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Primer Vector on Fixed-Time Impulsive Trajectories

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In this paper, the definition of the primer vector is extended to include nonoptimal as well as optimal trajectories. With this definition, simple tests are developed which determine how a given trajectory can be improved (in terms of velocity requirements). This problem arose in the study of the use of impulsive trajectories to generate approximate adjoint initial conditions for finite thrust vehicles. To do this, the optimum fixed-time impulsive trajectory must be found. However, since many mission analyses are done on an impulsive basis, a wider application is foreseen. Necessary conditions are developed for when an additional impulse can improve the trajectory; how interior impulses of a multi-impulse trajectory can be moved so as to decrease the cost; and when initial and/or final coasts improve the trajectory. In the case of transfers between circular, coplanar orbits a geometric interpretation is given. For the case of an inverse-square gravitational field, the components of the primer vector can be calculated analytically. Using Floquet theory, a convenient form of this solution is presented.

Introduction

THE term "primer vector" was introduced by Lawden[1] to denote the three adjoint variables associated with the velocity vector on an optimal trajectory. Lawden derived a necessary condition for the optimality of impulsive trajectories in terms of the magnitude of this vector. (Optimum in this paper is defined as minimum characteristic velocity.)

In this paper, the definition of the primer vector is extended to nonoptimal impulsive trajectories. It is shown that this primer gives a clear indication of how the original, or reference, trajectory can be improved in terms of decreasing characteristic velocity requirements. The two main results are:

1) The criterion for an additional impulse. Using this test indicates whether or not the reference trajectory can be improved by an additional midcourse impulse.

2) The transversality condition. Using this test one can determine how the interior (midcourse) impulses of the reference trajectory should be moved in both position and time so as to decrease the characteristic velocity. Also, it is possible to determine whether initial and/or final coasts will improve the trajectory.

For an inverse square gravitational field, the solution for the primer vector is known in analytical form. A particularly simple form of the solution is included here, which is considered useful for analytical work (although not appropriate for numerical calculations).

This work originated with the problem of using the adjoint variables calculated from impulsive trajectories as starting values for an iterative procedure for optimum finite thrust trajectories which satisfy the same boundary conditions. The success of this technique is demonstrated in Ref. 2 for two burn trajectories. As yet, the method has not been verified numerically for multiburn trajectories. A natural conjecture is that to solve these more complicated cases one should again begin with the corresponding optimal impulsive trajectory. Using the techniques discussed here, it is possible to devise an algorithm which will determine a locally optimum multi-impulse trajectory starting from any initial trajectory which satisfies the boundary conditions. Since this would be essentially a gradient technique, it would converge only to a minimum within the same "valley." The existence of other solutions is not ruled out.

A second use of these techniques lies in mission analyses using chemical rockets. Here, the impulsive approximation is sufficiently accurate for most purposes. By a simple calculation (given the Lambert solution) it is possible to tell in what direction the relevant parameters should be changed.

Formulation and Notation

The equation of motion is

$$\ddot{x} = \nabla\Phi \tag{1}$$

where Φ is the gravitational potential. For impulsive trajectories, the velocity vector $v = (v_1,v_2,v_3)$ can be altered discontinuously; however, the position vector $x = (x_1,x_2,x_3)$ must be continuous. The criterion of optimality is the sum of the magnitudes of the velocity increments,

$$J = \sum_k |\Delta v_k|$$

The optimization problem may then be stated as follows: given an initial state (v_0,x_0) and a final state (v_f,x_f), find the trajectory which connects these states in a given travel time t_f such that J is minimized. Assume that Γ is a trajectory which satisfies the boundary conditions and consider small perturbations about Γ. Let (x,v) and (x',v') denote the state vectors on Γ and on a perturbed trajectory Γ', respectively. Define

$$\begin{aligned} \delta x(t) &= x'(t) - x(t) \\ \delta v(t) &= v'(t) - v(t) \end{aligned} \tag{2}$$

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If Γ and Γ' are sufficiently close to justify a linear analysis, then $(\delta v, \delta x)$ are, to first order, the solutions of the following variational equations of (1):

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{v} \end{pmatrix} = \begin{pmatrix} O & I \\ G & O \end{pmatrix} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} \tag{3}$$

where I is the (3×3) identity matrix and G is the gravity-gradient matrix. The elements of G are given by

$$g_{ij} = (\partial^2\Phi/\partial x_i \partial x_j)$$

In second-order form (3) can be written

$$\delta\ddot{x} = G\delta x \tag{4}$$

The (6×6) transition matrix $\Omega(t,\tau)$ for this system can be partitioned into four (3×3) matrices as follows:

$$\Omega(t,\tau) = \begin{pmatrix} \Omega_{11}(t,\tau)\Omega_{12}(t,\tau) \\ \Omega_{21}(t,\tau)\Omega_{22}(t,\tau) \end{pmatrix} \tag{5}$$

The adjoint system to (3) is

$$\begin{pmatrix} \dot{\mu} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} O & -G \\ -I & O \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \end{pmatrix}$$

where μ and λ are 3-vectors. In second-order form this becomes

$$\ddot{\lambda} = G\lambda \tag{6}$$

identical to (4). Hence, the transition matrix for $(\lambda,\dot{\lambda})$ will be identical to (5):

$$\begin{pmatrix} \lambda(t) \\ \dot{\lambda}(t) \end{pmatrix} = \Omega(t,\tau)\begin{pmatrix} \lambda(t) \\ \dot{\lambda}(t) \end{pmatrix} \tag{7}$$

From the definition of the adjoint system, it can be shown that the identity

$$\lambda \cdot \delta v - \dot{\lambda} \cdot \delta x = \text{const} \tag{8}$$

holds everywhere on Γ. This equation is the basis for most of the analysis which follows.

Solution for the Transition Matrix in an Inverse Square Field

For the two-body case, it is possible to find an explicit solution for the transition matrix. Several such solutions have been published previously[1,3,4]; however, the solution presented here is considered to be particularly simple and useful for analytical work.

For an inverse square field ($\Phi = 1/r$) the elements of G have the form

$$g_{ij} = -(1/r^3)[\delta_{ij} - (3x_ix_j/r^2)]$$

If the reference trajectory Γ is elliptical, then (3) is linear with periodic coefficients and Floquet theory applies. That is, any solution matrix must have the form

$$X(t) = P(t)e^{Qt} \tag{9}$$

where $P(t + T) = P(t)$, a periodic matrix; T is the period of Γ; and Q is a constant matrix.

This solution can be written

$$P(t) = \begin{pmatrix} 0 & 0 & -x_2 & p_1 & v_1 & x_1 \\ 0 & 0 & x_1 & p_2 & v_2 & x_2 \\ x_2 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -v_2 & \dot{p}_1 & \dot{v}_1 & -\frac{1}{2}v_1 \\ 0 & 0 & v_1 & \dot{p}_2 & \dot{v}_2 & -\frac{1}{2}v_2 \\ v_2 & -v_1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$e^{Qt} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{2}t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

where

$$p_1 = h^2 + (x_2^2/r) \qquad p_2 = -x_1x_2/r \tag{12}$$

and h is the angular momentum.

In this form, the $x_1 - x_2$ plane is the plane of motion. Units have been normalized so that the gravitational constant is unity. For interplanetary trajectories, this would mean that, e.g., lengths are in a.u., velocities in EMOS (earth mean orbital speed) and times in TAU = a.u./EMOS.

The matrices (10) and (11) reflect the fact that one solution is secular and the remaining five are periodic. As written previously, these solutions are valid for hyperbolic as well as elliptical trajectories, as can be verified by direct substitution. This form is not valid for parabolic, circular, or rectilinear orbits. (The columns are still solutions of the variational equations; however, they become linearly dependent in these cases.)

For circular trajectories, the third and fifth columns become linearly dependent. The third column may, in this case, be replaced by the "symmetric complement" of (12):

$$q_1 = -x_1x_2/r \qquad q_2 = h^2 + x_1^2/r \tag{13}$$

plus the derivatives of these functions. The resulting system is then linearly independent for circular trajectories as well.

Given the solution matrix, the transition matrix is

$$\Omega(t,\tau) = P(t)e^{Q(t-\tau)}P^{-1}(\tau)$$

By taking the dot product of the solution to the adjoint equation $(-\dot{\lambda},\lambda)$ with the columns of $P(t)e^{Qt}$ one generates expressions which are constant along each two-impulse segment. The dot product of $(-\dot{\lambda},\lambda)$ with the first three columns can be written in vector form:

$$(\dot{\lambda} \times x) - (\lambda \times v) = c \tag{14}$$

which agrees with a result of Pines.[5] The out-of-plane (x_3) component of this vector is λ_θ, the adjoint variable associated with the central angle. The dot product of $(-\dot{\lambda},\lambda)$ with the fifth column gives the Hamiltonian

$$\lambda \cdot \dot{v} - \dot{\lambda} \cdot v = H \tag{15}$$

The dot product with the sixth column can be written in the form

$$\tfrac{2}{3}\dot{\lambda} \cdot x + \tfrac{1}{3}\lambda \cdot v - Ht = a$$

The dot product with the fourth column yields another constant;

$$(\dot{p}_1\lambda_1 + \dot{p}_2\lambda_2) - (p_1\dot{\lambda}_1 + p_2\dot{\lambda}_2) = b \tag{16}$$

In general, these expressions will be constant only on each two-impulse segment of a multi-impulse trajectory. On optimal trajectories, however, c and H are constant over the entire trajectory. Using the properties of the primer vector at impulse points (described in the next section), the jump in constant a over an impulse of an optimum is

$$(a^+ - a^-) = \tfrac{1}{3}\lambda \cdot \Delta v$$
$$= -\tfrac{1}{3}V. \log(m^+/m^-)$$

V_J is the jet velocity. Therefore, the following expression is constant over the entire (optimal) trajectory

$$\tfrac{2}{3}\dot{\lambda} \cdot x + \tfrac{1}{3}\lambda \cdot v - Ht + \tfrac{1}{3}V_J \log m = a' \tag{17}$$

again in agreement with Pines.[5]

The Primer Vector

Consider Γ, a two-impulse trajectory (or two-impulse segment of a multi-impulse trajectory), with impulses Δv_0 at t_0 and Δv_f at t_f. The primer vector λ is defined as the solution to system (6) which satisfies the following boundary conditions:

$$\lambda(t_0) = \lambda_0 = \Delta v_0/|\Delta v_0| \qquad \lambda(t) = \lambda_f = \Delta v_f/|\Delta v_f|$$

At the endpoints of Γ, λ has unit magnitude and is aligned with the velocity increment. A solution satisfying these boundary conditions can be found if $\Omega_{12}(t_f,t_0)$ is nonsingular; the initial value of $\dot{\lambda}$ is given by

$$\dot{\lambda}_0 = \Omega_{12}^{-1}(t_f,t_0)[\lambda_f - \Omega_{11}(t_f,t_0)\lambda_0]$$

The preceding definition of the primer vector is extended easily to multi-impulse trajectories. In such cases, the right-hand side of (6) is different on the different segments. At each impulse point, t_k, λ is again defined as a unit vector in the direction of the impulse

$$\lambda(t_k) = \Delta v_k/|\Delta v_k|$$

The solutions from different arcs are, therefore, joined together so that λ is continuous over the entire trajectory. The primer rate $\dot{\lambda}$ will, in general, be discontinuous at impulse points. However, for optimal trajectories, Lawden[1] has shown that $\dot{\lambda}$ is continuous and, at interior impulses, is orthogonal to λ. Lawden has also shown that a necessary condition for an impulse trajectory to be optimal is that the (normalized) primer magnitude $p = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{1/2}$ satisfy the equation

$$p(t) \leq 1$$

for $t_0 \leq t \leq t_f$.

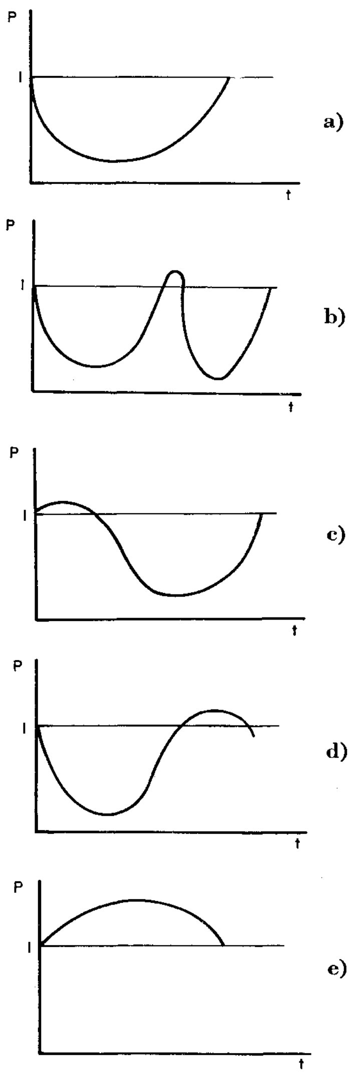

Fig. 1 Sample primer vectors.

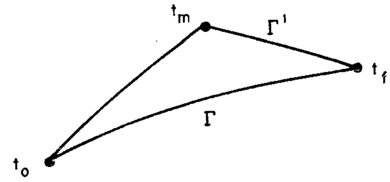

Fig. 2 Comparison trajectories for additional impulse.

In attempting to iterate to finite thrust trajectories from impulsive trajectories which satisfy the same boundary conditions, it has been observed from numerical results[2] that the primer magnitude behavior falls into one of the five categories shown in Fig. 1. These categories are meant to be representative rather than exhaustive; there may well be other types.

Of the five types, only (*A*) satisfies the necessary condition of Lawden. In the subsequent analysis, it will be shown that in each of the other cases, the primer gives an indication of how the reference trajectory can be improved.

Criterion for Additional Impulse

Consider the two-impulse trajectory Γ (shown schematically in Fig. 2) which goes from x_0 to x_f in the prescribed transit time. Γ may be a complete trajectory or a two-impulse segment of a multi-impulse trajectory. By Lambert's theorem[6] there are no other two-impulse trajectories (in the neighborhood of Γ) which satisfy these boundary conditions. There is, however, a four parameter family of three-impulse trajectories which do satisfy these conditions.

Assume that Γ passes through the point x_m at $t = t_m$. The four parameters used to describe a neighboring three-impulse trajectory Γ' will be the time of the midcourse impulse t_m and the position relative to Γ at this time $\delta x_m = x'(t_m) - x(t_m)$. The conditions under which Γ' can be realized are now developed. In particular, it must be established that Γ can be imbedded within a family of trajectories which satisfy (1) and the boundary conditions.

Assuming Γ does not pass through any singularities, the right-hand side of the equations of motion is analytic in the state variables. Trajectories which satisfy these equations are analytic in the initial conditions.[7] In particular, the position vector on Γ' can be described by

$$x'(t) = x(t) + (\partial x/\partial x_0)\Delta x_0 + (\partial x/\partial v_0)\Delta v_0 + \xi$$

where ξ represents higher order terms; i.e.,

$$\lim_{\substack{\Delta x_0 \to 0 \\ \Delta v_0 \to 0}} \frac{|\xi|}{|\Delta x_0| + |\Delta x_0|} = 0$$

For the prescribed boundary conditions, $\Delta x_0 = 0$. The velocity increment is chosen so that Γ' passes through $(x_m + \delta x_m)$ at t_m:

$$x'(t_m) - x(t_m) = \delta x_m = [\partial x(t_m)/\partial v_0]\Delta v_0 + \xi$$

Then let

$$\Delta v_0 = \delta v_0 + \delta v_0'$$

where

$$\delta v_0 = [\partial x(t_m)/\partial v_0]^{-1}\delta x_m$$

$$\delta v_0' = -[\partial x(t_m)/\partial v_0]^{-1}\xi$$

$\delta v_0'$ is clearly of higher order than first. A sufficient condition for such a solution to exist is that

$$\Omega_{12}(t_m,t_0) = \partial x(t_m)/\partial v_0$$

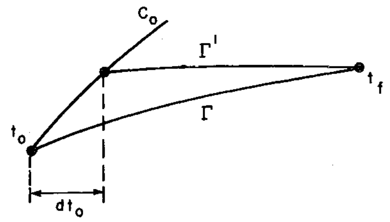

Fig. 3 Comparison trajectories for initial coast.

be nonsingular. The same argument applies on the second and final segment of Γ', so that an additional conditional is that

$$\Omega_{12}(t_m,t_f) = \partial x(t_m)/\partial v_f$$

be nonsingular. It has been shown by Stern[3] that the singularities of Ω_{12} occur only at discrete points. Since they are discrete, these singular points do not affect the arguments which follow.

The first and second segments of Γ' are constructed so that the trajectory is continuous in position. However, the velocities at the junction of these two segments are, in general, different. Therefore, a small midcourse impulse is required which, to first order, is $(\delta v_m^+ - \delta v_m^-)$.

Assuming the existence of a trajectory which satisfies the boundary conditions, then, it has been shown that a four parameter family $(t_m,\delta x_m)$ of neighboring three impulse trajectories exists which also satisfies (1) and the boundary conditions to first order. The difference between Γ and Γ' is described by the variational equations (3).

In what follows, a linear analysis is presented. However, it has been established that the family of trajectories is smooth and contains the reference trajectories. The conclusions drawn will therefore be valid for the nonlinear case, provided the neighboring trajectories considered are "sufficiently close" to the reference. (In such an event, the linear terms will dominate higher-order terms.)

The costs on Γ and Γ', dropping the higher-order terms $\delta v'$, are as follows:

on Γ $\quad J = |\Delta v_0| + |\Delta v_f|$

on Γ' $\quad J' = |\Delta v_0 + \delta v_0| + |\delta v_m^+ - \delta v_m^-| + |\Delta v_f + \delta v_f|$

The difference in cost, to first order, is

$$\delta J = (\Delta v_0/|\Delta v_0|)\delta v_0 + |\delta v_m + - \delta v_m^-| - (|\Delta v_f|/|\Delta v_f|)\delta v_f$$

From the definition of the primer vector,

$$\delta J = \lambda_0 \cdot \delta v_0 + |\delta v_m^+ - \delta v_m^-| - \lambda_f \cdot \delta v_f$$

Using (8) this becomes

$$\delta J = -\lambda_m \cdot (\delta v_m^+ - \delta v_m^-) + |\delta v_m^+ - \delta v_m^-|$$

This expression is homogeneous in $(\delta v_m^+ - \delta v_m^-)$.

Denoting the magnitude of the midcourse impulse by c, then

$$\delta J = c(1 - \lambda_m \cdot \eta)$$

where η is a unit vector in the direction of $(\delta v_m^+ - \delta v_m^-)$. If δJ can be made negative, then Γ' represents an improvement in cost over Γ. This can occur if, and only if, $p(t_m) > 1$.

Two conclusions can be drawn from this discussion:

1) This constitutes an alternative proof of Lawden's necessary condition for Γ to be an optimal trajectory.

2) If the primer magnitude is greater than one at any time, then there exists a three-impulse trajectory (segment) which has lower cost than the reference two-impulse trajectory (segment). The greatest improvement in cost (to first order) can be realized by applying the midcouse impulse at the time the primer magnitude reaches its maximum and in the direction of the primer vector.

Using the boundary conditions $\delta x(t_0) = 0$, $\delta x(t_f) = 0$, it can be shown that for any trajectory Γ' passing through $x(t_m) + \delta x_m$,

$$(\delta v_m^+ - \delta v_m^-) = A\delta x_m$$

where

$$A = \Omega_{22}(t_m t_f)\Omega_{12}^{-1}(t_m,t_f) - \Omega_{22}(t_m,t_0)\Omega_{12}^{-1}(t_m,t_0)$$

For maximum improvement, $(\delta v_m^+ - \delta v_m^-)$ should be parallel to λ_m. Therefore, assuming A is nonsingular, choose

$$\delta x_m = \epsilon A^{-1}\lambda_m$$

(assuming the inverse exists) where ϵ is a small parameter.

Type B of Fig. 1 represents a case where improvement by a third impulse is possible. Choosing δx_m as indicated previously, two Lambert problems are then solved: 1) connecting (x_0,t_0) with $(x_m + \delta x_m,t_m)$; 2) connecting $(x_m + \delta x_m,t_m)$ with (x_f,t_f). For ϵ "sufficiently small" this three-impulse trajectory will then represent an improvement.

Transversality

In this section, an expression is developed which gives the differential cost between two neighboring trajectories. This expression is the analog of the transversality condition of the calculus of variations. In this case, however, there is wider applicability since neither trajectory need be an optimum. This is in distinction to the finite thrust case, and is a result of the fact that cost is incurred only at discrete points.

Consider the two neighboring trajectories shown in Fig. 3. Both trajectories are initially on orbit C_0 at t_0. On Γ, impulses are applied at t_0 and t_f. Γ', on the other hand, remains on C_0 until t_1 $(= t_0 + dt_0)$ and then impulses are applied at t_1 and t_f. Both trajectories have terminal states corresponding to orbit C_f at t_f.

The symbol $d(\cdot)$ will be used to indicate noncontemporaneous variations, that is

$$dx = x'(t + dt) - x(t)$$

To first order, the relationship between dx and δx is

$$dx = \delta x + \dot{x}dt$$

On Γ, the cost is

$$J = |v_0^+ - v_0^-| + |v_f^+ - v_f^-|$$

and on Γ',

$$J' = |v_1^+ - v_1^-| + |v_f^+ - (v_f^- + \delta v_f)|$$

The differential cost is given by

$$\delta J = \lambda_0 \cdot (dv_0^+ - dv_0^-) - \lambda_f \cdot \delta v_f^- \tag{18}$$

where

$$dv_0^+ = v_1^+(t_0 + dt_0) - v_0^+(t_0)$$
$$= \delta v_0^+ + \dot{v}_0^+ dt_0$$
$$dv_0^- = \dot{v}_0^- dt_0$$

Since $\dot{v}$ is continuous (it depends only on position and time)

$$dv_0^+ - dv_0^- = \delta v_0^+$$

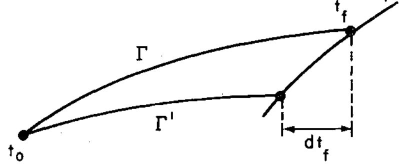

Fig. 4. Comparison trajectories for final coast.

Substituting in the aforementioned equation and using (8),

$$\delta J = \dot{\lambda}_0 \cdot \delta x_0$$

or, since $\delta x_0 = dx_0 - \dot{x}_0{}^+ dt_0$,

$$\delta J = \dot{\lambda}_0 \cdot dx_0 - (\dot{\lambda}_0 \cdot \dot{x}_0{}^+) dt_0 \tag{19}$$

This "transversality condition" represents the difference in cost between two neighboring trajectories whose initial points differ in position by dx_0 and in time by dt_0 (and whose final position and time are identical). Note that no considerations of optimality have been required.

If the final point differs by (dt_f, dx_f), the proper expression is

$$\delta J = -\dot{\lambda}_f \cdot dx_f + (\dot{\lambda}_f \cdot \dot{x}_f{}^+) dt_f \tag{20}$$

The trajectories being compared are shown in Fig. 4.

Since acceleration is continuous, Eq. (19) can be put in more familiar form by adding $\lambda_0 \cdot (v_0{}^+ - v_0{}^-) = 0$ and noting that $dv_0 = \dot{v}_0{}^- dt_0$. Equation (19) then becomes

$$\delta J = -\lambda_0 \cdot dv_0 + \dot{\lambda}_0 \cdot dx_0 + H dt_0 \tag{21}$$

which is exactly the same form as the usual transversality equation for finite thrust (optimal) trajectories. Similarly (20) becomes

$$\delta J = \lambda_f \cdot dv_f - \dot{\lambda}_f \cdot dx_f - H dt_f \tag{22}$$

Equations (19) and (20) are the fundamental form, however, since the differentials are independent.

Moving Interior Impulses

Consider the two three-impulse segments shown in Fig. 5. The differential cost between these two trajectories can be derived from Eqs. (19) and (20) and is given by

$$\begin{aligned}\delta J &= [\dot{\lambda}_m{}^+ \cdot dx_m - (\dot{\lambda}_m{}^+ \cdot \dot{x}_m{}^+) dt_m] - \\ &\qquad [\dot{\lambda}_m{}^- \cdot dx_m - (\dot{\lambda}_m{}^- \cdot x_m{}^-) dt_m] \\ &= (\dot{\lambda}_m{}^+ - \dot{\lambda}_m) \cdot dx_m + (H^+ - H^-) dt_m\end{aligned} \tag{23}$$

where the equation

$$(H^+ - H^-) = -(\dot{\lambda}_m{}^+ \cdot \dot{x}^+ - \dot{\lambda}_m{}^- \cdot \dot{x}^-)$$

has been used. dx_m and dt_m in Eq. (23) can be chosen independently.

The following conclusions can then be drawn:

1) If the reference trajectory is an optimum, then it is necessary that

$$\dot{\lambda}^+ = \dot{\lambda}^- \qquad \text{and} \qquad H^+ = H^-$$

Since H is constant on any segment, it is therefore constant along the entire trajectory. If H and $\dot{\lambda}$ are continuous, then

$$\dot{\lambda}_m \cdot (\dot{x}_m{}^+ - \dot{x}_m{}^{-1}) = 0$$

Since λ_m is aligned with the velocity impulse, this last equation becomes

$$\dot{\lambda}_m \cdot \lambda_m = 0$$

or $dp/dt = 0$ at impulse points. This represents alternate proofs of Lawden's results.

2) If Γ is not an optimum, then a neighboring trajectory with lower cost can be found by choosing

$$dx_m = -\epsilon[\dot{\lambda}_m + - \dot{\lambda}_m{}^-] \qquad dt_m = -\epsilon[H^+ - H^-]$$

If ϵ is sufficiently small, then the three-impulse trajectory which passes through $x_m + \epsilon dx_m$ at $t_m + \epsilon dt_m$ will represent an improvement over Γ. Therefore, Eq. (23) tells us how interior impulses should be moved in position and time so as to reduce characteristic velocity.

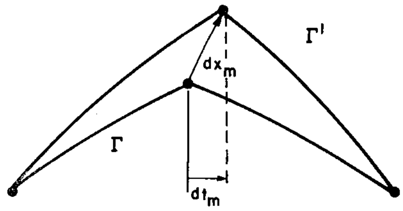

Fig. 5 Comparison trajectories for moving interior impulse.

Final and Initial Coasts

To test the effect of an initial coast, the reference trajectory is compared with a neighboring trajectory which has been allowed to coast ($dt_0 > 0$) in the initial orbit (Fig. 3). The existence of such a solution can be established in a similar manner, as in the previous section.

Using the transversality condition (19), the difference in cost to first order is

$$\delta J = \dot{\lambda}_0 \cdot dx_0 - (\dot{\lambda}_0 \cdot \dot{x}_0{}^+) dt_0$$

Substituting $dx_0 = \dot{x}_0{}^- dt_0$, (the superscript minus refers to increments along the initial orbit), this becomes

$$\delta J = \dot{\lambda}_0 \cdot (\dot{x}_0{}^- - \dot{x}_0{}^+) dt_0$$

Since $dt_0 > 0$, δJ will be negative if

$$\dot{\lambda}_0 \cdot (\dot{x}_0{}^- - \dot{x}_0{}^+) < 0$$

From the definition of the primer, λ_0 is parallel to Δv_0. Therefore, the last equation implies

$$-(\dot{\lambda}_0 \cdot \lambda_0) < 0$$

or

$$dp/dt|_{t=0} > 0$$

In other words, if the primer magnitude exceeds unity immediately after the initial impulse, an initial coast would lower the cost, for instance, type C in Fig. 1. Similarly, it can be shown that if

$$dp/dt|_{t=t_f} < 0$$

then a final coast improves the cost. The trajectories being compared in this case are shown in Fig. 4. For this case, it must be remembered that, to meet the time constraint, $dt_f < 0$. The primer of type D in Fig. 1 is an example. Type E of Fig. 1 represents a trajectory which is so far from optimal that almost anything (additional impulse, initial coast, final coast) will improve it.

Circular Coplanar Orbits

In the special case of transfers between circular coplanar orbits the aforesaid conditions have a simple geometric interpretation. It is more convenient here to shift to polar coordinates. Equation (22) for a final coast becomes

$$\delta J = \lambda_\theta d\theta_f{}^+ - H dt_f{}^+ < 0$$

and for an initial coast, Equation (21) becomes

$$\delta J = -\lambda_\theta d\theta_0{}^- + H dt_0{}^- < 0$$

where $\lambda_\theta = \dot{\lambda}_1 x_2 - \dot{\lambda}_2 x_1 - \lambda_1 \dot{x}_2 + \lambda_2 \dot{x}_1$.

Let ω_0 be the angular rate of the initial orbit and ω_f the angular rate of the final orbit. Then the aforementioned conditions can be written

$$\begin{aligned}&\text{Final coast} \quad (dt_f < 0){:} \quad H/\lambda_\theta > \omega_f \\ &\text{Initial coast} \quad (dt_0 > 0){:} \quad H/\lambda_\theta < \omega_0\end{aligned} \tag{24}$$

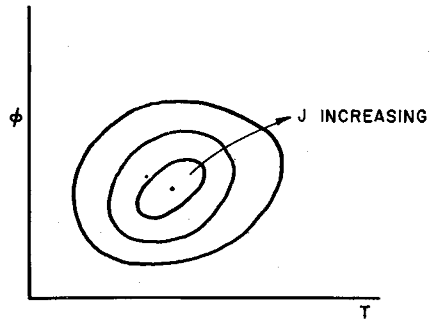

Fig. 6 Two impulse contours.

Now consider Fig. 6. Here, contours of constant J are plotted against the prescribed central angle (φ) and trip time (T). These contours are closed about the minimum J, or Hohmann, trajectory. The value of J increases going outward from this point.

Along the contours, since J is constant,

$$\delta J = \lambda_\theta d\varphi - Hdt = 0$$

(Note: $\lambda_\theta = \partial J/\partial \theta_f$; $H = -\partial J/\partial t_f$.)
The slope of the contours is thus given by

$$m = d\varphi/dT = H/\lambda_\theta$$

To interpret Eqs. (24) geometrically, consider Fig. 7. The original two-impulse contours are shown as broken lines. Straight lines with slope ω_0 and ω_f have been drawn tangent to these contours. The solid lines represent contours of constant J when initial and final coasts are considered. In region A, where $m < \omega_0$, initial coasts represent an improvement. In region C, where $m > \omega_f$, final coasts are an improvement.

For example, if a transfer corresponding to point P_1 is required, it is cheaper to coast initially through angle $\Delta\varphi$ (which takes time ΔT) and then perform the transfer corresponding to P_2.

Also note that the set of points on the $\varphi - T$ plane which can be reached for the minimum (Hohmann) cost is a wedge (cross-hatched on the figure). Any point in this wedge can be reached by a unique combination of initial and final coasts plus the 180° transfer. Other points may need either an initial or final coast, but not both. For other results and numerical calculations see Ref. 8.

Summary

Lawden's definition of the primer vector was extended to nonoptimal trajectories. It is shown that the primer gives an indication of how the original trajectory can be improved. Once the two-impulse or other initial solution has been calculated, the primer magnitude can be readily checked and its shape will determine the type, timing, and direction of correction needed (if any). Testing the magnitude of the primer determines the effect of an additional midcourse impulse. For multiple impulse trajectories, Eq. (23) shows how the interior impulses should be moved to improve the trajectory. The slope of the primer magnitude can be tested to determine the effect of terminal coasts. For inverse square fields, the analytical solution for the primer is known; a simple form of this solution is presented here.

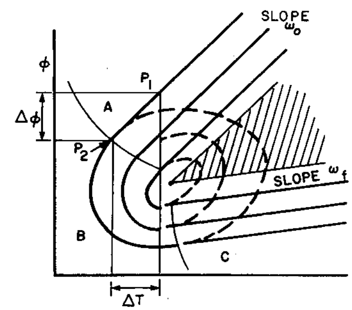

Fig. 7 Two impulse plus coast contours.

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