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OPTIMAL LOW-THRUST LIMITED POWER TRANSFER BETWEEN NEIGHBOURING QUASI-CIRCULAR ORBITS OF SMALL INCLINATIONS AROUND AN OBLATE PLANET

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Abstract—A complete analytic study about the influence of Earth s oblateness on the optimal low-thrust limited power transfer of small amplitude (orbit correction) between quasi-circular orbits of small inclinations is carried out up to the first order in a small parameter defined by the nondimensional thrust acceleration The coefficient for the second zonal harmonic J_{20} and the nondimensional thrust acceleration are supposed to be the same order of magnitude Hori's method for generalized canonical systems is applied in order to obtain the analytical solution for adjoint and state differential equations Simple analytic solutions are obtained explicitly for long-time transfer

1 INTRODUCTION

The problem of optimal low-thrust limited power transfer in a central Newtonian force field has been the subject of numerous studies (see Pitkin[1]) Most of these studies are concerned with numerical results, but analytic solutions exist for transfers between neighbouring orbits, i e orbit corrections[2–6] and for long-time transfers between arbitrary elliptical orbits[7,8] These analytic studies were carried out considering that the orbital changes due to the thrust are small and the thrust acceleration is small compared to the acceleration of gravity For orbit corrections a linearization of the equations of motion around a reference orbit could be made and the solution of the linearized problem could be calculated analytically For long-time transfer, where the fixed transfer duration is sufficiently long to allow a large number of revolutions around the planet, the determination of the optimal trajectories is facilitated by the use of the notion of "Mean Hamiltonian"[8]

In the present paper we will investigate the effects of Earth's oblateness on the optimal low-thrust limited power transfer between neighbouring quasi-circular orbits of small inclinations This optimal problem may be defined as follows it is proposed to transfer, between the prescribed initial time t_0 and the final time t_f also prescribed, a space vehicle from the initial orbit O_0 to the final orbit O_f such that the fuel consumption is minimum, considering an oblate gravity field The propulsion system that effects the orbital changes is assumed to be low-thrust limited power, i e it operates at constant exhaust power with variable ejection velocity The nondimensional thrust acceleration is supposed to be of the same order as the oblateness coefficient for the second zonal harmonic J_{20}

Hori's method for generalized canonical systems[9] is applied in order to obtain a complete analytic solution up to the first order in a small parameter defined by the nondimensional thrust acceleration Long-time transfer is analysed in detail and simple analytic solutions are obtained explicitly for three particular cases (i) modification of the semi-major axis, (ii) rotation of the eccentricity vector **e**, (iii) rotation of the orbital plane

2 EQUATIONS FOR OPTIMAL TRAJECTORIES

The optimal control problem related to the optimal transfer problem defined above will be formulated as a Mayer problem

The motion of a space vehicle in a quasi-circular orbit of small inclination is described by the following set of nonsingular variables[10]

$$a = \text{semi-major axis}$$
$$\lambda = M + \bar{\omega}$$
$$\xi = e\cos\bar{\omega}$$
$$\eta = e\sin\bar{\omega}$$
$$P = \sin\frac{I}{2}\cos\Omega$$
$$Q = \sin\frac{I}{2}\sin\Omega \qquad (1)$$

where a, e, I, Ω, ω, M are the known Keplerian elements and $\bar{\omega} = \omega + \Omega$

The fuel consumption is described by the following variable[11]

$$J = \frac{1}{2}\int_{t_0}^{t} \Gamma^2\,dt \qquad (2)$$

where Γ is the thrust acceleration, used as control variable The performance index to be maximized is then $-J_f$

Hence, the state vector $\mathbf{X}$ is defined by the nonsingular variables and the consumption variable, i e

$$\mathbf{X}^{\mathrm{T}} = [\mathbf{x}^{\mathrm{T}}\, J],$$

where

$$\mathbf{x}^{\mathrm{T}} = [a\, \lambda\, \xi\, \eta\, P\, Q]$$

The development of the equations of motion of a space vehicle in nonsingular variables, subject to the thrust acceleration and the geopotential can be obtained from Giacaglia's paper cited above This development is straightforward though tedious So, the state equations are given by

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{n} S \\
\frac{d\lambda}{dt} &= n\left[1 + 3J_{20}\left(\frac{a_e}{a}\right)^2\right] - \frac{2}{na} R \\
&\quad + \frac{W}{na}(P \sin\lambda - Q\cos\lambda) \\
\frac{d\xi}{dt} &= -\tfrac{3}{2} n J_{20}\left(\frac{a_e}{a}\right)^2 \sin\lambda \\
&\quad + \frac{R}{na}\sin\lambda + \frac{2S}{na}\cos\lambda \\
\frac{d\eta}{dt} &= \tfrac{3}{2} n J_{20}\left(\frac{a_e}{a}\right)^2 \cos\lambda \\
&\quad - \frac{R}{na}\cos\lambda + \frac{2S}{na}\sin\lambda \\
\frac{dP}{dt} &= \tfrac{3}{2} n J_{20}\left(\frac{a_e}{a}\right)^2 \\
&\quad \times (Q + Q\cos 2\lambda - P\sin 2\lambda) \\
&\quad + \frac{W}{2na}\cos\lambda \\
\frac{dQ}{dt} &= \tfrac{3}{2} n J_{20}\left(\frac{ae}{a}\right)^2 \\
&\quad \times (-P + P\cos 2\lambda + Q\sin 2\lambda) \\
&\quad + \frac{W}{2na}\sin\lambda \\
\frac{dJ}{dt} &= \tfrac{1}{2}(R^2 + S^2 + W^2) \qquad (3)
\end{aligned}$$

where R, S and W are the radial, circumferential and normal components of the thrust acceleration Γ, a_e is the Earth's equatorial radius and n is the mean motion

In this study we suppose that the nondimensional thrust acceleration γ and the coefficient for the second zonal harmonic J_{20} are of the same order of magnitude, i e $\gamma = \Gamma/n^2 a = O(J_{20})$

The adjoint vector $\mathbf{P}^{\mathrm{T}} = [\mathbf{p}^{\mathrm{T}}\, p_J]$, where $\mathbf{p}^{\mathrm{T}} = [p_a p_\lambda p_\xi p_\eta p_P p_Q]$ is introduced and the Hamiltonian function H is formed using eqns (3)

$$\begin{aligned}
H &= \frac{2}{n} S p_a + \left\{n + 2\epsilon - \frac{2}{na} R \right.\\
&\quad \left. + \frac{W}{na}(P\sin\lambda - Q\cos\lambda)\right\} p_\lambda \\
&\quad + \left\{-\epsilon\sin\lambda + \frac{R}{na}\sin\lambda + \frac{2S}{na}\cos\lambda\right\} p_\xi \\
&\quad + \left\{\epsilon\cos\lambda - \frac{R}{na}\cos\lambda + \frac{2S}{na}\sin\lambda\right\} p_\eta \\
&\quad + \left\{\epsilon(Q + Q\cos 2\lambda - P\sin 2\lambda) + \frac{W}{2na}\cos\lambda\right\} p_P \\
&\quad + \left\{\epsilon(-P + P\cos 2\lambda + Q\sin 2\lambda)\right. \\
&\quad \left. + \frac{W}{2na}\sin\lambda\right\} p_Q + \tfrac{1}{2}(R^2 + S^2 + W^2) p_J \qquad (4)
\end{aligned}$$

where

$$\epsilon = \tfrac{3}{2} n J_{20}\left(\frac{a_e}{a}\right)^2$$

The optimal control Γ^* must be selected in order to maximize the Hamiltonian

$$\Gamma^* = \arg\max H$$

Then, the optimal thrust acceleration is given by

$$\begin{aligned}
R^* &= -\frac{1}{p_J}\left\{-\frac{2}{na} p_\lambda + \frac{1}{na}(p_\xi \sin\lambda - p_\eta\cos\lambda)\right\} \\
S^* &= -\frac{1}{p_J}\left\{\frac{2}{n} p_a + \frac{2}{na}(p_\xi\cos\lambda + p_\eta\sin\lambda)\right\} \\
W^* &= -\frac{1}{p_J}\left\{\frac{p_\lambda}{na}(P\sin\lambda - Q\cos\lambda)\right. \\
&\quad \left. + \frac{1}{2na}(p_P\cos\lambda + p_Q\sin\lambda)\right\} \qquad (5)
\end{aligned}$$

The optimal trajectories are generated by the maximized Hamiltonian, obtained from eqns (4) and (5)

$$\begin{aligned}
H^* = \max_{\Gamma} H &= (n + 2\epsilon) p_\lambda - \epsilon\{p_\xi\sin\lambda - p_\eta\cos\lambda \\
&\quad - (Q + Q\cos 2\lambda - P\sin 2\lambda) p_P \\
&\quad - (-P + P\cos 2\lambda + Q\sin 2\lambda) p_Q\} \\
&\quad + (1 + \tfrac{1}{2} p_J)\Gamma^{*2} \qquad (6)
\end{aligned}$$

These optimal trajectories are obtained by solving the canonical equations

$$\dot{\mathbf{x}}^{\mathrm{T}} = \frac{\partial H^*}{\partial \mathbf{P}}, \quad \dot{\mathbf{P}}^{\mathrm{T}} = -\frac{\partial H^*}{\partial \mathbf{X}} \qquad (7)$$

An optimal trajectory must satisfy the boundary conditions at the initial time $t_0 = 0$, $\mathbf{X}(t_0) = \mathbf{X}_0$, and at the prescribed final time $t = t_f$, $\mathbf{X}(t_f) = \mathbf{X}_f$, as well as the transversality conditions that provide $p_{J_f} = -1$ and $p_{\lambda_f} = 0$ for transfer problem

The adjoint variable p_J is one of the first integrals for dynamical system (7), since J is an ignorable variable

$$p_J = \text{constant} = p_{J_f} = -1 \tag{8}$$

Then, from eqns (5) and (8) we obtain the optimal thrust acceleration

$$R^* = -\frac{2}{na}p_\lambda + \frac{1}{na}(p_\xi \sin\lambda - p_\eta \cos\lambda)$$

$$S^* = \frac{2}{n}p_a + \frac{2}{na}(p_\xi \cos\lambda + p_\eta \sin\lambda)$$

$$W^* = \frac{p_\lambda}{na}(P\sin\lambda - Q\cos\lambda) + \frac{1}{2na}(p_P\cos\lambda + p_Q\sin\lambda) \tag{9}$$

From eqns (6) and (9) we obtain the maximized Hamiltonian as a function of the adjoint and state variables

$$\begin{aligned} H^* = {} & (n+2\epsilon)p_\lambda - \epsilon\{p_\xi \sin\lambda - p_\eta\cos\lambda \\ & -(Q + Q\cos 2\lambda - P\sin 2\lambda)p_P \\ & -(-P + P\cos 2\lambda + Q\sin 2\lambda)p_Q\} \\ & + \frac{1}{2n^2a^2}\{4p_\lambda^2 + 4p_a^2a^2 + \tfrac{5}{2}(p_\xi^2 + p_\eta^2) \\ & + \tfrac{1}{2}p_\lambda^2(P^2+Q^2) + \tfrac{1}{8}(p_P^2 + p_Q^2) \\ & + \tfrac{1}{2}p_\lambda(Pp_Q - Qp_P) - (4p_\xi p_\lambda - 8ap_ap_\eta)\sin\lambda \\ & + (4P_\lambda p_\eta + 8ap_ap_\xi)\cos\lambda + [-PQp_\lambda^2 + \tfrac{1}{4}p_Pp_Q \\ & + \tfrac{1}{2}p_\lambda(Pp_P - Qp_Q) + 3p_\xi p_\eta]\sin 2\lambda \\ & + [\tfrac{3}{2}(p_\xi^2 - p_\lambda^2) + \tfrac{1}{2}p_\lambda^2(Q^2 - P^2) + \tfrac{1}{8}(p_P^2 - p_Q^2) \\ & - \tfrac{1}{2}p_\lambda(Qp_P + Pp_Q)]\cos 2\lambda\} \end{aligned} \tag{10}$$

This maximized Hamiltonian can be written as follows

$$H^* = H_0^* + H_{J_{20}}^* + H_{\Gamma}^*,$$

where H_0^* is of the first order in γ and is the undisturbed Hamiltonian Since the nondimensional thrust acceleration and the coefficient for the second zonal harmonic J_{20} are of the same order, $H_{J_{20}}^*$ is of the second order in γ and is the part of the maximized Hamiltonian due to Earth's oblateness H_Γ^* is also of the second order in γ and is the remaining part due to the optimal thrust acceleration $H_{J_{20}}^*$ and H_Γ^* are the disturbing part of the Hamiltonian

3 APPLICATION OF HORI'S METHOD FOR GENERALIZED CANONICAL SYSTEMS

In order to solve the canonical system defined by eqns (7) and (10), we will apply the generalized canonical form of Hori's method for non-canonical systems[9], since these equations are generalized canonical form of the differential system (3) for optimal trajectories

Although H_0^* is of the first order, and $H_{J_{20}}^*$ and H_Γ^* are of the second order in a small parameter, that may be defined as being of the same magnitude of γ, in order to apply Hori's method we will consider H_0^* as being of the zeroth order, and, $H_{J_{20}}^*$ and H_Γ^* as being of the first order in the small parameter, since the Hamiltonian H^* may be factored by this small parameter This device corresponds to define a new time scale, $dt' = \alpha dt$ where α is of the same order of the small parameter

A canonical transformation

$$(\mathbf{p}, \mathbf{x}) \xrightarrow{S} (\boldsymbol{\pi}, \boldsymbol{\chi})$$

will be made by the generating function $S(\boldsymbol{\pi}, \boldsymbol{\chi})$ that is a power series of the small parameter The equation of the transformation for the order $n \geqslant 1$ is given by[12]

$$\frac{dS^{(n)}}{d\tau} = \Psi^{(n)}(\tau) - H^{(n)} \tag{11}$$

where $\Psi^{(n)}$ is a known function from the preceding orders, $H^{(n)}$ is the new Hamiltonian and τ is an auxiliary parameter introduced by the Hori auxiliary system defined by the undisturbed dynamical system

$$\frac{d\boldsymbol{\chi}^T}{d\tau} = \frac{\partial H^{(0)}}{\partial \boldsymbol{\pi}},$$

$$\frac{d\boldsymbol{\pi}^T}{d\tau} = -\frac{\partial H^{(0)}}{\partial \boldsymbol{\chi}},$$

that is

$$\begin{aligned} \frac{d\chi_i}{d\tau} &= 0, \\ \frac{d\pi_1}{d\tau} &= \frac{3}{2}\sqrt{\frac{\mu}{\chi_1^5}}\pi_2, \\ \frac{d\chi_2}{d\tau} &= \sqrt{\frac{\mu}{\chi_1^3}}, \\ \frac{d\pi_j}{d\tau} &= 0 \end{aligned} \tag{12}$$

where μ is the gravitational parameter, $i = 1, 3, 4, 5, 6$ and $j = 2, 3, 4, 5, 6$ Note that by construction of Hori's method $H_0^* = H^{(0)}$ (see Hori[12])

The solution of the Hori auxiliary system is

$$\begin{aligned} \chi_i &= c_i, \\ \pi_1 &= \frac{3}{2}\sqrt{\frac{\mu}{c_1^5}}b_2\tau + b_1, \\ \chi_2 &= \sqrt{\frac{\mu}{c_1^3}}\tau + c_2, \\ \pi_j &= b_j \end{aligned} \tag{13}$$

where b_k and c_k, $k = 1, 2, \ldots, 6$, are arbitrary constants with respect to τ.

In order to solve eqn (11) Hori applied the averaging principle and obtained the following solutions

$$H^{(n)} = \langle \Psi^{(n)} \rangle_\tau, \tag{14}$$

$$S^{(n)} = \int (\Psi^{(n)} - H^{(n)})\, d\tau, \tag{15}$$

where the symbol $\langle\ \rangle_\tau$ means the average with respect to τ.

So, the new Hamiltonian H and the generating function are given up to the first order in the small parameter by the following equations

zeroth order

$$H^{(0)} = \sqrt{\frac{\mu}{c_1^3}}\, b_2 \tag{16}$$

first order

$$\begin{aligned} H^{(1)} = {} & 3\sqrt{\frac{\mu}{c_1^3}} J_{20}\left(\frac{a_e}{c_1}\right)^2 [b_2 - \tfrac{1}{2}(b_6 c_5 - b_5 c_6)] \\ & + \frac{c_1}{2\mu}\left\{4b_2^2 + 4c_1^2\left(\frac{3}{2}\sqrt{\frac{\mu}{c_1^5}}\, b_2 \tau + b_1\right)^2\right. \\ & + \tfrac{5}{2}(b_3^2 + b_4^2) + \tfrac{1}{2} b_2^2 (c_5^2 + c_6^2) \\ & + \tfrac{1}{8}(b_5^2 + b_6^2) + \tfrac{1}{2} b_2 (b_6 c_5 - b_5 c_6) \\ & + 12\sqrt{\frac{\mu}{c_1^3}}\, b_2 \tau \left[b_4 \sin\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right. \\ & \left.\left. + b_3 \cos\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right]\right\} \end{aligned} \tag{17}$$

$$\begin{aligned} S^{(1)} = {} & \tfrac{3}{2} J_{20}\left(\frac{a_e}{c_1}\right)^2 \left\{ b_3 \cos\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right. \\ & + b_4 \sin\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right) + \tfrac{1}{2}(b_5 c_6 + b_6 c_5) \\ & \times \sin 2\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right) + \tfrac{1}{2}(b_5 c_5 - b_6 c_6) \\ & \left.\times \cos 2\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right\} + \frac{1}{2}\sqrt{\frac{c_1^5}{\mu^3}} \\ & \times \left\{ 4b_2 \left[b_3 \cos\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right.\right. \\ & \left. + b_4 \sin\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right] \\ & + 8 b_1 c_1 \left[- b_4 \cos\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right. \\ & \left. + b_3 \sin\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right] \\ & - \tfrac{1}{2}[-b_2^2 c_5 c_6 + \tfrac{1}{4} b_5 b_6 \\ & + \tfrac{1}{2} b_2 (b_5 c_5 - b_6 c_6) + 3 b_3 b_4] \\ & \times \cos 2\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right) + \tfrac{1}{2}[\tfrac{3}{2}(b_3^2 - b_4^2) \\ & + \tfrac{1}{2} b_2^2 (c_6^2 - c_5^2) + \tfrac{1}{8}(b_5^2 - b_6^2) \\ & - \tfrac{1}{2} b_2 (b_5 c_6 + b_6 c_5)] \\ & \left. \times \sin 2\left(\sqrt{\frac{\mu}{c_1^3}}\,\tau + c_2\right)\right\} \end{aligned} \tag{18}$$

It should be noted that, by construction of Hori's method, the generating function is free from secular or mixed secular terms [eqn (18)] that appear in the new Hamiltonian [eqns (16) and (17)].

According to Hori's method for generalized canonical systems we must apply the Lagrange variational equations[13] in order to obtain the time variations of the "constants", $\mathbf{b}$ and $\mathbf{c}$, as well as, of the auxiliary parameter τ of the Hori auxiliary system. The set $(\mathbf{b}, \mathbf{c})$ is a new set of canonical variables, since the solution of Hori auxiliary system defines a new canonical transformation—a Mathieu transformation[9]

$$(\boldsymbol{\pi}, \boldsymbol{\chi}) \xrightarrow{\text{Mathieu}} (\mathbf{b}, \mathbf{c})$$

Then, the new dynamical system is given by

$$\frac{d\mathbf{c}^T}{dt} = \frac{\partial H^{(1)}}{\partial \mathbf{b}}, \quad \frac{d\mathbf{b}^T}{dt} = -\frac{\partial H^{(1)}}{\partial \mathbf{c}}, \tag{19}$$

and the time dependence of the parameter τ is given by

$$\frac{d\tau}{dt} = 1 + \left(\frac{\partial \tau}{\partial \boldsymbol{\chi}}\right)\left(\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{c}}\right)\left(\frac{\partial H^{(1)}}{\partial \mathbf{b}}\right)^T \tag{20}$$

It should be noted that in optimal control problems we also need to solve the adjoint equations of the new dynamical system, while for a generalized canonical system it is not necessary, because the adjoint variables are artificially introduced.

4. SOLUTION OF THE NEW DYNAMICAL SYSTEM

Since we will condiser only transfers of small amplitude (orbit corrections), the new Hamiltonian and the generating function defined by eqns (16)–(18) can be linearized around a reference orbit. Hence, the new dynamical system (19) and eqn (20) for the parameter τ are given by

$$\frac{dc_1}{dt} = 4\frac{\bar{c}_1^3}{\mu}\left[\frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^5}}\, b_2 \tau + b_1\right], \tag{21a}$$

$$\frac{dc_2}{dt} = 3\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2 + \frac{\bar{c}_1}{2\mu}$$
$$\times\left\{8b_2 + 12\sqrt{\frac{\mu}{\bar{c}_1}}\tau\left(\frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^5}}b_2\tau + b_1\right)\right.$$
$$+ b_2(c_5^2 + c_6^2) + \tfrac{1}{2}(c_5 b_6 - c_6 b_5)$$
$$+ 12\sqrt{\frac{\mu}{\bar{c}_1^3}}b_2\tau\left[b_4\sin\left(\sqrt{\frac{\mu}{\bar{c}_1^3}}\tau + c_2\right)\right.$$
$$\left.\left. + b_3\cos\left(\sqrt{\frac{\mu}{\bar{c}_1^3}}\tau + c_2\right)\right]\right\}, \tag{21b}$$

$$\frac{dc_3}{dt} = \frac{\bar{c}_1}{2\mu}\left\{5b_3 + 12\sqrt{\frac{\mu}{\bar{c}_1^3}}b_2\,\tau\cos\left(\sqrt{\frac{\mu}{\bar{c}_1^3}}\tau + c_2\right)\right\}, \tag{21c}$$

$$\frac{dc_4}{dt} = \frac{\bar{c}_1}{2\mu}\left\{5b_4 + 12\sqrt{\frac{\mu}{\bar{c}_1^3}}b_2\,\tau\sin\left(\sqrt{\frac{\mu}{\bar{c}_1^3}}\tau + c_2\right)\right\}, \tag{21d}$$

$$\frac{dc_5}{dt} = \frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2 c_6 + \frac{\bar{c}_1}{2\mu}\left(\tfrac{1}{4}b_5 - \tfrac{1}{2}b_2c_5\right), \tag{21e}$$

$$\frac{dc_6}{dt} = -\frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2 c_5 + \frac{\bar{c}_1}{2\mu}\left(\tfrac{1}{4}b_6 + \tfrac{1}{2}b_2c_5\right), \tag{21f}$$

$$\frac{db_i}{dt} = 0(\gamma^2), \quad i = 1, 2, \tag{21g}$$

$$\frac{db_j}{dt} = 0, \quad j = 3, 4, \tag{21h}$$

$$\frac{db_5}{dt} = \frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2 b_6 - \frac{\bar{c}_1}{2\mu}\left(b_2^2 c_5 + \tfrac{1}{2}b_2 b_6\right), \tag{21i}$$

$$\frac{db_6}{dt} = -\frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2 b_5 - \frac{\bar{c}_1}{2\mu}\left(b_2^2 c_6 - \tfrac{1}{2}b_2 b_5\right), \tag{21j}$$

and

$$\frac{d\tau}{dt} = 1 + \sqrt{\frac{\bar{c}_1^3}{\mu}}\frac{dc_2}{dt} - \frac{3}{2}\frac{\tau}{\bar{c}_1}\frac{dc_1}{dt} \tag{22}$$

where the overbar denotes the reference orbit

Now, we may integrate these set of differential equations by successive approximations It should be noted that the differential equations for b_5, b_6, c_5 and c_6 form a linear system of differential equations which coefficients depend on b_2 But two important results simplify the integration of these equations, since we will consider the first order solution only $(b_6c_5 - b_5c_6)$ is a first integral of this system of differential equations and $b_2 = 0$, because $p_{\lambda_f} = 0$ from the transversality condition So, we obtain

$$c_1 = c_{10} + 4\sqrt{\frac{\bar{c}_1^9}{\mu^3}}b_{10}\bar{\lambda}, \tag{23a}$$

$$c_2 = c_{20} + 3J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2\bar{\lambda} + \frac{\bar{c}_1}{2\mu}\left\{6\,b_{10}\sqrt{\frac{\bar{c}_1^5}{\mu}}\bar{\lambda}^2\right.$$
$$\left. + \frac{1}{2}\sqrt{\frac{\bar{c}_1^3}{\mu}}\bar{\lambda}(b_{60}c_{50} - b_{50}c_{60})\right\} \tag{23b}$$

$$c_3 = c_{30} + \tfrac{5}{2}b_{30}\sqrt{\frac{\bar{c}_1^5}{\mu^3}}\bar{\lambda}, \tag{23c}$$

$$c_4 = c_{40} + \tfrac{5}{2}b_{40}\sqrt{\frac{\bar{c}_1^5}{\mu^3}}\bar{\lambda}, \tag{23d}$$

$$c_5 = c_{50}\cos\bar{\varepsilon}t + c_{60}\sin\bar{\varepsilon}t + \frac{1}{8}\sqrt{\frac{\bar{c}_1^5}{\mu^3}}$$
$$\times\bar{\lambda}(b_{50}\cos\bar{\varepsilon}t + b_{60}\sin\bar{\varepsilon}t), \tag{23e}$$

$$c_6 = -c_{50}\sin\bar{\varepsilon}t + c_{60}\cos\bar{\varepsilon}t - \frac{1}{8}\sqrt{\frac{\bar{c}_1^5}{\mu^3}}$$
$$\times\bar{\lambda}(b_{50}\sin\bar{\varepsilon}t - b_{60}\cos\bar{\varepsilon}t) \tag{23f}$$

$$b_i = b_{i0}, \quad i = 1, 2, 3, 4, \tag{23g}$$

$$b_5 = b_{50}\cos\bar{\varepsilon}t + b_{60}\sin\bar{\varepsilon}t, \tag{23h}$$

$$b_6 = -b_{50}\sin\bar{\varepsilon}t + b_{60}\cos\bar{\varepsilon}t, \tag{23i}$$

where

$$\bar{\varepsilon} = \frac{3}{2}\sqrt{\frac{\mu}{\bar{c}_1^3}}J_{20}\left(\frac{a_e}{\bar{c}_1}\right)^2$$

and

$$\bar{\lambda} = \sqrt{\frac{\mu}{\bar{c}_1^3}}\,t$$

5 OPTIMAL TRAJECTORIES

The formal solution of the original dynamical system defined by eqns (7) and (10) may be obtained through the generating function $S(\boldsymbol{\pi}, \boldsymbol{\chi})$ and for that it is necessary to calculate the Poisson brackets $\{\boldsymbol{\pi}, S\}$ and $\{\boldsymbol{\chi}, S\}$ However from Hori's method the generating function is given as a function of **b**, **c** and τ (eqn (18)] Nevertheless, the application of Hori's method generates a canonical transformation from the variables (**p**, **x**) to the new variables (**b**, **c**) resulting from the composition of the canonical infinitesimal transformation S with the Mathieu transformation

$$(\mathbf{p}, \mathbf{x}) \xrightarrow{S} (\boldsymbol{\pi}, \boldsymbol{\chi}) \xrightarrow{\text{Mathieu}} (\mathbf{b}, \mathbf{c})$$

Therefore, the Poisson brackets $\{\boldsymbol{\pi}, S\}$ and $\{\boldsymbol{\chi}, S\}$ can be calculated directly in terms of the new set of canonical variables (**b**, **c**)

Since for transfer problems the final position in the orbit is unspecified, only the orbital elements a, ξ, η, P and Q need to be considered So, the optimal trajectory is given up to the first order in the small parameter by the following equations, obtained from eqns (18) and (23) by application of Hori's algorithm for generalized canonical systems [9]

$$a = a_0 + 4\sqrt{\frac{\bar{a}^9}{\mu^3}}p_{a_0}(\bar{\lambda} - \bar{\lambda}_0) + 4\sqrt{\frac{\bar{a}^7}{\mu^3}}$$
$$\times(-p_{\eta_0}\cos\bar{\lambda} + p_{\xi_0}\sin\bar{\lambda})_{\bar{\lambda}_0}^{\bar{\lambda}}, \tag{24a}$$

$$\xi = \xi_0 + \tfrac{5}{2}p_{\xi_0}\sqrt{\frac{\bar{a}^5}{\mu^3}}(\bar{\lambda} - \bar{\lambda}_0) + \tfrac{3}{2}J_{20}\left(\frac{a_e}{\bar{a}}\right)^2 \cos\bar{\lambda}\Big|_{\bar{\lambda}_0}^{\bar{\lambda}} + \frac{1}{2}\sqrt{\frac{\bar{a}^7}{\mu^3}}\{8\bar{a}p_{a_0}\sin\bar{\lambda} + \tfrac{3}{2}(-p_{\eta_0}\cos 2\bar{\lambda} + p_{\xi_0}\sin 2\bar{\lambda})\}_{\bar{\lambda}_0}^{\bar{\lambda}}, \tag{24b}$$

$$\eta = \eta_0 + \tfrac{5}{2}p_{\eta_0}\sqrt{\frac{\bar{a}^5}{\mu^3}}(\bar{\lambda} - \bar{\lambda}_0) + \tfrac{3}{2}J_{20}\left(\frac{a_e}{\bar{a}}\right)^2 \times \sin\bar{\lambda}\Big|_{\bar{\lambda}_0}^{\bar{\lambda}} + \frac{1}{2}\sqrt{\frac{\bar{a}^7}{\mu^3}}\{-8\bar{a}p_{a_0}\cos\bar{\lambda} - \tfrac{3}{2}(p_{\xi_0}\cos 2\bar{\lambda} + p_{\eta_0}\sin 2\bar{\lambda})\}_{\bar{\lambda}_0}^{\bar{\lambda}}, \tag{24c}$$

$$\begin{aligned}
P = \left(1 - \frac{\delta^2}{4}\right)^{-1} \Bigg\{ & P_0\Bigg[\cos\delta(\bar{\lambda} - \bar{\lambda}_0) - \frac{\delta}{2}\cos(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) + \frac{\delta}{2}\cos(\delta\bar{\lambda}_0 - (2+\delta)\bar{\lambda}) - \frac{\delta^2}{4}\cos((2+\delta)(\bar{\lambda} - \bar{\lambda}_0))\Bigg] \\
& + Q_0\Bigg[\sin\delta(\bar{\lambda} - \bar{\lambda}_0) + \frac{\delta}{2}\sin(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) - \frac{\delta}{2}\sin(\delta\bar{\lambda}_0 - (2+\delta)\bar{\lambda}) + \frac{\delta^2}{4}\sin((2+\delta)(\bar{\lambda} - \bar{\lambda}_0))\Bigg] \\
& - \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}p_{P_0}\Bigg[\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \times \cos\delta(\bar{\lambda} - \bar{\lambda}_0) - \tfrac{1}{2}\sin(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) - \frac{\delta}{4}\sin\delta(\bar{\lambda} - \bar{\lambda}_0) + \frac{\delta}{2}\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \\
& \times \cos(\delta\bar{\lambda}_0 - (2+\delta)\bar{\lambda}) - \frac{\delta}{4}\sin((2+\delta)(\bar{\lambda} - \bar{\lambda}_0)) - \frac{\delta^2}{8}\sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0)\Bigg] \\
& - \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}p_{Q_0}\Bigg[\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \times \sin\delta(\bar{\lambda} - \bar{\lambda}_0) - \tfrac{1}{2}\cos(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) + \frac{\delta}{4}\cos\delta(\bar{\lambda} - \bar{\lambda}_0) + \frac{\delta}{2}\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \\
& \times \sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) - \frac{\delta}{4}\cos((2+\delta)(\bar{\lambda} - \bar{\lambda}_0)) + \frac{\delta^2}{8}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0)\Bigg]\Bigg\} \\
& + \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}\{\bar{\lambda}[p_{P_0}\cos\delta(\bar{\lambda} - \bar{\lambda}_0) + p_{Q_0}\sin\delta(\bar{\lambda} - \bar{\lambda}_0)] + \frac{\delta}{2}\bar{\lambda}[p_{P_0}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) + p_{Q_0}\sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0)] \\
& + \tfrac{1}{2}[p_{P_0}\sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) - p_{Q_0}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0)]\},
\end{aligned} \tag{24d}$$

$$\begin{aligned}
Q = \left(1 - \frac{\delta^2}{4}\right)^{-1} \Bigg\{ & P_0\Bigg[-\sin\delta(\bar{\lambda} - \bar{\lambda}_0) + \frac{\delta}{2}\sin(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) + \frac{\delta}{2}\sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) - \frac{\delta^2}{4}\sin((2+\delta)(\bar{\lambda} - \lambda_0))\Bigg] \\
& + Q_0\Bigg[\cos\delta(\bar{\lambda} - \bar{\lambda}_0) + \frac{\delta}{2}\cos(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) - \frac{\delta}{2}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) - \frac{\delta^2}{4}\cos((2+\delta)(\bar{\lambda} - \bar{\lambda}_0))\Bigg] \\
& - \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}p_{P_0}\Bigg[-\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \times \sin\delta(\bar{\lambda} - \bar{\lambda}_0) - \tfrac{1}{2}\cos(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) - \frac{\delta}{4}\cos\delta(\bar{\lambda} - \bar{\lambda}_0) + \bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \\
& \times \frac{\delta}{2}\sin((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) + \frac{\delta}{4}\cos((2+\delta)(\bar{\lambda} - \bar{\lambda}_0)) + \frac{\delta^2}{8}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0)\Bigg] \\
& - \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}p_{Q_0}\Bigg[\bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right)\cos\delta(\bar{\lambda} - \bar{\lambda}_0) + \tfrac{1}{2}\sin(\delta\bar{\lambda} - (2+\delta)\bar{\lambda}_0) - \frac{\delta}{4}\sin\delta(\bar{\lambda} - \bar{\lambda}_0) - \bar{\lambda}_0\left(1 - \frac{\delta^2}{4}\right) \\
& \times \frac{\delta}{2}\cos((2+\delta)\bar{\lambda} - \delta\bar{\lambda}_0) - \frac{\delta}{4}\sin((2+\delta)(\bar{\lambda} - \bar{\lambda}_0))
\end{aligned}$$

$$+\frac{\delta^2}{8}\sin((2+\delta)\bar{\lambda}-\delta\bar{\lambda}_0)\Big]\Big\}$$

$$-\frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}\Big\{\bar{\lambda}[p_{P_0}\sin\delta(\bar{\lambda}-\bar{\lambda}_0)$$

$$-p_{Q_0}\cos\delta(\bar{\lambda}-\bar{\lambda}_0)]$$

$$-\frac{\delta}{2}\bar{\lambda}[p_{P_0}\sin((2+\delta)\bar{\lambda}-\delta\bar{\lambda}_0)$$

$$-p_{Q_0}\cos((2+\delta)\bar{\lambda}-\delta\bar{\lambda}_0)]$$

$$+\tfrac{1}{2}[p_{P_0}\cos((2+\delta)\bar{\lambda}-\delta\bar{\lambda}_0)$$

$$+p_{Q_0}\sin((2+\delta)\bar{\lambda}-\delta\bar{\lambda}_0)]\Big\}, \tag{24e}$$

$$p_a=p_{a_0},\quad p_\xi=p_{\xi_0},\quad p_\eta=p_{\eta_0}, \tag{24f}$$

$$p_P=p_{P_0}\cos\delta(\bar{\lambda}-\bar{\lambda}_0)+p_{Q_0}\sin\delta(\bar{\lambda}-\bar{\lambda}_0), \tag{24g}$$

$$p_Q=-p_{P_0}\sin\delta(\bar{\lambda}-\bar{\lambda}_0)+p_{Q_0}\cos\delta(\bar{\lambda}-\bar{\lambda}_0) \tag{24h}$$

where the subscript "0" indicates the initial values and

$$\delta=\tfrac{3}{2}J_{20}\left(\frac{a_e}{\bar{a}}\right)^2$$

These equations represent a complete first order solution for optimal transfer between neighbouring quasi-circular orbits of small inclinations in an oblate gravity field They contain five arbitrary constants, the initial values for the adjoint variables p_{a_0}, p_{ξ_0}, p_{η_0}, p_{P_0} and p_{Q_0}, that must be determined so as to satisfy the two-point boundary-value problem of going from the initial orbit O_0 at time $t_0=0$ to the final orbit O_f at the prescribed time t_f These equations are linear in the initial values for the adjoint variables $p_{a_0}, p_{\xi_0}, p_{\eta_0}, p_{P_0}, p_{Q_0}$, i e they can be written as follows

$$\Delta\mathbf{x}'=A\ \mathbf{p}_0'+B \tag{25}$$

where $\Delta\mathbf{x}'$ represents the imposed changes in the five orbital elements (a, ξ, η, P, Q), $\mathbf{p}_0'$ is the initial value of the adjoint vector, A is a 5×5 matrix and B is a vector Both, the matrix A and the vector B, include the Earth's oblateness effects Therefore, the two-point boundary-value problem can be solved by straightforward techniques

The general form of matrix A, that can be decomposed into two square matrices 3×3 and 2×2, shows that exists an uncoupling between the modifications in the orbital plane and the rotation of the orbital plane This general result is similar to the results obtained by Edelbaum[2] and Marec[5,6] for transfers between neighbouring elliptical orbits in a central gravity field

6 THE PRIMER LOCUS

Once the optimal trajectory has been calculated, i e the two-point boundary-value problem has been solved, we can write the optimal thrust acceleration

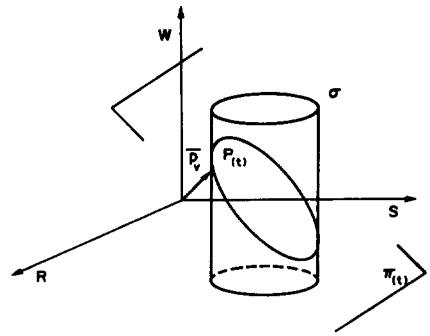


Fig 1 Primer locus ("polaire")

concerning to the linearized problem as follows

$$R^*=\frac{1}{\bar{n}\bar{a}}(p_\xi\sin\bar{\lambda}-p_\eta\cos\bar{\lambda})$$

$$S^*=\frac{2}{\bar{n}}p_a+\frac{2}{\bar{n}\bar{a}}(p_\xi\cos\bar{\lambda}+p_\eta\sin\bar{\lambda})$$

$$W^*=\frac{1}{2\bar{n}\bar{a}}(p_P\cos\bar{\lambda}+p_Q\sin\bar{\lambda}) \tag{26}$$

where p_a, p_ξ and p_η are constants and p_P and p_Q are given by eqns (24g) and (24h) as functions of $\bar{\lambda}$

These equations define at the time t an ellipse in the three-dimensional space RSW, named "primer locus" by Lawden[13] or "polaire" by Marec[5,6], resulting from the intersection between an elliptical cylinder σ which generatrices are parallel to the W axis and a plane π that passes through the intersection of the cylinder axis with the plane RS (Fig 1) The cylinder σ and the plane π are described by eqns (27) and (28), respectively It should be noted that the plane π is no longer fixed, it changes with the time, since the adjoint variables p_P and p_Q are time functions Therefore, we may define the primer locus as an "osculating" ellipse

$$R^{*2}+\frac{1}{4}\left(S^*-\frac{2}{\bar{n}}p_a\right)^2=\frac{1}{(\bar{n}\bar{a})^2}(p_\xi^2+p_\eta^2) \tag{27}$$

$$2(p_\xi p_Q-p_\eta p_P)R^*+(p_\eta p_Q+p_\xi p_P)\times\left(S^*-\frac{2}{\bar{n}}p_a\right)-4(p_\xi^2+p_\eta^2)\,W^*=0 \tag{28}$$

7 LONG-TIME TRANSFERS

As the time for transfer is much greater than the period of the reference orbit so that the transfer requires a large number of revolutions (see Edelbaum[2] and Marec[5,6]) eqns (24) can be greatly simplified and an explicit solution easily obtained This is done by neglecting the short periodic terms in comparison with the secular, long-periodic and mixed secular terms in eqns (24) It should be noted that the long periodic and mixed secular terms only appear in these equations due to the inclusion of Earth's oblateness effects Therefore, for long-time transfer, the changes in the state variables are given by

$$\Delta a = 4\sqrt{\frac{\bar{a}^9}{\mu^3}}\Delta\bar{\lambda}\, p_{a_0}, \tag{29a}$$

$$\Delta\xi = \frac{5}{2}\sqrt{\frac{\bar{a}^5}{\mu^3}}\Delta\bar{\lambda}\, p_{\xi_0}, \tag{29b}$$

$$\Delta\eta = \frac{5}{2}\sqrt{\frac{\bar{a}^5}{\mu^3}}\Delta\bar{\lambda}\, p_{\eta_0}, \tag{29c}$$

$$\begin{aligned}\Delta P &= P_0(\cos\delta\Delta\bar{\lambda} - 1) + Q_0\sin\delta\Delta\bar{\lambda} \\ &\quad + \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}\Delta\bar{\lambda}\,(p_{P_0}\cos\delta\Delta\bar{\lambda} \\ &\quad + p_{Q_0}\sin\delta\Delta\bar{\lambda}),\end{aligned} \tag{29d}$$

$$\begin{aligned}\Delta Q &= -P_0\sin\delta\Delta\bar{\lambda} + Q_0(\cos\delta\Delta\bar{\lambda} - 1) \\ &\quad - \frac{1}{8}\sqrt{\frac{\bar{a}^5}{\mu^3}}\Delta\bar{\lambda}(p_{P_0}\sin\delta\Delta\bar{\lambda} \\ &\quad - p_{Q_0}\cos\delta\Delta\bar{\lambda})\end{aligned} \tag{29e}$$

By solving these equations for the initial values of the adjoint variables p_{a_0}, p_{ξ_0}, p_{η_0}, p_{P_0} and p_{Q_0}, we may express explicitly the fuel consumption as a function of the changes in the state variables and the initial values for P and Q variables, that define the osculating orbital plane at the time t_0 Therefore,

$$\begin{aligned}\Delta J &= \frac{1}{2}\sqrt{\frac{\mu^3}{\bar{a}^5}}(\Delta\bar{\lambda})^{-1}\left\{\frac{1}{4}\left(\frac{\Delta a}{\bar{a}}\right)^2\right. \\ &\quad + \tfrac{2}{5}(\Delta\xi^2 + \Delta\eta^2) + 8(\Delta P^2 + \Delta Q^2) \\ &\quad - 2[(P_0^2 + Q_0^2)(\cos\delta\Delta\bar{\lambda} - 1) \\ &\quad + \sqrt{(\Delta P^2 + \Delta Q^2)(P_0^2 + Q_0^2)} \\ &\quad \times \cos(\delta\Delta\lambda - \theta - \phi) \\ &\quad \left. - (P_0\Delta P + Q_0\Delta Q)]\right\},\end{aligned} \tag{30}$$

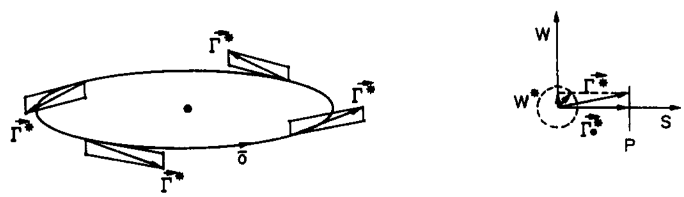
where $\tan\theta = -\Delta Q/\Delta P$ and $\tan\phi = Q_0/P_0$

Equation (30) may be written as follows

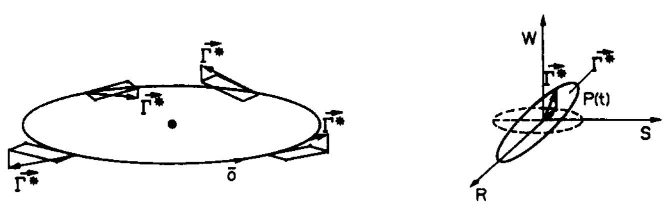
$$\Delta J = \Delta J_0 + \Delta J_{J_{20}}, \tag{31}$$

where ΔJ_0 denotes the fuel consumption for the undisturbed problem (i e the transfer problem in a central gravity field) and $\Delta J_{J_{20}}$ denotes the fuel consumption required to counteract the effects of Earth's oblateness

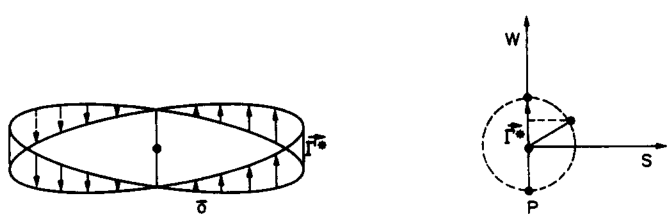
It should be noted that, for long-time transfers, if the variation of one of the orbital elements a, ξ, η, P



(a) Modification of the semi-major axis



(b) Rotation of the eccentricity vector $\bar{e}$



(c) Rotation of the orbital plane

Fig 2 Long-time transfers

and Q is imposed, then there is no variation induced in the four other elements

The optimum thrust acceleration program and the "primer locus"[14] for three particular transfers (modification of the semi-major axis, rotation of the eccentricity vector **e** and rotation of the orbital plane) are shown pictorially in Fig 2 These optimum programs show that exists a normal component of the optimum thrust acceleration $\boldsymbol{\Gamma}^*$ that counteracts the effects of Earth's oblateness (the changes on the orbital plane)

8 CONCLUSION

This paper has been concerned with the optimal low-thrust limited power transfer between neighbouring quasi-circular orbits of small inclinations in an oblate gravity field The complete analytic solution for optimum thrust program has been carried out up to the first order in the nondimensional thrust acceleration The general result obtained here is similar to the results obtained by Edelbaum[2] and Marec[5,6] for transfers between neighboring elliptical orbits in a central gravity field, i e there is an uncoupling between the modifications in the orbital plane and the rotation of the orbital plane The two-point boundary-value problem can be solved by straightforward techniques, since the equations for the changes in the orbital elements (a, ξ, η, P, Q) are linear in the initial values of the adjoint variables $(p_{a_0}, p_{\xi_0}, p_{\eta_0}, p_{P_0}, p_{Q_0})$ For long-time transfer the two-point boundary-value problem is easily solved Three particular long-time transfers have been analysed modification of the semi-major axis, rotation of the eccentricity vector **e** and rotation of the orbital plane Their optimum thrust acceleration programs show that exists a normal component of $\boldsymbol{\Gamma}^*$ that counteracts the effects of Earth's oblateness

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