SEMIANALYTIC SATELLITE THEORY

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Contents

1   Introduction

Modern space surveillance requires fast and accurate orbit predictions for myriads of objects in a broad range of Earth orbits. Conventional Special Perturbations orbit propagators, based on numerical integration of the osculating equations of motion, are accurate but extremely slow (typically requiring 100 or more steps per satellite revolution to give good predictions). Conventional General Perturbations orbit propagators, based on fully analytical orbit theories like those of Brouwer, are faster but contain large errors due to inherent approximations in the theories. New orbit generators based on Semianalytic Satellite Theory (SST) have been able to approach the accuracy of Special Perturbations propagators and the speed of General Perturbations propagators.

SST has been originated by P. J. Cefola and his colleagues, whose names are in the references at the end of this document. The theory is scattered throughout the listed conference preprints, published papers, technical reports, and private communications. Our purpose in this document is to simplify, assemble, unify, and extend the theory. This document includes truncation algorithms and corrects misprints in our earlier work [Danielson, Neta, and Early, 1994].

SST represents the orbital state of a satellite with an equinoctial element set $(a_1, \ldots, a_6)$. The first five elements $a_1, \ldots, a_5$ are slowly varying in time. The sixth element a_6 is the mean longitude λ and so is rapidly varying.

SST decomposes the *osculating* elements $\hat{a}_i$ into *mean* elements a_i plus a small remainder which is 2π *periodic* in the fast variable:

$$\hat{a}_i = a_i + \eta_i(a_1, \ldots, a_6, t) \tag{1}$$

(Here we use hats to distinguish the elements of the osculating ellipse from the elements of the averaging procedure. The values of a free index are assumed to be obvious from the context; e. g. , here i can have the values 1, 2, 3, 4, 5, or 6, so (1) represents 6 equations.) The mean elements a_i are governed by ordinary differential equations of the form

$$\frac{da_i}{dt} = n\delta_{i6} + A_i(a_1, \ldots, a_5, t) \tag{2}$$

Here t is the time, n is the (mean) mean motion, and δ_{i6} is the Kronecker delta (i. e. , $\delta_{16} = \delta_{26} = \delta_{36} = \delta_{46} = \delta_{56} = 0$, $\delta_{66} = 1$). The short-periodic variations η_i are expressable in Fourier series of the form

$$\eta_i = \sum_{j=1}^{\infty} [C_i^j(a_1, \ldots, a_5, t) \cos j\lambda + S_i^j(a_1, \ldots, a_5, t) \sin j\lambda] \tag{3}$$

Having formulas for the mean element rates A_i, we can integrate the mean equations (2) numerically using large step sizes (typically 1 day in length). The formulas for the Fourier coefficients C_i^j and S_i^j in (3) also only need to be evaluated at the integrator step times. Values of the osculating elements $\hat{a}_i$ at request times not coinciding with the integrator step times can be computed from (1) using interpolation formulas.

In subsequent chapters we will outline the methods of derivation and give explicit formulas for the terms A_i, C_i^j, S_i^j corresponding to various perturbing forces.

2 Mathematical Preliminaries

2.1 Equinoctial Elements

The generalized method of averaging can be applied to a wide variety of orbit element sets. The equinoctial elements were chosen for SST because the variational equations for the equinoctial elements are nonsingular for all orbits for which the generalized method of averaging is appropriate–namely, all elliptical orbits.

In this chapter we give an overview of the equinoctial elements, which are *osculating* (even though they do not have hats). They are discussed in more detail in [Broucke and Cefola, 1972], [Cefola, Long, and Holloway, 1974], [Long, McClain, and Cefola, 1975], [Cefola and Broucke, 1975], [McClain, 1977 and 1978], and [Shaver, 1980].

2.1.1 Definition of the Equinoctial Elements

There are six elements in the equinoctial element set:

$$
\begin{aligned}
a_1 &= a && \text{semimajor axis} \\
a_2 &= h && \left.\right\} \text{ components of the eccentricity vector} \\
a_3 &= k && \\
a_4 &= p && \left.\right\} \text{ components of the ascending node vector} \\
a_5 &= q && \\
a_6 &= \lambda && \text{mean longitude}
\end{aligned}
$$

The semimajor axis a is the same as the Keplerian semimajor axis. The eccentricity vector has a magnitude equal to the eccentricity and it points from the central body to perigee. Elements h and k are the $\mathbf{g}$ and $\mathbf{f}$ components, respectively, of the eccentricity vector in the equinoctial reference frame defined below. The ascending node vector has a magnitude which depends on the inclination and it points from the central body to the ascending node. Elements p and q are the $\mathbf{g}$ and $\mathbf{f}$ components, respectively, of the ascending node vector in the equinoctial reference frame.

There are actually two equinoctial element sets: the *direct* set and the *retrograde* set. As the names imply, the direct set is more appropriate for direct satellites and the retrograde set is more appropriate for retrograde satellites. It is possible, however, to use direct elements for retrograde satellites and vice versa, and for non-equatorial satellites this presents no problem. For equatorial satellites there are singularities which must be avoided by choosing the appropriate equinoctial element set. For direct elements

$$
\lim_{i \to \pi} \sqrt{p^2 + q^2} = \infty \tag{1}
$$

while for retrograde elements

$$
\lim_{i \to 0} \sqrt{p^2 + q^2} = \infty \tag{2}
$$

For each equinoctial element set there are three associated vectors $(\mathbf{f}, \mathbf{g}, \mathbf{w})$ which define the equinoctial reference frame. These vectors form a right-handed orthonormal triad with the following properties:
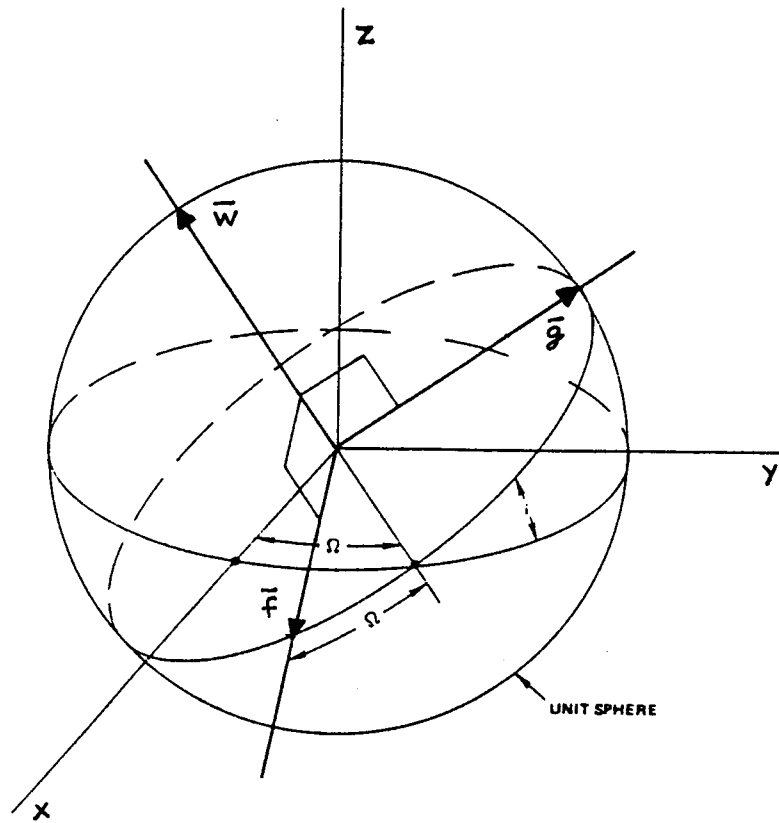
Figure 1: Direct Equinoctial Reference Frame
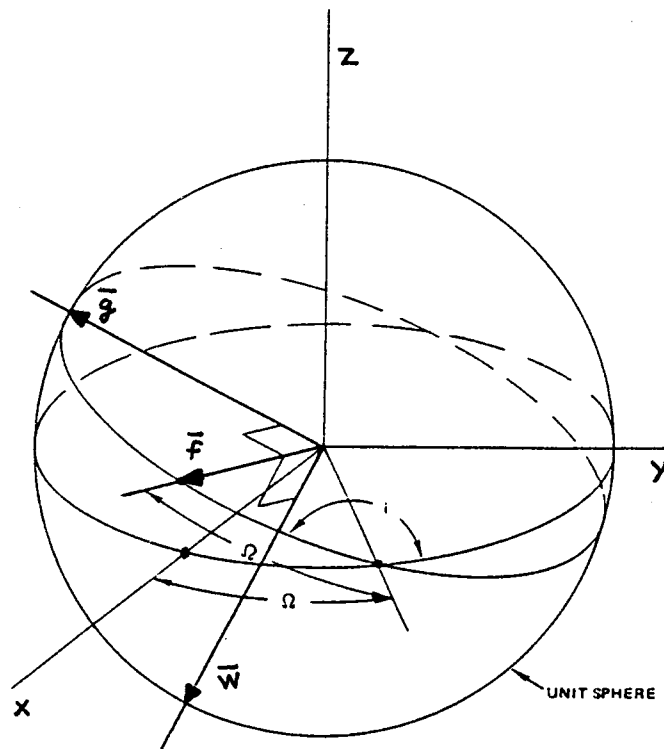
Figure 2: Retrograde Equinoctial Reference Frame

1. **f** and **g** lie in the satellite orbit plane.

2. **w** is parallel to the angular momentum vector of the satellite.

3. The angle between **f** and the ascending node is equal to the longitude of the ascending node.

This leaves two choices for **f** and **g**, one associated with the direct element set and one associated with the retrograde element set. The two sets of (**f, g, w**) are illustrated in Figures 1 and 2. In the Figures, and throughout this document, (x, y, z) denote a set of Cartesian coordinates whose origin moves with the center of mass of the central body and whose axes are nonrotating with respect to inertial space.

2.1.2 Conversion from Keplerian Elements to Equinoctial Elements

If a, e, i, Ω, ω, M denote the conventional Keplerian element set then the equinoctial elements are given by:

$$
\begin{aligned}
a &= a \\
h &= e \sin(\omega + I\Omega) \\
k &= e \cos(\omega + I\Omega) \\
p &= \left[\tan\left(\tfrac{i}{2} \right) \right]^I \sin\Omega \\
q &= \left[\tan\left(\tfrac{i}{2} \right) \right]^I \cos\Omega \\
\lambda &= M + \omega + I\Omega
\end{aligned}
\tag{1}
$$

The quantity I is called the retrograde factor and has two possible values:

$$
I = \begin{cases}
+1 & \text{for the direct equinoctial elements} \\
-1 & \text{for the retrograde equinoctial elements}
\end{cases}
\tag{2}
$$

There are two auxiliary longitudes associated with the equinoctial element set: the eccentric longitude F and the true longitude L. They are related to the Keplerian eccentric anomaly E and true anomaly f by the equations:

$$
\begin{aligned}
F &= E + \omega + I\Omega \\
L &= f + \omega + I\Omega
\end{aligned}
\tag{3}
$$
$$
\tag{4}
$$

These auxiliary longitudes are used in converting from equinoctial elements to position and velocity. In addition, certain perturbations are modeled with Fourier series expansions in F or L.

2.1.3   Conversion from Equinoctial Elements to Keplerian Elements

In order to convert from equinoctial to Keplerian elements, it is first necessary to compute an auxiliary angle ζ, which is defined by:

$$
\begin{aligned}
\sin \zeta &= \frac{h}{\sqrt{h^2 + k^2}} \\[2mm]
\cos \zeta &= \frac{k}{\sqrt{h^2 + k^2}}
\end{aligned}
\tag{1}
$$

The Keplerian elements are then given by:

$$
\begin{aligned}
a &= a \\[1mm]
e &= \sqrt{h^2 + k^2} \\[1mm]
i &= \pi\left(\frac{1 - I}{2}\right) + 2I \arctan \sqrt{p^2 + q^2} \\[1mm]
\sin \Omega &= \frac{p}{\sqrt{p^2 + q^2}} \\[1mm]
\cos \Omega &= \frac{q}{\sqrt{p^2 + q^2}} \\[1mm]
\omega &= \zeta - I\Omega \\[1mm]
M &= \lambda - \zeta
\end{aligned}
\tag{2}
$$

where I is defined by (2.1.2-2).

The Keplerian eccentric and true anomalies are given by:

$$
\begin{aligned}
E &= F - \zeta \\
f &= L - \zeta
\end{aligned}
\tag{3}
$$

2.1.4   Conversion from Equinoctial Elements to Position and Velocity

The first step in converting from equinoctial elements to position and velocity is to determine the equinoctial reference frame basis vectors $(\mathbf{f}, \mathbf{g}, \mathbf{w})$. Their components in the (x, y, z) system are

$$
\mathbf{f} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2Ip \end{bmatrix}
$$

$$
\mathbf{g} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 2Ipq \\ (1 + p^2 - q^2)I \\ 2q \end{bmatrix}
\tag{1}
$$

$$
\mathbf{w} = \frac{1}{1 + p^2 + q^2} \begin{bmatrix} 2p \\ -2q \\ (1 - p^2 - q^2)I \end{bmatrix}
$$

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The second step is to find the eccentric and true longitudes F and L, respectively. To find the eccentric longitude F, one must solve the equinoctial form of Kepler's Equation (see Section 7.1)):

$$\lambda = F + h \cos F - k \sin F \tag{2}$$

Then define two auxiliary quantities, the Kepler mean motion n and a quantity called b:

$$n = \sqrt{\frac{\mu}{a^3}} \tag{3}$$

$$b = \frac{1}{1 + \sqrt{1 - h^2 - k^2}} \tag{4}$$

Here, and throughout this document, μ is the gravitational constant GM of the central body. The true longitude L is then given by:

$$
\begin{aligned}
\sin L &= \frac{(1 - k^2 b) \sin F + hkb \cos F - h}{1 - h \sin F - k \cos F} \\
\cos L &= \frac{(1 - h^2 b) \cos F + hkb \sin F - k}{1 - h \sin F - k \cos F}
\end{aligned}
\tag{5}
$$

The third step is to compute the position and velocity components (X, Y) and $(\dot{X}, \dot{Y})$ of the satellite in the equinoctial reference frame. The radial distance of the satellite is given by:

$$r = a(1 - h \sin F - k \cos F) = \frac{a(1 - h^2 - k^2)}{1 + h \sin L + k \cos L} \tag{6}$$

The position components are then given by:

$$
\begin{aligned}
X &= a[(1 - h^2 b) \cos F + hkb \sin F - k] = r \cos L \\
Y &= a[(1 - k^2 b) \sin F + hkb \cos F - h] = r \sin L
\end{aligned}
\tag{7}
$$

The velocity components are then given by:

$$
\begin{aligned}
\dot{X} &= \frac{na^2}{r}[hkb \cos F - (1 - h^2 b) \sin F] = -\frac{na(h + \sin L)}{\sqrt{1 - h^2 - k^2}} \\
\dot{Y} &= \frac{na^2}{r}[(1 - k^2 b) \cos F - hkb \sin F] = \frac{na(k + \cos L)}{\sqrt{1 - h^2 - k^2}}
\end{aligned}
\tag{8}
$$

Here dots denote differentiation with respect to the time t.

The final step is now to compute the position and velocity vectors:

$$
\begin{aligned}
\mathbf{r} &= X\mathbf{f} + Y\mathbf{g} \\
\dot{\mathbf{r}} &= \dot{X}\mathbf{f} + \dot{Y}\mathbf{g}
\end{aligned}
\tag{9}
$$

2.1.5   Conversion from Position and Velocity to Equinoctial Elements

The first step in converting from position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ is to compute the semimajor axis a, which is obtained by inverting the well-known energy integral for the two-body problem:

$$a = \frac{1}{\dfrac{2}{|\mathbf{r}|} - \dfrac{|\dot{\mathbf{r}}|^2}{\mu}} \tag{1}$$

The second step is to compute the basis vectors $(\mathbf{f}, \mathbf{g}, \mathbf{w})$ of the equinoctial reference frame. The $\mathbf{w}$ vector is obtained by normalizing the angular momentum vector:

$$\mathbf{w} = \frac{\mathbf{r} \times \dot{\mathbf{r}}}{|\mathbf{r} \times \dot{\mathbf{r}}|} \tag{2}$$

Equinoctial elements p and q are then given by:

$$
\begin{aligned}
p &= \frac{w_x}{1 + I w_z} \\
q &= -\frac{w_y}{1 + I w_z}
\end{aligned} \tag{3}
$$

Vectors $\mathbf{f}$ and $\mathbf{g}$ are then computed using elements p and q in equations (2.1.4-1a, 1b).

The third step is to compute the eccentricity-related quantities. The eccentricity vector $\mathbf{e}$ is given by:

$$\mathbf{e} = -\frac{\mathbf{r}}{|\mathbf{r}|} + \frac{\dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}})}{\mu} \tag{4}$$

Equinoctial elements h and k are then given by:

$$
\begin{aligned}
h &= \mathbf{e} \cdot \mathbf{g} \\
k &= \mathbf{e} \cdot \mathbf{f}
\end{aligned} \tag{5}
$$

The final step is to compute the mean longitude λ. First compute the position coordinates of the satellite in the equinoctial reference frame:

$$
\begin{aligned}
X &= \mathbf{r} \cdot \mathbf{f} \\
Y &= \mathbf{r} \cdot \mathbf{g}
\end{aligned} \tag{6}
$$

Then compute the eccentric longitude F:

$$
\begin{aligned}
\sin F &= h + \frac{(1 - h^2 b)Y - hkbX}{a\sqrt{1 - h^2 - k^2}} \\
\cos F &= k + \frac{(1 - k^2 b)X - hkbY}{a\sqrt{1 - h^2 - k^2}}
\end{aligned} \tag{7}
$$

where b is defined by (2.1.4-4). The mean longitude λ is then given by the equinoctial form (2.1.4-2) of Kepler's equation.

2.1.6 Partial Derivatives of Position and Velocity with Respect to the Equinoctial Elements

Let

$$
\begin{aligned}
A &= na^2 = \sqrt{\mu a} \\
B &= \sqrt{1 - h^2 - k^2} = \frac{1}{b} - 1 \\
C &= 1 + p^2 + q^2
\end{aligned}
\tag{1}
$$

The partial derivatives of the position vector $\mathbf{r}$ with respect to the equinoctial elements are then given by:

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial a} &= \frac{\mathbf{r}}{a} \\
\frac{\partial \mathbf{r}}{\partial h} &= \frac{\partial X}{\partial h}\mathbf{f} + \frac{\partial Y}{\partial h}\mathbf{g} \\
\frac{\partial \mathbf{r}}{\partial k} &= \frac{\partial X}{\partial k}\mathbf{f} + \frac{\partial Y}{\partial k}\mathbf{g} \\
\frac{\partial \mathbf{r}}{\partial p} &= \frac{2[Iq(Y\mathbf{f} - X\mathbf{g}) - X\mathbf{w}]}{C} \\
\frac{\partial \mathbf{r}}{\partial q} &= \frac{2I[p(X\mathbf{g} - Y\mathbf{f}) + Y\mathbf{w}]}{C} \\
\frac{\partial \mathbf{r}}{\partial \lambda} &= \frac{\dot{\mathbf{r}}}{n}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
\frac{\partial X}{\partial h} &= -\frac{k\dot{X}}{n(1+B)} + \frac{aY\dot{Y}}{AB} \\
\frac{\partial Y}{\partial h} &= -\frac{k\dot{Y}}{n(1+B)} - \frac{aX\dot{Y}}{AB} - a \\
\frac{\partial X}{\partial k} &= \frac{h\dot{X}}{n(1+B)} + \frac{aY\dot{X}}{AB} - a \\
\frac{\partial Y}{\partial k} &= \frac{h\dot{Y}}{n(1+B)} - \frac{aX\dot{X}}{AB}
\end{aligned}
\tag{3}
$$

The partial derivatives of the velocity vector $\dot{\mathbf{r}}$ with respect to the equinoctial elements

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are given by:

$$\frac{\partial \dot{\mathbf{r}}}{\partial a} = -\frac{\dot{\mathbf{r}}}{2a}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial h} = \frac{\partial \dot{X}}{\partial h}\mathbf{f} + \frac{\partial \dot{Y}}{\partial h}\mathbf{g}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial k} = \frac{\partial \dot{X}}{\partial k}\mathbf{f} + \frac{\partial \dot{Y}}{\partial k}\mathbf{g}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial p} = \frac{2[Iq(\dot{Y}\mathbf{f} - \dot{X}\mathbf{g}) - \dot{X}\mathbf{w}]}{C} \tag{4}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial q} = \frac{2I[p(\dot{X}\mathbf{g} - \dot{Y}\mathbf{f}) + \dot{Y}\mathbf{w}]}{C}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial \lambda} = -\frac{na^3\mathbf{r}}{r^3}$$

where

$$\frac{\partial \dot{X}}{\partial h} = \frac{a\dot{Y}^2}{AB} + \frac{A}{r^3}\left(\frac{akX}{1+B} - \frac{Y^2}{B}\right)$$

$$\frac{\partial \dot{Y}}{\partial h} = -\frac{a\dot{X}\dot{Y}}{AB} + \frac{A}{r^3}\left(\frac{akY}{1+B} + \frac{XY}{B}\right)$$

$$\frac{\partial \dot{X}}{\partial k} = \frac{a\dot{X}\dot{Y}}{AB} - \frac{A}{r^3}\left(\frac{ahX}{1+B} + \frac{XY}{B}\right) \tag{5}$$

$$\frac{\partial \dot{Y}}{\partial k} = -\frac{a\dot{X}^2}{AB} - \frac{A}{r^3}\left(\frac{ahY}{1+B} - \frac{X^2}{B}\right)$$

2.1.7 Partial Derivatives of Equinoctial Elements with Respect to Position and Velocity

Let $\frac{\partial a_i}{\partial \mathbf{r}}$ and $\frac{\partial a_i}{\partial \dot{\mathbf{r}}}$ represent the vectors whose components in the (x, y, z) system are the partial derivatives of the element a_i with respect to (x, y, z) and $(\dot{x}, \dot{y}, \dot{z})$, respectively;

$$\frac{\partial a_i}{\partial \mathbf{r}} = \left[\frac{\partial a_i}{\partial x}\ \frac{\partial a_i}{\partial y}\ \frac{\partial a_i}{\partial z}\right]$$

$$\frac{\partial a_i}{\partial \dot{\mathbf{r}}} = \left[\frac{\partial a_i}{\partial \dot{x}}\ \frac{\partial a_i}{\partial \dot{y}}\ \frac{\partial a_i}{\partial \dot{z}}\right] \tag{1}$$

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The partial derivatives of the equinoctial elements with respect to position are then given by

$$
\begin{aligned}
\frac{\partial a}{\partial \mathbf{r}} &= \frac{2a^2 \mathbf{r}}{r^3} \\
\frac{\partial h}{\partial \mathbf{r}} &= -\frac{abhB\mathbf{r}}{r^3} + \frac{k(p\dot{X} - Iq\dot{Y})\mathbf{w}}{AB} - \frac{B}{A}\frac{\partial \dot{\mathbf{r}}}{\partial k} \\
\frac{\partial k}{\partial \mathbf{r}} &= -\frac{abkB\mathbf{r}}{r^3} - \frac{h(p\dot{X} - Iq\dot{Y})\mathbf{w}}{AB} + \frac{B}{A}\frac{\partial \dot{\mathbf{r}}}{\partial h} \\
\frac{\partial p}{\partial \mathbf{r}} &= -\frac{C\dot{Y}\mathbf{w}}{2AB} \\
\frac{\partial q}{\partial \mathbf{r}} &= -\frac{C\dot{X}\mathbf{w}}{2AB} \\
\frac{\partial \lambda}{\partial \mathbf{r}} &= -\frac{\dot{\mathbf{r}}}{A} + \frac{(p\dot{X} - Iq\dot{Y})\mathbf{w}}{AB} - \frac{bB}{A}\left(h\frac{\partial \dot{\mathbf{r}}}{\partial h} + k\frac{\partial \dot{\mathbf{r}}}{\partial k}\right)
\end{aligned}
\tag{2}
$$

The partial derivatives of the equinoctial elements with respect to velocity are given by:

$$
\begin{aligned}
\frac{\partial a}{\partial \dot{\mathbf{r}}} &= \frac{2\dot{\mathbf{r}}}{n^2 a} \\
\frac{\partial h}{\partial \dot{\mathbf{r}}} &= \frac{(2\dot{X}Y - X\dot{Y})\mathbf{f} - X\dot{X}\mathbf{g}}{\mu} + \frac{k(IqY - pX)\mathbf{w}}{AB} \\
\frac{\partial k}{\partial \dot{\mathbf{r}}} &= \frac{(2X\dot{Y} - \dot{X}Y)\mathbf{g} - Y\dot{Y}\mathbf{f}}{\mu} - \frac{h(IqY - pX)\mathbf{w}}{AB} \\
\frac{\partial p}{\partial \dot{\mathbf{r}}} &= \frac{CY\mathbf{w}}{2AB} \\
\frac{\partial q}{\partial \dot{\mathbf{r}}} &= \frac{ICX\mathbf{w}}{2AB} \\
\frac{\partial \lambda}{\partial \dot{\mathbf{r}}} &= -\frac{2\mathbf{r}}{A} + \frac{k\frac{\partial h}{\partial \dot{\mathbf{r}}} - h\frac{\partial k}{\partial \dot{\mathbf{r}}}}{1 + B} + \frac{(IqY - pX)\mathbf{w}}{A}
\end{aligned}
\tag{3}
$$

2.1.8   Poisson Brackets

The Poisson brackets of the element set $(a_1, \ldots, a_6)$ are defined by the equations

$$
(a_i,\, a_j) = \frac{\partial a_i}{\partial \mathbf{r}} \cdot \frac{\partial a_j}{\partial \dot{\mathbf{r}}} - \frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \frac{\partial a_j}{\partial \mathbf{r}}
\tag{1}
$$

It is immediately evident that

$$
\begin{aligned}
(a_i,\, a_i) &= 0 \\
(a_i,\, a_j) &= -(a_j,\, a_i)
\end{aligned}
\tag{2}
$$

The fifteen independent Poisson brackets for the equinoctial element set $(a,\ h,\ k,\ p,\ q,\ \lambda)$ are given by

$$
\begin{aligned}
(a,\ h) &= 0 & (h,\ \lambda) &= \frac{hB}{A(1+B)} \\[2mm]
(a,\ k) &= 0 & (k,\ p) &= \frac{hpC}{2AB} \\[2mm]
(a,\ p) &= 0 & (k,\ q) &= \frac{hqC}{2AB} \\[2mm]
(a,\ q) &= 0 & (k,\ \lambda) &= \frac{kB}{A(1+B)} \\[2mm]
(a,\ \lambda) &= -\frac{2}{na} & (p,\ q) &= -\frac{C^2}{4AB}I \\[2mm]
(h,\ k) &= -\frac{B}{A} & (p,\ \lambda) &= \frac{pC}{2AB} \\[2mm]
(h,\ p) &= -\frac{kpC}{2AB} & (q,\ \lambda) &= \frac{qC}{2AB} \\[2mm]
(h,\ q) &= -\frac{kqC}{2AB}
\end{aligned}
\tag{3}
$$

2.1.9 Direction Cosines (α, β, γ)

The conservative perturbations are more conveniently described by the direction cosines (α, β, γ) of the symmetry axis rather than the equinoctial elements p and q. For central-body gravitational spherical harmonics, let $\mathbf{z}_B$ be the unit vector from the center of mass to the geographic north pole of the central-body. For third-body point mass effects and shadowless solar radiation pressure, let $\mathbf{z}_B$ be the unit vector from the center of mass to the third-body. The direction cosines of $\mathbf{z}_B$ with respect to the equinoctial reference frame $(\mathbf{f}, \mathbf{g}, \mathbf{w})$ are then given by

$$
\begin{aligned}
\alpha &= \mathbf{z}_B \cdot \mathbf{f} \\
\beta &= \mathbf{z}_B \cdot \mathbf{g} \\
\gamma &= \mathbf{z}_B \cdot \mathbf{w}
\end{aligned}
\tag{1}
$$

The quantities (α, β, γ) are not independent but related by the equation

$$
\alpha^2 + \beta^2 + \gamma^2 = 1
\tag{2}
$$

Note that (α, β, γ) are functions of p and q, since the unit vectors $(\mathbf{f},\ \mathbf{g},\ \mathbf{w})$ are functions of p and q through equations (2.1.4-1). Note also that (α, β, γ) are functions of t, since $\mathbf{z}_B$ is a varying function of time. If the vector $\mathbf{z}_B$ along the geographic axis of the central-body is parallel at epoch to the z-axis of Figures 1 and 2, then the direction cosines of $\mathbf{z}_B$ are at epoch

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$$\begin{aligned}
\alpha &= -\frac{2Ip}{1+p^2+q^2} \\
\beta &= \frac{2q}{1+p^2+q^2} \\
\gamma &= \frac{(1-p^2-q^2)I}{1+p^2+q^2}
\end{aligned} \tag{3}$$

The partial derivatives of (α, β, γ) with respect to p and q are:

$$\begin{aligned}
\frac{\partial\alpha}{\partial p} &= -\frac{2(Iq\beta+\gamma)}{C} \\
\frac{\partial\alpha}{\partial q} &= \frac{2Ip\beta}{C} \\
\frac{\partial\beta}{\partial p} &= \frac{2Iq\alpha}{C} \\
\frac{\partial\beta}{\partial q} &= -\frac{2I(p\alpha-\gamma)}{C} \\
\frac{\partial\gamma}{\partial p} &= \frac{2\alpha}{C} \\
\frac{\partial\gamma}{\partial q} &= -\frac{2I\beta}{C}
\end{aligned} \tag{4}$$

2.2 Variation-of-Parameters (VOP) Equations of Motion

The Cartesian equations of motion for an artificial satellite in an inertial coordinate system are [Battin, 1987]:

$$\ddot{\mathbf{r}} = -\frac{\mu\mathbf{r}}{|\mathbf{r}|^3} + \mathbf{q} + \nabla\mathcal{R} \tag{1}$$

Here $\mathbf{r}$ is the position vector from the center of mass of the central body to the satellite, $\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$ is the acceleration vector, $\mu = GM$ is the gravitational constant of the central body, $\mathbf{q}$ is the acceleration due to nonconservative perturbing forces (atmospheric drag, solar radiation pressure), and $\mathcal{R}$ is a potential-like function called the disturbing function from which one can derive the acceleration due to conservative perturbing forces (central-body spherical harmonics, third-body point-mass). If m and Π are the mass and potential energy, respectively, of the satellite, then the disturbing function $\mathcal{R}$ is given by:

$$\mathcal{R} = -\frac{\Pi}{m} \tag{2}$$

In order to apply the generalized method of averaging, it is necessary to convert the equations of motion into a form giving the rates of change of the satellite orbit elements as a function of the orbit elements themselves. The equations of motion resulting from this conversion are called the Variation-of-Parameters (VOP) equations of motion. The derivation of these equations is discussed in some detail in [Cefola, Long, and Holloway, 1974] and [McClain, 1977].

In this section we let $(a_1, \ldots, a_6) = (a, h, k, p, q, \lambda)$ denote the *osculating* equinoctial elements (even though they do not have hats). Then the VOP equations of motion turn out to be

$$\dot{a}_i = n\delta_{i6} + \frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q} - \sum_{j=1}^{6} (a_i,\, a_j) \frac{\partial \mathcal{R}}{\partial a_j} \tag{3}$$

Here $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ is the velocity vector, $n = \sqrt{\frac{\mu}{a^3}}$ is the Kepler mean motion, and δ_{i6} is the Kronecker delta. The partial derivatives $\partial a_i / \partial \dot{\mathbf{r}}$ are given by equations (2.1.7-3), and the Poisson brackets $(a_i,\, a_j)$ are given by equations (2.1.8-2,3).

The VOP equations of motion (3) include three contributions to the orbit element rates of change. The two-body part is:

$$\dot{a}_i = n\delta_{i6} \tag{4}$$

The Gaussian or nonconservative part is:

$$\dot{a}_i = \frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q} \tag{5}$$

The Lagrangian or conservative part is:

$$\dot{a}_i = -\sum_{j=1}^{6} (a_i,\, a_j) \frac{\partial \mathcal{R}}{\partial a_j} \tag{6}$$

In the remainder of this document it will be convenient to discuss these contributions separately, but they must be added together to obtain the total orbit element rates of change.

The Lagrangian part of the VOP equations of motion contains the partial derivatives of the disturbing function $\mathcal{R}$ with respect to p and q. The perturbations which contribute to $\mathcal{R}$ are not conveniently described in terms of p and q, however. For these functions, it is better to write $\mathcal{R}$ as a function of (a, h, k, λ) and the direction cosines (α, β, γ) of the symmetry axis of the perturbation. The partial derivatives of the disturbing function $\mathcal{R}$ with respect to p and q can then be obtained by applying the Chain Rule:

$$\begin{aligned} \frac{\partial \mathcal{R}}{\partial p} &= \frac{\partial \mathcal{R}}{\partial \alpha}\frac{\partial \alpha}{\partial p} + \frac{\partial \mathcal{R}}{\partial \beta}\frac{\partial \beta}{\partial p} + \frac{\partial \mathcal{R}}{\partial \gamma}\frac{\partial \gamma}{\partial p} \\ \frac{\partial \mathcal{R}}{\partial q} &= \frac{\partial \mathcal{R}}{\partial \alpha}\frac{\partial \alpha}{\partial q} + \frac{\partial \mathcal{R}}{\partial \beta}\frac{\partial \beta}{\partial q} + \frac{\partial \mathcal{R}}{\partial \gamma}\frac{\partial \gamma}{\partial q} \end{aligned} \tag{7}$$

The partial derivatives of (α, β, γ) with respect to p and q are given by (2.1.9-4). To simplify the notation, let us again use the auxiliary quantities A, B, C defined by (2.1.6-1). Also, let us define the cross-derivative operator

$$\mathcal{R},_{\alpha\beta} = \alpha\frac{\partial \mathcal{R}}{\partial \beta} - \beta\frac{\partial \mathcal{R}}{\partial \alpha} \tag{8}$$

Note that $\mathcal{R},_{\alpha\beta} = -\mathcal{R},_{\beta\alpha}$. Then the partial derivatives of $\mathcal{R}$ with respect to p and q turn out to be

$$\begin{aligned} \frac{\partial \mathcal{R}}{\partial p} &= \frac{2}{C}\Big(\mathcal{R},_{\alpha\gamma} + Iq\mathcal{R},_{\alpha\beta}\Big) \\ \frac{\partial \mathcal{R}}{\partial q} &= -\frac{2I}{C}\Big(\mathcal{R},_{\beta\gamma} + p\mathcal{R},_{\alpha\beta}\Big) \end{aligned} \tag{9}$$

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With this notation, equations (6) for the Lagrangian part of the VOP equations of motion become:

$$
\begin{aligned}
\dot{a} &= \frac{2a}{A}\frac{\partial\mathcal{R}}{\partial\lambda} \\
\dot{h} &= \frac{B}{A}\frac{\partial\mathcal{R}}{\partial k} + \frac{k}{AB}\left(p\mathcal{R}_{,\alpha\gamma} - Iq\mathcal{R}_{,\beta\gamma}\right) - \frac{hB}{A(1+B)}\frac{\partial\mathcal{R}}{\partial\lambda} \\
\dot{k} &= -\left[\frac{B}{A}\frac{\partial\mathcal{R}}{\partial h} + \frac{h}{AB}\left(p\mathcal{R}_{,\alpha\gamma} - Iq\mathcal{R}_{,\beta\gamma}\right) + \frac{kB}{A(1+B)}\frac{\partial\mathcal{R}}{\partial\lambda}\right] \\
\dot{p} &= \frac{C}{2AB}\left[p\left(\mathcal{R}_{,hk} - \mathcal{R}_{,\alpha\beta} - \frac{\partial\mathcal{R}}{\partial\lambda}\right) - \mathcal{R}_{,\beta\gamma}\right] \\
\dot{q} &= \frac{C}{2AB}\left[q\left(\mathcal{R}_{,hk} - \mathcal{R}_{,\alpha\beta} - \frac{\partial\mathcal{R}}{\partial\lambda}\right) - I\mathcal{R}_{,\alpha\gamma}\right] \\
\dot{\lambda} &= -\frac{2a}{A}\frac{\partial\mathcal{R}}{\partial a} + \frac{B}{A(1+B)}\left(h\frac{\partial\mathcal{R}}{\partial h} + k\frac{\partial\mathcal{R}}{\partial k}\right) + \frac{1}{AB}\left(p\mathcal{R}_{,\alpha\gamma} - Iq\mathcal{R}_{,\beta\gamma}\right)
\end{aligned}
\tag{10}
$$

2.3   Equations of Averaging

The Generalized Method of Averaging may be used to divide the VOP equations of motion (2.2–3) into a short-periodic part which can be integrated analytically and a slowly-varying part which can be integrated numerically using time steps several orders of magnitude longer than the time steps appropriate for integrating the untransformed equations of motion. The Generalized Method of Averaging and other perturbation techniques are discussed in [Nayfeh, 1973]. Only a summary of the application of this procedure to (2.2–3) will be given here. More details can be found in [Cefola, Long, and Holloway, 1974], [McClain, 1977], [McClain, Long, and Early, 1978] and [Green, 1979].

To apply the Generalized Method of Averaging we first assume that the osculating orbit elements $\hat{a}_i$ are related to a set of mean elements a_i by a near-identity transformation:

$$
\hat{a}_i = a_i + \sum_{j=1}^{\infty} \epsilon^j \eta_i^j(a, h, k, p, q, \lambda, t)
\tag{1}
$$

Here again, the indexed variables $(a_1, \ldots, a_6)$ refer to the equinoctial orbit elements (a, h, k, p, q, λ) and hats distinguish the osculating elements from the mean elements. The quantity $\epsilon^j \eta_i^j$ represents a small short-periodic variation of order j in element i. The quantity ϵ is called the "small parameter" and plays the role of a variational parameter in deriving the Equations of Averaging. (Note that the superscript j is used in the symbol ϵ^j to designate a power and in the symbol η_i^j to designate an index.)

The short-periodic variations are assumed to contain all of the high-frequency components in the osculating elements $\hat{a}_i$, so that the mean elements a_i vary slowly with time. This requirement can be expressed by the following two sets of inequalities:

$$\frac{1}{n}\left|\frac{da}{dt}\right| \ll a$$

$$\frac{1}{n}\left|\frac{da_i}{dt}\right| \ll 1 \qquad \text{for} \quad i = 2, 3, 4, 5 \tag{2}$$

$$\frac{1}{n}\left|\frac{d\lambda}{dt} - n\right| \ll 1$$

where n is the Kepler mean motion, and

$$\Delta^{k+1}\left|\frac{d^{k+1}a}{dt^{k+1}}\right| \ll a$$

$$\tag{3}$$

$$\Delta^{k+1}\left|\frac{d^{k+1}a_i}{dt^{k+1}}\right| \ll 1 \qquad \text{for} \quad i = 2, 3, 4, 5, 6$$

where k is the order and Δ is the step size of the numerical integrator. Inequalities (2) ensure that second-order effects will be small, while inequalities (3) ensure that the integrator errors will be small.

Using the variational parameter ϵ, we can write the osculating Cartesian equations of motion (2.2-1) as

$$\frac{d^2\hat{\mathbf{r}}}{dt^2} = -\frac{\mu\hat{\mathbf{r}}}{|\hat{\mathbf{r}}|^3} + \epsilon(\mathbf{q} + \nabla\mathcal{R}) \tag{4}$$

As ϵ increases from 0 to 1 the resulting motion varies smoothly from two-body motion to the actual motion. The osculating VOP equations of motion (2.2–3) then become

$$\frac{d\hat{a}_i}{dt} = n(\hat{a})\delta_{i6} + \epsilon\left[\frac{\partial\hat{a}_i}{\partial\hat{\mathbf{r}}}\cdot\mathbf{q} - \sum_{j=1}^{6}(\hat{a}_i, \hat{a}_j)\frac{\partial\mathcal{R}}{\partial\hat{a}_j}\right] \tag{5}$$

which can be written in the form

$$\frac{d\hat{a}_i}{dt} = n(\hat{a})\delta_{i6} + \epsilon F_i(\hat{a}, \hat{h}, \hat{k}, \hat{p}, \hat{q}, \hat{\lambda}, t) \tag{6}$$

The terms ϵF_i give the osculating element rates of change due to the perturbing forces as functions of the osculating elements.

We assume the following form for the mean VOP equations of motion:

$$\frac{da_i}{dt} = n(a)\delta_{i6} + \sum_{j=1}^{\infty}\epsilon^j A_i^j(a, h, k, p, q, t) \tag{7}$$

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The terms $\epsilon^j A_i^j$ give the mean element rates of change due to the perturbing forces as functions of the mean elements. For most perturbations, the A_i^j are independent of the fast variable λ as indicated in (7). For central-body resonant tesseral harmonics, the A_i^j are also slowly-varying functions of $j\lambda - m\theta$ (see section 3.3).

The osculating rate functions F_i and the mean rate functions A_i^j may be explicit functions of the time t because the perturbing forces may change with time when the satellite position and velocity are held constant, e. g., due to motion of the Moon.

We now expand the osculating rate functions in a variational power series about the mean elements:

$$F_i(\hat{a}, \hat{h}, \hat{k}, \hat{p}, \hat{q}, \hat{\lambda}, t) = F_i(a, h, k, p, q, \lambda, t) + \sum_{j=1}^{\infty} \epsilon^j f_i^j(a, h, k, p, q, \lambda, t) \tag{8}$$

Here

$$f_i^1 = \sum_{j=1}^{6} \frac{\partial F_i}{\partial a_j} \eta_j^1 \tag{9}$$

$$f_i^2 = \sum_{j=1}^{6} \frac{\partial F_i}{\partial a_j} \eta_j^2 + \frac{1}{2} \sum_{j=1}^{6} \sum_{k=1}^{6} \frac{\partial^2 F_i}{\partial a_j \partial a_k} \eta_j^1 \eta_k^1 \tag{10}$$

$$\vdots$$

Similarly, we expand the osculating Kepler mean motion about the mean semimajor axis:

$$n(\hat{a}) = n(a) + \sum_{j=1}^{\infty} \epsilon^j N^j(a) \tag{11}$$

Here

$$N^1 = -\frac{3}{2} \frac{\eta_1^1}{a} n(a) \tag{12}$$

$$N^2 = \left[-\frac{3}{2} \frac{\eta_1^2}{a} + \frac{15}{8} \frac{(\eta_1^1)^2}{a^2} \right] n(a) \tag{13}$$

$$N^3 = \left[-\frac{3}{2} \frac{\eta_1^3}{a} + \frac{15}{4} \frac{\eta_1^1 \eta_1^2}{a^2} - \frac{35}{16} \frac{(\eta_1^1)^3}{a^3} \right] n(a) \tag{14}$$

$$\vdots$$

Having all the necessary expansions, we can now derive the Equations of Averaging. First, differentiate (1) with respect to t and use (7) to obtain one expression for the osculating element rates. Next, expand the functions on the right side of (6) using (8) through (14) to obtain another expression for the osculating rates. Then equate the two expansions and require that they be equal for all values of ϵ between 0 and 1. Since the powers of ϵ are

linearly independent, the coefficients of ϵ^j must be equal. Taking $j = 1, 2, 3, \ldots$ yields the Equations of Averaging of order $1, 2, 3, \ldots$ respectively:

$$A_i^1 + \frac{\partial \eta_i^1}{\partial \lambda} n(a) + \frac{\partial \eta_i^1}{\partial t} = F_i(a, h, k, p, q, \lambda, t) + N^1 \delta_{i6} \tag{15}$$

$$A_i^2 + \frac{\partial \eta_i^2}{\partial \lambda} n(a) + \frac{\partial \eta_i^2}{\partial t} = f_i^1 + N^2 \delta_{i6} - \sum_{j=1}^{6} \frac{\partial \eta_i^1}{\partial a_j} A_j^1 \tag{16}$$

$$A_i^3 + \frac{\partial \eta_i^3}{\partial \lambda} n(a) + \frac{\partial \eta_i^3}{\partial t} = f_i^2 + N^3 \delta_{i6} - \sum_{j=1}^{6} \left(\frac{\partial \eta_i^2}{\partial a_j} A_j^1 + \frac{\partial \eta_i^1}{\partial a_j} A_j^2 \right) \tag{17}$$

$$\vdots$$

In the Equations of Averaging shown above, the osculating rate functions F_i, the mean rate functions A_i^j, and the short-periodic variations η_i^j contain effects due to many perturbations. In order to obtain practical expressions for A_i^j and η_i^j, it is convenient to partition the Equations of Averaging to separate the effects of different perturbations.

The first step in partitioning the Equations of Averaging is to express the osculating rate functions F_i in terms of the contributions $F_{i\alpha}$ of the separate perturbations:

$$\epsilon F_i = \sum_\alpha \nu_\alpha F_{i\alpha} \tag{18}$$

The sum is taken over all perturbations of interest. The parameters ν_α are variational parameters, one for each perturbation. Each ν_α can vary from 0, at which the perturbation is turned off, to 1, at which the perturbation has its actual strength. The partitioned Equations of Averaging are required to be valid for all values of the ν_α.

Substituting equation (18) into the first-order Equations of Averaging (15) leads to the following expressions for A_i^1 and η_i^1:

$$\epsilon A_i^1 = \sum_\alpha \nu_\alpha A_{i\alpha} \tag{19}$$

$$\epsilon \eta_i^1 = \sum_\alpha \nu_\alpha \eta_{i\alpha} \tag{20}$$

Substituting equations (18)–(20) into the second-order Equations of Averaging (16) leads to the following expressions for A_i^2 and η_i^2:

$$\epsilon^2 A_i^2 = \sum_{\alpha\beta} \nu_\alpha \nu_\beta A_{i\alpha\beta} \tag{21}$$

$$\epsilon^2 \eta_i^2 = \sum_{\alpha\beta} \nu_\alpha \nu_\beta \eta_{i\alpha\beta} \tag{22}$$

Substituting equations (18)–(22) into the third-order Equations of Averaging (17) leads to the following expressions for A_i^3 and η_i^3:

$$\epsilon^3 A_i^3 = \sum_{\alpha\beta\gamma} \nu_\alpha \nu_\beta \nu_\gamma A_{i\alpha\beta\gamma} \tag{23}$$

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$$\epsilon^3 \eta_i^3 = \sum_{\alpha\beta\gamma} \nu_\alpha \nu_\beta \nu_\gamma \eta_{i\alpha\beta\gamma} \tag{24}$$

Similar expressions exist at higher orders. (Remember that the first index in $F_{i\alpha}$, $A_{i\alpha}$, $\eta_{i\alpha}$, etc. refers to the orbit element.)

If we substitute (18)–(20) into (15) and then equate the coefficients of each ν_α, we obtain the partitioned form of the first-order Equations of Averaging. Similar procedures with (16), (17), ... lead to higher-order equations. The partitioned Equations of Averaging *of arbitrary order* can be written in the concise form

$$A_i + \frac{\partial \eta_i}{\partial \lambda} n(a) + \frac{\partial \eta_i}{\partial t} = G_i - \frac{3}{2} \frac{\eta_1}{a} n(a) \delta_{i6} \tag{25}$$

Explicit expressions for the G_i functions up to order 3 are:

$$G_{i\alpha} = F_{i\alpha}(a, h, k, p, q, \lambda, t) \tag{26}$$

$$G_{i\alpha\beta} = \sum_{j=1}^{6} \frac{\partial F_{i\alpha}}{\partial a_j} \eta_{j\beta} + \frac{15}{8} \frac{\eta_{1\alpha}\eta_{1\beta}}{a^2} n(a) \delta_{i6} - \sum_{j=1}^{6} \frac{\partial \eta_{i\alpha}}{\partial a_j} A_{j\beta} \tag{27}$$

$$G_{i\alpha\beta\gamma} = \sum_{j=1}^{6} \frac{\partial F_{i\alpha}}{\partial a_j} \eta_{j\beta\gamma} + \frac{1}{2} \sum_{j=1}^{6} \sum_{k=1}^{6} \frac{\partial^2 F_{i\alpha}}{\partial a_j \partial a_k} \eta_{j\beta} \eta_{k\gamma} + \left(\frac{15}{4} \frac{\eta_{1\alpha}\eta_{1\beta\gamma}}{a^2} - \frac{35}{16} \frac{\eta_{1\alpha}\eta_{1\beta}\eta_{1\gamma}}{a^3} \right) n(a) \delta_{i6}$$

$$- \sum_{j=1}^{6} \left(\frac{\partial \eta_{i\alpha\beta}}{\partial a_j} A_{j\gamma} + \frac{\partial \eta_{i\alpha}}{\partial a_j} A_{j\beta\gamma} \right) \tag{28}$$

Comparing the partitioned Equations of Averaging (25)–(28) with the full Equations of Averaging (15)–(17), we see that the first-order equations are identical. The second-order and higher-order partitioned equations include auto-coupling equations (e.g., $\alpha = \beta = \ldots = 1$) which are identical to the full equations, but in addition include cross-coupling equations (e.g., $\alpha = 1, \beta = 2$).

The partitioned Equations of Averaging (25)–(28) give the fundamental relations which can be used to derive expressions for the mean element rates $A_{i\alpha}$, $A_{i\alpha\beta}$, $A_{i\alpha\beta\gamma}$ and the short-periodic variations $\eta_{i\alpha}$, $\eta_{i\alpha\beta}$, $\eta_{i\alpha\beta\gamma}$. Then the total mean element rates A_i^1, A_i^2, A_i^3 and short-periodic variations η_i^1, η_i^2, η_i^3 can be obtained from the decomposition relations (19)–(24).

2.4   Averaged Equations of Motion

The Equations of Averaging (2.3-25) can be solved for the mean element rates $A_{i\alpha}, A_{i\alpha\beta}, A_{i\alpha\beta\gamma} \ldots$ by applying an averaging operator $< \cdots >$, to be defined in this section, to both sides of each equation. The resulting expressions for the $A_{i\alpha}, A_{i\alpha\beta}, A_{i\alpha\beta\gamma} \ldots$ can then be added together as shown in equations (2.3-19, 21, 23) and substituted into equations (2.3-7) with $\epsilon = \nu_\alpha = 1$ to form the mean, or averaged, equations of motion:

$$\frac{da_i}{dt} = n(a) \delta_{i6} + \sum_\alpha A_{i\alpha} + \sum_{\alpha,\beta} A_{i\alpha\beta} + \sum_{\alpha,\beta,\gamma} A_{i\alpha\beta\gamma} + \ldots \tag{1}$$

These equations can then be integrated with a numerical integrator to obtain values for the mean elements a_i at a given time.

The averaging operator is required to be linear; that is, if ρ and σ are any two real numbers and f and g are any two real piecewise continuous functions of the mean elements, then:

$$< \rho f + \sigma g >= \rho < f > + \sigma < g > \tag{2}$$

The averaging operator is also required to be idempotent; that is, if f is any real piecewise continuous function of the mean elements, then:

$$<< f >>=< f > \tag{3}$$

To make the averaging transformation useful, we also require the averaging operator to have the property that solving the Equations of Averaging with it yields slowly-varying mean element rates and small short-periodic variations.

In order to be able to divide the Equations of Averaging into separate equations for the mean element rates and the short-periodic variations, we impose the following conditions:

$$< \eta_i >= 0 \tag{4}$$

It is not immediately obvious from the Equations of Averaging (2.3-25) that the short-periodic variations can be required to average to zero (equations (4)). Let us first observe from (2.3-26, 27, 28) that, at any order, G_i are predetermined functions of the osculating rate functions $F_{i\alpha}$ and the solutions of the lower-order Equations of Averaging. Therefore G_i are fixed, while we can vary A_i and η_i in any manner which satisfies (2.3-25). Let us assume that the short-periodic variations η_i do not average to zero, and write

$$< \eta_i >= k_i \tag{5}$$

Let us then define

$$A'_i = A_i + n\frac{\partial k_i}{\partial \lambda} + \frac{\partial k_i}{\partial t} + \frac{3}{2}\frac{n}{a}k_1\delta_{i6} \tag{6}$$

$$\eta'_i = \eta_i - k_i \tag{7}$$

Solving (6)–(7) for A_i and η_i and substituting the resulting expressions into (2.3-25) yields

$$A'_i + n\frac{\partial \eta'_i}{\partial \lambda} + \frac{\partial \eta'_i}{\partial t} = G_i - \frac{3}{2}\frac{\eta'_1}{a}n\delta_{i6} \tag{8}$$

We see that A'_i and η'_i are solutions to the Equations of Averaging, and we can thus choose A'_i and η'_i to be the preferred solutions. Averaging (7) and applying (2, 3, 5) yields

$$< \eta'_i >= 0 \tag{9}$$

We can therefore require the short-periodic variations to average to zero (equations (4)).

If the osculating rate functions $F_{i\alpha}$ for a given perturbation are small, 2π-periodic in the satellite mean longitude λ, and slowly-varying in time when the satellite orbit elements are held fixed, then the single-averaging operator has the required properties:

$$< f >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a, h, k, p, q, \lambda, t)d\lambda \tag{10}$$

Most of the perturbations commonly acting on a satellite can be single-averaged:

1. Central-body gravitational zonal harmonics.

2. Third-body gravitational point mass effects.

3. Atmospheric drag.

4. Solar radiation pressure.

Some perturbations are quickly-varying when expressed as a function of time but slowly-varying when expressed as a function of both time and a perturbing-body phase angle θ which varies linearly with time. If the osculating rate functions $F_{i\alpha}$ for such a perturbation are small, 2π-periodic in both λ and θ, and slowly-varying in time when θ and the satellite orbit elements are held fixed, then the double-averaging operator has the required properties:

$$\begin{aligned} < f > \ = \ & \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, p, q, \lambda, \theta, t)d\lambda d\theta \\ & + \frac{1}{2\pi^2} \sum_{(j,m)\in\mathcal{B}} \left[\cos(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, p, q, \lambda', \theta', t) \cos(j\lambda' - m\theta')d\lambda'd\theta' \right. \\ & \left. + \sin(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(a, h, k, p, q, \lambda', \theta', t) \sin(j\lambda' - m\theta')d\lambda'd\theta' \right] \end{aligned} \tag{11}$$

(Equation (11) may be written in alternate forms, one of which is used in (3.3-2).) Here $\mathcal{B}$ is the set of all ordered pairs (j, m) with the following properties:

$$m \geq 1 \tag{12}$$

$$|j\dot{\lambda} - m\dot{\theta}| < \frac{2\pi}{\tau} \tag{13}$$

where τ is the minimum period desired for perturbations included in the averaged equations of motion. The same minimum period should be used for all double-averaged perturbations. Inequality (13) is called the resonance condition and denotes the satellite frequencies j and perturbing-body frequencies m which are in resonance with each other. It often happens that the satellite has no resonances with the perturbing body, in which case set $\mathcal{B}$ is empty.

The minimum period τ should obey the following inequalities:

$$\tau \ \geq \ 3\tau_\lambda \tag{14}$$

$$\tau \ \geq \ 3\tau_\theta \tag{15}$$

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where τ_λ and τ_θ are the periods of the satellite and the perturbing-body phase angle θ respectively. In addition, τ should usually be at least 8 times as long as the step size used in the numerical integrator, and much longer if accurate integration of perturbations with this period is desired. It is dangerous to make τ too long, however. Components of the double-averaged perturbation which have periods equal to τ or shorter will be excluded from the averaged equations of motion and treated as short-periodic variations. If τ is too long and deep resonance exists, some of these short-periodic variations may be large enough to cause large second-order coupling effects, making the averaging expansions (2.3-1) and (2.3-7) diverge. For this reason, τ should be less than 100 times the integrator step size. Making τ small enough to ensure convergence of the averaging expansions takes priority over inequality (15), which must be dropped if the perturbing-body phase angle θ varies too slowly. If θ is the rotation angle of the Earth, this will not be necessary, since the Earth rotates quickly. If θ is the rotation angle of Mercury or Venus, it may be necessary to drop condition (15), depending on how large the m-daily ($j = 0$) short-periodic variations due to the gravitational tesseral harmonics are. The Moon is a borderline case.

The following perturbations can be double-averaged:

1. Central-body gravitational sectoral and tesseral harmonics. For this perturbation, θ is the rotation angle of the central body. (For a more precise definition of θ, see Section 2.7.1) The double-averaged central-body gravitational spherical harmonic model in SST is fully-developed and will be discussed in Section 3.3.

2. Third-body point-mass effects, if the third body orbits the central body. For this perturbation, θ is the equinoctial mean longitude of the third body. The current double-averaged third-body model assumes that the third body orbits the central body in a slowly-varying Keplerian ellipse. Methods for predicting the effects of short-periodic variations in the third-body orbit on the satellite orbit have yet to be developed. For Earth satellites, the short-periodic variations in the orbit of the Moon are substantial. If they are included in the Lunar ephemeris used with the double-averaged third-body model, the integrator step size will be driven down to values appropriate for the single-averaged third-body model, thus destroying the usefulness of the double-averaging expansion. The step size reduction is avoided by using a smoothed ephemeris for the Moon, but this creates an error in the computed satellite mean element rates due to the Lunar perturbation, and the size of this error is not known. Because of these limitations, double-averaged third-body perturbation models will not be discussed in detail in this document. For a complete description of the current model, see [Collins, 1981].

There are some perturbations for which no averaging operator with the required properties can be found. These perturbations are called non-averageable and include:

1. Atmospheric drag, with an asymmetric spacecraft and fast, non-periodic attitude variation.

2. Solar radiation pressure, with an asymmetric spacecraft and fast, non-periodic attitude variation.

3. Continuous thrust, with fast, non-periodic changes in direction.

4. Impulsive thrust.

These perturbations are typical of directed flight, which in general is not required to be either slowly-varying or 2π-periodic in λ or any other phase angle. Semianalytic Satellite Theory cannot predict the effects of these perturbations, with the exception of impulsive thrust (see below), and should not be used unless they are small enough to ignore.

Some types of directed flight are averageable, and these include many scenarios of practical interest:

1. A drag-perturbed satellite whose solar panels always point directly at the Sun. The attitude of the satellite relative to the atmosphere will be 2π-periodic in λ and will vary slowly in time as the Earth revolves about the Sun.

2. A spacecraft with a solar sail which is feathered when approaching the Sun and perpendicular to the Sun line when receding. The attitude of the sail will be 2π-periodic in λ and will vary slowly in time as the Earth revolves about the Sun.

3. A spacecraft with a constant-thrust ion engine whose thrust is always parallel to the orbit. The direction of the resulting acceleration will be 2π-periodic in λ and the magnitude will vary slowly in time as reaction mass is depleted and the spacecraft gets lighter.

Of course, the perturbations remain averageable only as long as the orbit remains elliptical. Parabolic and hyperbolic orbits are beyond the scope of this document.

Impulsive thrust is not averageable, but its effects can be predicted using the following procedure:

1. Integrate the averaged equations of motion (1) up to the time of the impulsive maneuver.

2. Compute the short-periodic variations as functions of the mean elements using equations (2.3-20, 22, 24) and the equations in Section 2.5.

3. Add the short-periodic variations to the mean elements to get the osculating elements (equations (2.3-1)).

4. Convert the osculating equinoctial orbit elements to position and velocity using the equations in Section 2.1.4.

5. Add the velocity change $\Delta\mathbf{v}$ caused by the impulsive maneuver to the satellite velocity.

6. Convert the satellite position and velocity to osculating equinoctial orbit elements using the equations in Section 2.1.5.

7. Invert equations (2.3-1) to convert the osculating elements to mean elements (see Section 6).

This procedure is always valid, but if impulsive thrusts occur more often than once per orbit, it may be too expensive to make the use of averaging worthwhile. For additional methods for modeling impulsive maneuvers, as well as continuous thrust, see McClain [1982].

Some perturbations are usually averageable, but cannot be averaged in certain circumstances because of large second-order effects. These include:

1. Third-body point-mass effects, for a satellite whose orbit comes too close to the orbit of the third body. This perturbation will be non-averageable even if resonance locking ensures that the satellite will always remain far from the third body.

2. Third-body point-mass effects, for a satellite whose orbit comes too close to the boundary of the central body's gravitational Sphere of Influence.

3. Atmospheric drag effects, during the terminal stage of reentry.

Semianalytical Satellite Theory should not be used under these circumstances.

It is worth considering at this point whether the averaging operators (10) and (11) actually have the required properties (2)–(3). It is immediately clear from equations (10) and (11) that both operators are linear (equation (2)). It is also clear from equation (10) that the single-averaging operator is idempotent (equation (3)). To show that the double-averaging operator is idempotent, we double-average both sides of equation (11) to obtain

$$
\begin{aligned}
<< f >> = & \; \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} < f > d\lambda \, d\theta \\
& + \frac{1}{2\pi^2} \sum_{(j,m)\in\mathcal{B}} [\cos(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} < f > \cos(j\lambda' - m\theta')d\lambda'd\theta' \\
& \qquad\qquad + \sin(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} < f > \sin(j\lambda' - m\theta')d\lambda'd\theta'] \\
= & \; \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f d\lambda d\theta \\
& + \frac{1}{4\pi^4} \sum_{(j,m)\in\mathcal{B}} [\cos(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos^2(j\lambda' - m\theta')d\lambda'd\theta' \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \cos(j\lambda'' - m\theta'')d\lambda''d\theta'' \\
& \qquad\qquad + \sin(j\lambda - m\theta) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin^2(j\lambda' - m\theta')d\lambda'd\theta' \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f \sin(j\lambda'' - m\theta'')d\lambda''d\theta''] \\
= & \; < f >
\end{aligned}
$$

(16)

Note that, if f is independent of θ, the double-averaging operator (11) reduces to the single-averaging operator (10). Thus we can without inconsistencies apply the single-averaging operator (10) to perturbations that do not depend upon θ, and the double-averaging operator (11) to perturbations that do depend upon θ.

If the perturbations can be averaged with the operators (10) or (11), then the Equations of Averaging (2.3-25) can be solved for the mean element rates by averaging both sides of each equation. Note that the averages are always taken with (a, h, k, p, q, t) held constant. Using (4), we thereby obtain for the mean element rates A_i at any order

$$ A_i = < G_i > \tag{17} $$

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Substituting (2.3-26, 27, 28) into (17), we can obtain explicit expressions for the mean element rates at each order.

The explicit equations for the first-order mean element rates have the same form for both single-averaged and double-averaged perturbations:

$$A_{i\alpha} = < F_{i\alpha}(a, h, k, p, q, \lambda, t) > \tag{18}$$

The averaging operator used is single in the first case and double in the second.

The explicit equations for the second-order mean element rates involve coupling terms between two perturbations, which may be two different perturbations (cross-coupling) or copies of the same perturbation (auto-coupling). If either perturbation is single-averaged, the equations take the form

$$A_{i\alpha\beta} = \sum_{j=1}^{6} \left\langle \frac{\partial F_{i\alpha}}{\partial a_j} \eta_{j\beta} \right\rangle + \frac{15}{8} \frac{n}{a^2} \langle \eta_{1\alpha}\eta_{1\beta} \rangle \delta_{i6} \tag{19}$$

If both perturbations are single-averaged, a single-averaging operator is used. If either perturbation is double-averaged, a double-averaging operator is used. If both perturbations are double-averaged, the equations take the form

$$A_{i\alpha\beta} = \sum_{j=1}^{6} \left\langle \frac{\partial F_{i\alpha}}{\partial a_j} \eta_{j\beta} \right\rangle + \frac{15}{8} \frac{n}{a^2} \langle \eta_{1\alpha}\eta_{1\beta} \rangle \delta_{i6} - \sum_{j=1}^{6} \left\langle \frac{\partial \eta_{i\alpha}}{\partial a_j} A_{j\beta} \right\rangle \tag{20}$$

If both perturbations use the same perturbing-body phase angle θ, then a double-averaging operator is used. However, if the perturbations use different perturbing-body phase angles θ and θ', then a triple-averaging operator is used.

Derivation of the explicit equations for the third-order mean element rates will be left as an exercise for the reader.

2.5   Short-Periodic Variations

Once the mean element rates $A_{i\alpha}$, $A_{i\alpha\beta}$, $A_{i\alpha\beta\gamma}$, ... are known, the Equations of Averaging (2.3-25) can be solved for the short-periodic variations $\eta_{i\alpha}$, $\eta_{i\alpha\beta}$, $\eta_{i\alpha\beta\gamma}$, ... by using Fourier series expansions for the functions $G_{i\alpha}, G_{i\alpha\beta}, G_{i\alpha\beta\gamma}, \dots$. The resulting expressions for the $\eta_{i\alpha}, \eta_{i\alpha\beta}, \eta_{i\alpha\beta\gamma}, \dots$ can then be added together as shown in equations (2.3-20, 22, 24) with $\epsilon = \nu_\alpha = 1$ and substituted into equations (2.3-1) with $\epsilon = 1$ to give the osculating elements:

$$\hat{a}_i = a_i + \sum_\alpha \eta_{i\alpha} + \sum_{\alpha\beta} \eta_{i\alpha\beta} + \sum_{\alpha\beta\gamma} \eta_{i\alpha\beta\gamma} + \dots \tag{1}$$

Of course, using Fourier series expansions for the functions G_i assumes that the osculating rate functions $F_{i\alpha}$ are 2π-periodic in the phase angles of the expansions. Most perturbations can be expressed with more than one kind of Fourier series expansion. In the following subsections we shall give formulas for the short-periodic variations corresponding to several possible expansions.

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Some of the general expressions presented in this section are new, so enough steps in the derivations will be presented to enable the reader to rederive them. Our derivations are based on the work of [Cefola and McClain, 1978], [Green, 1979], [McClain and Slutsky, 1980], [Slutsky, 1980], [Slutsky and McClain, 1981], and [McClain, 1982].

2.5.1 General η_i Expansions in λ

Recall that the Equations of Averaging of arbitrary order can be written as

$$A_i + n\frac{\partial \eta_i}{\partial \lambda} + \frac{\partial \eta_i}{\partial t} = G_i - \frac{3}{2}\frac{n}{a}\delta_{i6}\eta_1 \tag{1}$$

where G_i are predetermined functions (equations (2.3-26, 27, 28)) and

$$A_i = <G_i> \tag{2}$$

At order m, the Equations of Averaging contain m perturbations, which may all be different or may include multiple copies of the same perturbation.

If all of the perturbations in the Equations of Averaging (1) are single-averaged, then the equations can be solved for the short-periodic variations η_i by integrating over the satellite mean longitude λ. It is convenient to first define the short-periodic kernels ξ_i:

$$\xi_i = \frac{1}{n}\int(G_i - A_i)d\lambda \tag{3}$$

The constant of integration is specified by requiring

$$<\xi_i> = 0 \tag{4}$$

In the absence of explicit time-dependence, the full short-periodic variations η_i are then given by:

$$\eta_i = \xi_i - \frac{3}{2a}\delta_{i6}\int \xi_1 \, d\lambda \tag{5}$$

The conditions (2.4-4) and (4) require

$$\left\langle \int \xi_1 \, d\lambda \right\rangle = 0 \tag{6}$$

In the presence of explicit time-dependence, the full short-periodic variations η_i are given by:

$$\begin{aligned}
\eta_i &= \xi_i + \sum_{k=1}^{K}\frac{(-1)^k}{n^k}\int_k \frac{\partial^k \xi_i}{\partial t^k}d\lambda^k \\
&\quad -\frac{3}{2a}\delta_{i6}\left[\int \xi_1 d\lambda + \sum_{k=1}^{K}(k+1)\frac{(-1)^k}{n^k}\int_{k+1}\frac{\partial^k \xi_1}{\partial t^k}d\lambda^{k+1}\right]
\end{aligned} \tag{7}$$

Here K is the order of the highest order partial derivatives desired in the expansions. The conditions (2.4-4) and (4) require

$$\left\langle \int_k \xi_i d\lambda^k \right\rangle = 0 \text{ for all } k \geq 1 \tag{8}$$

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Here the following somewhat unusual notation is used for the operator which is the kth indefinite integral (the inverse of the kth derivative operator):

$$\int_k f(\lambda)d\lambda^k = \underbrace{\int \cdots \int}_{k} f(\lambda)\underbrace{d\lambda \cdots d\lambda}_{k} \tag{9}$$

Alternately, knowing the short-periodic variations η_i^0 in the absence of explicit time-dependence, we can compute recursively the short-periodic variations η_i^k including the kth order time derivatives:

$$
\begin{aligned}
\eta_i^0 &= \xi_i - \frac{3}{2a}\delta_{i6}\int \xi_1 d\lambda \\
\eta_i^k &= \eta_i^{k-1} + \frac{(-1)^k}{n^k}\int_k \frac{\partial^k \eta_i^0}{\partial t^k}d\lambda^k + \frac{3k}{2an^k}(-1)^{k+1}\delta_{i6}\int_{k+1}\frac{\partial^k \eta_1^0}{\partial t^k}d\lambda^{k+1}
\end{aligned}
\tag{10}
$$

Explicitly, we suppose that the functions G_i can be written as a Fourier series in the mean longitude λ:

$$
\begin{aligned}
G_i(a,h,k,p,q,\lambda,t) &= \mathcal{C}_i^0(a,h,k,p,q,t) \\
&+ \sum_{j=1}^{\infty}\left[\mathcal{C}_i^j(a,h,k,p,q,t)\cos j\lambda + \mathcal{S}_i^j(a,h,k,p,q,t)\sin j\lambda\right]
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
\mathcal{C}_i^0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}G_i\,d\lambda \\
\mathcal{C}_i^j &= \frac{1}{\pi}\int_{-\pi}^{\pi}G_i\cos j\lambda\,d\lambda \\
\mathcal{S}_i^j &= \frac{1}{\pi}\int_{-\pi}^{\pi}G_i\sin j\lambda\,d\lambda
\end{aligned}
\tag{12}
$$

Using (2.4-10) and (2), we can obtain the mean element rates A_i:

$$A_i = \mathcal{C}_i^0 \tag{13}$$

We can then obtain the short-periodic variations η_i by performing the integrals (7, 8). The final result of these calculations is:

$$\eta_i = \sum_{j=1}^{\infty}[C_i^j\cos j\lambda + S_i^j\sin j\lambda] \tag{14}$$

where

$$
\begin{aligned}
C_i^j &= -\frac{1}{jn}\left[\mathcal{S}_i^j - \frac{3}{2}\frac{1}{a}\frac{\delta_{i6}}{j}\mathcal{C}_1^j\right] + \frac{1}{(jn)^2}\left[\frac{\partial \mathcal{C}_i^j}{\partial t} + \frac{3}{2}\frac{2}{a}\frac{\delta_{i6}}{j}\frac{\partial \mathcal{S}_1^j}{\partial t}\right] \\
&+ \frac{1}{(jn)^3}\left[\frac{\partial^2 \mathcal{S}_i^j}{\partial t^2} - \frac{3}{2}\frac{3}{a}\frac{\delta_{i6}}{j}\frac{\partial^2 \mathcal{C}_1^j}{\partial t^2}\right] - \frac{1}{(jn)^4}\left[\frac{\partial^3 \mathcal{C}_i^j}{\partial t^3} + \frac{3}{2}\frac{4}{a}\frac{\delta_{i6}}{j}\frac{\partial^3 \mathcal{S}_1^j}{\partial t^3}\right] - \cdots \\
S_i^j &= \frac{1}{jn}\left[\mathcal{C}_i^j + \frac{3}{2}\frac{1}{a}\frac{\delta_{i6}}{j}\mathcal{S}_1^j\right] + \frac{1}{(jn)^2}\left[\frac{\partial \mathcal{S}_i^j}{\partial t} - \frac{3}{2}\frac{2}{a}\frac{\delta_{i6}}{j}\frac{\partial \mathcal{C}_1^j}{\partial t}\right] \\
&- \frac{1}{(jn)^3}\left[\frac{\partial^2 \mathcal{C}_i^j}{\partial t^2} + \frac{3}{2}\frac{3}{a}\frac{\delta_{i6}}{j}\frac{\partial^2 \mathcal{S}_1^j}{\partial t^2}\right] - \frac{1}{(jn)^4}\left[\frac{\partial^3 \mathcal{S}_i^j}{\partial t^3} - \frac{3}{2}\frac{4}{a}\frac{\delta_{i6}}{j}\frac{\partial^3 \mathcal{C}_1^j}{\partial t^3}\right] + \cdots
\end{aligned}
\tag{15}
$$

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2.5.2   General η_i Expansions in F

In this section we suppose that the functions G_i are expanded as a finite Fourier series in the eccentric longitude F:

$$G_i(a, h, k, p, q, F, t) = \mathcal{C}_i^0(a, h, k, p, q, t) \tag{1}$$

$$+ \sum_{j=1}^{J} \left[\mathcal{C}_i^j(a, h, k, p, q, t) \cos jF + \mathcal{S}_i^j(a, h, k, p, q, t) \sin jF \right]$$

where

$$
\begin{aligned}
\mathcal{C}_i^0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G_i \, dF \\
\mathcal{C}_i^j &= \frac{1}{\pi} \int_{-\pi}^{\pi} G_i \cos jF \, dF \\
\mathcal{S}_i^j &= \frac{1}{\pi} \int_{-\pi}^{\pi} G_i \sin jF \, dF
\end{aligned}
\tag{2}
$$

Again we may use (2.4-10) and (2.5.1-2, 3, 4, 7, 8) to obtain the short periodic variations η_i. To do this, it will be necessary to convert the integrals over λ into Fourier series expansions over F. We begin by supposing $f(\lambda)$ has a finite Fourier series expansion in F with known coefficients and it averages to zero:

$$f(\lambda) = \mathcal{C}^0 + \sum_{j=1}^{J} (\mathcal{C}^j \cos jF + \mathcal{S}^j \sin jF) \tag{3}$$

$$< f(\lambda) >= 0 \tag{4}$$

Using

$$\int f(\lambda) d\lambda = \int f(\lambda) \frac{\partial \lambda}{\partial F} dF \tag{5}$$

the following consequence of the equinoctial form (2.1.4-2) of Kepler's Equation

$$\frac{\partial \lambda}{\partial F} = \frac{r}{a} = 1 - h \sin F - k \cos F \tag{6}$$

and the following well-known trigonometric identities

$$
\begin{aligned}
\cos jF \cos kF &= \frac{\cos(j-k)F + \cos(j+k)F}{2} \\
\cos jF \sin kF &= \frac{\sin(j+k)F - \sin(j-k)F}{2} \\
\sin jF \cos kF &= \frac{\sin(j+k)F + \sin(j-k)F}{2} \\
\sin jF \sin kF &= \frac{\cos(j-k)F - \cos(j+k)F}{2}
\end{aligned}
\tag{7}
$$

we can convert the right side of (5) into a Fourier series expansion in F. Note in particular that the condition (4) implies that the constant term $\mathcal{C}^0$ in (3) must be related to the Fourier coefficients $\mathcal{C}^1$ and $\mathcal{S}^1$ by

$$\mathcal{C}^0 = \frac{k}{2}\mathcal{C}^1 + \frac{h}{2}\mathcal{S}^1 \tag{8}$$

Higher-order integrals can be computed using recursion formulas obtained from the equation

$$\int_{m+1} f(\lambda)d\lambda^{m+1} = \int \int_m f(\lambda)d\lambda^m d\lambda \tag{9}$$

We summarize the conversion for the general multiple integral with the following notation:

$$\int_m f(\lambda)d\lambda^m = U_m^0(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma) + \sum_{j=1}^{J+m}[U_m^j(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma)\cos jF + V_m^j(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma)\sin jF] \tag{10}$$

$$< \int_m f(\lambda)d\lambda^m >= 0 \tag{11}$$

Here the arguments $(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma)$ denote that the functions U_m^j and V_m^j depend upon the coefficients $\mathcal{C}^j$ and $\mathcal{S}^j$ appearing in (3) through the relations:

for $1 \le j \le J$:

$$U_0^j = \mathcal{C}^j$$

$$V_0^j = \mathcal{S}^j$$

for $m \ge 0$:

$$U_m^0 = \frac{k}{2}U_m^1 + \frac{h}{2}V_m^1$$

$$U_{m+1}^1 = \frac{hk}{2}U_m^1 - \left(1 - \frac{h^2}{2}\right)V_m^1 - \frac{h}{2}U_m^2 + \frac{k}{2}V_m^2$$

$$V_{m+1}^1 = \left(1 - \frac{k^2}{2}\right)U_m^1 - \frac{hk}{2}V_m^1 - \frac{k}{2}U_m^2 - \frac{h}{2}V_m^2$$

$$\tag{12}$$

for $2 \le j \le J + m + 1$:

$$U_{m+1}^j = \frac{1}{j}\left(-V_m^j + \frac{h}{2}U_m^{j-1} + \frac{k}{2}V_m^{j-1} - \frac{h}{2}U_m^{j+1} + \frac{k}{2}V_m^{j+1}\right)$$

$$V_{m+1}^j = \frac{1}{j}\left(U_m^j - \frac{k}{2}U_m^{j-1} + \frac{h}{2}V_m^{j-1} - \frac{k}{2}U_m^{j+1} - \frac{h}{2}V_m^{j+1}\right)$$

for $j \ge J + m + 1$:

$$U_m^j = 0$$
$$V_m^j = 0$$

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With this notation, we can write the mean element rates A_i and the short-periodic variations η_i corresponding to (1):

$$A_i = \mathcal{C}_i^0 - \frac{k}{2}\mathcal{C}_i^1 - \frac{h}{2}\mathcal{S}_i^1 \tag{13}$$

$$\eta_i = C_i^0 + \sum_{j=1}^{J+K+2} \left[C_i^j \cos jF + S_i^j \sin jF \right] \tag{14}$$

where

$$C_i^0 = \frac{k}{2}C_i^1 + \frac{h}{2}S_i^1$$

$$
\begin{aligned}
C_i^j &= \frac{1}{n}U_1^j\left(\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta\right) + \sum_{k=1}^{K}\frac{(-1)^k}{n^{k+1}}U_{k+1}^j\left(\frac{\partial^k \mathcal{C}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_i^\zeta}{\partial t^k}\right) \\
&\quad -\frac{3}{2a}\delta_{i6}\left[\frac{1}{n}U_2^j\left(\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta\right) + \sum_{k=1}^{K}(k+1)\frac{(-1)^k}{n^{k+1}}U_{k+2}^j\left(\frac{\partial^k \mathcal{C}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_1^\zeta}{\partial t^k}\right)\right] \\
S_i^j &= \frac{1}{n}V_1^j\left(\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta\right) + \sum_{k=1}^{K}\frac{(-1)^k}{n^{k+1}}V_{k+1}^j\left(\frac{\partial^k \mathcal{C}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_i^\zeta}{\partial t^k}\right) \\
&\quad -\frac{3}{2a}\delta_{i6}\left[\frac{1}{n}V_2^j\left(\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta\right) + \sum_{k=1}^{K}(k+1)\frac{(-1)^k}{n^{k+1}}V_{k+2}^j\left(\frac{\partial^k \mathcal{C}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_1^\zeta}{\partial t^k}\right)\right]
\end{aligned}
\tag{15}
$$

2.5.3   General η_i Expansions in L

In this section we suppose that the functions G_i are expanded as a finite *modified* Fourier series in the true longitude L:

$$
\begin{aligned}
G_i(a,h,k,p,q,L,t) &= \mathcal{C}_i^0(a,h,k,p,q,t) + \sum_{m=1}^{M}\mathcal{D}_i^m(a,h,k,p,q,t)(L-\lambda)^m \\
&\quad + \sum_{j=1}^{J}\left[\mathcal{C}_i^j(a,h,k,p,q,t)\cos jL + \mathcal{S}_i^j(a,h,k,p,q,t)\sin jL\right]
\end{aligned}
\tag{1}
$$

Here the quantities $(L-\lambda)^m$ are written separately, rather than by replacing them with their Fourier series expansions recorded below.

The equation of the center may be calculated from the Fourier series expansion

$$L - \lambda = \sum_{j=1}^{\infty}\frac{2}{j}(\sigma_j \cos jL - \rho_j \sin jL) \tag{2}$$

Here

$$
\begin{aligned}
\rho_j &= \langle \cos jL \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos jL \, d\lambda \\
\sigma_j &= \langle \sin jL \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin jL \, d\lambda
\end{aligned}
\tag{3}
$$

The auxiliary quantities ρ_j and σ_j can be evaluated using the equations [*]

$$
\begin{aligned}
\rho_j &= (1+jB)(-b)^j C_j(k,h) \\
\sigma_j &= (1+jB)(-b)^j S_j(k,h)
\end{aligned}
\tag{4}
$$

where b and B are given by (2.1.4-4) and (2.1.6-1b), and

$$
C_j(k,h) + iS_j(k,h) = (k+ih)^j
\tag{5}
$$

are obtained from the recursion formulas

$$
\begin{aligned}
C_{j+1}(k,h) &= kC_j(k,h) - hS_j(k,h) \ , \ C_0 = 1 \\
S_{j+1}(k,h) &= hC_j(k,h) + kS_j(k,h) \ , \ S_0 = 0
\end{aligned}
\tag{6}
$$

The quantities $(L-\lambda)^m$ may be calculated from the expansion

$$
(L-\lambda)^m = \kappa_m^0 + \sum_{j=1}^{\infty}(\kappa_m^j \cos jL + \psi_m^j \sin jL)
\tag{7}
$$

Here the first-order coefficients κ_1^j and ψ_1^j are obtained immediately from (2):

$$
\begin{aligned}
\kappa_1^0 &= 0 \\
\kappa_1^j &= \frac{2\sigma_j}{j} \\
\psi_1^j &= -\frac{2\rho_j}{j}
\end{aligned}
\tag{8}
$$

Higher-order coefficients κ_m^j and ψ_m^j are obtained using the following recursion formulas, obtained by multiplying the series on the right sides of (2) and (7):

$$
\kappa_{m+1}^0 = \sum_{k=1}^{\infty} \frac{1}{k}(\sigma_k \kappa_m^k - \rho_k \psi_m^k)
$$

$$
\begin{aligned}
\kappa_{m+1}^j &= \frac{2\sigma_j}{j}\kappa_m^0 + \sum_{k=1}^{\infty} \frac{1}{j+k}(\sigma_{j+k}\kappa_m^k - \rho_{j+k}\psi_m^k) \\
&+ \sum_{k=1}^{\infty} \frac{1}{k}(\sigma_k \kappa_m^{j+k} - \rho_k \psi_m^{j+k}) + (1-\delta_{j1}) \sum_{k=1}^{j-1} \frac{1}{j-k}(\sigma_{j-k}\kappa_m^k + \rho_{j-k}\psi_m^k) \\
\psi_{m+1}^j &= -\frac{2\rho_j}{j}\kappa_m^0 - \sum_{k=1}^{\infty} \frac{1}{j+k}(\sigma_{j+k}\psi_m^k + \rho_{j+k}\kappa_m^k) \\
&+ \sum_{k=1}^{\infty} \frac{1}{k}(\sigma_k \psi_m^{j+k} + \rho_k \kappa_m^{j+k}) + (1-\delta_{j1}) \sum_{k=1}^{j-1} \frac{1}{j-k}(\sigma_{j-k}\psi_m^k - \rho_{j-k}\kappa_m^k)
\end{aligned}
\tag{9}
$$

---

[*]Let $Z = \exp(iL)$ and use the Residue Theorem to integrate

$$
\rho_j + i\sigma_j = \frac{(1-h^2-k^2)^{3/2}}{2\pi} \int_{-\pi}^{\pi} \frac{\exp(ijL)}{(1+h\sin L + k\cos L)^2}dL
$$

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Note that if m is odd, then $(L - \lambda)^m$ is antisymmetric about perigee. Therefore we have

$$\kappa_{2k+1}^0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (L - \lambda)^{2k+1} dL = 0 \tag{10}$$

i. e. , only even powers of $(L - \lambda)$ have nonzero constant terms.

Again we may use (2.4-10) and (2.5.1-2,3,4,7,8) to obtain the short-periodic variations η_i. To do this, it will be necessary to convert the integrals over λ into modified Fourier series expansions in L. We begin by supposing $f(\lambda)$ has a modified Fourier series expansion in L with known coefficients and it averages to zero:

$$f(\lambda) = \mathcal{C}^0 + \sum_{m=1}^{M} \mathcal{D}^m (L - \lambda)^m + \sum_{j=1}^{J} (\mathcal{C}^j \cos jL + \mathcal{S}^j \sin jL) \tag{11}$$

$$< f(\lambda) >= 0 \tag{12}$$

Using

$$\int f(\lambda) d\lambda = \int f(\lambda) \frac{\partial \lambda}{\partial L} dL , \tag{13}$$

the following consequences of (2.1.4-2,5,6) and (2)

$$\frac{\partial \lambda}{\partial L} = \frac{1}{\sqrt{1 - h^2 - k^2}} \left(\frac{r}{a} \right)^2 = \frac{(1 - h^2 - k^2)^{3/2}}{(1 + h \sin L + k \cos L)^2} = 1 + 2 \sum_{j=1}^{\infty} (\rho_j \cos jL + \sigma_j \sin jL), \tag{14}$$

the integral

$$\int (L - \lambda)^m d\lambda = -\frac{(L - \lambda)^{m+1}}{m + 1} + \int (L - \lambda)^m dL , \tag{15}$$

the identities (2.5.2-7) and expansions (7), we can convert the right side of (13) into a Fourier series expansion in L. Note in particular that equations (3) and (12) imply that the constant term $\mathcal{C}^0$ in (11) must be related to the Fourier coefficients $\mathcal{C}^i, \mathcal{S}^i$ and $\mathcal{D}^i$ by

$$\mathcal{C}^0 = - \sum_{m=1}^{M} \mathcal{D}^m \kappa_m^0 - \sum_{j=1}^{J} (\mathcal{C}^j \rho_j + \mathcal{S}^j \sigma_j) \tag{16}$$

Note also that we need the following formulas for the product of two Fourier series:

$$
\sum_{j=1}^{J} (\mathcal{C}^j \cos jL + \mathcal{S}^j \sin jL) \sum_{k=1}^{\infty} (\rho_k \cos kL + \sigma_k \sin kL) = \frac{1}{2} \sum_{j=1}^{\infty} \Big\{ \mathcal{I}_1^J(j)(\mathcal{C}^j \rho_j + \mathcal{S}^j \sigma_j)
$$
$$
+ \Big[(1 - \delta_{j1}) \sum_{\ell=\max(j-J,1)}^{j-1} (\mathcal{C}^{j-\ell} \rho_\ell - \mathcal{S}^{j-\ell} \sigma_\ell) + \mathcal{I}_1^{J-1}(j) \sum_{\ell=1}^{J-j} (\mathcal{C}^{j+\ell} \rho_\ell + \mathcal{S}^{j+\ell} \sigma_\ell)
$$
$$
+ \sum_{\ell=1}^{J} (\mathcal{C}^\ell \rho_{j+\ell} + \mathcal{S}^\ell \sigma_{j+\ell}) \Big] \cos jL \tag{17}
$$
$$
+ \Big[(1 - \delta_{j1}) \sum_{\ell=\max(j-J,1)}^{j-1} (\mathcal{C}^{j-\ell} \sigma_\ell + \mathcal{S}^{j-\ell} \rho_\ell) + \mathcal{I}_1^{J-1}(j) \sum_{\ell=1}^{J-j} (-\mathcal{C}^{j+\ell} \sigma_\ell + \mathcal{S}^{j+\ell} \rho_\ell)
$$
$$
+ \sum_{\ell=1}^{J} (\mathcal{C}^\ell \sigma_{j+\ell} - \mathcal{S}^\ell \rho_{j+\ell}) \Big] \sin jL \Big\}
$$

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Here $\mathcal{I}_r^s(j)$ is the inclusion operator defined by

$$\mathcal{I}_r^s(j) = \begin{cases} 1 & \text{if } r \leq j \leq s \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Higher-order integrals can be computed using recursion formulas obtained from equation (2.5.2-9). We summarize the conversion for the general multiple integral with the following notation:

$$
\begin{aligned}
\int_k f(\lambda)d\lambda^k &= U_k^0(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma, \mathcal{D}^\varsigma) + \sum_{m=1}^{M+k} W_k^m(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma, \mathcal{D}^\varsigma)(L - \lambda)^m \\
&\quad + \sum_{j=1}^{\infty} \left[U_k^j(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma, \mathcal{D}^\varsigma) \cos jL + V_k^j(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma, \mathcal{D}^\varsigma) \sin jL \right]
\end{aligned} \tag{19}
$$

$$\left\langle \int_k f(\lambda)d\lambda^k \right\rangle = 0 \tag{20}$$

Here the arguments $(\mathcal{C}^\varsigma, \mathcal{S}^\varsigma, \mathcal{D}^\varsigma)$ denote that the functions U_k^j, V_k^j, and W_k^m depend upon the coefficients $\mathcal{C}^j, \mathcal{S}^j$, and $\mathcal{D}^m$ appearing in (11) through the relations
for $j \geq 1$:

$$
\begin{aligned}
U_1^j &= -\frac{1}{j} \left[\mathcal{I}_1^J(j)\mathcal{S}^j + \sum_{m=1}^M \mathcal{D}^m \psi_m^j + (1 - \delta_{j1}) \sum_{k=\max(j-J,1)}^{j-1} (\mathcal{C}^{j-k}\sigma_k + \mathcal{S}^{j-k}\rho_k) \right. \\
&\quad \left. + \mathcal{I}_1^{J-1}(j) \sum_{k=1}^{J-j} (-\mathcal{C}^{j+k}\sigma_k + \mathcal{S}^{j+k}\rho_k) + \sum_{k=1}^J (\mathcal{C}^k \sigma_{j+k} - \mathcal{S}^k \rho_{j+k}) \right] \\
V_1^j &= \frac{1}{j} \left[\mathcal{I}_1^J(j)\mathcal{C}^j + \sum_{m=1}^M \mathcal{D}^m \kappa_m^j + (1 - \delta_{j1}) \sum_{k=\max(j-J,1)}^{j-1} (\mathcal{C}^{j-k}\rho_k - \mathcal{S}^{j-k}\sigma_k) \right. \\
&\quad \left. + \mathcal{I}_1^{J-1}(j) \sum_{k=1}^{J-j} (\mathcal{C}^{j+k}\rho_k + \mathcal{S}^{j+k}\sigma_k) + \sum_{k=1}^J (\mathcal{C}^k \rho_{j+k} + \mathcal{S}^k \sigma_{j+k}) \right] \\
W_1^1 &= -\mathcal{C}^0 \\
W_1^m &= -\frac{\mathcal{D}^{m-1}}{m}
\end{aligned} \tag{21}
$$

for $k \geq 1$:

$$
\begin{aligned}
U_k^0 &= -\sum_{m=1}^{M+k} W_k^m \kappa_m^0 - \sum_{j=1}^{\infty} (U_k^j \rho_j + V_k^j \sigma_j) \\
U_{k+1}^j &= -\frac{1}{j} \left[V_k^j + \sum_{m=1}^{M+k-1} W_k^m \psi_m^j + (1 - \delta_{j1}) \sum_{\ell=1}^{j-1} (U_k^{j-\ell}\sigma_\ell + V_k^{j-\ell}\rho_\ell) \right. \\
&\quad \left. + \sum_{\ell=1}^{\infty} (-U_k^{j+\ell}\sigma_\ell + U_k^\ell \sigma_{j+\ell} + V_k^{j+\ell}\rho_\ell - V_k^\ell \rho_{j+\ell}) \right]
\end{aligned}
$$

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$$V_{k+1}^j = \frac{1}{j}\left[U_k^j + \sum_{m=1}^{M+k-1} W_k^m \kappa_m^j + (1 - \delta_{j1}) \sum_{\ell=1}^{j-1} (U_k^{j-\ell} \rho_\ell - V_k^{j-\ell} \sigma_\ell) \right.$$
$$\left. + \sum_{\ell=1}^{\infty} (U_k^{j+\ell} \rho_\ell + U_k^\ell \rho_{j+\ell} + V_k^{j+\ell} \sigma_\ell + V_k^\ell \sigma_{j+\ell}) \right]$$
$$W_{k+1}^1 = -U_k^0$$
$$W_{k+1}^m = -\frac{W_k^{m-1}}{m}$$

With this notation, we can write the mean element rates A_i and the short-periodic variations η_i corresponding to (1):

$$A_i = \mathcal{C}_i^0 + \sum_{m=1}^{M} \mathcal{D}_i^m \kappa_m^0 + \sum_{j=1}^{J} (\mathcal{C}_i^j \rho_j + \mathcal{S}_i^j \sigma_j) \tag{22}$$

$$\eta_i = C_i^0 + \sum_{m=1}^{M+K+2} D_i^m (L - \lambda)^m + \sum_{j=1}^{\infty} (C_i^j \cos jL + S_i^j \sin jL) \tag{23}$$

where

$$C_i^0 = -\sum_{m=1}^{M+K+2} D_i^m \kappa_m^0 - \sum_{j=1}^{\infty} (C_i^j \rho_j + S_i^j \sigma_j)$$

$$C_i^j = \frac{1}{n} U_1^j (\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta, \mathcal{D}_i^\zeta) + \sum_{k=1}^{K} \frac{(-1)^k}{n^{k+1}} U_{k+1}^j \left(\frac{\partial^k \mathcal{C}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_i^\zeta}{\partial t^k} \right)$$
$$- \frac{3}{2a} \delta_{i6} \left[\frac{1}{n} U_2^j (\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) + \sum_{k=1}^{K} (k+1) \frac{(-1)^k}{n^{k+1}} U_{k+2}^j \left(\frac{\partial^k \mathcal{C}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_1^\zeta}{\partial t^k} \right) \right]$$

$$S_i^j = \frac{1}{n} V_1^j (\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta, \mathcal{D}_i^\zeta) + \sum_{k=1}^{K} \frac{(-1)^k}{n^{k+1}} V_{k+1}^j \left(\frac{\partial^k \mathcal{C}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_i^\zeta}{\partial t^k} \right) \tag{24}$$
$$- \frac{3}{2a} \delta_{i6} \left[\frac{1}{n} V_2^j (\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) + \sum_{k=1}^{K} (k+1) \frac{(-1)^k}{n^{k+1}} V_{k+2}^j \left(\frac{\partial^k \mathcal{C}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_1^\zeta}{\partial t^k} \right) \right]$$

$$D_i^m = \mathcal{I}_1^{M+1}(m) \frac{1}{n} W_1^m (\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta, \mathcal{D}_i^\zeta) + \sum_{k=1}^{K} \mathcal{I}_1^{M+k+1}(m) \frac{(-1)^k}{n^{k+1}} W_{k+1}^m \left(\frac{\partial^k \mathcal{C}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_i^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_i^\zeta}{\partial t^k} \right)$$
$$- \frac{3}{2a} \delta_{i6} \left[\mathcal{I}_1^{M+2}(m) \frac{1}{n} W_2^m (\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) \right.$$
$$\left. + \sum_{k=1}^{K} \mathcal{I}_1^{M+k+2}(m)(k+1) \frac{(-1)^k}{n^{k+1}} W_{k+2}^m \left(\frac{\partial^k \mathcal{C}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{S}_1^\zeta}{\partial t^k}, \frac{\partial^k \mathcal{D}_1^\zeta}{\partial t^k} \right) \right]$$

In the absence of explicit time-dependence, equations (23)–(24) can be simplified. The short-periodic variations become:

$$\eta_i = C_i^0 + \sum_{m=1}^{M+2} D_i^m (L - \lambda)^m + \sum_{j=1}^{\infty} (C_i^j \cos jL + S_i^j \sin jL) \tag{25}$$

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where

$$
\begin{aligned}
C_i^0 &= -\sum_{m=1}^{M+2} D_i^m \kappa_m^0 - \sum_{j=1}^{\infty}(C_i^j \rho_j + S_i^j \sigma_j) \\
C_i^j &= \frac{1}{n}U_1^j(\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta, \mathcal{D}_i^\zeta) - \frac{3}{2an}\delta_{i6}U_2^j(\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) \\
S_i^j &= \frac{1}{n}V_1^j(\mathcal{C}_i^\zeta, \mathcal{S}_i^\zeta, \mathcal{D}_i^\zeta) - \frac{3}{2an}\delta_{i6}V_2^j(\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) \\
D_i^1 &= -\frac{1}{n}C_i^0 + \frac{3}{2an}\delta_{i6}U_1^0(\mathcal{C}_1^\zeta, \mathcal{S}_1^\zeta, \mathcal{D}_1^\zeta) \\
D_i^2 &= -\frac{1}{2n}\mathcal{D}_i^1 - \frac{3}{4an}\delta_{i6}\mathcal{C}_1^0 \\
D_i^m &= -\frac{1}{mn}\mathcal{D}_i^{m-1} - \frac{3}{2am(m-1)n}\delta_{i6}\mathcal{D}_1^{m-2}
\end{aligned}
\tag{26}
$$

2.5.4   General η_i Expansions in λ, θ

If one or more perturbations are double-averaged, then the functions G_i can be written as a double Fourier series in the mean longitude λ and the perturbing-body phase angle θ:

$$
G_i(a,h,k,p,q,\lambda,\theta,t) = \sum_{j,m}[\mathcal{C}_i^{jm}(a,h,k,p,q,t)\cos(j\lambda-m\theta) + \mathcal{S}_i^{jm}(a,h,k,p,q,t)\sin(j\lambda-m\theta)]
\tag{1}
$$

where

$$
\begin{aligned}
\mathcal{C}_i^{00} &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G_i\, d\lambda d\theta \\
\mathcal{C}_i^{jm} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G_i \cos(j\lambda-m\theta)\, d\lambda d\theta \\
\mathcal{S}_i^{jm} &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} G_i \sin(j\lambda-m\theta)\, d\lambda d\theta
\end{aligned}
\tag{2}
$$

Using (2.4-11) and (2.5.1-2), we can obtain the mean element rates A_i:

$$
A_i = \sum_{(j,m)\in\mathcal{B}}[\mathcal{C}_i^{jm}\cos(j\lambda-m\theta) + \mathcal{S}_i^{jm}\sin(j\lambda-m\theta)]
\tag{3}
$$

We can then obtain the short-periodic variations η_i by integrating the Equations of Averaging (2.5.1-1) in a manner similar to the single-averaged case in Section 2.5.1. Assuming that the coefficients $\mathcal{C}_i^{jm}$, $\mathcal{S}_i^{jm}$ and the rotation rate $\dot\theta$ do not explicitly depend upon time, we obtain

$$
\eta_i = \sum_{(j,m)\notin\mathcal{B}}[C_i^{jm}\cos(j\lambda-m\theta) + S_i^{jm}\sin(j\lambda-m\theta)]
\tag{4}
$$

where

$$
\begin{aligned}
C_i^{jm} &= -\frac{1}{jn-m\dot\theta}\left[\mathcal{S}_i^{jm} - \frac{3}{2}\frac{n}{a}\frac{\delta_{i6}}{jn-m\dot\theta}\mathcal{C}_1^{jm}\right] \\
S_i^{jm} &= \frac{1}{jn-m\dot\theta}\left[\mathcal{C}_i^{jm} + \frac{3}{2}\frac{n}{a}\frac{\delta_{i6}}{jn-m\dot\theta}\mathcal{S}_1^{jm}\right]
\end{aligned}
\tag{5}
$$

2.5.5 First-Order $\eta_{i\alpha}$ for Conservative Perturbations

For conservative perturbations, it may be advantageous to use an alternate solution for the first-order short-periodic variations $\eta_{i\alpha}$ which avoids having to obtain the osculating rate functions $F_{i\alpha}$. If a perturbation α is conservative, then the osculating rate functions $F_{i\alpha}$ can be expressed as (2.2-6):

$$F_{i\alpha} = -\sum_{j=1}^{6}(a_i, a_j)\frac{\partial \mathcal{R}}{\partial a_j} \tag{1}$$

where $(a_1 \ldots a_6)$ are the equinoctial elements (a, h, k, p, q, λ), the quantities (a_i, a_j) are the Poisson brackets given by (2.1.8-2,3), and $\mathcal{R}$ is the osculating disturbing function.

If the perturbation α is single-averaged, we can define the mean disturbing function U:

$$U = <\mathcal{R}> = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{R}(a, h, k, p, q, \lambda, t)d\lambda \tag{2}$$

Averaging both sides of (1), we obtain the first-order mean element rates due to this perturbation:

$$A_{i\alpha} = -\sum_{j=1}^{5}(a_i, a_j)\frac{\partial U}{\partial a_j} \tag{3}$$

The first-order short-periodic variations caused by this perturbation can be obtained from a potential-like function S called the short-periodic generating function:

$$S = \int(\mathcal{R} - U)d\lambda \tag{4}$$

$$<S> = 0 \tag{5}$$

Using (2.3-26), (2.4-10), and (1)–(4), we can obtain the first-order short-periodic kernels $\xi_{i\alpha}$ by performing the integrals (2.5.1-2, 3) (with the subscript i replaced by $i\alpha$):

$$\xi_{i\alpha} = -\frac{1}{n}\sum_{j=1}^{6}(a_i, a_j)\frac{\partial S}{\partial a_j} \tag{6}$$

From (5) it is clear that:

$$\langle\xi_{i\alpha}\rangle = 0 \tag{7}$$

Combining (2.1.8-2,3) and (6)–(7), we obtain:

$$\xi_{1\alpha} = \frac{2}{n^2a}\frac{\partial S}{\partial \lambda} \tag{8}$$

$$\int\xi_{1\alpha}d\lambda = \frac{2}{n^2a}S \tag{9}$$

Substituting (6) and (9) into (2.5.1-5), we obtain the following expression for the first-order short-periodic variations $\eta_{i\alpha}$ in the absence of explicit time-dependence:

$$\eta_{i\alpha} = -\frac{1}{n}\left[\sum_{j=1}^{6}(a_i, a_j)\frac{\partial S}{\partial a_j} + \frac{3}{na^2}\delta_{i6}S\right] \tag{10}$$

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In the presence of explicit time-dependence, the first-order short-periodic variations $\eta_{i\alpha}$ are given by:

$$
\begin{aligned}
\eta_{i\alpha} \;=\; & -\frac{1}{n}\left\{ \sum_{j=1}^{6}(a_i,a_j)\frac{\partial S}{\partial a_j} \right. \\
& + \sum_{k=1}^{K}\frac{(-1)^k}{n^k}\left[\sum_{j=1}^{5}(a_i,a_j)\frac{\partial}{\partial a_j}\int_k \frac{\partial^k S}{\partial t^k}d\lambda^k + (a_i,\lambda)\int_{k-1}\frac{\partial^k S}{\partial t^k}d\lambda^{k-1}\right] \\
& \left. + \frac{3}{na^2}\delta_{i6}\left[S + \sum_{k=1}^{K}(k+1)\frac{(-1)^k}{n^k}\int_k \frac{\partial^k S}{\partial t^k}d\lambda^k\right]\right\}
\end{aligned}
\tag{11}
$$

2.5.6 Second-Order $\eta_{i\alpha\beta}$ for Two Perturbations Expanded in λ

In this section we suppose that the osculating rate functions $F_{i\alpha}$ and the first-order short-periodic variations $\eta_{i\alpha}$ can be written as Fourier series in the mean longitude λ:

$$
\begin{aligned}
F_{i\alpha}(a,h,k,p,q,\lambda) \;=\; & \mathcal{C}^0_{i\alpha}(a,h,k,p,q,t) \\
& + \sum_{j=1}^{\infty}[\mathcal{C}^j_{i\alpha}(a,h,k,p,q,t)\cos j\lambda + \mathcal{S}^j_{i\alpha}(a,h,k,p,q,t)\sin j\lambda]
\end{aligned}
\tag{1}
$$

$$
\eta_{i\alpha} \;=\; \sum_{j=1}^{\infty}(C^j_{i\alpha}\cos j\lambda + S^j_{i\alpha}\sin j\lambda)
\tag{2}
$$

From (2.3-27), the second-order functions $G_{i\alpha\beta}$ are

$$
G_{i\alpha\beta} = \sum_{r=1}^{6}\frac{\partial F_{i\alpha}}{\partial a_r}\eta_{r\beta} + \frac{15n}{8a^2}\delta_{i6}\eta_{1\alpha}\eta_{1\beta} - \sum_{r=1}^{6}\frac{\partial\eta_{i\alpha}}{\partial a_r}A_{r\beta}
\tag{3}
$$

Substituting (1)–(2) into (3), and using (2.5.3-17) with $J = \infty$, we can write $G_{i\alpha\beta}$ as a Fourier series in λ:

$$
G_{i\alpha\beta} = \mathcal{C}^0_{i\alpha\beta} + \sum_{j=1}^{\infty}(\mathcal{C}^j_{i\alpha\beta}\cos j\lambda + \mathcal{S}^j_{i\alpha\beta}\sin j\lambda)
\tag{4}
$$

where

$$
\mathcal{C}^0_{i\alpha\beta} = \frac{1}{2}\sum_{j=1}^{\infty}\left[\sum_{r=1}^{5}\left(\frac{\partial\mathcal{C}^j_{i\alpha}}{\partial a_r}C^j_{r\beta} + \frac{\partial\mathcal{S}^j_{i\alpha}}{\partial a_r}S^j_{r\beta}\right) + j\mathcal{S}^j_{i\alpha}C^j_{6\beta} - j\mathcal{C}^j_{i\alpha}S^j_{6\beta} + \frac{15n}{8a^2}\delta_{i6}(C^j_{1\alpha}C^j_{1\beta} + S^j_{1\alpha}S^j_{1\beta})\right]
$$

$$
\mathcal{C}^j_{i\alpha\beta} = \sum_{r=1}^{5}\left[\frac{\partial\mathcal{C}^o_{i\alpha}}{\partial a_r}C^j_{r\beta} - \mathcal{C}^0_{r\beta}\frac{\partial C^j_{i\alpha}}{\partial a_r} + \frac{1-\delta_{j1}}{2}\sum_{k=1}^{j-1}\left(\frac{\partial\mathcal{C}^{j-k}_{i\alpha}}{\partial a_r}C^k_{r\beta} - \frac{\partial\mathcal{S}^{j-k}_{i\alpha}}{\partial a_r}S^k_{r\beta}\right)\right.
$$

$$
+ \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{\partial\mathcal{C}^{j+k}_{i\alpha}}{\partial a_r}C^k_{r\beta} + \frac{\partial\mathcal{C}^k_{i\alpha}}{\partial a_r}C^{j+k}_{r\beta} + \frac{\partial\mathcal{S}^{j+k}_{i\alpha}}{\partial a_r}S^r_{k\beta} + \frac{\partial\mathcal{S}^k_{i\alpha}}{\partial a_r}S^{j+k}_{k\beta}\right)\Bigg]
$$

$$
- j\mathcal{C}^o_{6\beta}S^j_{i\alpha} + \frac{1-\delta_{j1}}{2}\sum_{k=1}^{j-1}\left[(j-k)(\mathcal{S}^{j-k}_{i\alpha}C^k_{6\beta} + \mathcal{C}^{j-k}_{i\alpha}S^k_{6\beta}) + \frac{15n}{8a^2}\delta_{i6}(C^{j-k}_{1\alpha}C^k_{1\beta} - S^{j-k}_{1\alpha}S^k_{1\beta})\right]
$$

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$$+\frac{1}{2}\sum_{k=1}^{\infty}\left[(j+k)(\mathcal{S}_{i\alpha}^{j+k}C_{6\beta}^{k}-\mathcal{C}_{i\alpha}^{j+k}S_{6\beta}^{k})+k(\mathcal{S}_{i\alpha}^{k}C_{6\beta}^{j+k}-\mathcal{C}_{i\alpha}^{k}S_{6\beta}^{j+k})\right.$$

$$\left.+\frac{15n}{8a^2}\delta_{i6}(C_{1\alpha}^{j+k}C_{1\beta}^{k}+C_{1\alpha}^{k}C_{1\beta}^{j+k}+S_{1\alpha}^{j+k}S_{1\beta}^{k}+S_{1\alpha}^{k}S_{1\beta}^{j+k})\right] \tag{5}$$

$$\mathcal{S}_{i\alpha\beta}^{j}=\sum_{r=1}^{5}\left[\frac{\partial\mathcal{C}_{i\alpha}^{o}}{\partial a_r}S_{r\beta}^{j}-\mathcal{C}_{r\beta}^{0}\frac{\partial S_{i\alpha}^{j}}{\partial a_r}+\frac{1-\delta_{j1}}{2}\sum_{k=1}^{j-1}\left(\frac{\partial\mathcal{C}_{i\alpha}^{j-k}}{\partial a_r}S_{r\beta}^{k}+\frac{\partial\mathcal{S}_{i\alpha}^{j-k}}{\partial a_r}C_{r\beta}^{k}\right)\right.$$

$$\left.+\frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{\partial\mathcal{C}_{i\alpha}^{j+k}}{\partial a_r}S_{r\beta}^{k}+\frac{\partial\mathcal{C}_{i\alpha}^{k}}{\partial a_r}S_{r\beta}^{j+k}+\frac{\partial\mathcal{S}_{i\alpha}^{j+k}}{\partial a_r}C_{r\beta}^{k}-\frac{\partial\mathcal{S}_{i\alpha}^{k}}{\partial a_r}C_{r\beta}^{j+k}\right)\right]$$

$$+j\mathcal{C}_{6\beta}^{o}C_{i\alpha}^{j}+\frac{1-\delta_{j1}}{2}\sum_{k=1}^{j-1}\left[(j-k)(\mathcal{S}_{i\alpha}^{j-k}S_{6\beta}^{k}-\mathcal{C}_{i\alpha}^{j-k}C_{6\beta}^{k})+\frac{15n}{8a^2}\delta_{i6}(C_{1\alpha}^{j-k}S_{1\beta}^{k}+S_{1\alpha}^{j-k}C_{1\beta}^{k})\right]$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}\left[-(j+k)(\mathcal{S}_{i\alpha}^{j+k}S_{6\beta}^{k}+\mathcal{C}_{i\alpha}^{j+k}C_{6\beta}^{k})+k(\mathcal{S}_{i\alpha}^{k}S_{6\beta}^{j+k}+\mathcal{C}_{i\alpha}^{k}C_{6\beta}^{j+k})\right.$$

$$\left.+\frac{15n}{8a^2}\delta_{i6}(-C_{1\alpha}^{j+k}S_{1\beta}^{k}+C_{1\alpha}^{k}S_{1\beta}^{j+k}+S_{1\alpha}^{j+k}C_{1\beta}^{k}-S_{1\alpha}^{k}C_{1\beta}^{j+k})\right]$$

Then the second-order mean element rates are

$$A_{i\alpha\beta}=\mathcal{C}_{i\alpha\beta}^{0} \tag{6}$$

and the second-order short-periodic variations are

$$\eta_{i\alpha\beta}=\sum_{j=1}^{\infty}(C_{i\alpha\beta}^{j}\cos j\lambda+S_{i\alpha\beta}^{j}\sin j\lambda) \tag{7}$$

where the $C_{i\alpha\beta}^{j}$ and $S_{i\alpha\beta}^{j}$ are given by (2.5.1-15) (with the subscript i replaced by $i\alpha\beta$) in terms of $\mathcal{C}_{i\alpha\beta}^{j}$ and $\mathcal{S}_{i\alpha\beta}^{j}$. The formulas in this section were published in [Danielson, March 1993].

2.5.7 Second-Order $\eta_{i\alpha\beta}$ for Two Perturbations Expanded in L

In this section we suppose that the osculating rate functions $F_{i\alpha}$ and the first-order short-periodic variations $\eta_{i\alpha}$ can be written as finite modified Fourier series in the true longitude L:

$$\begin{aligned}F_{i\alpha}(a,h,k,p,q,L,t) &= \mathcal{C}_{i\alpha}^{0}(a,h,k,p,q,t)\\ &+\sum_{j=1}^{J^{\alpha}}[\mathcal{C}_{i\alpha}^{j}(a,h,k,p,q,t)\cos jL+\mathcal{S}_{i\alpha}^{j}(a,h,k,p,q,t)\sin jL]\end{aligned} \tag{1}$$

$$
\begin{aligned}
\eta_{i\alpha}(a,h,k,p,q,L,t) \;=\;& C^0_{i\alpha}(a,h,k,p,q,t) + \sum_{m=1}^{M^\alpha} D^m_{i\alpha}(a,h,k,p,q,t)(L-\lambda)^m \\
&+ \sum_{k=1}^{K^\alpha}[C^k_{i\alpha}(a,h,k,p,q,t)\cos kL + S^k_{i\alpha}(a,h,k,p,q,t)\sin kL]
\end{aligned}
\tag{2}
$$

The second-order functions $G_{i\alpha\beta}$ are again given by (2.5.6-3). Differentiating (1)–(2), we can obtain the needed partials

$$
\frac{\partial F_{i\alpha}}{\partial a_r} \;=\; \mathcal{C}^{or}_{i\alpha} + \sum_{j=1}^{J^{\alpha r}}(\mathcal{C}^{jr}_{i\alpha}\cos jL + \mathcal{S}^{jr}_{i\alpha}\sin jL)
\tag{3}
$$

$$
\frac{\partial \eta_{i\alpha}}{\partial a_r} \;=\; C^{or}_{i\alpha} + \sum_{m=1}^{M^{\alpha r}} D^{mr}_{i\alpha}(L-\lambda)^m + \sum_{k=1}^{K^{\alpha r}}(C^{kr}_{i\alpha}\cos kL + S^{kr}_{i\alpha}\sin kL)
\tag{4}
$$

The product of two Fourier series can be converted into a single Fourier series with the formula

$$
\begin{aligned}
&\sum_{j=1}^{J}(\mathcal{C}^j\cos jL + \mathcal{S}^j\sin jL)\sum_{k=1}^{K}(C^k\cos kL + S^k\sin kL) \\
&= \frac{1}{2}\sum_{j=1}^{J+K}\Big\{\mathcal{I}_1^{\min(J,K)}(j)(\mathcal{C}^j C^j + \mathcal{S}^j S^j) \\
&\quad + [\mathcal{I}_2^{J+K}(j)\sum_{k=\max(j-J,1)}^{\min(j-1,K)}(\mathcal{C}^{j-k}C^k - \mathcal{S}^{j-k}S^k) + \mathcal{I}_1^{J-1}(j)\sum_{k=1}^{\min(J-j,K)}(\mathcal{C}^{j+k}C^k + \mathcal{S}^{j+k}S^k) \\
&\quad + \mathcal{I}_1^{K-1}(j)\sum_{k=1}^{\min(K-j,J)}(C^k\mathcal{C}^{j+k} + S^k\mathcal{S}^{j+k})]\cos jL \\
&\quad + [\mathcal{I}_2^{J+K}(j)\sum_{k=\max(j-J,1)}^{\min(j-1,K)}(\mathcal{C}^{j-k}S^k + \mathcal{S}^{j-k}C^k) + \mathcal{I}_1^{J-1}(j)\sum_{k=1}^{\min(J-j,K)}(-\mathcal{C}^{j+k}S^k + \mathcal{S}^{j+k}C^k) \\
&\quad + \mathcal{I}_1^{K-1}(j)\sum_{k=1}^{\min(K-j,J)}(C^k\mathcal{S}^{j+k} - S^k\mathcal{C}^{j+k})]\sin jL\Big\}
\end{aligned}
\tag{5}
$$

With the use of (2.5.3-7), we then obtain

$$
G_{i\alpha\beta} = \mathcal{C}^0_{i\alpha\beta} + \sum_{m=1}^{M^{\alpha r}+M^\beta}\mathcal{D}^m_{i\alpha\beta}(L-\lambda)^m + \sum_{j=1}^{\infty}(\mathcal{C}^j_{i\alpha\beta}\cos jL + \mathcal{S}^j_{i\alpha\beta}\sin jL)
\tag{6}
$$

where

$$
\begin{aligned}
\mathcal{C}^0_{i\alpha\beta} = \sum_{r=1}^{6} & \left[\mathcal{C}^{or}_{i\alpha} C^0_{r\beta} + \frac{1}{2} \sum_{j=1}^{J^{\alpha r}} \sum_{m=1}^{M^\beta} (\mathcal{C}^{jr}_{i\alpha} \kappa^j_m + \mathcal{S}^{jr}_{i\alpha} \psi^j_m) D^m_{r\beta} \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{\min(J^{\alpha r}, K^\beta)} (\mathcal{C}^{jr}_{i\alpha} C^j_{r\beta} + \mathcal{S}^{jr}_{i\alpha} S^j_{r\beta}) - A_{r\beta} \mathcal{C}^{or}_{i\alpha} \right] \\
+ \frac{15n}{8a^2} \delta_{i6} & \left[C^0_{1\alpha} C^0_{1\beta} + \frac{1}{2} \sum_{j=1}^{J^\alpha} \sum_{m=1}^{M^\beta} (C^j_{1\alpha} \kappa^j_m + S^j_{1\alpha} \psi^j_m) D^m_{1\beta} + \frac{1}{2} \sum_{j=1}^{J^\beta} \sum_{m=1}^{M^\alpha} (C^j_{1\beta} \kappa^j_m + S^j_{1\beta} \psi^j_m) D^m_{1\alpha} \right. \\
& \left. + \frac{1}{2} \sum_{j=1}^{\min(J^\alpha, K^\beta)} (C^j_{1\alpha} C^j_{1\beta} + S^j_{1\alpha} S^j_{1\beta}) \right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}^j_{i\alpha\beta} = \sum_{r=1}^{6} & \Big\{ \mathcal{I}^{K^\beta}_1(j) \mathcal{C}^{or}_{i\alpha} C^j_{r\beta} + \mathcal{I}^{J^{\alpha r}}_1(j) \mathcal{C}^{jr}_{i\alpha} C^0_{r\beta} \\
& + \frac{1}{2} \mathcal{I}^{J^{\alpha r} + K^\beta}_2(j) \sum_{k=\max(j-J^{\alpha r}, 1)}^{\min(j-1, K^\beta)} (\mathcal{C}^{j-k,r}_{i\alpha} C^k_{r\beta} - \mathcal{S}^{j-k,r}_{i\alpha} S^k_{r\beta}) \\
& + \frac{1}{2} \mathcal{I}^{J^{\alpha r}-1}_1(j) \sum_{k=1}^{\min(J^{\alpha r}-j, K^\beta)} (\mathcal{C}^{j+k,r}_{i\alpha} C^k_{r\beta} + \mathcal{S}^{j+k,r}_{i\alpha} S^k_{r\beta}) \\
& + \frac{1}{2} \mathcal{I}^{K^\beta-1}_1(j) \sum_{k=1}^{\min(K^\beta-j, J^{\alpha r})} (\mathcal{C}^{kr}_{i\alpha} C^{j+k}_{r\beta} + \mathcal{S}^{kr}_{i\alpha} S^{j+k}_{r\beta}) \\
& - \mathcal{I}^{K^{\alpha r}}_1(j) A_{r\beta} \mathcal{C}^{jr}_{i\alpha} + \sum_{m=1}^{M^\beta} D^m_{r\beta} [\mathcal{I}^{J^{\alpha r}}_1(j) \mathcal{C}^{jr}_{i\alpha} \kappa^0_m \\
& + \frac{1}{2} (1 - \delta_{j1}) \sum_{k=\max(j-J^{\alpha r}, 1)}^{j-1} (\mathcal{C}^{j-k,r}_{i\alpha} \kappa^k_m - \mathcal{S}^{j-k,r}_{i\alpha} \psi^k_m) \\
& + \frac{1}{2} \mathcal{I}^{J^{\alpha r}-1}_1(j) \sum_{k=1}^{J^{\alpha r}-j} (\mathcal{C}^{j+k,r}_{i\alpha} \kappa^k_m + \mathcal{S}^{j+k,r}_{i\alpha} \psi^k_m) + \frac{1}{2} \sum_{k=1}^{J^{\alpha r}} (\mathcal{C}^{kr}_{i\alpha} \kappa^{j+k}_m + \mathcal{S}^{kr}_{i\alpha} \psi^{j+k}_m)] \Big\}
\end{aligned}
$$

$$+\frac{15n}{8a^2}\delta_{i6}\Big\{\mathcal{I}_1^{K^\beta}(j)C_{1\alpha}^0C_{1\beta}^j+\mathcal{I}_1^{K^\alpha}(j)C_{1\beta}^0C_{1\alpha}^j$$

$$+\frac{1}{2}\mathcal{I}_2^{K^\alpha+K^\beta}(j)\sum_{k=\max(j-K^\alpha,1)}^{\min(j-1,K^\beta)}(C_{1\alpha}^{j-k}C_{1\beta}^k-S_{1\alpha}^{j-k}S_{1\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\alpha-1}(j)\sum_{k=1}^{\min(K^\alpha-j,K^\beta)}(C_{1\alpha}^{j+k}C_{1\beta}^k+S_{1\alpha}^{j+k}S_{1\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\beta-1}(j)\sum_{k=1}^{\min(K^\beta-j,K^\alpha)}(C_{1\alpha}^kC_{1\beta}^{j+k}+S_{1\alpha}^kS_{1\beta}^{j+k})$$

$$+\sum_{m=1}^{M^\beta}D_{1\beta}^m[\mathcal{I}_1^{K^\alpha}(j)C_{1\alpha}^j\kappa_m^0+\frac{1}{2}(1-\delta_{j1})\sum_{k=\max(j-K^\alpha,1)}^{j-1}(C_{1\alpha}^{j-k}\kappa_m^k-S_{1\alpha}^{j-k}\psi_m^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\alpha-1}(j)\sum_{k=1}^{K^\alpha-j}(C_{1\alpha}^{j+k}\kappa_m^k+S_{1\alpha}^{j+k}\psi_m^k)+\frac{1}{2}\sum_{k=1}^{K^\alpha}(C_{1\alpha}^k\kappa_m^{j+k}+S_{1\alpha}^k\psi_m^{j+k})]$$

$$+\sum_{m=1}^{M^\alpha}D_{1\alpha}^m[\mathcal{I}_1^{K^\beta}(j)C_{1\beta}^j\kappa_m^0+\frac{1}{2}(1-\delta_{j1})\sum_{k=\max(j-K^\beta,1)}^{j-1}(C_{1\beta}^{j-k}\kappa_m^k-S_{1\beta}^{j-k}\psi_m^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\beta-1}(j)\sum_{k=1}^{K^\beta-j}(C_{1\beta}^{j+k}\kappa_m^k+S_{1\beta}^{j+k}\psi_m^k)+\frac{1}{2}\sum_{k=1}^{K^\beta}(C_{1\beta}^k\kappa_m^{j+k}+S_{1\beta}^k\psi_m^{j+k})]\Big\}$$

$$\tag{7}$$

$$\mathcal{S}_{i\alpha\beta}^j=\sum_{r=1}^{6}\Big\{\mathcal{I}_1^{K^\beta}(j)\mathcal{C}_{i\alpha}^{or}S_{r\beta}^j+\mathcal{I}_1^{J^{\alpha r}}(j)\mathcal{S}_{i\alpha}^{jr}C_{r\beta}^0$$

$$+\frac{1}{2}\mathcal{I}_2^{J^{\alpha r}+K^\beta}(j)\sum_{k=\max(j-J^{\alpha r},1)}^{\min(j-1,K^\beta)}(\mathcal{C}_{i\alpha}^{j-k,r}S_{r\beta}^k+\mathcal{S}_{i\alpha}^{j-k,r}C_{r\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{J^{\alpha r}-1}(j)\sum_{k=1}^{\min(J^{\alpha r}-j,K^\beta)}(-\mathcal{C}_{i\alpha}^{j+k,r}S_{r\beta}^k+\mathcal{S}_{i\alpha}^{j+k,r}C_{r\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\beta-1}(j)\sum_{k=1}^{\min(K^\beta-j,J^{\alpha r})}(\mathcal{C}_{i\alpha}^{kr}S_{r\beta}^{j+k}-\mathcal{S}_{i\alpha}^{kr}C_{r\beta}^{j+k})$$

$$-\mathcal{I}_1^{K^{\alpha r}}(j)A_{r\beta}S_{i\alpha}^{jr}+\sum_{m=1}^{M^\beta}D_{r\beta}^m[\mathcal{I}_1^{J^{\alpha r}}(j)\mathcal{S}_{i\alpha}^{jr}\kappa_m^0$$

$$+\frac{1}{2}(1-\delta_{j1})\sum_{k=\max(j-J^{\alpha r},1)}^{j-1}(\mathcal{C}_{i\alpha}^{j-k,r}\psi_m^k+\mathcal{S}_{i\alpha}^{j-k,r}\kappa_m^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{J^{\alpha r}-1}(j)\sum_{k=1}^{J^{\alpha r}-j}(-\mathcal{C}_{i\alpha}^{j+k,r}\psi_m^k+\mathcal{S}_{i\alpha}^{j+k,r}\kappa_m^k)+\frac{1}{2}\sum_{k=1}^{J^{\alpha r}}(\mathcal{C}_{i\alpha}^{kr}\psi_m^{j+k}-\mathcal{S}_{i\alpha}^{kr}\kappa_m^{j+k})]\Big\}$$

$$+\frac{15n}{8a^2}\delta_{i6}\Big\{\mathcal{I}_1^{K^\beta}(j)C_{1\alpha}^0 S_{1\beta}^j + \mathcal{I}_1^{K^\alpha}(j)C_{1\beta}^0 S_{1\alpha}^j$$

$$+\frac{1}{2}\mathcal{I}_2^{K^\alpha+K^\beta}(j)\sum_{k=\max(j-K^\alpha,1)}^{\min(j-1,K^\beta)}(C_{1\alpha}^{j-k}S_{1\beta}^k + S_{1\alpha}^{j-k}C_{1\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\alpha-1}(j)\sum_{k=1}^{\min(K^\alpha-j,K^\beta)}(-C_{1\alpha}^{j+k}S_{1\beta}^k + S_{1\alpha}^{j+k}C_{1\beta}^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\beta-1}(j)\sum_{k=1}^{\min(K^\beta-j,K^\alpha)}(C_{1\alpha}^k S_{1\beta}^{j+k} - S_{1\alpha}^k C_{1\beta}^{j+k})$$

$$+\sum_{m=1}^{M^\beta}D_{1\beta}^m[\mathcal{I}_1^{K^\alpha}(j)S_{1\alpha}^j\kappa_m^0 + \frac{1}{2}(1-\delta_{j1})\sum_{k=\max(j-K^\alpha,1)}^{j-1}(C_{1\alpha}^{j-k}\psi_m^k + S_{1\alpha}^{j-k}\kappa_m^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\alpha-1}(j)\sum_{k=1}^{K^\alpha-j}(-C_{1\alpha}^{j+k}\psi_m^k + S_{1\alpha}^{j+k}\kappa_m^k) + \frac{1}{2}\sum_{k=1}^{K^\alpha}(C_{1\alpha}^k\psi_m^{j+k} - S_{1\alpha}^k\kappa_m^{j+k})]$$

$$+\sum_{m=1}^{M^\alpha}D_{1\alpha}^m[\mathcal{I}_1^{K^\beta}(j)S_{1\beta}^j\kappa_m^0 + \frac{1}{2}(1-\delta_{j1})\sum_{k=\max(j-K^\beta,1)}^{j-1}(C_{1\beta}^{j-k}\psi_m^k + S_{1\beta}^{j-k}\kappa_m^k)$$

$$+\frac{1}{2}\mathcal{I}_1^{K^\beta-1}(j)\sum_{k=1}^{K^\beta-j}(-C_{1\beta}^{j+k}\psi_m^k + S_{1\beta}^{j+k}\kappa_m^k) + \frac{1}{2}\sum_{k=1}^{K^\beta}(C_{1\beta}^k\psi_m^{j+k} - S_{1\beta}^k\kappa_m^{j+k})]\Big\}$$

$$\mathcal{D}_{i\alpha\beta}^m = \sum_{r=1}^{6}[\mathcal{I}_1^{M^\beta}(m)\mathcal{C}_{i\alpha}^{or}D_{r\beta}^m - \mathcal{I}_1^{M^{\alpha r}}(m)A_{r\beta}D_{i\alpha}^{mr}]$$

$$+\frac{15n}{8a^2}\delta_{i6}\Big[\mathcal{I}_1^{M^\beta}(m)C_{1\alpha}^0 D_{1\beta}^m + \mathcal{I}_1^{M^\alpha}(m)C_{1\beta}^0 D_{1\alpha}^m + \mathcal{I}_2^{M^\alpha+M^\beta}(m)\sum_{j=1}^{m-1}D_{1\alpha}^{m-j}D_{1\beta}^j\Big]$$

Then the second-order mean element rates are

$$A_{i\alpha\beta} = \mathcal{C}_{i\alpha\beta}^0 + \sum_{m=1}^{M^{\alpha r}+M^\beta}\mathcal{D}_{i\alpha\beta}^m\kappa_m^0 + \sum_{j=1}^{\infty}(\mathcal{C}_{i\alpha\beta}^j\rho_j + \mathcal{S}_{i\alpha\beta}^j\sigma_j) \tag{8}$$

and the second-order short-periodic variations are

$$\eta_{i\alpha\beta} = C_{i\alpha\beta}^0 + \sum_{m=1}^{M^{\alpha r}+M^\beta+K+2}D_{i\alpha\beta}^m(L-\lambda)^m + \sum_{j=1}^{\infty}(C_{i\alpha\beta}^j\cos jL + S_{i\alpha\beta}^j\sin jL) \tag{9}$$

where the $C_{i\alpha\beta}^j$, $S_{i\alpha\beta}^j$, $D_{i\alpha\beta}^m$ are given by (2.5.3-24) (with the subscript i replaced by $i\alpha\beta$ and M replaced by $M^{\alpha r}+M^\beta$) in terms of $\mathcal{C}_{i\alpha\beta}^j$ and $\mathcal{S}_{i\alpha\beta}^j$. The formulas in this section were published in [Danielson, August 1993].

2.6 Partial Derivatives for State Estimation

Observational data may be used to improve the estimate of a satellite's state. Some differential correction algorithms which have been used in conjunction with SST are described in

[Green, 1979], [Taylor, 1982] and [Long, Capellari, Velez, and Fuchs, 1989]. It is our purpose here only to explain how to obtain the partial derivatives needed in such filters.

We let O_k denote the value of the kth observed quantity computed with the SST orbital generator. The SST state variables are the initial mean elements $a_i(t_0)$ and various constant parameters c_i (the geopotential coefficients C_{nm} and S_{nm} in (2.7-1), the drag coefficient C_D in (3.4-3), the solar radiation pressure coefficient C_R in (3.5-6), etc.). Required for a batch filter are the partial derivatives of the O_k with respect to the state variables $a_i(t_0)$ and c_i.

The actual observations are commonly of position and velocity components in a local coordinate frame fixed on the surface of the Earth. However, through transformations these components may be expressed in terms of the orbital elements, so we can regard the O_k as an implicit function of the osculating elements $\hat{a}_j$. Application of the chain rule then produces

$$\frac{\partial O_k}{\partial a_i(t_0)} = \sum_{j=1}^{6} \frac{\partial O_k}{\partial \hat{a}_j} \frac{\partial \hat{a}_j}{\partial a_i(t_0)} \tag{1}$$

$$\frac{\partial O_k}{\partial c_i} = \sum_{j=1}^{6} \frac{\partial O_k}{\partial \hat{a}_j} \frac{\partial \hat{a}_j}{\partial c_i} \tag{2}$$

Assuming we can obtain the partials $\frac{\partial O_k}{\partial \hat{a}_j}$ analytically, our remaining task is to calculate the partials $\frac{\partial \hat{a}_j}{\partial a_i(t_0)}$ and $\frac{\partial \hat{a}_j}{\partial c_i}$. Differentiating the decomposition (1-1) yields

$$\frac{\partial \hat{a}_j}{\partial a_i(t_0)} = \sum_{k=1}^{6} \left(\delta_{jk} + \frac{\partial \eta_j}{\partial a_k} \right) \frac{\partial a_k}{\partial a_i(t_0)} \tag{3}$$

$$\frac{\partial \hat{a}_j}{\partial c_i} = \sum_{k=1}^{6} \left(\delta_{jk} + \frac{\partial \eta_j}{\partial a_k} \right) \frac{\partial a_k}{\partial c_i} + \frac{\partial \eta_j}{\partial c_i} \tag{4}$$

The partials $\frac{\partial a_k}{\partial a_i(t_0)}$ and $\frac{\partial a_k}{\partial c_i}$ are often arranged to form a matrix $\mathbf{\Phi}$, referred to as the state transition matrix:

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} \frac{\partial a_1}{\partial a_1(t_0)} & \cdots & \frac{\partial a_1}{\partial a_6(t_0)} & \frac{\partial a_1}{\partial c_1} & \cdots & \cdots & \frac{\partial a_1}{\partial c_\ell} \\ \vdots & & \vdots & \vdots & & & \vdots \\ \frac{\partial a_6}{\partial a_1(t_0)} & \cdots & \frac{\partial a_6}{\partial a_6(t_0)} & \frac{\partial a_6}{\partial c_1} & \cdots & \cdots & \frac{\partial a_6}{\partial c_\ell} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & 1 & & \vdots \\ \vdots & & \vdots & \vdots & & 1 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{5}$$

Here ℓ is the number of parameters c_i. Differentiating (5) with respect to t and interchanging the ordinary and partial derivatives, we can obtain the following initial value problem for $\mathbf{\Phi}(t, t_0)$:

$$\dot{\mathbf{\Phi}}(t, t_0) = \mathbf{F}\mathbf{\Phi}(t, t_0), \qquad \mathbf{\Phi}(t_0, t_0) = \mathbf{I} \tag{6}$$

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Here **I** is the identity matrix and

$$
\mathbf{F} =
\begin{bmatrix}
\dfrac{\partial \dot{a}_1}{\partial a_1} & \cdots & \dfrac{\partial \dot{a}_1}{\partial a_5} & 0 & \dfrac{\partial \dot{a}_1}{\partial c_1} & \cdots & \cdots & \dfrac{\partial \dot{a}_1}{\partial c_\ell} \\
\vdots & & \vdots & \vdots & \vdots & & & \vdots \\
\dfrac{\partial \dot{a}_6}{\partial a_1} & \cdots & \dfrac{\partial \dot{a}_6}{\partial a_5} & 0 & \dfrac{\partial \dot{a}_6}{\partial c_1} & \cdots & \cdots & \dfrac{\partial \dot{a}_6}{\partial c_\ell} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & & & \vdots \\
\vdots & & & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\tag{7}
$$

The $\mathbf{\Phi}$ matrix is a function only of the five slowly varying mean elements, and therefore the numerical integration of (6) can be done with the same large step size as used in the integration of equations (1)-(2) for the mean element rates. Values of Φ at observation times not coinciding with the integrator step times can be obtained by interpolation.

Our task has thus been reduced to the calculation of the partial derivatives $\frac{\partial \dot{a}_j}{\partial a_i}, \frac{\partial \dot{a}_j}{\partial c_i}, \frac{\partial \eta_j}{\partial a_i}$, and $\frac{\partial \eta_j}{\partial c_i}$. These same partials are also needed in a sequential Kalman filter. For the two-body part of the mean element rates

$$
\dot{a}_i = n\delta_{i6}
\tag{8}
$$

the only nonzero partial in (7) is

$$
\frac{\partial \dot{a}_6}{\partial a_1} = -\frac{3n}{2a}
\tag{9}
$$

and thus

$$
\mathbf{\Phi} =
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & & & & & & \vdots \\
0 & & & & & & & \vdots \\
0 & & & & & & & \vdots \\
0 & & & & & & & \vdots \\
-\dfrac{3n(t - t_0)}{2a} & & & & & & & \vdots \\
0 & & & & & & & \vdots \\
\vdots & & & & & & & \vdots \\
\vdots & & & & & & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\tag{10}
$$

Although the differential correction algorithm for updating the initial mean elements may converge with only the two-body partials (10), the speed of convergence can be improved by including the dominant perturbation partials. Analytical formulas have been obtained for the partial derivatives with respect to the mean elements a_i of the J_2 contribution to the mean element rates $\dot{a}_j$ (equations (3.1-12)), the partial derivatives with respect to the geopotential coefficients C_{nm} and S_{nm} of the resonant tesseral contribution to the mean

element rates $\dot{a}_j$ (equations (3.2-9)), and the partial derivatives with respect to the mean elements a_i of the J_2 contribution to the short periodic variations η_j (from the expressions in Section 4.1).

2.7   Central-Body Gravitational Potential

The well-known expression for the disturbing function due to the gravitational field of the central body is [Battin, 1987]:

$$\mathcal{R}(r,\phi,\psi) = \frac{\mu}{r} \sum_{n=2}^{N} \sum_{m=0}^{\min(n,M)} \left(\frac{R}{r}\right)^n P_{nm}(\sin\phi)(C_{nm}\cos m\psi + S_{nm}\sin m\psi) \tag{1}$$

Here

$$
\begin{aligned}
r &= \text{radial distance from center of mass of central body} \\
\phi &= \text{geocentric latitude} \\
\psi &= \text{geographic longitude} \\
\mu &= \text{central-body gravitational constant} \\
R &= \text{central-body mean equatorial radius} \\
P_{nm} &= \text{associated Legendre function of order } m \text{ and degree } n \\
C_{nm},\, S_{nm} &= \text{geopotential constant coefficients} \\
M &= \text{maximum order of geopotential field } (M \leq N) \\
N &= \text{maximum degree of geopotential field}
\end{aligned}
$$

In this section all elements are *osculating* (even though they do not have hats).

In the first subsection we shall outline the development of the central-body gravitational potential into the form used in SST. Complete details are to be found in [Cefola, 1976], [McClain, 1978], [Proulx, McClain, Early, and Cefola, 1981], and [Proulx, 1982]. Then in the remaining subsections we describe methods for calculating the various functions used in the expansion.

2.7.1   Expansion of the Geopotential in Equinoctial Variables

We start by writing (2.7-1) in the complex form

$$\mathcal{R} = \text{Re}\left\{\frac{\mu}{r} \sum_{n=2}^{N} \sum_{m=0}^{\min(n,M)} \left(\frac{R}{r}\right)^n P_{nm}(\sin\phi)(C_{nm} - iS_{nm})\exp(im\psi)\right\} \tag{1}$$

Here $i = \sqrt{-1}$ and $\text{Re}\{z\}$ is the real part of z. With the goal of expressing (1) in terms of the equinoctial elements, we set

$$\psi = \alpha_B - \theta \tag{2}$$

where θ is the central body rotation angle and α_B is the right ascension. If we let $(\mathbf{x}_B, \mathbf{y}_B, \mathbf{z}_B)$ denote a right-handed orthonormal triad fixed in the central body, with $\mathbf{x}_B$ pointing to the prime meridian and $\mathbf{z}_B$ to the geographic north pole, then θ may be calculated from [Early, 1982]:

$$\sin\theta \;=\; \frac{-\mathbf{f}\cdot\mathbf{y}_B + I\mathbf{g}\cdot\mathbf{x}_B}{1 + I\gamma} \tag{3}$$

$$\cos\theta \;=\; \frac{\mathbf{f}\cdot\mathbf{x}_B + I\mathbf{g}\cdot\mathbf{y}_B}{1 + I\gamma} \tag{4}$$

(Remember that γ is defined by (2.1.9-1c).)

Next the spherical harmonics $P_{nm}(\sin\phi)\exp(im\alpha_B)$ are expanded as a Fourier series in the true longitude L, using a rotational transformation theorem for the spherical harmonics:

$$P_{nm}(\sin\phi)\exp(im\alpha_B) = \sum_{s=-n}^{n} V_{ns}^{m} S_n^{ms}\exp(isL) \tag{5}$$

The V_{ns}^{m} coefficients are defined by:

$$V_{ns}^{m} = \begin{cases} \dfrac{(-1)^{\frac{n-s}{2}}}{2^n}\dfrac{(n+s)!(n-s)!}{(n-m)!(\frac{n+s}{2})!(\frac{n-s}{2})!} & \text{if}\quad n-s\ \text{is even} \\[4mm] 0 & \text{if}\quad n-s\ \text{is odd} \end{cases} \tag{6}$$

The rotation functions $S_n^{ms}(\alpha,\beta,\gamma)$ may be expressed in terms of the dot products (α,β,γ) of the $\mathbf{z}_B$ vector with the equinoctial reference triad $(\mathbf{f},\mathbf{g},\mathbf{w})$:

$$S_n^{ms} = I^m \begin{cases} (-1)^{m-s}2^s(\alpha+i\beta)^{Im-s}(1+I\gamma)^{-Im}P_{n+s}^{m-s,-m-s}(\gamma) & \text{if}\quad s \le -m \\[3mm] (-1)^{m-s}2^{-m}\dfrac{(n+m)!(n-m)!}{(n+s)!(n-s)!}(\alpha+iI\beta)^{m-Is}(1+I\gamma)^{Is}P_{n-m}^{m-s,m+s}(\gamma) & \text{if}\quad |s| \le m \\[3mm] 2^{-s}(\alpha-i\beta)^{s-Im}(1+I\gamma)^{Im}P_{n-s}^{s-m,s+m}(\gamma) & \text{if}\quad s \ge m \end{cases} \tag{7}$$

(Remember that I is defined by (2.1.2-2).) Here $P_\ell^{vw}(\gamma)$ are Jacobi polynomials. (Note that commas are used to separate indices in a symbol such as $P_{n+s}^{m-s,-m-s}$ in order to prevent ambiguities.)

Next the product $(\frac{r}{a})^n\exp(isL)$ is expanded in a Fourier series in the *mean* longitude λ (the sixth equinoctial element a_6):

$$\left(\frac{r}{a}\right)^n \exp(isL) = \sum_{j=-\infty}^{\infty} Y_j^{ns}\exp(ij\lambda) \tag{8}$$

Now the Hansen coefficients X_j^{ns} are defined by [Hansen, 1855]

$$\left(\frac{r}{a}\right)^n \exp(isf) = \sum_{j=-\infty}^{\infty} X_j^{ns}\exp(ijM) \tag{9}$$

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and the kernel K_j^{ns} of the Hansen coefficient X_j^{ns} is defined by

$$K_j^{ns}(e) = e^{-|s-j|} X_j^{ns}(e) \tag{10}$$

where f is the true anomaly, M is the mean anomaly, and in (10) e is the orbital eccentricity. Hence, remembering equations (2.1.2–1, 4), we can express the functions Y_j^{ns} as

$$Y_j^{ns}(h,k) = [k + ih\,\mathrm{sgn}\,(s-j)]^{|s-j|} K_j^{ns} \tag{11}$$

Here

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{12}$$

The last step is simply a rearrangement of the order of summation, so as to isolate the total disturbing potential due to the phase angle $j\lambda - m\theta$, and to facilitate the use of stable recursion formulas. We also introduce some new notation, to enable the results to be written concisely. First, define the functions

$$\Gamma_{ns}^m(\gamma) = \begin{cases} (-1)^{m-s} 2^s (1 + I\gamma)^{-Im} & \text{if } s \leq -m \\ (-1)^{m-s} 2^{-m} \dfrac{(n+m)!(n-m)!}{(n+s)!(n-s)!} (1 + I\gamma)^{Is} & \text{if } |s| \leq m \\ 2^{-s} (1 + I\gamma)^{Im} & \text{if } s \geq m \end{cases} \tag{13}$$

Next, put

$$G_{ms}^j + iH_{ms}^j = \begin{cases} [k + ih\,\mathrm{sgn}(s-j)]^{|s-j|}(\alpha + iI\beta)^{m-Is} & \text{if } |s| \leq m \\ [k + ih\,\mathrm{sgn}(s-j)]^{|s-j|}[\alpha - i\beta\,\mathrm{sgn}(s-m)]^{|s-Im|} & \text{if } |s| \geq m \end{cases} \tag{14}$$

Then define the Jacobi polynomial P_ℓ^{vw} indices by

$$\begin{aligned} \ell &= \begin{cases} n-m & \text{if } |s| \leq m \\ n-|s| & \text{if } |s| > m \end{cases} \\ v &= |m-s| \\ w &= |m+s| \end{aligned} \tag{15}$$

The disturbing function can now finally be written as

$$\begin{aligned} \mathcal{R} = \mathrm{Re}\Big\{ \frac{\mu}{a} \sum_{j=-\infty}^{\infty} \sum_{m=0}^{M} \sum_{s=-N}^{N} \sum_{n=\max(2,m,|s|)}^{N} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw} \\ (G_{ms}^j + iH_{ms}^j)(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)] \Big\} \end{aligned} \tag{16}$$

Note that the functions G_{ms}^j and H_{ms}^j defined by (14) are of degree $|s-j|$ in the eccentricity. The power $|s-j|$ has been called a *D'Alembert characteristic* (not in accordance with its original connotation [Brouwer, 1961]).

2.7.2 Calculation of V_{ns}^m Coefficients

The V_{ns}^m coefficients are defined by (2.7.1-6). Since

$$V_{n,s}^m = (-1)^s V_{n,-s}^m \tag{1}$$

we can restrict our discussion to the case $s \geq 0$ without loss of generality. Furthermore, since $V_{n,s}^m = 0$ when $n - s$ is odd, we need only consider the case when $n - s$ is even. Also, note that the lowest value of the degree n in the summations (2.7.1-16) is greater than or equal to $2, m$, and $|s|$.

Suitable recurrence relations are

$$
\begin{aligned}
V_{n+2,s}^m &= -\frac{(n+s+1)(n-s+1)}{(n-m+2)(n-m+1)} V_{n,s}^m \\
V_{n,s}^{m+1} &= (n-m) V_{n,s}^m
\end{aligned}
\tag{2}
$$

Appropriate initialization is provided by

$$
\begin{aligned}
V_{0,0}^0 &= 1 \\
V_{s+1,s+1}^0 &= \frac{(2s+1)}{(s+1)} V_{s,s}^0
\end{aligned}
\tag{3}
$$

That is, to calculate the V_{ns}^m coefficients, first use (3) to get values for $m = 0$ and $n = s$ for $s = 0, 1, \ldots$. Then use (2b) for $m > 0$ and still $n = s$. Finally, use (2a) for increasing n with any nonnegative m and s.

2.7.3 Calculation of Kernels K_j^{ns} of Hansen Coefficients

From the definitions (2.7.1-9, 10), the kernels of the Hansen coefficients are given by

$$K_j^{ns}(e) = \frac{e^{-|s-j|}}{2\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n \cos(sf - jM) dM \tag{1}$$

The kernels of the Hansen coefficients are thus functions of the orbital eccentricity e. Note from (1) that

$$K_j^{n,s} = K_{-j}^{n,-s} \tag{2}$$

so we can restrict our discussion to the case $s \geq 0$ without loss of generality.

For the special case $j = 0$, the kernels may be evaluated in a form algebraically closed in the eccentricity. This is because the Hansen coefficients with $j = 0$ are related to the associated Legendre functions by [McClain, 1978]

$$
\begin{aligned}
X_0^{-n-1,s} &= \frac{(n-1)!\chi^n}{(n+s-1)!} P_{n-1}^s(\chi) \quad \text{for } n \geq 1 \\
X_0^{n,s} &= \frac{(-1)^s(n-s+1)!\chi^{-n-1}}{(n+1)!} P_{n+1}^s(\chi) \quad \text{for } n \geq 0
\end{aligned}
\tag{3}
$$

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where

$$\chi = \frac{1}{\sqrt{1 - e^2}} = \frac{1}{\sqrt{1 - h^2 - k^2}} = \frac{1}{B} \tag{4}$$

and many recursion formulas are available for the associated Legendre functions, which for arguments in the range $1 \leq \chi < \infty$ are defined by

$$P_n^s(\chi) = \frac{1}{2^n n!} (\chi^2 - 1)^{s/2} \frac{d^{n+s}}{d\chi^{n+s}} (\chi^2 - 1)^n \tag{5}$$

For the special kernels with the first superscript negative (needed in Sections 3.1 and 4.1), appropriate recursion formulas are

$$K_0^{-n-1,s} = \begin{cases} 0 & \text{for } n = s \geq 0 \\[2mm] \dfrac{\chi^{1+2s}}{2^s} & \text{for } n = s + 1 \geq 1 \\[2mm] \dfrac{(n-1)\chi^2}{(n+s-1)(n-s-1)} \left[(2n-3)K_0^{-n,s} - (n-2)K_0^{-n+1,s} \right] & \text{for } n \geq s + 2 \geq 2 \end{cases} \tag{6}$$

For the special kernels with the first superscript nonnegative (needed in Section 3.2), appropriate recursion formulas are

$$K_0^{n,s} = \begin{cases} -\dfrac{(2s-1)}{s} K_0^{s-2,s-1} & \text{for } n = s - 1 \geq 1 \\[2mm] \dfrac{(2s+1)}{s+1} K_0^{s-1,s} & \text{for } n = s \geq 1 \\[2mm] \dfrac{2n+1}{n+1} K_0^{n-1,s} - \dfrac{(n+s)(n-s)}{n(n+1)\chi^2} K_0^{n-2,s} & \text{for } n \geq s + 1 \geq 2 \end{cases} \tag{7}$$

with initializations

$$\begin{aligned} K_0^{0,0} &= 1 \\ K_0^{0,1} &= -1 \end{aligned} \tag{8}$$

The general kernels $K_j^{-n-1,s}$ may be computed from the following recurrence relation [Proulx, McClain, Early, and Cefola, 1981]:

$$K_j^{-n-1,s} = \frac{\chi^2}{(3-n)(1-n+s)(1-n-s)} \left\{ (3-n)(1-n)(3-2n)K_j^{-n,s} \right. \\ \left. -(2-n)[(3-n)(1-n) + \frac{2js}{\chi}]K_j^{-n+1,s} + j^2(1-n)K_j^{-n+3,s} \right\} \tag{9}$$

To initialize this recurrence relation, we need the values of the four kernels $K_j^{-n,s}$, $K_j^{-n+1,s}$, $K_j^{-n+2,s}$, and $K_j^{-n+3,s}$ at $n = \max(2, m, s)$. These latter kernels are calculated by infinite series representations. In a study of the various possibilities [Proulx and McClain, 1988], it was found that the expansion of choice is

$$K_j^{ns} = (1 - e^2)^{n+\frac{3}{2}} \sum_{\alpha=0}^{\infty} Y_{\alpha+a, \alpha+b}^{ns} e^{2\alpha} \tag{10}$$

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Here

$$
\begin{aligned}
a &= \max(j - s, 0) \\
b &= \max(s - j, 0)
\end{aligned}
\tag{11}
$$

The $Y_{\rho\sigma}^{ns}$ terms are called modified Newcomb operators and may be computed by the recurrence relation

$$
\begin{aligned}
4(\rho + \sigma)Y_{\rho,\sigma}^{n,s} &= 2(2s - n)Y_{\rho-1,\sigma}^{n,s+1} + (s - n)Y_{\rho-2,\sigma}^{n,s+2} - 2(2s + n)Y_{\rho,\sigma-1}^{n,s-1} \\
&\quad -(s + n)Y_{\rho,\sigma-2}^{n,s-2} + 2(2\rho + 2\sigma + 2 + 3n)Y_{\rho-1,\sigma-1}^{n,s}
\end{aligned}
\tag{12}
$$

Recursion formula (12) is initialized by

$$
Y_{0,0}^{n,s} = 1
\tag{13}
$$

and by treating quantities with negative subscripts as identically zero. That is, to calculate the $Y_{\rho\sigma}^{ns}$ coefficients for any n, use (12) -(13) to get values for $s = 0, \ldots, n$ and each successive value of $\rho = 0, 1, \ldots$ and $\sigma = 0, 1, \ldots$ Note that the $Y_{\rho\sigma}^{ns}$ terms are rational constants, and therefore they can be computed once and stored for all later applications.

2.7.4   Calculation of Jacobi Polynomials P_ℓ^{vw}

The Jacobi polynomials appear in the expression (2.7.1-7) for the rotation functions. $P_\ell^{vw}(\gamma)$ is a polynomial of degree ℓ in γ, which from (2.1.9-1c) is the cosine of the angle between a vector from the geographic south to north pole of the central body and the angular momentum vector of the satellite. The Jacobi polynomials $P_\ell^{vw}(\gamma)$ with $m = 0$ in the indices (2.7.1-15), i.e. $\ell = n - s \geq 0$ and $v = w = s \geq 0$, are related to the associated Legendre functions $P_{ns}(\gamma)$ by

$$
P_{n-s}^{ss}(\gamma) = 2^s \frac{n!}{(n + s)!}(1 - \gamma^2)^{-s/2}P_{ns}(\gamma)
\tag{1}
$$

The Jacobi polynomials can be computed from the standard recurrence relation [Szegö, 1959]:

$$
\begin{aligned}
&2\ell(\ell + v + w)(2\ell + v + w - 2)P_\ell^{vw}(\gamma) = \\
&(2\ell + v + w - 1)[(2\ell + v + w)(2\ell + v + w - 2)\gamma + v^2 - w^2]P_{\ell-1}^{vw}(\gamma) \\
&-2(\ell + v - 1)(\ell + w - 1)(2\ell + v + w)P_{\ell-2}^{vw}(\gamma)
\end{aligned}
\tag{2}
$$

This recursion formula is initialized by

$$
\begin{aligned}
P_0^{vw} &= 1 \\
P_{-1}^{vw} &= 0
\end{aligned}
\tag{3}
$$

2.7.5 Calculation of G_{ms}^j and H_{ms}^j Polynomials

From the definitions (2.7.1-14), the functions G_{ms}^j and H_{ms}^j are polynomials in the equinoctial elements h, k and the direction cosines α, β.

These polynomials may all be calculated from one set of generic recurrence formulas, based on the C_j and S_j polynomials obtained from (2.5.3-6):

$$
G_{ms}^j = \begin{cases} C_{|s-j|}(k,h)C_{m-Is}(\alpha,\beta) - I\,\mathrm{sgn}(s-j)S_{|s-j|}(k,h)S_{m-Is}(\alpha,\beta) & \text{for } |s| \le m \\ C_{|s-j|}(k,h)C_{|s-Im|}(\alpha,\beta) + \mathrm{sgn}(s-j)\mathrm{sgn}(s-m)S_{|s-j|}(k,h)S_{|s-Im|}(\alpha,\beta) & \text{for } |s| \ge m \end{cases}
$$

(1)

$$
H_{ms}^j = \begin{cases} IC_{|s-j|}(k,h)S_{m-Is}(\alpha,\beta) + \mathrm{sgn}(s-j)S_{|s-j|}(k,h)C_{m-Is}(\alpha,\beta) & \text{for } |s| \le m \\ -\mathrm{sgn}(s-m)C_{|s-j|}(k,h)S_{|s-Im|}(\alpha,\beta) + \mathrm{sgn}(s-j)S_{|s-j|}(k,h)C_{|s-Im|}(\alpha,\beta) & \text{for } |s| \ge m \end{cases}
$$

(2)

2.8 Third-Body Gravitational Potential

The disturbing function due to the gravitational field of a third-body point mass is [Battin, 1987]:

$$
\mathcal{R}(r, \phi, t) = \frac{\mu_3}{R_3}\left(\frac{R_3}{|\mathbf{R}_3 - \mathbf{r}|} - \frac{r\cos\phi}{R_3}\right)
$$

(1)

Here

$\mathbf{r}$ = vector from the center of mass of the central body to the satellite

$\mathbf{R}_3(t)$ = vector from the center of mass of the central body to the third body

ϕ = angle between the vectors $\mathbf{r}$ and $\mathbf{R}_3$

μ_3 = third-body gravitational constant

In this section all elements are *osculating* (even though they do not have hats).

The quantity $\frac{R_3}{|\mathbf{R}_3 - \mathbf{r}|}$ can be expanded in the following series:

$$
\frac{R_3}{|\mathbf{R}_3 - \mathbf{r}|} = \frac{1}{\sqrt{1 - \frac{2r\cos\phi}{R_3} + \frac{r^2}{R_3^2}}} = \sum_{n=0}^{\infty}\left(\frac{r}{R_3}\right)^n P_n(\cos\phi)
$$

(2)

where P_n is the Legendre polynomial of degree n. Hence the third-body disturbing function (1) can be written as

$$
\mathcal{R} = \frac{\mu_3}{R_3}\sum_{n=2}^{N}\left(\frac{r}{R_3}\right)^n P_n(\cos\phi)
$$

(3)

where N is the maximum power of the parallax factor $\frac{r}{R_3}$ to be retained in the expansion.

The further development of the third-body potential into the form used in SST is similar to that of the central-body potential in Section 2.7. Complete details are to be found in [Cefola and Broucke, 1975], [McClain, 1978], [Cefola and McClain, 1978], and [Slutsky, 1983].

2.8.1 Expansion of Third-Body Potential in Equinoctial Variables

With the goal of expressing (2.8-3) in terms of the equinoctial elements, we set

$$\cos\phi = \alpha\cos L + \beta\sin L \qquad (1)$$

where L is the true longitude, and now (α, β, γ) are the dot products of the unit vector $\frac{\mathbf{R_3}}{R_3}$ with the equinoctial reference triad $(\mathbf{f}, \mathbf{g}, \mathbf{w})$. Due to the motion of the third body, (α, β, γ) are slowly varying functions of the time t and are the source of weak time-dependence effects.

Next the expression $P_n(\alpha\cos L + \beta\sin L)$ is expanded into a Fourier series, using an addition formula for the Legendre polynomials:

$$P_n(\alpha\cos L + \beta\sin L) = \sum_{s=0}^{n}(2 - \delta_{0s})V_{ns}Q_{ns}(\gamma)[C_s(\alpha, \beta)\cos sL + S_s(\alpha, \beta)\sin sL] \qquad (2)$$

Here δ_{0s} is the Kronecker delta, and the polynomials $C_s(\alpha, \beta)$ and $S_s(\alpha, \beta)$ are the same as those in (2.7.5-1, 2). The new coefficients introduced in (2) are defined by:

$$V_{ns} = \begin{cases} \dfrac{(-1)^{\frac{n-s}{2}}}{2^n}\dfrac{(n-s)!}{(\frac{n+s}{2})!(\frac{n-s}{2})!} & \text{if } n-s \text{ is even} \\[2mm] 0 & \text{if } n-s \text{ is odd} \end{cases} \qquad (3)$$

$$Q_{ns}(\gamma) = \frac{d^s P_n(\gamma)}{d\gamma^s} \qquad (4)$$

After substituting (2) into (2.8-3), we can write the result in the complex form

$$\mathcal{R} = \text{Re}\left\{\frac{\mu_3}{R_3}\sum_{n=2}^{N}\sum_{s=0}^{n}(2-\delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns}Q_{ns}[C_s(\alpha, \beta) - iS_s(\alpha, \beta)]\left(\frac{r}{a}\right)^n\exp(isL)\right\} \qquad (5)$$

The last steps are simply the replacement of the product $\left(\frac{r}{a}\right)^n e^{isL}$ by the expansion (2.7.1-8), and a rearrangement of the order of summation. The disturbing function can then be written as

$$\mathcal{R} = \text{Re}\left\{\frac{\mu_3}{R_3}\sum_{j=-\infty}^{\infty}\sum_{s=0}^{N}\sum_{n=\max(2,s)}^{N}(2-\delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns}Y_j^{ns}Q_{ns}[C_s(\alpha, \beta) - iS_s(\alpha, \beta)]exp(ij\lambda)\right\} \qquad (6)$$

2.8.2 Calculation of V_{ns} Coefficients

The V_{ns} coefficients are defined by (2.8.1-3). Since $V_{ns} = 0$ when $n - s$ is odd, we need only consider the case when $n - s$ is even. (Note from (2.7.1-6) that $V_{ns}^m = \frac{(n+s)!}{(n-m)!}V_{ns}$.)

A suitable recurrence relation is [Cefola and Broucke, 1975]

$$V_{n+2,s} = -\frac{(n-s+1)}{(n+s+2)}V_{n,s} \qquad (1)$$

Appropriate initialization is provided by

$$V_{0,0} = 1$$
$$V_{s+1,s+1} = \left(\frac{1}{2s+2}\right) V_{s,s} \tag{2}$$

2.8.3 Calculation of Q_{ns} Polynomials

The Q_{ns} polynomials defined by (2.8.1-4) are derivatives of the Legendre polynomials evaluated at γ, which from (2.2-7c) is the cosine of the angle between $\mathbf{R}_3$ and the angular momentum vector of the satellite. The polynomial Q_{ns} can also be expressed in terms of the associated Legendre function P_{ns}:

$$Q_{ns}(\gamma) = (1 - \gamma^2)^{-s/2} P_{ns}(\gamma) \tag{1}$$

(Note from (2.7.1-6) that $V_{ns}^s = Q_{ns}(0)$ and from (2.7.4-1) that $P_{n-s}^{ss}(\gamma) = 2^s \frac{n!}{(n+s)!} Q_{ns}(\gamma)$.)

Recursion formulas for the Q_{ns} polynomials follow directly from standard recursion formulas for the P_{ns} functions [Cefola and Broucke, 1975]:

$$Q_{n,s}(\gamma) = \begin{cases} (2s-1)Q_{s-1,s-1}(\gamma) & \text{for } n = s \\ (2s+1)\gamma Q_{s,s}(\gamma) & \text{for } n = s+1 \\ \dfrac{(2n-1)\gamma Q_{n-1,s}(\gamma) - (n+s-1)Q_{n-2,s}(\gamma)}{(n-s)} & \text{for } n > s+1 \end{cases} \tag{2}$$

These recursion formulas are simply initialized by

$$Q_{0,0} = 1 \tag{3}$$

3 First-Order Mean Element Rates

As we have seen, the first-order mean element rates $A_{i\alpha}$ are given by equations (2.4-18). The osculating rate functions $F_{i\alpha}$ for a conservative perturbation are given by (2.2-10) and for a nonconservative perturbation by (2.2-5). In this chapter we record the specific form of these equations for each of several perturbations.

3.1 Central-Body Gravitational Zonal Harmonics

For the central-body gravitational zonal harmonics, the appropriate averaging operator $< \cdots >$ is (2.4-10) and the appropriate disturbing function $\mathcal{R}$ is (2.7.1-16) with $m = 0$. Further details of the reduction of the averaged equations to the form recorded here may be found in [Cefola and Broucke, 1975] and [McClain, 1978].

The first-order contribution of the central-body gravitational zonal harmonics to the averaged equations of motion (2.4-1) is

$$\frac{da}{dt} = 0$$

$$\frac{dh}{dt} = \frac{B}{A}\frac{\partial U}{\partial k} + \frac{k}{AB}(pU_{,\alpha\gamma} - IqU_{,\beta\gamma})$$

$$\frac{dk}{dt} = -\frac{B}{A}\frac{\partial U}{\partial h} - \frac{h}{AB}(pU_{,\alpha\gamma} - IqU_{,\beta\gamma})$$

$$\frac{dp}{dt} = -\frac{C}{2AB}U_{,\beta\gamma}$$

$$\frac{dq}{dt} = -\frac{IC}{2AB}U_{,\alpha\gamma}$$

$$\frac{d\lambda}{dt} = -\frac{2a}{A}\frac{\partial U}{\partial a} + \frac{B}{A(1+B)}\left(h\frac{\partial U}{\partial h} + k\frac{\partial U}{\partial k}\right) + \frac{1}{AB}(pU_{,\alpha\gamma} - IqU_{,\beta\gamma})$$

(1)

Here (a, h, k, p, q, λ) are now the *mean* elements and U is the *mean* disturbing function (2.5.5-2). In deriving (1) from (2.2-10), we have made use of the following property of the cross-derivatives for the mean disturbing function:

$$U_{,hk} - U_{,\alpha\beta} = 0 \tag{2}$$

The mean disturbing function reduces to

$$U = -\frac{\mu}{a}\sum_{s=0}^{N-2}\sum_{n=s+2}^{N}(2 - \delta_{0s})\left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} Q_{ns} G_s \tag{3}$$

Here

$$
\begin{array}{lll}
J_n = -C_{n0} & = & \text{geopotential coefficients} \\
V_{ns} & = & \text{coefficients calculated from (2.8.2-1,2)} \\
K_0^{-n-1,s} & = & \text{kernels of Hansen coefficients calculated from (2.7.3-6)} \\
Q_{ns}(\gamma) & = & \text{functions calculated from (2.8.3-2,3)} \\
G_s = G_{0s}^0 & = & \text{polynomials calculated in Section 2.7.5}
\end{array}
$$

Since

$$G_s + iH_s = G_{0s}^0 + iH_{0s}^0 = (k+ih)^s(\alpha - i\beta)^s = [C_s(k,h) + iS_s(k,h)][C_s(\alpha,\beta) - iS_s(\alpha,\beta)] \tag{4}$$

an alternate set of recursion formulas for the G_s polynomials is

$$
\begin{array}{llll}
G_s & = & (k\alpha + h\beta)G_{s-1} - (h\alpha - k\beta)H_{s-1} & , \quad G_0 = 1 \\
H_s & = & (h\alpha - k\beta)G_{s-1} + (k\alpha + h\beta)H_{s-1} & , \quad H_0 = 0
\end{array}
\tag{5}
$$

Note that the G_s polynomials are of degree s in the eccentricity; for small eccentricity orbits, the series (3) may be truncated by prescribing the maximum possible value of the D'Alembert characteristic s.

In equations (1), we need the partial derivatives of U with respect to $(a, h, k, \alpha, \beta, \gamma)$. These are easily obtained by differentiating equation (3):

$$
\begin{aligned}
\frac{\partial U}{\partial a} &= \frac{\mu}{a^2} \sum_{s,n} (2 - \delta_{0s})(n+1) \left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} Q_{ns} G_s \\
\frac{\partial U}{\partial h} &= -\frac{\mu}{a} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} Q_{ns} \left(K_0^{-n-1,s} \frac{\partial G_s}{\partial h} + h\chi^3 G_s \frac{dK_0^{-n-1,s}}{d\chi} \right) \\
\frac{\partial U}{\partial k} &= -\frac{\mu}{a} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} Q_{ns} \left(K_0^{-n-1,s} \frac{\partial G_s}{\partial k} + k\chi^3 \frac{dK_0^{-n-1,s}}{d\chi} \right) \\
\frac{\partial U}{\partial \alpha} &= -\frac{\mu}{a} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} Q_{ns} \frac{\partial G_s}{\partial \alpha} \\
\frac{\partial U}{\partial \beta} &= -\frac{\mu}{a} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} Q_{ns} \frac{\partial G_s}{\partial \beta} \\
\frac{\partial U}{\partial \gamma} &= -\frac{\mu}{a} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} \frac{dQ_{ns}}{d\gamma} G_s
\end{aligned}
\tag{6}
$$

Here we have obtained the partial derivatives with respect to h and k of $K_0^{-n-1,s}(\chi)$ from the chain rule and the definition (2.7.3-4).

Recursion formulas for $\frac{dK_0^{-n-1,s}(\chi)}{d\chi}$ are obtained by differentiating the recursion formulas (2.7.3-5):

$$
\frac{dK_0^{-n-1,s}}{d\chi} =
\begin{cases}
0 & \text{for } n = s \\
\dfrac{(1+2s)\chi^{2s}}{2^s} & \text{for } n = s+1 \\
\dfrac{(n-1)\chi^2}{(n+s-1)(n-s+1)} \left[(2n-3)\dfrac{dK_0^{-n,s}}{d\chi} - (n-2)\dfrac{dK_0^{-n+1,s}}{d\chi} \right] & \\
\quad + \dfrac{2}{\chi} K_0^{-n-1,s} & \text{for } n > s+1
\end{cases}
\tag{7}
$$

Recursion formulas for $\frac{dQ_{ns}(\gamma)}{d\gamma}$ are obtained by differentiating (2.8.1-4):

$$
\frac{dQ_{n,s}}{d\gamma}(\gamma) = Q_{n,s+1}(\gamma)
\tag{8}
$$

Recursion formulas for the partial derivatives of G_s may be obtained by differentiating (4):

$$
\begin{aligned}
\frac{\partial G_s}{\partial h} &= s\beta G_{s-1} - s\alpha H_{s-1} \\
\frac{\partial G_s}{\partial k} &= s\alpha G_{s-1} + s\beta H_{s-1} \\
\frac{\partial G_s}{\partial \alpha} &= sk G_{s-1} - sh H_{s-1} \\
\frac{\partial G_s}{\partial \beta} &= sh G_{s-1} + sk H_{s-1}
\end{aligned}
\tag{9}
$$

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If we retain only the J_2 term in the expansion (3), the central-body gravitational disturbing function simplifies to

$$U = \frac{J(\gamma^2 - \frac{1}{3})}{a^3(1 - h^2 - k^2)^{3/2}} \tag{10}$$

where

$$J = \frac{3\mu R^2 J_2}{4}$$

The contribution of (10) to the averaged equation of motion (1) is

$$
\begin{aligned}
\frac{da}{dt} &= 0 \\
\frac{dh}{dt} &= \frac{Jk[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{AB^4 a^3} \\
\frac{dk}{dt} &= -\frac{Jh[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{AB^4 a^3} \\
\frac{dp}{dt} &= -\frac{CJ\beta\gamma}{AB^4 a^3} \\
\frac{dq}{dt} &= -\frac{ICJ\alpha\gamma}{AB^4 a^3} \\
\frac{d\lambda}{dt} &= \frac{J[(1 + B)(3\gamma^2 - 1) + 2\gamma(p\alpha - Iq\beta)]}{AB^4 a^3}
\end{aligned}
\tag{11}
$$

In order to update the orbital elements in a differential corrections procedure, it is necessary to compute the partial derivatives with respect to the mean orbital elements of the mean element rates (the $\frac{\partial \dot{a}_j}{\partial a_i}$ in the matrix $\mathbf{F}$ defined by (2.6-7)). The partial derivatives of the J_2 contribution (11) to the mean element rates are easily obtained, using (2.1.6-1) and (2.1.9-4). The nonzero derivatives are

$$\frac{\partial \dot{h}}{\partial a} = -\frac{7Jk[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{2a^4 AB^4}$$

$$\frac{\partial \dot{h}}{\partial h} = \frac{4Jhk[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{a^3 AB^6}$$

$$\frac{\partial \dot{h}}{\partial k} = \frac{J(1 - h^2 + 3k^2)[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{a^3 AB^6}$$

$$\frac{\partial \dot{h}}{\partial p} = \frac{2Jk[6\alpha\gamma + 2p(\alpha^2 - \gamma^2) - 2q^2\alpha\gamma - 2Iq\beta(\alpha + p\gamma) + C\alpha\gamma]}{a^3 AB^4 C}$$

$$\frac{\partial \dot{h}}{\partial q} = -\frac{2IJk[6\beta\gamma + 2p\alpha\gamma + 2Iq\gamma^2 - 2Ipq\alpha\gamma - 2p^2\beta\gamma + C\beta\gamma]}{a^3 AB^4 C}$$

$$\frac{\partial \dot{k}}{\partial a} = \frac{7Jh[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{2a^4 AB^4}$$

$$\frac{\partial \dot{k}}{\partial h} = -\frac{(1 - k^2 + 3h^2)J[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{a^3 AB^6}$$

$$\frac{\partial \dot{k}}{\partial k} = -\frac{4Jhk[3\gamma^2 - 1 + 2\gamma(p\alpha - Iq\beta)]}{a^3 AB^6}$$

$$\frac{\partial \dot{k}}{\partial p} = -\frac{2Jh[6\alpha\gamma + 2p(\alpha^2 - \gamma^2) - 2q^2\alpha\gamma - 2Iq\beta(\alpha + p\gamma) + C\alpha\gamma]}{a^3 AB^4 C}$$

$$\frac{\partial \dot{k}}{\partial q} = \frac{2IJh[6\beta\gamma + 2p\alpha\gamma + 2Iq\gamma^2 - 2Ipq\alpha\gamma - 2p^2\beta\gamma + C\beta\gamma]}{a^3 AB^4 C}$$

$$\frac{\partial \dot{p}}{\partial a} = \frac{7CJ\beta\gamma}{2a^4AB^4}$$

$$\frac{\partial \dot{p}}{\partial h} = -\frac{4CJh\beta\gamma}{a^3AB^6}$$

$$\frac{\partial \dot{p}}{\partial k} = -\frac{4CJk\beta\gamma}{a^3AB^6}$$

$$\frac{\partial \dot{p}}{\partial p} = -\frac{2J[p\beta\gamma + \alpha(\beta + Iq\gamma)]}{a^3AB^4}$$

$$\frac{\partial \dot{p}}{\partial q} = \frac{2IJ[-ICq\beta\gamma + \beta^2 - \gamma^2 + p\alpha\gamma]}{a^3AB^4C}$$

$$\frac{\partial \dot{q}}{\partial a} = \frac{7CIJ\alpha\gamma}{2a^4AB^4}$$

$$\frac{\partial \dot{q}}{\partial h} = -\frac{4CIJh\alpha\gamma}{a^3AB^6}$$

$$\frac{\partial \dot{q}}{\partial k} = -\frac{4CIJk\alpha\gamma}{a^3AB^6} \qquad (12)$$

$$\frac{\partial \dot{q}}{\partial p} = -\frac{2IJ[p\alpha\gamma + \alpha^2 - \gamma^2 - Iq\beta\gamma]}{a^3AB^4}$$

$$\frac{\partial \dot{q}}{\partial q} = -\frac{2J[qCI\alpha\gamma + \beta(\alpha - p\gamma)]}{a^3AB^4C}$$

$$\frac{\partial \dot{\lambda}}{\partial a} = -\frac{7J[(1+B)(3\gamma^2-1) + 2\gamma(p\alpha - Iq\beta)]}{2a^4AB^4}$$

$$\frac{\partial \dot{\lambda}}{\partial h} = \frac{Jh[(3\gamma^2-1)(4+5B) + 8\gamma(p\alpha - Iq\beta)]}{a^3AB^6}$$

$$\frac{\partial \dot{\lambda}}{\partial k} = \frac{Jk[(3\gamma^2-1)(4+3B) + 8\gamma(p\alpha - Iq\beta)]}{a^3AB^6}$$

$$\frac{\partial \dot{\lambda}}{\partial p} = \frac{2J[6(1+B)\alpha\gamma + 2p(\alpha^2-\gamma^2) - 2q^2\alpha\gamma - 2Ipq\beta\gamma + C\alpha\gamma]}{a^3AB^4C}$$

$$\frac{\partial \dot{\lambda}}{\partial q} = -\frac{2IJ[6(1+B)\beta\gamma + 2p\beta(\alpha - p\gamma) + 2Iq\gamma(\gamma - p\alpha) + C\beta\gamma]}{a^3AB^4C}$$

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3.2 Third-Body Gravitational Potential

For a third-body point mass, the appropriate averaging operator is (2.4-10) and the appropriate disturbing function $\mathcal{R}$ is (2.8.1-6). Further details of the reduction of the averaged equations to the form recorded here may be found in [Cefola and Broucke, 1975] and [McClain, 1978].

The first-order contribution of the third-body gravitational disturbing function to the averaged equations of motion is identical in form to equations (3.1-1) for the central-body zonal harmonics. Of course, the direction cosines (α, β, γ) have different interpretations for the two perturbations.

The mean disturbing function is now

$$U = \frac{\mu_3}{R_3} \sum_{s=0}^{N} \sum_{n=\max(2,s)}^{N} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} G_s \tag{1}$$

Here

$$
\begin{array}{lll}
V_{ns} & = & \text{coefficients calculated from (2.8.2-1, 2)} \\
K_0^{ns} & = & \text{kernels of the Hansen coefficients calculated from (2.7.3-7, 8)} \\
Q_{ns}(\gamma) & = & \text{polynomials calculated from (2.8.3-2, 3)} \\
G_s & = & \text{polynomials which can be calculated from (3.1-5)}
\end{array}
$$

The partial derivatives of U needed in equations (3.1-1) are easily obtained by differentiating equation (1):

$$
\begin{aligned}
\frac{\partial U}{\partial a} &= \frac{\mu_3}{R_3 a} \sum_{s,n} (2 - \delta_{0s}) n \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} G_s \\
\frac{\partial U}{\partial h} &= \frac{\mu_3}{R_3} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} \left(K_0^{ns} \frac{\partial G_s}{\partial h} + h \chi^3 G_s \frac{dK_0^{ns}}{d\chi}\right) \\
\frac{\partial U}{\partial k} &= \frac{\mu_3}{R_3} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} \left(K_0^{ns} \frac{\partial G_s}{\partial k} + k \chi^3 G_s \frac{dK_0^{ns}}{d\chi}\right) \\
\frac{\partial U}{\partial \alpha} &= \frac{\mu_3}{R_3} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} \frac{\partial G_s}{\partial \alpha} \\
\frac{\partial U}{\partial \beta} &= \frac{\mu_3}{R_3} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} \frac{\partial G_s}{\partial \beta} \\
\frac{\partial U}{\partial \gamma} &= \frac{\mu_3}{R_3} \sum_{s,n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} \frac{dQ_{ns}}{d\gamma} G_s
\end{aligned} \tag{2}
$$

Recursion formulas for $\frac{dK_0^{ns}}{d\chi}$ are obtained by differentiating the recursion formulas (2.7.3-6, 7):

$$\frac{dK_0^{ns}}{d\chi} = \begin{cases} 0 & \text{for } n = s - 1 \text{ or } n = s \\ \dfrac{2n+1}{n+1} \dfrac{dK_0^{n-1,s}}{d\chi} - \dfrac{(n+s)(n-s)}{n(n+1)\chi^2} \dfrac{dK_0^{n-2,s}}{d\chi} + \dfrac{2(n+s)(n-s)}{n(n+1)\chi^3} K_0^{n-2,s} & \text{for } n > s \end{cases} \tag{3}$$

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Recursion formulas for the derivatives of Q_{ns} and G_s are given by (3.1-8, 9).

If we wish to retain only the dominant terms in the expansion (1), we should include at least the first two terms, since the $n = 2$ term vanishes in the $\frac{dh}{dt}$ and $\frac{dk}{dt}$ equations for zero eccentricity orbits, leaving the $n = 3$ term dominant [Collins and Cefola, 1978].

3.3   Central-Body Gravitational Resonant Tesserals

For the central-body gravitational tesseral harmonics, the appropriate averaging operator is (2.4-11) and the appropriate disturbing function is (2.7.1-16) with $m \neq 0$. Further details of the reduction of the averaged equations to the form here may be found in [Proulx, McClain, Early, and Cefola, 1981] and [Proulx, 1982].

The first-order contribution of the central-body gravitational tesseral harmonics to the averaged equations of motion (2.4-1) is identical to (2.2-10), except that $\mathcal{R}$ is replaced by U:

$$
\begin{aligned}
\dot{a} &= \frac{2a}{A}\frac{\partial U}{\partial \lambda} \\
\dot{h} &= \frac{B}{A}\frac{\partial U}{\partial k} + \frac{k}{AB}\left(pU_{,\alpha\gamma} - IqU_{,\beta\gamma}\right) - \frac{hB}{A(1+B)}\frac{\partial U}{\partial \lambda} \\
\dot{k} &= -\left[\frac{B}{A}\frac{\partial U}{\partial h} + \frac{h}{AB}\left(pU_{,\alpha\gamma} - IqU_{,\beta\gamma}\right) + \frac{kB}{A(1+B)}\frac{\partial U}{\partial \lambda}\right] \\
\dot{p} &= \frac{C}{2AB}\left[p\left(U_{,hk} - U_{,\alpha\beta} - \frac{\partial U}{\partial \lambda}\right) - U_{,\beta\gamma}\right] \\
\dot{q} &= \frac{C}{2AB}\left[q\left(U_{,hk} - U_{,\alpha\beta} - \frac{\partial U}{\partial \lambda}\right) - IU_{,\alpha\gamma}\right] \\
\dot{\lambda} &= -\frac{2a}{A}\frac{\partial U}{\partial a} + \frac{B}{A(1+B)}\left(h\frac{\partial U}{\partial h} + k\frac{\partial U}{\partial k}\right) + \frac{1}{AB}\left(pU_{,\alpha\gamma} - IqU_{,\beta\gamma}\right)
\end{aligned}
\tag{1}
$$

Here (a, h, k, p, q, λ) are now the *mean* elements and U is the *mean* disturbing function defined by

$$
\begin{aligned}
U &= <\mathcal{R}> = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathcal{R}(a, h, k, p, q, \lambda, \theta, t)d\lambda\, d\theta \\
&+ \mathrm{Re}\left\{\frac{1}{2\pi^2}\sum_{\substack{(j,m)\in\mathcal{B}}}[\exp[i(j\lambda - m\theta)]\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathcal{R}(a, h, k, p, q, \lambda', \theta', t)\exp[-i(j\lambda' - m\theta')]d\lambda'\, d\theta']\right\}
\end{aligned}
\tag{2}
$$

(Remember that $\mathcal{B}$ here is the set of all ordered pairs (j, m) with the properties (2.4-12, 13)).

The mean disturbing function is identical to (2.7.1-16), except that now only the resonant tesserals are included in the summations:

$$
U = \mathrm{Re}\Big\{\frac{\mu}{a}\sum_{\substack{j \\ (j,m)\in\mathcal{B}}}\sum_{m=1}^{M}\sum_{s=-N}^{N}\sum_{n=\max(2,m,|s|)}^{N}\left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}
$$
$$
(G_{ms}^j + iH_{ms}^j)(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)]\Big\}
\tag{3}
$$

In equations (1), we need the partial derivatives of U with respect to $(a, h, k, \lambda, \alpha, \beta, \gamma)$. These are easily obtained by differentiating equation (3):

$$\frac{\partial U}{\partial a} = \text{Re}\Big\{-\frac{\mu}{a^2} \sum_{j,m,s,n} (n+1)\left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}(G_{ms}^j + iH_{ms}^j)$$

$$(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)]\Big\}$$

$$\frac{\partial U}{\partial h} = \text{Re}\Big\{\frac{\mu}{a} \sum_{j,m,s,n} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m P_\ell^{vw}(C_{mn} - iS_{nm})$$

$$[K_j^{-n-1,s}\left(\frac{\partial G_{ms}^j}{\partial h} + i\frac{\partial H_{ms}^j}{\partial h}\right) + 2h(G_{ms}^j + H_{ms}^j)\frac{dK_j^{-n-1,s}}{de^2}]\exp[i(j\lambda - m\theta)]\Big\}$$

$$\frac{\partial U}{\partial k} = \text{Re}\Big\{\frac{\mu}{a} \sum_{j,m,s,n} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m P_\ell^{vw}(C_{mn} - iS_{nm})$$

$$[K_j^{-n-1,s}\left(\frac{\partial G_{ms}^j}{\partial k} + i\frac{\partial H_{ms}^j}{\partial k}\right) + 2k(G_{ms}^j + H_{ms}^j)\frac{dK_j^{-n-1,s}}{de^2}]\exp[i(j\lambda - m\theta)]\Big\} \qquad (4)$$

$$\frac{\partial U}{\partial \lambda} = \text{Re}\Big\{\frac{\mu i}{a} \sum_{j,m,s,n} j\left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}(G_{ms}^j + iH_{ms}^j)(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)]\Big\}$$

$$\frac{\partial U}{\partial \alpha} = \text{Re}\Big\{\frac{\mu}{a} \sum_{j,m,s,n} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}\left(\frac{\partial G_{ms}^j}{\partial \alpha} + i\frac{\partial H_{ms}^j}{\partial \alpha}\right)(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)]\Big\}$$

$$\frac{\partial U}{\partial \beta} = \text{Re}\Big\{\frac{\mu}{a} \sum_{j,m,s,n} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}\left(\frac{\partial G_{ms}^j}{\partial \beta} + i\frac{\partial H_{ms}^j}{\partial \beta}\right)(C_{nm} - iS_{nm})\exp[i(j\lambda - m\theta)]\Big\}$$

$$\frac{\partial U}{\partial \gamma} = \text{Re}\Big\{\frac{\mu}{a} \sum_{j,m,s,n} \left(\frac{R}{a}\right)^n I^m V_{ns}^m K_j^{-n-1,s}(G_{ms}^j + iH_{ms}^j)(C_{nm} - iS_{nm})$$

$$\left(P_\ell^{vw}\frac{d\Gamma_{ns}^m}{d\gamma} + \Gamma_{ns}^m\frac{dP_\ell^{vw}}{d\gamma}\right)\exp[i(j\lambda - m\theta)]\Big\}$$

Here we have obtained the partial derivatives with respect to h and k of $K_0^{-n-1,s}(e^2)$ from the chain rule and the relation $h^2 + k^2 = e^2$, where e is the orbital eccentricity.

Recursion formulas for $\frac{dK_j^{ns}}{de^2}$ are obtained by differentiating the expansion (2.7.3-10):

$$\frac{dK_j^{ns}}{de^2} = -\frac{(n+\frac{3}{2})}{(1-e^2)}K_j^{ns} + (1-e^2)^{n+\frac{3}{2}} \sum_{\alpha=1}^{\infty} \alpha Y_{\alpha+a,\alpha+b}^{ns} e^{2(\alpha-1)} \qquad (5)$$

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Recursion formulas for $\frac{dP_\ell^{vw}}{d\gamma}$ are obtained by differentiating the recursion formulas (2.7.4-2, 3):

$$2\ell(\ell + v + w)(2\ell + v + w - 2)\frac{dP_\ell^{vw}}{d\gamma}(\gamma) =$$

$$(2\ell + v + w - 1)[(2\ell + v + w)(2\ell + v + w - 2)\gamma + v^2 - w^2]\frac{dP_{\ell-1}^{vw}}{d\gamma}(\gamma)$$

$$-2(\ell + v - 1)(\ell + w - 1)(2\ell + v + w)\frac{dP_{\ell-2}^{vw}}{d\gamma}(\gamma)$$

$$+(2\ell + v + w - 1)(2\ell + v + w)(2\ell + v + w - 2)P_{\ell-1}^{vw}(\gamma)$$

$$\frac{dP_0^{vw}}{d\gamma} = 0$$

$$\frac{dP_{-1}^{vw}}{d\gamma} = 0$$

$$(6)$$

Recursion formulas for the partial derivatives of G_{ms}^j and H_{ms}^j are obtained by differentiating (2.5.3-5) and (2.7.5-1, 2):

$$\frac{\partial G_{ms}^j}{\partial k} = \begin{cases} |s - j|C_{|s-j|-1}(k,h)C_{m-Is}(\alpha, \beta) - I(s - j)S_{|s-j|-1}(k,h)S_{m-Is}(\alpha, \beta) \ \text{ for } |s| \leq m \\ |s - j|C_{|s-j|-1}(k,h)C_{|s-Im|}(\alpha, \beta) - (s - j)\text{sgn}(m - s)S_{|s-j|-1}(k,h)S_{|s-Im|}(\alpha, \beta) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } |s| \geq m \end{cases}$$

$$(7)$$

etc. Formulas for $\frac{d\Gamma_{ns}^m}{d\gamma}$ are obtained by differentiating (2.7.1-13).

In order to update the geopotential coefficients C_{nm} and S_{nm} in a differential corrections procedure, it is necessary to compute the partial derivatives with respect to the coefficients of the mean element rates (the $\frac{\partial \dot{a}_j}{\partial c_i}$ in the matrix $\mathbf{F}$ defined by (2.6-7)). These are easily obtained by partial differentiating (1) and (4) with respect to C_{nm} and S_{nm}. Introducing the parameter

$$\zeta = \begin{cases} 1 & \text{if } \max(2, m, |s|) \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$(8)$$

we can write the results in the compact form

$$\frac{\partial \dot{a}}{\partial C_{nm}} + i\frac{\partial \dot{a}}{\partial S_{nm}} = \frac{2\mu i}{A} \sum_{\substack{j=-\infty \\ (j,m)\in\mathcal{B}}}^{\infty} \sum_{s=-N}^{N} \zeta j \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw}(G_{ms}^j + iH_{ms}^j)\exp[i(j\lambda - m\theta)]$$

$$(9)$$

etc.

If we assume that the orbital eccentricity is zero, the averaged equations of motion for the resonant tesserals significantly simplify. See [Collins and Cefola, 1978].

3.4 Atmospheric Drag

For atmospheric drag, the appropriate averaging operator is (2.4-10), and the first-order mean element rates are obtained by substituting (2.2-5) into (2.4-18). To avoid having to

solve Kepler's equation, and to smooth the perturbation around perigee, we can convert the integrals over the mean longitude λ into integrals over the eccentric longitude F or true longitude L by use of (2.5.2-6) or (2.5.3-14). The first-order contribution of drag to the averaged equations of motion can then be written in either of the forms

$$\frac{da_i}{dt} = \frac{1}{2\pi} \int_{F_1}^{F_2} \left(\frac{r}{a}\right) \left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q}\right) dF \tag{1a}$$

or

$$\frac{da_i}{dt} = \frac{1}{2\pi\sqrt{1 - h^2 - k^2}} \int_{L_1}^{L_2} \left(\frac{r}{a}\right)^2 \left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q}\right) dL \tag{1b}$$

The quantities $\frac{r}{a}$ and $\frac{\partial a_i}{\partial \dot{\mathbf{r}}}$ are given in terms of the equinoctial elements by equations (2.1.4-1, 6, 7, 8, 9), (2.1.6-1), and (2.1.7-3).

The limits (F_1, F_2) in (1a) or (L_1, L_2) in (1b) indicate the values of F or L at atmosphere entry and exit. If the satellite enters and leaves the atmosphere at a critical distance $\overline{r}$ from the center of the central body, then

$$\begin{aligned} F_1 &= -\overline{E} + \omega + I\Omega \\ F_2 &= \overline{E} + \omega + I\Omega \end{aligned} \tag{2a}$$

where

$$\overline{E} = \arccos\left[\frac{1 - \frac{\overline{r}}{a}}{e}\right]$$

or

$$\begin{aligned} L_1 &= -\overline{f} + \omega + I\Omega \\ L_2 &= \overline{f} + \omega + I\Omega \end{aligned} \tag{2b}$$

where

$$\overline{f} = \arccos\left[\frac{\frac{a(1-e^2)}{\overline{r}} - 1}{e}\right]$$

Of course, if the satellite remains totally within the atmosphere, the limits of integration in (1) can be taken to be $(-\pi, \pi)$.

The perturbing acceleration due to atmospheric drag is commonly modeled by the formula [Escobal, 1965]:

$$\mathbf{q} = \frac{C_D A}{2m} \rho |\mathbf{v} - \dot{\mathbf{r}}|(\mathbf{v} - \dot{\mathbf{r}}) \tag{3}$$

Here

$\quad C_D =$ drag coefficient of satellite
$\quad\quad$ (Assuming total specular reflection,
$\quad\quad C_D = 2$ for a sphere,
$\quad\quad C_D = 4$ for a flat plate perpendicular to $(\mathbf{v} - \dot{\mathbf{r}})$.)
$\quad A =$ cross sectional area of satellite
$\quad m =$ mass of satellite
$\quad \rho =$ density of atmosphere

$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = $ velocity of satellite

$\mathbf{v} = $ velocity of atmosphere

If we assume that the atmosphere rotates with an angular rate equal to the angular velocity ω of the central body, then $\mathbf{v} = \omega \times \mathbf{r}$. The vector $\mathbf{q}$ is resolved along the (x, y, z) axes of Figures 1 and 2 for use in the quadratures (1).

The developers of SST have used various density models for the upper atmosphere of the Earth. One of these is the modified Harris-Priester atmosphere (described in [Long, Capellari, Velez, and Fuchs, 1989]):

$$\rho = \rho_{\min} + (\rho_{\max} - \rho_{\min}) \cos^N\left(\frac{\phi_b}{2}\right) \tag{4}$$

where

$$\rho_{\min}(H) = \rho_{\min_1} \exp\left(\frac{H_1 - H}{H_{\min}}\right)$$

$$\rho_{\max}(H) = \rho_{\max_1} \exp\left(\frac{H_1 - H}{H_{\max}}\right)$$

$$H_{\min} = \frac{H_1 - H_2}{\ln\left(\frac{\rho_{\min_2}}{\rho_{\min_1}}\right)}$$

$$H_{\max} = \frac{H_1 - H_2}{\ln\left(\frac{\rho_{\max_2}}{\rho_{\max_1}}\right)}$$

Here $\rho_{\min}(H)$ and $\rho_{\max}(H)$ denote the minimum and maximum densities at a height H above a reference ellipsoid ($H_1 \le H \le H_2$), H_1, H_2, $\rho_{\min_1}$, $\rho_{\max_1}$, $\rho_{\min_2}$, $\rho_{\max_2}$, N are constants whose values are available from Tables, and ϕ_b denotes the angle between the diurnal bulge and the satellite. If $\mathbf{b}$ denotes a unit vector pointing from the Earth's center to the diurnal bulge, then

$$\cos \phi_b = \frac{\mathbf{b} \cdot \mathbf{r}}{r} \tag{5}$$

The diurnal bulge follows the path of the Sun but, because of the Earth's rotation, lags the sub-solar point by an angle θ_b (approximately 30° at 2 P.M. local time). We can obtain the vector $\mathbf{b}$ by rotating the vector $\mathbf{z}_B$ (pointing from the Earth's center to the sun) through an angle θ_b about the Earth's axis of rotation. Letting $\mathbf{R}$ denote the 3x3 matrix whose elements are direction cosines between the (x, y, z) axes and an Earth-fixed set of Cartesian axes, we can write the transformation law between the (x, y, z) components of $\mathbf{b}$ and $\mathbf{z}_B$ as

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{R} \begin{bmatrix} \cos \theta_b & -\sin \theta_b & 0 \\ \sin \theta_b & \cos \theta_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{R}^T \begin{bmatrix} z_{Bx} \\ z_{By} \\ z_{Bz} \end{bmatrix} \tag{6}$$

Here $\mathbf{R}^T$ denotes the transpose of the matrix $\mathbf{R}$ (see [Danielson, 1991]).

3.5 Solar Radiation Pressure

The general equations expressing the first-order contribution of solar radiation pressure to the averaged equations of motion are formally identical to the equations (3.4-1) for atmospheric drag.

The limits (F_1, F_2) in (1a) or (L_1, L_2) in (1b) now indicate the values of F or L at shadow exit and entry. If we assume the shadow is a circular cylinder in shape, the shadow exit and entry longitudes are determined by the shadow equation (as explained in [Escobal, 1965] and [Cefola, Long, and Holloway, 1974]):

$$S = 0 \tag{1}$$

Here

$$S = 1 - \mathcal{M}(1 + k\cos L + h\sin L)^2 - (\alpha\cos L + \beta\sin L)^2$$

$$\mathcal{M} = \frac{R_\oplus^2}{a^2(1 - h^2 - k^2)}$$

$$\alpha = \frac{\mathbf{R}_3}{R_3}\cdot\mathbf{f}$$

$$\beta = \frac{\mathbf{R}_3}{R_3}\cdot\mathbf{g}$$

where $R_\oplus$ is the central-body radius, and $\mathbf{R}_3$ is the vector from the center of mass of the central body to the sun. To obtain the solutions to equation (1), the following quartic equation must be solved:

$$\mathcal{A}_0\cos^4 L + \mathcal{A}_1\cos^3 L + \mathcal{A}_2\cos^2 L + \mathcal{A}_3\cos L + \mathcal{A}_4 = 0 \tag{2}$$

where

$$
\begin{aligned}
\mathcal{A}_0 &= 4\mathcal{B}^2 + \mathcal{C}^2 \\
\mathcal{A}_1 &= 8\mathcal{B}\,\mathcal{M}h + 4\mathcal{C}\,\mathcal{M}k \\
\mathcal{A}_2 &= -4\mathcal{B}^2 + 4\mathcal{M}^2h^2 - 2\mathcal{D}\,\mathcal{C} + 4\mathcal{M}^2k^2 \\
\mathcal{A}_3 &= -8\mathcal{B}\,\mathcal{M}h - 4\mathcal{D}\mathcal{M}k \\
\mathcal{A}_4 &= -4\mathcal{M}^2h^2 + \mathcal{D}^2 \\
\mathcal{B} &= \alpha\beta + \mathcal{M}hk \\
\mathcal{C} &= \alpha^2 - \beta^2 + \mathcal{M}(k^2 - h^2) \\
\mathcal{D} &= 1 - \beta^2 - \mathcal{M}(1 + h^2)
\end{aligned}
$$

The real roots of (2) must be sorted to eliminate extraneous roots and to determine the exit and entry values of true longitude L. The correct values of L must satisfy (1) as well as the condition

$$\frac{\mathbf{R}_3}{R_3}\cdot\frac{\mathbf{r}}{r} = \cos\phi = \alpha\cos L + \beta\sin L < 0 \tag{3}$$

At exit from shadow

$$\frac{\partial \mathcal{S}}{\partial L} > 0 \qquad (4)$$

while at entry into shadow

$$\frac{\partial \mathcal{S}}{\partial L} < 0 \qquad (5)$$

Of course, if the satellite remains totally within sunlight, the limits of integration in (3.4-1) can be taken to be $(-\pi, \pi)$.

The perturbing acceleration due to solar radiation pressure is [Cefola, 1982]:

$$\mathbf{q} = \frac{C_R A}{2m} \frac{\mathcal{L}}{c} R_\odot^2 \frac{(\mathbf{r} - \mathbf{R}_3)}{|\mathbf{r} - \mathbf{R}_3|^3} \qquad (6)$$

Here

C_R = radiation pressure coefficient of satellite

          (Assuming total specular reflection,

          $C_R = 2$ for a spherical mirror or black body,

          $C_R = 4$ for a flat mirror perpendicular to $(\mathbf{r} - \mathbf{R}_3)$.)

A = cross sectional area of satellite

m = mass of satellite

$\mathcal{L}$ = mean solar flux at surface of sun

c = speed of light

$R_\odot$ = radius of sun

$\mathbf{r} - \mathbf{R}_3$ = position vector from sun to satellite

If we suppose that the satellite is always in sunlight, and that the parameter

$$\mathcal{T} = \frac{C_R A}{2m} \frac{\mathcal{L}}{c} R_\odot^2 \qquad (7)$$

is constant, we can derive (6) from the disturbing function

$$\mathcal{R} = -\frac{\mathcal{T}}{|\mathbf{R}_3 - \mathbf{r}|} \qquad (8)$$

Use of the expansion (2.8-2) then leads to

$$\mathcal{R} = -\frac{\mathcal{T}}{R_3} \sum_{n=1}^{N} \left(\frac{r}{R_3}\right)^n P_n(\cos \phi) \qquad (9)$$

The radiation disturbing function (9) is identical to the third-body disturbing function (2.8-3), except that the factor μ_3 is replaced by $-\mathcal{T}$ and the summation starts from $n = 1$. Hence we can immediately write down the mean radiation disturbing function by analogy with (3.2-1):

$$U = -\frac{\mathcal{T}}{R_3} \sum_{s=0}^{N} \sum_{n=\max(1,s)}^{N} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} G_s \qquad (10)$$

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If we retain only the first nonzero term in the expansion (10), the mean radiation disturbing function simplifies to

$$U = \mathcal{V}(k\alpha + h\beta) \tag{11}$$

where

$$\mathcal{V} = \frac{3\mathcal{T}a}{2R_3^2}$$

The contribution of (11) to the averaged equations of motion (3.1-1) is

$$
\begin{aligned}
\frac{da}{dt} &= 0 \\
\frac{dh}{dt} &= \frac{B\mathcal{V}\alpha}{A} - \frac{\mathcal{V}k\gamma}{AB}(kp - Ihq) \\
\frac{dk}{dt} &= -\frac{B\mathcal{V}\beta}{A} + \frac{\mathcal{V}h\gamma}{AB}(kp - Ihq) \\
\frac{dp}{dt} &= \frac{C\mathcal{V}h\gamma}{2AB} \\
\frac{dq}{dt} &= \frac{IC\mathcal{V}k\gamma}{2AB} \\
\frac{d\lambda}{dt} &= -\frac{(2 + B)\mathcal{V}(k\alpha + h\beta)}{A(1 + B)} - \frac{\mathcal{V}\gamma}{AB}(kp - Ihq)
\end{aligned}
\tag{12}
$$

4   First-Order Short-Periodic Variations

Knowing Fourier series expansions for the osculating rate functions $F_{i\alpha}$ or the osculating disturbing function $\mathcal{R}$, we can construct the first-order short-periodic variations $\eta_{i\alpha}$ from the results in Section 2.5. In this chapter we record the specific forms of the expansions for each of several perturbations.

4.1   Central-Body Gravitational Zonal Harmonics

For the central-body gravitational zonal harmonics, the appropriate disturbing function $\mathcal{R}$ is (2.7.1-16) with $m = 0$. From the results in Sections 2.5.1 or 2.5.5, we can construct a Fourier series expansion in the mean longitude λ for the first-order short-periodic variations $\eta_{i\alpha}$, as was done by Green [1979].

However, for the central-body zonal harmonics, it is possible to construct a *finite modified* Fourier series in the true longitude L for the $\eta_{i\alpha}$. For single-averaged perturbing forces which increase rapidly as the satellite approaches the central body, notably central-body zonal harmonics and atmospheric drag, a Fourier series expansion in L will converge faster than equivalent expansions in F or λ when the eccentricity of the satellite orbit is large. This happens because the magnitude of such a perturbation is strongly peaked around perigee when considered as a function of λ. Transforming the independent variable to L broadens the peak considerably, making it easier to approximate with a finite Fourier series. In this Section we outline the construction of this most desirable expansion in L. Further details may

be found in [Cefola and McClain, 1978], [Kaniecki, 1979], [McClain and Slutsky, 1980], and [Slutsky, 1980].

The disturbing function less it's mean can be written as

$$\mathcal{R} - U = \text{Re}\left\{ -\frac{\mu}{a} \sum_{n=2}^{N} \sum_{s=0}^{n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} Q_{ns} [C_s(\alpha, \beta) + i S_s(\alpha, \beta)] \right.$$
$$\left. \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} Y_j^{-n-1,-s} \exp(ij\lambda) \right\} \tag{1}$$

Here

$$
\begin{array}{llll}
J_n = -C_{n0} & = & \text{geopotential coefficients} \\
V_{ns} & = & \text{coefficients calculated from (2.8.2-1,2)} \\
Q_{ns}(\gamma) & = & \text{polynomials calculated from (2.8.3-2, 3)} \\
C_s(\alpha, \beta), \ S_s(\alpha, \beta) & = & \text{polynomials calculated from (2.5.3-6)} \\
Y_j^{-n-1,-s} & = & \text{coefficients of the expansion (2.7.1-8):}
\end{array}
$$

$$\left(\frac{a}{r}\right)^{n+1} \exp(-isL) = \sum_{j=-\infty}^{\infty} Y_j^{-n-1,-s} \exp(ij\lambda) \tag{2}$$

The short-periodic generating function (2.5.5-4, 5) is easily obtained by integrating (1):

$$S = \text{Re}\left\{ -\frac{\mu}{a} \sum_{n=2}^{N} \sum_{s=0}^{n} (2 - \delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} Q_{ns} [C_s(\alpha, \beta) + i S_s(\alpha, \beta)] \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \frac{Y_j^{-n-1,-s} \exp(ij\lambda)}{ij} \right\} \tag{3}$$

The infinite series in the mean longitude λ in (3) can be replaced by a finite modified series in the true longitude L. To see this, first integrate both sides of (2) with respect to λ to obtain

$$\sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \frac{Y_j^{-n-1,-s} \exp(ij\lambda)}{ij} = \int^{\lambda} \left(\frac{a}{r}\right)^{n+1} \exp(-isL) d\lambda - \lambda Y_0^{-n-1,-s} \tag{4}$$

Next perform the integral in (4) by using the expansion

$$\left(\frac{a}{r}\right)^n = \sqrt{1 - h^2 - k^2} \sum_{j=-n}^{n} Y_0^{-n-2,-j} \exp(ijL) \tag{5}$$

and the change of variable (from 2.5.3-14)

$$d\lambda = \frac{1}{\sqrt{1 - h^2 - k^2}} \left(\frac{r}{a}\right)^2 dL \tag{6}$$

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The infinite series then becomes

$$
\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{Y_j^{-n-1,-s}\exp(ij\lambda)}{ij} = \sum_{\substack{j=-(n-1) \\ j\neq s}}^{n-1} \frac{Y_0^{-n-1,-j}\exp[i(j-s)L]}{i(j-s)} + Y_0^{-n-1,-s}(L-\lambda) \tag{7}
$$

Replacing the infinite series in (3) with (7), and introducing the kernels $K_0^{-n-1,j}$ of the Hansen coefficients through (2.7.1-11), we obtain

$$
S = U(L-\lambda) - \mathrm{Re}\left\{ \frac{\mu}{a} \sum_{n=2}^{N} \sum_{s=0}^{n} (2-\delta_{0s}) \left(\frac{R}{a}\right)^n J_n V_{ns} Q_{ns} \right.
$$
$$
\left. \sum_{\substack{j=-(n-1) \\ j\neq s}}^{n-1} \frac{[C_s(\alpha,\beta)+iS_s(\alpha,\beta)][C_{|j|}(k,h)-i\mathrm{sgn}(j)S_{|j|}(k,h)]K_0^{-n-1,j}\exp[i(j-s)L]}{i(j-s)} \right\} \tag{8}
$$

Since all of the symbols in (8) except for $i=\sqrt{-1}$ are real, we can easily cast S into the real form

$$
S = U(L-\lambda) - \frac{\mu}{a} \sum_{n=2}^{N} \sum_{s=0}^{n} (2-\delta_{0s}) J_n L_n^s \left\{ \sum_{\substack{j=0 \\ j\neq s}}^{n-1} K_0^{-n-1,j} \left[\frac{H_{js}\cos(j-s)L + G_{js}\sin(j-s)L}{j-s}\right] \right.
$$
$$
\left. + \sum_{j=1}^{n-1} K_0^{-n-1,j} \left[\frac{-I_{js}\cos(j+s)L + J_{js}\sin(j+s)L}{j+s}\right] \right\} \tag{9}
$$

where

$$
\begin{aligned}
L_n^s(\gamma) &= \left(\frac{R}{a}\right)^n V_{ns} Q_{ns}(\gamma) \\
G_{js} &= C_j(k,h)C_s(\alpha,\beta) + S_j(k,h)S_s(\alpha,\beta) \\
H_{js} &= C_j(k,h)S_s(\alpha,\beta) - S_j(k,h)C_s(\alpha,\beta) \\
I_{js} &= C_j(k,h)S_s(\alpha,\beta) + S_j(k,h)C_s(\alpha,\beta) \\
J_{js} &= C_j(k,h)C_s(\alpha,\beta) - S_j(k,h)S_s(\alpha,\beta)
\end{aligned} \tag{10}
$$

The last step is a redefinition of indices, so as to isolate the coefficients of $\cos jL$ and $\sin jL$, and a rearrangement of the order of summation. The short-periodic generating function can then finally be written as the finite modified Fourier series

$$
S = \mathcal{C}^0 + U(L-\lambda) + \sum_{j=1}^{2N-1} (\mathcal{C}^j \cos jL + \mathcal{S}^j \sin jL) \tag{11}
$$

Here the coefficient $\mathcal{C}^0$ is

$$
\mathcal{C}^0 = - \sum_{j=1}^{2N-1} (\mathcal{C}^j \rho_j + \mathcal{S}^j \sigma_j) \tag{12}
$$

where ρ_j and σ_j are given by (2.5.3-3, 4). The coefficients $\mathcal{C}^j$ are

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$$\mathcal{C}^j = -\frac{\mu}{aj}\left\{\mathcal{I}_1^{N-1}(j)\left[\sum_{s=j}^{N-1}\sum_{n=s+1}^{N}(2-\delta_{0,s-j})J_nH_{s,s-j}K_0^{-n-1,s}L_n^{s-j}\right.\right. \tag{13}$$

$$\left.\left.-\sum_{s=0}^{N-j}\sum_{n=\max(j+s,j+1)}^{N}(2-\delta_{0,j+s})J_nH_{s,j+s}K_0^{-n-1,s}L_n^{j+s}\right]-2\mathcal{I}_2^N(j)J_jH_{0,j}K_0^{-j-1,0}L_j^j\right\}$$

$$+\frac{\mu}{aj}\left\{\mathcal{I}_2^N(j)\left[\sum_{s=1}^{j-1}(2-\delta_{0,j-s})J_jI_{s,j-s}K_0^{-j-1,s}L_j^{j-s}\right]\right.$$

$$\left.+\mathcal{I}_1^{N-1}(j)\left[\sum_{s=1}^{j}\sum_{n=j+1}^{N}(2-\delta_{0,j-s})J_nI_{s,j-s}K_0^{-n-1,s}L_n^{j-s}\right]+\mathcal{I}_3^{2N-1}(j)\textstyle\sum_1^j\right\}$$

where

$$\sum_1^j = \sum_{s=j-\min(j-1,N)}^{\frac{j}{2}-1}\sum_{n=j-s}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nI_{s,j-s}K_0^{-n-1,s}L_n^{j-s}$$

$$+\sum_{s=\frac{j}{2}}^{\min(j-1,N)-1}\sum_{n=s+1}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nI_{s,j-s}K_0^{-n-1,s}L_n^{j-s}\qquad\text{for } j \text{ even} \tag{14a}$$

$$\sum_1^j = \sum_{s=\frac{j-1}{2}}^{\min(j-1,N)-1}\sum_{n=s+1}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nI_{s,j-s}K_0^{-n-1,s}L_n^{j-s}$$

$$+\begin{cases}0 & \text{for } N=2,3\\[2em]\mathcal{I}_5^{2N-3}(j)\displaystyle\sum_{s=j-\min(j-1,N)}^{\frac{j-3}{2}}\sum_{n=j-s}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nI_{s,j-s}K_0^{-n-1,s}L_n^{j-s} & \text{for } N\geq4\end{cases}\quad\text{for } j \text{ odd}$$

$$\tag{14b}$$

Similarly, the coefficients $\mathcal{S}^j$ are

$$\mathcal{S}^j = -\frac{\mu}{aj}\left\{\mathcal{I}_1^{N-1}(j)\left[\sum_{s=j}^{N-1}\sum_{n=s+1}^{N}(2-\delta_{0,s-j})J_nG_{s,s-j}K_0^{-n-1,s}L_n^{s-j}\right.\right.$$

$$\left.\left.+\sum_{s=0}^{N-j}\sum_{n=\max(j+s,j+1)}^{N}(2-\delta_{0,j+s})J_nG_{s,j+s}K_0^{-n-1,s}L_n^{j+s}\right]+2\mathcal{I}_2^N(j)J_jG_{0,j}K_0^{-j-1,0}L_j^j\right\}$$

$$\tag{15}$$

$$-\frac{\mu}{aj}\left\{\mathcal{I}_2^N(j)\left[\sum_{s=1}^{j-1}(2-\delta_{0,j-s})J_jJ_{s,j-s}K_0^{-j-1,s}L_j^{j-s}\right]\right.$$

$$\left.+\mathcal{I}_1^{N-1}(j)\left[\sum_{s=1}^{j}\sum_{n=j+1}^{N}(2-\delta_{0,j-s})J_nJ_{s,j-s}K_0^{-n-1,s}L_n^{j-s}\right]+\mathcal{I}_3^{2N-1}(j)\Sigma_2^j\right\}$$

where

$$\Sigma_2^j = \sum_{s=j-\min(j-1,N)}^{\frac{j}{2}-1}\sum_{n=j-s}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nJ_{s,j-s}K_0^{-n-1,s}L_n^{j-s}$$

$$+\sum_{s=\frac{j}{2}}^{\min(j-1,N)-1}\sum_{n=s+1}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nJ_{s,j-s}K_0^{-n-1,s}L_n^{j-s}\qquad\text{for }j\text{ even}$$

(16a)

$$\Sigma_2^j = \sum_{s=\frac{j-1}{2}}^{\min(j-1,N)-1}\sum_{n=s+1}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nJ_{s,j-s}K_0^{-n-1,s}L_n^{j-s}$$

$$+\begin{cases}0\qquad\text{for }N=2,3\\[2em]\mathcal{I}_5^{2N-3}(j)\displaystyle\sum_{s=j-\min(j-1,N)}^{\frac{j-3}{2}}\sum_{n=j-s}^{\min(j-1,N)}(2-\delta_{0,j-s})J_nJ_{s,j-s}K_0^{-n-1,s}L_n^{j-s}\text{for }N\geq 4\end{cases}\qquad\text{for }j\text{ odd}$$

(16b)

Note that the first index of the G, H, I, J polynomials defined in (10) indicates their degree in the eccentricity; for small eccentricity orbits, the series (14)–(16) may be truncated by prescribing the maximum possible value of s.

The first-order short-periodic variations $\eta_{i\alpha}$ generated by the function S given by (11) can be derived using equations (2.2-10), (2.5.2-7), (2.5.5-10), and the following:

$$\frac{\partial L}{\partial h} = -\frac{1}{B^3}\left\{\frac{3}{2}kb+kb\left(1+\frac{h^2+k^2}{2}\right)+2(B+k^2b)\cos L\right.$$

$$\left.+2hkb\sin L+\frac{k}{2}[B+(k^2-h^2)b]\cos 2L+\frac{h}{2}(B+2k^2b)\sin 2L\right\}$$

$$\frac{\partial L}{\partial k} = \frac{1}{B^3}\left\{\frac{3}{2}hB+hb\left(1+\frac{h^2+k^2}{2}\right)+2hkb\cos L\right.$$

(17)

$$\left.+2(B+h^2b)\sin L+\frac{h}{2}[-B+(k^2-h^2)b]\cos 2L+\frac{k}{2}(B+2h^2b)\sin 2L\right\}$$

$$\frac{\partial L}{\partial \lambda} = \frac{1}{B^3}\left\{\frac{2+h^2+k^2}{2}+2k\cos L+2h\sin L+\frac{k^2-h^2}{2}\cos 2L+hk\sin 2L\right\}$$

Here, as usual, h and k are equinoctial orbital elements and b and B are defined by (2.1.4-4) and (2.1.6-1b).

In the absence of explicit time-dependence, the first-order short-periodic variations can be written as the finite modified Fourier series

$$\eta_{i\alpha} = C_i^0 + D_i(L - \lambda) + \sum_{j=1}^{2N+1} (C_i^j \cos jL + S_i^j \sin jL) \tag{18}$$

(Remember that the equation of the center $(L - \lambda)$ may be calculated from (2.5.3-2,3,4).) Expressions for the coefficients in (18) are given below in terms of the following quantities:

1. The equinoctial elements (a, h, k, p, q) and the retrograde factor I (equation (2.1.2-2)).

2. The direction cosines (α, β, γ) of the perturbation symmetry axis in the equinoctial reference frame (equations (2.1.9-1)).

3. The Kepler mean motion n (equation (2.1.4-3)).

4. The auxiliary parameter χ (equation (2.7.3-4)).

5. The mean disturbing function U of the perturbation (equation (3.1-3)).

6. The coefficients $\mathcal{C}^j$ and $\mathcal{S}^j$ of the modified Fourier series expansion in L for the short-periodic generating function S (equations (13)–(20)).

7. The cross-derivative operator (equation (2.2-8)).

8. The inclusion operator (equation (2.5.3-18)).

The constant terms in the expansions (18) are given by:

$$C_i^0 = - \sum_{j=1}^{2N+1} (C_i^j \rho_j + S_i^j \sigma_j) \tag{19}$$

The short-periodic coefficients for the semimajor axis a are given by:

$$
\begin{aligned}
C_1^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi^3}{n^2a}\Big(4kU - hk\mathcal{C}^1 + \frac{k^2-h^2}{2}\mathcal{S}^1\Big)\Big] \\
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi^3}{n^2a}(k^2-h^2)U\Big] \\
&\quad +\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi^3}{n^2a}(j+2)\Big(-hk\mathcal{C}^{j+2} + \frac{k^2-h^2}{2}\mathcal{S}^{j+2}\Big)\Big] \\
&\quad +\mathcal{I}_1^{2N-2}(j)\Big[\frac{2\chi^3}{n^2a}(j+1)\Big(-h\mathcal{C}^{j+1} + k\mathcal{S}^{j+1}\Big)\Big] \\
&\quad +\mathcal{I}_1^{2N-1}(j)\Big[\frac{3\chi^3-\chi}{n^2a}j\mathcal{S}^j\Big] \\
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{2\chi^3}{n^2a}(j-1)\Big(h\mathcal{C}^{j-1} + k\mathcal{S}^{j-1}\Big)\Big] \\
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi^3}{n^2a}(j-2)\Big(hk\mathcal{C}^{j-2} + \frac{k^2-h^2}{2}\mathcal{S}^{j-2}\Big)\Big] \\
S_1^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi^3}{n^2a}\Big(4hU + \frac{k^2-h^2}{2}\mathcal{C}^1 + hk\mathcal{S}^1\Big)\Big] \\
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi^3}{n^2a}2hkU\Big] \\
&\quad -\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi^3}{n^2a}(j+2)\Big(\frac{k^2-h^2}{2}\mathcal{C}^{j+2} + hk\mathcal{S}^{j+2}\Big)\Big] \\
&\quad -\mathcal{I}_1^{2N-2}(j)\Big[\frac{2\chi^3}{n^2a}(j+1)\Big(k\mathcal{C}^{j+1} + h\mathcal{S}^{j+1}\Big)\Big] \\
&\quad -\mathcal{I}_1^{2N-1}(j)\Big[\frac{3\chi^3-\chi}{n^2a}j\mathcal{C}^j\Big] \\
&\quad -\mathcal{I}_2^{2N}(j)\Big[\frac{2\chi^3}{n^2a}(j-1)\Big(k\mathcal{C}^{j-1} - h\mathcal{S}^{j-1}\Big)\Big] \\
&\quad -\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi^3}{n^2a}(j-2)\Big(\frac{k^2-h^2}{2}\mathcal{C}^{j-2} + hk\mathcal{S}^{j-2}\Big)\Big] \\
D_1 &= 0
\end{aligned}
\tag{20}
$$

The short-periodic coefficients for element h are given by:

$$
\begin{aligned}
C_2^j &= -\mathcal{I}_1^1(j)\Big[\frac{\chi}{4n^2a^2}\big(k\mathcal{C}^1 + h\mathcal{S}^1\big)\Big] \\
&\quad -\mathcal{I}_2^2(j)\Big[\frac{\chi}{2n^2a^2}hU\Big] \\
&\quad -\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi}{4n^2a^2}(j+2)\big(k\mathcal{C}^{j+2} + h\mathcal{S}^{j+2}\big)\Big] \\
&\quad -\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi}{n^2a^2}(j+1)\mathcal{C}^{j+1}\Big] \\
&\quad +\mathcal{I}_1^{2N-1}(j)\Big[\frac{k\chi}{n^2a^2}\big(p\mathcal{C}_{,\alpha\gamma}^j - Iq\mathcal{C}_{,\beta\gamma}^j\big) + \frac{1}{\chi n^2a^2}\frac{\partial \mathcal{C}^j}{\partial k} + \frac{3h\chi}{2n^2a^2}j\mathcal{S}^j\Big] \\
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{\chi}{n^2a^2}(j-1)\mathcal{C}^{j-1}\Big] \\
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi}{4n^2a^2}(j-2)\big(k\mathcal{C}^{j-2} - h\mathcal{S}^{j-2}\big)\Big] \\
S_2^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi}{4n^2a^2}\big(8U - h\mathcal{C}^1 + k\mathcal{S}^1\big)\Big] \\
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi}{2n^2a^2}kU\Big] \\
&\quad +\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi}{4n^2a^2}(j+2)\big(h\mathcal{C}^{j+2} - k\mathcal{S}^{j+2}\big)\Big] \\
&\quad -\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi}{n^2a^2}(j+1)\mathcal{S}^{j+1}\Big] \\
&\quad +\mathcal{I}_1^{2N-1}(j)\Big[\frac{k\chi}{n^2a^2}\big(p\mathcal{S}_{,\alpha\gamma}^j - Iq\mathcal{S}_{,\beta\gamma}^j\big) + \frac{1}{\chi n^2a^2}\frac{\partial \mathcal{S}^j}{\partial k} - \frac{3h\chi}{2n^2a^2}j\mathcal{C}^j\Big] \\
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{\chi}{n^2a^2}(j-1)\mathcal{S}^{j-1}\Big] \\
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi}{4n^2a^2}(j-2)\big(h\mathcal{C}^{j-2} + k\mathcal{S}^{j-2}\big) \\
D_2 &= \frac{1}{\chi n^2a^2}\frac{\partial U}{\partial k} + \frac{k\chi}{n^2a^2}\big(pU_{,\alpha\gamma} - IqU_{,\beta\gamma}\big)
\end{aligned}
\tag{21}
$$

The short-periodic coefficients for element k are given by:

$$
\begin{aligned}
C_3^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi}{4n^2a^2}\big(8U - h\mathcal{C}^1 + k\mathcal{S}^1\big)\Big] \\
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi}{2n^2a^2}kU\Big] \\
&\quad +\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi}{4n^2a^2}(j+2)\big(-h\mathcal{C}^{j+2} + k\mathcal{S}^{j+2}\big)\Big] \\
&\quad +\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi}{n^2a^2}(j+1)\mathcal{S}^{j+1}\Big] \\
&\quad -\mathcal{I}_1^{2N-1}(j)\Big[\frac{h\chi}{n^2a^2}\big(p\mathcal{C}_{,\alpha\gamma}^{j} - Iq\mathcal{C}_{,\beta\gamma}^{j}\big) + \frac{1}{\chi n^2a^2}\frac{\partial\mathcal{C}^j}{\partial h} - \frac{3k\chi}{2n^2a^2}j\mathcal{S}^j\Big] \\
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{\chi}{n^2a^2}(j-1)\mathcal{S}^{j-1}\Big] \\
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi}{4n^2a^2}(j-2)\big(h\mathcal{C}^{j-2} + k\mathcal{S}^{j-2}\big)\Big] \\
S_3^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi}{4n^2a^2}\big(k\mathcal{C}^1 + h\mathcal{S}^1\big)\Big] \\
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi}{2n^2a^2}hU\Big] \\
&\quad -\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi}{4n^2a^2}(j+2)\big(k\mathcal{C}^{j+2} + h\mathcal{S}^{j+2}\big)\Big] \\
&\quad -\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi}{n^2a^2}(j+1)\mathcal{C}^{j+1}\Big] \\
&\quad -\mathcal{I}_1^{2N-1}(j)\Big[\frac{h\chi}{n^2a^2}\big(p\mathcal{S}_{,\alpha\gamma}^{j} - Iq\mathcal{S}_{,\beta\gamma}^{j}\big) + \frac{1}{\chi n^2a^2}\frac{\partial\mathcal{S}^j}{\partial h} + \frac{3k\chi}{2n^2a^2}j\mathcal{C}^j\Big] \\
&\quad -\mathcal{I}_2^{2N}(j)\Big[\frac{\chi}{n^2a^2}(j-1)\mathcal{C}^{j-1}\Big] \\
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi}{4n^2a^2}(j-2)\big(-k\mathcal{C}^{j-2} + h\mathcal{S}^{j-2}\big)\Big] \\
D_3 &= -\frac{1}{\chi n^2a^2}\frac{\partial U}{\partial h} - \frac{h\chi}{n^2a^2}(pU_{,\alpha\gamma} - IqU_{,\beta\gamma})
\end{aligned}
\tag{22}
$$

The short-periodic coefficients for element p are given by:

$$
\begin{aligned}
C_4^j &= \mathcal{I}_1^{2N-1}(j)\frac{(1+p^2+q^2)\chi}{2n^2a^2}\Big[-\mathcal{C}_{,\beta\gamma}^{j} + p\big(\mathcal{C}_{,hk}^{j} - \mathcal{C}_{,\alpha\beta}^{j} - j\mathcal{S}^j\big)\Big] \\
S_4^j &= \mathcal{I}_1^{2N-1}(j)\frac{(1+p^2+q^2)\chi}{2n^2a^2}\Big[-\mathcal{S}_{,\beta\gamma}^{j} + p\big(\mathcal{S}_{,hk}^{j} - \mathcal{S}_{,\alpha\beta}^{j} + j\mathcal{C}^j\big)\Big] \\
D_4 &= -\frac{(1+p^2+q^2)\chi U_{,\beta\gamma}}{2n^2a^2}
\end{aligned}
\tag{23}
$$

The short-periodic coefficients for element q are given by:

$$
\begin{aligned}
C_5^j &= \mathcal{I}_1^{2N-1}(j)\frac{(1+p^2+q^2)\chi}{2n^2a^2}\Big[-I\mathcal{C}_{,\alpha\gamma}^{j} + q\big(\mathcal{C}_{,hk}^{j} - \mathcal{C}_{,\alpha\beta}^{j} - j\mathcal{S}^j\big)\Big] \\
S_5^j &= \mathcal{I}_1^{2N-1}(j)\frac{(1+p^2+q^2)\chi}{2n^2a^2}\Big[-I\mathcal{S}_{,\alpha\gamma}^{j} + q\big(\mathcal{S}_{,hk}^{j} - \mathcal{S}_{,\alpha\beta}^{j} + j\mathcal{C}^j\big)\Big] \\
D_5 &= -\frac{(1+p^2+q^2)\chi I U_{,\alpha\gamma}}{2n^2a^2}
\end{aligned}
\tag{24}
$$

The short-periodic coefficients for the mean longitude λ are given by:

$$
\begin{aligned}
C_6^j &= -\mathcal{I}_1^1(j)\Big[\frac{\chi^2}{2n^2a^2(1+\chi)}\Big(4hU + \frac{k^2-h^2}{2}\mathcal{C}^1 + hk\mathcal{S}^1\Big)\Big] \\[2mm]
&\quad -\mathcal{I}_2^2(j)\Big[\frac{\chi^2}{n^2a^2(1+\chi)}hkU\Big] \\[2mm]
&\quad -\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi^2(j+2)}{2n^2a^2(1+\chi)}\Big(\frac{k^2-h^2}{2}\mathcal{C}^{j+2} + hk\mathcal{S}^{j+2}\Big)\Big] \\[2mm]
&\quad -\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi^2(j+1)}{n^2a^2(1+\chi)}\Big(k\mathcal{C}^{j+1} + h\mathcal{S}^{j+1}\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_1^{2N-1}(j)\Big[-\frac{2}{n^2a}\frac{\partial\mathcal{C}^j}{\partial a} + \frac{1}{n^2a^2(1+\chi)}\Big(h\frac{\partial\mathcal{C}^j}{\partial h} + k\frac{\partial\mathcal{C}^j}{\partial k}\Big) + \frac{\chi}{n^2a^2}\Big(p\mathcal{C}_{,\alpha\gamma}^j - Iq\mathcal{C}_{,\beta\gamma}^j\Big) - \frac{3}{n^2a^2}\mathcal{C}^j\Big] \\[2mm]
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{\chi^2(j-1)}{n^2a^2(1+\chi)}\Big(k\mathcal{C}^{j-1} - h\mathcal{S}^{j-1}\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi^2(j-2)}{2n^2a^2(1+\chi)}\Big(\frac{k^2-h^2}{2}\mathcal{C}^{j-2} - hk\mathcal{S}^{j-2}\Big)\Big] \\[4mm]
S_6^j &= \mathcal{I}_1^1(j)\Big[\frac{\chi^2}{2n^2a^2(1+\chi)}\Big(4kU - hk\mathcal{C}^1 + \frac{k^2-h^2}{2}\mathcal{S}^1\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_2^2(j)\Big[\frac{\chi^2}{n^2a^2(1+\chi)}\frac{k^2-h^2}{2}U\Big] \\[2mm]
&\quad +\mathcal{I}_1^{2N-3}(j)\Big[\frac{\chi^2(j+2)}{2n^2a^2(1+\chi)}\Big(hk\mathcal{C}^{j+2} - \frac{k^2-h^2}{2}\mathcal{S}^{j+2}\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_1^{2N-2}(j)\Big[\frac{\chi^2(j+1)}{n^2a^2(1+\chi)}\Big(h\mathcal{C}^{j+1} - k\mathcal{S}^{j+1}\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_1^{2N-1}(j)\Big[-\frac{2}{n^2a}\frac{\partial\mathcal{S}^j}{\partial a} + \frac{1}{n^2a^2(1+\chi)}\Big(h\frac{\partial\mathcal{S}^j}{\partial h} + k\frac{\partial\mathcal{S}^j}{\partial k}\Big) + \frac{\chi}{n^2a^2}\Big(p\mathcal{S}_{,\alpha\gamma}^j - Iq\mathcal{S}_{,\beta\gamma}^j\Big) - \frac{3}{n^2a^2}\mathcal{S}^j\Big] \\[2mm]
&\quad +\mathcal{I}_2^{2N}(j)\Big[\frac{\chi^2(j-1)}{n^2a^2(1+\chi)}\Big(h\mathcal{C}^{j-1} + k\mathcal{S}^{j-1}\Big)\Big] \\[2mm]
&\quad +\mathcal{I}_3^{2N+1}(j)\Big[\frac{\chi^2(j-2)}{2n^2a^2(1+\chi)}\Big(hk\mathcal{C}^{j-2} + \frac{k^2-h^2}{2}\mathcal{S}^{j-2}\Big)\Big] \\[4mm]
D_6 &= -\frac{2}{n^2a}\frac{\partial U}{\partial a} + \frac{1}{n^2a^2(1+\chi)}\Big(h\frac{\partial U}{\partial h} + k\frac{\partial U}{\partial k}\Big) + \frac{\chi}{n^2a^2}\Big(pU_{,\alpha\gamma} - IqU_{,\beta\gamma}\Big)
\end{aligned}
\tag{25}
$$

Note that the D_i coefficients are simply related to the first-order mean element rates $A_{i\alpha}$ (given by the right sides of (3.1-1)):

$$D_i = \frac{A_{i\alpha}}{n} \tag{26}$$

From the central-body gravitational potential, there are three possible sources of explicit time-dependence:

1. Motion of the central-body symmetry axis. For the Earth, this is a combination of precession of the equinoxes, nutation, and polar motion.

2. Variations in the central-body rotation rate.

3. Tidal potential.

The principle effects of 1 and 2 are accounted for in SST by using at each time step the epoch triad $(\mathbf{x}_B, \mathbf{y}_B, \mathbf{z}_B)$ to evaluate the direction cosines (α, β, γ) from (2.1.9-1) and the rotation angle θ from (2.7.1-3,4). However, for the Earth the above sources are thought to be too small or two slowly varying to cause significant explicit time-dependence effects.

In order to update the orbital elements in a differential corrections procedure, it is necessary to compute the partial derivatives with respect to the mean elements of the short-periodic variations (the $\frac{\partial \eta_j}{\partial a_k}$ in (2.6-3, 4)). The partial derivatives of the J_2 contribution to (18) are currently available in the SST code, for the special case of zero eccentricity and replacement of (α, β, γ) with the explicit formulas (2.1.9-3) in p and q (thus motion of the central-body symmetry axis is neglected).

4.2 Third-Body Gravitational Potential

For a third-body point mass, the appropriate disturbing function $\mathcal{R}$ is (2.8.1-6). From the results in Sections (2.5.1) or (2.5.5), we can construct a Fourier series expansion in the mean longitude λ for the first-order short-periodic variations $\eta_{i\alpha}$, as was done by Green [1979].

However, for the third-body disturbing function, it is possible to construct a *finite* Fourier series in the eccentric longitude F for the $\eta_{i\alpha}$. Since the D'Alembert characteristics are bounded, this solution is of closed form in the eccentricity. In this Section we outline the construction of this most desirable expansion in F. Further details can be found in [McClain, 1978], [Cefola and McClain, 1978], [Slutsky and McClain, 1981], and [Slutsky, 1983].

The disturbing function less its mean can be written as

$$\mathcal{R} - U = \mathrm{Re}\left\{\frac{\mu_3}{R_3}\sum_{n=2}^{N}\sum_{s=0}^{n}(2-\delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns}Q_{ns}[C_s(\alpha,\beta)-iS_s(\alpha,\beta)]\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} Y_j^{ns}\exp(ij\lambda)\right\}$$
$$\tag{1}$$

Here

$$
\begin{array}{lcl}
V_{ns} & = & \text{coefficients calculated from (2.8.2-1, 2)} \\
Q_{ns}(\gamma) & = & \text{polynomials calculated from (2.8.3-2, 3)} \\
C_s(\alpha, \beta),\ S_s(\alpha, \beta) & = & \text{polynomials calculated from (2.5.3-6)} \\
Y_j^{ns} & = & \text{coefficients of the expansion (2.7.1-8)}
\end{array}
$$

The short-periodic generating function (2.5.5-4, 5) is easily obtained by integrating (1):

$$
S = \mathrm{Re}\left\{ \frac{\mu_3}{R_3} \sum_{n=2}^{N} \sum_{s=0}^{n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns}[C_s(\alpha, \beta) - iS_s(\alpha, \beta)] \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{Y_j^{ns}\exp(ij\lambda)}{ij} \right\} \quad (2)
$$

The infinite series in the mean longitude λ in (2) can be replaced by a finite series in the eccentric longitude F. To see this, first integrate both sides of (2.7.1-8) to obtain

$$
\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{Y_j^{ns}\exp(ij\lambda)}{ij} = \int^{\lambda} \left(\frac{r}{a}\right)^n \exp(isL)d\lambda - \lambda Y_0^{ns} \quad (3)
$$

Next perform the integral in (3) by using the expansion

$$
\left(\frac{r}{a}\right)^n \exp(isL) = \sum_{j=-n}^{n} W_j^{ns}\exp(ijF) \quad (4)
$$

where

$$
W_0^{ns} = Y_0^{n-1,s} \quad (5)
$$

and by using the change of variable (from 2.5.2-6))

$$
d\lambda = \left(\frac{r}{a}\right) dF \quad (6)
$$

The infinite series then becomes

$$
\sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{Y_j^{ns}\exp(ij\lambda)}{ij} = \sum_{\substack{j=-(n+1) \\ j\neq 0}}^{n+1} \frac{W_j^{n+1,s}\exp(ijF)}{ij} + Y_0^{ns}(F - \lambda) \quad (7)
$$

Replacing the infinite series in (2) with (7), recalling the relationships (2.5.3-5), (2.7.1-11), (3.1-4), and introducing the mean disturbing function U through (3.2-1), we obtain

$$
\begin{aligned}
S \;=\; & U(F - \lambda) \\
& +\mathrm{Re}\left\{ \frac{\mu_3}{R_3} \sum_{n=2}^{N} \sum_{s=0}^{n} (2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} \sum_{\substack{j=-(n+1) \\ j\neq 0}}^{n+1} \frac{[C_s(\alpha, \beta) - iS_s(\alpha, \beta)]W_j^{n+1,s}\exp(ijF)}{ij} \right\}
\end{aligned}
$$

$$
(8)
$$

The coefficients W_j^{ns} may be expressed in the form

$$W_j^{ns} = e^{-|j-s|} w_j^{ns} [C_{|j-s|}(k,h) - i\mathrm{sgn}(j-s) S_{|j-s|}(k,h)] \tag{9}$$

The functions $w_j^{ns}(e)$ possess the Jacobi polynomial representation

$$w_j^{ns}(e) = \left(\frac{1-c^2}{1+c^2}\right)^n (-c)^{|j-s|} \begin{cases} \dfrac{(n+s)!(n-s)!}{(n+j)!(n-j)!}(1-c^2)^{-|s|} P_{n-|s|}^{|j-s|,|j+s|}(\chi) & \text{for } |s| \geq |j| \\[2mm] (1-c^2)^{-|j|} P_{n-|j|}^{|j-s|,|j+s|}(\chi) & \text{for } |s| \leq |j| \end{cases} \tag{10}$$

Here again χ is defined by (2.7.3-4), e is the orbital eccentricity, and

$$c = \frac{e}{1+\sqrt{1-e^2}} = \frac{\sqrt{h^2+k^2}}{1+\sqrt{1-h^2-k^2}} \tag{11}$$

After substituting (9) into (8), we can easily cast S into its real form. The indices may be rearranged as usual, and Kepler's equation (2.1.4) may be used to replace $(F - \lambda)$ in (8) by

$$F - \lambda = k\sin F - h\cos F \tag{12}$$

The short-periodic generating function can then finally be written as the finite Fourier series

$$S = \mathcal{C}^0 + U(k\sin F - h\cos F) + \sum_{j=1}^{N+1} (\mathcal{C}^j \cos jF + \mathcal{S}^j \sin jF) \tag{13}$$

Here the coefficient $\mathcal{C}^0$ is (implied by $< S >= 0$ and (2.5.2-8))

$$\mathcal{C}^0 = \frac{k}{2}\mathcal{C}^1 + \frac{h}{2}\mathcal{S}^1 \tag{14}$$

The coefficients $\mathcal{C}^j$ and $\mathcal{S}^j$ are

$$\mathcal{C}^j = \sum_{s=0}^{N} \sum_{n=\max(2,j-1,s)}^{N} \frac{G_{ns}}{j} \left\{ -e^{-|j-s|} w_j^{n+1,s} [\mathrm{sgn}(j-s) C_s(\alpha,\beta) S_{|j-s|}(k,h) + S_s(\alpha,\beta) C_{|j-s|}(k,h)] \right.$$
$$\left. +e^{-(j+s)} w_{-j}^{n+1,s} [-C_s(\alpha,\beta) S_{j+s}(k,h) + S_s(\alpha,\beta) C_{j+s}(k,h)] \right\} \tag{15}$$

$$\mathcal{S}^j = \sum_{s=0}^{N} \sum_{n=\max(2,j-1,s)}^{N} \frac{G_{ns}}{j} \left\{ e^{-|j-s|} w_j^{n+1,s} [C_s(\alpha,\beta) C_{|j-s|}(k,h) - \mathrm{sgn}(j-s) S_s(\alpha,\beta) S_{|j-s|}(k,h)] \right.$$
$$\left. +e^{-(j+s)} w_{-j}^{n+1,s} [C_s(\alpha,\beta) C_{j+s}(k,h) + S_s(\alpha,\beta) S_{j+s}(k,h)] \right\} \tag{16}$$

where

$$G_{ns} = \frac{\mu_3}{R_3}(2 - \delta_{0s}) \left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} \tag{17}$$

The first-order short-periodic variations $\eta_{i\alpha}$ generated by the function S given by (13) can be derived using equations (2.1.8-3), (2.5.5-10), and the following (obtained by differentiating (2.1.4-2)):

$$
\begin{aligned}
\frac{\partial F}{\partial h} &= -\frac{a}{r}\cos F \\
\frac{\partial F}{\partial k} &= \frac{a}{r}\sin F \\
\frac{\partial F}{\partial \lambda} &= \frac{a}{r}
\end{aligned}
\tag{18}
$$

where r is given by (2.1.4-6). The partial derivatives of S with respect to the elements a, α, β, γ follow by straightforward differentiation of (13):

$$
\frac{\partial S}{\partial(a,\alpha,\beta,\gamma)} = \frac{\partial \mathcal{C}^0}{\partial(a,\alpha,\beta,\gamma)} + \frac{\partial U}{\partial(a,\alpha,\beta,\gamma)}(k\sin F - h\cos F)
$$

$$
+ \sum_{j=1}^{N+1}\left(\frac{\partial \mathcal{C}^j}{\partial(a,\alpha,\beta,\gamma)}\cos jF + \frac{\partial \mathcal{S}^j}{\partial(a,\alpha,\beta,\gamma)}\sin jF\right)
\tag{19}
$$

The coefficients C^j, S^j and the mean disturbing function U are functions of the semimajor axis a through the powers of the parallax factor alone, functions of the direction cosines α and β through the polynomials $C_s(\alpha,\beta)$, $S_s(\alpha,\beta)$, and functions of the direction cosine γ through the polynomials $Q_{ns}(\gamma)$. A finite Fourier series representation for $\frac{\partial S}{\partial \lambda}$ may be obtained by partial differentiation of (2) with the infinite series replaced by (2.7.1-8), followed by substitution of (4) and (9) and the usual reduction to the real form

$$
\frac{\partial S}{\partial \lambda} = \mathcal{C}^0_{,\lambda} + \sum_{j=1}^{N}(\mathcal{C}^j_{,\lambda}\cos jF + \mathcal{S}^j_{,\lambda}\sin jF)
\tag{20}
$$

Here the coefficients are

$$
\mathcal{C}^0_{,\lambda} = \frac{k}{2}\mathcal{C}^1_{,\lambda} + \frac{h}{2}\mathcal{S}^1_{,\lambda}
$$

$$
\mathcal{C}^j_{,\lambda} = \sum_{s=0}^{N}\sum_{n=\max(2,j,s)}^{N} G_{ns}\left\{e^{-|j-s|}w_j^{ns}[C_s(\alpha,\beta)C_{|j-s|}(k,h) - \mathrm{sgn}(j-s)S_s(\alpha,\beta)S_{|j-s|}(k,h)]\right.
$$

$$
\left. + e^{-(j+s)}w_{-j}^{ns}[C_s(\alpha,\beta)C_{j+s}(k,h) + S_s(\alpha,\beta)S_{j+s}(k,h)]\right\}
\tag{21}
$$

$$
\mathcal{S}^j_{,\lambda} = \sum_{s=0}^{N}\sum_{n=\max(2,j,s)}^{N} G_{ns}\left\{e^{-|j-s|)}w_j^{ns}[\mathrm{sgn}(j-s)C_s(\alpha,\beta)S_{|j-s|}(k,h) + S_s(\alpha,\beta)C_{|j-s|}(k,h)]\right.
$$

$$
\left. + e^{-(j+s)}w_{-j}^{ns}[C_s(\alpha,\beta)S_{j+s}(k,h) - S_s(\alpha,\beta)C_{j+s}(k,h)]\right\}
\tag{22}
$$

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The partial derivatives of S with respect to the elements h and k may be obtained by differentiation of (13) and use of (18) and (20):

$$
\frac{\partial S}{\partial h} = \frac{\partial \mathcal{C}^0}{\partial h} + k \frac{\partial U}{\partial h} \sin F - \left[\frac{\partial (hU)}{\partial h} + \mathcal{C}^0_{,\lambda} \right] \cos F \tag{23}
$$
$$
+ \sum_{j=1}^{N+1} \left[\left(\frac{\partial \mathcal{C}^j}{\partial h} - \frac{1}{2} \mathcal{C}^{j-1}_{,\lambda} - \frac{1}{2} \mathcal{C}^{j+1}_{,\lambda} \right) \cos jF + \left(\frac{\partial \mathcal{S}^j}{\partial h} - \frac{1}{2} \mathcal{S}^{j-1}_{,\lambda} - \frac{1}{2} \mathcal{S}^{j+1}_{,\lambda} \right) \sin jF \right]
$$

$$
\frac{\partial S}{\partial k} = \frac{\partial \mathcal{C}^0}{\partial k} + \left[\frac{\partial (kU)}{\partial k} + \mathcal{C}^0_{,\lambda} \right] \sin F - h \frac{\partial U}{\partial k} \cos F \tag{24}
$$
$$
+ \sum_{j=1}^{N+1} \left[\left(\frac{\partial \mathcal{C}^j}{\partial k} - \frac{1}{2} \mathcal{S}^{j-1}_{,\lambda} + \frac{1}{2} \mathcal{S}^{j+1}_{,\lambda} \right) \cos jF + \left(\frac{\partial \mathcal{S}^j}{\partial k} + \frac{1}{2} \mathcal{C}^{j-1}_{,\lambda} - \frac{1}{2} \mathcal{C}^{j+1}_{,\lambda} \right) \sin jF \right]
$$

In the summations in (23)–(24) $\mathcal{C}^j_{,\lambda}$ and $\mathcal{S}^j_{,\lambda}$ are defined to be zero for values of j outside the range $1 \leq j \leq N$. The dependence of $\mathcal{C}^j$, $\mathcal{S}^j$ and U on the elements h and k is through the polynomials $C_\ell(k,h), S_\ell(k,h)$, the eccentricity e, and the coefficients w_j^{ns}, K_0^{ns}. In the absence of explicit time-dependence, the first-order short-periodic variations can thereby be written as the finite Fourier series

$$
\eta_{i\alpha} = C_i^0 + \sum_{j=1}^{N+1} (C_i^j \cos jF + S_i^j \sin jF) \tag{25}
$$

where the coefficients C_i^0 are given by (2.5.2-15a).

Green [1979] studied the effects of explicit time-dependence by including several terms in the formulas (2.5.1-15) for the coefficients C_i^j and S_i^j of the λ-expansions (2.5.1-14) of the short-periodic variations. He used finite difference formulas to compute the partial derivatives with respect to time of the coefficients $\mathcal{C}_i^j$ and $\mathcal{S}_i^j$ in the λ-expansions (2.5.1-11) of the osculating rate functions. For medium or high altitude Earth satellites, he found that the Lunar point mass perturbation varies quickly enough for explicit time-dependence effects to be significant. McClain and Slutsky [1988] also found that the inclusion of explicit time-dependence effects due to the Moon and Sun improved the performance of SST for high altitude Earth satellites.

We can include the effects of explicit time-dependence in the F-expansions of the present Section by using (2.5.1-10) or (2.5.5-11) and expressions in Section 2.5.2.1, the first-order short-periodic variations $\eta_{i\alpha}^k$ including the kth order time derivatives are

$$
\eta_{i\alpha}^k = C_i^{0,k} + \sum_{j=1}^{N+k+1} (C_i^{j,k} \cos jF + S_i^{j,k} \sin jF) \tag{26}
$$

where

$$
\begin{aligned}
C_i^{0,k} &= C_i^{0,k-1} + \frac{(-1)^k}{n^k} U_k^0 \left(\frac{\partial^k C_i^\zeta}{\partial t^k}, \frac{\partial^k S_i^\zeta}{\partial t^k} \right) + \frac{3k}{2a}(-1)^{k+1} \delta_{i6} U_k^0 \left(\frac{\partial^k C_1^\zeta}{\partial t^k}, \frac{\partial^k S_1^\zeta}{\partial t^k} \right) \\
C_i^{j,k} &= \mathcal{I}_1^{N+k}(j) C_i^{j,k-1} + \frac{(-1)^k}{n^k} U_k^j \left(\frac{\partial^k C_i^\zeta}{\partial t^k}, \frac{\partial^k S_i^\zeta}{\partial t^k} \right) + \frac{3k}{2a}(-1)^{k+1} \delta_{i6} U_k^j \left(\frac{\partial^k C_1^\zeta}{\partial t^k}, \frac{\partial^k S_1^\zeta}{\partial t^k} \right) \\
S_i^{j,k} &= \mathcal{I}_1^{N+k}(j) S_i^{j,k-1} + \frac{(-1)^k}{n^k} V_k^j \left(\frac{\partial^k C_i^\zeta}{\partial t^k}, \frac{\partial^k S_i^\zeta}{\partial t^k} \right) + \frac{3k}{2a}(-1)^{k+1} \delta_{i6} V_k^j \left(\frac{\partial^k C_1^\zeta}{\partial t^k}, \frac{\partial^k S_1^\zeta}{\partial t^k} \right)
\end{aligned}
\tag{27}
$$

The recursions (27) are initialized by

$$
\begin{aligned}
C_i^{0,0} &= C_i^0 \\
C_i^{j,0} &= C_i^j \\
S_i^{j,0} &= S_i^j
\end{aligned}
\tag{28}
$$

where the C_i^0, C_i^j, S_i^j are the coefficients in the expansion (25). The quantities U_k^j, V_k^j in (27) are given by the relations (2.5.2-12) with $J = N + 1$ and $m = k$, and $\mathcal{I}_r^s(j)$ is the inclusion operator (2.5.3-18).

4.3 Central-Body Gravitational Tesserals

For the central-body gravitational tesseral harmonics, the appropriate disturbing function $\mathcal{R}$ is (2.7.1-16) with $m \neq 0$. From the results in Section 2.5.4, we can construct the first-order short-periodic variations $\eta_{i\alpha}$. Further details beyond those given here may be found in [Proulx, McClain, Early, and Cefola, 1981] and [Proulx, 1981].

In the absence of explicit time-dependence, the first-order short-periodic variations are given by (2.5.4-4):

$$
\eta_{i\alpha} = \sum_{\substack{j=-\infty \\ (j,m)\notin\mathcal{B}}}^{\infty} \sum_{m=1}^{\infty} [C_i^{jm} \cos(j\lambda - m\theta) + S_i^{jm} \sin(j\lambda - m\theta)]
\tag{1}
$$

The C_i^{jm} and S_i^{jm} are given by (2.5.4-5) in terms of Fourier coefficients $\mathcal{C}_i^{jm}$ and $\mathcal{S}_i^{jm}$ of the osculating rate functions (2.5.4-1). These latter coefficients are easily constructed by substituting the disturbing function (2.7.1-16) with $m \neq 0$ into (2.2-10). The needed partial derivatives of $\mathcal{R}$ have the same form as (3.3-4). To illustrate, we give below the final formulas for the coefficients $\mathcal{C}_1^{jm}$ and $\mathcal{S}_1^{jm}$:

$$
\begin{aligned}
\mathcal{C}_1^{jm} &= \frac{2\mu j}{A} \sum_{s=-N}^{N} \sum_{n=\max(2,m,|s|)}^{N} \left(\frac{R}{a} \right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw} (G_{ms}^j S_{nm} - H_{ms}^j C_{nm}) \\
\mathcal{S}_1^{jm} &= -\frac{2\mu j}{A} \sum_{s=-N}^{N} \sum_{n=\max(2,m,|s|)}^{N} \left(\frac{R}{a} \right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_\ell^{vw} (G_{ms}^j C_{nm} + H_{ms}^j S_{nm})
\end{aligned}
\tag{2}
$$

(Remember that A is defined by (2.1.6-1a).)

This theory has been implemented up to a 50x50 gravity field model [Fonte, 1993].

4.4 Atmospheric Drag

For atmospheric drag, the osculating rate functions $F_{i\alpha}$ are given by (2.2-5) with perturbing acceleration $\mathbf{q}$ given by (3.4-3). From the results in Section 2.5.1, we can construct an expansion in the mean longitude λ for the first-order short-periodic variations:

$$\eta_{i\alpha} = \sum_{j=1}^{\infty}(C_i^j \cos j\lambda + S_i^j \sin j\lambda) \tag{1}$$

where C_i^j and S_i^j are given by (2.5.1-15) in terms of Fourier coefficients $\mathcal{C}_i^j$ and $\mathcal{S}_i^j$ of the λ-expansion (2.5.1-11) of the osculating rate functions:

$$\begin{aligned}
\mathcal{C}_i^j &= \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}}\cdot\mathbf{q}\right)\cos j\lambda \, d\lambda \\
\mathcal{S}_i^j &= \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}}\cdot\mathbf{q}\right)\sin j\lambda \, d\lambda
\end{aligned} \tag{2}$$

Of course, the limits (λ_1, λ_2) indicate the values of λ at atmosphere entry and exit, and we convert the integrals over λ into integrals over F or L using (2.5.2-6) or (2.5.3-14). Green [1979] used the λ-expansion (1) for atmospheric drag, but indicated the desirability of an expansion in another geometric variable.

From the results in Section 2.5.2, we can construct an alternate expansion in the eccentric longitude F:

$$\eta_{i\alpha} = C_i^0 + \sum_{j=1}^{\infty}(C_i^j \cos jF + S_i^j \sin jF) \tag{3}$$

where C_i^j and S_i^j are given by (2.5.2-14) in terms of Fourier coefficients $\mathcal{C}_i^j$ and $\mathcal{S}_i^j$ of the F-expansion (2.5.2-1) of the osculating rate functions:

$$\begin{aligned}
\mathcal{C}_i^j &= \frac{1}{\pi}\int_{F_1}^{F_2}\left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}}\cdot\mathbf{q}\right)\cos jF \, dF \\
\mathcal{S}_i^j &= \frac{1}{\pi}\int_{F_1}^{F_2}\left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}}\cdot\mathbf{q}\right)\sin jF \, dF
\end{aligned} \tag{4}$$

But, as was indicated in Section 4.1, the most desirable expansion for atmospheric drag is in terms of the true longitude L. From (2.5.3-23)

$$\eta_{i\alpha} = C_i^0 + \sum_{m=1}^{K+2} D_i^m(L-\lambda)^m + \sum_{j=1}^{\infty}(C_i^j \cos jL + S_i^j \sin jL) \tag{5}$$

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where C_i^j, S_i^j, D_i^m are given by (2.5.3-24) (with the index $M = 0$) in terms of the coefficients $\mathcal{C}_i^j, \mathcal{S}_i^j, \mathcal{D}_i^m$ defined by

$$
\begin{aligned}
\mathcal{C}_i^j &= \frac{1}{\pi} \int_{L_1}^{L_2} \left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q} \right) \cos jL \, dL \\
\mathcal{S}_i^j &= \frac{1}{\pi} \int_{L_1}^{L_2} \left(\frac{\partial a_i}{\partial \dot{\mathbf{r}}} \cdot \mathbf{q} \right) \sin jL \, dL \\
\mathcal{D}_i^m &= 0
\end{aligned}
\tag{6}
$$

From atmospheric drag, there are five possible sources of explicit time-dependence:

1. Variations in the solar extreme ultraviolet (EUV) flux.

2. Geomagnetic storms.

3. Seasonal latitudinal variations in the atmosphere density.

4. Motion of the diurnal bulge in the atmosphere.

5. Motion of atmospheric tides raised by the Sun and Moon.

Variations in the solar EUV flux can cause the atmospheric density at a given altitude to vary by up to three orders of magnitude and are the dominant cause of error in lifetime predictions for near-Earth satellites. The primary source of variation in the solar EUV flux is the 11-year sunspot cycle, and it might be assumed that variations with a period of 11 years would be too slow to cause appreciable explicit time-dependence effects. The sunspot cycle is far from sinusoidal, however. At the beginning of each cycle, there is a steep rise in solar EUV flux which could conceivably be fast enough to cause significant explicit time-dependence effects. In addition, there is a secondary variation in solar EUV flux with a period of about 27 days caused by the rotation of the Sun, which brings sunspots alternately into and out of view around the limb of the Sun. A period of 27 days is comparable to the 27.3-day period of the Lunar point-mass perturbation, which is known to have significant explicit time-dependence effects. Geomagnetic storms can cause large variations in the atmosphere density over time scales of a day or less. The potential for significant explicit time-dependence effects is obvious. The remaining sources of explicit time-dependence effects for atmospheric drag are thought to be too small or too slowly-varying to be significant. However, explicit time-dependence effects from atmospheric drag have not yet been studied with SST.

4.5   Solar Radiation Pressure

The first-order short-periodic variations $\eta_{i\alpha}$ due to solar radiation pressure are formally identical to the equations in Section 4.4 for atmospheric drag, where the perturbing acceleration $\mathbf{q}$ is given by (3.5-6). For solar radiation pressure, the simpler expansions (4.4-1) in the mean longitude λ or (4.4-3) in the eccentric longitude F should be adequate. Also, Green [1979] found that the explicit time-dependence effects from solar radiation pressure were minor.

As we have seen in Section 3.5, if the satellite is always in sunlight, the perturbing acceleration $\mathbf{q}$ can be derived from a disturbing function $\mathcal{R}$ which is nearly identical to the

third-body disturbing function. Hence we can immediately obtain a finite Fourier series in the eccentric longitude F for the $\eta_{i\alpha}$ by analogy with the results in Section 4.2.

5   Higher-Order Terms

Generally, the algebraic complexity greatly increases when we try to compute higher-order mean element rates and short-periodic variations. Also, it is assumed that higher-order terms due to most perturbations are usually negligible. In this chapter we report on those few higher-order effects which have been studied to date with SST.

5.1   Second-Order $A_{i\alpha\beta}$ and $\eta_{i\alpha\beta}$ Due to Gravitational Zonals and Atmospheric Drag

The second-order mean element rates $A_{i\alpha\beta}$ and short-period variations $\eta_{i\alpha\beta}$ due to two perturbations expanded in λ may be constructed as shown in Section 2.5.6. We can calculate analytically the Fourier coefficients $\mathcal{C}_{i1}^j$ and $\mathcal{S}_{i1}^j$ of the expansions in λ for the osculating rate functions F_{i1} due to the central-body gravitational zonal harmonics by substituting the disturbing function (2.7.1-16) with $m = 0$ into equations (2.2-10). We can calculate numerically the Fourier coefficients $\mathcal{C}_{i2}^j$ and $\mathcal{S}_{i2}^j$ of the expansions in λ for the osculating rate functions F_{i2} due to atmospheric drag from (4.4-2). The partial derivatives $\frac{\partial \mathcal{C}_{i\alpha}^j}{\partial a_r}$ and $\frac{\partial \mathcal{S}_{i\alpha}^j}{\partial a_r}$ needed in (2.5.6-5) can be calculated analytically for the central-body gravitational zonals and by numerical quadrature for atmospheric drag.

   At the present time, the only terms in these analytical formulas which are available in the SST code are:

1. The J_2-squared auto-coupling mean element rates A_{i11}, correct to first power of the eccentricity and with (α, β, γ) replaced by the explicit formulas (2.1.9-3) in p and q. These terms were constructed with the MACSYMA algebraic utilities of [Zeis, 1978] and [Bobick, 1981].

2. The J_2- squared auto-coupling short-periodic variations η_{i11}, correct to zero power of the eccentricity and with (α, β, γ) replaced by the explicit formulas (2.1.9-3) in p and q. These terms were constructed in [Zeis, 1978] and corrected in [Green, 1979].

3. The J_2/drag cross-coupling mean elements rates A_{i12}, correct to zero power of the eccentricity and with the retrograde factor $I = 1$ and (α, β, γ) replaced by the explicit formulas (2.1.9-3) in p and q. These terms were constructed in [Green, 1979].

   Green [1979] studied the second-order effects of J_2 and drag using a combination of analytical and numerical methods. He found that the drag/J_2 cross-coupling terms A_{i21} cause significant effects for low altitude satellites. He found that Izsak's J_2 height correction (1) applied to the density determination in the formulas (3.4-1, 3, 4) gave a good approximation to the A_{i21} terms. The following expression added to the height H is the J_2 short-periodic correction (from Section 4.1) to the radial distance r:

$$\Delta r = \frac{J_2 R^2}{4(1-e^2)a}\left[\sin^2 i \cos 2(f+\omega) + (3\sin^2 i - 2)\left(1 + \frac{e\cos f}{1+\sqrt{1-e^2}} + \frac{2\sqrt{1-e^2}}{1+e\cos f}\right)\right] \quad (1)$$

Green's explanation of why Izsak's J_2 height correction works is that adding the J_2 short periodics to the elements in the drag osculating rate functions and then averaging is equivalent to adding the drag/J_2 cross-coupling terms to the first-order mean element rates, if we neglect products of short-periodic variations:

$$<F_{i2}(a_1+\eta_{11},\ldots,a_6+\eta_{6i})> \approx <F_{i2}(a_1,\ldots,a_6)> + \sum_j <\frac{\partial F_{i2}}{\partial a_j}(a_1,\ldots,a_6)\eta_{ji}>$$

$$\approx A_{i2} + A_{i21}$$

$$(2)$$

5.2 Second-Order $\eta_{i\alpha\beta}$ Cross-Coupling Between Secular Gravitational Zonals and Tesseral Harmonics

In high-order shallow resonance orbits, the tesseral harmonics which contribute the most significant short-period motion are likely to be those with degree and order centered around the resonant order. For such orbits, the second-order short-periodic variations due to cross-coupling between these tesseral harmonics and the J_2 secular terms may also be significant. In this Section we outline how to construct these critical short periodics. For further details and a discussion of numerical results see [Cefola, 1981] and [Cefola and Proulx, 1991].

For the present purpose, we retain in the expansions (2.5.1-10) of the osculating rate functions F_{i1} due to the central-body gravitational zonal harmonics only the mean element rates A_{i1} given by (3.1-1):

$$F_{i1} \approx A_{i1} \tag{1}$$

Furthermore, we completely neglect the first-order short-periodic variations η_{i1} due to the zonal harmonics:

$$\eta_{i1} \approx 0 \tag{2}$$

The osculating rate functions F_{i2} due to the tesseral harmonics may be expanded in the Fourier series

$$F_{i2} = \sum_{j,m}[\mathcal{C}_i^{jm}\cos(j\lambda - m\theta) + \mathcal{S}_i^{jm}\sin(j\lambda - m\theta)] \tag{3}$$

The first-order short-periodic variations due to the tesseral harmonics are then given by (2.5.4-4):

$$\eta_{i2} = \sum_{(j,m)\notin\mathcal{B}}[C_i^{jm}\cos(j\lambda - m\theta) + S_i^{jm}\sin(j\lambda - m\theta)] \tag{4}$$

where, in the absence of explicit time-dependence, the C_i^{jm} and S_i^{jm} are related to the Fourier coefficients $\mathcal{C}_i^{jm}$ and $\mathcal{S}_i^{jm}$ by (2.5.4-5).

To obtain the second-order cross-coupling terms, we need to construct the functions $G_{i12} + G_{i21}$ from (2.3-27):

$$
\begin{aligned}
G_{i12} + G_{i21} &= \sum_{r=1}^{6} \left(\frac{\partial F_{i1}}{\partial a_r} \eta_{r2} + \frac{\partial F_{i2}}{\partial a_r} \eta_{r1} \right) + \frac{15n}{4a^2} \delta_{i6} \eta_{11} \eta_{12} \\
&\quad - \sum_{r=1}^{6} \left(\frac{\partial \eta_{i1}}{\partial a_r} A_{r2} + \frac{\partial \eta_{i2}}{\partial a_r} A_{r1} \right)
\end{aligned}
\tag{5}
$$

Substituting (1)–(4) into (5) yields

$$
G_{i12} + G_{i21} \approx \sum_{(j,m) \notin \mathcal{B}} [\overline{\mathcal{C}}_i^{jm} \cos(j\lambda - m\theta) + \overline{\mathcal{S}}_i^{jm} \sin(j\lambda - m\theta)]
\tag{6}
$$

where

$$
\begin{aligned}
\overline{\mathcal{C}}_i^{jm} &= \sum_{r=1}^{5} \left(\frac{\partial A_{i1}}{\partial a_r} C_r^{jm} - A_{r1} \frac{\partial C_i^{jm}}{\partial a_r} \right) - j A_{61} S_i^{jm} \\
\overline{\mathcal{S}}_i^{jm} &= \sum_{r=1}^{5} \left(\frac{\partial A_{i1}}{\partial a_r} S_r^{jm} - A_{r1} \frac{\partial S_i^{jm}}{\partial a_r} \right) + j A_{61} C_i^{jm}
\end{aligned}
\tag{7}
$$

The cross-coupling short-periodics are then

$$
\eta_{i12} + \eta_{i21} = \sum_{(j,m) \notin \mathcal{B}} [\overline{C}_i^{jm} \cos(j\lambda - m\theta) + \overline{S}_i^{jm} \sin(j\lambda - m\theta)]
\tag{8}
$$

where the coefficients $\overline{C}_i^{jm}$ and $\overline{S}_i^{jm}$ are given in terms of $\overline{\mathcal{C}}_i^{jm}$ and $\overline{\mathcal{S}}_i^{jm}$ by the relations (2.5.4-5). The partials $\frac{\partial A_{i1}}{\partial a_r}$ of the J_2 mean element rates needed in (7) are given by equations (3.1-12). The partials $\frac{\partial C_i^{jm}}{\partial a_r}$ and $\frac{\partial S_i^{jm}}{\partial a_r}$ of the tesseral first-order short-periodic coefficients are related as usual to the partials $\frac{\partial \mathcal{C}_i^{jm}}{\partial a_r}$ and $\frac{\partial \mathcal{S}_i^{jm}}{\partial a_r}$ of the tesseral osculating rate coefficients, which may be obtained by differentiating formulas such as (4.3-2).

Code based on this approximate theory has been developed only for the J_2 secular/m-daily terms ($j = 0$ in (4)), with finite differences used to obtain the partials of the m-daily coefficients. Construction and programming of a complete second-order theory for a double-averaged perturbation expanded in λ, θ has yet to be accomplished.

6   Truncation Algorithms

Semianalytic Satellite Theory contains many long series expansions. Some of the series are infinite and hence must be truncated, whereas others are finite but are truncated to reduce the computing cost. Automatic truncation algorithms are currently used in the SST code for three of these series expansions:

(i) The Hansen kernels are initialized by the infinite series representation (2.7.3-10) in powers of e^2. Convergence of the series has been investigated by [Proulx and McClain, 1988] (see also [Sabol, 1994]). The automatic truncation of the series is straightforward and need not be discussed here.

(ii) The expansion of the mean disturbing function (3.2-1) for a third-body point mass is automatically truncated by a procedure described by [Long and Early, 1978]. In the first part of this chapter we document this procedure.

(iii) The expansion of the mean disturbing function (3.1-1) for the central-body zonal harmonics is automatically truncated as part of a procedure described by [Long and Early, 1978] for the truncation of the averaged nonresonant central-body disturbing function. However, nonresonant tesserals are currently excluded from the averaged equations of motion. In the second part of this chapter we propose an improved automatic truncation algorithm for this case.

All other series expansions in the SST code are currently truncated manually by using tables for various orbit classifications as described by [Cefola, 1993]. In the final parts of this chapter we propose automatic truncation algorithms for all other series which must presently be manually truncated by an experienced user. The formulas in this chapter were published in [Danielson and Sagovac, 1995].

An automatic truncation algorithm removes from an expansion all terms whose absolute values are less than a certain truncation tolerance. To do this, the algorithm evaluates a close upper bound for the absolute value of each term in the series and sets the indices of the expansion to include only those terms whose upper bounds are greater than the truncation tolerance. The upper bounds are evaluated using the parameters existing at the initial epoch of the integration span.

For the truncation procedure to be reliable, the upper bounds must satisfy the following two conditions stated in [Long and Early, 1978]:

1. The upper bounds must be upper bounds throughout the integration span. As the orbit evolves and the perturbing bodies move, the absolute value of each term in the series must remain less than or approximately equal to the corresponding upper bound. This condition can be satisfied by choosing upper bounds which depend only on slowly varying dynamic parameters. For typical Earth satellites and typical integration spans, the slowly varying parameters are usually the equinoctial orbital elements (a, h, k, p, q), the distances R_3 from the center of mass of the central body to the third bodies, and the direction cosines (α, β, γ) of the central-body rotation axis. For eccentric drag-perturbed satellites the eccentricity e may be a rapidly decreasing parameter, so the upper bounds should be increasing functions of e.

2. The upper bounds must eventually monotonically decrease as the truncatable indices of the expansion increase. Each term in the series can be factored into a product of constants, non-zero functions of the dynamic parameters, and oscillating functions of the dynamic parameters. The positions and numbers of zeros of an oscillating function vary as the indices of the function increase. To avoid premature truncation of the series, a smooth upper envelope is used as the upper bound for each oscillating factor.

Our automatic truncation algorithms require as inputs only the values ϵ and $\bar{\epsilon}$ of the truncation tolerances for the central-body and third-body gravitational potentials, respectively. In the current SST code, it is suggested to increase the truncation tolerance $\bar{\epsilon}$ for distant eccentric satellites because the expansion for the third-body mean gravitational potential converges slowly for these satellites, and the extra accuracy achieved by using a smaller tolerance is expensive [Long and Early, 1978]. However, implementation of our proposed

truncation algorithms may make economical the use of a single relatively small number for ϵ and $\bar{\epsilon}$ to yield high accuracy.

6.1   Third-Body Mean Gravitational Potential

The mean disturbing function due to the gravitational field of a third-body point mass is from (3.2-1)

$$U = \sum_{s=0}^{S \leq N} \sum_{n=\max(2,s)}^{N} U_{sn} \tag{1}$$

where

$$U_{sn} = \frac{\mu_3}{R_3}(2 - \delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns} K_0^{ns} Q_{ns} G_s \tag{2}$$

The expansion (1) has two truncatable indices: n giving the power of $\frac{a}{R_3}$, and s giving the power of e. The purpose of the truncation algorithm is to determine the maximum values N and S of n and s, respectively.

Now each term (2) in the series (1) is less than or equal to the product of the absolute values of its factors:

$$U_{sn} \leq \frac{2\mu_3}{R_3}\left(\frac{a}{R_3}\right)^n |V_{ns}|\,|K_0^{ns}|\,|Q_{ns}|\,|G_s|$$

From (2.8.1-3), we can easily obtain the constants $|V_{ns}|$. From (2.7.3-3b,5), $|K_0^{ns}(e)|$ are clearly positive for all $0 \leq e < 1$. The functions $|Q_{ns}(\gamma)|$ are bounded for all $-1 \leq \gamma \leq 1$ by $Q_{ns}(1)$:

$$|Q_{ns}(\gamma)| \leq Q_{ns}(1) = \frac{(n+s)!}{2^s s!(n-s)!} \qquad (n \geq s) \tag{3}$$

From (3.1-4) and the relation $k^2 + h^2 = e^2$, the upper bound on $|G_s|$ is

$$|G_s| \leq (k^2 + h^2)^{s/2}(\alpha^2 + \beta^2)^{s/2} = e^s(\alpha^2 + \beta^2)^{s/2} \leq e^s \tag{4}$$

Multiplying the upper bounds of all these factors together, we finally obtain

$$U_{sn} \leq |U_{sn}|_{Bound} = \frac{2\mu_3}{R_3}\left(\frac{a}{2R_3}\right)^n \frac{(n+s)!}{s!\left(\frac{n+s}{2}\right)!\left(\frac{n-s}{2}\right)!}|K_0^{ns}|\left(\frac{e}{2}\right)^s \tag{5}$$

The truncation algorithm requires the calculation of $|U_{sn}|_{Bound}$ for each $s = 0, 1, \ldots$ and $n = \max(2, s), \ldots$ for $n - s$ even. Then $N(s)$ is the greatest integer for which $|U_{sN}|_{Bound} > \bar{\epsilon}$, and S is the greatest integer for which $|U_{Sn}|_{Bound} > \bar{\epsilon}$. Here $\bar{\epsilon}$ is the prescribed truncation tolerance for the third-body gravitational potential. Note that if the n summation in (1) is done first for each successive s, then N depends on s as indicated above; whereas if it is desired to perform the s summation in (1) first for each successive n, then S will depend on n.

This algorithm has been successful and is used in the current SST code. However the index N is taken to be the maximum $N(s)$ amongst all s, resulting in the retention of negligible terms in the series for U. The upper bound (5) can be simplified considerably, but at a cost of significantly overestimating the size of $|U_{sn}|_{Bound}$ in most cases [Long and Early, 1978].

6.2   Central-Body Mean Zonal Harmonics

The mean disturbing function due to the gravitational zonal harmonics of the central body is from (3.1-3)

$$U = \sum_{s=0}^{S \leq N-2} \sum_{n=s+2}^{N} U_{sn} \tag{1}$$

where

$$U_{sn} = -\frac{\mu}{a}(2 - \delta_{0s})\left(\frac{R}{a}\right)^n J_n V_{ns} K_0^{-n-1,s} Q_{ns} G_s \tag{2}$$

The expansion (1) has two truncatable indices: n giving the order of the geopotential coefficients and s giving the power of e. The purpose of the truncation algorithm is to determine the maximum values N and S of n and s, respectively.

Now each term (2) in the series (1) is less than or equal to the product of the absolute values of its factors:

$$U_{sn} \leq \frac{2\mu}{a}\left(\frac{R}{a}\right)^n |J_n||V_{ns}|K_0^{-n-1,s}|Q_{ns}||G_s|$$

From (2.8.1-3) we can easily obtain the constants $|V_{ns}|$. From (2.7.3-3a,5), $K_0^{-n-1,s}(e)$ are clearly positive for all $0 \leq e < 1$. The functions $|Q_{ns}(\gamma)|$ can be replaced by the upper bound $|Q_{ns}(\gamma)|_{Bound}$, given in [Danielson and Sagovac, 1995, Appendix A]:

$$|Q_{ns}(\gamma)|_{Bound} = \left[[Q_{ns}(\gamma)]^2 + \frac{(1 - \gamma^2)}{[n(n+1) - s(s+1)]}\left[\frac{d}{d\gamma}Q_{ns}(\gamma)\right]^2 \right]^{1/2} \tag{3}$$

From (6.1-4) and the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, the upper bound on $|G_s|$ may now be taken to be

$$|G_s| \leq e^s(\alpha^2 + \beta^2)^{s/2} = e^s(1 - \gamma^2)^{s/2}$$

Note that since γ is now a slowly varying parameter, it need not be removed from the upper bounds as in (6.1-3,4). Multiplying the upper bounds of all these factors together, we finally obtain

$$U_{sn} \leq |U_{sn}|_{Bound} = \frac{2\mu}{a}\left(\frac{R}{2a}\right)^n |J_n|\frac{(n-s)!}{\left(\frac{n+s}{2}\right)!\left(\frac{n-s}{2}\right)!}K_0^{-n-1,s}|Q_{ns}|_{Bound}(1 - \gamma^2)^{s/2}e^s$$

The truncation algorithm requires the calculation of $|U_{sn}|_{Bound}$ for each $s = 0, 1, \ldots$ and $n = s+2, s+3, \ldots$ for $n-s$ even. Then $N(s)$ is the greatest integer for which $|U_{sN}|_{Bound} > \epsilon$, and S is the greatest integer for which $|U_{Sn}|_{Bound} > \epsilon$. Here ϵ is the prescribed truncation tolerance for the central-body gravitational potential. Of course, N can be no larger than the index of the highest available geopotential coefficient J_N.

6.3 Central-Body Tesseral Harmonics

The disturbing function due to the gravitational tesseral harmonics of the central body is from (2.7.1-16)

$$R = \sum_{j=J_1}^{J_2} \sum_{m=1}^{M \leq N} \sum_{s=S_1 \geq -N}^{s=S_2 \leq N} \sum_{n=\max(2,m,|s|)}^{N} R_{jmsn} \tag{1}$$

where

$$R_{jmsn} = \mathrm{Re}\left\{ \frac{\mu}{a} \left(\frac{R}{a}\right)^n I^m V_{ns}^m \Gamma_{ns}^m K_j^{-n-1,s} P_l^{vw} (C_{nm} - iS_{nm})(G_{ms}^j + iH_{ms}^j)\exp[i(j\lambda - m\theta)] \right\} \tag{2}$$

The expansion (1) has four truncatable indices: n giving the order of the geopotential coefficients, s, m giving the degree of the geopotential coefficients, and j giving the frequency of λ. The purpose of the truncation algorithm is to determine the maximum value N of n, the minimum and maximum values S_1 and S_2 of s, the maximum value M of m, and the minimum and maximum values J_1 and J_2 of j.

Now each term (2) in the series (1) is less than or equal to the product of the absolute values of its factors:

$$|R_{jmsn}| \leq \frac{\mu}{a} \left(\frac{R}{a}\right)^n |V_{ns}^m||\Gamma_{ns}^m||K_j^{-n-1,s}||P_l^{vw}||C_{nm} - iS_{nm}||G_{ms}^j + iH_{ms}^j|$$

From (2.7.1-6,13) we can easily obtain $|V_{ns}^m|$ and $|\Gamma_{ns}^m|$. The Hansen kernels can be replaced by the upper bound [Danielson and Sagovac, 1995, Appendix B]

$$|K_j^{ns}(e)| \leq |K_j^{ns}(e)|_{Bound} = (1 - e^2)^{n+3/2} \max_{e=0 \; or \; e=1} \mathcal{K}_j^{ns}(e)$$

where $\mathcal{K}_j^{ns}(e) = |K_j^{ns}(e)| / (1 - e^2)^{n+3/2}$. Here the values of K_j^{ns} at $e = 0$ may be calculated recursively from (2.7.3-10,11,12,13) or directly from

$$K_j^{ns}(0) = \begin{cases} \left(-\dfrac{1}{2}\right)^{s-j} \displaystyle\sum_{k=0}^{s-j} \dfrac{(n+j+k+2)_{s-j-k}\, j^k}{k!\,(s-j-k)!} & \text{for } s \geq j \\[4mm] \left(-\dfrac{1}{2}\right)^{j-s} \displaystyle\sum_{k=0}^{j-s} \dfrac{(n-j+k+2)_{j-s-k}\,(-j)^k}{k!\,(j-s-k)!} & \text{for } s \leq j \end{cases}$$

where $(\alpha)_k$ are the Pochhammer symbols defined by $(\alpha)_0 = 1$ and $(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$, and the values of $\mathcal{K}_j^{ns}$ at $e = 1$ may be calculated from

$$\mathcal{K}_j^{ns}(1) = \sum_{k=0}^{-n-2} \binom{-n-2}{k} [1 + (-1)^{k+s}] \begin{cases} 0 & k < s \\ 2^{-k-1}\dbinom{k}{(k-s)/2} & s \leq k \end{cases}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ are the binomial coefficients. A smooth upper bound for the Jacobi polynomials was given in [Long and Early, 1978]:

$$|P_l^{vw}(\gamma)| \leq |P_l^{vw}|_{Bound} = \sqrt{[P_l^{vw}(\gamma)]^2 + \frac{1-\gamma^2}{l(v+w+l+1)}\left[\frac{d}{d\gamma}P_l^{vw}(\gamma)\right]^2} \qquad (3)$$

The upper bound for $|C_{nm} - iS_{nm}|$ is simply

$$|C_{nm} - iS_{nm}| \leq \sqrt{C_{nm}^2 + S_{nm}^2}$$

From (2.7.1-14) the upper bound on $|G_{ms}^j + iH_{ms}^j|$ is

$$|G_{ms}^j + iH_{ms}^j| \leq (k^2+h^2)^{|s-j|/2}(\alpha^2+\beta^2)^{|s-Im|/2} = e^{|s-j|}(1-\gamma^2)^{|s-Im|/2}$$

Multiplying all of these upper bounds together, we finally obtain

$$R_{jmsn} \leq |R_{jmsn}|_{Bound} = \frac{\mu}{a}\left(\frac{R}{a}\right)^n |V_{ns}^m||\Gamma_{ns}^m||K_j^{-n-1,s}|_{Bound}$$

$$|P_l^{vw}|_{Bound}\sqrt{C_{nm}^2 + S_{nm}^2}(1-\gamma^2)^{|s-Im|/2}e^{|s-j|}$$

The truncation algorithm requires the calculation of $|R_{jmsn}|_{Bound}$ for each $j = 0, \pm 1, \pm 2, \ldots$ and $m = 1, 2, \ldots$ and $s = j, j \pm 1, j \pm 2, \ldots$ and $n = \max(2, m, |s|), \ldots$ for $n - s$ even. Then $N(j, m, s)$ is the greatest integer for which $|R_{jmsN}|_{Bound} > \epsilon$, $S_1(j, m)$ and $S_2(j, m)$ are the smallest and greatest integers for which $|R_{jmSn}|_{Bound} > \epsilon$, $M(j)$ is the greatest integer for which $|R_{jMsn}|_{Bound} > \epsilon$, and J_1 and J_2 are the smallest and greatest integers for which $|R_{Jmsn}|_{Bound} > \epsilon$. Here again ϵ is the truncation tolerance for the central-body gravitational potential. Of course, N and M can be no larger than the indices of the highest available geopotential coefficients C_{NM} and S_{NM}.

6.4 Central-Body Zonal Harmonics Short-Periodics

The short-periodic generating function due to the gravitational zonal harmonics of the central body is from (4.1-11):

$$S = U(L - \lambda) + \mathcal{C}^0 + \sum_{j=1}^{J}(\mathcal{C}^j \cos jL + \mathcal{S}^j \sin jL) \qquad (1)$$

Here U is given by (6.2-1) and

$$U(L - \lambda) = \sum_{m=1}^{M} \frac{2U}{m}(\sigma_m \cos mL - \rho_m \sin mL) \qquad (2)$$

Other Fourier coefficients in (1) are from (4.1-12, 13, 14a)

$$\mathcal{C}^0 = -\sum_{j=1}^{J}(\mathcal{C}^j \rho_j + \mathcal{S}^j \sigma_j)$$

$$\mathcal{C}^j = \mathcal{I}_1^{J_1 \leq N_1 - 1}(j) \sum_{s=j}^{S_1 \leq N_1 - 1} \sum_{n=s+1}^{N_1} \mathcal{C}_{sn}^{j1} + \mathcal{I}_1^{J_2 \leq N_2 - 1}(j) \sum_{s=0}^{S_2 \leq N_2 - j} \sum_{n=\max(j+s,j+1)}^{N_2} \mathcal{C}_{sn}^{j2}$$

$$+\mathcal{I}_2^{J_3 \leq N_3}(j) \mathcal{C}^{j3} + \mathcal{I}_2^{J_4 \leq N_4}(j) \sum_{s=1}^{S_4 \leq j-1} \mathcal{C}_s^{j4} + \mathcal{I}_1^{J_5 \leq N_5 - 1}(j) \sum_{s=1}^{S_5 \leq j} \sum_{n=j+1}^{N_5} \mathcal{C}_{sn}^{j5} \qquad (3)$$

$$+\mathcal{I}_3^{J_6 \leq 2N_6 - 1}(j) \sum_{s=j-\min(j-1,N_6)}^{S_6 \leq \frac{j}{2} - 1} \sum_{n=j-s}^{\min(j-1,N_6)} \mathcal{C}_{sn}^{j6}$$

$$+\mathcal{I}_3^{J_7 \leq 2N_7 - 1}(j) \sum_{s=j/2}^{S_7 \leq \min(j-1,N_7) - 1} \sum_{n=s+1}^{\min(j-1,N_7)} \mathcal{C}_{sn}^{j7}$$

where

$$\mathcal{C}_{sn}^{j1} = -\frac{\mu}{aj}(2 - \delta_{0,s-j}) J_n H_{s,s-j} K_0^{-n-1,s} L_n^{s-j}$$

$$\mathcal{C}_{sn}^{j2} = \frac{\mu}{aj}(2 - \delta_{0,j+s}) J_n H_{s,s+j} K_0^{-n-1,s} L_n^{j+s}$$

$$\mathcal{C}^{j3} = \frac{2\mu}{aj} J_j H_{0,j} K_0^{-j-1,0} L_j^j \qquad (4)$$

$$\mathcal{C}_s^{j4} = \frac{\mu}{aj}(2 - \delta_{0,j-s}) J_j I_{s,j-s} K_0^{-j-1,s} L_j^{j-s}$$

$$\mathcal{C}_{sn}^{j5} = \mathcal{C}_{sn}^{j6} = \mathcal{C}_{sn}^{j7} = \frac{\mu}{aj}(2 - \delta_{0,j-s}) J_n I_{s,j-s} K_0^{-n-1,s} L_n^{j-s}$$

For brevity, we have in the last two series in (3) only shown the limits appropriate for j even, and we have not shown the expansions for the Fourier coefficients $\mathcal{S}^j$ in (1). However, the maximum values of the truncatable indices of the series we do not show are identical to the ones determined by the procedure outlined here.

The expansion (2) has the one truncatable index M giving the frequency of L. The expansions in (3) have up to three truncatable indices: n giving the order of the geopotential coefficients, s giving the power of e, and j giving the frequency of L. The purpose of the truncation algorithm is to determine the maximum value M of m, the maximum values $N_1, \ldots, N_7$ of n, the maximum values $S_1, \ldots, S_7$ of s, and the maximum values $J_1, \ldots, J_7$ of j.

Now each Fourier coefficient in the series (2) is less than or equal to its absolute value, which using (2.5.3-4,5) is

$$\frac{2U\sigma_m}{m} \leq \frac{2|U|\,|\sigma_m|}{m} = \frac{2|U|}{m}(1 + mB)|b|^m |S_m(k,h)|$$

$$\leq \frac{2|U|}{m}(1 + mB)|b|^m |C_m(k,h) + iS_m(k,h)|$$

$$= \frac{2|U|}{m} \frac{(1+mB)}{(1+B)^m} (k^2 + h^2)^{m/2}$$

$$= \frac{2|U|}{m} \frac{(1+m\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^m} e^m \tag{5}$$

The upper bound of the Fourier coefficient $-\frac{2U\rho_m}{m}$ is the same.

Each of the Fourier coefficients (4) is less than or equal to the product of the absolute values of its factors, which using (4.1-10a) are

$$|\mathcal{C}_{sn}^{j1}| \le \frac{2\mu}{aj} \left(\frac{R}{a}\right)^n |J_n| \, |H_{s,s-j}| \, |V_{n,s-j}| \, K_0^{-n-1,s} \, |Q_{n,s-j}|$$

$$|\mathcal{C}_{sn}^{j2}| \le \frac{2\mu}{aj} \left(\frac{R}{a}\right)^n |J_n| \, |H_{s,j+s}| \, |V_{n,j+s}| \, K_0^{-n-1,s} \, |Q_{n,j+s}|$$

$$|\mathcal{C}^{j3}| \le \frac{2\mu}{aj} \left(\frac{R}{a}\right)^j |J_j| \, |H_{0,j}| \, |V_{j,j}| \, K_0^{-j-1,0} \, |Q_{j,j}|$$

$$|\mathcal{C}_s^{j4}| \le \frac{2\mu}{aj} \left(\frac{R}{a}\right)^j |J_j| \, |I_{s,j-s}| \, |V_{j,j-s}| \, K_0^{-j-1,s} \, |Q_{j,j-s}|$$

$$|\mathcal{C}_{sn}^{j5}| \le \frac{2\mu}{aj} \left(\frac{R}{a}\right)^n |J_n| \, |I_{s,j-s}| \, |V_{n,j-s}| \, K_0^{-j-1,s} \, |Q_{n,j-s}|$$

From (2.8.1-3), we can easily obtain the constants $|V_{ns}|$. The functions $K_0^{-n-1,s}(e)$ are positive, and the functions $|Q_{n,s}(\gamma)|$ may be replaced by the upper bound (6.2-3). From the definitions (2.5.3-5) and (4.1-10c), the upper bound on H_{js} is

$$|H_{js}| = |\text{Re}\{[C_j(k,h) + i\, S_j(k,h)]\, [S_s(\alpha,\beta) + i\, C_s(\alpha,\beta)]\}|$$

$$\le |[C_j(k,h) + i\, S_j(k,h)]\, [S_s(\alpha,\beta) + i\, C_s(\alpha,\beta)]|$$

$$= |C_j(k,h) + i\, S_j(k,h)|\, |C_s(\alpha,\beta) - i\, S_s(\alpha,\beta)|$$

$$= (k^2 + h^2)^{j/2}(\alpha^2 + \beta^2)^{s/2} = e^j\, (1 - \gamma^2)^{s/2}$$

The upper bound on I_{js} is the same. Multiplying the upper bounds of all these factors together, we finally obtain

$$|\mathcal{C}_{sn}^{j1}| \le |\mathcal{C}_{sn}^{j1}|_{Bound} = \frac{2\mu}{aj} \left(\frac{R}{a}\right)^n |J_n| \, |V_{n,s-j}| \, K_0^{-n-1,s} |Q_{n,s-j}|_{Bound} \, (1 - \gamma^2)^{\frac{s-j}{2}} \, e^s$$

$$|\mathcal{C}_{sn}^{j2}| \le |\mathcal{C}_{sn}^{j2}|_{Bound} = \frac{2\mu}{aj} \left(\frac{R}{a}\right)^n |J_n| \, |V_{n,j+s}| \, K_0^{-n-1,s} |Q_{n,j+s}|_{Bound} \, (1 - \gamma^2)^{\frac{j+s}{2}} \, e^s$$

$$|\mathcal{C}^{j3}| \le |\mathcal{C}_{sn}^{j3}|_{Bound} = \frac{2\mu}{aj} \left(\frac{R}{a}\right)^j |J_j| \, |V_{j,j}| \, K_0^{-j-1,0} |Q_{j,j}|_{Bound} \, (1 - \gamma^2)^{j/2}$$

$$|\mathcal{C}_s^{j4}| \le |\mathcal{C}_s^{j4}|_{Bound} = \frac{2\mu}{aj}\left(\frac{R}{a}\right)^j |J_j|\,|V_{j,j-s}|\,K_0^{-j-1,s}|Q_{j,j-s}|_{Bound}\,(1-\gamma^2)^{\frac{i-s}{2}}\,e^s$$

$$|\mathcal{C}_{sn}^{j5}| \le |\mathcal{C}_{sn}^{j5}|_{Bound} = \frac{2\mu}{aj}\left(\frac{R}{a}\right)^n |J_n|\,|V_{n,j-s}|\,K_0^{-j-1,s}|Q_{n,j-s}|_{Bound}\,(1-\gamma^2)^{\frac{i-s}{2}}\,e^s$$

To determine the tolerance for the Fourier coefficients in the short-periodic generator S, we note from (2.5.5-4) that

$$\frac{\partial S}{\partial \lambda} = R - U \tag{6}$$

and again let ϵ be the truncation tolerance for the central-body gravitational potential. It follows from (6) that a term $\mathcal{C}^j \cos jL$ in S may be significant only if

$$j\,|\mathcal{C}^j|\,\frac{\partial L}{\partial \lambda} > \epsilon \tag{7}$$

Now from (2.5.3-14)

$$\frac{\partial L}{\partial \lambda} = \sqrt{1-e^2}\left(\frac{a}{r}\right)^2 = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}$$

$$< \frac{(1+e)^2}{(1-e^2)^{3/2}} = \frac{\sqrt{1+e}}{(1-e)^{3/2}} \tag{8}$$

From (7)-(8)

$$|\mathcal{C}^j| > \frac{(1-e)^{3/2}}{j\sqrt{1+e}}\,\epsilon = \epsilon_j$$

so ϵ_j is the appropriate truncation tolerance for the Fourier coefficients in S.

The truncation algorithm for the series (2) requires the calculation of the upper bound (5) for each $m = 1, 2, 3, \ldots$. Then M is the greatest integer for which

$$2\,|U|\,\frac{\sqrt{1+e}}{(1-e)^{3/2}}\,\frac{(1+M\sqrt{1-e^2})}{(1+\sqrt{1-e^2})^M}\,e^M > \epsilon$$

The truncation algorithm for the first series in (3) requires the calculation of $|\mathcal{C}_{sn}^{j1}|_{Bound}$ for each $j = 1, 2, \ldots$ and $s = j, j+1, \ldots$ and $n = s+1, s+2, \ldots$ for $n-s$ even. Then $N_1(j, s)$ is the greatest integer for which $|\mathcal{C}_{sN_1}^{j1}|_{Bound} > \epsilon_j$, $S_1(j)$ is the greatest integer for which $|\mathcal{C}_{S_1 n}^{j1}|_{Bound} > \epsilon_j$, and J_1 is the greatest integer for which $|\mathcal{C}_{sn}^{J_1 1}|_{Bound} > \epsilon_j$. The maximum indices $N_2, \ldots, N_7$ and $S_2, \ldots, S_7$ and $J_2, \ldots, J_7$ are similarly determined. Of course, the indices $N_1, \ldots, N_7$ can be no larger than the index of the highest available geopotential coefficient J_N. The index J in (1) is the maximum index amongst $J_1, \ldots, J_7$.

The first-order short-periodic variations are then given by equations (4.1-18) through (4.1-25) with the index N replaced by $\frac{1+J}{2}$.

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6.5 Third-Body Short-Periodics

The short-periodic generating function due to the gravitational field of a third-body point mass is from (4.2-13)

$$S = \mathcal{C}^0 + U(k \sin F - h \cos F) + \sum_{j=1}^{J} (\mathcal{C}^j \cos jF + \mathcal{S}^j \sin jF) \tag{1}$$

Here U is given by (6.1-1) and

$$\mathcal{C}^0 = \frac{k}{2}\mathcal{C}^1 + \frac{h}{2}\mathcal{S}^1$$

$$\mathcal{C}^j = \mathcal{I}_1^{J_1 \leq N_1+1}(j) \sum_{s=0}^{S_1 \leq N_1} \sum_{n=\max(2,j-1,s)}^{N_1} \mathcal{C}_{sn}^{j1} + \mathcal{I}_1^{J_2 \leq N_2+1}(j) \sum_{s=0}^{S_2 \leq N_2} \sum_{n=\max(2,j-1,s)}^{N_2} \mathcal{C}_{sn}^{j2} \tag{2}$$

where now

$$\mathcal{C}_{sn}^{j1} = \frac{-\mu_3}{jR_3}(2-\delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} e^{-|j-s|} w_j^{n+1,s}$$
$$[\mathrm{sgn}(j-s) C_s(\alpha,\beta) S_{|j-s|}(k,h) + S_s(\alpha,\beta) C_{|j-s|}(k,h)] \tag{3}$$
$$\mathcal{C}_{sn}^{j2} = \frac{\mu_3}{jR_3}(2-\delta_{0s})\left(\frac{a}{R_3}\right)^n V_{ns} Q_{ns} e^{-(j+s)} w_{-j}^{n+1,s}$$
$$[-C_s(\alpha,\beta) S_{j+s}(k,h) + S_s(\alpha,\beta) C_{j+s}(k,h)]$$

For brevity, we have not shown the expansions for $\mathcal{S}^j$ in (1). However, the maximum values of the truncatable indices of the series we do not show are identical to the ones determined by the procedure outlined here.

The expansions in (2) have three truncatable indices: n giving the power of $\frac{a}{R_3}$, s, and j giving the frequency of F. The purpose of the truncation algorithm is to determine the maximum values N, S, and J of n, s, and j, respectively.

Each of the Fourier coefficients (3) is less than or equal to the product of the absolute values of its factors, which using (2.5.3-5) is:

$$|\mathcal{C}_{sn}^{j1}| \leq \frac{2\mu}{jR_3}\left(\frac{a}{R_3}\right)^n |V_{ns}| |Q_{ns}(1)||w_j^{n+1,s}|_{Bound}$$

$$|\mathcal{C}_{sn}^{j2}| \leq \frac{2\mu}{jR_3}\left(\frac{a}{R_3}\right)^n |V_{ns}| |Q_{ns}(1)||w_{-j}^{n+1,s}|_{Bound}$$

From (2.8.1-3) and (6.1-3) we can obtain the constants $|V_{ns}|$ and $|Q_{ns}(1)|$. From (4.2-10) we can obtain the upper bounds on $|w_j^{ns}(e)|$ (note that the Jacobi polynomials $P_n^{\alpha\beta}(\chi) > 0$ for argument $\chi > 1$).

To determine the tolerance for the Fourier coefficients in the short-periodic generator S, we again note from (6.4-6) that a term $\mathcal{C}^j \cos jF$ in S may be significant only if

$$j\,|\mathcal{C}^j|\,\frac{\partial F}{\partial \lambda} > \bar{\epsilon} \tag{4}$$

where $\bar{\epsilon}$ is the tolerance for the third-body gravitational potential. Now from (4.2-18c)

$$\frac{\partial F}{\partial \lambda} = \frac{a}{r} = \frac{1 + e \cos f}{1 - e^2}$$

$$< \frac{1 + e}{1 - e^2} = \frac{1}{1 - e} \tag{5}$$

From (4)-(5)

$$|\mathcal{C}^j| > \frac{(1 - e)}{j}\bar{\epsilon} = \bar{\epsilon}_j$$

so $\bar{\epsilon}_j$ is the appropriate truncation tolerance for the Fourier coefficients in S.

The truncation algorithm for the first series in (2) requires the calculation of $|\mathcal{C}_{sn}^{j1}|_{Bound}$ for each $j = 1, 2, \ldots$ and $s = 0, 1, \ldots$ and $n = \max(2, j - 1, s), \ldots$ for $n - s$ even. Then $N_1(j, s)$ is the greatest integer for which $|\mathcal{C}_{sN_1}^{j1}|_{Bound} > \bar{\epsilon}_j$, $S_1(j)$ is the greatest integer for which $|\mathcal{C}_{S_1 n}^{j1}|_{Bound} > \bar{\epsilon}_j$, and J_1 is the greatest integer for which $|\mathcal{C}_{sn}^{J_1 1}|_{Bound} > \bar{\epsilon}_j$. The maximum indices N_2, S_2, and J_2 are similarly determined. The index J is the maximum index amongst J_1 and J_2.

The first-order short-periodic variations are then obtained from S by the procedure outlined in Section 4.2.

6.6 Nonconservative Short-Periodics and Second-Order Expansions

The first-order short-periodic variations for a nonconservative perturbation have the form (4.4-1)

$$\eta_{i\alpha} = \sum_{j=1}^{J}(\mathcal{C}_i^j \cos j\lambda + \mathcal{S}_i^j \sin j\lambda) \tag{1}$$

The Fourier coefficients in (1) are determined by numerical integration of the osculating rate functions. The second-order short-periodic variations have expansions analogous to (1), with the Fourier coefficients related to products of the osculating rate functions and first-order short-periodic variations. Although we have shown a λ-expansion here, it may be preferable to use alternate expansions in L or F.

The purpose of the truncation algorithm is to determine the maximum value J of the index j in (1). We propose to simply retain all Fourier coefficients which are greater (in absolute value) than the largest (in absolute value) Fourier coefficient which has been dropped from the first-order short-periodic variations for the conservative perturbations. This latter coefficient is the largest (in absolute value) of all the Fourier coefficients in the short-periodic variations neglected when applying the truncation procedures outlined in the preceding three sections.

The second-order mean element rates are also obtained from expansions of products of the osculating rate functions and first-order short-periodic variations. We propose to truncate

these expansions by retaining all terms which are greater (in absolute value) than the largest (in absolute value) term which has been dropped from the first-order mean element rates for the conservative perturbations. This latter coefficient is the largest (in absolute value) of all the terms in the mean element rates neglected when applying the truncation procedures outlined in the first three sections.

7 Numerical Methods

The numerical methods which are currently used in SST are standard. In this chapter we record the essential mathematical formulas. Further details may be found in any numerical analysis textbook (e. g., [Ferziger, 1981]).

7.1 Numerical Solution of Kepler's Equation

The equinoctial form of Kepler's Equation is (2.1.4-2):

$$\lambda = F + h\cos F - k\sin F \tag{1}$$

This equation can be solved iteratively using Newton's method:

$$
\begin{aligned}
F_0 &= \lambda \\
F_{i+1} &= F_i - \left(\frac{F_i + h\cos F_i - k\sin F_i - \lambda}{1 - h\sin F_i - k\cos F_i} \right) \text{ for } i = 0, 1, 2, \ldots
\end{aligned}
\tag{2}
$$

7.2 Numerical Differentiation

We need to differentiate functions in order to obtain the mean element rates, short-periodic variations, and partial derivatives for state estimation. Analytical formulas are preferable if possible to obtain, because of their greater precision. However, the derivatives of a function can be approximated by finite difference schemes.

We suppose $f(x)$ is a smooth function of x. Then the central difference approximation for the derivative of $f(x)$ is

$$\frac{df}{dx}(x) \approx \frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} \tag{1}$$

The error in this approximation is

$$\frac{df}{dx}(x) - \left[\frac{f(x + \Delta) - f(x - \Delta)}{2\Delta} \right] = -\frac{\Delta^2}{6}\frac{d^3 f}{dx^3}(\xi), \qquad x - \Delta \le \xi \le x + \Delta \tag{2}$$

For example, Green [1979] used the central difference approximation (1) to calculate the partial derivatives needed for state estimation (see Section 2.6). He obtained good results with a step size of $\Delta = 10^{-5}x$, using double precision.

7.3 Numerical Quadrature

We need to integrate functions in order to obtain the mean element rates and short-periodic coefficients. Analytical formulas are preferable if possible to obtain, because of their greater precision. Also, they are more computationally efficient since analytical formulas need to be evaluated only once per mean equations integration step, whereas numerical integration requires evaluation at each abscissa of the quadrature [Long and McClain, 1976]. However, numerical evaluation of integrals of the type

$$\int_a^b f(x)dx \tag{1}$$

is mandatory for the computation of the mean element rates and short-periodic coefficients involving atmospheric drag or solar radiation pressure with eclipsing. Since the substitution

$$\xi = \frac{2x - (a+b)}{b - a} \tag{2}$$

transforms the integral (1) into

$$\int_a^b f(x)dx = \frac{b - a}{2} \int_{-1}^1 f(\xi)d\xi \tag{3}$$

we can restrict our discussion to integrals with limits between -1 to $+1$ without loss of generality.

A quadrature formula approximates an integral by a weighted sum of the values of the integrand at points on the interval of integration:

$$\int_{-1}^1 f(\xi)d\xi \approx \sum_{i=1}^n w_i f(\xi_i), \qquad -1 \leq \xi_1 < \xi_2 < \dots \xi_n \leq 1 \tag{4}$$

An evaluation of different quadrature formulas has shown the Gaussian quadrature formulas to be generally efficient [Early, 1975]. The weight factors w_i for Gaussian quadratures have been tabulated, and the abscissas ξ_i are simply the zeros of the Legendre polynomial of degree n. The error in the Gaussian quadrature formula is

$$\int_{-1}^1 f(\xi)d\xi - \sum_{i=1}^n w_i f(\xi_i) \approx \frac{2^{2n+1}(n!)^4}{(2n+1)(2n!)^3} \frac{d^{2n} f(\xi)}{d\xi^{2n}}, \quad -1 \leq \xi \leq 1 \tag{5}$$

A polynomial of degree $2n - 1$ is integrated exactly.

The appropriate number n of abscissas in the Gaussian quadrature formulas needed for SST can vary from 12 to 96, depending on the highest frequency components contained in the function to be integrated. For example, Green [1979] found that if the first 10 pairs of short-periodic coefficients are to be retained in (2.5.1-13), the number n for the integrals (2.5.1-11) must be at least 48.

7.4 Numerical Integration of Mean Equations

The averaged equations of motion (1-2) may be solved with a Runge-Kutta numerical integration method. We consider the following system of ordinary differential equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

Here $\mathbf{x}$ denotes the column matrix of mean elements, and $\mathbf{f}$ denotes the column matrix of mean element rates. We divide the t-axis into points $(t_1, t_2, \ldots)$ of equal width h, and let $\mathbf{x}_i = \mathbf{x}(t_i)$. Then the standard fourth-order Runge Kutta algorithm is

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \quad \text{for } i = 1, 2, 3, \ldots \tag{2}$$

where

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_i, t_i) \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{x}_i + \frac{\Delta}{2}\mathbf{k}_1, t_i + \frac{\Delta}{2}) \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{x}_i + \frac{\Delta}{2}\mathbf{k}_2, t_i + \frac{\Delta}{2}) \\
\mathbf{k}_4 &= \mathbf{f}(\mathbf{x}_i + \Delta\mathbf{k}_3, t_i + \Delta)
\end{aligned} \tag{3}
$$

The error in the formulas (2) is bounded by

$$C\Delta^5 \frac{d^5\mathbf{x}_i}{dt^5} \tag{4}$$

where C is a constant.

Since the mean element rates depend only on slowly varying quantities, step sizes Δ of a day or more can usually be used. The integrator time step Δ should be $\frac{1}{8}$ or less of the minimum period τ of the oscillations included in the mean equations of motion. Some limitations are the period of orbital precession due to J_2 and the period of the moon.

Initial values of the mean elements $a_i(t_1)$ can be obtained from initial values of the osculating elements $\hat{a}_i(t_1)$ by either of two methods:

1. Numerically integrate the VOP equations of motion over a time interval at least as long as the period of the largest significant short-periodic effect (usually one or two satellite orbits - see [McClain and Slutsky, 1980]), and then use a differential correction procedure to find the initial mean elements which give the best least-squares fit between the SST trajectory and the Cowell trajectory.

2. Use successive substitution into the near-identity transformation (1-1) until a specified agreement is reached:

$$
\begin{aligned}
a_i^0(t_1) &= \hat{a}_i(t_1) \\
a_i^{k+1}(t_1) &= \hat{a}_i(t_1) - \eta_i[a_1^k(t_1), \ldots, a_6^k(t_1), t_1] \quad \text{for } k = 0, 1, 2, \ldots
\end{aligned} \tag{5}
$$

This method is faster than method 1, but may require the inclusion of a comprehensive set of short-periodic variations to avoid a large bias in the initial mean elements.

It should be pointed out that the time averages of the osculating elements over some time interval are generally *not* a good approximation to the mean elements [Early, 1986].

7.5   Interpolation

Since the mean elements and short-periodic coefficients are slowly varying, their values at desired times not coinciding with the mean equation step times can be computed by relatively low order interpolation formulas.

First, suppose that at distinct times $(t_1, \ldots, t_n)$ we know the values $[f(t_1), \ldots, f(t_n)]$ of a smooth function $f(t)$. In Lagrange interpolation we approximate $f(t)$ by a polynomial of degree $n-1$ passing through the known values:

$$f(t) \approx \sum_{i=1}^{n} f(t_i) L_i(t) \tag{1}$$

Here

$$L_i(t) = \frac{(t - t_1) \cdots (t - t_{i-1})(t - t_{i+1}) \cdots (t - t_n)}{(t_i - t_1) \cdots (t_i - t_{i-1})(t_i - t_{i+1}) \cdots (t_i - t_n)} \tag{2}$$

Note that

$$L_i(t_j) = \delta_{ij} \tag{3}$$

The error in the Lagrange interpolation formula is

$$f(t) - \sum_{i=1}^{n} f(t_i) L_i(t) = \frac{(t - t_1) \cdots (t - t_n)}{n!} \frac{d^n f}{d\xi^n}(\xi), \quad t_1 < \xi < t_n \tag{4}$$

Lagrange interpolation is currently used to interpolate the short-periodic coefficients, the velocity vector, and the partial derivatives needed for differential correction. An adequate order $n-1$ has been found to be 3 (4 interpolator points) [Taylor, 1978].

Next, suppose that at distinct times $(t_1, \ldots, t_n)$ we know both the values $[f(t_1), \ldots, f(t_n)]$ and the derivatives $[\dot{f}(t_1), \ldots, \dot{f}(t_n)]$ of a smooth function $f(t)$. In Hermite interpolation we approximate $f(t)$ by a polynomial of degree $2n-1$ passing through the known values and derivatives:

$$f(t) \approx \sum_{i=1}^{n} \left\{ [1 - 2(t - t_i)\dot{L}_i(t_i)][L_i(t)]^2 f(t_i) + (t - t_i)[L_i(t)]^2 \dot{f}(t_i) \right\} \tag{5}$$

Here again $L_i(t)$ are the Lagrange basis functions (2). The error in the Hermite interpolation formula is

$$\frac{[(t - t_1) \cdots (t - t_n)]^2}{(2n)!} \frac{d^{2n} f}{d\xi^{2n}}(\xi), \quad t_1 < \xi < t_n \tag{6}$$

Hermite interpolation is currently used to interpolate the mean elements and the position vector. An adequate order $2n-1$ has been found to be 5 (3 interpolator points).

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