1. Introduction

Let $\mathcal{V}$ be 2-dimensional real vector space. We fix coordinates and identify $\mathcal{V}$ with $\mathbb{R}^2$. For every $r \in \mathbb{R}$, let $\sigma_r : \mathbb{R}^2 \to \mathbb{R}^2$ be $x \mapsto rx$.

Definition 1.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function, $V \subseteq \mathbb{R}^2$ and $q \in \mathbb{Q}$. We say that *strong minimality is violated by f on V by q-lines* if there are $k \in \mathbb{N}$ and $p \in \mathbb{R}^2$, such that for every open $X \subseteq \mathbb{R}^2$ containing p, the set

$$A = \big\{x \in X : \big|\Gamma(x + \sigma_q) \cap \Gamma(f) \cap (V \times \mathbb{R}^2)\big| \geq k\big\}$$

is infinite and co-infinite in X (that is, both X and $X \setminus A$ are infinite). We simply say that *strong minimality is violated by f on V* if there is $q \in \mathbb{Q}$ such that strong minimality is violated by f on V by q-lines.

We reduce the problem of counting points of intersection of surfaces in $\mathbb{R}^4$ to counting points of intersection of curves in $\mathbb{R}^2$, as follows. For $f : \mathbb{R}^2 \to \mathbb{R}^2$, write $f = (f^1, f^2)$. We first observe that for every $f, g : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\pi\big(\Gamma(f) \cap \Gamma(g)\big) = \pi\big(\Gamma(f^1) \cap \Gamma(g^1)\big) \cap \pi\big(\Gamma(f^2) \cap \Gamma(g^2)\big). \tag{1}$$

In particular, the number of points in $\Gamma(f) \cap \Gamma(g)$ (in case this intersection is finite) is equal to the number of points in the right-hand side of (1). We also observe the following.

Fact 1.2. *Let $t \in \mathbb{R}$, $W \subseteq \mathbb{R}^2$ and $q \in \mathbb{Q}$. Then*

$$\big|\Gamma((0, t) + \sigma_q) \cap \Gamma(f) \cap (V \times \mathbb{R}^2)\big| =$$

$$= \big|\pi\big(\Gamma(\sigma_q^1) \cap \Gamma(f^1)\big) \cap \pi\big(\Gamma(t + \sigma_q^2) \cap \Gamma(f^2)\big) \cap V\big|.$$

Given two functions $f, g : \mathbb{R}^m \to \mathbb{R}^n$, we denote by $f \perp_a g$ the fact that the graphs of f and g intersect transversally at a.

2. Preliminaries

2.1. **Eigenvalues.** Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator with $L(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$.

Fact 2.1. *A real r is an eigenvalue for L if and only if $\Gamma(\sigma_r) \cap \Gamma(L)$ is infinite.*

Proof. We have: r is an eigenvalue for L, if and only if the matrix $\begin{pmatrix} a - r & b \\ c & d - r \end{pmatrix}$ has rank < 2, if and only if the set of all points (x, y) satisfying

$$ax + by = rx$$

$$cx + dy = ry$$

is infinite, if and only if $\Gamma(\sigma_r) \cap \Gamma(\tau)$ is infinite. $\square$

For every $r \in \mathbb{R}$ with $L^1 \perp \sigma_r^1$ and $L^2 \perp \sigma_r^2$, we denote for $i = 1, 2$,

$$s_r^i := \text{ the slope of } \pi(\Gamma(\sigma_r^i) \cap \Gamma(L^i)), \text{ possibly } \infty.$$

Claim 2.2. *Assume r_0 is a real eigenvalue for L such that $L^1 \perp \sigma_{r_0}^1$ and $L^2 \perp \sigma_{r_0}^2$. If $s_{r_0}^1 \neq \infty$, then $s_r^1 - s_r^2$ changes sign at r_0.*

Proof. We have that $\pi(\Gamma(\sigma_r^1) \cap \Gamma(L^1))$ consists of the points (x, y) satisfying

$$ax + by = rx$$

and $\pi(\Gamma(\sigma_r^2) \cap \Gamma(L^2))$ consists of the points (x, y) satisfying

$$cx + dy = ry$$

Then $s_r^1 = \frac{r-a}{b}$ and $s_r^2 = \frac{c}{r-d}$. By Fact 2.1, $s_{r_0}^1 = s_{r_0}^2$, and hence the condition $s_{r_0}^1 \neq \infty$ implies $b \neq 0$ and $r_0 \neq d$. It follows that $s_r^1 - s_r^2 = \frac{(r-a)(r-d)-bc}{b(r-d)}$ changes sign at r_0 if and only if the map

$$\Delta(r) = (r - a)(r - d) - bc$$

is strictly monotone at r_0. But r_0 is a root of $\Delta(r)$, so the last statement is true unless $r_0 = a = d$ and $bc = 0$, which is excluded since $r_0 \neq d$. $\square$

2.2. **Facts from o-minimality.** We work in an o-minimal expansion of the reals, and collect some basic facts about definable curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$.

Fact 2.3. *Let $I = (-a, a) \subseteq \mathbb{R}$. Let $\{g_r\}_{r \in J}$ be a continuous definable family of functions $g_r : I \to \mathbb{R}$ with $g_r(0) = 0$. Assume that for all $r > r_0$ in J, we have $g_r'(0) > 0$ and that for all $r < r_0$ in J, we have $g_r'(0) < 0$. Then there is $q \in \mathbb{Q}$ and $b \in (0, a)$ such that $g_q(b) = 0$.*

Proof. Assume not. For all $q > r_0$ in $\mathbb{Q} \cap J$, since $g_q'(0) > 0$, there is $c \in (0, a)$ with $g(c) > 0$. By the Intermediate Value Theorem, continuity of g_q and our assumption, g_q is everywhere positive on $(0, a)$. Similarly, for all $q < r_0$ in $\mathbb{Q} \cap J$, g_q is everywhere negative on $(0, a)$. Now let c be any element in $(0, a)$. By continuity of the family $\{g_r\}_{r \in \mathbb{R}}$ and the Intermediate Value Theorem again, there is $r \in \mathbb{R}$ such that $g_r(c) = 0$ (in fact, $r = r_0$). Contradiction. $\square$

Fact 2.4. *Let $Y \subseteq \mathbb{R}$ be an open interval containing 0 and $S \subseteq \mathbb{R}^2$ an open set containing 0. Assume $h : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with $h(0) = 0$, and $\tau : \mathbb{R}^2 \to \mathbb{R}$ a linear map. Then for every $s < 0 < t$ in Y there is an open box $V \subseteq S$ containing 0 such that*

$$\pi\big(\Gamma(s + \tau) \cap \Gamma(h)\big) \cap V = \pi\big(\Gamma(t + \tau) \cap \Gamma(h)\big) \cap V = \emptyset.$$

Proof. Since $s < 0 < t$, none of $\Gamma(s + \tau)$ and $\Gamma(t + \tau)$ passes through the origin. Since h is continuous, $h(cl(S))$ is closed. Hence, for any $x \in \mathbb{R}$, $B_x := \pi\big(\Gamma(x + \tau) \cap \Gamma(h_{\restriction cl(S)})\big)$ is closed and does not contain 0.. By regularity axiom for $\mathbb{R}^2$, there is an open box $V \subseteq S$ containing 0 and not intersecting B_s nor B_t. $\square$

Note: we will only use the existence of *some* $s < 0 < t$ in Y with the above property.

Fact 2.5. *Let $I, K, Y \subseteq \mathbb{R}$ be two open intervals with center 0 and let $V = I \times K$. Let $\{\beta_t\}_{t \in Y}$ be a continuous definable family of maps $\beta_t : I \to \mathbb{R}$, and $g : I \to \mathbb{R}$ a continuous definable map. Assume that:*

(1) *the graph of g is contained in V,*
(2) *for every $s < t$ in Y and all $x \in I$, $\beta_s(x) < \beta_t(x)$,*
(3) *$f_0(0) = 0$ and there are $r < 0 < s$ in Y such that $\Gamma(\beta_s) \cap V = \Gamma(\beta_t) \cap V = \emptyset$.*

Then the set

$$\{x \in Y : |\Gamma(\beta_x) \cap \Gamma(g) \cap V| \leq 1\}$$

is infinite.

Proof. Let c_1 and c_2 be the endpoints of $\Gamma(g)$. Consider $x_1 := \inf\{x \in Y : \Gamma(\beta_x) \cap V \neq \emptyset\}$ and $x_2 := \sup\{x \in Y : \Gamma(\beta_x) \cap V \neq \emptyset\}$. By (2) and (3), $x_1, x_2 \in cl(Y)$. Moreover, the sets $\Gamma(\beta_{x_1}) \cap C$ and $\Gamma(\beta_{x_2}) \cap C$ are disjoint. Therefore, one of these sets must contain at most one of c_1 and c_2 (which would happen if any of these points were a corner of U). Without loss of generality assume that $\Gamma(\beta_{x_1}) \cap C$ is that set. By continuity, there is $\varepsilon > 0$ such that for all $x \in (x_1, x_1+\varepsilon)$, $|\Gamma(\beta_x) \cap \Gamma(g)| \leq 1$. $\square$

3. Result

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an $\mathbb{R}$-differentiable function, definable in some o-minimal expansion of the real field such that $\langle \mathbb{R}^2, +, f\rangle$ is non-locally modular. Assume that the matrix of df_z has form $\begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix}$, $z \in \mathbb{R}^2$. We first handle a special case.

Lemma 3.1. *Suppose there is an open $W \subseteq \mathbb{R}^2$ such that b_z is zero everywhere on W or c_z is zero everywhere on W. Then strong minimality is violated by f on an open subset of W.*

Proof. Let us assume that df_z has form $\begin{pmatrix} a_z & b_z \\ 0 & d_z \end{pmatrix}$ everywhere on W. The case where b_z is zero everywhere on W can be handled symmetrically.

We first observe that d_z cannot be fixed on any open subset of W. Indeed, assume it is. Then f^2 is affine on W, and thus f is affine on an infinite set: indeed, if we pick $t \in \mathbb{R}$ with all but its third coordinate zero and so that $\Gamma(t+\sigma^1_{d_z}) \cap \Gamma(f^1)$ is infinite, then the solution set of $(t, 0) + \sigma_{d_z}(x, y) = f(x, y)$ is infinite. This contradicts non-local modularity.

This observation on d_z and the condition $\frac{\partial f^2}{\partial x} = 0$ on W imply that the map f^2 has form $f^2(x, y) = g(y)$ on W, for some function $g : I \to \mathbb{R}$ which is nowhere linear. Hence there is $z \in W$ such that $g'(z_2) = q \in \mathbb{Q}$ and g' is strictly monotone around z_2, say on an open interval J. Let $V = (\mathbb{R} \times J) \cap W$. By translating $\Gamma(f)$, we may assume that $(z, f(z)) = 0$. Now, for every $t \in \mathbb{R}$, let

$$F_t = \Gamma(t + \sigma^2_q)$$

$$H_t = \Gamma((0, t) + \sigma_q)$$

By the description of f^2 on V, it follows that for a sufficiently small interval $(a, b) \subseteq \mathbb{R}$ containing 0:

- for every $t \in (a, 0)$, $F_t \cap \Gamma(f^2) \cap (V \times \mathbb{R})$ contains infinite segments of two lines whose projections are parallel to the x-axis, and
- for every $s \in (0, b)$, $F_t \cap \Gamma(f^2) \cap (V \times \mathbb{R}) = \emptyset$.

We also have that $F_0 \cap \Gamma(f^2) \cap (W' \times \mathbb{R})$ contains an infinite segment of the x-axis around 0.

On the other hand, we split into two possibilities for f^1: either $\pi\big(\Gamma(\sigma^1_q) \cap \Gamma(f^1)\big) \cap V$ contains an infinite subset of the x-axis, or it does not. In the former case, we contradict local modularity, because f is linear on the infinite solution set of $\sigma_q(x, y) = f(x, y)$. In the latter case, the intersection of $\Gamma(\sigma^1_q)$ with $\Gamma(f^1)$ close to 0 is a curve γ which is not parallel to the x-axis. It easily follows, using Fact 1.2, that for sufficiently small interval $(a, b) \subseteq \mathbb{R}$ containing 0:

- for every $t \in (a, 0)$, $H_t \cap \Gamma(f) \cap (V \times \mathbb{R}^2)$ has exactly two points, and
- for every $s \in (0, b)$ $H_s \cap \Gamma(f) \cap (V \times \mathbb{R}^2) = \emptyset$.

That is, strong minimality is violated by f on V. $\square$

Proposition 3.2. *Suppose there is an open $W \subseteq \mathbb{R}^2$ such that for every $z \in W$, df_z has real eigenvalue(s). Then strong minimality is violated by f on an open subset of W.*

Proof. The case where one of b_z and c_z is zero on an open subset of W is handled by Lemma 3.1. So assume otherwise. By o-minimality, there is an open $S \subseteq W$ on which b_z is nowhere zero. We may assume $S = W$. By translating $\Gamma(f)$, we may also assume that $0 \in W$ and $f(0) = 0$. Denote by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of df_0.

Claim 1. *If r is a real eigenvalue of df_0, then $f^1 \perp_0 \sigma_r^1$ and $f^2 \perp_0 \sigma_r^2$.*

Proof of Claim. Since $b \neq 0$, clearly for every $r \in R$, the graphs of $\sigma_r^1 : (x, y) \mapsto rx$ and $(x, y) \mapsto ax + by$ intersect on a one-dimensional set. Since $c \neq 0$, so do the graphs of $\sigma_r^2 : (x, y) \mapsto ry$ and $(x, y) \mapsto cx + dy$. $\square$

Fix a real eigenvalue r_0 for df_0. We prove the stronger conclusion that, for every $q \in \mathbb{Q}$ sufficiently close to r_0, strong minimality is violated by f on W by q-lines; namely, for every open $X \subseteq \mathbb{R}^2$ containing 0, the set

$$A = \big\{t \in X : \big|\Gamma(t + \sigma_q) \cap \Gamma(f) \cap (W \times \mathbb{R}^2)\big| \geq 2\big\}$$

is infinite and co-infinite in X. Fix an open box $X \subseteq \mathbb{R}^2$ containing 0. Denote, for every $r \in \mathbb{R}$ and $i = 1, 2$,

$$s_r^i := \text{ the slope of } \pi(\Gamma(\sigma_r^i) \cap \Gamma(df_0^i)).$$

Since $b \neq 0$, we have $s_{r_0}^1 \neq \infty$. By Fact 2.1, $s_{r_0}^1 = s_{r_0}^2$ and we denote this common value by s_{r_0}. By Claim 2.2, $s_r^1 - s_r^2$ changes sign at r_0. We are interested in the following two families of curves: for $i = 1, 2$ and $r \in \mathbb{R}$, let

$$A_r^i := \pi\big(\Gamma(\sigma_r^i) \cap \Gamma(f^i)\big).$$

Let $Y \subseteq X$ be the set of all $t \in X$ whose first coordinate is zero. We identify Y with its projection onto the second coordinate. For every $t \in Y$ and $r \in \mathbb{R}$, let

$$B_{t,r} = \pi\big(\Gamma(t + \sigma_r^2) \cap \Gamma(f^2)\big).$$

In particular, $B_{0,r} = A_r^2$, for all $r \in \mathbb{R}$. Since $\{A_r^i\}_{r \in \mathbb{R}}$ and $\{B_{t,r}\}_{(t,r) \in Y \times \mathbb{R}}$ are both continuous families of curves and $s_{r_0} \neq \infty$, there are an open interval I with center 0, an open interval $J \subseteq \mathbb{R}$ with $r_0 \in J$, and an open subset $Y' \subseteq Y$ with $0 \in Y'$, such that for all $r \in J$ and $t \in Y'$,

- A_r^i close to 0 is the graph of a definable continuous map $\alpha_r^i : I \to \mathbb{R}$
- $B_{t,r}$ close to 0 is the graph of a definable continuous map $\beta_{t,r} : I \to \mathbb{R}$.

By reducing I further, if necessary, we may assume that the graph of every α_r^i is contained in V. We also replace Y by Y'.

Claim 2. *For every open $V \subseteq \mathbb{R}^2$ containing 0, there is $q \in \mathbb{Q}$ so that A_q^1 and A_q^2 intersect inside V at only finitely many points and at least twice (that is, at least at one more point than 0). Moreover, q can be chosen arbitrarily close to r_0.*

Proof of Claim 2. For $i = 1, 2$ and $r \in J$, $(\alpha_r^i)'(0) = s_r^i$. Indeed, $\Gamma(\sigma_r^i) \cap \Gamma(df_0^i)$ is tangent to $\Gamma(\sigma_r^i) \cap \Gamma(f^i)$ at 0, and thus $\pi(\Gamma(\sigma_r^i) \cap \Gamma(df_0^i))$ is tangent to $\pi(\Gamma(\sigma_r^i) \cap \Gamma(f^i))$ at 0.

Since $s_r^1 - s_r^2$ changes sign at r_0, it follows that either

- for all $r < r_0$, $(\alpha_r^2)'(0) < s_{r_0} < (\alpha_r^1)'(0)$ and
- for all $r > r_0$, $(\alpha_r^2)'(0) > s_{r_0} > (\alpha_r^1)'(0)$

or the reverse inequalities hold. Without loss of generality, let us assume that the above inequalities hold. Hence the conditions of Fact 2.3 hold for $g_r = \alpha_r^2 - \alpha_r^1$. It follows that there is $q \in \mathbb{Q}$, arbitrarily close to r_0, and $0 \neq b \in I$ such that $\alpha_q^1(b) = 0 = \alpha_q^2(b)$. So $b \in A_q^1 \cap A_q^2$. On the other hand, by non-local modularity, $A_q^1 \cap A_q^2$ is finite (because so is $\Gamma(\sigma_q) \cap \Gamma(f)$) by Observation (1) from Introduction. $\square$

For every $r \in \mathbb{R}$, the sets in the family $\{B_{t,r}\}_{t\in Y}$ are pairwise disjoint. Hence, one of the following holds:

$$\text{for all } s < t \text{ and } x \in I,\ \beta_{s,r}(x) < \beta_{t,r}(x), \tag{2}$$

$$\text{for all } s < t \text{ and } x \in I,\ \beta_{s,r}(x) > \beta_{t,r}(x).$$

Without loss of generality, let us assume the former holds.

Claim 3. *There are $q \in \mathbb{Q}$, $s < 0 < t \in Y$, and an open box $V \subseteq \pi(W)$ containing 0, such that*

(1) *$A_q^1 \cap V$ is the graph of a continuous function.*
(2) *$B_{s,q} \cap V = B_{t,q} \cap V = \emptyset$*
(3) *A_q^1 and A_q^2 intersect inside V at only finitely many points and at least twice.*

Proof of Claim 3. By Fact 2.4 for $h = f^2$, $\tau = \sigma_{r_0}^2$ and $S = \pi(W)$, there are $s < 0 < t$ in Y and open $V \subseteq \pi(W)$ containing 0 such that

$$B_{s,r_0} \cap V = B_{t,r_0} \cap V = \emptyset.$$

We may replace V by a smaller open box containing 0, so that $A_{r_0}^1$ is the graph of a continuous function.

Since the family of planes $\{\Gamma(\sigma_q)\}_{q\in\mathbb{R}}$ is continuous, we also have that for $q \in \mathbb{R}$ sufficiently close to r_0,

$$B_{s,q} \cap V = B_{t,q} \cap V = \emptyset.$$

Since the family $\{A_r^1\}_{r\in\mathbb{R}}$ is continuous, we also have that for $q \in \mathbb{R}$ sufficiently close to r_0, A_q^1 is the graph of a continuous function.

We may thus pick any $q \in \mathbb{Q}$ as in Claim 2, sufficiently close to r_0. $\square$

Now let q, s, t and V be as in Claim 3. By statement (2) above and Claim 3(1) and (2), the assumptions of Fact 2.5 hold. Thus the set

$$\{t \in Y : |\Gamma(\alpha_q^1) \cap \Gamma(\beta_{t,q}) \cap V| \leq 1\}$$

is infinite. By Fact 1.2, so is the set $\{t \in Y : |\Gamma((0,t) + \sigma_q) \cap \Gamma(f) \cap (V \times \mathbb{R}^2)| \leq 1\}$. On the other hand, by Claim 3(3), and by continuity of the family $\{f_t\}_{t\in Y}$, the set

$$\{t \in Y : |\Gamma(\alpha_q^1) \cap \Gamma(\beta_{t,q}) \cap V| \geq 2\}$$

is also infinite. By Fact 1.2, so is the set $\{t \in Y : |\Gamma((0,t) + \sigma_q) \cap \Gamma(f) \cap (V \times \mathbb{R}^2)| \geq 2\}$. That is, strong minimality is violated by f on V. $\square$

Remark 3.3. The conclusion of Lemma 3.1 and Proposition 3.2 is in fact stronger: there is an open subset V of W, such that for every $z \in V$, there is a real eigenvalue r_0 of df_z, such that for $q \in \mathbb{Q}$ sufficiently close to r_0, strong minimality is violated by f on V by q-lines. Indeed, in both proofs q was chosen arbitrarily close to an eigenvalue of df_0.