Optimal Multiple-Impulse Time-Fixed Rendezvous Between Circular Orbits

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Minimum-fuel, multiple-impulse, time-fixed solutions are obtained for circle-to-circle rendezvous. The coplanar case and a restricted class of noncoplanar cases are analyzed. For several initial phase angles of the target relative to the vehicle, optimal solutions are obtained for a range of fixed transfer times and terminal radii. Primer vector theory is used to obtain the optimal number of impulses, their times and positions, and the presence of initial or final coasting arcs. For sufficiently large transfer times, the optimal solutions become the well-known time-open solutions, such as the Hohmann transfer for the coplanar case. The results obtained can be used to perform a time vs fuel trade-off for missions which have operational time constraints, such as space rescue operations and avoidance maneuvers.

I. Introduction

A RENDEZVOUS of an orbiting vehicle with a target body in another orbit can be accomplished in a specified time by two thrust impulses. The first impulse places the vehicle on a trajectory which intercepts the target at the specified time; the second impulse matches the vehicle velocity with the target velocity to complete the rendezvous. For the simplest case of a coplanar circle-to-circle rendezvous, the minimum-fuel solution is the well-known two-impulse Hohmann transfer (if the ratio of the terminal radii is less than 11.94) *only if* the specified transfer time is sufficiently large. For the Hohmann transfer to be the optimal rendezvous, the specific transfer time must be equal to or greater than the sum of a) the time required for the correct phasing to occur between the vehicle and the target in order that the transfer ellipse intercept the target, and b) the Hohmann transfer time for the specified terminal orbits. The time (a) required for the correct phasing represents an initial coast arc on the optimal solution. If the specified transfer time is greater than the sum of (a) and (b), the optimal solution is the Hohmann transfer followed by a final coast arc after the rendezvous is accomplished until the specified time has elapsed. The Hohmann transfer is the time-open solution for the optimal rendezvous.

The more interesting case addressed in this study occurs when the specified transfer time is less than the sum of (a) and (b) described above. In this case the Hohmann transfer is not the optimal solution because it violates the time constraint. Applications which might require this constrained transfer time are space rescue operations, avoidance maneuvers, and other missions involving operational constraints on transfer time. A simple examples is rendezvous between close orbits, in which case the phasing time (a) can be excessive due to the large synodic period.

In order to accomplish minimum-fuel, time-fixed rendezvous between circular orbits in both the coplanar and noncoplanar case, the optimal solutions often require the use of additional impulses. The object of this study is to determine optimal, multiple-impulse, time-fixed solutions for coplanar circle-to-circle rendezvous and for a restricted class of noncoplanar rendezvous.

The complete details of this study are presented in Ref. 1 and represent an extension to the nonlinear and noncoplanar cases of the optimal time-fixed solutions obtained by Prussing[2,3] for linearized coplanar rendezvous between close circular orbits. In the linearized coplanar case, as many as four impulses are required for an optimal time-fixed rendezvous. In the noncoplanar linearized case, the maximum number of impulses required is six. An analysis of the time-open linearized problem is given by Jones.[4]

Earlier studies of circle-to-circle rendezous include a comparison of the Hohmann transfer and the bielliptic transfer for the coplanar case by Billik and Roth[5] and a study of the noncoplanar case by Baker.[6]

II. Necessary Conditions for an Optimal Transfer

The equations of motion of a spacecraft which is thrusting in a gravitational field can be written in terms of the orbital radius vector $\boldsymbol{r}$ as[7,8]:

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{1}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{g}(\boldsymbol{r}) + \Gamma \boldsymbol{u} \tag{2}$$

$$\dot{J} = \Gamma \tag{3}$$

The variable Γ is the thrust acceleration magnitude ($0 \le \Gamma \le \Gamma_{max}$), $\boldsymbol{u}$ is a unit vector in the thrust direction, and J is the characteristic velocity to be minimized. Equations (1-3) describe the behavior of the state vector $\boldsymbol{x}^T = [\boldsymbol{r}^T \boldsymbol{v}^T J]$ under the influence of the gravitational acceleration $\boldsymbol{g}(\boldsymbol{r})$ and the control variables Γ and $\boldsymbol{u}$. For a high-thrust engine one can make the impulsive thrust approximation by assuming the thrust

Submitted June 13, 1984; presented as Paper 84-2036 at the AIAA/AAS Astrodynamics Conference, Seattle, WA, Aug. 20-22, 1984; revision received April 25, 1985.

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magnitude is unbounded ($\Gamma_{max} \to \infty$). In this case the engine is either off ($\Gamma = 0$) or provides an impulsive thrust of infinitesimal time duration. The solution for impulsive thrusts then requires the determination of the times, locations, and directions of the thrust impulses which satisfy the specified boundary conditions for an orbit transfer, interception, or rendezvous.

The determination of a minimum-fuel solution requires the solution of an optimal control problem over the time interval $t_0 \leq t \leq t_f$ which minimizes the final value of J and satisfies the equations of motion and the orbital boundary conditions of the problem.

The Hamiltonian function, which is to be maximized, is then obtained using the equations of motion Eqs. (1-3) as

$$H = \lambda_r^T v + \lambda_v^T (g + \Gamma u) + \lambda_J \Gamma \tag{4}$$

The adjoint equations for the problem are

$$\dot{\lambda}_r^T = -\partial H/\partial r = -\lambda_v^T G(r) \tag{5}$$

$$\dot{\lambda}_v^T = -\partial H/\partial v = -\lambda_r^T \tag{6}$$

$$\dot{\lambda}_J = -\partial H/\partial J = 0 \tag{7}$$

where $G(r)$ is the (symmetric) gravity gradient matrix. The boundary conditions on λ_r and λ_v depend on the state terminal constraints $r(t_F)$ and $v(t_F)$, but, because the characteristic velocity is unconstrained, the constant value of its adjoint variable is

$$\lambda_J(t) = -1 \tag{8}$$

The Hamiltonian H of Eq. (4) is maximized over the choice of thrust direction u by maximizing the dot product $\lambda_v^T u$, i.e., by aligning the thrust vector with the adjoint to the velocity λ_v. Because of the significance of the vector λ_v, it has been termed the *primer vector* by Lawden[9] in his pioneering work in minimum-fuel orbit transfer. Denoting the primer vector by p and incorporating the fact that the optimal thrust direction is aligned with p, the adjoint Eqs. (5-7) and the Hamiltonian Eq. (4) can be expressed as

$$\ddot{p} = G(r)p \tag{9}$$

$$H = p^T g - \dot{p}^T v + (p-1)\Gamma \tag{10}$$

where p is the magnitude of the primer vector. Note that the primer vector satisfies the same Eq. (9) as the first-order perturbation δr in the radius vector about a reference no-thrust orbit. Convenient forms of the solution to this equation for an inverse-square gravitional field are given by Glandorf[10] and by Gravier, Marchal and Culp.[11]

From the Hamiltonian, Eq. (10), one identifies the switching function for the thrust magnitude as $p-1$. In the continuous thrust case, the Hamiltonian is maximized by choosing $\Gamma = 0$ when $p < 1$, and $\Gamma = \Gamma_{max}$ when $p > 1$. In the impulsive case $\Gamma = 0$ when $p < 1$, with the impulses occurring at those instants at which $p(t)$ is tangent to $p = 1$ from below.[9]

The necessary conditions for an optimal impulsive transfer, first derived by Lawden,[9] can be expressed entirely in terms of the primer vector as:

1) The primer vector satisfies Eq. (9) and must be continuous with continuous first derivative.

2) The magnitude $p \leq 1$ during the transfer with impulses occurring at those instants for which $p = 1$.

3) At an impulse time the primer vector is a unit vector in the optimal thrust direction.

4) As a consequence of condition 2, $\dot{p} = \dot{p}^T p = 0$ at all interior impulses (not at the initial or final time).

Figure 1 displays a primer vector magnitude time history for an optimal four-impulse solution having a transfer time equal to 2.3 inner orbit periods. Figure 2 depicts an optimal solution of duration 2.8 which is comprised of three-impulses with both an initial coast arc of duration 0.5 and a final cost arc of duration 0.8.

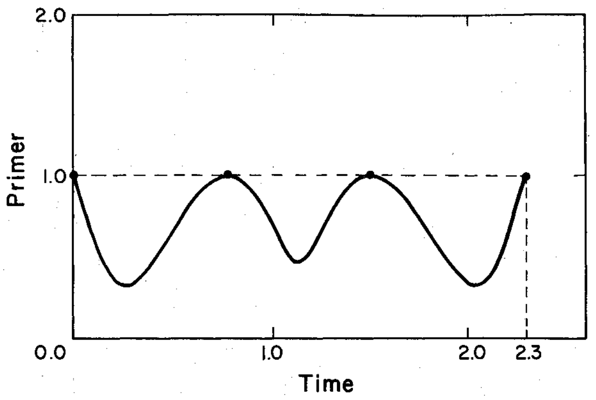

Fig. 1 Optimal four-impulse primer magnitude.

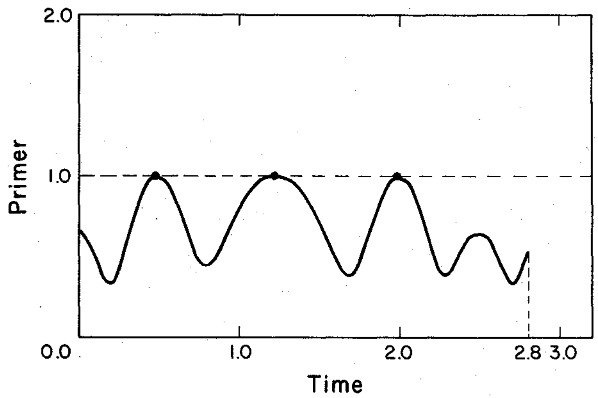

Fig. 2 Optimal three-impulse with terminal coasts primer magnitude.

III. The Minimization Procedure

Based on these necessary conditions, Lion and Handelsman[12] formulated a procedure for obtaining optimal time-fixed solutions which is used to obtain the optimal rendezvous solutions in this study. This procedure has been applied by Jezewski and Rozendaal,[13] Gross and Prussing[14] and others. Briefly described, the primer vector is first evaluated along the two-impulse solution which satisfies the orbital boundary conditions, enforcing the necessary condition that the primer vector at the impulse times is a unit vector in the thrust direction:

$$p(t_0) = \Delta v_0 / \Delta v_0 \tag{11}$$

$$p(t_f) = \Delta v_f / \Delta v_f \tag{12}$$

The required velocity changes in Eqs. (11-12) are obtained by solving Lambert's Problem for the specified transfer time and orbital boundary conditions. A very efficient Lambert algorithm is that of Battin and Vaughan.[15] If the solution violates the necessary conditions for an optimal solution, the manner in which the conditions are violated provides information on how to obtain a neighboring solution which provides a first-order decrease in the cost. This is accomplished by either adding an additional midcourse impulse, adjusting impulse locations or times, or by including an initial or final coasting arc. The trajectory so modified becomes a new reference trajectory along which the primer vector is again calculated to

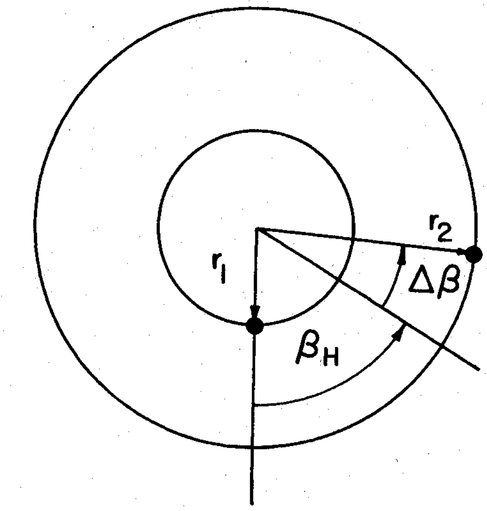

Fig. 3 Target initial phase angle.

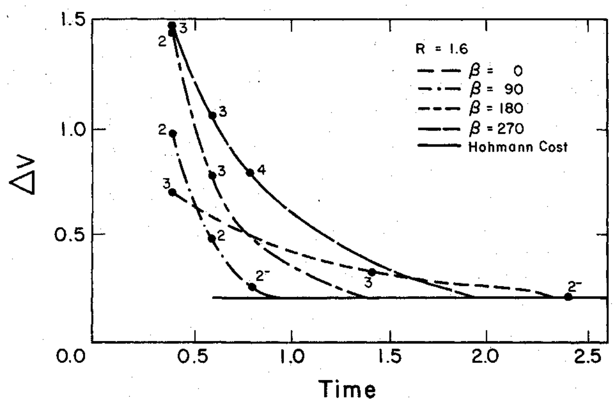

Fig. 4 ΔV vs time plot for $R=1.16$.

determine how to improve the trajectory. An iterative process is thus developed which incrementally decreases the cost on each iteration, converging to a solution which satisfies the necessary conditions. The optimal number of impulses and their positions and times are automatically determined by the iterative process.

The criterion for including an additional impulse is obtained by comparing a two-impulse trajectory (or segment of a multiple-impulse trajectory) with a neighboring three-impulse trajectory having a midcourse impulse of magnitude Δv_m at an arbitrary time t_m in the direction u_m. The first order change in the cost obtained by including the additional impulse is given by[12]:

$$\delta J = \Delta v_m (1 - p_m^T u_m) \tag{13}$$

This change in cost can be made negative if the primer vector magnitude exceeds unity at any time along the two-impulse trajectory or segment. This constitutes an alternative proof of Lawden's condition that $p \leq 1$ on an optimal solution. The greatest decrease in the cost is obtained in Eq. (13) when t_m is chosen to be at the time at which the primer magnitude is a maximum and the impulse direction u_m is chosen to be aligned with p_m.

Once an additional impulse has been included, each two-impulse segment is analyzed in the manner described above. The criteria for iteratively adjusting the time t_m and position r_m of each midcourse impulse are obtained from the expression for the differential cost between two neighboring three-impulse trajectories, for which[12]

$$\delta J = (\dot{p}_{m+}^T - \dot{p}_{m-}^T)\mathrm{d}r_m + (H^+ - H^-)\mathrm{d}t_m \tag{14}$$

where the notations + and − refers to conditions immediately after and prior to the midcourse impulse. Equation (14) provides the gradients of the cost with respect to changes in the position r_m and the time t_m of the midcourse impulse, which can be used in an iterative minimization algorithm.

The criterion for including an initial coasting arc is obtained by considering an initial coast of duration $\mathrm{d}t_0 > 0$ prior to the first impulse. The differential change in cost is[12]

$$\delta J = -\Delta v_0 \dot{p}_0^T p_0 \mathrm{d}t_0 \tag{15}$$

This change in cost is negative if the necessary conditions are violated by $\dot{p}_0^T p_0 > 0$, which then becomes the condition under which an initial coast will decrease the cost.

The criterion for including a final coast represented by $\mathrm{d}t_f < 0$ is obtained analogously. The differential change in cost is

$$\delta J = -\Delta v_f \dot{p}_f^T p_f \mathrm{d}t_f \tag{16}$$

Because $\mathrm{d}t_f < 0$, the condition under which a final coast will decrease the cost is $\dot{p}_f^T p_f < 0$. Complete descriptions of these conditions and their applications are given in Refs. 1, 12, and 13.

IV. Optimal Coplanar Circle-to-Circle Rendezvous

As mentioned in the Introduction, the time-open minimum-fuel rendezvous solution between coplanar circular orbits of radii r_1 and r_2 is the Hohmann transfer if $1 < R < 11.94$, where $R = r_2/r_1$ and it is assumed that $r_2 > r_1$. In all of the discussion and results which follow, it is assumed that the initial orbit is the inner orbit, i.e., the transfers are from the inner to outer orbit. The corresponding outer-to-inner transfers are obtained by simply considering the final time to be initial time, running time backwards $(\tau - \tau_0 = t_F - t)$, and reversing the directions of the thrust impulses.

The total time required for the optimal time-open (Hohmann) transfer depends on the terminal orbit radii and on the initial phase angle of the target relative to the vehicle. This phase angle varies at a constant rate with period equal to the synodic period S of the terminal orbits, given by

$$S/P_1 = P_2/(P_2 - P_1) = R^{3/2}/(R^{3/2} - 1) \tag{17}$$

where P_1 and P_2 are the periods of the terminal orbits. As shown in Fig. 3, an arbitrary phase angle β can be described as $\beta = \beta_H + \Delta\beta$, where β_H is the phase angle required for the Hohmann transfer to intercept the target and $\Delta\beta$ the deviation from this value $(0 \leq \Delta\beta < 2\pi)$. The value of β_H, which is always a "lead angle" for inner-to-outer transfers, is given by

$$\beta_H = \pi\{1 - [(1+R)/2R]^{3/2}\} \tag{18}$$

which, for $1 < R < \infty$ increases monotonically with R between the limits $0 < \beta_H < 116.36$ deg.

If $\beta = \beta_H$ at the initial time $(\Delta\beta = 0)$, a Hohmann transfer is available immediately and will intercept the target in a transfer time T_H:

$$T_H/P_1 = [(1+R)^3/2^5]^{1/2} \tag{19}$$

If, on the other hand, $\beta \neq \beta_H$, a waiting time is required prior to the first Hohmann impulse until the deviation $\Delta\beta = \beta - \beta_H$ decreases to zero. This waiting time T_w is given by

$$T_w/P_1 = (\Delta\beta/2\pi)R^{3/2}/(R^{3/2} - 1) \tag{20}$$

That is, the waiting time is proportional to $\Delta\beta$ with the proportionality constant being the synodic period divided by 2π.

As seen in Eq. (20) for a given $\Delta\beta$, $T_w \to \infty$ as $R \to 1$. For a given R, $T_w \to S$ as $\Delta\beta \to 2\pi$. As a result, the total transfer time required for a rendezvous using a Hohmann transfer $T = T_w + T_H$ can be excessively long, especially for close orbits and large initial phase angles. For close orbits T_w tends to be large because the synodic period is large, e.g., for $R = 1.1$, $\Delta\beta = \pi$, the total time $T = 4.3P_1$, of which 87% is waiting time. For distant orbits, the synodic period is smaller, but the Hohmann transfer time is longer, e.g., for $R = 6.6$, $\Delta\beta = 3\pi/2$, $T = 4.5P_1$, of which only 18% is waiting time.

For these reasons, alternatives to the Hohmann transfer have been investigated for use in interception and rendezvous missions. A class of conceptually simple nonoptimal alternatives which require shorter times was analyzed in Ref. 5. In the current study, minimum-fuel solutions are obtained over a range of fixed transfer times (which are less than the time required for the Hohmann transfer) for specified initial phase angles β and ratios of terminal orbit radii R. Plots of total ΔV vs transfer time are made and can be used for a time vs fuel tradeoff. One can, for example, 1) specify the transfer time and determine the minimum-fuel cost for the rendezvous, or 2) determine the shortest transfer time for a given ΔV budget.

Figure 4 displays optimal coplanar rendezvous solutions for $R = 1.6$ for four values of initial target phase angle, $\beta = 0$, 90, 180, and 270 deg. The ΔV cost is plotted in units of inner circular orbit speed, and the transfer time is plotted in units of inner orbit period P_1. The solid horizontal line at $\Delta V = 0.21$ represents the Hohmann transfer cost, which is the absolute minimum cost and is attained along each curve for the sufficiently large value of transfer time described previously. As seen in Fig. 4, the optimal time-fixed solutions require two, three or four impulses, depending on the transfer time and the phase angle. The symbol 2^- designates a two-impulse with initial coast solution. The point at which each curve meets the Hohmann cost line is a 2^- transfer with the initial coast being the waiting time. For large transfer times the optimal solution is a $^-2^+$ transfer, that is, a two-impulse (Hohmann) transfer with both an initial and final coast. Similar plots for other values of R are given in Ref. 1.

Plots of $\Delta\beta$ vs transfer time, such as Fig. 4, provide valuable fuel vs time tradeoff information for a given initial target position. These curves are nonincreasing functions of transfer time because the cost of any optimal solution, represented by a point on a particular curve, can be extrapolated horizontally to large transfer times merely by adding a final coast arc to the solution (after the rendezvous has been accomplished). Thus no optimal transfer for a transfer time larger than a given optimal transfer can have a higher cost than the given optimal transfer.

Fig. 5 displays, for $R = 1.2$ and $\beta = \pi$, a comparison of the optimal solutions with the two impulse solutions requiring the same transfer times. For $T \leq 0.4$ the optimal solutions are two-impulse solutions, and for $T \geq 2.4$ the optimal solution is the Hohmann transfer. For $0.4 < T < 2.4$ a significant amount of fuel can be saved over two-impulse solutions by using three and four impulse transfers as shown. For $0.54 < T < 1.4$ the best (nonoptimal) two-impulse solution available is the solution at $T = 0.54$ followed by a final coast, with a cost of $\Delta V = 0.75$. By contrast, the optimal solutions over this same range of transfer times utilize either three or four impulses with the cost decreasing from approximately $\Delta V = 0.7$ down to 0.2 as shown, representing a savings of from 7% to 73%. Four-impulse solutions remain the optimal solutions, using less fuel than two-impulse solutions until $T = 2.4$, when the Hohmann transfer becomes available. The reason that the dashed two-impulse curve in Fig. 5 does not descend to the 2^- Hohmann Transfer is that transfers on the dashed curve have no coast periods.

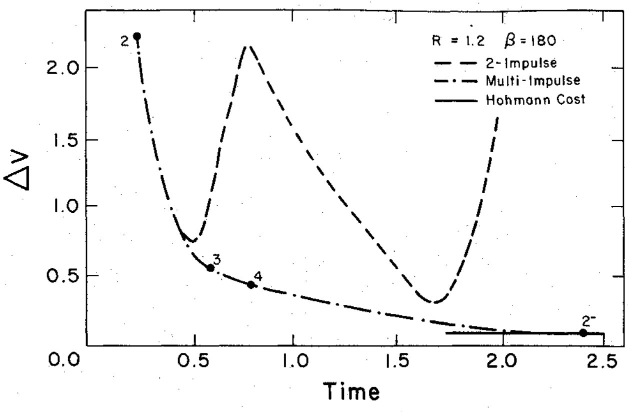

Fig. 5 Comparison of optimal and two-impulse solutions.

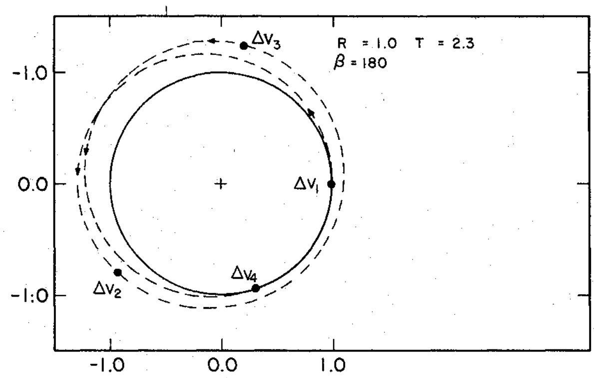

Fig. 6 Example four-impulse optimal rendezvous.

Figure 6 shows the optimal trajectory for the problem of rendezvous with a target in the same circular orbit ($R = 1$), having an initial phase angle $\beta = \pi$. Depending on the transfer time, the optimal solutions for all $R = 1$ rendezvous require either two or four impulses,[1] in agreement with the linearized results of Refs. 2 and 3. For a transfer time $T = 2.3$, the optimal solution is the multiple-revolution, four-impulse trajectory shown in Fig. 6. The vehicle departs the initial orbit tangentially and orbits outside the target orbit, allowing the target to catch up with the vehicle. The vehicle then intercepts the target orbit tangentially to complete the optimal rendezvous. The cost of this rendezvous is $\Delta V = 0.189$. By comparison, the best available two-impulse rendezvous has a cost of $\Delta V = 0.224$.

For the (nonoptimal) two-impulse rendezvous the vehicle departs the initial orbit nearly tangentially, going into an elliptical orbit having a period of slightly less than 1.5 target orbit periods. After one vehicle orbit the vehicle intercepts the target nearly tangentially. A second impulse completes the rendezvous at a cost which is 19% larger than the optimal, four-impulse solution.

V. Optimal Noncoplanar Circle-to-Circle Rendezvous

The problem of rendezvous between noncoplanar orbits has received less attention than the coplanar case due to the increased difficulty of considering the out-of-plane motion. The excellent survey article by Gobetz and Doll[16] discusses the basic modes which have usually been considered: 1) the (two-impulse) Hohmann transfer with plane change, 2) the (three-impulse) bielliptic transfer with plane change, and 3) the (three-impulse) modified Hohman transfer (in-plane Hohmann transfer to the outer radius followed by plane change at the line of nodes).

A restricted class of noncoplanar rendezvous is considered in this study. The restriction is that the vehicle is on the line of

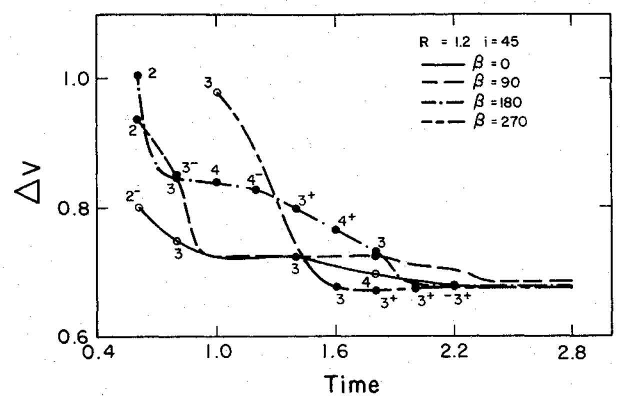

Fig. 7 ΔV **vs time plot for** $R = 1.2$, $i = 45$ **deg.**

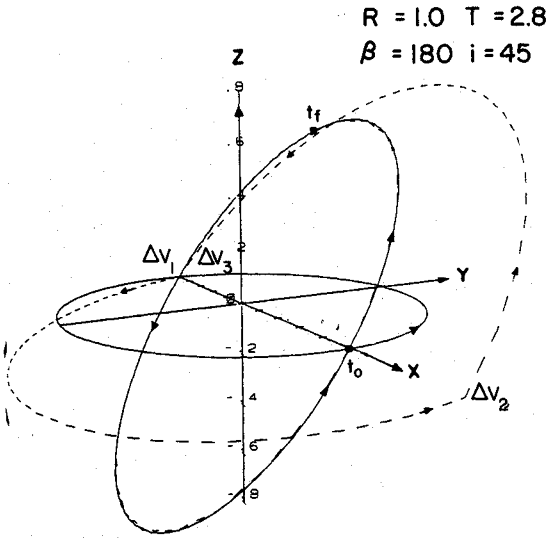

Fig. 8 Example optimal rendezvous for $R = 1$, $i = 45$ **deg.**

nodes at the initial time. Each rendezvous can then be described in terms of the terminal radius ratio R, the initial phase angle of the target with respect to the vehicle β, the inclination of the target orbit i, and the transfer time. Another way of describing this same initial geometry is that the initial argument of latitude of the vehicle (angle between the nodal and radius vectors) is zero. The initial argument of latitude of the target is the initial phase angle β.

Figure 7 shows plots of ΔV vs transfer time for $R = 1.2$ and $i = 45$ deg for four values of β. The curves are nonincreasing functions of transfer time, although slight plotting inaccuracies make it appear otherwise. The optimal solution consists of two, three, or four impulses, in some cases with coasting periods. The solutions for all four values of β shown approach the same time-open solution for a sufficiently long transfer time, namely, the bielliptic transfer with plane change. This result is consistent with Fig. 7 of Ref. 16, which summarizes time-open noncoplanar circle-to-circle transfers. Results for different values of R and i are given in Ref. 1.

Figure 8 shows the optimal trajectory for rendezvous with a target in a circular orbit of the same radius ($R = 1$) inclined at 45 deg for a value of $\beta = \pi$ and a time of 2.8 orbit periods. This time is sufficiently long that the optimal solution is the $^{-}3^{+}$ time-open bielliptic transfer with plane change, having both initial and final coasting arcs. The primer magnitude for this is shown in Fig. 2. The total plane change is optimally split among the impulses. Following an initial coast of 0.5 initial orbit periods, 4.8-deg plane change occurs at the first impulse. An additional 35.4-deg plane change occurs at the second impulse at radius 1.62, and a final plane change of 4.8 deg occurs

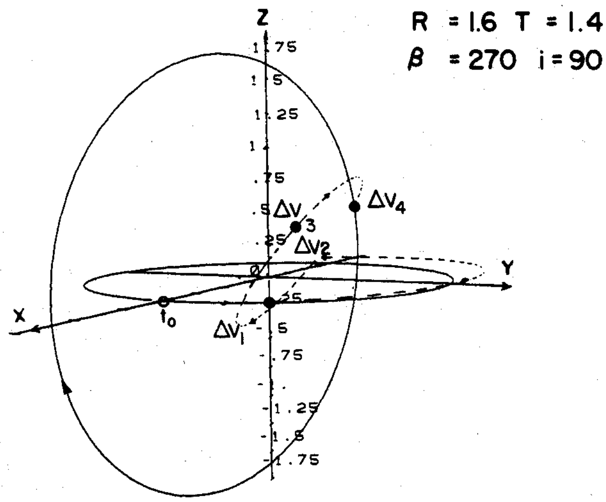

Fig. 9 Example optimal rendezvous for $R = 1.6$, $i = 90$ **deg.**

at the third and final impulse. A final coasting period of 0.8 then satisfies the time-fixed value of 2.8.

This optimal solution, composed of an initial coast of duration 0.5, a transfer of duration 1.5, and a final coast of duration 0.8 is nonunique. The same solution can be used with no initial coast and a final coast of 1.3 or with an initial coast of 1.0 and a final coast of 0.3. The cost of this optimal solution is $\Delta V = 0.703$, which agrees with Fig. 7 of Ref. 16. By comparison, the velocity change necessary to perform the 45-deg plane change at the initial radius is greater, $\Delta V = 0.765$. Thus, even for $\beta = 0$, for which a one-impulse solution is available, the time-open optimal solution is the three impulse, bielliptic transfer with plane change. This is the optimal rendezvous solution if the fixed transfer time is consistent with Ref. 16 and also with the last figure of Ref. 17 (which is mistitled and should read "Transfer Between Inclined Circular Orbits of Equal Radius").

Figure 9 shows an extreme case of a 90-deg plane change for $R = 1.6$, $\beta = 3\pi/2$, and a relatively short transfer time of 1.4 initial orbit periods. The optimal solution requires four impulses with a short initial coast arc of duration 0.05. The plane change is optimally split among the impulses as follows: 2.5 deg at the first impulse, 60.9 deg at the second impulse at radius 1.44, essentially zero at the third impulse at radius 0.33, and 26.6 deg at the fourth impulse. It is interesting to note that, although the specified transfer time is relatively short (approximately 70% of the target orbit period), a short initial coast is optimal, and four impulses are required. The time-open optimal solution for this case is the biparabolic transfer through infinity,[16] which requires infinite time.

For the restricted class of noncoplanar transfers considered, the typical number of impulses required for optimal rendezvous is two, three, or four. In a few isolated cases five impulses are required.[1] It is thought that in the general noncoplanar case (nonzero initial vehicle argument of latitude) a larger number of impulses may become more prevalent for optimal rendezvous. Only in the linearized case can a theoretical maximum be determined, which is equal to six.[17]

VI. Concluding Remarks

For both coplanar and a restricted class of noncoplanar circle-to-circle rendezvous, optimal, time-fixed, multiple-impulse solutions have been obtained using primer vector theory. The results are useful for a time vs fuel trade-off for missions which have operational time constraints, such as space rescue missions or avoidance maneuvers.

For sufficiently large transfer times, the solutions become the well-known time-open solutions, such as the Hohmann transfer for the coplanar case. For specified transfer times which are less than the time-open values, optimal time-fixed solutions are obtained, including the optimal number of impulses, their times and locations, and the existence and duration of initial or final coasting arcs. Optimal time-fixed solutions often use additional impulses to minimize fuel, and can provide a significant fuel savings over two-impulse solutions. Another interesting aspect of the results is that even in cases where the specified transfer time is relatively small, the optimal solution may employ initial or final coasting arcs in order to take advantage of more favorable geometry.

The question of global optimality of the minimizing solutions is an important consideration. If the optimal time-fixed solutions approach the known time-open global optimal solutions as the transfer time is increased, one can be fairly confident that the solutions are globally optimal. However, the iteration algorithm can and sometimes does converge to only a local minimum, which does not tend toward the known time-open optimal solution. In this case, one must introduce an additional impulse or a coast period to cause the algorithm to explore other possibilities in its convergence to an optimal solution.

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