Roots of polynomials

- We know the number of roots, complex roots included.
- High order polynomials are often highly conditioned. (Even evaluations need care.)

Best use specialized methods instead of general-purpose ones.

Roots of polynomials: Companion matrix

Consider an n-by-n matrix

$$\mathbf{A}_n = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \ddots & & -a_2 \\ & \ddots & \ddots & 0 & \vdots \\ & & 0 & 1 & -a_{n-1} \end{pmatrix}$$

Its secular equation

$$0 = \det(x\mathbf{1} - \mathbf{A}) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Root-finding $\iff$ eigenvalue problem for $\mathbf{A}_n$

Roots of polynomials: Companion matrix

- Solve the eigenvalue problem for the companion matrix
- (optionally) polish the roots with several Newton steps

Companion matrix: proof by induction

The base of induction: $n = 2$

$$\det(x\mathbf{1} - \mathbf{A}_2) = \det \begin{pmatrix} x & a_0 \\ -1 & x + a_1 \end{pmatrix}$$

Companion matrix: proof by induction

The base of induction: $n = 2$

$$\begin{aligned}\det(x\mathbf{1} - \mathbf{A}_2) &= \det\begin{pmatrix} x & a_0 \\ -1 & x + a_1 \end{pmatrix} \\ &= x(x + a_1) + a_0 = x^2 + a_1 x + a_0\end{aligned}$$

Companion matrix: proof by induction

The step of induction: Suppose that $\det(x\mathbf{1} - \mathbf{A}_{n-1})$ is a required monic polynomial.

$$\det(x\mathbf{1} - \mathbf{A}_n) = \det \begin{pmatrix} x & & & & a_0 \\ -1 & x & & & a_1 \\ 0 & -1 & \ddots & & a_2 \\ & \ddots & \ddots & x & \vdots \\ & & 0 & -1 & x + a_{n-1} \end{pmatrix}$$

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Use the cofactor expansion over the first row:

$$\det(x\mathbf{1} - \mathbf{A}_n) = x \det(x\mathbf{1} - \mathbf{A}_{n-1}) +$$

Companion matrix: proof by induction

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Use the cofactor expansion over the first row:

$$\det(x\mathbf{1} - \mathbf{A}_n) = x\det(x\mathbf{1} - \mathbf{A}_{n-1}) + a_0(-1)^{n+1}(-1)^{n-1}$$

Companion matrix: proof by induction

The step of induction: Suppose that $\det(x\mathbf{1} - \mathbf{A}_{n-1})$ is a required monic polynomial.

$$\det(x\mathbf{1} - \mathbf{A}_n) = \det \begin{pmatrix} x & & & & a_0 \\ -1 & x & & & a_1 \\ 0 & -1 & \ddots & & a_2 \\ & \ddots & \ddots & x & \vdots \\ & & 0 & -1 & x + a_{n-1} \end{pmatrix}$$

Use the cofactor expansion over the first row:

$$\begin{aligned} \det(x\mathbf{1} - \mathbf{A}_n) &= x \det(x\mathbf{1} - \mathbf{A}_{n-1}) + a_0(-1)^{n+1}(-1)^{n-1} \\ &= x \left(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1\right) + a_0 \end{aligned}$$