Householder Algorithm

This method allows us to change a *symmetric* $n \times n$ matrix $A = (a_{ij})$ into a *tridiagonal* matrix with the same set of eigenvalues.

Let v be a column vector with $||v||_2 = 1$. The *Householder transformation* corresponding to the vector v is the orthogonal matrix

$$H = I_n - 2vv^t.$$

Algorithm:

Step 1. Set $k = 1$ and initialize $B = A$.

Step 2. Compute

$$s = \sqrt{\sum_{i=k+1}^{n} b_{ik}^2}.$$

If $s = 0$, then set $k = k + 1$ and recompute s.

Step 3. Set

$$SG = \begin{cases} -1 & \text{if} \quad b_{k+1,k} < 0 \\ +1 & \text{if} \quad b_{k+1,k} \geq 0 \end{cases}$$

Step 4. Set

$$z = \frac{1}{2}(1 + SG.b_{k+1,k}/s)$$

Step 5. Set $v_i = 0$ for $i = 1, 2, \cdots, k$, set $v_{k+1} = \sqrt{z}$, and set

$$v_i = \tfrac{SG\,.b_{ki}}{2v_{k+1}s} \qquad i = k+2, \cdots, n$$

Step 6. Set $v = (v_1, v_2, \cdots, v_n)^t$ and let $H = I_n - 2vv^t$.

Step 7. Compute $A = HBH$.

Step 8. If $k = n - 2$, then output A and stop.

Step 9. Set $k = k + 1$, $B = A$, and go to step 2.

Example. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 2 \\ -1 & 2 & 1 & -1 \\ 2 & 1 & 3 & 2 \\ 2 & -1 & 2 & 1 \end{pmatrix}.$$

Then $B = (b_{ij}) = A$,

$$s = \sqrt{(-1)^2 + 2^2 + 2^2} = 3,$$

$$z = \frac{1}{2}(1 + 1/3) = \frac{2}{3}, \text{ and}$$

$$v^t = \left[\, v_1 = 0, \quad v_2 = \sqrt{z} = \tfrac{2}{\sqrt{6}}, \quad v_3 = \tfrac{-b_{13}}{2v_2(3)} = \tfrac{-1}{\sqrt{6}}, \quad v_4 = \tfrac{-b_{14}}{2v_2(3)} = \tfrac{-1}{\sqrt{6}} \,\right] = \left[0, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right].$$

Then matrix

$$H = I_4 - 2vv^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2/3 & -1/3 & -1/3 \\ 0 & -1/3 & 1/6 & 1/6 \\ 0 & -1/3 & 1/6 & 1/6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 2/3 & 2/3 \\ 0 & 2/3 & 2/3 & -1/3 \\ 0 & 2/3 & -1/3 & 2/3 \end{pmatrix}.$$

Next we find

$$HBH = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 34/9 & 7/9 & 1/9 \\ 0 & 7/9 & 25/9 & 10/9 \\ 0 & 1/9 & 10/9 & -5/9 \end{pmatrix} = A$$

We now repeat the procedure to determine the vector v as if the matrix $B = A$ were the 3 x 3 matrix

$$\begin{pmatrix} 34/9 & 7/9 & 1/9 \\ 7/9 & 25/9 & 10/9 \\ 1/9 & 10/9 & -5/9 \end{pmatrix}.$$

In this case, we assume that v^t is of the form $[0, v_2, v_3]$. We now find that

$$s = \sqrt{(7/2)^2 + (1/9)^2} = \sqrt{50/81} = 0.7856742 \quad \text{and}$$

$$z = \frac{1}{2}(1 + \frac{7}{9}(1/0.7856742)) = 0.99497475 \ .$$

Next

$$v^t = \left[\, 0, \quad \sqrt{z}, \quad \tfrac{1/9}{2v_2 s} \,\right] = [\,0, \quad 0.99748421, \quad 0.07088902\,].$$

The 3 x 3 matrix $I_3 - 2vv^t$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.98994950 & -0.14142136 \\ 0 & -0.14142136 & 0.005025253 \end{pmatrix}.$$

Now we let H be the 4 x 4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.98994950 & -0.14142136 \\ 0 & 0 & -0.14142136 & 0.005025253 \end{pmatrix}$$

and compute product HBH. The result is the matrix

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 3.7777777 & -0.7856743 & 0 \\ 0 & -0.7856743 & 3.0222222 & -0.6000000 \\ 0 & 0 & -0.6000000 & -0.8000000 \end{pmatrix}$$

which is the desired tridiagonal matrix.