Optimal Impulsive Orbital Maneuver Synthesis Through Direct Optimization

Pedro Kuntz Puglia[1] Willer Santos[2] Emilien Flayac[3]

[1]ITA, Student
[2]ITA, Professor (AESP)
[3]ISAE-SUPAERO, Professor (DISC)

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Context

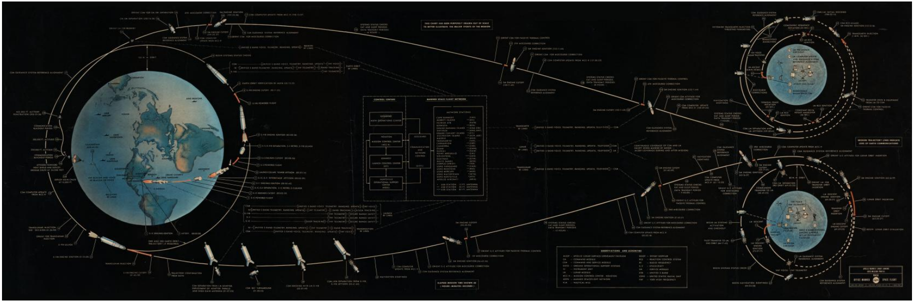
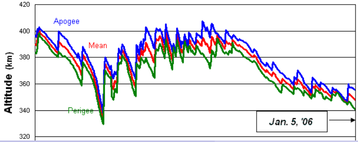
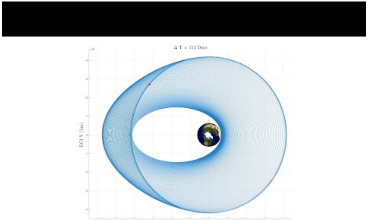


Figure: [12]





Problem Statement

Central Question

What is the most efficient sequence of maneuvers that takes a spacecraft from an initial state to a final state in a given time?

- Efficient: least propellant usage
- General case in mind (no particular analytical solutions)
- How much time? Feasibility, trade-offs?
- How many impulses?
- Is it optimal?

Hypotheses

- Choice for *impulsive propulsion* $\rightarrow$ reducible to parameter optimization
- Good numerical solvers: Ipopt[16]
- Many local optima (non-convex problem)
- Expect *Primer vector* theory provides (some) solutions

Objectives

- Apply primer vector theory;
- Study how much time fo transfer, and how to find it;
- Compare numerical and analytical results;
- Discuss applications in common scenarios;

Some institutions already know how to optimize orbital maneuvers. Why study it again?

(a) Orekit library provides maneuver analysis, and indirect optimization (outdated).

(b) CNES Patrius library provides analysis, not Synthesis.

Available Downloads

There are no available downloads for this record.

(c) NASA's Mystic software (Dawn mission) is not available for download.

No widely available orbital maneuver optimization software.

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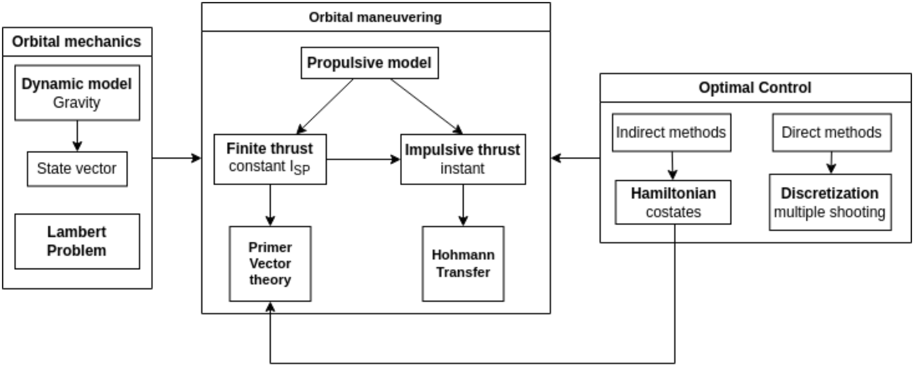
Theory components



Two-Body Motion

Keplerian dynamics:

$$\ddot{\mathbf{r}} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} \tag{1}$$

Closed, periodic elliptical trajectory for negative energy (bound satellite).

State vector choice:

- **Cartesian**: $\mathbf{x} = \begin{bmatrix} \mathbf{r} \ \textit{(position)} \\ \mathbf{v} \ \textit{(velocity)} \end{bmatrix}$
- **Keplerian** (elliptical orbit only):

$$\mathbf{x} = \begin{bmatrix} a \ \textit{(semi-major axis)} \\ e \ \textit{(eccentricity)} \\ i \ \textit{(inclination)} \\ \Omega \ \textit{(RAAN)} \\ \omega \ \textit{(argument of perigee)} \\ \theta \ \textit{(true anomaly)} \end{bmatrix}$$

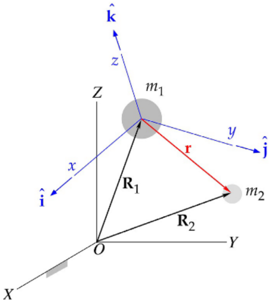


Figure: [6]

Statement

What is the orbit of a satellite that passes by position $\mathbf{r}_2$ at a time Δt after being in position $\mathbf{r}_1$?

- **Importance**: auxiliary role in orbital maneuvering (feasible transfer);
- If $\mathbf{v_1}$ is found, problem is solved;
- Issue with $\mathbf{r}_1 \parallel \mathbf{r}_2$ (eg, perigee & apogee): orbit plane unknown
- Universal variable formulations: simple, cannot handle indetermination [6][14]
- Cartesian formulation [13]

Generic optimal control problem

Given a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$, a fixed initial condition $\mathbf{x}(0) = \mathbf{x}_i$, a total time t_f and a final condition $\mathbf{x}(t_f) = \mathbf{x}_f$, find control trajectory $\mathbf{u}(t)$ minimizing (max.) objective $J[\mathbf{x}(t), \mathbf{u}(t)] = h(\mathbf{x}(t_f)) + \int_0^{t_f} L(x(t), u(t))dt$.

Indirect method

- Hamiltonian [1]: $H = L(\mathbf{x}, \mathbf{u}) + \lambda^T f(\mathbf{x}, \mathbf{u})$
- costate λ, adjoint equations
- Pontryagin's Minimum Principle:

$$\mathbf{u} = \arg\min_{\mathbf{u} \in \mathcal{U}} H[\mathbf{x}, \mathbf{u}, \lambda]. \qquad (2)$$

- Solve for $\mathbf{x}$, λ, $\mathbf{u}$

Direct method

- Discretize time, diff. eq.
- Numerical integration: Euler, RK4, RK8
- Trajectories $\rightarrow \mathbf{x}_k, \mathbf{u}_k$
- Parameter optimization
- Solve for $\mathbf{x}$, $\mathbf{u}$

Extended dynamics

$$\ddot{\mathbf{r}} = -\frac{\mu}{\|\mathbf{r}\|^3}\mathbf{r} + \frac{\mathbf{F}}{m}. \tag{3}$$

Thrust **F** related to mass m through *propulsion model*.

Finite Thrust

- $F = -\dot{m}v_e$
- constant exhaust velocity v_e and specific impulse $v_e = I_{sp}g_0$ (CSI)
- $F \leq F_{\text{max}}$
- Extra state: mass

Impulsive thrust ($F_{\text{max}} \to \infty$)

- Discrete impulses, *coasting arcs*
- Tsiolkovsky's Equation [6]:

$$\Delta v = v_e \ln\left(\frac{m_i}{m_f}\right) \tag{4}$$

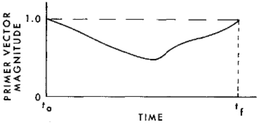
- $\mathbf{v}(t_i^+) = \mathbf{v}(t_i^-) + \Delta\mathbf{v}$
- $\min \int -\dot{m}dt \leftrightarrow \min \sum \Delta v_i$

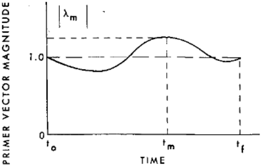
Primer vector theory

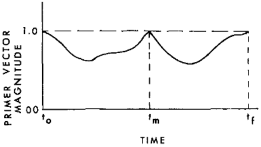
- Apply Hamiltonian to finite thrust CSI case [5] ($\int \dot{m} dt \ll m(0)$)
- *primer vector* $\mathbf{p} = -\lambda_v$ (velocity costate)
- Optimal thrust statisfies (*bang-bang*)

$$\mathbf{F} = \begin{cases} F_{\max}\frac{\mathbf{p}}{p} & , p > 1 \\ 0 & , p < 1 \end{cases} \tag{5}$$

- Extension to impulsive case
  1. $\mathbf{p}(t)$ and $\dot{\mathbf{p}}(t)$ continuous;
  2. $\|\mathbf{p}\| \leq 1$, impulses happen when $\|\mathbf{p}\| = 1$;
  3. $\mathbf{p}$ has the direction of impulse at the impulse instants;
  4. $\frac{d\|\mathbf{p}\|}{dt} = 0$ at impulses at $t \in (0, t_f)$.
- $p(t)$ analytical in coasting arcs [7]







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Numerical Tools

- Julia [2] language
- `SatelliteToolbox` [3]: INPE's own orbit propagator
- `JuMP` [9]: optimization modelling sub-language
- Ipopt [16]: robust nonlinear optimizer. Local, deterministic, grandient-based

Modelling

- Coasting arc model
  - Dynamics discretized with RK4 $\mathbf{x}_{\text{next}} = f_{RK}(\mathbf{x}_{\text{prev}}, \Delta t)$, N steps
  - $N+1$ state vectors $\mathbf{x}^j$ related by

$$\mathbf{x}^{j+1} = f_{RK}(\mathbf{x}^j, \frac{t_p}{N}), j = 1, \dots, N \tag{6}$$

  - 6 DoF left
  - $\mathbf{x}^1$ given: propagation
  - $\mathbf{r}^1$, $\mathbf{r}^{N+1}$ given: Cartesian Lambert solver
- Two impulse model
  - 3 coasts $\mathbf{x}_c^j$, $j = 1, \dots, N+1$, $c = 1, 2, 3$
  - Δv_i, $\hat{\mathbf{u}}_i$, $i = 1, 2$ such that $\|\hat{\mathbf{u}}_i\| = 1$ and $\mathbf{x}_{i+1}^1 = \mathbf{x}_i^{N+1} + \begin{bmatrix} 0_{3\times 1} \\ \Delta v_i \hat{\mathbf{u}}_i \end{bmatrix}, m = 1, 2$
  - variable impulse intervals s.t. $\Delta t_1 + \Delta t_2 \leq t_f$
  - Objective: $\min \Delta v_1 + \Delta v_2$

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Expected Results

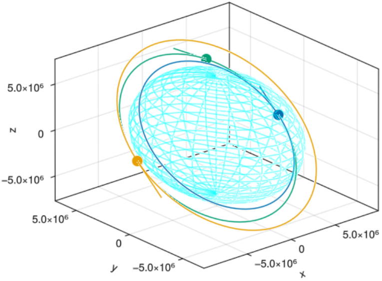
- More robust code
  - perturbed orbital dynamics (easy)
  - multiple impulses (medium);
  - better convergence guarantees (unknown);
  - multiple revolution transfers (hard)
- Estimation of transfer time t_f instead of arbitrary input [13]
- Practical orbital transfer cases under appropriate models:
  1. LEO maintenance: inclination and altitude;
  2. SSO maintenance: inclination, RAAN and semimajor-axis;
  3. Constellation phasing: change phase along orbit
  4. LEO to GEO transfer: multiple impulse, small plane change, numerical challenge (different time and spatial scales between the start and the end)

Goal

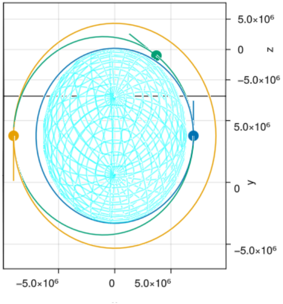
Reproduce Hohmann transfer numerically.

Element	Initial	Final
a	7000 km	9000 km
e	0	0
i	51°	51°
Ω	0°	0°
ω	0°	0°
θ	0°	180°

Table: Orbital elements used for the Hohmann transfer case analysis



(a) 3D view.

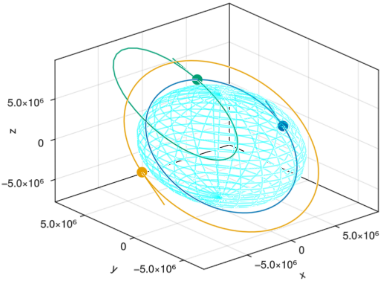


(b) Orbital plane view.

Variables: $t_f = 3560.54$ s, $\sum \|\Delta \mathbf{v}\| = 887.56$ m/s

Lambert Problem solution

- t_f set to analytical time (cheating!)
- Initial guesses Δt_1 and Δt_2: $\frac{t_f}{3}$
- Coast, impulse, Lambert solution, impulse, coast: feasible but non-optimal $\rightarrow$ use as initial guess for optimizer



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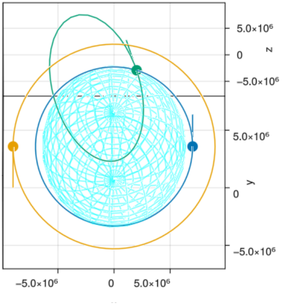
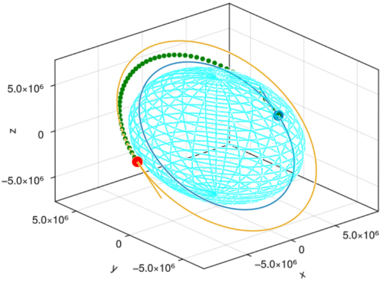


(b) Orbital plane view.

Optimized solution

$N = 50$

Variable	Analytical	Solver
Δt_1	0s	8e-4s
Δt_2	3560.54s	3560.538s
$\sum \lVert \Delta \mathbf{v} \rVert$	$887.56 m/s$	$887.56 m/s$



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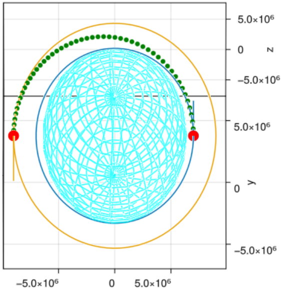

(b) Orbital plane view.

References I

Dimitri P. Bertsekas.
Dynamic Programming and Optimal Control Vol. I.
Athena Scientific, 1st edition, 1995.

Jeff Bezanson, Alan Edelman, Stefan Karpinski, and Viral B Shah.
Julia: A fresh approach to numerical computing.
SIAM Review, 59(1):65–98, 2017.

Ronan Arraes Jardim Chagas, Thatcher Chamberlin, Helge Eichhorn, Yakir Luc Gagnon, Alberto Mengali, Chris Binz, Yasushi SHOJI, Federico Stra, Forrest Gasdia, Hiroaki Yamazoe, Julia TagBot, Srikumar, chenyongzhi, and justbyoo.
Juliaspace/satellitetoolbox.jl: v1.0.0, January 2025.

Vladimir A. Chobotov.
Orbital Mechanics Third Edition.
American Institute of Aeronautics and Astronautics, Inc., 2002.

B. Conway.
Spacecraft Trajectory Optimization.
Cambridge Aerospace Series. Cambridge University Press, 2010.

H.D. Curtis.
Orbital Mechanics: For Engineering Students.
Aerospace Engineering. Butterworth-Heinemann, 2020.

DAVID R. GLANDORF.
Lagrange multipliers and the state transition matrix for coasting arcs.
AIAA Journal, 7(2):363–365, 1969.

K. R. Korneev and S. P. Trofimov.
Low-thrust trajectory optimization in kustaanheimo–stiefel variables.
Cosmic Research, 62(3):256–265, Jun 2024.

References III

Miles Lubin, Oscar Dowson, Joaquim Dias Garcia, Joey Huchette, Benoît Legat, and Juan Pablo Vielma.
JuMP 1.0: Recent improvements to a modeling language for mathematical optimization.
Mathematical Programming Computation, 2023.

NASA.
Iss altitude.
Available at https://commons.wikimedia.org/wiki/File:ISS_altitude.png, 2006.

NASA.
Centaur program.
Available at https://www1.grc.nasa.gov/historic-facilities/rockets-systems-area/centaur-program/, 2025.

NASA.
Lunar mission flight path.
Available at https://airandspace.si.edu/multimedia-gallery/5317hjpg, 2025.

David Ottesen and Ryan Russell.
Unconstrained direct optimization of spacecraft trajectories using many embedded lambert problems.
Journal of Optimization Theory and Applications, 191, 12 2021.

Alexander Sukhanov.
Lectures on atsrodynamics, 2010.

J. D. Williams.
Hall effect thrusters.
Available at `https://projects-web.engr.colostate.edu/ionstand/research/research.php`, 2025.

References V

Andreas Wächter and Lorenz Biegler.
On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming.
Mathematical programming, 106:25–57, 03 2006.