

INSTITUTO TECNOLÓGICO DE AERONÁUTICA



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**OPTIMAL IMPULSIVE ORBITAL MANEUVER
SYNTHESIS THROUGH DIRECT OPTIMIZATION
AND NECESSARY CONDITIONS VERIFICATION**

Trabalho de Graduação
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OPTIMAL IMPULSIVE ORBITAL MANEUVER SYNTHESIS THROUGH DIRECT OPTIMIZATION AND NECESSARY CONDITIONS VERIFICATION

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dedicar...

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“Pointy end up, flamey end down.”
— TIM DODD, EVERYDAY ASTRONAUT

Resumo

RESUMO

Abstract

This work presents the development and characterization process of a cold gas thruster vectorization system. The motor is required to have a thrust of 2 N and a chamber pressure of 5 bar. The chosen vectorization method for testing was the jet vane. The constructed motor had slight deviations from the requirements, with a specific impulse of 46.6 s. This motor was mounted on a control mechanism of the deflecting blade, and this assembly was coupled to a three-component scale for force and moment characterization. As a final result, the control derivatives for lateral force and moment were obtained. Finally, the methodological issues encountered and engineering trade-offs identified for the system were presented.

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Lista de Símbolos

F	Empuxo propulsivo
\dot{m}	Vazão mássica
v_e	Velocidade de exaustão média
p_c	Pressão de câmara
p_e	Pressão de saída média
p_{amb}	Pressão ambiente
A_c	Área da seção transversal da câmara
A_e	Área da seção transversal da saída da tubeira
A_t	Área da seção transversal da garganta
ε	Razão de expansão
I_{sp}	Impulso específico
C_F	Coefficiente de empuxo
C^*	Velocidade característica
F_x	Força horizontal, transversal ao motor foguete
F_y	Força vertical, na direção do empuxo propulsivo
M	Torque resultante
δ	Deflexão da lâmina (<i>jet vane</i>)
$F_{x\delta}$	Derivada da força lateral em relação à deflexão da lâmina
M_δ	Derivada de momento em relação à deflexão da lâmina

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1 Introduction

1.1 Context

Space exploration relies on clever resource management, since satellites have a finite amount of resources (propellant and other consumables) to fulfill their mission. Up to this date, all space hardware is expendable, that is, when the consumables required for mission maintenance are finished, the mission ends, marking the end of the exploration of a very expensive engineered system. Thus the need for optimization arises in this domain.

Contrary to science fiction, where spaceships seem to be constantly propelled by their thrusters, real life satellites change their courses in discrete moments of maximum thrust application, surrounded by (usually long) coasting periods. This is due to the relatively high power delivered by traditional rocket engines, which can, in the matter of seconds or minutes, greatly alter a satellite's orbit. Certain more modern propulsion systems, such as electric rocket engines, are somewhat of an exception; this technicality will be discussed in further sections.

Orbital maneuvers are necessary in all stages of a satellite's lifecycle. In the beginning of a mission, the satellite is released by the launch vehicle in an orbit that is usually not the mission's orbit. Therefore, an *injection maneuver* is necessary to bring the satellite into an operational orbit. This is usually the biggest maneuver a satellite must execute during its lifecycle, consuming a high fraction of its propellant storage.

During a mission, the satellite must perform sporadic *maintenance maneuvers*, which are small course correction maneuvers to mitigate external perturbations such as atmospheric drag, oblateness effects (if undesired), influence of celestial bodies, and solar radiation pressure. Their frequency and magnitude vary depending on mission requirements, and in industrial applications, must not conflict with other, possibly simultaneous events (such as observation of a ground target), as well as taking into account pointing constraints, since a spacecraft might have sensitive sensors that must not be pointed at the sun, or have solar panels that need uninterrupted illumination. Those are by far the most common type of maneuver, and a loose, non-exhaustive classification arises naturally.

The simplest type of maneuver is that of *orbit raising*, which consists in bringing the satellite from a (often circular) orbit and increasing its semimajor-axis (and thus, its period) until a desired value is reached. This maneuver is commonly found in LEO applications, due to the presence of atmospheric drag; notably, it is performed by the ISS about once a month (NASA, 2009). From a theoretical standpoint, it presents a simple, introductory case, often restricted to two dimensions instead of three. There are plenty of theoretical results about it, most notably the Hohmann transfer, a two-impulse maneuver which is known to be the two-impulse optimal from a plethora of theoretical tools. Other more elaborate results include the bielliptic transfer, which can be shown to surpass Hohmann's performance in certain conditions by allowing a third impulse. Another scenario that falls under this category is that of high orbit injections, such as LEO to GEO or LEO to MEO.

A second type of maneuver is a *plane change* maneuver. Satellites move (approximately) in a plane which contains its position and velocity vectors and the center of Earth. By changing the direction of the velocity, this plane may be change. Common cases include an inclination change during orbital insertion, which may be required if the inclination of the target orbit is different to the latitude of the launch center CITE. Another plane change instance is that of a change in the right ascension of the ascending node (RAAN), which is especially useful for SSOs. SSO injection requires that the orbit be placed approximately perpendicular to the Sun; this requires careful positioning of the ascending node. Another interesting case is that of a combined plane change and orbit raising maneuver, such as that starting from an inclined LEO orbit targeting a GEO (thus, equatorial) orbit. A clever combination of both requirements can allow for great performance gains as compared to sequential maneuvers.

A final type of maneuver is the *phasing* maneuver. This maneuver consists in changing the position occupied by the satellite within the same orbit at a certain time. This maneuver is very important for *orbital rendez-vous*, where not only it is required that two vessels share the same orbit, but also they must have the same position and velocity at the same time. Another application for this type of maneuver is that of *rideshare injection*, where a swarm of satellites is carried by a launcher hub and they must be distributed around a shared orbit, with certain angular intervals in between. The execution of such a maneuver usually involves placing the satellite in an intermediate orbit with slightly different period than the initial one, and waiting multiple revolutions for the convergence of the satellite and the (mobile) target. A notable, recurring example of this is the rendez-vous of the Soyuz capsule with the ISS, which can take up to 3 days CITE.

Finally, at the end-of-life, there are legal constraints on where a satellite may be disposed of. LEO missions have a deadline for deorbiting into Earth's atmosphere, while GEO satellites are usually placed into a cemetery orbit which does not intersect the highly

prized GEO region. As an end-of-life procedure, feasibility is of utmost importance, while ensuring optimality increases the lifespan of the mission.

1.2 Problem statement

This work aims to develop modern numerical methods for orbital maneuver optimization in Earth orbit. Combinations of propulsive and orbital models are to be paired with adequate numerical schemes and theoretical tools to produce feasible maneuvers that also satisfy certain optimality conditions. The main deliverable shall be a code package capable of generating and optimizing maneuvers between an initial and final orbital state, for an allowed time of flight in between, as well as the mathematical formulation and derivation of such a problem.

The main models to be studied are those of two body Keplerian dynamics and impulsive maneuvers, as they offer the most opportunities for validation with analytical results. Further models to be studied, if time allows it, are continuous thrust propulsion models and two body dynamics with oblateness perturbations (J2 effects).

It is desired to validate the numerical algorithms with certain known analytical results, such as the Hohmann transfer, reproduce certain methods from the literature, and apply some of the formalism of optimal control (in the form of primer vector theory) to the solutions obtained.

It is not in the scope of this work to compare different numerical schemes; a sufficient one shall be found and exploited throughout. However, a novel, experimental method for optimal control synthesis based on polynomial optimization may be attempted if time allows it CITE.

The problem of orbital maneuvers is very general and it is possible to abstract it from the specifics of a particular satellite's hardware by reasoning with position and (changes in) velocity. Therefore, application cases shall be representative of classes of maneuvers, instead of restricting their application to the specifics of one mission. This work focuses on Earth exploration activities, thus excluding lunar and interplanetary transfers.

1.3 Hypotheses

All hardware restrictions such as need for contact with a ground station, pointing constraints, mission objectives (such as observation of a ground target), and possibility of hardware failure and imprecisions are neglected. Attitude dynamics are assumed to be as fast as needed and always precise. Further assumptions depend on the propulsion and

orbital model chosen and shall be discussed in future sections.

1.4 Objectives

The objective of this work is to implement an impulsive orbital maneuver optimizer. Given an initial orbital state, a final orbital state, and a transfer time, the goal is to characterize the control history that optimally satisfies these requirements, spending the least amount of propellant possible. Secondary objectives include:

- Apply primer vector theory to the solutions found. This is a central tool in the field, and provides analytical necessary conditions for verifying optimality;
- Discuss different parameterizations for the problem;
- Compare some of the numerical results with known analytical results, namely the Hohmann transfer;
- Discuss some instances of application of this method to common aerospace scenarios;
- **Optional:** expand the work to continuous thrust propulsive models;
- **Optional:** include orbital perturbations, most notably oblateness effects;

1.5 Justification

The strict performance requirements in the space domain make it paramount to use orbital resources sparingly. Optimization of orbital maneuvers increases the envelope of possible missions, be it in terms of mission lifespan, which increases profitability, or in terms of mission design, allowing for bolder, high-profile missions.

In the context of Brazil's space industry, ITASAT-2 is a formation-flying mission which is subject to orbital disturbances, such as atmospheric drag and oblateness effects (FRANCO; SANTOS, 2020). Thus the need for efficient orbital maneuvers arises.

1.6 Work structure

This work is organized in chapters as follows:

1. Introduction, where preliminary contextualization is given;

-
2. Theory and fundamentals, where the mathematical description of the problem is given;
 3. Bibliographic survey, where some previous results in the field are discussed;
 4. Methodology, where implementation details are given;
 5. Preliminary and expected results, where some baseline results are exposed, and future, expected results are enumerated;
 6. Planning, where a timeline of future work is given;

2 Theory and fundamentals

2.1 Optimal Control

Optimal control is the area of control theory which tries to find the best control action to satisfy some requirements, such as altering a system's state in some way desired way. Here, "best" is defined as maximizing or minimizing some performance metric. In practice, and in particular in the scope of this work, this can be interpreted as attaining a target orbit in a certain amount of time, while minimizing fuel consumption.

The mathematical nature of an optimal control problem varies greatly depending on the nature of the system, the requirements, and the objective. Here, a selected subset of this vast theory shall be presented. Suppose a continuous time dynamical system operating on times $t \in [0, t_f]$, where $t_f \in \mathbb{R}$, given by

$$\dot{X}(t) = f(X(t), u(t)) \quad (2.1)$$

where $X(t) : \mathbb{R} \rightarrow \mathcal{X} \subset \mathbb{R}^n$ is the state vector trajectory describing the system state, $u(t) : \mathbb{R} \rightarrow \mathcal{U} \subset \mathbb{R}^m$ is the control vector trajectory and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the function describing its temporal dynamics.

In addition, the control vector might be subject to some inequality constraints, representing for instance saturation of actuators. Therefore, an admissible control set \mathcal{U} is defined by a vector of constraint functions $g(u)$ as

$$\mathcal{U} = \{u \in \mathbb{R}^m \mid g(u) \leq 0\} \quad (2.2)$$

where the inequality is understood to hold component-wise.

At the initial time, the system is supposed to be in a given state X_i such that

$$X(0) = X_i. \quad (2.3)$$

At the final time t_f , some components of the final state vector are specified, while

others are subject to optimization. Let the index set \mathcal{K} be the set of state variables that are fixed at the final time, such that

$$X_k(t_f) = X_{fk}, k \in \mathcal{K} \quad (2.4)$$

for some given values X_{fk} .

To complete the optimal control problem, a performance metric needs to be introduced. In general, any functional of the form $J[X(t), u(t)]$ may be taken as this performance metric; however, a common form with desirable properties, which shall be adopted in this work, is given by

$$J[X(t), u(t)] = h(X(t_f)) + \int_0^{t_f} L(X(t), u(t)) dt \quad (2.5)$$

where the functions $h(X)$ and $L(X, u)$ are respectively called the *terminal cost* and the *temporal cost* functions.

The optimal control problem is then that of finding a control trajectory $u(t)$ that minimizes (or maximizes) the performance metric. Here the problem shall be presented as a minimization problem; but the formulation is perfectly analogous for a maximization problem. That said, the complete optimal control problem may be stated as finding the function $u(t)$ such that

$$u(t) = \arg \min_{u(t), X(t)} J[X(t), u(t)] \quad (2.6)$$

subject to

$$\dot{X}(t) = f(X(t), u(t)) \quad (2.7)$$

$$X(0) = X_i \quad (2.8)$$

$$X_k(t_f) = X_{fk}, k \in \mathcal{K} \quad (2.9)$$

In general, this is a very hard problem. The optimization variable $u(t)$ is not merely a vector of parameters but a whole trajectory of them; thus, the search space is enormous. There are techniques to turn this problem into a simple parameter optimization problem, which are known as *direct methods*, which shall be discussed later. There are however tools for extracting necessary conditions for the solution of this problem at all points in time. These are known as *indirect methods*.

One of this tools is the Hamiltonian, a quantity that describes the ensemble of objectives and constraints. It shall be defined for a minimization problem, and maximization problems can be adapted by changing the sign of the performance metric. Given a sys-

tem of the form in equation (2.1), constraints in the forms of (2.3) and (2.4), and a cost function in the form (2.5), the Hamiltonian H is defined as (BERTSEKAS, 1995)

$$H(X(t), u(t), \lambda(t)) = L(X, u) + \lambda(t)^T f(X, u) \quad (2.10)$$

for all times t , state and control vectors X and u along a trajectory. $\lambda(t)$ is the costate trajectory, a new set of variables introduced as the continuous-time equivalent of Lagrangian multipliers. These new variables are subject to the adjoint equations

$$\dot{\lambda} = - \left(\frac{\partial H}{\partial X} \right)^T = - \left(\frac{\partial f}{\partial X} \right)^T \lambda - \left(\frac{\partial L}{\partial X} \right)^T \quad (2.11)$$

and boundary conditions (BRYSON; HO, 1975)

$$\lambda_k(t_f) = \frac{\partial h(X(t_f))}{\partial X_k}, k \notin \mathcal{K} \quad (2.12)$$

To complete the Hamiltonian approach, Pontryagin's Minimum Principle is introduced. It states that a necessary condition for attaining the minimum in equation (2.6) is that, at all times t , and along the optimal trajectory,

$$u(t) = \arg \min_{u \in \mathcal{U}} H[X(t), u, \lambda(t)]. \quad (2.13)$$

With the control trajectory obtained as a function of $X(t)$ and $\lambda(t)$ from equation (2.13), there are $2n$ variables, the state and costate trajectories, and $2n$ boundary conditions, the initial and final states. Thus, the problem is well-posed and configures a Two Point Boundary Value Problem (TPBVP) (BRYSON; HO, 1975).

2.2 Orbital Mechanics

Orbital mechanics concerns itself with the motion of bodies in space subject to gravitational and disturbance forces. A variety of models exist, differing in precision and availability of analytical tools. The simpler the model, the more analytical tools are available, and the smaller the precision. The simplest model of all, and the basis for all others, is the two body problem, where a central massive body is supposed to be stationary while a moving satellite is subject to its gravitational attraction, also known as Keplerian motion.

2.2.1 Two Body Motion

Let r be the 3-dimensional position of a satellite, and μ the gravitational parameter of the central body. The dynamics of the satellite's position are given by (CURTIS, 2020)

$$\ddot{r} = g(r) = -\frac{\mu}{\|r\|^3}r, \quad (2.14)$$

where $g(r)$ represents the gravitational acceleration field, thus configuring a 6-dimensional state vector $X = \begin{bmatrix} r^T & v^T \end{bmatrix}^T$, where v is the satellite's velocity. The system contains a singularity at the states with $\|r\| = 0$, which configures a non-convex domain. In practice, this point is rarely encountered as it lies inside of the central body, thus far from the regions of interest. It is proven that no analytical solution exists for this differential equation; however, much is known about its solutions.

In this model, the possible trajectories are known to be conics, and therefore restricted to a plane. For bound satellites, that is, those in orbit around the central body, this trajectory is an ellipse where the central body lies on one of its foci. Mathematically, a “bound” satellite is one whose specific energy (mechanical energy over mass of the satellite), given by (CURTIS, 2020)

$$\epsilon = -\frac{\mu}{\|r\|} + \frac{v^2}{2}, \quad (2.15)$$

is negative. The trajectory is closed, and the movement is periodic with period (CURTIS, 2020)

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \quad (2.16)$$

where a is the semi-major axis of the ellipse.

In this case, an alternative state vector may be introduced in the form of the Keplerian elements. These are (CURTIS, 2020):

- a : semi-major axis of the ellipse;
- e : excentricity of the ellipse;
- i : inclination of the orbit's plane with respect to the Equatorial plane;
- Ω : right ascension of the ascending node, that is, angle between the vernal equinox direction and the direction where the satellite crosses the Equatorial plane from South to North;
- ω : argument of perigee, or angle, in the plane of the orbit, between the ascending node and the perigee (point of smallest distance to the central body);

- θ : true anomaly, or angle between the perigee and the current position of the satellite.

These elements are related to the Cartesian state vector through ADD PERIFOCAL EQUATIONS.

In this formulation, all elements but the true anomaly are constant in time. The true anomaly can be related to time implicitly through two other quantities, the mean anomaly M and the excentric anomaly E (CURTIS, 2020):

$$M = 2\pi \frac{t - t_p}{T} \quad (2.17)$$

$$E - e \sin E = M \quad (2.18)$$

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \quad (2.19)$$

where t_p is the time of the last perigee passage. By computing the mean anomalies in an initial and a final time, and solving the notorious Kepler's equation (2.18), and finally finding a suitable true anomaly with (2.19), a semi-analytical temporal solution can be found. The process of finding the position of a satellite in the future is called *orbit propagation*. Define an orbit propagator as a function $p_o(X_i, t)$ such that

$$X_f = p_o(X_i, t) \quad (2.20)$$

where X_f is the satellite's final state after a time t , with initial state X_i .

2.2.2 Lambert's Problem

An important problem in orbital mechanics is that of the determination of the initial and final velocities of a satellite that passes through two points in space r_1 and r_2 with a time interval Δt in between. This problem first arose in the field of orbit determination but also finds application in the context of orbital maneuvers. Namely, Lambert's Problem seeks to find a feasible solution to a TPBVP, which is of interest to the optimal control TPBVP.

In general, this problem can have multiple solutions, corresponding to prograde and retrograde trajectories, with less than one or multiple revolutions. The resulting orbit can, in general, be elliptic, parabolic or hyperbolic.

EXPLAIN FORMULATIONS HERE

Many formulations exist, including CURTIS, SUKHANOV, and numerical integration.

They all suffer from a physical indetermination in the case of collinear r_1 and r_2 : the plane of the orbit is indeterminate. In this case, one can find many feasible solutions but determining exact velocities requires extra information about the plane of the orbit.

2.3 Orbital Maneuvers

When a satellite is able to maneuver, the Keplerian dynamics of equation (2.14) need to be augmented with the thrust control vector F , which applies a propulsion force on the satellite. Supposing that m is the total current mass of the spacecraft, the dynamics are given by

$$\ddot{r} = -\frac{\mu}{\|r\|^3}r + \frac{F}{m}. \quad (2.21)$$

The generation of thrust F is tied to the consumption of propellant, according to some propulsion model. Three main models exist. The first is the continuous specific impulse continuous thrust model, adequate for chemical engines. The second is the impulsive thrust model, the limiting case of the previous model where a burn is considered to happen instantly. And the last one is the variable specific impulse model, which models electric rocket engines and shall not be explored in this work.

The application of optimal control to the field of orbital maneuvering is mainly concerned with the preservation of propellant. Suppose a satellite has an orbital state X_i and is required to maneuver to an orbital state X_f in a time t_f , and it is desired to minimize the amount of propellant used. A convenient way of expressing this is that it is desired to maximize the final mass of the spacecraft, with constraints:

$$\max_{F(t)} m(t_f) \quad (2.22)$$

$$X(0) = X_i \quad (2.23)$$

$$X(t_f) = X_f \quad (2.24)$$

2.3.1 Constant specific impulse model

Chemical and cold gas thrusters are characterized by an exhaust velocity v_e at which the propellant flow is ejected from the spacecraft. With a propellant flow rate \dot{m}_p , the thrust F is given by

$$\|F\| = v_e \dot{m}_p \quad (2.25)$$

The propellant flow is deducted from the spacecraft's mass; therefore it can be stated that $\dot{m} = -\dot{m}_p$. Thus, in this model, the spacecraft's mass is a seventh state variable. A

new state vector $X_m = \begin{bmatrix} r^T & v^T & m \end{bmatrix}^T$ is defined and subject to the dynamics

$$\begin{bmatrix} \dot{r} \\ \dot{v} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} v \\ -\frac{\mu}{\|r\|^3} r + \frac{F}{m} \\ -\frac{\|F\|}{v_e} \end{bmatrix} \quad (2.26)$$

In addition, thrusters have limited flow rates, which imposes a maximum thrust magnitude F_{\max} :

$$\|F\| \leq F_{\max} \quad (2.27)$$

The objective (2.22) can be developed for this model by integrating \dot{m} as

$$m(t_f) = m(0) - \int_0^{t_f} \frac{\|F\|}{v_e} dt \quad (2.28)$$

Let Γ be the acceleration due to thrust such that $\Gamma = \frac{F}{m}$. The literature (SPACECRAFT..., 2010) then suggests considering that the propellant consumption is small compared to the total mass of the satellite, such that it can be stated that

$$m(t_f) \approx m(0) - \frac{m(0)}{v_e} \int_0^{t_f} \|\Gamma\| dt \quad (2.29)$$

Therefore the objective can be restated as

$$\min_{\Gamma(t)} \int_0^{t_f} \|\Gamma\| dt \quad (2.30)$$

subject to $\|\Gamma(t)\| \leq \gamma_{\max}$ at all times, with $\gamma_{\max} = \frac{F_{\max}}{m(0)}$.

If the control variable is set to be Γ instead of F , this leads to a mass-independent problem. This approximation leads to primer vector theory, and is therefore important. However, the assumption of constant mass should be challenged.

2.3.2 Impulsive propulsion model

A very simple propulsion model that allows for easier solution of the orbital maneuvering problem supposes that the propulsive forces are much greater and operate much faster than the gravitational force, introducing discontinuities in velocity. This is called *impulsive thrust*. The propulsion model relies on Tsiolkovsky's equation (SPACECRAFT..., 2010),

$$\Delta v = v_e \ln \left(\frac{m_i}{m_f} \right), \quad (2.31)$$

where Δv is the magnitude of an instantaneous change in velocity, v_e is the engine's exhaust velocity (which is treated as a known parameter), m_i is the initial spacecraft mass and m_f , the final mass. Supposing a burn happens at time t_b , the propulsion model can then be expressed through a Dirac delta as

$$\left. \frac{F}{m} \right|_{t=t_b} = \delta(t - t_b) V_e \ln \left(\frac{m(t_b^-)}{m(t_b^+)} \right) = \delta(t - t_b) \Delta v, \quad (2.32)$$

which yields a velocity discontinuity

$$\|v(t_b^+) - v(t_b^-)\| = V_e \ln \left(\frac{m(t_b^-)}{m(t_b^+)} \right) = \Delta v. \quad (2.33)$$

Now, considering a generic maneuver with n burns, there are $n + 1$ coasting segments related by the change in velocity $\Delta \vec{v}_j$ associated with the j -th burn. Considering burn times t_j , with $t_f \geq t_{j+1} \geq t_j \geq 0$, and the initial and final times 0 and t_f , the system is subject to boundary conditions

$$X(0) = X_i \quad (2.34)$$

$$r(t_j^+) = r(t_j^-), \quad \forall j = 1, \dots, n \quad (2.35)$$

$$v(t_j^+) = v(t_j^-) + \Delta \vec{v}_j, \quad \forall j = 1, \dots, n \quad (2.36)$$

$$X(t_f) = X_f \quad (2.37)$$

and to dynamical equation (2.14) in the intermediate times. Using the concept of orbit propagator, this adds the constraints

$$X(t_1^-) = p_o(X(0), t_1) \quad (2.38)$$

$$X(t_{j+1}^-) = p_o(X(t_j^+), t_{j+1} - t_j), \forall j = 1, \dots, n - 1 \quad (2.39)$$

$$X(t_f) = p_o(X(t_n^+), t_f - t_n) \quad (2.40)$$

to the previous list. Each impulse is described by its time t_j and its velocity change vector Δv_j . Accounting for $X(0)$, all of the intermediate $r(t_j^-)$, $r(t_j^+)$, $v(t_j^-)$ and $v(t_j^+)$, and also the final state $X(t_f)$, plus the impulse parameters, there are $6 + 12n + 6 + 4n = 16n + 12$ unknowns. At the same time, there are $6 + (3 + 3)n + 6 + 6 + 6(n - 1) + 6 = 12n + 18$ constraints. Therefore, for general initial and final conditions, the *minimal number of impulses* is 2.

Tsiolkovsky's equation can also be applied between the initial time and the final time, thus relating mass at time t_f with the total velocity change, which is the sum of all n

burns executed during the transfer:

$$m(t_f) = m(0) \exp \left(-\frac{\sum_{i=1}^n \Delta v_i}{v_e} \right) \quad (2.41)$$

Since v_e and $m(0)$ are not subject to optimization, the objective (2.22) is equivalent to minimizing the sum of magnitudes of impulses used during the transfer:

$$\min \sum_{i=1}^n \Delta v_i. \quad (2.42)$$

Thus, in the impulsive case, the problem is *independent of spacecraft mass*, and it can be eliminated from the state vector. However, the introduction of a discrete parameter, the number of burns n , is worth discussing. The optimization of non linear problems with mixed continuous and discrete variables is called Mixed Integer Non-Linear Programming (MINLP), and is much more complicated than regular non-linear programming. Although the implementation of such a solver might be of interest, some simple reasoning and the theory of primer vectors (exposed in the next section) can help determine the number of impulses needed (LUO *et al.*, 2010).

2.3.3 Primer vector theory

MENTION CONTINUITY OF PRIMER VECTOR SOMEWHERE

The application of optimal control theory to orbital maneuvers dates back to the 1960s, with pioneering work by Lawden (SPACECRAFT..., 2010). In particular, he coined the term "primer vector" as an analogy with the fact that the costate trajectory imposes a necessary condition for firing the engines, thus acting as a "primer". This theory is based on the indirect necessary conditions provided by the Hamiltonian formalism previously exposed, and is explored in (SPACECRAFT..., 2010).

The acceleration due to thrust Γ can be split into its magnitude and direction as $\Gamma = \gamma \hat{u}$, $\gamma = \|\Gamma\|$, $\|\hat{u}\| = 1$ so that objective (2.30) can be rewritten as

$$\min_{\gamma(t), \hat{u}(t)} \int_0^{t_f} \gamma dt \quad (2.43)$$

and the Hamiltonian of the system is the given by

$$H = \gamma + \begin{bmatrix} \lambda_r^T & \lambda_v^T \end{bmatrix} \begin{bmatrix} v \\ g(r) + \gamma \hat{u} \end{bmatrix} \quad (2.44)$$

The costate is subject to the adjoint equations given in block matrix form as

$$\begin{bmatrix} \dot{\lambda}_r \\ \dot{\lambda}_v \end{bmatrix} = \begin{bmatrix} 0_{3 \times 3} & -\left(\frac{\partial g(r)}{\partial r}\right)^T \\ -I_{3 \times 3} & 0 \end{bmatrix} \begin{bmatrix} \lambda_r \\ \lambda_v \end{bmatrix} \quad (2.45)$$

which is linear in the costate variables.

Rearranging the Hamiltonian to factor γ ,

$$H = (1 + \lambda_v^T \hat{u})\gamma + \lambda_r^T v + \lambda_v^T g(r). \quad (2.46)$$

By applying Pontryagin's Minimum Principle, the thrust magnitude γ is given piecewise by analyzing the sign of its coefficient:

$$\begin{cases} \gamma_{\max} & , 1 + \lambda_v^T \hat{u} < 0 \\ 0 & , 1 + \lambda_v^T \hat{u} > 0 \\ \text{intermediate} & , \text{otherwise} \end{cases} \quad (2.47)$$

The case where $1 + \lambda_v^T \hat{u} = 0$ on a finite interval of time will not be considered here CHECK WHAT LAWDEN HAS TO SAY. When $\gamma = \gamma_{\max}$, its coefficient should be as negative as possible, which happens when \hat{u} has the opposite direction to λ_v :

$$\hat{u} = -\frac{\lambda_v}{\|\lambda_v\|}. \quad (2.48)$$

This direction, the *optimal thrust direction* given by $-\lambda_v$ is what constitutes the primer vector p , which is defined as $p = -\lambda_v$. The engine firing conditions can be stated solely as a function of the primer vector as (SPACECRAFT..., 2010)

$$\begin{cases} \gamma_{\max} & , \|p\| > 1 \\ 0 & , \|p\| < 1 \\ \text{intermediate} & , \text{otherwise} \end{cases} \quad (2.49)$$

The primer vector differential equations can be deduced from the adjoint equations and, due to linearity, admit a solution through the form of a state transition matrix $\Phi(t - t_0)$ such that

$$\begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} = \Phi(t - t_0) \begin{bmatrix} p(t_0) \\ \dot{p}(t_0) \end{bmatrix}. \quad (2.50)$$

For an inverse square gravitational field, a closed form for $\Phi(t - t_0)$ exists (GLAN-

DORF, 1969). For problems with final position and velocity constraints, no boundary conditions are given for the adjoint equations. EXPLAIN M N T MATRICES? later in implementation maybe

This theory can be extended to impulsive maneuvers by considering that since impulses are instantaneous, and the primer vector trajectory is continuous, $\|p\| \leq 1$, and impulses happen at the instants when equality is attained.

Thus, the necessary conditions offered by primer vector theory for an impulsive maneuver scenario are:

1. $p(t)$ and $\dot{p}(t)$ are continuous;
2. $\|p\| \leq 1$, and impulses happen when $\|p\| = 1$;
3. p has the direction of \hat{u} at the impulse instants;
4. $\frac{d\|p\|}{dt} = 0$ at impulses between the initial and final times (non-inclusive).

3 Bibliographic survey

glandorf

interactive primer vector

battin impl lambbert

how many impulses?

n impulse trajectories

infinitely many optimal

lion handelsman

switching curve, min time, impulsive as limit case

multiple impulse rendezvous

multiple impulse multi rev()

4 Methodology

All code was implemented in the Julia language due to the availability of *packages* for subproblems of this work. In the next section, several components of the solution to the problem of optimization of impulsive maneuvers are detailed, along with its full formulation.

4.1 Orbit Propagation

The implementation of an orbit propagator concerns itself with the implementation of the function $p_o(X, t)$ introduced in equation (2.20). Two different cases are to be considered: "explicit"propagation, where numerical inputs are available and a numerical output is desired; and "implicit"propagation, where the propagation step is a part of a larger solver.

Brazil's INPE developed a package for orbit propagation and analysis with several models (Kepler, J2 semi-analytical secular and short term, among others) called **SatelliteToolbox**. It provides quite convient functions for converting between the Cartesian state vector $\begin{bmatrix} r^T & v^T \end{bmatrix}^T$ and the Keplerian elements, as well as functions for the propagation of orbits by some specified amount of time t_p . Its algorithms were chosen with precision around edge cases in mind (SCHWARZ, 2014), making it numerically precise but unsuitable for nonlinear solvers, which expect differentiable functions everywhere. The functions in this package are also limited to elliptic orbits.

Therefore, this is an auxiliary package used for verification, initial guess generation, and direct numerical propagation whenever required. When propagation is required in the statement of a nonlinear optimization problem, another method for orbit propagation is required. Discretized numerical integration in Cartesian coordinates was chosen for this. Discretized integration inside a numerical solver is done via collocation, direct shooting or multiple shooting (also known as relaxation). Multiple shooting was chosen since it leads to a sparse problem, which benefits the performance of the chosen solver (see next section)

Let $X_{\text{next}} = f_{RK}(X_{\text{prev}}, \Delta t)$ be the (two-body) dynamics function discretized through a fourth order Runge Kutta method. Then a number N of integration steps is chosen and $N + 1$ state vector variables $X^j, j = 1, \dots, N + 1$ are created belonging to an array $\chi \in \mathbb{R}^{6 \times (N+1)}$. They are subject to the constraints

$$X^{j+1} = f_{RK}(X^j, \frac{t_p}{N}), j = 1, \dots, N. \quad (4.1)$$

This leaves $\dim X = 6$ degrees of freedom, which are to be specified with a boundary condition. This boundary condition can be an initial condition, a final condition or relation to another coasting segment through an impulse, as will be discussed in Section 4.5. Thus, this parameterization of orbital propagation is *isoconstrained*.

4.2 Nonlinear solver

Nonlinear solvers are algorithms which iteratively approximate the solution to a system of nonlinear equations or a constrained optimization problem. Many algorithms, and many different algorithms exist. Broadly speaking, there are two classes of algorithms: gradient-free and gradient based. Gradient-free methods do not rely on derivatives for finding a solution; prototypical examples are Simplex methods, evolutionary algorithms, and simulated annealing. They are quite general but do not fully exploit a problem's structure when derivative information is available. Gradient-based methods include gradient descent, sequential quadratic programming (SQP), Newton-Raphson, trust-region methods and interior point methods. A sophisticated implementation of an interior point method is found in Ipopt, an open-source solver with interfaces in many languages (Wächter; Biegler, 2006).

A further distinction between algorithms is whether they are local or global solvers. Local solvers will converge (usually quickly) to a local optimum, which is not certified to be the global optimum (and in general, should not be assumed to be). Thus they rely on good initial guesses, to be provided to the solver, to arrive at an extremum. Global solvers can discover multiple local extrema and select the best among them; they sometimes rely on multiple (possibly random) initial values for a local optimizer. Ipopt is a local solver.

Julia offers a multitude of nonlinear solvers, each with different scope, interface, and algorithms. This work has chosen to use JuMP, a package which offers a modelling language for optimization problems that is quite close to mathematical notation. The package's *tech stack* can be seen in figure 4.1. Internally, JuMP relies on an intermediary package, MathOptInterface, which standardizes solver interfaces. This in turn allows for the usage of Ipopt_jll, an unofficial wrapper for the Ipopt solver. Ipopt is an open-source nonlinear solver widely recognized for its speed and precision, outperforming many competitors and

being quite flexible. It is especially well suited to problems with many variables (up to thousands) with sparse constraints (that is, constraints that depend only on small subsets of variables). It is capable of handling equality and inequality constraints.

This solver allows for the specification of lower and upper bounds of variables separately to the specification of inequality constraints. Variable bounds are guaranteed to be respected at all iterations; inequality constraints are only guaranteed to be satisfied at the converged solution, if the problem is feasible. This distinction is important because some constraints define the domain of problem and should never be violated; other constraints are problem-based and therefore can be violated during the iteration process.

Finally, Ipopt is built with sparsity in mind. Complex calculations should be broken into simpler constraints, each depending on less variables. This may even increase the number of variables but Ipopt's performance profits from this structure.

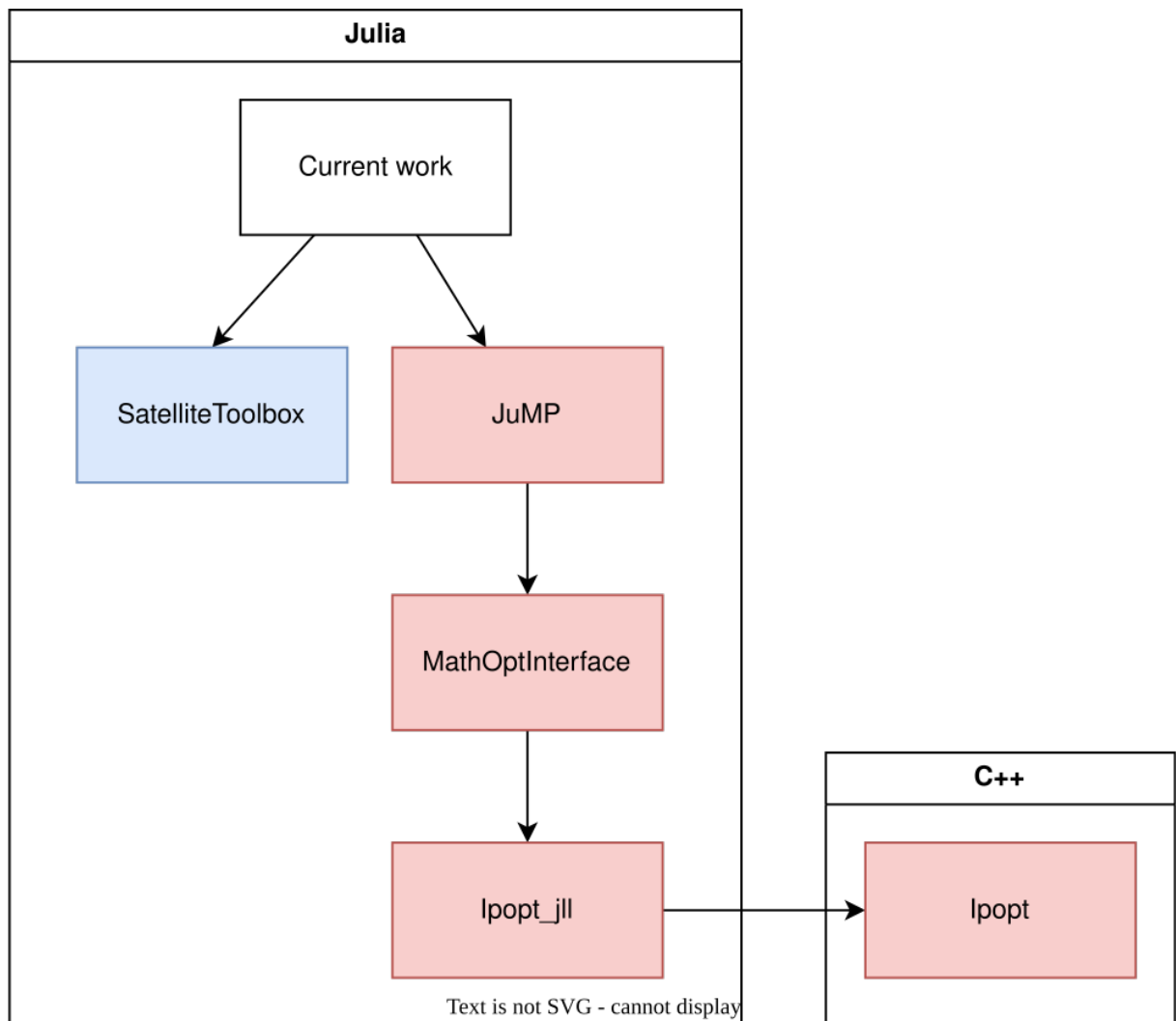


FIGURA 4.1 – Relationship of code components

4.3 Lambert problem implementation

The formulations stated in the previous chapter make for one-dimensional nonlinear programs, which leads to high performance. However, they do not handle the singularity case of $r_1 \parallel r_2$, which is of particular importance to orbital maneuvers as they often happen at periapsis and apoapsis. Sukhanov's formulation actually gives expressions for the initial radial and normal velocity when the input positions are collinear; the plane of the orbit should then be adequately chosen afterwards. However, this was found to be very numerically sensitive and another algorithm was used in the rest of this work. An implicit orbit propagation algorithm, as described in Section 4.1 is setup with boundary conditions

$$r_{(j=1)} = r_1 \quad (4.2)$$

$$r_{(j=N+1)} = r_2 \quad (4.3)$$

which account for the 6 boundary conditions needed. In order to help convergence in the collinear case (and neighboring cases), two inequality constraints may be added:

$$r_j^T r_j \geq R_{\text{Earth}}^2 \quad (4.4)$$

$$r_j \times v_j \geq 0 \quad (4.5)$$

where the second constraint must be inverted if the desired orbit is retrograde.

4.4 Primer vector calculation

Verification of the necessary conditions for optimality requires knowing the primer vector history. There are no boundary conditions on the primer vector, but for an orbital transfer with two impulses, Δv_1 at time t_1 and Δv_2 at time t_2 , two primer vector values are known:

$$p(t_1) = \frac{\Delta v_1}{\|\Delta v_1\|} \quad (4.6)$$

$$p(t_2) = \frac{\Delta v_2}{\|\Delta v_2\|} \quad (4.7)$$

The state transition between these two instants can be stated as

$$\begin{bmatrix} p(t_2) \\ \dot{p}(t_2) \end{bmatrix} = \Phi(t_2 - t_1) \begin{bmatrix} p(t_1) \\ \dot{p}(t_1) \end{bmatrix} = \begin{bmatrix} M(t_2 - t_1) & N(t_2 - t_1) \\ S(t_2 - t_1) & T(t_2 - t_1) \end{bmatrix} \begin{bmatrix} p(t_1) \\ \dot{p}(t_1) \end{bmatrix} \quad (4.8)$$

where M , N , S and T are square matrices. From now on the times shall be denoted as indices for conciseness. If p and \dot{p} are known for a certain time, the entire trajectory can be found. It is easy to isolate $\dot{p}_1 = \dot{p}(t_1)$:

$$\dot{p}_1 = N_{21}^{-1} (p_2 - M_{21}p_1). \quad (4.9)$$

N_{21} is invertible except for isolated values of t (SPACECRAFT..., 2010); in these cases, the primer vector trajectory is assumed to lie in the orbital plane, spanned by $\begin{bmatrix} r_1 & v_1 \end{bmatrix}$. Thus, for singular N ,

$$\dot{p}_1 = \begin{bmatrix} r_1 & v_1 \end{bmatrix} (N \begin{bmatrix} r_1 & v_1 \end{bmatrix})^\dagger (p_2 - M_{21}p_1) \quad (4.10)$$

where A^\dagger denotes the pseudo-inverse of a rectangular matrix.

4.5 Optimal impulsive maneuver problem statement

The full optimization problem, as it is implemented in code, shall be stated in this section. Due to JuMP's intuitive modelling language, the problem is coded almost as it is stated here. The input parameters are:

1. r_1, v_1 : initial orbital position and velocity;
2. r_2, v_2 : final orbital position and velocity;
3. t_f : transfer time;
4. N : number of integration steps per coasting arc.

The problem comprises the following variables, where inequalities represent variable bounds:

1. $\Delta t_1 \geq 0, \Delta t_2 \geq 0$: intervals between 0 and the first maneuver and between the first and second maneuver;
2. $\Delta v_1 \geq 0, \Delta v_2 \geq 0^1$: magnitudes of the first and second impulses;
3. $\hat{u}_1, \hat{u}_2 \in \mathbb{R}^3$: directions of the first and second impulses;

¹The problem could have been parameterized with vector quantities for the changes in velocities, $\Delta \vec{v}$, but the objective function would then be stated $\sum \sqrt{\Delta \vec{v}^T \Delta \vec{v}}$, which is not differentiable at $\Delta \vec{v} = 0$, which is inconvenient.

4. $\chi_1, \chi_2, \chi_3 \in \mathbb{R}^{6 \times (N+1)}$: arrays of state variables for each coasting arc. They shall be indexed as χ_c^{kj} , $c = 1, 2, 3$ is the coasting arc index, $k = 1, \dots, 6$ is the component index, $j = 1, \dots, N + 1$ is the state vector index. X_c^j, r_c^j, v_c^j denote the j -th state, position and velocity vectors of the c -th coasting arc: $(X_c^j)_k = \chi_c^{kj}$.

These variables are then subjected to constraints:

$$\text{Total time less than transfer time} \quad \Delta t_1 + \Delta t_2 \leq t_f \quad (4.11)$$

$$\text{Unit directions} \quad \Delta \hat{u}_m^T \Delta \hat{u}_m = 1, m = 1, 2 \quad (4.12)$$

$$\text{Initial state} \quad \chi_1^{k,1} = \begin{bmatrix} r_1 \\ v_1 \end{bmatrix}_k, k = 1, \dots, 6 \quad (4.13)$$

$$\text{Final state} \quad \chi_3^{k,N+1} = \begin{bmatrix} r_2 \\ v_2 \end{bmatrix}_k, k = 1, \dots, 6 \quad (4.14)$$

$$\text{Maneuver boundary conditions} \quad \chi_{m+1}^{k,1} = \chi_m^{k,N+1} + \begin{bmatrix} 0_{3 \times 1} \\ \Delta v_m \hat{u}_m \end{bmatrix}, m = 1, 2 \quad (4.15)$$

$$\text{Propagation of coasting arcs} \quad X_c^{j+1} = f_{RK}(X_c^j, \Delta t_c/N), c = 1, 2, 3, j = 1, \dots, N \quad (4.16)$$

$$\text{where} \quad \Delta t_3 = t_f - \Delta t_1 - \Delta t_2 \quad (4.17)$$

$$\text{Prograde orbit} \quad r_c^j \times v_c^j \geq 0, \forall j, c \quad (4.18)$$

$$\text{Non-intersection with Earth} \quad (r_c^j)^T r_c^j \geq R_{\text{Earth}}^2 \quad (4.19)$$

Finally, the objective is given by

$$\min \Delta v_1 + \Delta v_2 \quad (4.20)$$

5 RESULTADOS E DISCUSSÃO

5.1 Preliminary direct optimization results

5.1.1 Primer vector theory application

5.2 Future results

6 CONCLUSÃO

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Apêndice A - Future Planning

FOLHA DE REGISTRO DO DOCUMENTO

1. CLASSIFICAÇÃO/TIPO TC	2. DATA 25 de março de 2015	3. DOCUMENTO Nº DCTA/ITA/DM-018/2015	4. Nº DE PÁGINAS 40
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