8. FEEDBACK CONTROL SYSTEMS

Topics:

* Transfer functions, block diagrams and simplification
* Feedback controllers
* Control system design

Objectives:

* To be able to represent a control system with block diagrams.
* To be able to select controller parameters to meet design objectives.

8.1 INTRODUCTION

Every engineered component has some function. A function can be described as a transformation of inputs to outputs. For example it could be an amplifier that accepts a signal from a sensor and amplifies it. Or, consider a mechanical gear box with an input and output shaft. A manual transmission has an input shaft from the motor and from the shifter. When analyzing systems we will often use transfer functions that describe a system as a ratio of output to input.

8.2 TRANSFER FUNCTIONS

Transfer functions are used for equations with one input and one output variable. An example of a transfer function is shown below in Figure 8.1. The general form calls for output over input on the left hand side. The right hand side is comprised of constants and the 'D' operator. In the example 'x' is the output, while 'F' is the input.
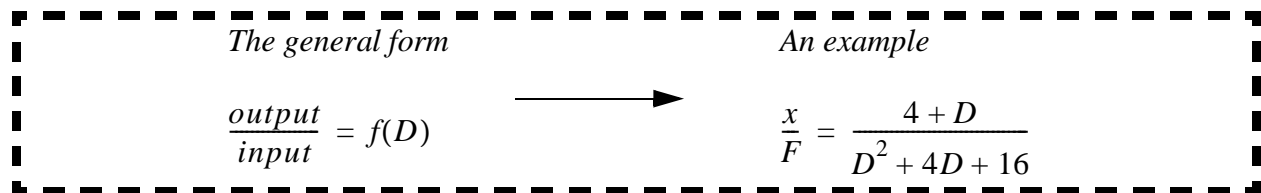
$$\textit{The general form} \qquad\qquad \textit{An example}$$

$$\frac{output}{input} = f(D) \qquad\longrightarrow\qquad \frac{x}{F} = \frac{4+D}{D^2 + 4D + 16}$$

Figure 8.1   A transfer function example

If both sides of the example were inverted then the output would become 'F', and the input 'x'. This ability to invert a transfer function is called reversibility. In reality many systems are not reversible.

There is a direct relationship between transfer functions and differential equations. This is shown for the second-order differential equation in Figure 8.2. The homogeneous equation (the left hand side) ends up as the denominator of the transfer function. The non-homogeneous solution ends up as the numerator of the expression.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f}{M}$$

$$xD^2 + 2\zeta\omega_n xD + \omega_n^2 x = \frac{f}{M}$$

$$x(D^2 + 2\zeta\omega_n D + \omega_n^2) = \frac{f}{M}$$

$$\frac{x}{f} = \frac{\left(\frac{1}{M}\right)}{D^2 + 2\zeta\omega_n D + \omega_n^2}$$

particular

homogeneous
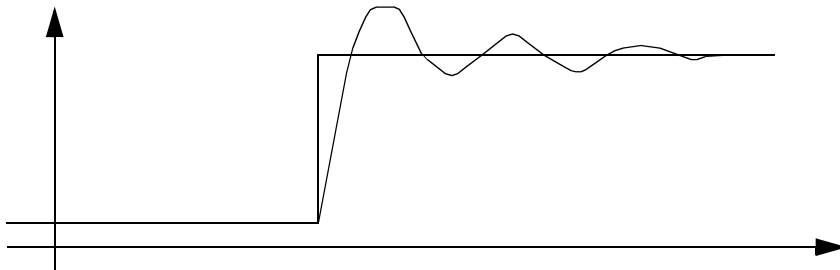
ω_n *Natural frequency of system - Approximate frequency of control system oscillations.*

ξ *Damping factor of system - If < 1 then underdamped, and the system will oscillate. If =1 critically damped. If < 1 overdamped, and never any oscillation (more like a first-order system). As damping factor approaches 0, the first peak becomes infinite in height.*

Figure 8.2   The relationship between transfer functions and differential equations for a mass-spring-damper example

The transfer function for a first-order differential equation is shown in Figure 8.3. As before the homogeneous and non-homogeneous parts of the equation becomes the denominator and the numerator of the transfer function.
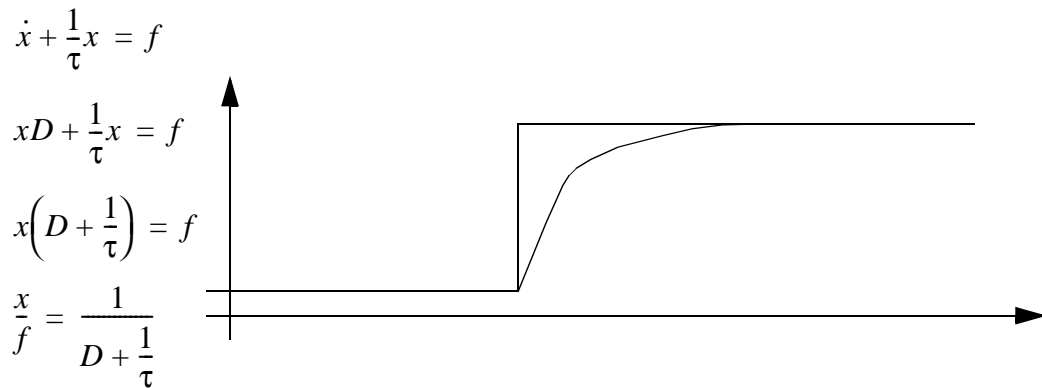
$$\dot{x} + \frac{1}{\tau}x = f$$

$$xD + \frac{1}{\tau}x = f$$

$$x\left(D + \frac{1}{\tau}\right) = f$$

$$\frac{x}{f} = \frac{1}{D + \frac{1}{\tau}}$$

Figure 8.3   A first-order system response

8.3 CONTROL SYSTEMS

Figure 8.4 shows a transfer function block for a car. The input, or control variable is the gas pedal angle. The system output, or result, is the velocity of the car. In standard operation the gas pedal angle is controlled by the driver. When a cruise control system is engaged the gas pedal must automatically be adjusted to maintain a desired velocity setpoint. To do this a control system is added, in this figure it is shown inside the dashed line. In this control system the output velocity is subtracted from the setpoint to get a system error. The subtraction occurs in the summation block (the circle on the left hand side). This error is used by the controller function to adjust the control variable in the system. Negative feedback is the term used for this type of controller.
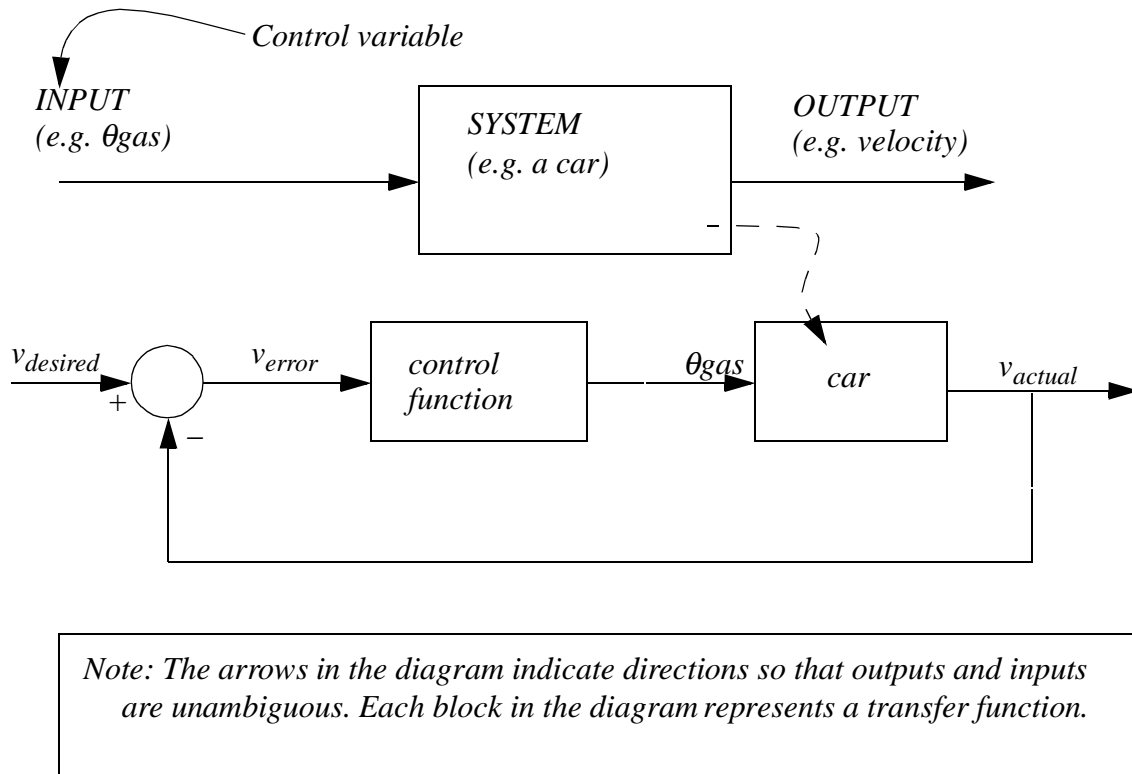
Figure 8.4   An automotive cruise control system

There are two main types of feedback control systems: negative feedback and positive feedback. In a positive feedback control system the setpoint and output values are added. In a negative feedback control the setpoint and output values are subtracted. As a rule negative feedback systems are more stable than positive feedback systems. Negative feedback also makes systems more immune to random variations in component values and inputs.

The control function in Figure 8.4 can be defined many ways. A possible set of rules for controlling the system is given in Figure 8.5. Recall that the system error is the difference between the setpoint and actual output. When the system output matches the setpoint the error is zero. Larger differences between the setpoint and output will result in larger errors. For example if the desired velocity is 50mph and the actual velocity 60mph, the error is -10mph, and the car should be slowed down. The rules in the figure give a general idea of how a control function might work for a cruise control system.

Human rules to control car (also like expert system/fuzzy logic):

1. If v_{error} is not zero, and has been positive/negative for a while, increase/decrease θ_{gas}
2. If v_{error} is very big/small increase/decrease θ_{gas}
3. If v_{error} is near zero, keep θ_{gas} the same
4. If v_{error} suddenly becomes bigger/smaller, then increase/decrease θ_{gas}.
5. etc.

Figure 8.5   Example control rules

In following sections we will examine mathematical control functions that are easy to implement in actual control systems.

8.3.1 PID Control Systems

The Proportional Integral Derivative (PID) control function shown in Figure 8.6 is the most popular choice in industry. In the equation given the 'e' is the system error, and there are three separate gain constants for the three terms. The result is a control variable value.

$$u = K_p e + K_i \int e\, dt + K_d \left(\frac{de}{dt} \right)$$

Figure 8.6   A PID controller equation

Figure 8.7 shows a basic PID controller in block diagram form. In this case the potentiometer on the left is used as a voltage divider, providing a setpoint voltage. At the output the motor shaft drives a potentiometer, also used as a voltage divider. The voltages from the setpoint and output are subtracted at the summation block to calculate the feedback error. The resulting error is used in the PID function. In the proportional branch the error is multiplied by a constant, to provide a longterm output for the motor (a ballpark guess). If an error is largely positive or negative for a while the integral branch value will become large and push the system towards zero. When there is a sudden change occurs in the error value the differential branch will give a quick response. The results of all three branches are added together in the second summation block. This result is then amplified to drive the motor. The overall performance of the system can be changed by adjusting the gains in the three branches of the PID function.
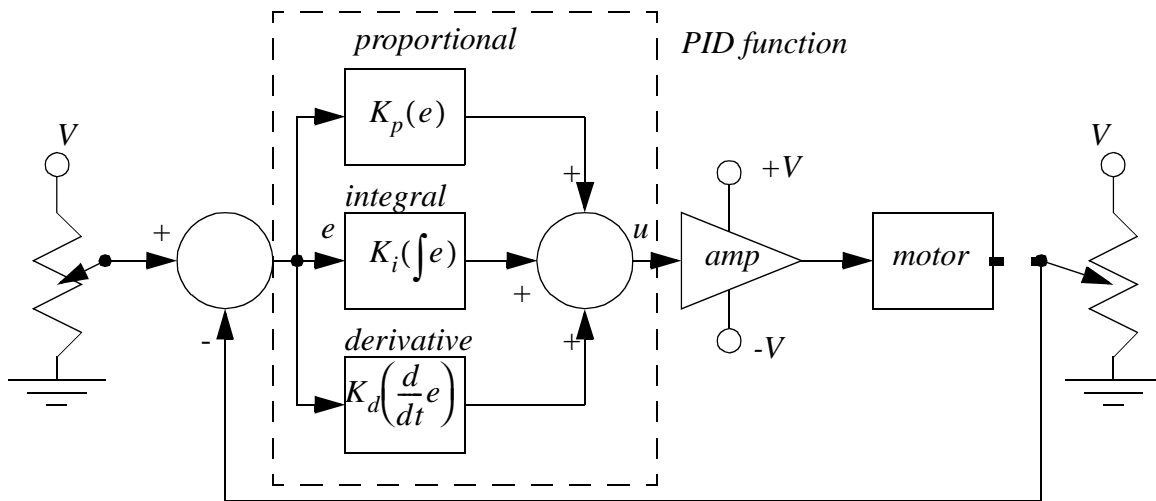
Figure 8.7   A PID control system

There are other variations on the basic PID controller shown in Figure 8.8. A PI controller results when the derivative gain is set to zero. (Recall the second order response.) This controller is generally good for eliminating long term errors, but it is prone to overshoot. In a P controller only the proportional gain in non-zero. This controller will generally work, but often cannot eliminate errors. The PD controller does not deal with longterm errors, but is very responsive to system changes.

For a PI Controller

$$\theta_{gas} = K_p v_{error} + K_i \int v_{error} dt$$

For a P Controller

$$\theta_{gas} = K_p v_{error}$$

For a PD Controller

$$\theta_{gas} = K_p v_{error} + K_d \left(\frac{dv_{error}}{dt} \right)$$

Figure 8.8   Some other control equations

Aside: The manual process for tuning a PID controlled is to set all gains to zero. The proportional gain is then adjusted until the system is responding to input changes without excessive overshoot. After that the integral gain is increased until the longterm errors disappear. The differential gain will be increased last to make the system respond faster.

8.3.2 Manipulating Block Diagrams

A block diagram for a system is not unique, meaning that it may be manipulated into new forms. Typically a block diagram will be developed for a system. The diagram will then be simplified through a process that is both graphical and algebraic. For example, equivalent blocks for a negative feedback loop are shown in Figure 8.9, along with an algebraic proof.
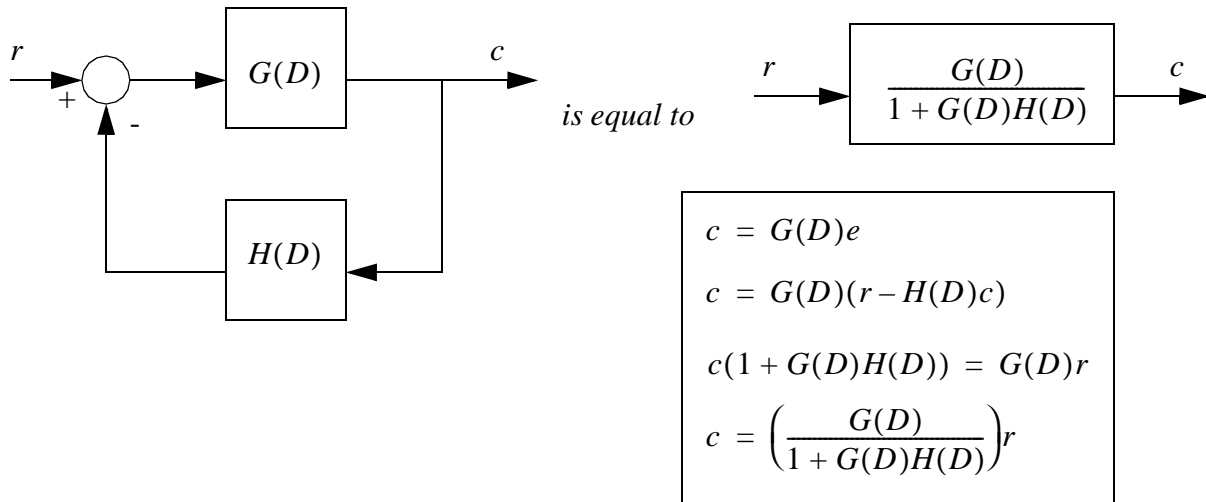

$$c = G(D)e$$

$$c = G(D)(r - H(D)c)$$

$$c(1 + G(D)H(D)) = G(D)r$$

$$c = \left(\frac{G(D)}{1 + G(D)H(D)}\right)r$$

Figure 8.9   A negative feedback block reduction

Other block diagram equivalencies are shown in Figure 8.10 to Figure 8.16. In all cases these operations are reversible. Proofs are provided, except for the cases where the equivalence is obvious.
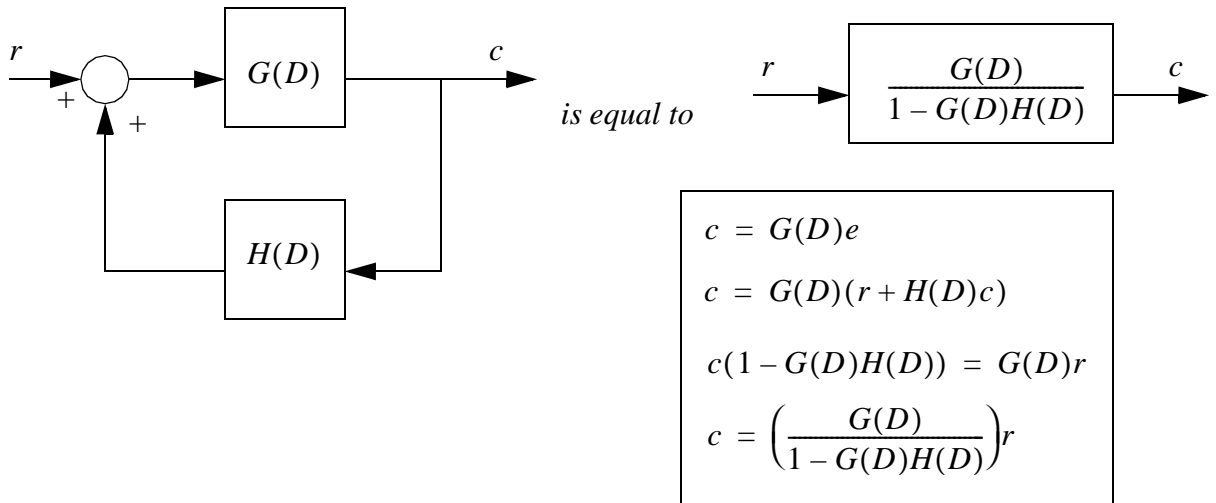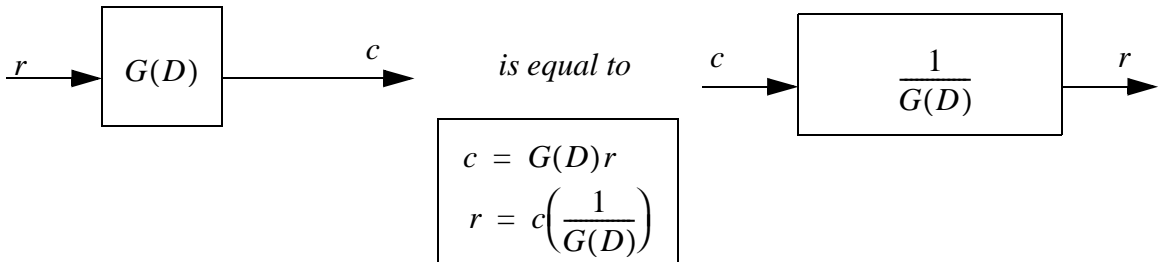
The diagram shows a positive feedback loop with input r into a summing junction (both inputs marked $+$), then into block $G(D)$, producing output c. The output feeds back through $H(D)$ to the summing junction. This *is equal to* a single block with input r and output c:

$$\frac{G(D)}{1 - G(D)H(D)}$$

$$c = G(D)e$$

$$c = G(D)(r + H(D)c)$$

$$c(1 - G(D)H(D)) = G(D)r$$

$$c = \left(\frac{G(D)}{1 - G(D)H(D)}\right)r$$

Figure 8.10   A positive feedback block reduction

Input r into block $G(D)$ producing output c *is equal to* input c into block $\frac{1}{G(D)}$ producing output r.

$$c = G(D)r$$

$$r = c\left(\frac{1}{G(D)}\right)$$

Figure 8.11   Reversal of function blocks

Input r into block $G(D)$ with output c that branches to a second c output, *is equal to* input r branching, with one path into block $G(D)$ producing output c and the branch path into another block $G(D)$ producing output c.
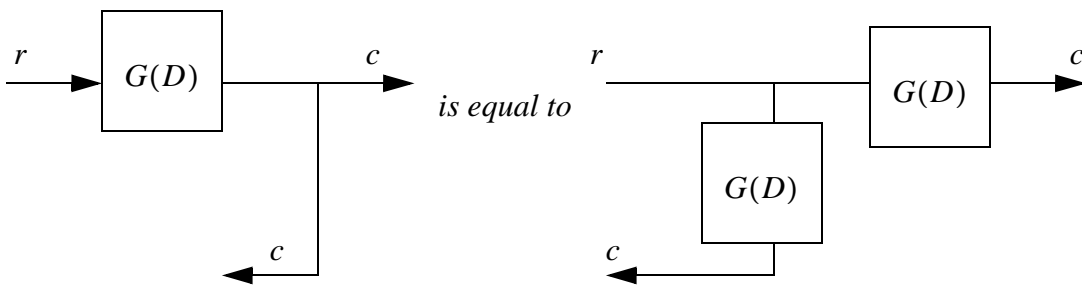
Figure 8.12   Moving branches before blocks

$$x = G(D)r$$
$$b = H(D)x$$
$$b = H(D)G(D))r$$
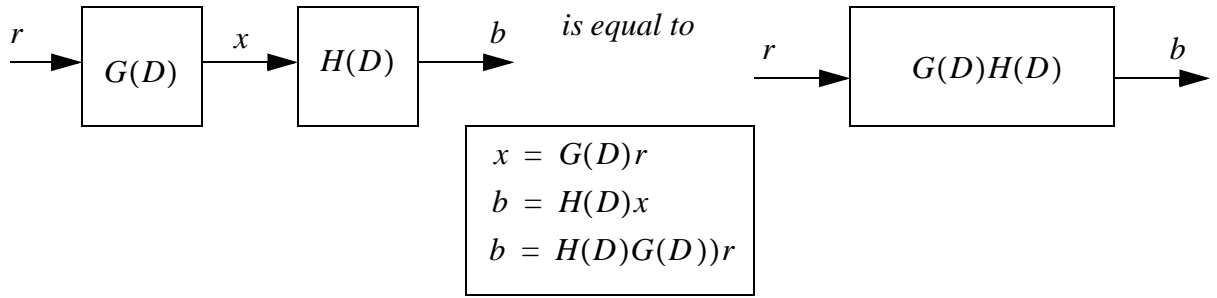
Figure 8.13   Combining sequential function blocks

is equal to
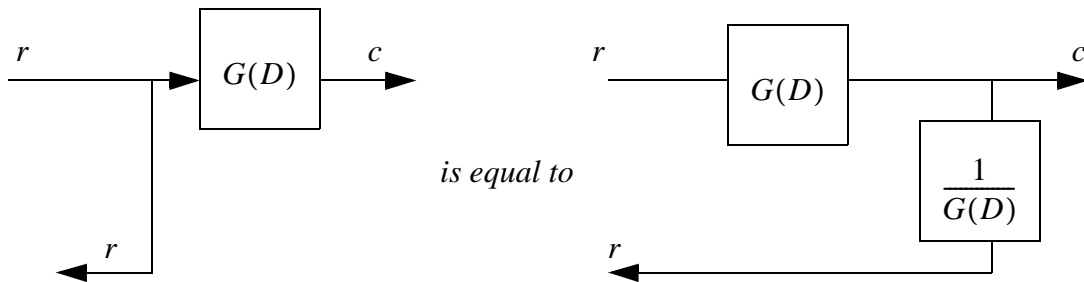
Figure 8.14   Moving branches after blocks

is equal to

$$e = \pm G(D)r \pm b$$
$$e = G(D)\left(\pm r \pm \frac{b}{\pm G(D)}\right)$$
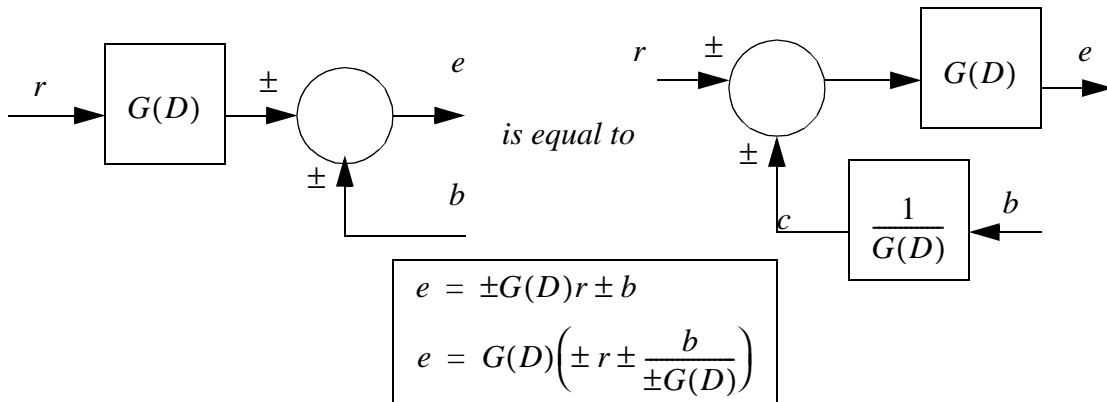
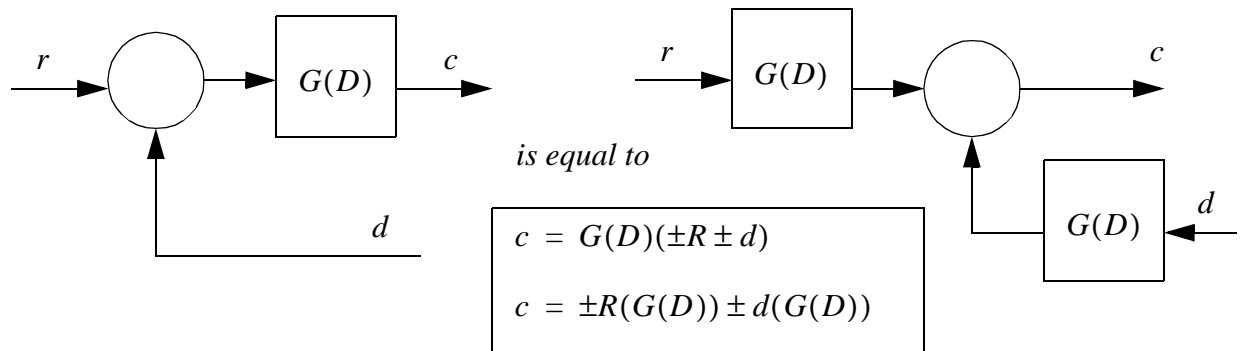*Figure 8.15*   Moving summation functions before blocks

Figure 8.16 Moving summation function past blocks

Recall the example of a cruise control system for an automobile presented in Figure 8.4. This example is extended in Figure 8.17 to include mathematical models for each of the function blocks. This block diagram is first simplified by multiplying the blocks in sequence. The feedback loop is then reduced to a single block. Notice that the feedback line doesn't have a function block on it, so by default the function is '1' - everything that goes in, comes out.
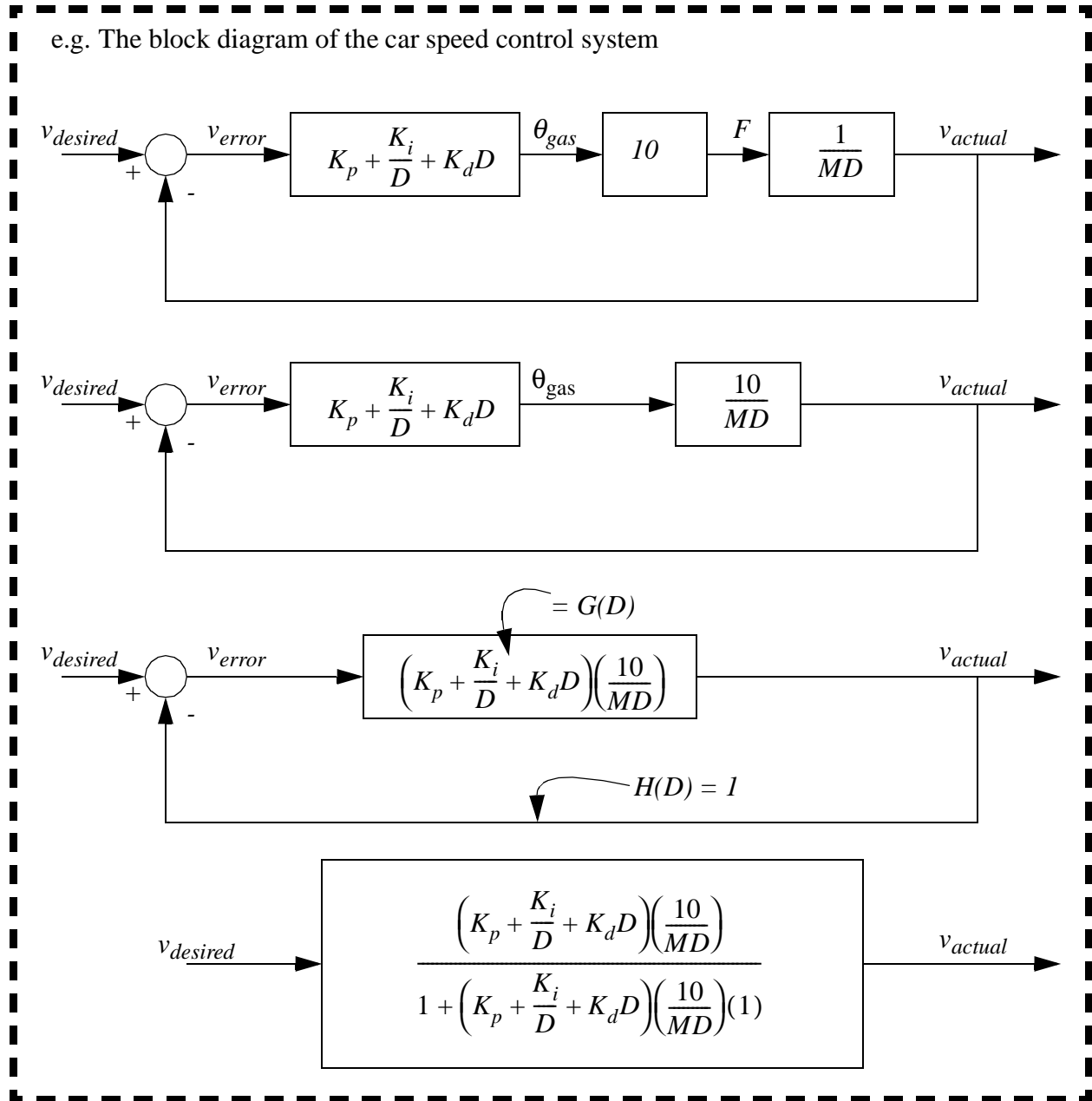
e.g. The block diagram of the car speed control system

$v_{desired}$ +   - v_{error} $K_p + \dfrac{K_i}{D} + K_dD$ θ_{gas} 10 F $\dfrac{1}{MD}$ v_{actual}

$v_{desired}$ +   - v_{error} $K_p + \dfrac{K_i}{D} + K_dD$ θ_{gas} $\dfrac{10}{MD}$ v_{actual}

$= G(D)$

$v_{desired}$ +   - v_{error} $\left(K_p + \dfrac{K_i}{D} + K_dD\right)\left(\dfrac{10}{MD}\right)$ v_{actual}

$H(D) = 1$

$v_{desired}$ $\dfrac{\left(K_p + \dfrac{K_i}{D} + K_dD\right)\left(\dfrac{10}{MD}\right)}{1 + \left(K_p + \dfrac{K_i}{D} + K_dD\right)\left(\dfrac{10}{MD}\right)(1)}$ v_{actual}

Figure 8.17   An example of simplifying a block diagram

The function block is further simplified in Figure 8.18 to a final transfer function for the whole system.

$$\left(\frac{\left(K_p + \dfrac{K_i}{D} + K_dD \right)}{\dfrac{Ms}{10} + \left(K_p + \dfrac{K_i}{D} + K_dD \right)} \right)$$

$v_{desired}$ → [block] → v_{actual}

$$\left(\frac{D^2(K_d) + D(K_p) + (K_i)}{D^2\left(\dfrac{M}{10} + K_d \right) + D(K_p) + (K_i)} \right)$$

$v_{desired}$ → [block] → v_{actual}

$$\frac{v_{actual}}{v_{desired}} = \left(\frac{D^2(K_d) + D(K_p) + (K_i)}{D^2\left(\dfrac{M}{10} + K_d \right) + D(K_p) + (K_i)} \right)$$
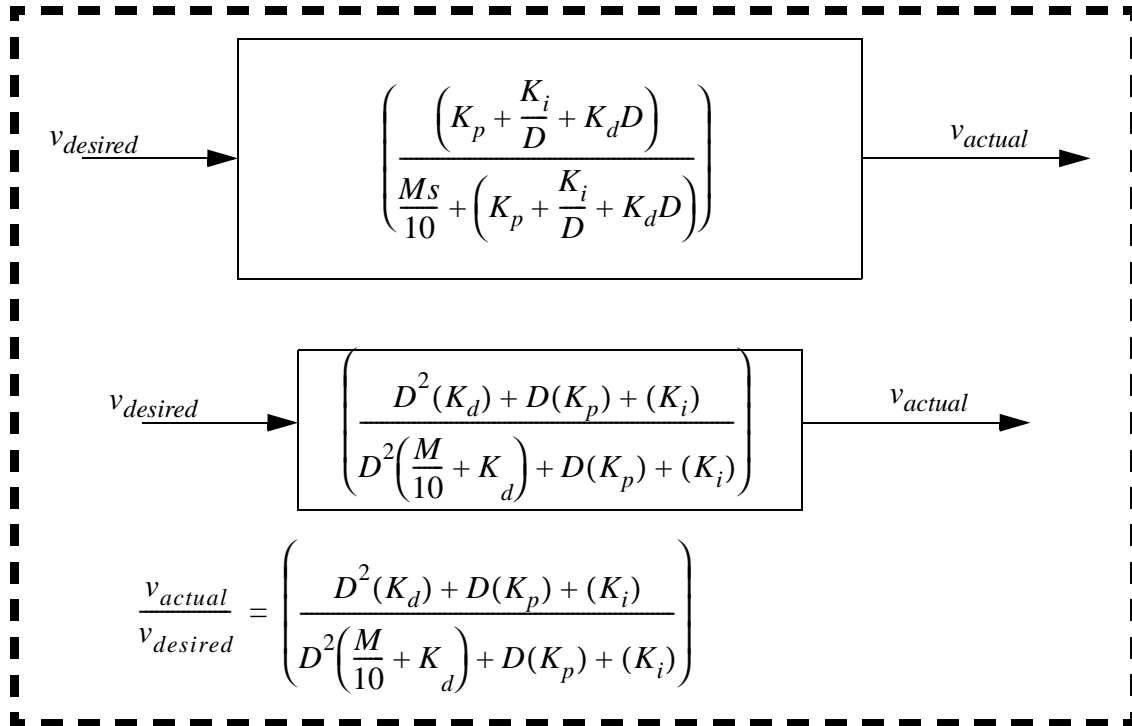
Figure 8.18   An example of simplifying a block diagram (continued)

8.3.3 A Motor Control System Example

Consider the example of a DC servo motor controlled by a computer. The purpose of the controller is to position the motor. The system in Figure 8.19 shows a reasonable control system arrangement. Some elements such as power supplies and commons for voltages are omitted for clarity.
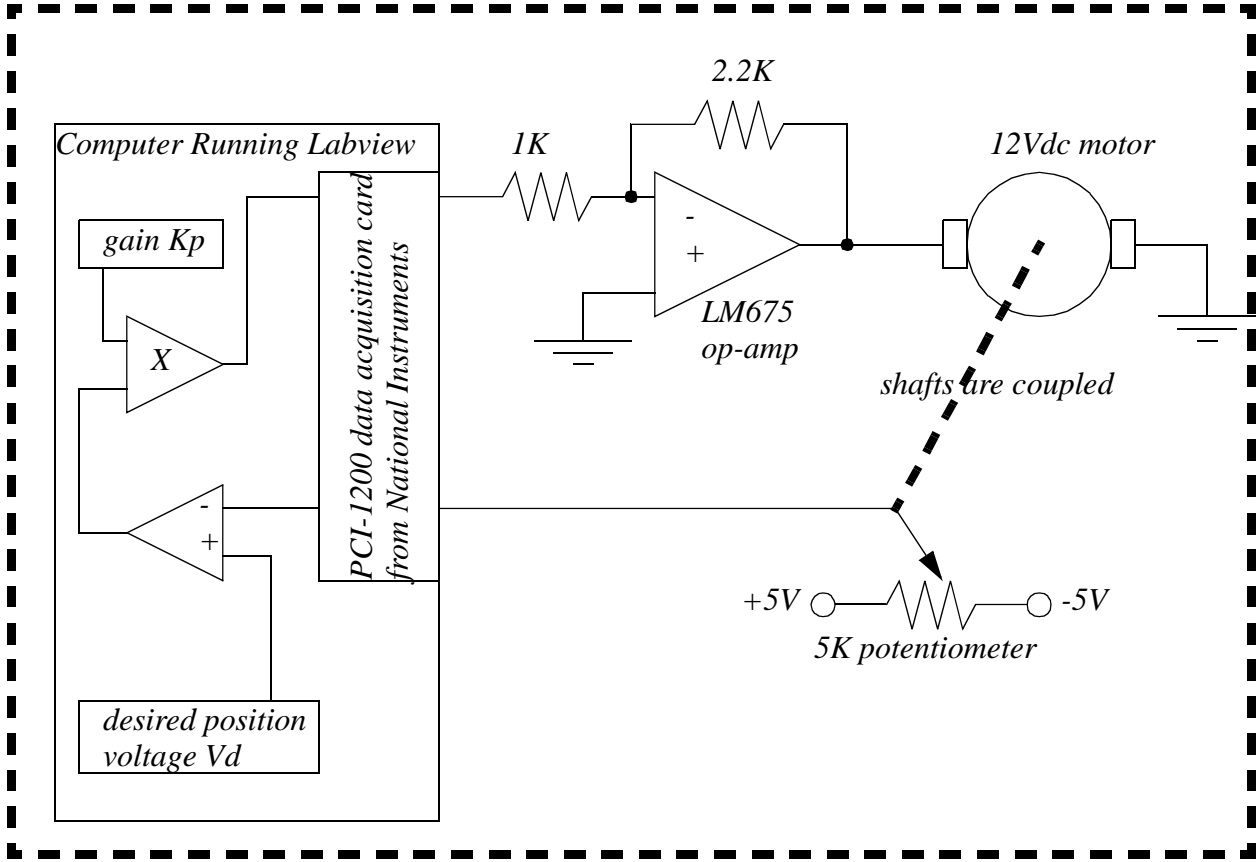
Figure 8.19   A motor feedback control system

The feedback controller can be represented with the block diagram in Figure 8.20.
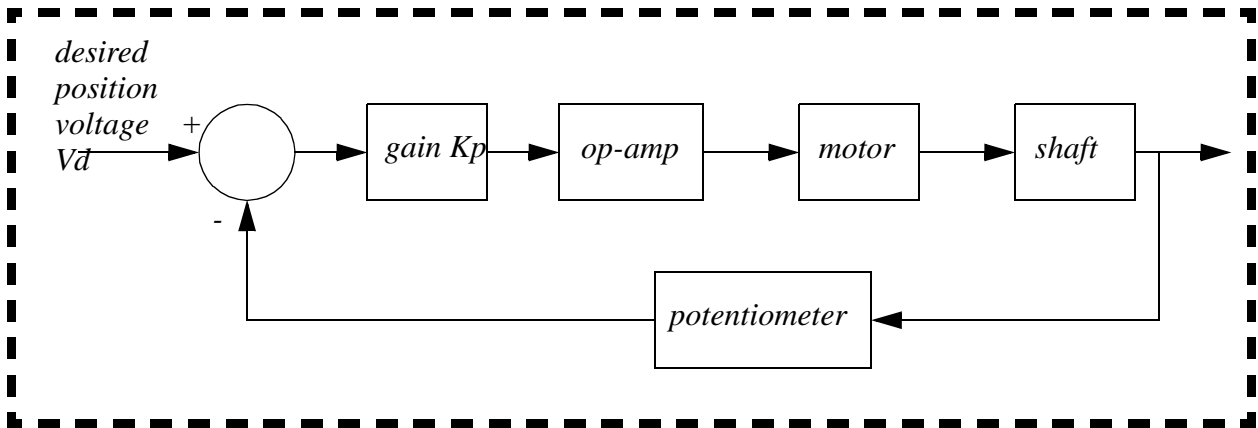
Figure 8.20   A block diagram for the feedback controller

The transfer functions for each of the blocks are developed in Figure 8.21. Two of the values must be provided by the system user. The op-amp is basically an inverting amplifier with a fixed gain of -2.2 times. The potentiometer is connected as a voltage divider and the equation relates angle to voltage. Finally the velocity of the shaft is integrated to give position.

Given or selected values:
>   *- desired potentiometer voltage Vd*
>   *- gain K*

For the op-amp:

$$\sum I_{V+} = \frac{V_+ - V_i}{1K} + \frac{V_+ - V_o}{2.2K} = 0 \qquad V_+ = V_- = 0V$$

$$\frac{-V_i}{1K} + \frac{-V_o}{2.2K} = 0$$

$$\frac{V_o}{V_i} = -2.2$$

For the potentiometer assume that the potentiometer has a range of 10 turns and 0 degrees is in the center of motion. So there are 5 turns in the negative and positive direction.

$$V_o = 5V\left(\frac{\theta}{5(2\pi)}\right)$$

$$\frac{V_o}{\theta} = 0.159 Vrad^{-1}$$

For the shaft, it integrates the angular velocity into position:

$$\omega = \frac{d}{dt}\theta$$

$$\frac{\theta}{\omega} = \frac{1}{D}$$

Figure 8.21   Transfer functions for the power amplifier, potentiometer and motor shaft

The basic equation for the motor is derived in Figure 8.22 using experimental data. In this case the motor was tested with the full inertia on the shaft, so there is no need to calculate 'J'.

For the motor use the differential equation and the speed curve when Vs=10V is applied:

1400 RPM

1s   2s   3s

$$\left(\frac{d}{dt}\right)\omega + \left(\frac{K^2}{JR}\right)\omega = \left(\frac{K}{JR}\right)V_s$$
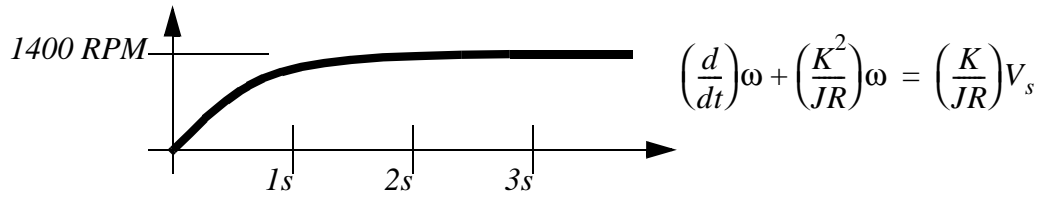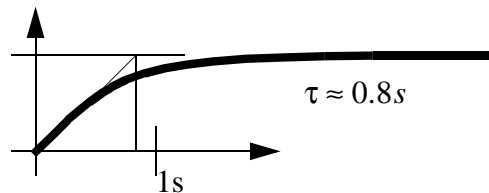
For steady-state

$$\left(\frac{d}{dt}\right)\omega = 0 \qquad \omega = 1400RPM = 146.6rads^{-1}$$

$$0 + \left(\frac{K^2}{JR}\right)146.6 = \left(\frac{K}{JR}\right)10$$

$$K = 0.0682$$

$$\tau \approx 0.8s$$

1s

$$\left(\frac{K^2}{JR}\right) = \frac{1}{\tau}$$

$$0.0682\left(\frac{K}{JR}\right) = \frac{1}{0.8s}$$

$$\frac{K}{JR} = 18.328$$

$$D\omega + \frac{1}{0.8}\omega = 18.33V_s$$

$$\frac{\omega}{V_s} = \frac{18.33}{D + 1.25}$$

Figure 8.22   Transfer function for the motor

The individual transfer functions for the system are put into the system block diagram in Figure 8.23. The block diagram is then simplified for the entire system to a single transfer function relating the desired voltage (setpoint) to the angular position (output). The transfer function contains the unknown gain value 'Kp'.
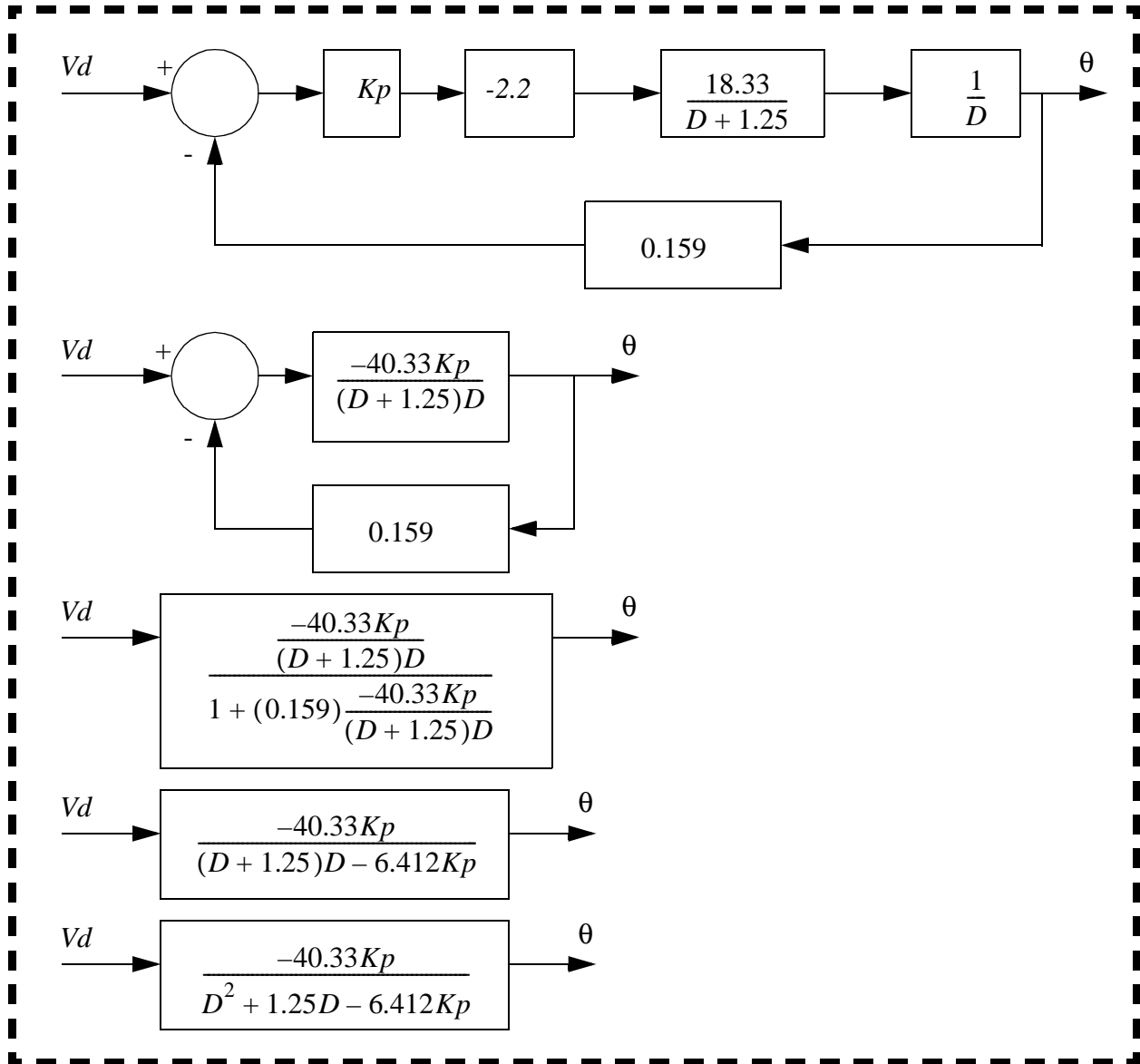
Figure 8.23   The system block diagram, and simplification

The value of 'Kp' can be selected to 'tune' the system performance. In Figure 8.24 the gain value is calculated to give the system an overall damping factor of 1.0, or critically damped. This is done by recognizing that the bottom (homogeneous) part of the transfer function is second-order and then extracting the damping factor and natural frequency. The final result of 'Kp' is negative, but this makes sense when the negative gain on the op-amp is considered.

We have specified, or been given the damping coefficient as a design objective.

$$\xi = 1.0$$

The denominator of the system transfer function can be compared to the standard second-order response.

$$D^2 + 1.25D - 6.412K_p = x'' + 2\xi\omega_n x' + \omega_n^2 x$$

$$1.25 = 2\xi\omega_n$$

$$1.25 = 2(1.0)\omega_n$$

$$\omega_n = 0.625$$

$$-6.412K_p = \omega_n^2$$

$$-6.412K_p = 0.625^2$$

$$K_p = -0.0609$$

Figure 8.24   Calculating a gain Kp

8.3.4 System Error

System error is often used when designing control systems. The two common types of error are system error and feedback error. The equations for calculating these errors are shown in Figure 8.25. If the feedback function 'H' has a value of '1' then these errors will be the same.

System error, $S = r - c$

Feedback error, $e = r - b$
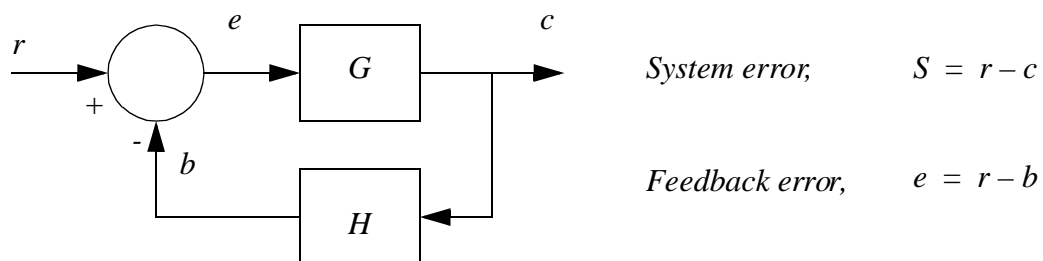
Figure 8.25   Controller errors

An example of calculating these errors is shown in Figure 8.26. The system is a simple integrator, with a unity feedback loop. The overall transfer function for the system is calculated and then used to find the system response. The response is then compared to the input to find the system error. In this case the error will go to zero as time approaches infinity.

Given,

$$G(D) = \frac{K_p}{D} \qquad H(D) = 1$$

$$\therefore \frac{c}{r} = \frac{G}{1 + GH} = \frac{K_p}{D + K_p}$$

$$\therefore \dot{c} + K_p c = K_p r$$

$$r = A$$

The homogeneous solution is,

$$c_h = C_1 e^{-K_p t}$$

The particular solution is found with a guess,

$$c_p = C_2$$
$$\dot{c}_p = 0$$
$$0 + K_p C_2 = K_p A \qquad\qquad C_2 = A$$

The solutions can be combined and the remaining unknown found for the system at rest initially.

$$c = c_h + c_p = C_1 e^{-K_p t} + A$$

$$0 = C_1 e^0 + A$$

$$C_1 = -A$$

$$c = -A e^{-K_p t} + A$$

The error can now be calculated.

$$S = r - c$$

$$S = A - (-A e^{-K_p t} + A) = A e^{-K_p t}$$

Figure 8.26 System error calculation example for a step input

Solve the previous problem using a ramp input,

$$r = At$$

Figure 8.27   Drill problem: Calculate the system error for a ramp input

Find the system error 'e' for the given ramp input, R.

$$G(D) = \frac{1}{D^2 + 4D + 5} \qquad\qquad H(D) = 5 \qquad\qquad r = 4t$$

Figure 8.28   Drill problem: Calculate the errors

8.3.5 Controller Transfer Functions

The PID controller, and simpler variations were discussed in earlier sections. A more complete table is given in Figure 8.29.

Type	*Transfer Function*
Proportional (P)	$G_c = K$
Proportional-Integral (PI)	$G_c = K\left(1 + \dfrac{1}{\tau D}\right)$
Proportional-Derivative (PD)	$G_c = K(1 + \tau D)$
Proportional-Integral-Derivative (PID)	$G_c = K\left(1 + \dfrac{1}{\tau D} + \tau D\right)$
Lead	$G_c = K\left(\dfrac{1 + \alpha\tau D}{1 + \tau D}\right) \qquad \alpha > 1$
Lag	$G_c = K\left(\dfrac{1 + \tau D}{1 + \alpha\tau D}\right) \qquad \alpha > 1$
Lead-Lag	$G_c = K\left[\left(\dfrac{1 + \tau_1 D}{1 + \alpha\tau_1 D}\right)\left(\dfrac{1 + \alpha\tau_2 D}{1 + \tau_2 D}\right)\right] \qquad \begin{matrix}\alpha > 1 \\ \tau_1 > \tau_2\end{matrix}$

Figure 8.29　Standard controller types

8.3.6 State Variable Control Systems

State variable matrices were introduced before. These can also be used to form a control system, as shown in Figure 8.30.

$$\frac{d}{dt}X = AX + BU$$

$$Y = CX + D$$
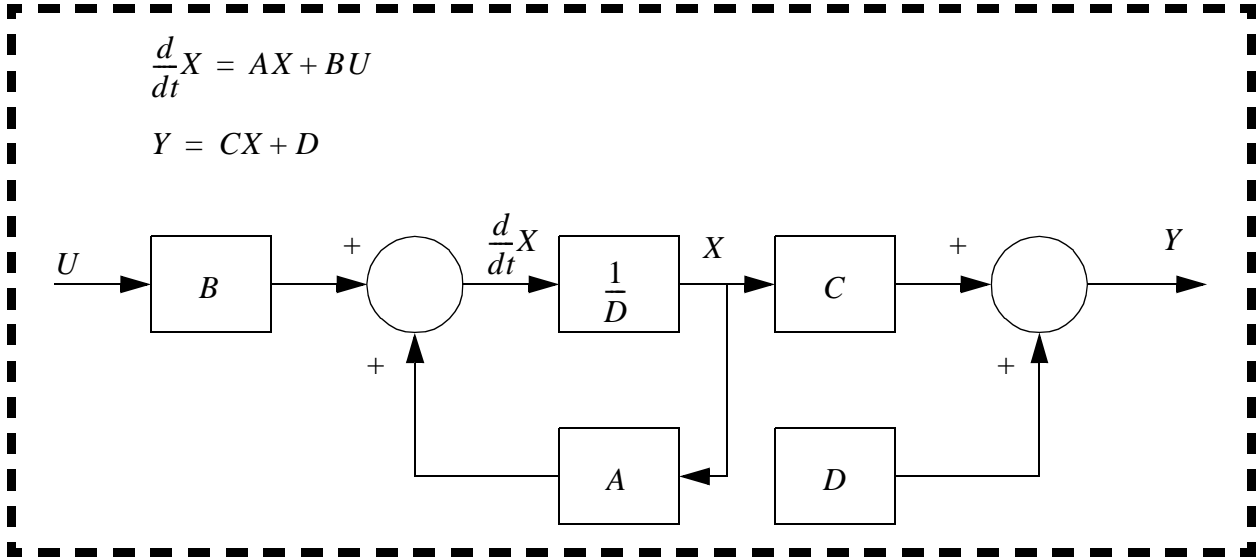
Figure 8.30 A state variable control system

8.3.7 Feedforward Controllers

When a model of a system is well known it can be used to improve the performance of a control system by adding a feedforward function, as pictured in Figure 8.31. The feedforward function is basically an inverse model of the process. When this is used together with a more traditional feedback function the overall system can outperform more traditional controllers function, such as the PID controller.
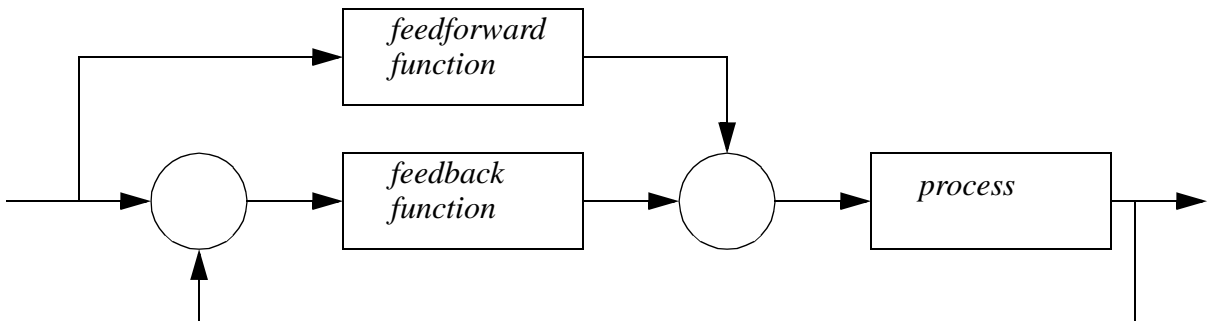
Figure 8.31 A feed forward controller

8.3.8 Cascade Controllers

When controlling a multistep process a cascade controller can allow refined control of sub-loops within the larger control system. Most large processes will have some form of cascade control. For example, the inner loop may be for a heating oven, while the outer loop controls a conveyor feeding parts into the oven.
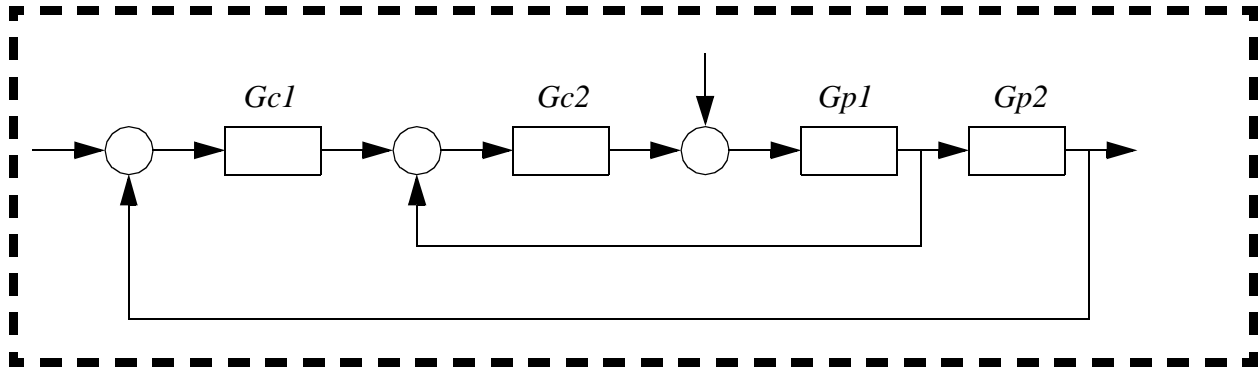
Figure 8.32   A cascade controller

8.4 SUMMARY

- Transfer functions can be used to model the ratio of input to output.
- Block diagrams can be used to describe and simplify systems.
- Controllers can be designed to meet criteria, such as damping ratio and natural frequency.
- System errors can be used to determine the long term stability and accuracy of a controlled system.
- Other control types are possible for more advanced systems.

8.5 PRACTICE PROBLEMS

1. Develop differential equations and then transfer functions for the mechanical system below. There is viscous damping between the block and the ground. A force is applied to cause the

mass the accelerate.
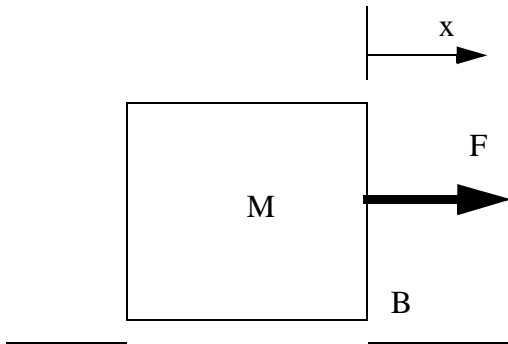
2. Develop a transfer function for the system below. The input is the force 'F' and the output is the voltage 'Vo'. The mass is suspended by a spring and a damper. When the spring is undeflected y=0. The height is measured with an ultrasonic proximity sensor. When y = 0, the output Vo=0V. If y=20cm then Vo=2V and if y=-20cm then Vo=-2V. Neglect gravity.

$$K_s = 10\frac{N}{m} \qquad K_d = 5N\frac{s}{m} \qquad M = 0.5Kg$$
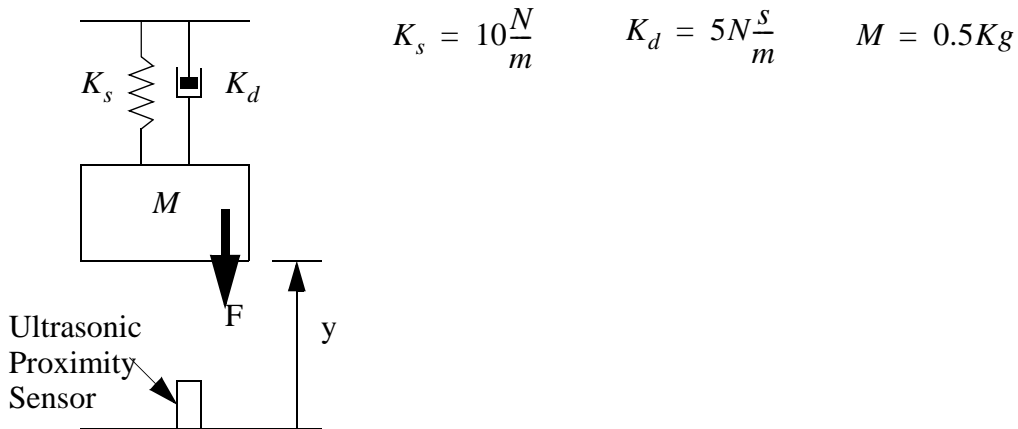
3. Find the transfer functions for the systems below. Here the input is a torque, and the output is the angle of the second mass.
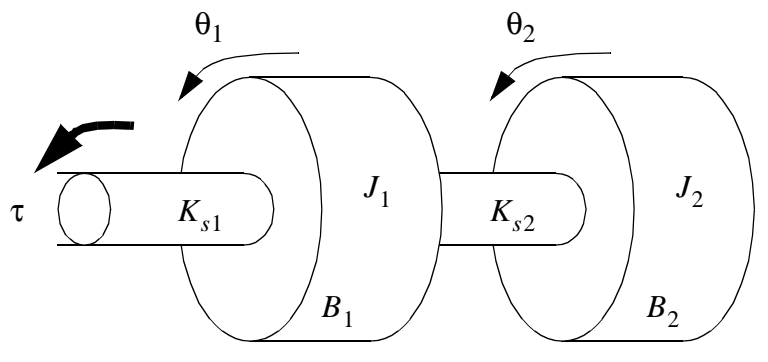
4. Find the transfer functions for the system below where Vi is the input and Vo is the output.
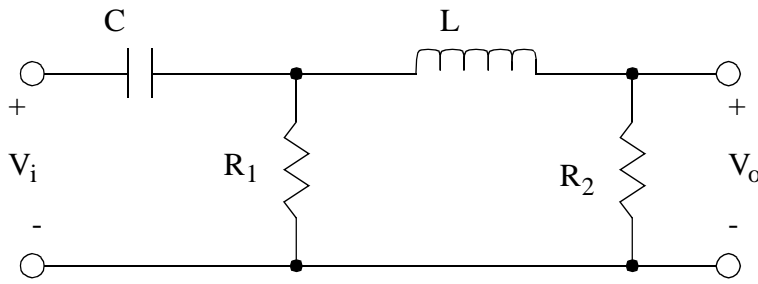
5. Given the transfer function, G(s), determine the time response output Y(t) to a step input X(t).

$$G = \frac{4}{D+2} = \frac{Y}{X}$$

$$X(t) = 20 \quad \text{When } t >= 0$$

6. Given the transfer function below, develop a mechanical system that it could be for. (Hint: Differential Equations).

$$\frac{x(D)}{F(D)} = \frac{1}{10 + 20D}$$

where: x = displacement
F = force

7. Given a mass supported by a spring and damper, find the displacement of the supported mass over time if it is released from neutral at t=0sec, and gravity pulls it downward.

        a) find the transfer function y/F.
        b) find the input function F.
        c) solve the input output equation symbolically to find the position as a function of
            time for Ks = 10N/m, Kd = 5Ns/m, M=10kg.
        d) solve part c) numerically.

8. a) What is a Setpoint, and what is it used for? b) What does feedback do in control systems?
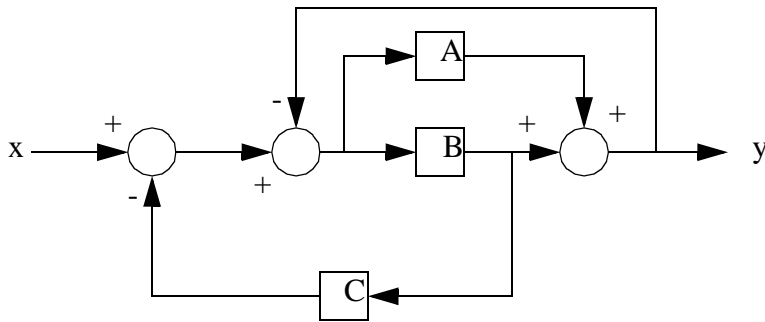
9. Simplify the block diagram below.
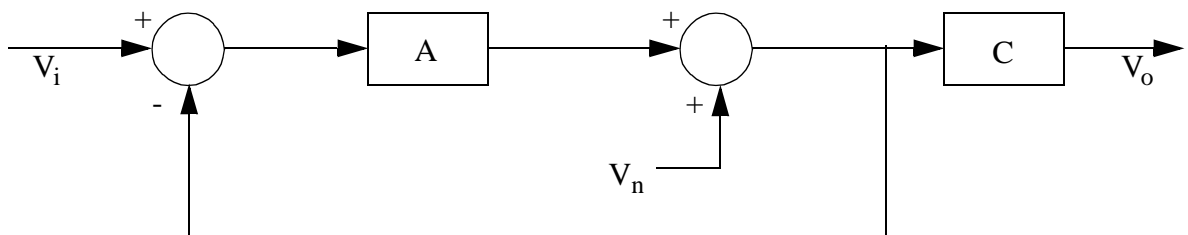
10. The block diagram below is for a servo motor position control system. The system uses a proportional controller.

a) Draw a sketch of what the actual system might look like. Identify components.
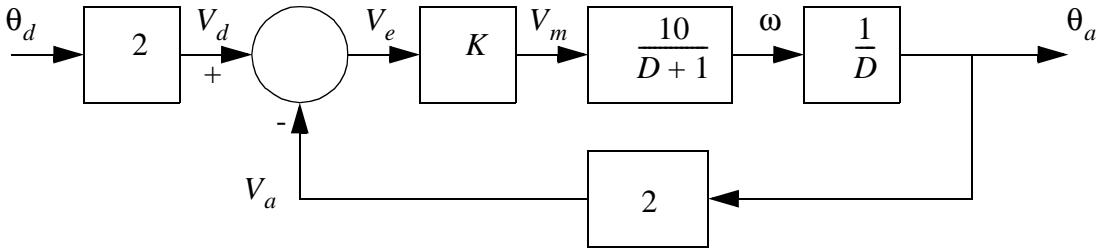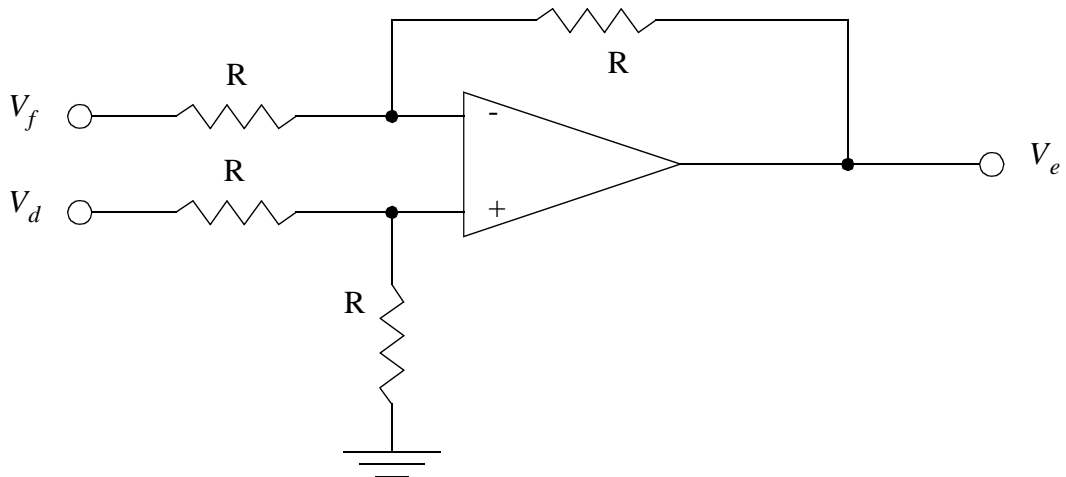

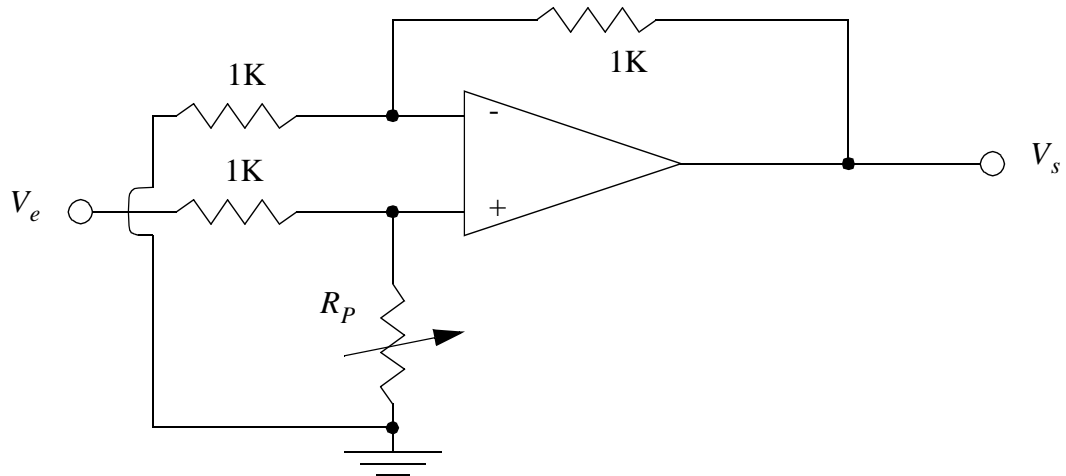b) Convert the system to a transfer function.

11. Simplify the block diagram below to a transfer function.

12. a) Develop an equation for the system below relating the two inputs to the output. (Hint: think of a summation block.)
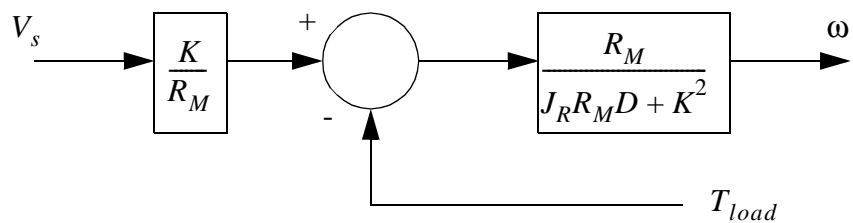
b) Develop an equation for the system below relating the input to the output.

c) The equation below can be used to model a permanent magnet DC motor with an applied torque. An equivalent block diagram is given. Prove that the block diagram is equivalent to the equation.

$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$
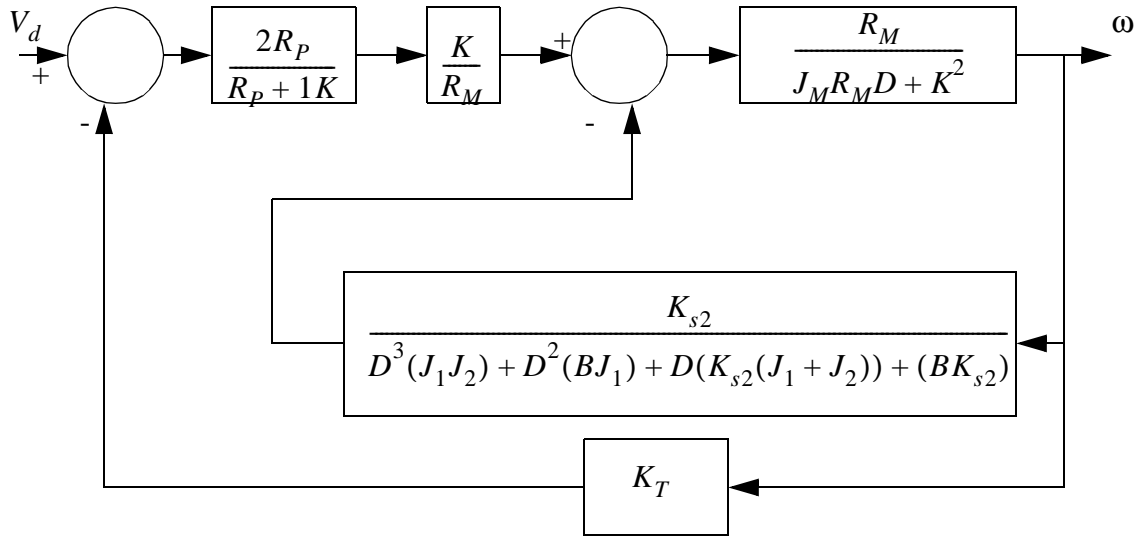

d) Write the transfer function for the system below relating the input torque to the output angle theta2. Then write the transfer function for the angular velocity of mass 2.
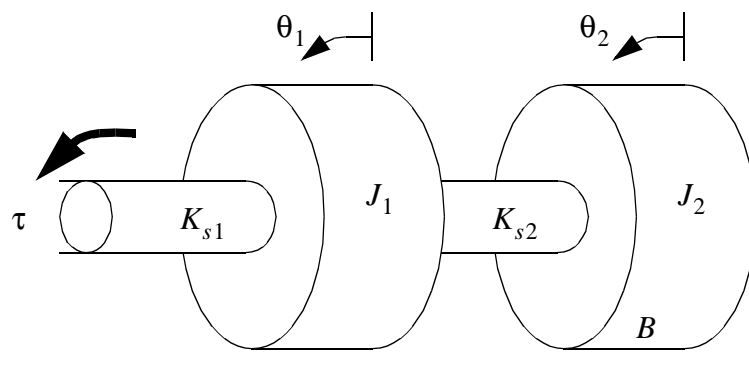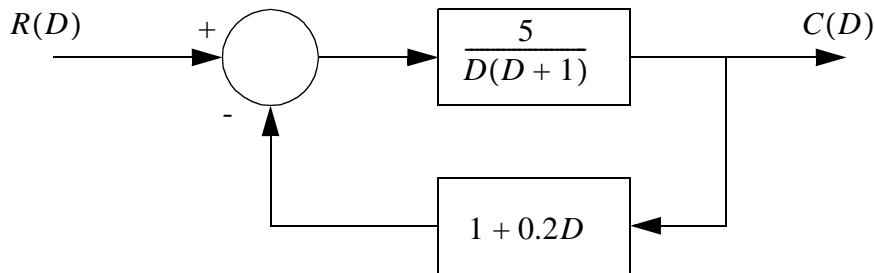

e) The system below is a combination of previous components, and a tachometer

for velocity feedback. Simplify the block diagram.

$$V_d \xrightarrow{+} \bigcirc \xrightarrow{-} \boxed{\frac{2R_P}{R_P + 1K}} \to \boxed{\frac{K}{R_M}} \xrightarrow{+} \bigcirc \xrightarrow{-} \boxed{\frac{R_M}{J_M R_M D + K^2}} \to \omega$$

$$\boxed{\frac{K_{s2}}{D^3(J_1 J_2) + D^2(BJ_1) + D(K_{s2}(J_1 + J_2)) + (BK_{s2})}}$$

$$\boxed{K_T}$$

13. Find the system error when the input is a ramp with the function r(t) = 0.5t. Sketch the system error as a function of time.

$$R(D) \xrightarrow{+} \bigcirc \xrightarrow{-} \boxed{\frac{5}{D(D+1)}} \to C(D)$$

$$\boxed{1 + 0.2D}$$

14. Given the block diagram below, select a system gain K that will give the overall system a damping ratio of 0.7 (for a step input). What is the resulting undamped natural frequency of the system?

$$\xrightarrow{+} \bigcirc \xrightarrow{-} \boxed{K} \to \boxed{\frac{2}{D^2 + 3D + 2}} \to$$
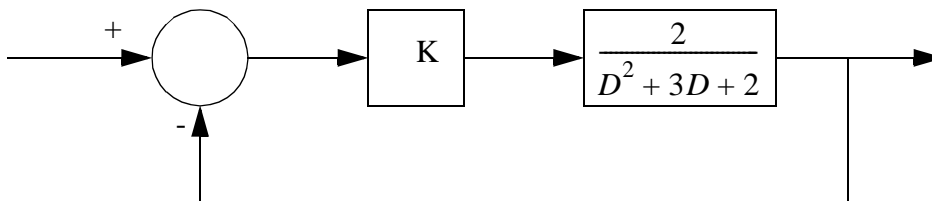
15. The following system is a feedback controller for an elevator. It uses a desired height 'd' provided by a user, and the actual height of the elevator 'h'. The difference between these two is called the error 'e'. The PID controller will examine the value 'e' and then control the speed of the lift motor with a control voltage 'c'. The elevator and controller are described with transfer functions, as shown below. all of these equations can be combined into a single system transfer

equation as shown.

$$e = d - h \quad \text{error}$$

$$\frac{c}{e} = K_p + \frac{K_i}{D} + K_d D = \frac{2D + 1 + D^2}{D} \quad \text{PID controller} \qquad \frac{h}{c} = \frac{10}{D^2 + D} \quad \text{elevator}$$

combine the transfer functions

$$\left(\frac{c}{e}\right)\left(\frac{h}{c}\right) = \frac{h}{e} = \frac{2D + 1 + D^2}{D} \frac{10}{D^2 + D} = \frac{(D+1)^2}{D} \frac{10}{D(D+1)} = \frac{10(D+1)}{D^2}$$

$$\frac{h}{d - h} = \frac{10(D+1)}{D^2} \qquad\qquad \text{eliminate 'e'}$$

$$h = \left(\frac{10(D+1)}{D^2}\right)(d - h)$$

$$h\left(1 + \frac{10(D+1)}{D^2}\right) = \left(\frac{10(D+1)}{D^2}\right)(d)$$

$$\frac{h}{d} = \left(\frac{\dfrac{10(D+1)}{D^2}}{1 + \dfrac{10(D+1)}{D^2}}\right) = \frac{10D + 10}{D^2 + 10D + 10} \qquad \text{system transfer function}$$

a) Find the response of the final equation to a step input. The system starts at rest on the ground floor, and the input (desired height) changes to 20 as a step input.

b) Calculate the damping coefficient and natural frequency of the results in part a).

16. Study the circuit below. Assume that for t<0s the circuit is discharged and off. Starting at t=0s an input of Vi=5sin(100,000t) is applied.
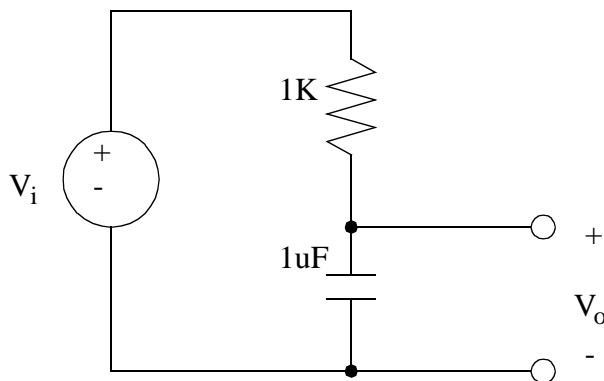
a) Write a differential equation and then a transfer function describing the circuit.

b) Find the output of the circuit using explicit integration (i.e., homogeneous and particular solutions).

8.6 PRACTICE PROBLEM SOLUTIONS

1.

$$\frac{x}{F} = \frac{1}{D(1 - DM)}$$

2.

$$\frac{V_o}{F} = \frac{V_o}{y}\frac{y}{F} = \frac{10}{-MD^2 - K_dD - K_s}$$

3.

$$\frac{\theta_2}{\tau} = \frac{K_{s2}}{D^4(J_1J_2) + D^3(B_1J_2 + B_2J_1) + D^2(K_{s2}(J_1 + J_2) + B_1B_2) + D(K_{s2}(B_1 + B_2))}$$
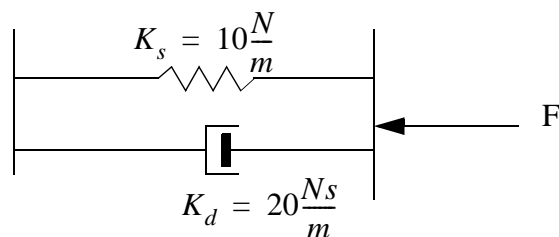
4.

$$\frac{V_o}{V_i} = \frac{D^2(R_1R_2CL) + D(R_1R_2^2C)}{D^3(R_1L^2C) + D^2(R_1R_2L(1 + 2C)) + D(2R_2L + R_1L + CR_1R_2^2) + (R_1R_2 + R_2^2)}$$

5.

$$y(t) = -40e^{-0.5t} + 40$$

6.

7.

a) $$\frac{y}{F} = \frac{1}{D^2M + DK_d + K_s}$$

b) $$F = Mg$$

c) $$y(t) = -10.13e^{-0.25t}\cos(0.968t - 0.253) + 9.81$$

d) $$\begin{bmatrix} y(t+h) \\ v(t+h) \end{bmatrix} = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + h\left(\begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 9.81 \end{bmatrix}\right)$$
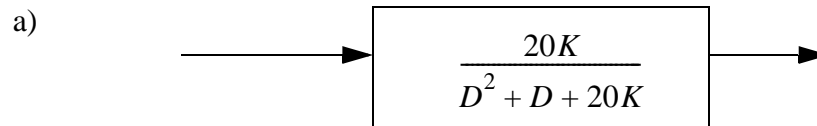
8.

a) A setpoint is a system input that correlates to a desired output for a system with negative feedback control.

b) Feedback in a control system allows an out to be compared to an input. The difference, the error, is used to adjust the control variable and increase the accuracy.
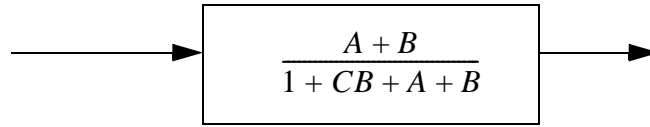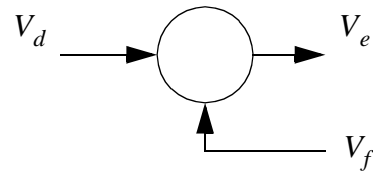
9.

10.

a)

b)

11.

$$\boxed{\dfrac{A + B}{1 + CB + A + B}}$$
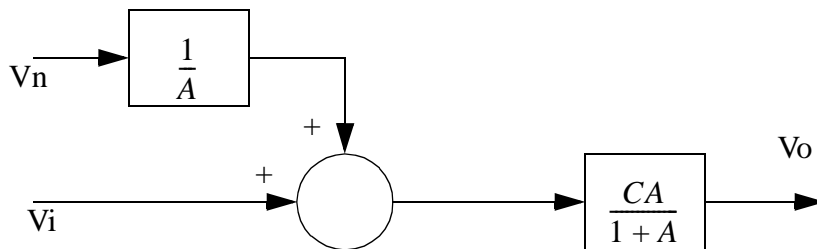
12.

a) $V_+ = V_d\left(\dfrac{R}{R + R}\right) = 0.5V_d$

$$\sum I_{V_-} = \dfrac{V_- - V_f}{R} + \dfrac{V_- - V_e}{R} = 0$$

$$V_e = 2V_- - V_f$$

$$V_e = 2(0.5V_d) - V_f$$

$$V_e = V_d - V_f$$

b)

$$\sum I_{V_-} = \dfrac{V_- - 0}{1K} + \dfrac{V_- - V_s}{1K} = 0$$

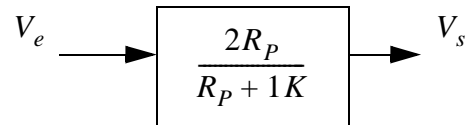$$V_- = 0.5V_s \qquad\qquad (1)$$

$$\sum I_{V_+} = \dfrac{V_+ - V_e}{1K} + \dfrac{V_+ - 0}{R_P} = 0$$

$$V_+\left(\dfrac{1}{1K} + \dfrac{1}{R_P}\right) = \dfrac{V_e}{1K}$$

substitute in (1)

$$0.5V_s\left(\dfrac{R_P + 1K}{R_P(1K)}\right) = \dfrac{V_e}{1K}$$

$$\dfrac{V_s}{V_e} = \dfrac{2R_P}{R_P + 1K}$$

$$\boxed{\dfrac{2R_P}{R_P + 1K}}$$

note: a constant set by the variable resistor Rp

c)
$$\left(\frac{d}{dt}\right)\omega + \omega\left(\frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega\left(D + \frac{K^2}{J_R R_M}\right) = V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}$$

$$\omega = \left(\frac{J R_M}{J_R R_M D + K^2}\right)\left(V_s\left(\frac{K}{J_R R_M}\right) - \frac{T_{load}}{J_R}\right)$$

$$\omega = \left(\frac{R_M}{J_R R_M D + K^2}\right)\left(V_s\left(\frac{K}{R_M}\right) - T_{load}\right)$$

d)

$$K_{s2}(\theta_1 - \theta_2)$$

τ

J_1

$$\sum M = \tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\tau - K_{s2}(\theta_1 - \theta_2) = J_1\theta_1 D^2$$

$$\theta_1(J_1 D^2 + K_{s2}) + \theta_2(-K_{s2}) = \tau \qquad (1)$$

$$K_{s2}(\theta_1 - \theta_2)$$

J_2

$$B\theta_2 D$$

$$\sum M = K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$K_{s2}(\theta_1 - \theta_2) - B\theta_2 D = J_2\theta_2 D^2$$

$$\theta_2(J_2 D^2 + K_{s2} + BD) = \theta_1(K_{s2})$$

$$\theta_1 = \theta_2\left(\frac{J_2 D^2 + K_{s2} + BD}{K_{s2}}\right) \qquad (2)$$

substitute (2) into (1)

$$\theta_2\left(\frac{J_2 D^2 + BD + K_{s2}}{K_{s2}}\right)(J_1 D^2 + K_{s2}) + \theta_2(-K_{s2}) = \tau$$

$$\theta_2\left(\frac{D^4(J_1 J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}{K_{s2}}\right) = \tau$$

$$\frac{\theta_2}{\tau} = \frac{K_{s2}}{D^4(J_1 J_2) + D^3(BJ_1) + D^2(K_{s2}(J_1 + J_2)) + D(BK_{s2})}$$

$$\omega_2 = D\theta_2$$

$$\frac{\omega_2}{\tau} = \frac{K_{s2}}{D^3(J_1 J_2) + D^2(BJ_1) + D(K_{s2}(J_1 + J_2)) + (BK_{s2})}$$

13.

$$S(t) = (-0.225 e^{-t})\cos\left(\frac{\sqrt{15}}{2}t - 0.477\right) + \frac{1}{5}$$

14.

$$\omega_d = 1.530\frac{rad}{s}$$

15.

a) $h(t) = 2.91e^{-8.873t} - 22.91e^{-1.127t} + 20$
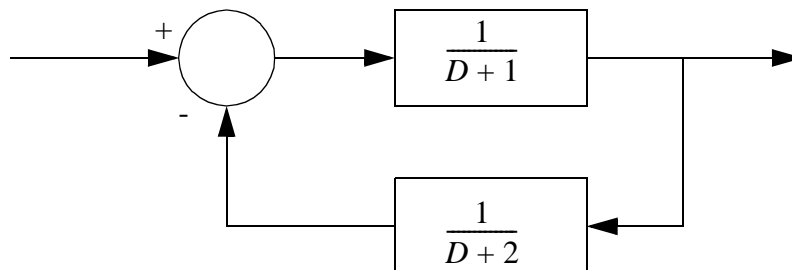
b) $\zeta = 1.58$ $\omega_n = 3.162\frac{rad}{s}$

16.

a) $\dfrac{V_o}{V_i} = \dfrac{1000}{D + 1000}$

b) $V_o = 0.050e^{-1000t} + 0.050\sin(10^5 t + 1.561)$

8.7 ASSIGNMENT PROBLEMS

1. Simplify the following block diagram.

2. Simplify the following block diagram.