In the case of an extremely high altitude plume, the scale factor ρ_a is large, resulting in relatively small values of R. Therefore, the transcendental terms can be replaced by $1/\rho_a$ and R, respectively, resulting in an explicit relationship of x/r_j as a function of r/r_j. For example, utilizing Eq. (4) and deleting all terms with powers greater than $(1/\rho_a)^2$ result in

$$\left(\frac{x}{r_j}\right)_0 = \frac{1}{\phi} - \frac{\phi}{3}\left[1 - \frac{1}{3\rho_a^2}\right] - \frac{\phi^3}{45}\left[1 - \frac{1}{\rho_a^2}\right] \quad (7)$$

Applying the same criteria to Eq. (3) results in a cubic equation; i.e.,

$$\left(\frac{r}{r_j}\right)^3 + \left[\frac{\rho_a^2(45 - 15\phi^2 - \phi^4)}{5\phi^2 + \phi^4}\right]\left(\frac{r}{r_j}\right) - \left[\frac{45\rho_a^2[x/r_j + (x/r_j)_0]}{5\phi + \phi^3}\right] = 0 \quad (8)$$

which is of the form

$$(r/r_j)^3 + a(r/r_j) + b[b = f(x/r_j)] = 0 \quad (9)$$

The solution to Eq. (9), for the range of values of interest, consists of one real root and two imaginary roots. Equations (5) and (6) can also be expressed as

$$\frac{r}{r_j} = 1 + \frac{3\phi(x/r_j)}{(3 - \phi^2)} \qquad \left(\frac{x}{r_j}\right)_0 = \frac{3 - \phi^2}{3\phi} \quad (10)$$

where r/r_j is a linear function of x/r_j once ϕ has been specified and is applicable for extremely large values of ρ_a.

The present solution, Eqs. (3) and (4), has been applied to the case discussed in Charwat's analysis; Fig. 3 shows the comparison among the present results and those obtained by the method of characteristics and by the approximate method of Adamson and Nicholls.[3] The two-term series representation is an excellent approximation ($\phi = 0.61885$), and is also within $\frac{1}{2}\%$ of the method-of-characteristics results. Also shown in Fig. 3 is the limiting form of Eqs. (3) and (4), which has been applied to a large pressure ratio case[4] ($P_T/P_\infty = 0.27 \times 10^5$, $\rho_a = 36.17$). The results of Charwat,[1] in addition to the method-of-characteristics solution, are also shown. As can be seen, Eqs. (7) and (8) are in excellent agreement with Charwat's predictions.

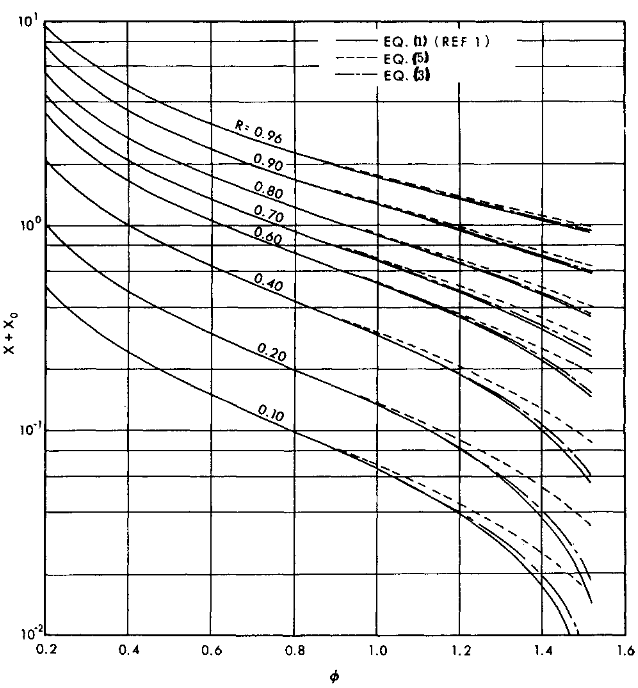

Fig. 2 Comparison of generalized jet boundary coordinates with approximate solutions.

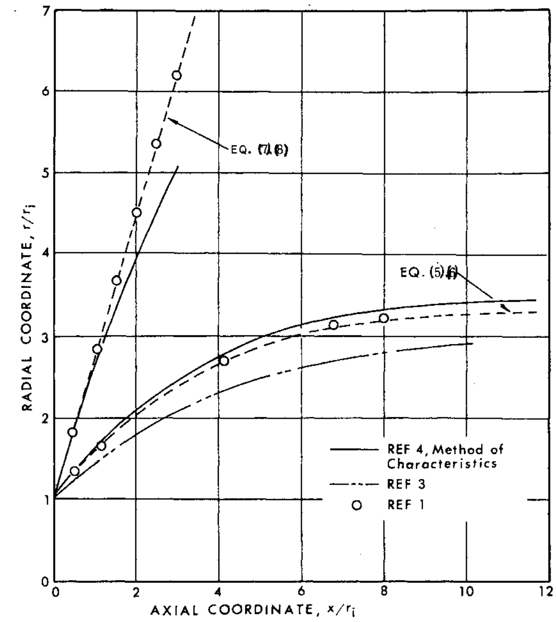

Fig. 3 Comparison of approximate solutions and method of characteristics.

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Lagrange Multipliers and the State Transition Matrix for Coasting Arcs

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Nomenclature

a_i = constant of integration ($i = 1, \ldots, 6$)
e = eccentricity
H = angular momentum vector
h = magnitude of H
p = semilatus rectum
R = position vector
r = magnitude of R
t = time
V = velocity vector
θ = true anomaly
λ = Lagrange multiplier vector
μ = gravitational parameter

Subscript

0 = reference condition

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1. Introduction

THE basis for solving optimal rocket trajectory problems has been shown by Lawden[3] and others to be the evaluation of the Lagrange multipliers. In many cases, the solution includes coasting periods in an inverse-square law gravitational field. If impulsive thrusting is allowed, the entire flight usually consists of coasting arcs separated by impulses. Then it is desirable to find for the coasting periods a closed-form solution of the Lagrange multipliers. Several forms of such solutions have been found. Hempel[2] and Eckenwiler[1] have derived solutions for two-dimensional flight. More recently, Lion and Handelsman[4] have derived a three-dimensional solution in which two of the coordinate-system axes lie in the plane of motion. For problems that involve multiple-coasting arcs in different orbital planes, the use of their results requires a coordinate-system transformation for each coasting period.

The solution presented here has the advantage that it can be used for flight in three dimensions with an arbitrarily oriented, Cartesian coordinate system. In addition, a single set of equations describes the solution for circular, elliptic, and hyperbolic trajectories. A change in the expression for one function results in a form that can be used for parabolic trajectories. The remaining case of rectilinear flight is not considered here.

2. Analysis

2.1 Equations of motion

The motion of a vehicle in an inverse-square-law gravitational field is governed by

$$\ddot{R} + (\mu/r^3)R = 0 \tag{1}$$

The solution to Eq. (1) can be written as

$$\begin{bmatrix} R \\ V \end{bmatrix} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{2}$$

where ξ and η are vectors of three elements, each of which can be determined from a reference state as

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \frac{1}{h} \begin{bmatrix} F_4 & -F_2 \\ -F_3 & F_1 \end{bmatrix}_{t_0} \begin{bmatrix} R_0 \\ V_0 \end{bmatrix} \tag{3}$$

and

$$\begin{aligned} F_1 &= r\cos\theta \qquad & F_3 &= -(h/p)\sin\theta \\ F_2 &= r\sin\theta \qquad & F_4 &= (h/p)(e+\cos\theta) \end{aligned} \tag{4}$$

The angular momentum vector is known to be constant during any coasting arc and is required for the analysis that follows. It is easily calculated from the reference state as

$$H = R_0 \times V_0 \tag{5}$$

2.2 Adjoint equations

The Lagrange multipliers are a solution to the adjoint equation

$$\ddot{\lambda} = -(\mu/r^3)\lambda + (3\mu/r^5)(R\cdot\lambda)R \tag{6}$$

Before proceeding with the derivation of the solution to Eq. (6), the motivation that led to the development of the form of the solution may be helpful. By forming certain vector dot and cross-products of λ and $\dot{\lambda}$ with $\ddot{R}$, R and $\dot{R}$ with $\ddot{\lambda}$, and combining the results, Pines[5] was able to derive five constants relating λ, $\dot{\lambda}$, R, and $\dot{R}$. Through a two-dimensional form of the state transition matrix, Lion and Handelsman[4] verified the results of Pines[5] and derived a sixth constant involving the in-plane components of λ and $\dot{\lambda}$. For all cases except rectilinear flight, R, V, and H are linearly independent, so that any three-dimensional vector can be expressed in terms of their unit vectors. An analysis of the six equations relating λ, $\dot{\lambda}$, R, and $\dot{R}$ led to the speculation that a closed-form solution for λ in terms of R, V, and H could be found. With such motivation the derivation proceeds formally.

Assume that λ can be written as

$$\lambda = \alpha R + \beta V + \gamma H \tag{7}$$

where α, β, and γ are scalar functions to be determined.

Successive differentiation of Eq. (7) gives

$$\dot{\lambda} = [\dot{\alpha} - (\mu/r^3)\beta]R + (\dot{\beta} + \alpha)V + \dot{\gamma}H \tag{8}$$

$$\ddot{\lambda} = \left(\ddot{\alpha} - \frac{2\mu}{r^3}\dot{\beta} + \frac{3\mu}{r^4}\dot{r}\beta - \frac{\mu}{r^3}\alpha\right)R + \left(\ddot{\beta} + 2\dot{\alpha} - \frac{\mu}{r^3}\beta\right)V + \ddot{\gamma}H \tag{9}$$

Substitution of Eq. (7) into the right-hand side of Eq. (6) gives

$$\ddot{\lambda} = \left(\frac{2\mu}{r^3}\alpha + \frac{3\mu}{r^4}\dot{r}\beta\right)R - \frac{\mu}{r^3}\beta V - \frac{\mu}{r^3}\gamma H \tag{10}$$

Equating the coefficients of R, V, and H in Eqs. (9) and (10) gives

$$\ddot{\alpha} - (2\mu/r^3)\dot{\beta} - (3\mu/r^3)\alpha = 0 \tag{11}$$

$$\ddot{\beta} + 2\dot{\alpha} = 0 \tag{12}$$

$$\ddot{\gamma} + (\mu/r^3)\gamma = 0 \tag{13}$$

The solution to the preceding three equations provides the desired result. Integration of Eq. (12) gives

$$\dot{\beta} = a_3 - 2\alpha \tag{14}$$

Substitution of Eq. (14) into Eq. (11) gives

$$\ddot{\alpha} + (\mu/r^3)\alpha = (2\mu/r^3)a_3 \tag{15}$$

which has as its solution

$$\alpha = a_1 r\cos\theta + a_2 r\sin\theta + 2a_3 \tag{16}$$

Substitution of Eq. (16) into Eq. (14) gives

$$\dot{\beta} = -2a_1 r\cos\theta - 2a_2 r\sin\theta - 3a_3 \tag{17}$$

Now let

$$g = 2\int r\cos\theta dt \quad , \quad f = 2\int r\sin\theta dt \tag{18}$$

The solution to Eq. (17) is then given by

$$\beta = -a_1 g - a_2 f - 3a_3 t + a_4 \tag{19}$$

It can be verified that the two integrals given by Eq. (18) are

$$f = -(p/h)(p+r)r\cos\theta \tag{20}$$

$$g = [(p/h)(p+r)r\sin\theta - 3ept]/(1-e^2) \qquad e \neq 1 \tag{21}$$

For the case of a parabolic orbit, it can be shown that the expression for g given by Eq. (21) becomes indeterminate. The proper expression for such cases is

$$g = \tfrac{2}{5}[3pt - (r^3/h)\sin\theta] \qquad e = 1 \tag{22}$$

The solution of Eq. (13) is contained in the solution of Eq, (15) given by Eq. (16). Thus,

$$\gamma = a_5 r\cos\theta + a_6 r\sin\theta \tag{23}$$

The desired solution is now complete in terms of the six constants of integration $(a_1\text{–}a_6)$. From Eq. (8) it is seen that expressions for $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ are required in order to evaluate $\dot{\lambda}$. The expression for $\dot{\beta}$ is given by Eq. (17), whereas the expressions for $\dot{\alpha}$ and $\dot{\gamma}$ are found by differentiating Eqs. (16) and (23):

$$\dot{\alpha} = -a_1(h/p)\sin\theta + a_2(h/p)(e+\cos\theta) \tag{24}$$

$$\dot{\gamma} = -a_5(h/p)\sin\theta + a_6(h/p)(e + \cos\theta) \tag{25}$$

Now let

$$\begin{aligned} F_5 &= \mu/r^3 \qquad F_6 = 3t \qquad F_7 = 3\mu t/r^3 \\ F_8 &= F_3 + gF_5 \qquad F_9 = F_4 + fF_5 \end{aligned} \tag{26}$$

and define $P(t)$ as

$$P(t) = \begin{bmatrix} F_1R - gV & F_2R - fV & 2R - F_6V & V & F_1H & F_2H \\ F_8R - F_1V & F_9R - F_2V & F_7R - V & -F_5R & F_3H & F_4H \end{bmatrix} \tag{27}$$

Equations (7) and (8) now can be written as

$$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix} = P(t)A \tag{28}$$

where A is a column vector of the six constants of integration.

For a given set of initial conditions for R, V, λ, and $\dot{\lambda}$, the integration constants can be found from Eq. (29):

$$A = P^{-1}(t_0)\begin{bmatrix} \lambda_0 \\ \dot{\lambda}_0 \end{bmatrix} \tag{29}$$

Once A has been determined, λ and $\dot{\lambda}$ can be evaluated from Eqs. (7) and (8).

2.3 *State transition matrix*

A perturbation in the state at some reference time t_0 is propagated through the state transition matrix $\Phi(t, t_0)$ as

$$\begin{bmatrix} \delta R \\ \delta V \end{bmatrix} = \Phi(t,t_0)\begin{bmatrix} \delta R_0 \\ \delta V_0 \end{bmatrix} \tag{30}$$

The solution to Eq. (6) can be written as

$$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix} = \Phi(t,t_0)\begin{bmatrix} \lambda_0 \\ \dot{\lambda}_0 \end{bmatrix} \tag{31}$$

Substitution of Eq. (29) into Eq. (28) gives

$$\begin{bmatrix} \lambda \\ \dot{\lambda} \end{bmatrix} = P(t)P^{-1}(t_0)\begin{bmatrix} \lambda_0 \\ \dot{\lambda}_0 \end{bmatrix} \tag{32}$$

Comparison of Eqs. (31) and (32) reveals that the state transition matrix can be expressed as

$$\Phi(t,t_0) = P(t)P^{-1}(t_0) \tag{33}$$

An analytic inversion of the matrix $P(t)$ was determined by calculating the determinant and the co-factors. The details are omitted here for brevity but the result is given by Eqs. (34) and (35)†

$$P^{-1}(t) = \frac{1}{h^3}\begin{bmatrix} F_2F_5\sigma - F_4w & 2F_4\sigma - F_2w \\ -F_1F_5\sigma + F_3w & -2F_3\sigma + F_1w \\ hw & -h\sigma \\ -b_1\sigma + b_2w & -b_3\sigma + b_4w \\ F_4H & -F_2H \\ -F_3H & F_1H \end{bmatrix} \tag{34}$$

$$\left.\begin{aligned} \sigma &= R \times H & w &= V \times H \\ b_1 &= h + F_5(F_1f - F_2g) & b_2 &= F_6h + F_3f - F_4g \\ b_3 &= F_6h + 2(F_3f - F_4g) & b_4 &= F_1f - F_2g \end{aligned}\right\} \tag{35}$$

It is interesting to note that the determinant of $P(t)$ was found to be h^6.

3. Summary

A simple analytic solution for the Lagrange multipliers has been derived in terms of flight time, Cartesian position, velocity and angular momentum vectors. The solution is valid for all types of conic arcs, except rectilinear, with a special form of the expression for one function for parabolic trajectories. The solution for the Lagrange multipliers was used to find the state transition matrix.

† The vectors σ, w, and H as used in Eq. (34) are row vectors.

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Experimental Identification of Shock Tunnel Flow Regimes

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Introduction

Smith[1] and Glick et al.[2] have proposed analytical models for the flow processes that occur during the starting of a hypersonic nozzle. In Smith's model an "unsteady expansion" wave dominates the flow starting process and the double shock system plays a minor role. In the analysis of Glick et al. the double shock system dominates the starting process. Smith[3] has obtained limited experimental data that show fair agreement with and tend to verify the flow model proposed in Ref. 1. The data of Ref. 3 were obtained at low enthalpy conditions. In the present study, tests were conducted in a high enthalpy, electrically arc heated helium driver shock tunnel at an M_s value of 13.6 and at P_1 = 150 torr (air). With this electrically arc heated helium driver, nearly tailored operation was achieved at this $M_s - P_1$ condition. This was the basic test condition for an MHD nozzle accelerator.[4] It was found during the course of acquiring measurements for

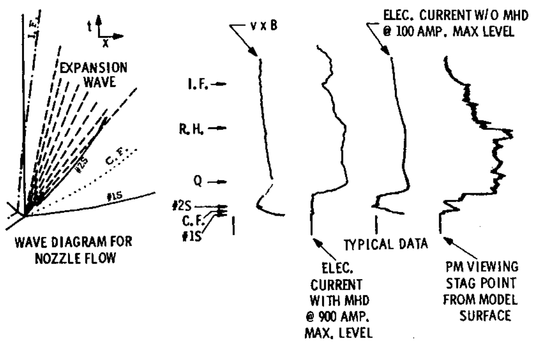

Fig. 1 Composite of some of the types of data used to define the nozzle flow regimes. The wave diagram is from Smith.[1] The traces shown were obtained at different nozzle stations and their time scales are accordingly adjusted so that similar flow events, such as R.H. and I.F., are aligned on the figure.

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