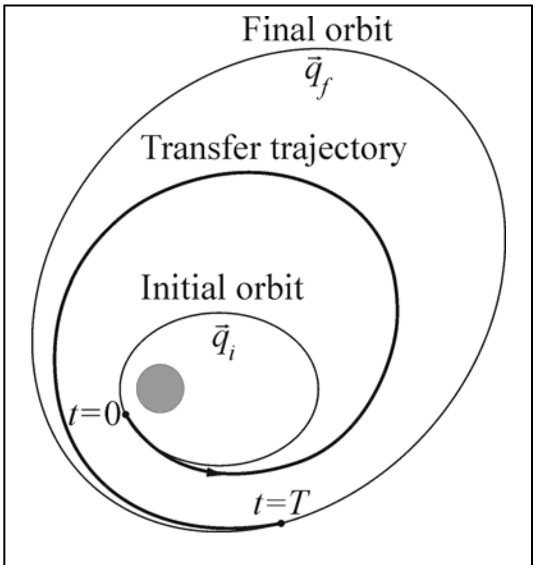


Fig. 17.1

17.4. Neighboring orbits

Assume the initial and final orbits close to each other and define vector

$$\vec{\xi} = \vec{\xi}(t) = \vec{x}(t) - \vec{y}_i(t) \tag{17.11}$$

Linearized equation of motion is

$$\boxed{\dot{\vec{\xi}} = F\vec{\xi} + \vec{g}} \tag{17.12}$$

where matrix F given by Eq. (17.3) is calculated in the initial orbit (see also Chapter 15). Solution to the equation (17.12) is given by Cauchy formula:

$$\vec{\xi} = \int_0^t \Phi(t,\tau)\vec{g}\,d\tau \tag{17.13}$$

where $\Phi(t, \tau)$ is state transition matrix (see previous Section and Chapter 11).

Eq. (17.12) is non-autonomous since matrix F is a function of time. Hence Hamiltonian of the linearized problem is

$$H = -\frac{\alpha^2}{2W_e} + \vec{p}^{\,T} F\vec{\xi} + \vec{p}_v^{\,T}\vec{\alpha} + p_t \tag{17.14}$$

where

$$\vec{p} = \{\vec{p}_r, \vec{p}_v\} \tag{17.15}$$

is vector of adjoint (costate) variables, $\vec{p}_v$ is Lawden's primer vector, p_t is an adjoint variable for the additional equation $\dot{t} = 1$ making the system autonomous (see Chapters 12, 13, 15).

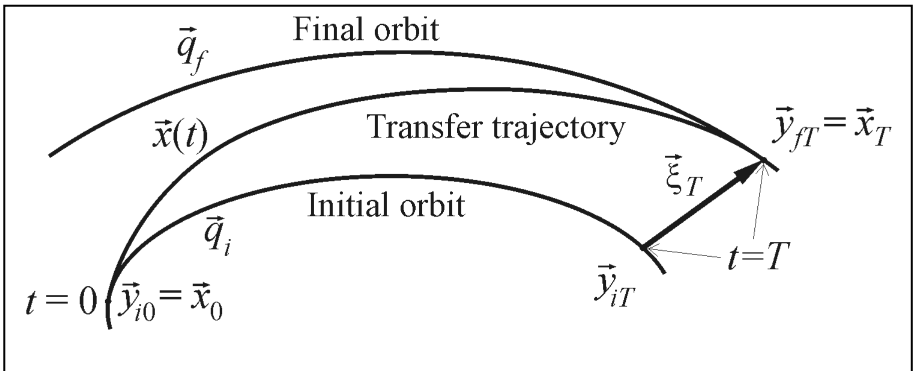


Fig. 17.2

The optimal thrust vector giving maximum to the Hamiltonian is

$$\boxed{\vec{\alpha} = W_e \vec{p}_v} \tag{17.16}$$

Vector $\vec{p}$ satisfies adjoint (costate) variational equation

$$\dot{\vec{p}}^T = -\frac{\partial H}{\partial \vec{x}} = -\vec{p}^T F \tag{17.17}$$

The costate transition matrix Ψ given by Eqs. (17.4 – 17.6) is a general solution to the equation (17.17). Then

$$\vec{p} = \Psi^T \vec{\beta} \tag{17.18}$$

where $\vec{\beta}$ is a constant 6-dimensional vector. Dividing matrix Ψ into two 6×3-dimensional submatrices

$$\Psi = \left[\Psi_r \;\; \Psi_v\right] \tag{17.19}$$

the primer vector can be represented as

$$\vec{p}_v = \Psi_v^T \vec{\beta} \tag{17.20}$$

Using Eqs. (17.16, 17.18) and properties of matrices $\Phi, \Psi,$ Eq. (17.13) becomes

$$\vec{\xi} = \Phi S \vec{\beta} \tag{17.21}$$

where $\Phi = \Phi(t, 0)$,

$$S = S(0,t) = \int_0^t W_e \Psi_v \Psi_v^T d\tau, \quad S_T = S(0,T) \tag{17.22}$$

Due to Eqs. (17.11, 17.16, 17.18, 17.21) in order to find the optimal thrust vector and the state vector of the transfer trajectory it is sufficient to find vector $\vec{\beta}$.

Since the final state vector $\vec{y}_{fT}$ of the transfer is not given the transversality condition is

$$\vec{p}_T = \vec{p}(T) = \left(\frac{\partial \vec{q}_{fT}}{\partial \vec{y}_{fT}}\right)^T \vec{\sigma} \tag{17.23}$$

where $\vec{\sigma}$ is any constant 5-dimensional vector. Due to the closeness of the initial and final orbits $\partial \vec{q}_f / \partial \vec{y}_f \approx U$ where

$$U = U(t) = \frac{\partial \vec{q}_i}{\partial \vec{y}_i} \tag{17.24}$$

$\Rightarrow$ Eq. (17.23) becomes

$$\vec{p}_T \approx U_T^T \vec{\sigma}, \quad U_T = U(T) = \frac{\partial \vec{q}_{iT}}{\partial \vec{y}_{iT}} \tag{17.25}$$

Define vector

$$\Delta \vec{q}(t) = \vec{q}_f(t) - \vec{q}_i(t), \tag{17.26}$$

$$\Delta \vec{q}_T = \Delta \vec{q}(T) \approx U_T \vec{\xi}_T, \quad \vec{\xi}_T = \vec{\xi}(T) = \vec{y}_{fT} - \vec{y}_{iT} \tag{17.27}$$

On the other hand

$$\Delta \vec{q}_T = \frac{\partial \vec{q}_{iT}}{\partial \vec{q}_{i0}} \Delta \vec{q}_0, \quad \Delta \vec{q}_0 = \Delta \vec{q}(0) \tag{17.28}$$

Using Eq. (17.21, 17.24, 17.27) and equation (17.6) Eq. (17.28) gives

$$\Delta \vec{q}_0 = \left(\frac{\partial \vec{q}_{iT}}{\partial \vec{q}_{i0}}\right)^{-1} \Delta \vec{q}_T = \frac{\partial \vec{q}_{i0}}{\partial \vec{q}_{iT}} \frac{\partial \vec{q}_{iT}}{\partial \vec{y}_{iT}} \frac{\partial \vec{y}_{iT}}{\partial \vec{y}_{i0}} S_T \vec{\beta} = \frac{\partial \vec{q}_{i0}}{\partial \vec{y}_{i0}} S_T \vec{\beta} = U_0 S_T \vec{\beta} \tag{17.29}$$

where

$$U_0 = U(0) = \frac{\partial \vec{q}_{i0}}{\partial \vec{y}_{i0}} \tag{17.30}$$

Eq. (17.18) for $t = T$ and Eq. (17.25) give

$$\Psi_T^T \vec{\beta} = U_T^T \vec{\sigma} \tag{17.31}$$

where $\Psi_T = \Psi(T, 0)$. From (17.31) taking into account equation (17.7) obtain

$$\vec{\beta} = \Phi_T^T U_T^T \vec{\sigma} \tag{17.32}$$

Substituting (17.31) into (17.29) using Eqs. (17.6, 17.24) gives

$$\Delta\vec{q}_0 = U_0 S_T \left(U_T \Phi_T\right)^T \vec{\sigma} = U_0 S_T \left(\frac{\partial \vec{q}_{iT}}{\partial \vec{y}_{iT}} \frac{\partial \vec{y}_{iT}}{\partial \vec{y}_{i0}}\right)^T \vec{\sigma}$$

$$= U_0 S_T \left(\frac{\partial \vec{q}_{iT}}{\partial \vec{q}_{i0}} \frac{\partial \vec{q}_{i0}}{\partial \vec{y}_{i0}}\right)^T \vec{\sigma} = U_0 S_T U_0^T Q^T \vec{\sigma} \tag{17.33}$$

where

$$Q = \frac{\partial \vec{q}_{iT}}{\partial \vec{q}_{i0}} \tag{17.34}$$

Then from Eq. (17.33)

$$\vec{\sigma} = Q^{-1^T} \left(U_0 S_T U_0^T\right)^{-1} \Delta\vec{q}_0 \tag{17.35}$$

and Eqs. (17.32, 17.6, 17.25, 17.30, 17.35) give

$$\vec{\beta} = \left(Q^{-1} U_T \Phi_T\right)^T \left(U_0 S_T U_0^T\right)^{-1} \Delta\vec{q}_0$$

$$= \left(\frac{\partial \vec{q}_{i0}}{\partial \vec{q}_{iT}} \frac{\partial \vec{q}_{iT}}{\partial \vec{y}_{iT}} \frac{\partial \vec{y}_{iT}}{\partial \vec{y}_{i0}}\right)^T \left(U_0 S_T U_0^T\right)^{-1} \Delta\vec{q}_0 = \left(\frac{\partial \vec{q}_{i0}}{\partial \vec{y}_{i0}}\right)^T \left(U_0 S_T U_0^T\right)^{-1} \Delta\vec{q}_0$$

Finally due to (17.30)

$$\boxed{\vec{\beta} = U_0^T V^{-1} \Delta\vec{q}_0} \tag{17.36}$$

where

$$\boxed{V = U_0 S_T U_0^T} \tag{17.37}$$

is a matrix of 5th order.

Then Eqs. (17.11, 17.16, 17.21) give

$$\boxed{\vec{\alpha} = W_e \Psi_v^T U_0^T V^{-1} \Delta \vec{q}_0} \tag{17.38}$$

$$\boxed{\vec{x} = \vec{y}_i + \Phi S U_0^T V^{-1} \Delta \vec{q}_0} \tag{17.39}$$

Due to Eq. (17.10) the state vector of the insertion into the final orbit can be found from Eq. (17.39) as follows

$$\boxed{\vec{y}_{fT} = \vec{y}_{iT} + \Phi_T S_T U_0^T V^{-1} \Delta \vec{q}_0} \tag{17.40}$$

Eqs. (17.9, 17.38, 17.22) give the performance index:

$$J = \frac{1}{2}\int_0^T W_e \Delta \vec{q}_0^T V^{-1^T} U_0 \Psi_v \Psi_v^T U_0^T V^{-1} \Delta \vec{q}_0 dt$$

$$= \frac{1}{2}\Delta \vec{q}_0^T V^{-1} U_0 \int_0^T W_e \Psi_v \Psi_v^T dt U_0^T V^{-1} \Delta \vec{q}_0 = \frac{1}{2}\Delta \vec{q}_0^T V^{-1} U_0 S U_0^T V^{-1} \Delta \vec{q}_0$$

$\Rightarrow$ due to (17.37)

$$\boxed{J = \frac{1}{2}\Delta \vec{q}_0^T V^{-1} \Delta \vec{q}_0} \tag{17.41}$$

17.5. Arbitrary orbits

Arbitrary initial and final orbits are considered here.

The time interval is divided into n subintervals and n – 1 reference orbits between initial and final orbits are specified somehow. For instance elements $\vec{q}_1,...,\vec{q}_{n-1}$ of the reference orbits may be defined in the following way

$$\vec{q}_j = \vec{q}_i + \frac{j}{n}\Delta\vec{q}_0 \quad \left(j = 1,...,n-1\right) \tag{17.42}$$

where vector $\Delta\vec{q}_0$ is given by (17.26, 17.28). The approach described in the previous section is applied to each subinterval. Defining vectors

$$\Delta\vec{q}_j = \vec{q}_j\left(0\right) - \vec{q}_{j-1}\left(0\right) \quad \left(j = 1,...,n\right), \quad \vec{q}_0 = \vec{q}_i, \ \vec{q}_n = \vec{q}_f, \tag{17.43}$$

at *j*th subinterval due to (17.41) obtain

$$J_j = \frac{1}{2}\Delta\vec{q}_j^T V_j^{-1}\Delta\vec{q}_j \tag{17.44}$$

where matrices

$$V_j = U_j S_j U_j^T, \ U_j = \frac{\partial\vec{q}_j\left(0\right)}{\partial\vec{y}_j\left(0\right)}, \ S_j = \int\limits_{t_{j-1}}^{t_j} W_e \Psi_{jv} \Psi_{jv}^T dt, \ \Psi_{jv} = \frac{\partial\vec{y}_j\left(0\right)}{\partial\vec{v}_j\left(t\right)} \tag{17.45}$$

are calculated in the j – 1st reference orbit (see Eqs. (17.37, 17.30, 17.22, 17.6, 17.19)).

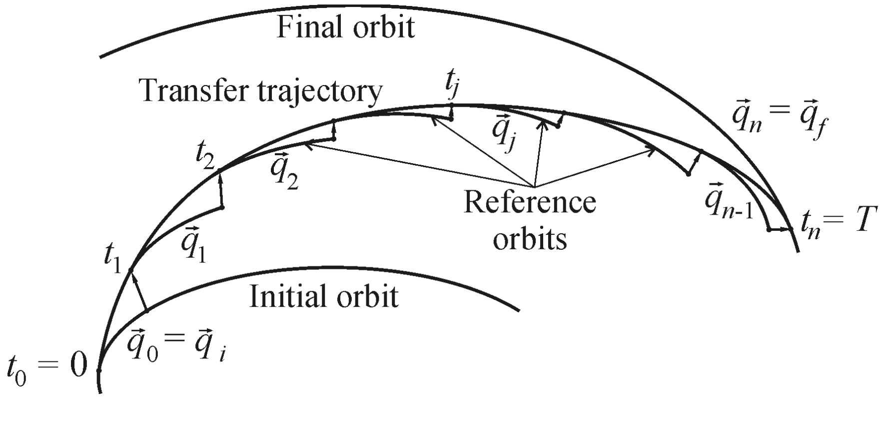


Fig. 17.3

Performance index of the whole problem is

$$\boxed{J = \sum_{j=1}^{n} J_j} \tag{17.46}$$

The following condition is fulfilled:

$$\boxed{\sum_{j=1}^{n} \Delta\vec{q}_j = \Delta\vec{q}_0} \tag{17.47}$$

New elements of the reference orbits forming optimal transfer trajectory can be found from $\min J$ under the condition (17.47). Helping function of the problem is

$$L = J - \vec{\lambda}^T \left(\sum_{j=1}^{n} \Delta\vec{q}_j - \Delta\vec{q} \right) \tag{17.48}$$

where $\vec{\lambda}$ is Lagrange multiplier. Necessary condition of minimum of function (17.48) due to (17.44) is

$$\left(\frac{\partial L}{\partial \Delta\vec{q}_j} \right)^T = V_j^{-1} \Delta\vec{q}_j - \vec{\lambda} = \vec{0} \tag{17.49}$$

Thus,

$$\Delta\vec{q}_j = V_j \vec{\lambda} \tag{17.50}$$

and Eqs. (17.47, 17.50) give

$$\vec{\lambda} = \left(\sum_{j=1}^{n} V_j \right)^{-1} \Delta\vec{q}_0 \tag{17.51}$$

Now new elements of the reference orbits subsequently can be found as follows:

$$\boxed{\vec{q}_j = \vec{q}_{j-1} + \Delta\vec{q}_j} \tag{17.52}$$

Vector (17.36) for *j*th subinterval is

$$\vec{\beta}_j = U_j^T V_j^{-1} \Delta\vec{q}_j \tag{17.53}$$

Using Eqs. (17.50, 17.51) vector (17.53) becomes

$$\boxed{\vec{\beta}_j = U_j^T \left(\sum_{j=1}^{n} V_j \right)^{-1} \Delta\vec{q}_0} \tag{17.54}$$

Due to Eqs. (17.16, 17.20, 17.36, 17.39) optimal thrust and state vector in *j*th subinterval are

$$\boxed{\vec{\alpha}(t) = W_e \Psi_{jv}^T(t)\vec{\beta}_j} \tag{17.55}$$

$$\boxed{\vec{x}(t) = \vec{y}_j(t) + \Phi_j(t) S_j(t)\vec{\beta}_j} \tag{17.56}$$

where $t_{j-1} \le t \le t_j$, $j = 1,..,n$. Eqs. (17.46, 17.44, 17.50, 17.51) give the performance index of the whole problem as follows

$$\boxed{J = \Delta\vec{q}_0^T \left(\sum_{j=1}^{n} V_j \right)^{-1} \Delta\vec{q}_0} \tag{17.57}$$

17.6. Calculation Procedure

Let $\vec{y}_j^0, \vec{y}_j^1$ be state vectors of the jth reference orbit at the beginning and end of the jth time subinterval (i.e. at times t_{j-1}, t_j respectively, $j = 1, \ldots, n$). Solution of the considered transfer problem may be obtained by means of the following calculation procedure:

1. $n - 1$ intermediate orbits are specified somehow (for instance, using Eq. (17.42)). A launch position in the initial orbit is specified (i.e. state $\vec{y}_i^0 = \vec{y}_i(0)$ is given) and respective initial state vector of the transfer trajectory is $\vec{x}_0 = \vec{y}_i^0$.
2. Vector $\vec{y}_j^0$ is calculated for $j = 1$ using the following equation similar to Eq. (17.40):
$$\vec{y}_j^0 = \vec{y}_{j-1}^1 + \Phi_j S_j U_j^T V_j^{-1} \Delta\vec{q}_j \tag{17.58}$$
where $\vec{y}_0 = \vec{y}_i$, matrices S_j, U_j^T, V_j^{-1} given by (17.45) and matrix $\Phi_j = \Phi\left(t_j, t_{j-1}\right)$ are calculated in the j – 1st reference orbit. Since Eq. (17.58) is approximate, vectors $\vec{q}_j(0), \Delta\vec{q}_j$ should be recalculated using state (17.58).
3. Step 2 is repeated for $j = 2, \ldots, n$.
4. Vector (17.51) is found and new vectors (17.50) are calculated. Then new reference orbits with elements $\vec{q}_{j+1} = \vec{q}_j + \Delta\vec{q}_j \;\; \left(j = 0, \ldots, n-1\right)$ are determined.
5. Performance index is calculated using Eq. (17.57) and steps 2–4 are repeated until decrement ΔJ of the performance index gets smaller than a given parameter $\varepsilon > 0$. As soon as $\left|\Delta J\right| < \varepsilon$ the optimal thrust vector and state vector of the transfer trajectory are calculated using Eqs. (17.54 – 17.56).

17.7. Transfer Types

The method described in Sections 17.4 – 17.6 may be used for the following low-thrust transfers:

1. From a given state vector to a given orbit with obtaining an optimal arrival point

   Solution to this problem is given in Sections 17.4 – 17.6.

2. From a given orbit to a given state vector with obtaining an optimal start point

   To solve this problem the method described in Sections 17.4 – 17.6 should be applied in the backward direction with retrograde time, i.e. the process starts in the final orbit at $t = T$ and ends in the initial orbit at $t = 0$.

3. Between two given orbits with obtaining optimal start and arrival points

   In this case odd iterations of the calculation procedure (see Section 17.6) are as in the first case and even iterations are as in the second case

17.8. Partly Given Final Orbit

Let us assume only m final orbital elements given with $m < 5$ and other $5–m$ elements free. Then vectors $\vec{q}_i, \vec{q}_j$ $\left(j = 1,\ldots,n-1 \right)$ in the equations of Sections 17.3 – 17.5 also are m-dimensional and the method described in these Sections may be applied with m-dimensional vectors $\vec{q}_i, \vec{q}_f, \vec{q}_j$ and matrices U_j, V_j given by (17.45) of dimensions $m\times6$ and $m\times m$ respectively.

17.9. Constraints on the Thrust Direction

Let us consider linear homogeneous equality constraint on the thrust direction

$$B\vec{\alpha} = \vec{0} \tag{17.59}$$

where $B = B(\vec{r}, \vec{v}, t)$ is $k\times3$-matrix, $k \geq 1$ (see Section 14.8). The projective matrix for this constraint is

$$P = I - B^T \left(BB^T\right)^{-1} B \tag{17.60}$$

Rank of matrix P is $R \leq 2$. Like it has been done in Section 15.12 the matrix integral (17.22) and optimal thrust vector can be found as follows:

$$S = S(0,t) = \int_0^t W_e \Psi_v P \Psi_v^T d\tau\,, \tag{17.61}$$

$$\boxed{\vec{\alpha} = W_e P \Psi_v^T \vec{\beta}} \tag{17.62}$$

where vector $\vec{\beta}$ is given by Eq. (17.36).

Note that matrix (17.22) is non-singular for any time interval of integration (see Theorem 15.1), but matrix (17.61) may be singular.

Non-singularity of matrix (17.61) is a sufficient condition of feasibility of a particular transfer under the constraint (17.59).

17.10. Special Cases of the Gravity Field

Oblate planet.

Motion near an oblate planet with no more perturbations is considered. Taking into account only secular perturbations, five orbital elements defining orbit can be found as follows (see Eq. (5.27))

Semimajor axis:	$a = a_0$
Eccentricity:	$e = e_0$
Inclination:	$i = i_0$
Longitude of the ascending node:	$\Omega = \Omega_0 - \frac{3}{2} J_2 n \left(\frac{R_e}{p}\right)^2 \cos i \cdot t$
Argument of pericenter:	$\omega = \omega_0 + \frac{3}{4} J_2 n \left(\frac{R_e}{p}\right)^2 \left(5\cos^2 i - 1\right) \cdot t$

where

J_2 = coefficient of the 2nd zonal harmonic (see Section 5.7),

$n = \frac{\sqrt{\mu}}{a^{3/2}}$ = mean motion,

R_e = equatorial radius of the planet,

$p = a\left(1 - e^2\right)$ = semilatus rectum.

Mean anomaly necessary for calculation of motion in the reference orbits is

$$M = M_0 + \left(\frac{3}{4} J_2 \left(\frac{R_e}{p}\right)^2 \sqrt{1-e^2}\left(3\cos^2 i - 1\right) + 1\right) nt$$

where $M_0 = M(0)$.

<u>Two-body problem</u>.

Non-perturbed Keplerian motion is considered here. In this case initial, final and intermediate reference orbits are Keplerian and

$$\vec{q}(t) = \vec{q}_0 = \text{const},$$

i.e. the orbital elements are first integrals of the motion. Assuming the final and reference orbits m-dimensional with $m \le 5$ (i.e. $\vec{q} = \{q_1, ..., q_m\}$ for these orbits, see Section 17.7), introduce extended vector

$$\tilde{\vec{q}} = \{q_1, ..., q_6\} = \{\vec{q}, q_{m+1}, ..., q_6\}$$

Then the extension of matrix (17.24) is

$$\tilde{U} = \tilde{U}(t) = \frac{\partial \tilde{\vec{q}}}{\partial \vec{y}} \tag{17.63}$$

where $\vec{y} = \vec{y}(t)$ is the state vector of the initial or a reference orbit. Note that matrix (17.63) is another designation for matrix A considered in Chapter 11 (see Section 11.5). Thus, Eq. (11.24) gives

$$\Phi = \Phi(t, 0) = \tilde{U}^{-1}\tilde{U}_0,$$

$$\Psi = \Phi^{-1} = \tilde{U}_0^{-1}\tilde{U}$$

Matrix $\tilde{U} = A$ is inverted in Section 11.9.
Representing vector $\vec{y}$ as $\vec{y} = \{\vec{r}, \vec{v}\}$ and using Eqs. (17.63, 17.22) matrix (17.37) becomes

$$V = U_0 \int_0^T W_e \Psi_v \Psi_v^T d\tau \, U_0^T = \int_0^T W_e \frac{\partial \vec{q}}{\partial \vec{y}_0} \frac{\partial \vec{y}_0}{\partial \vec{v}} \left(\frac{\partial \vec{q}}{\partial \vec{y}_0} \frac{\partial \vec{y}_0}{\partial \vec{v}} \right)^T d\tau$$

$$= \int_0^T W_e \frac{\partial \vec{q}}{\partial \vec{v}} \left(\frac{\partial \vec{q}}{\partial \vec{v}} \right)^T d\tau \tag{17.64}$$

If the orbital elements are given by Eq. (11.30) then matrix $\partial\vec{q}/\partial\vec{v}$ in (17.64) consists of rows of matrix Q defined in Eq. (11.26). Hence, matrix (17.64) is a part of the matrix integral (15.24) calculated analytically in Section 15.4.

Thus, matrices $\tilde{U}^{-1}, \Phi, \Psi, V$ are calculated analytically for the two-body problem.

17.11. Local Optimality of the Method

Neighboring orbits in the two-body problem are considered here for simplicity. The case is considered when a unique element of the final orbit is specified, i.e. $m = 1$ and $\vec{q}_f = q_f$ is a given scalar; then $\vec{q}_i = q_i$ and $\Delta\vec{q} = \Delta q = q_f - q_i$ also are scalars.

$\Rightarrow$ matrix U_0 given by Eq. (17.30) is a row and matrix V given by Eq. (17.37) is a scalar

$\Rightarrow$ Eq. (17.38) becomes

$$\vec{\alpha} = W_e \Psi_v^T U_0^T V^{-1} \Delta q = s\left(U_0 \Psi_v\right)^T = s\left(\frac{\partial q_i}{\partial \vec{y}_{i0}} \frac{\partial \vec{y}_{i0}}{\partial \vec{v}_i}\right)^T = s\left(\frac{\partial q_i}{\partial \vec{v}_i}\right)^T \quad (17.65)$$

where $s = \dfrac{W_e \Delta q}{V}$ is a scalar parameter. As is shown in Section 13.5 vector $\vec{p} = \left(\dfrac{\partial q}{\partial \vec{v}}\right)^T$ defines a locally optimal thrust vector

$\Rightarrow$ due to Eq. (17.65) <u>the suggested method gives a locally optimal solution</u> in the linear approximation.

17.12. Conclusions

Disadvantages of the suggested method

- It is applicable only to the LP thrust
- It does not give globally optimal solution

Advantages of the suggested method

- The method is good for
  - any gravity field
  - transfers with any number of orbits
- The method is
  - semi-analytical (analytical for the two-body problem)
  - simple
  - fast
- The method can be applied to the following transfer types
  - point to orbit
  - orbit to point
  - orbit to orbit
- The method can be used in the cases of
  - partly given final orbit
  - constraints put on the thrust direction

The suggested method can be used at early phases of the mission design

Annex A. Comparison of two-impulse transfers between circular and elliptic orbits

Final orbit is outside the initial one

Eq. (6.22) gives

$$\frac{\Delta v_\pi - \Delta v_\alpha}{v_0} = \eta \tag{A.1}$$

where

$$\eta = \sqrt{2}\left[(1-\xi_1)\sqrt{\frac{\xi_0}{\xi_1(1+\xi_1)}} + \frac{\xi_1 - \xi_0}{\sqrt{\xi_1(\xi_0+\xi_1)}} - \frac{1-\xi_0}{\sqrt{1+\xi_0}}\right], \tag{A.2}$$

ξ_0, ξ_1 subject to (6.20).
As is seen in (A.1, A.2)

$$\Delta v_\pi = \Delta v_\alpha$$

if $\xi_1 = \xi_0$ or $\xi_1 = 1$.
Fig. A.1 shows values of (A.1) versus ξ_0 for different values of ξ_1

$$\Rightarrow \quad \boxed{\Delta v_\pi \geq \Delta v_\alpha} \tag{A.3}$$

for all ξ_0, ξ_1 subject to (6.20).

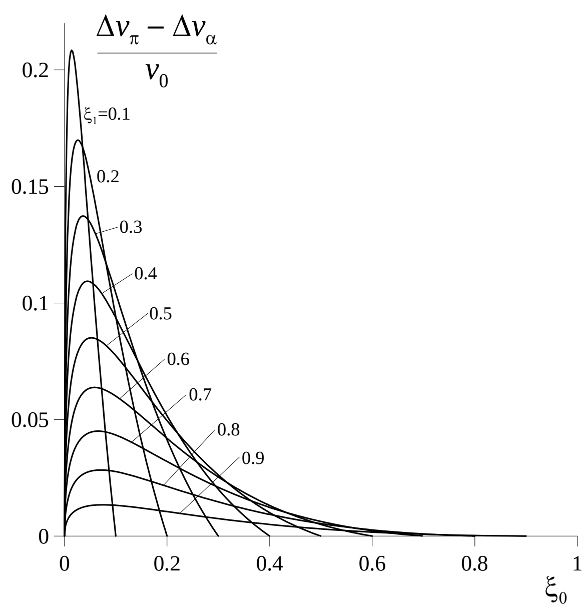


Fig. A.1

Final orbit is inside the initial one

Eq. (6.26) gives

$$\frac{\Delta v_\alpha - \Delta v_\pi}{v_0} = \eta \tag{A.4}$$

where η is given by (A.2), ξ_0, ξ_1 subject to (6.24).
As is seen in (A.2, A.4)

$$\Delta v_\pi = \Delta v_\alpha$$

if $\xi_0 = 1$ or $\xi_1 = 1$.

Fig. A.2 shows values of (A.4) versus $1/\xi_0$ for different values of ξ_1

$$\Rightarrow \quad \boxed{\Delta v_\pi \le \Delta v_\alpha} \qquad \text{(A.5)}$$

for all ξ_0, ξ_1 subject to (6.24).

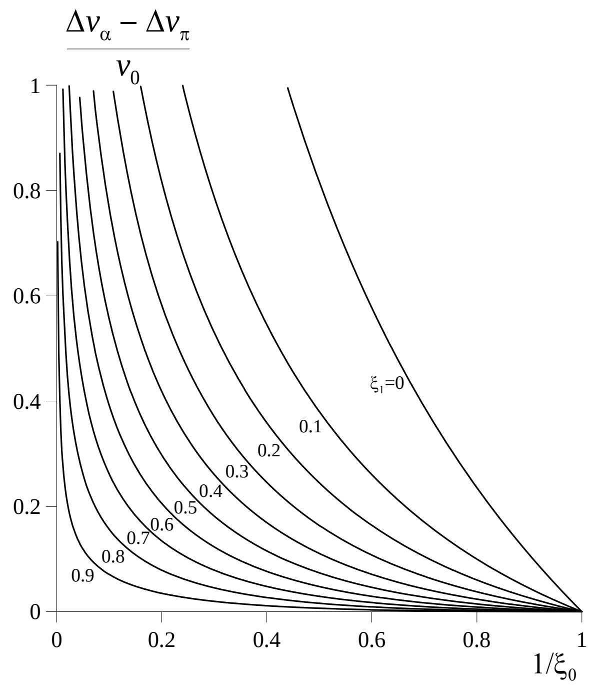


Fig. A.2

Initial and final orbits intersect

Eq. (6.29, 6.30) give

$$\frac{\Delta v_\pi - \Delta v_\alpha}{\sqrt{2}v_0} = \sqrt{2} - \sqrt{1+\xi_0} + \sqrt{\frac{\xi_0}{\xi_1(1+\xi_1)}} + \sqrt{\frac{\xi_0\xi_1}{1+\xi_1}} - \sqrt{\frac{\xi_0+\xi_1}{\xi_1}}$$

$$= \sqrt{2} - \sqrt{1+\xi_0} + \sqrt{\frac{\xi_0(1+\xi_1)}{\xi_1}} - \sqrt{\frac{\xi_0+\xi_1}{\xi_1}} \qquad \text{(A.6)}$$

Multiplying and dividing first and second members of (A.6) by $\sqrt{2}+\sqrt{1+\xi_0}$ and third and fourth members of (A.6) by $\sqrt{\xi_0(1+\xi_1)}+\sqrt{\xi_0+\xi_1}$ gives

$$\frac{\Delta v_\pi - \Delta v_\alpha}{\sqrt{2}v_0} = \frac{1-\xi_0}{\sqrt{2}+\sqrt{1+\xi_0}} - \frac{(1-\xi_0)\sqrt{\xi_1}}{\sqrt{\xi_0(1+\xi_1)}+\sqrt{\xi_0+\xi_1}}$$

$$= (1-\xi_0)\frac{\sqrt{\xi_0(1+\xi_1)}+\sqrt{\xi_0+\xi_1}-\sqrt{2\xi_1}-\sqrt{\xi_1(1+\xi_0)}}{\left(\sqrt{2}+\sqrt{1+\xi_0}\right)\left[\sqrt{\xi_0(1+\xi_1)}+\sqrt{\xi_0+\xi_1}\right]} \qquad \text{(A.7)}$$

Due to (6.28) $1-\xi_0 \ge 0$, $\sqrt{\xi_0+\xi_0\xi_1} \ge \sqrt{\xi_1+\xi_0\xi_1}$, $\sqrt{\xi_0+\xi_1} \ge \sqrt{2\xi_1}$

$\Rightarrow$ numerator and denominator in (A.7) are positive

$$\Rightarrow \quad \boxed{\Delta v_\pi \ge \Delta v_\alpha} \qquad \text{(A.8)}$$

and $\Delta v_\pi = \Delta v_\alpha$ if $\xi_1 = \xi_0$ or $\xi_1 = 1$.

Annex B. Projections onto sets and their properties

Let us consider a vector space with elements $\vec{a}, \vec{b}, \ldots$ and with norm $\|\vec{a}\| = a = \sqrt{\vec{a}^T \vec{a}}$ and metric $\rho(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\|$. Superscript "0" will denote unit vectors, i.e. $\vec{a}^0 = \vec{a}/a$.

Definition. Vector projection (or projection) of $\vec{b}$ onto $\vec{a}$ is the vector

$$\vec{b}_a = (\vec{b}^T \vec{a}^0)\vec{a}^0 = \vec{a}^0 \vec{a}^{0T} \vec{b} \tag{B.1}$$

$\Rightarrow$ matrix

$$P = \vec{a}^0 \vec{a}^{0T} \tag{B.2}$$

projects any vector onto $\vec{a}$, i.e.

$$\vec{b}_a = P\vec{b} \tag{B.3}$$

Matrix P is projective matrix, or projector, and has the following properties:

$$P^T = P^2 = P \tag{B.4}$$

Definition. Projection $\vec{b}_A$ of vector $\vec{b}$ onto a closed set of vectors A is a projection onto a vector $\vec{a} \in A$ on which $d = \max \vec{b}^T \vec{a}^0$ (or $\min\|\vec{b} - \vec{a}^0\|$) is reached if $d > 0$ and $\vec{b}_A = \vec{0}$ if $d \le 0$.

Let $\vec{a}^0$ be unit vector of $\vec{b}_A$ if $\vec{b}_A \ne \vec{0}$ and $\vec{a}^0 = \vec{0}$ if $\vec{b}_A = \vec{0}$

$\Rightarrow$ matrix (B.2) projects $\vec{b}$ onto A.

Definition. Non-zero projection $\vec{b}_A$ will be called also absolute projection onto the set.

Note. Projection onto a set may not belong to the set. For example, projection onto a set of unit vectors not necessarily is unit vector.

<u>Lemma B.1</u>. a) If $\exists \; x > 0$: $x\vec{b} \in A$ then $\vec{b}_A = \vec{b}$.

b) If $x\vec{b} \notin A$ for any $x > 0$ then $\vec{b}_A$ is reached in the boundary of the set A.

<u>Proof</u>

a) Due to the definition of the projection onto a set if $x\vec{b} \in A$ then $\max \vec{b}^T \vec{a}^0$ is reached when
$\vec{a} = x\vec{b} \;\Rightarrow\; \vec{a}^0 = \vec{b}^0 \;\Rightarrow\; \vec{b}_A = \vec{a}^0 \vec{a}^{0T} \vec{b} = \vec{b}$.

b) Assume that the projection $\vec{b}_A$ is reached at a vector $\vec{a}$ which is an intrinsic point of the set A $\Rightarrow$ $\vec{a}$ is contained in A together with its vicinity $\Rightarrow$ there is a vector $\vec{a}_\varepsilon$ in the vicinity forming a smaller angle with $\vec{b}$ than vector $\vec{a}$, i.e. $\vec{b}^T \vec{a}^0 < \vec{b}^T \vec{a}_\varepsilon^0$ what contradicts the definition of the projection onto a set.

Lemma B.1 is illustrated by Fig. B.1.

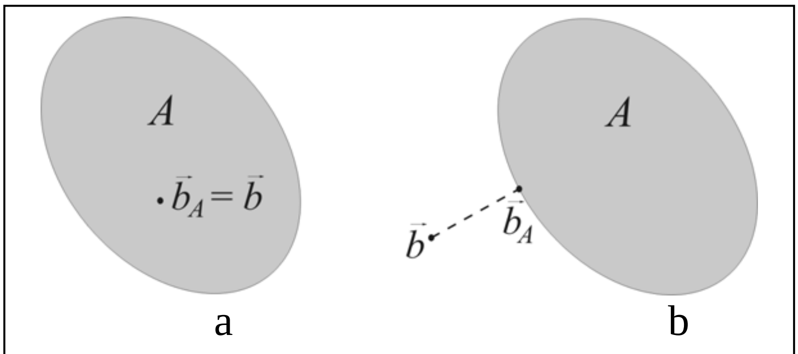


Fig. B.1

<u>Definition</u>. Projection $\vec{b}_i$ of $\vec{b}$ onto a vector $\vec{a}_i \in A$ $(i = 1, 2, \ldots)$ is a <u>local projection</u> onto A if there is a vicinity of $\vec{a}_i$ such that for any vector $\vec{a}_\varepsilon$ belonging to A and this vicinity

$$\vec{b}^T \vec{a}_\varepsilon^0 \le \vec{b}^T \vec{a}_i^0 \qquad \text{(B.5)}$$

(In other words a local $\max \vec{b}^T \vec{a}_i^0$ is reached at the local projection.)

Absolute and local projections are shown in Fig. B.2.

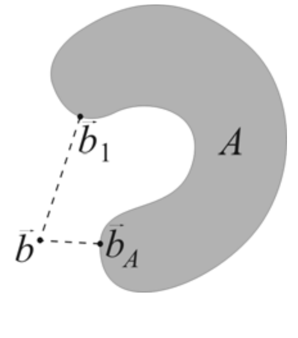


Fig. B.2

Let us consider intersection of s sets:

$$A = \bigcap_{j=1}^{s} A_j \tag{B.6}$$

<u>Lemma B.2</u>. Absolute projection onto the intersection of sets (B.6) is reached either at a projection (absolute or local) onto one of the sets or at the intersection of boundaries of at least two sets.

<u>Proof</u>

If $\exists \; x > 0$ such that $x\vec{b} \in A$ then due to lemma B.1 $\vec{b}_A = \vec{b}$ and $\vec{b}_A$ is absolute projection of $\vec{b}$ for each of the sets A_j.

Assume that $x\vec{b} \notin A$ for any $x > 0 \;\Rightarrow$ due to lemma B.1 projection $\vec{b}_A$ is reached in the boundary of the set A, i.e. in the boundary of one set (case 1) or a few (case 2) sets A_j. Case 2 corresponds to the lemma statement.

Assume that vector $\vec{a} \in A$ in which $\vec{b}_A$ is reached belongs to the boundary of a set A_i $(1 \le i \le s)$ and is an intrinsic point for other sets $\Rightarrow \; \exists$ a vicinity V of $\vec{a}$: $V \subset A_j, j \neq i$. Assume that $\vec{b}_A$ is not a projection onto $A_i \;\Rightarrow\; \exists \;\; \vec{a}_\varepsilon \in A_i,\; \vec{a}_\varepsilon \in V$:

$$\vec{b}^T \vec{a}_\varepsilon^0 > \vec{b}^T \vec{a}^0 \tag{B.7}$$

On the other hand $\vec{a}_\varepsilon \in A \;\Rightarrow$ due to (B.7) $\vec{b}_A$ is not reached in $\vec{a}$. This contradiction proves the lemma.

Lemma B.2 is illustrated by Fig. B.3 where $\vec{b}_1, \vec{b}_{A1}, \vec{b}_{A2}$ are local projection of $\vec{b}$ onto A_1 and absolute projections of $\vec{b}$ onto A_1, A_2.

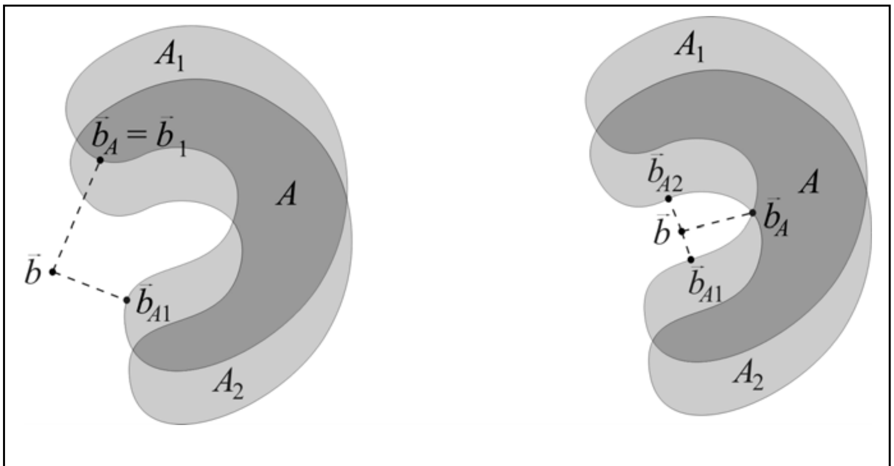


Fig. B.3

Lemma B.3. Let $\vec{b}_{Ai}$ be absolute projection of $\vec{b}$ onto set A_i reaching in vector $\vec{a}_i$ and $\vec{a}_i \in A$ where set A is intersection of the sets $A_1, \ldots, A_s$. Then $\vec{b}_{Ai}$ is absolute projection of $\vec{b}$ onto A.

Proof

Absolute projection of $\vec{b}$ onto A_i is reached in $\vec{a}_i$

$$\Rightarrow \quad \vec{b}^T \vec{a}^0 \le \vec{b}^T \vec{a}_j^0 \tag{B.8}$$

for any $\vec{a} \in A_j \Rightarrow$ (B.8) is fulfilled for any $\vec{a} \in A$ what proves lemma.

Lemma B.4. If there is unique projection of $\vec{b}$ onto each of the sets $A_1, \ldots, A_s$ then:

1) If $\exists\ A_i$ $(1 \le i \le s)$ such that projection $\vec{b}_i$ of $\vec{b}$ onto A_i is reached in vector $\vec{a}_i \in A$ then such a set is unique and $\vec{b}_A = \vec{b}_i$.
2) If among sets $A_1, \ldots, A_s$ there is no one projection of $\vec{b}$ on which is reached in vector $\vec{a}_i \in A$ then projection of $\vec{b}$ onto A is reached in intersection of at least two sets.

Proof

1) Projection $\vec{b}_i$ onto A_i is at the same time projection of $\vec{b}$ onto A due to lemma B.3. Let us assume that $\exists\ A_j$ $(j \ne i)$ such that projection of $\vec{b}$ onto A_j is reached in vector $\vec{a}_j \in A$. This is only possible if

$$\vec{b}^T \vec{a}_i^0 = \vec{b}^T \vec{a}_j^0 \tag{B.9}$$

However, if $\vec{a}_j \in A$ then $\vec{a}_j \in A_i \Rightarrow$ (B.9) contradicts uniqueness of projection onto A_i.

2) This statement follows from lemma B.2.

Lemmas B.3 and B.4 are illustrated by Fig. B.4.

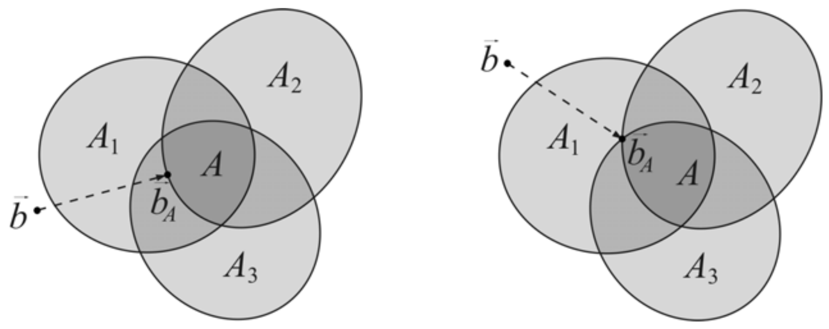


Fig. B.4

Annex C. Calculation of integral $\int\limits_{t_0}^{t} \frac{dt}{r^{n/2}}$

Designate

$$x = \frac{m}{m_0} = 1 - \frac{\dot{m}_p \tau}{m_0} \qquad \text{(C.1)}$$

From (16.10, 16.15, C.1) obtain:

$$\ln x = \eta - \eta_0 \qquad \text{(C.2)}$$

Consider indefinite integrals:

$$l_n = l_n(t) = m_0 \int (\eta_0 + \ln x)^n \, dx \quad (n = 0, 1, 2, ...) \qquad \text{(C.3)}$$

These integrals can be calculated by parts:

$$l_n = m_0 x (\eta_0 + \ln x)^n - m_0 \int x \, d(\eta_0 + \ln x)^n$$

$$= m(\eta_0 + \eta - \eta_0) - n\, m_0 \int (\eta_0 + \ln x)^{n-1} dx$$

$$\Rightarrow \quad \boxed{l_0 = m_0 x = m, \quad l_n = m\eta^n - n l_{n-1}} \qquad \text{(C.4)}$$

From (C.4) obtain l_n in another form:

$$l_n = m\left[\eta^n - n\eta^{n-1} + n(n-1)\eta^{n-2} - ... - (-1)^n n!\eta + (-1)^n n!\right] \qquad \text{(C.5)}$$

(the arbitrary constant is ignored in (C.4, C.5)).
Now consider the definite integral

$$L_n = m_0 \int\limits_{t_0}^{t} (\eta_0 + \ln x)^n \, dx \quad (n = 0, 1, 2, ...) \qquad \text{(C.6)}$$

Due to (C.4)

$$\boxed{\begin{aligned} L_0 &= l_0(t) - l_0(t_0) = m - m_0 \\ L_n &= l_n(t) - l_n(t_0) = m\eta^n - m_0\eta_0^n - nL_{n-1} \end{aligned}} \qquad \text{(C.7)}$$

Eqs. (16.7, 16.13, 16.15, C.1, C.6) give

$$\int_{t_0}^{t} \frac{dt}{r^{n/2}} = \frac{m_0}{r_0^{n/2} \eta_0^n \dot{m}_p} \int_{x_0}^{x} (\eta_0 + \ln x)^n \, dx = \mp \frac{u^n L_n}{r_0^{n/2} \dot{m}_p \sqrt{(\mu/r_0)^n}} \tag{C.8}$$

Taking into account (C.1, C.2) it can be shown that:

<u>for odd *n*</u>:

$$L_n = \begin{cases} <0 \ \text{ if } \ 0 < \eta < \eta_0 \ \text{ (acceleration)} \\ >0 \ \text{ if } \ \eta < \eta_0 < 0 \ \text{ (braking)} \end{cases} \tag{C.9}$$

<u>for even *n*</u>:

$$L_n < 0 \tag{C.10}$$

Eqs. (C.8–C.10) finally give

$$\boxed{\int_{t_0}^{t} \frac{dt}{r^{n/2}} = \frac{u^n |L_n|}{\mu^{n/2} \dot{m}_p}} \tag{C.11}$$