

STRUCTURE THEOREMS IN TAME EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

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ABSTRACT. We study sets and groups definable in tame expansions of o-minimal structures. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of an o-minimal \mathcal{L} -structure \mathcal{M} by a dense set P , such that three tameness conditions hold. We prove a structure theorem for definable sets and functions in analogy with the influential cell decomposition theorem known for o-minimal structures. The structure theorem advances the state-of-the-art in all known examples of $\widetilde{\mathcal{M}}$, as it achieves a decomposition of definable sets into *unions* of ‘cones’, instead of only boolean combinations of them. We also develop the right dimension theory in the tame setting. Applications include: (i) the dimension of a definable set coincides with a suitable pregeometric dimension, and it is invariant under definable bijections, (ii) every definable map is given by an \mathcal{L} -definable map off a subset of its domain of smaller dimension, and (iii) around generic elements of a definable group, the group operation is given by an \mathcal{L} -definable map.

1. INTRODUCTION

Definable groups in models of first-order theories have been at the core of model theory for at least a period of three decades (see, for example, [5, 33, 41]) and have been crucially used in important applications of model theory to other areas of mathematics (such as in [28]). An indispensable tool in their analysis has been a structure theorem for the definable sets and types: analyzability of types and the existence of a rank in the stable category, and a cell decomposition theorem and the associated topological dimension in the o-minimal setting. In this paper we establish a structure theorem for definable sets and functions in tame expansions of o-minimal structures, introduce and analyze the relevant notion of dimension and establish a local theorem for definable groups in this setting. Our structure theorem is inspired by a cone decomposition theorem known for semi-bounded o-minimal structures ([15, 17, 34]), which was also vitally used in the analysis of definable groups therein ([21]). The structure theorem has opened the way to other applications of the tame setting, beyond the study of definable groups, such as the point counting theorems in [19].

Let us briefly discuss the tame setting. O-minimal structures were introduced and first studied by van den Dries [10] and Knight-Pillay-Steinhorn [32, 40] and have since provided a rigid framework to study real algebraic and analytic geometry. They have enjoyed a wide spectrum of applications reaching out even to number theory

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and Diophantine geometry (such as in Pila’s solution of certain cases of the André-Oort Conjecture [37]). However, o-minimality can only be used to model phenomena that are at least locally finite, or more precisely, objects that have only finitely many connected components. Tame expansions of o-minimal structures can further model phenomena that escape from the o-minimal context, but yet exhibit tame geometric behavior. They have recently seen significant growth ([1, 3, 6, 8, 12, 14, 25, 31]) and are by now divided into two important classes of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. We establish our cone decomposition theorem in the former category. In the second category, a relevant structure theorem has already been obtained in [44], benefiting largely by the presence of definable choice in that setting (absent here).

We now fix our setting and describe the results of this paper. Let \mathcal{M} be an o-minimal expansion of an ordered group with underlying language \mathcal{L} . Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of \mathcal{M} by a dense set P so that certain tameness conditions hold (those are listed in Section 2.1). For example, $\widetilde{\mathcal{M}}$ can be a dense pair ([12]), or P can be an independent set ([9]) or a multiplicative group with the Mann Property ([14]). To establish our structure theorem below, we introduce a new invariant for definable sets, the ‘large dimension’, which turns out to coincide with the combinatorial dimension coming from a pregeometry in [3]. These results are in the spirit of some standard and recent literature. In an o-minimal structure, the cell decomposition theorem ([13, 32]) is used to show that the associated ‘topological dimension’ equals the combinatorial dimension coming from the dcl-pregeometry ([38]). In a semi-bounded structure, the cone decomposition theorem ([15, 17, 34]) is used to show that the associated ‘long dimension’ equals the dimension coming from the short closure pregeometry ([17]). In both settings, the equivalence of the two dimensions has proven extremely powerful in many occasions and in particular in the analysis of definable groups (see, for example, [17, 21, 22, 39]). Here, we apply the strategy from the semi-bounded setting to that of tame expansions of o-minimal structures and establish the analogous results in $\widetilde{\mathcal{M}}$.

In Sections 2 and 3 we include some preliminaries and do preparatory work for what follows. In Section 4, we introduce the notions of a *cone* and *large dimension*. Although the definitions appear to be rather technical, we show in subsequent work that they are in fact optimal (see Section 5.2, Question 5.14 and [18]). In Section 5, we prove the following theorem.

Structure Theorem (5.1).

- (1) *Let $X \subseteq M^n$ be an A -definable set. Then X is a finite union of A -definable cones.*
- (2) *Let $f : X \rightarrow M$ be an A -definable function. Then there is a finite collection \mathcal{C} of A -definable cones, whose union is X and such that f is fiber \mathcal{L}_A -definable with respect to each cone in \mathcal{C} .*

We then conclude that the large dimension is invariant under definable bijections (Corollary 5.3). The above Structure Theorem is a substantial improvement of the ‘near-model completeness’ results established in known cases (such as [1, 12, 14]) in that it achieves a decomposition of definable sets into *unions* (instead of boolean combinations) of cones. It also includes definable maps $f : M^n \rightarrow M$ for any n (instead of only $n = 1$). To illustrate the last point, let us consider the following example of a map for $n = 1$ from [12]. Consider a dense pair $\langle \mathcal{M}, P \rangle$ of real closed

fields and let $\alpha \notin P$. So \mathcal{M} could be the real field, P the field of real algebraic numbers, and $\alpha = \pi$. Let $f : M \rightarrow M$ be the definable map given by

$$f(x) = \begin{cases} r & \text{if } x = r + \alpha s \text{ for some (unique) } r, s \in P \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the graph of f is dense in M^2 , and hence f is not as tame as an \mathcal{L} -definable map. However, [12, Theorem 3] establishes that every definable map $f : M \rightarrow M$ is given by an \mathcal{L} -definable map off a small set (here, the \mathcal{L} -definable map is 0 and the small set is $P + \alpha P$). A far reaching application of our structure theorem is the following generalization of this phenomenon.

Theorem 5.7. *Every A -definable map $f : M^n \rightarrow M$ is given by an $\mathcal{L}_{A \cup P}$ -definable map off a set of large dimension $< n$.*

We expect that this theorem will be useful in the future and already manifest one of its immediate corollaries here. Namely, we answer a question by Dolich-Miller-Steinhorn [9]: in dense pairs, the graph of a \emptyset -definable unary function is nowhere dense (Proposition 5.8).

In Section 6, we compare the large dimension of a definable set to the scl-dimension coming from [3]. In [3], the authors work under three similar tameness conditions on $\widetilde{\mathcal{M}}$ and prove that the *small closure* operator scl defines a pregeometry under further assumptions on \mathcal{M} ([3, Corollary 77]). Here, we observe that those further assumptions are in fact unnecessary (Corollary 6.4) and derive the equivalence of the two dimensions (Proposition 6.9), always.

In Section 7, we exploit this equivalence and set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$. Indeed, making use of desirable properties of ‘scl-generic’ elements (Fact 6.13), we achieve the following result.

Local theorem for definable groups (7.6). *Let $G = \langle G, * \rangle$ be a definable group of large dimension k . Then for every scl-generic element a in G , there is a $2k$ -cone $C \subseteq G \times G$, whose topological closure contains (a, a) , and on which the operation*

$$(x, y) \mapsto x * a^{-1} * y$$

is given by an \mathcal{L} -definable map.

We note that an analogous local theorem for semi-bounded groups was proved in [17, Theorem 6.3] and was then vitally used in the global analysis of semi-bounded groups in [21]. We expect that the present local theorem will be as crucial in forthcoming analysis of definable groups in $\widetilde{\mathcal{M}}$, and we list a series of open questions in the end of Section 7. The ultimate goal would be to understand definable groups in terms of \mathcal{L} -definable groups and small groups (Conjecture 7.8). Note that \mathcal{L} -definable groups have been exhaustively studied and are well-understood, some of the main results being proved in [7, 16, 21, 22, 29, 30, 35].

We next indicate some of the key aspects of this paper. Both the definition of the large dimension, as well as that of a cone, are based on the notion of a *supercone* given in Section 4, which in its turn is based on the notion of a *large* subset of M coming from [3] or [14]. Namely, a supercone J in M^n is defined, recursively on n , as a union of a specific family of large fibers over a supercone in M^{n-1} . The

large dimension of a definable set X is then the maximum k such that a supercone from M^k can be *embedded* into X . The nature of this embedding is crucial: while the definition of the large dimension is given via a strong notion of embedding, proving its invariance under definable bijections in Corollary 5.3 requires an equivalent definition via a weaker notion of embedding. We establish that equivalence in Corollary 4.22.

Let us now describe the main idea behind the proof of the Structure Theorem in Section 5 that also explains the role of large dimension in it and motivates all the preparatory work done in Sections 3 and 4. The notion of a large/small set is defined in Section 2 and that of a k -cone in Section 4. Roughly speaking, a k -cone is a set of the form

$$h \left(\bigcup_{g \in S} \{g\} \times J_g \right),$$

where h is an \mathcal{L} -definable continuous map with each $h(g, -)$ injective, $S \subseteq M^m$ is a small set, and $\{J_g\}_{g \in S}$ a definable family of supercones in M^k . The proof of the Structure Theorem runs by simultaneous induction on n for three statements, Theorem 5.1 (1) - (3). For (1), in the inductive step, let $X \subseteq M^{n+1}$. By the inductive hypothesis, we may assume that the projection $\pi(X)$ onto the first n coordinates is a k -cone, and by definability of smallness (Remark 3.4(a)), we may separate two cases. If all fibers of X above $\pi(X)$ are large, then we can simply follow the definition of a cone and, using $(2)_n$ and Lemma 4.10, we conclude that X is a $k+1$ -cone. If all fibers of X above $\pi(X)$ are small, then we first need to turn X into a small union of (\mathcal{L} -definable images of) subsets $J_g \subseteq \pi(X)$ as above. This is achieved using Lemma 3.7 and it is illustrated in Example 3.9. Unfortunately, the sets J_g obtained are not necessarily supercones, but we can remedy the situation by applying a uniform version of $(1)_{n-1}$, namely $(3)_{n-1}$. We derive $(3)_n$ from $(1)_n$ using a standard compactness argument. We derive $(2)_n$ from $(1)_n$ by first applying Corollary 3.27 to obtain \mathcal{L} -definability of f outside a subset of $\pi(X)$ of *smaller large dimension*. We then conclude it by sub-induction on large dimension.

In Section 5.2, we explore the optimality of our Structure Theorem. We prove that a stronger version where the notion of a cone is strengthened by requiring that h is injective on $\bigcup_{g \in S} \{g\} \times J_g$ is not possible. This is essentially due to the lack of definable choice in our setting (see, for example, [8, Section 5.5]). In Section 5.3, however, we isolate a key ‘choice property’ that implies a strengthened version of Lemma 3.7 (see Lemma 5.11), which in turn guarantees a Strong Structure Theorem (5.12). This study suggests a new line of research where the behavior of \mathcal{L} -definable maps on small sets is pending to be explored. A list of open questions is included, whereas further optimality results are established in subsequent work [18].

It is an important feature of this work that we keep track of all parameters. If X is an A -definable set then, by Lemma 2.5 below, its closure is $\mathcal{L}_{A \cup P}$ -definable. However, our Structure Theorem establishes that every A -definable set is a finite union of A -definable sets (the cones) whose closures are actually \mathcal{L}_A -definable. We warn the reader that we make a slight abuse of terminology in the interests of keeping the text succinct: an A -definable cone will be assumed to have its closure \mathcal{L}_A -definable; see Section 4.1 for more details.

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2. THE SETTING

Throughout this paper, we fix an o-minimal theory T expanding the theory of ordered abelian groups with a distinguished positive element 1. We also fix the language \mathcal{L} of T and $\mathcal{L}(P)$ the language \mathcal{L} augmented by a unary predicate symbol P . Let \tilde{T} be an $\mathcal{L}(P)$ -theory expanding T . If $\mathcal{M} = \langle M, <, +, \dots \rangle \models T$, then $\tilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of \mathcal{M} that models \tilde{T} . By ‘ A -definable’ we mean ‘definable in $\tilde{\mathcal{M}}$ with parameters from A ’. By ‘ \mathcal{L}_A -definable’ we mean ‘definable in \mathcal{M} with parameters from A ’. We omit the index A if we do not want to specify the parameters.

For a subset $X \subseteq M$, we write $\text{dcl}(X)$ for the definable closure of X in \mathcal{M} , and $\text{dcl}_{\mathcal{L}(P)}(X)$ for the definable closure of X in $\tilde{\mathcal{M}}$. By the o-minimality of T , the operation that maps $X \subseteq M$ to $\text{dcl}(X)$ is a pregeometry on M . For an \mathcal{L} -definable set $X \subseteq M^n$, we denote by $\dim(X)$ the corresponding pregeometric dimension.

The following definition is taken essentially from [14].

Definition 2.1. Let $X \subseteq M^n$ be a definable set. We call X *large* if there is some m and an \mathcal{L} -definable function $f : M^{nm} \rightarrow M$ such that $f(X^m)$ contains an open interval in M . We call X *small* if it is not large.

Note that if $X \subseteq M$ is small and I an interval in M , then $I \setminus X$ is large (with a proof identical to that of [3, Lemma 20]). We will use this observation throughout this paper. In Lemma 3.11 and Corollary 3.12 below we prove that smallness is equivalent to P -internality, in the usual sense of geometric stability theory.

Definition 2.2. If $X, Z \subseteq M^n$ are definable, we say that X is *small in Z* if $X \cap Z$ is small. We say that X is *co-small in Z* if $Z \setminus X$ is small.

2.1. Assumptions. We assume that \tilde{T} satisfies the following three tameness conditions: for every model $\tilde{\mathcal{M}} \models \tilde{T}$,

- (I) P is small.
- (II) (Near model-completeness) Every A -definable set $X \subseteq M^n$ is a boolean combination of sets of the form

$$\{x \in M^n : \exists z \in P^m \varphi(x, z)\},$$

where $\varphi(x, z)$ is an \mathcal{L}_A -formula.

- (III) (Open definable sets are \mathcal{L} -definable) For every parameter set A such that $A \setminus P$ is dcl-independent over P , and for every A -definable set $V \subset M^s$, its topological closure $cl(V) \subseteq M^s$ is \mathcal{L}_A -definable.

From now on, and unless stated otherwise, \tilde{T} satisfies Assumptions (I)-(III) and $\tilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ is a sufficiently saturated model of \tilde{T} .

Remark 2.3. (i) Assumptions (I)-(III) are analogous to Assumptions (1)-(3) from [3, Theorem 3]. Here, however, we insist on having some control on the defining parameters. Moreover, an easy argument shows that under our assumptions, (3) from [3, Theorem 3] holds, but without the additional condition that the set S mentioned there be \emptyset -definable.

(ii) Assumption (III) indeed guarantees that open definable sets are \mathcal{L} -definable, see Lemma 2.5 below.

(iii) We do not know whether assumptions (I) and (III) imply (II).

Notation-terminology. The topological closure of a set $X \subseteq M^n$ is denoted by $cl(X)$. If $X, Y \subseteq M$ and $b = (b_1, \dots, b_n)$, we sometimes write $X \cup b$ or Xb for $X \cup \{b_1, \dots, b_n\}$, and XY for $X \cup Y$. If $\varphi(x, y)$ is an $\mathcal{L}(P)$ -formula and $a \in M^n$, then we write $\varphi(M^m, a)$ for

$$\{b \in M^m : \widetilde{\mathcal{M}} \models \varphi(b, a)\}.$$

Similarly, given any subset $X \subseteq M^m \times M^n$ and $a \in M^n$, we write X_a for

$$\{b \in M^m : (b, a) \in X\}.$$

For convenience, we sometimes write $f(t, X)$ for $f(\{t\} \times X)$. If $m \leq n$, then $\pi_m : M^n \rightarrow M^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1} , unless stated otherwise. By an open box in M^k , or a k -box, we mean a set $I_1 \times \dots \times I_k \subseteq M^k$, where each $I_j \subseteq M$ is an open interval. By dimension of an \mathcal{L} -definable set we mean its usual o-minimal dimension, and the notions of dcl-independence, dcl-rank and dcl-generics are the usual notions attached to the dcl-pregeometry (see, for example, [39]). A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g \in S} \{g\} \times J_g$ is definable, disjoint if every two elements of it are disjoint, and small if S is small. We often identify \mathcal{J} with $\bigcup_{g \in S} \{g\} \times J_g$. If for each $t \in T$, $\mathcal{J}_t = \{J_{g,t}\}_{g \in S_t}$ is a family of sets, we call $\{\mathcal{J}_t\}_{t \in T}$ definable if $\bigcup_{t \in T, g \in S_t} \{(g, t)\} \times J_{g,t}$ is definable.

Our examples are often given for structures over the reals (such as Example 4.20 and the counterexample in Section 5.2). But they can easily be adopted to the current, saturated setting, by moving to an elementary extension.

2.2. Examples.

Dense pairs. The first example we wish to consider is dense pairs of o-minimal structures. A *dense pair* $\langle \mathcal{M}, \mathcal{N} \rangle$ is a pair of models of T such that $\mathcal{N} \neq \mathcal{M}$, but \mathcal{N} is dense in \mathcal{M} . Let $\widetilde{T} = T^d$ be the theory of dense pairs in the language $\mathcal{L}(P)$. By [12], T^d is complete and every model of T^d satisfies (I) and (II) ([12, Lemma 4.1] and [12, Theorem 1], respectively).

It is left to explain why (III) holds in dense pairs. Here we apply [6, Corollary 3.1]. Let A be a parameter set such that $A \setminus N$ is dcl-independent over N . Set

$$D := \{a \in M : a \text{ is dcl-independent over } N \cup A\}.$$

It is easy to see that D and A satisfy Assumptions (1) and (2) of [6, Corollary 3.1]. It is left to show that also the third assumption of that corollary holds. Towards that goal, recall the following notation from [12]. Given $\mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{Q} \models T$ with $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{Q}$ and $\mathcal{M} \subseteq \mathcal{O} \subseteq \mathcal{Q}$, we say that \mathcal{N} and \mathcal{O} are *free over* \mathcal{M} (in \mathcal{Q}) if every subset $Y \subseteq N$ that is dcl-independent over \mathcal{M} is also dcl-independent over \mathcal{O} .

Proposition 2.4. *Let $a \in D$. Then the $\mathcal{L}(P)$ -type of a over A is implied by the \mathcal{L} -type over A and the fact that $a \in D$.*

Proof. Let $\langle \mathcal{M}, \mathcal{N} \rangle \models T^d$ be κ -saturated, where $\kappa > |T|$. Let Γ be the set of all isomorphisms $i : \langle \mathcal{M}_1, \mathcal{N}_1 \rangle \rightarrow \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ between substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$ such that $|M_1| < \kappa, |M_2| < \kappa$, \mathcal{M}_1 and \mathcal{N} are free over \mathcal{N}_1 and \mathcal{M}_2 and \mathcal{N} are free over \mathcal{N}_2 . By [12, Claim on p. 67], Γ has the back-and-forth property. Let $a, b \in D$ such that a and b satisfy the same \mathcal{L} -type over A . Then there is an \mathcal{L} -isomorphism

$$i : \text{dcl}(a \cup A) \rightarrow \text{dcl}(b \cup A).$$

Since both a and b are dcl-independent over $N \cup A$, the isomorphism expands to an isomorphisms

$$i : \langle \text{dcl}(a \cup A), \text{dcl}(A) \cap \mathcal{N} \rangle \rightarrow \langle \text{dcl}(b \cup A), \text{dcl}(A) \cap \mathcal{N} \rangle$$

of substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$. Since $a \cup (A \setminus N)$ is dcl-independent over N , $\text{dcl}(a \cup A)$ and \mathcal{N} are free over $\text{dcl}(N) \cap \mathcal{N}$. By the same argument $\text{dcl}(b \cup A)$ and \mathcal{N} are free over $\text{dcl}(A) \cap \mathcal{N}$. Hence $i \in \Gamma$. Since Γ is a back-and-forth system, a and b satisfy the same $\mathcal{L}(P)$ -type over A . \square

Groups with the Mann property. Let Γ be a dense subgroup of $\mathbb{R}_{>0}$ that has the *Mann property*, that is for every $a_1, \dots, a_n \in \mathbb{Q}^\times$, there are finitely many $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $a_1\gamma_1 + \dots + a_n\gamma_n = 1$ and $\sum_{i \in I} a_i\gamma_i \neq 0$ for every nonempty subset I of $\{1, \dots, n\}$. Every multiplicative subgroup of finite rank in $\mathbb{R}_{>0}$ has the Mann property, see [23].

We assume that for every prime number p , the subgroup of p -th powers in Γ has finite index in Γ . Let \mathcal{L} be the language of ordered rings augmented by a constant symbol for each $\gamma \in \Gamma$. Let T be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma} \rangle$ in that language and let $\tilde{T} = T(\Gamma)$ be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma}, \Gamma \rangle$ in the language $\mathcal{L}(P)$. By [14, Theorem 7.5], every model of $T(\Gamma)$ satisfies (II). A proof that every model satisfies (I) is in [25, Proposition 3.5].

Again, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P \rangle \models T(\Gamma)$. Let A for every parameter set A such that $A \setminus P$ is dcl-independent over P . Set

$$D := \{a \in M : a \text{ is dcl-independent over } P \cup A\}.$$

One can check easily that assumptions (1) and (2) of [6, Corollary 3.1] follow from the o-minimality of T . Finally it is easy to see that almost the same proof as for Proposition 2.4, just using the back-and-forth system in the proof of Theorem 7.1 in [14] instead of [12, Claim on p. 67], shows that assumption (3) of [6, Corollary 3.1] is satisfied as well.

There are several other closely related examples. In [27] proper o-minimal expansions \mathcal{R} of the real field and finite rank subgroups Γ of $\mathbb{R}_{>0}$ are constructed such that the structure (\mathcal{R}, Γ) satisfies Assumptions (I)-(III). Indeed, the fact that these structures satisfy Assumptions (I) and (II) is immediate from results in [27]. Assumption (III) follows by the same argument as above. In [1, 25] certain expansions of the real field by subgroups of either the unit circle or an elliptic curve are studied. One can easily show using the above argument that these structures satisfy Assumptions (I)-(III) after adjusting their statements for the fact that P now lies in a 1-dimensional semialgebraic set in \mathbb{R}^2 . Since no significant new argument is involved, we leave it to the reader to verify that our main results also hold in this slightly more general setting.

Independent sets. Let $\tilde{T} = T^{\text{indep}}$ be an $\mathcal{L}(P)$ -theory extending T by axioms stating that P is dense and dcl-independent. By [9], T^{indep} is complete and every model of T^{indep} satisfies (I) and (II) by [9, 2.1] and [9, 2.9], respectively. As usual, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P \rangle \models T^{\text{indep}}$. Let A be a parameter set such that $A \setminus P$ is dcl-independent over P . Set

$$D := \{a \in M : a \text{ is dcl-independent over } P \cup A\}.$$

From the o-minimality of T , assumptions (1) and (2) of [6, Corollary 3.1] follow easily as above. By [9, 2.12], assumption (3) of [6, Corollary 3.1] holds as well.

Non-examples.

(1) By Assumption (III), P must be dense in a finite union of open intervals and points. Indeed, the closure of P has to be \mathcal{L} -definable. Therefore, tame expansions of \mathcal{M} by discrete sets, such as $\langle \mathbb{R}, 2^{\mathbb{Z}} \rangle$, do not belong to this setting.

(2) We do not know whether the theory of every expansion $\langle \mathcal{M}, P \rangle$ of an o-minimal structure \mathcal{M} with o-minimal open core [8, 31] satisfies Assumptions (II) or (III). Assumption (I) does not hold in case P is a generic predicate.

(3) If $\widetilde{\mathcal{M}} = \langle M, <, +, \mathcal{P} \rangle$ is semi-bounded, that is, a pure ordered group expanded by the structure of a real closed field $\mathcal{P} = \langle P, \oplus, \otimes \rangle$ on some bounded open interval $P \subseteq M$, then Assumptions (II) and (III) hold by [17], but (I) does not.

2.3. \mathcal{L} -definability. In general, an \mathcal{L} -definable set X which is also A -definable need not be \mathcal{L}_A -definable. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, and $x, y \in M \setminus P$ such that there are (unique) $g, h \in P$ with $x = g + hy$. Then $\{g\}$ is \mathcal{L} -definable and $\{x, y\}$ -definable, but in general not $\mathcal{L}_{\{x, y\}}$ -definable. The following lemma, however, implies, in particular, that every such X is always $\mathcal{L}_{A \cup P}$ -definable.

Lemma 2.5. *Let $X \subseteq M^n$ be an A -definable set. Then there is a finite $B \subseteq A$ such that X is $B \cup P$ -definable and B is dcl-independent over P . Hence, by Assumption (III), $\text{cl}(X)$ is $\mathcal{L}_{A \cup P}$ -definable. In particular, if X is closed (or open), then it is $\mathcal{L}_{A \cup P}$ -definable.*

Proof. Without loss of generality, we can assume that A is finite. Let $B \subseteq A$ be a maximal subset of A that is dcl-independent over P . Suppose $B = \{b_1, \dots, b_k\}$. Hence for every $a \in A \setminus B$, there are $g_a \in P^l$ and an \mathcal{L}_\emptyset -definable map $h : M^{l+k} \rightarrow M$ such that $h(g_a, b_1, \dots, b_k) = a$. Set $H = \{g_a : a \in A \setminus B\}$. Since X is A -definable, it is also $B \cup H$ -definable. \square

A positive answer to the following open question would give better control to the set of parameters (see also after Corollary 3.23 below).

Question 2.6. *For X as above, are there finite $B \subseteq A$ and $H \subseteq P \cap \text{dcl}_{\mathcal{L}(P)}(A)$, such that X is $B \cup H$ -definable and B is dcl-independent over P ?*

By [9, 2.26], Question 2.6 admits a positive answer when $\widetilde{T} = T^{\text{indep}}$. However, we do not know the answer even when $\widetilde{T} = T^d$.

The reader might wonder whether for every definable subset X of P^l there is an \mathcal{L} -definable set $Y \subseteq M^n$ such that $X = Y \cap P^n$. While this is true for dense pairs by [12, Theorem 2(2)], this fails in examples arising from groups with the Mann property (see [3, Proposition 57]).

Although all our known examples that satisfy Assumptions (I)-(III) have NIP (see [2, 26]), the following question stands open.

Question 2.7. *Do Assumptions (I)-(III) imply that \widetilde{T} has NIP?*

2.4. Basic facts for \mathcal{L} -definable and small sets. We include some basic facts that will be used in the sequel.

Fact 2.8. *Let $f : X \subseteq M^m \rightarrow M^n$ be a finite-to-one \mathcal{L} -definable function. Then there is a finite partition $X = X_1 \cup \dots \cup X_k$ into definable sets such that each $f|_{X_i}$ is injective.*

Proof. Standard. □

Fact 2.9. *Let $f : A \subseteq M^m \rightarrow M^n$ be an \mathcal{L} -definable function. Let*

$$X_f = \{a \in A : f^{-1}(f(a)) \text{ is finite}\}.$$

Then $\dim f(A \setminus X_f) < \dim A$.

Proof. Let $R = f(A \setminus X_f)$. By definition of X_f , for every $r \in R$, $f^{-1}(r)$ has dimension > 0 . Since $A \setminus X_f$ equals the disjoint union $\bigcup_{r \in R} f^{-1}(r)$, we have by standard properties of dimension:

$$\dim(A \setminus X_f) \geq \min_r \dim f^{-1}(r) + \dim R.$$

Hence, $\dim A \geq 1 + \dim R$ and $\dim R < \dim A$. □

Fact 2.10. *If $X, Z, I \subseteq M^m$ are definable sets, and X is co-small in Z , then $X \cap I$ is co-small in $Z \cap I$.*

Proof. Immediate from the definitions. □

3. SMALL SETS

In this section we establish properties of small sets that will be important in the proof of the Structure Theorem. The two most crucial results are Lemma 3.7 and Corollary 3.27 below.

3.1. Families of small sets and P -boundness. With the exception of Lemma 3.7 below, the results of this section were either established in [3] or are minor improvements of results in [3]. Since the assumptions in [3] differ from ours, we reprove the results here. Most of the proofs are direct adjustments from those in [3], but are included for the convenience of the reader. They often involve induction on formulas whose base step deals with a ‘basic’ set defined next.

Definition 3.1. A subset $X \subseteq M^n$ is called *basic over A* if it is of the form

$$\bigcup_{g \in P^m} \varphi(M, g),$$

for some \mathcal{L}_A -formula. We say X is *basic* if it is basic over some parameter set A .

Note that by Assumption (II) every definable set is a boolean combination of basic sets.

Lemma 3.2. *Let $p \in \mathbb{N}$. For $j = 1, \dots, p$, let $\{S_{1,j,t}\}_{t \in M^l}, \{S_{2,j,t}\}_{t \in M^l}$ be A -definable families of subsets of P^n . Let $f_1, \dots, f_p, h_1, \dots, h_p : M^{n+l} \rightarrow M$ be A -definable*

functions. Then there are A -definable families $\{Q_{j,t}\}_{t \in M^l}, \{R_{j,t}\}_{t \in M^l}$ of subsets of P^n , for $j = 1, \dots, p$, such that for every $t \in M^l$,

$$\bigcup_j f_j(S_{1,j,t}, t) \cap \bigcup_j h_j(S_{2,j,t}, t) = \bigcup_j f_j(Q_{j,t}, t),$$

$$\left(M \setminus \bigcup_j f_j(S_{1,j,t}, t)\right) \cup \bigcup_j h_j(S_{2,j,t}, t) = M \setminus \bigcup_j f_j(R_{j,t}, t).$$

Proof. Set

$$Q_{j,t} := \{g \in S_{1,j,t} : \bigvee_{i=1}^p \exists g' \in S_{2,i,t} \ h_i(g', t) = f_j(g, t)\}$$

and

$$R_{j,t} := \{g \in S_{1,j,t} : \bigwedge_{i=1}^p \forall g' \in S_{2,i,t} \ h_i(g', t) \neq f_j(g, t)\}.$$

□

Lemma 3.3. *Let $\{X_t\}_{t \in M^l}$ be an A -definable family of subsets of M . Then there are $m, n, p \in \mathbb{N}$ and for each $i = 1, \dots, m$ there are*

- *an A -definable family $\{S_{i,j,t}\}_{t \in M^l}$ of subsets of P^n , for each $j = 1, \dots, p$,*
- *\mathcal{L}_A -definable functions $h_{i,1}, \dots, h_{i,p} : M^{n+l} \rightarrow M$,*
- *an A -definable function $a_i : M^l \rightarrow M \cup \{\infty\}$,*

such that for $t \in M^l$,

- (i) *$-\infty = a_0(t) \leq a_1(t) \leq \dots \leq a_m(t) = \infty$ is a decomposition of M , and*
- (ii) *one of the following holds:*
 - (a) *$[a_{i-1}(t), a_i(t)] \cap X_t = V_{i,t}$ or*
 - (b) *$[a_{i-1}(t), a_i(t)] \cap X_t = (M \setminus V_{i,t}) \cap [a_{i-1}(t), a_i(t)]$,**where $V_{i,t} = \bigcup_j h_{i,j}(S_{i,j,t}, t)$.*

Proof. First consider a definable family of basic sets, say $(\mathbb{D}_t)_{t \in M^l}$, that is a definable family of the form

$$\mathbb{D}_t = \bigcup_{g \in P^n} \varphi(M, g, t),$$

where $\varphi(x, y, z)$ is an \mathcal{L}_A -formula and $t \in M^l$. By cell decomposition, there are two finite sets J_1, J_2 , \mathcal{L}_A -definable cells $(Y_{1,j})_{j \in J_1}$ and $(Y_{2,j})_{j \in J_2}$ in M^{n+l} and \mathcal{L}_A -definable functions $(f_{1,j})_{j \in J_1}$, $(f_{2,j})_{j \in J_2}$ and $(f_{3,j})_{j \in J_2}$ from M^{n+l} to M such that

$$\mathbb{D}_t = \bigcup_{j \in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t) \cup \bigcup_{j \in J_2} \bigcup_{g \in Y_{2,j,t} \cap P^n} (f_{2,j}(g, t), f_{3,j}(g, t)).$$

Without loss of generality, we can assume that $|J_1| = |J_2|$. Set $p := |J_1|$ and assume that $J_1 = J_2 = \{1, \dots, p\}$. Set $U_t := \bigcup_{j \in J_2} \bigcup_{g \in Y_{2,j,t} \cap P^n} (f_{2,j}(g, t), f_{3,j}(g, t))$. Note that U_t is open. By Assumption (III), U_t is a finite union of open intervals. Since finitely many intervals only have finitely many endpoints and U_t is At -definable, the endpoints of the intervals of U_t are At -definable. Let V_t be the topological closure of $\bigcup_{j \in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t)$. By Assumption (III) again, V_t is \mathcal{L} -definable. Hence it is a finite union of intervals and points. Since there are only finitely many endpoints and V_t is At -definable, these endpoints are At -definable. Hence we have a decomposition of M

$$-\infty = a_0(t) \leq a_1(t) \leq \dots \leq a_m(t) = \infty$$

such that either

- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t))$ or
- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t)) \cap \bigcup_{j \in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t)$.

In the first case set $S_{i,j,t} := \emptyset$ and set $h_{i,j}(x, y) = 0$ for all $(x, y) \in M^{n+l}$. In the second case set

$$S_{i,j,t} := \{g \in Y_{1,j,t} \cap P^n : f_{1,j}(g, t) \in (a_{i-1}(t), a_i(t))\},$$

and set $h_{i,j} = f_{1,j}$. By compactness, we can find an $m \in \mathbb{N}$ that works for every $t \in M^l$. Hence (i)-(ii) holds for $(\mathbb{D}_t)_{t \in M^n}$.

By Assumption (II) it is enough to check that if the statement of the Lemma holds for two definable $(X_t)_{t \in M^l}$ and $(Z_t)_{t \in M^l}$, then it also holds for $(M \setminus X_t)_{t \in M^l}$ and $(X_t \cup Z_t)_{t \in M^l}$. So suppose that the statement holds for $(X_t)_{t \in M^l}$ and $(Z_t)_{t \in M^l}$. It is immediate that the conclusion holds for $(M \setminus X_t)_{t \in M^l}$ as well. It is easy to check that Lemma 3.2 implies that the conclusion also holds for $(X_t \cup Z_t)_{t \in M^l}$. \square

Remark 3.4. The sets $V_{i,t}$ above are small, since P is small (Assumption (I)). Hence:

(a) the set

$$\{t \in M^n : X_t \cap [a_{i-1}(t), a_i(t)] \text{ is small}\}$$

is equal to

$$\{t \in M^n : X_t \cap [a_{i-1}(t), a_i(t)] = V_{i,t}\}.$$

Hence, it is A -definable. In particular, the set of all $t \in M^n$ such that X_t is small is A -definable.

(b) the set of $(t, a_i(t))$ for which X_t is small in $(a_{i-1}(t), a_i(t))$ is A -definable.

We will make use of the following consequence of Lemma 3.3.

Corollary 3.5. *Let $\{X_t\}_{t \in I}$ be an A -definable family of subsets of M , where each $X_t \subseteq M$ is small and $I \subseteq M^n$. Then there are $m \in \mathbb{N}$, \mathcal{L}_A -definable continuous functions $h_j : V_j \subseteq M^{m+n} \rightarrow M$ and A -definable families $\{S_{j,t}\}_{t \in I}$ of sets $S_{j,t} \subseteq P^m$, $j = 1, \dots, p$, such that for every $t \in I$, $X_t = \bigcup_j h_j(S_{j,t}, t)$.*

Proof. Without requiring the continuity of the h_j 's, the statement is immediate from Lemma 3.3. Now, to get the continuity, apply the cell decomposition theorem for o-minimal structures to get, for each j , cells $V_{j,1}, \dots, V_{j,s(j)}$ such that h_j is continuous on each $V_{j,i}$. Let $S'_{j,i,t} := S_{j,t} \cap V_{j,i} \subseteq P^m$. We have

$$X_t = \bigcup_{j,i} h_j(S'_{j,i,t}, t),$$

as required. \square

The following example shows that in the last corollary the set $S_{j,t}$ has to depend on t .

Example 3.6. Let $\widetilde{\mathcal{M}} \models T^d$. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and

$$X = \bigcup_{a \in M^{>0}} \{a\} \times X_a.$$

Let h_j and $S_{j,a}$ be as in Corollary 3.5, and assume towards a contradiction that all $S_{j,a}$'s equal some S_j . So for every $a \in M^{>0}$,

$$(*) \quad (0, a) \cap P = \bigcup_j h_j(S_j, a).$$

Take $p \in S_j$. By o-minimality, $h_j(p, -)$ is eventually continuous close to 0. Since $h_j(p, M^{>0}) \subseteq P$ by $(*)$ and P is codense in M , $h_j(p, -)$ is eventually constant close to 0. That is, there is $a_p > 0$ and $c_p \in P$, such that for every $0 < a < a_p$, $h_j(p, a) = c_p$. Thus, if $0 < a < c_p$, we have $h_j(p, a) = c_p \notin (0, a) \cap P$, a contradiction.

We now derive a few corollaries of the above results. The next lemma shows how to turn a family $X = \{X_a\}_{a \in C}$ of small sets into a small family of subsets Z_g of C . This will be a crucial step in the proof of the Structure Theorem. There, we will further need to replace Z_{ig} by “cones”, which are defined in Section 4.

Lemma 3.7. *Let $X = \bigcup_{a \in C} \{a\} \times X_a$ be A -definable where each $X_a \subseteq M$ is small, non-empty, and $C \subseteq M^n$. Then there are $l, m \in \mathbb{N}$, and for each $i = 1, \dots, l$,*

- *an \mathcal{L}_A -definable continuous function $h_i : V_i \subseteq M^{m+n} \rightarrow M^{n+1}$,*
- *an A -definable small set $S_i \subseteq M^m$, and*
- *an A -definable set $Z_i \subseteq S_i \times C$ contained in V_i ,*

such that for

$$U_i = h_i \left(\bigcup_{g \in S_i} \{g\} \times Z_{ig} \right)$$

we have

- (1) $X = U_1 \cup \dots \cup U_l$ is a disjoint union,
- (2) for every i and $g \in S_i$, $h_i(g, -) : V_{ig} \subseteq M^n \rightarrow M^{n+1}$ is injective,
- (3) $C = \bigcup_{i,g} Z_{ig}$.

Proof. We first observe that there are $m, p \in \mathbb{N}$, \mathcal{L}_A -definable continuous functions $h_i : V_i \subseteq M^{m+n} \rightarrow M$ and A -definable families Y_i of small sets $Y_{ia} \subseteq P^m$, $i = 1, \dots, p$, such that for every $a \in I$,

- (1) $X_a = \bigcup_i h_i(Y_{ia}, a)$
- (2) $\{h_i(Y_{ia}, a)\}_{i=1, \dots, p}$ are disjoint.

Indeed, this follows from Corollary 3.5; for (2), recursively replace Y_{ia} , $1 < i \leq p$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z, a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a \in C} \{a\} \times X_a = \bigcup_i \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a).$$

For every i , let $S_i = P^m$. For every i and $g \in P^m$, let

$$U_i = \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a),$$

which are also disjoint, and

$$Z_{ig} = \{a \in C : g \in Y_{ia}\}.$$

Since h_i and $\{Y_{ia}\}_{a \in C}$ are A -definable, so are U_i and $\{Z_{ig}\}_{g \in S_i}$. We have $C = \bigcup_{i,g} Z_{ig}$. Consider now the \mathcal{L}_A -definable continuous map $\hat{h}_i : V_i \subseteq M^{m+n} \rightarrow M^{n+1}$ with

$$\hat{h}_i(g, a) = (a, h_i(g, a)).$$

Then

$$U_i = \hat{h}_i \left(\bigcup_{g \in S_i} \{g\} \times Z_{ig} \right)$$

works. □

Remark 3.8. As the last proof shows, in fact we obtain $S_i = P^m$. We decided, however, to keep the current formulation because the proof can then be adopted in similar situations (such as in Lemma 5.11 below). Had we kept the stronger formulation ($S_i = P^m$), what follows would result to a Structure Theorem 5.1 where in Definition 4.3 of a cone we could require $S \subseteq P^m$. However, we recover this information anyway, see Remark 4.5(7).

Let us illustrate Lemma 3.7 with an example.

Example 3.9. Let $\widetilde{\mathcal{M}} \models T^d$. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and

$$X = \bigcup_{a \in M^{>0}} \{a\} \times X_a.$$

Then we can turn X into a small union of (\mathcal{L} -definable images of) large subsets of M , as follows. For every $g \in P$, let

$$J_g = \{a \in M : a > g\}.$$

Then

$$X = h \left(\bigcup_{g \in P} \{g\} \times J_g \right),$$

where $h : M^2 \rightarrow M^2$ switches the coordinates, $h(x, y) = (y, x)$. In this case, X is in fact seen to be 1-cone (according to Definition 4.3 below).

We now turn to examine better the notion of smallness.

Definition 3.10. A set $X \subseteq M^n$ is *P-bound* over A , if there is an \mathcal{L}_A -definable function $f : M^m \rightarrow M^n$ such that $X \subseteq f(P^m)$. We omit A if we do not want to specify the parameters.

Lemma 3.11. *An A-definable set is small if and only if it is P-bound over A.*

Proof. Since P is small, it follows immediately that every P -bound set is small. For the other direction, observe first that, by Corollary 3.5, every A -definable small subset of M is P -bound over A . Now let $X \subseteq M^n$ be A -definable, and let $\pi_i : M^n \rightarrow M$ be the projection onto the i -th coordinate. If X is small, so is $\pi_i(X)$ for $i = 1, \dots, n$. Since each A -definable small subset of M is P -bound over A , so is $\pi_i(X)$. Hence $\prod_{i=1}^n \pi_i(X)$ is P -bound over A and so is $X \subseteq \prod_{i=1}^n \pi_i(X)$. \square

We show that in the definition of largeness and P -boundedness, we can replace \mathcal{L} -definability by definability. Recall from geometric stability theory that given two definable sets $X \subseteq M^n$ and $Y \subseteq M^k$, X is called *Y-internal* over A if there is an A -definable $f : M^{mk} \rightarrow M^n$ such that $X \subseteq f(Y^m)$.

Corollary 3.12. *Let X be a definable set.*

- (1) *X is P-bound over A if and only if it is P-internal over A .*
- (2) *X is large if and only if an open interval is X -internal.*

Proof. By Lemma 3.11, Definition 2.1 and Assumption (I), it is easy to see that (1) implies (2). For (1), let $F : M^k \rightarrow M^n$ be A -definable such that $X \subseteq F(P^k)$. Without loss of generality, we may assume that $A \setminus P$ is dcl-independent over P . For each $g \in P^k$, the singleton $\{F(g)\}$ equals its topological closure. Since $F(g)$ is definable over $A \cup g$ and $(A \cup g) \setminus P$ is dcl-independent over P , we get by Assumption

(III) that $\{F(g)\}$ is $\mathcal{L}_{A \cup g}$ -definable. Hence, by compactness, there are finitely many \mathcal{L}_A -functions F_1, \dots, F_l such that for all $g \in P^k$, $F(g) = F_i(g)$ for some i . Hence

$$F(P^k) \subseteq \bigcup_i F_i(P^k).$$

However, the right hand side is P -bound over A , and hence so is $F(P^k)$. \square

The following is then immediate.

Corollary 3.13. *Let $f : X \rightarrow M^n$ be a definable injective function. Then X is small if and only if $f(X)$ is small.*

A stronger version of the Corollary 3.13 is provided by the invariance result in Corollary 5.3 below. Here are three more corollaries of Lemma 3.11.

Corollary 3.14. *Let $Y \subseteq M^m$ be small and let $(X_t)_{t \in Y}$ be a definable family of small sets of M^n . Then $\bigcup_{t \in Y} X_t$ is small.*

Proof. By Lemma 3.11 and compactness, there is a definable family of \mathcal{L} -definable functions $(f_t)_{t \in Y}$ such that $X_t \subseteq f_t(P^k)$ for each $t \in Y$. Again by Lemma 3.11, there is also an \mathcal{L} -definable function $g : P^l \rightarrow M^m$ such that $Y \subseteq g(P^l)$. Set $h : M^{k+l} \rightarrow M^n$ be the function that takes (x, y) to $f_{g(y)}(x)$. Then $\bigcup_{t \in Y} X_t \subseteq h(P^{k+l})$ and hence is P -bound. \square

Corollary 3.15. *The union and cartesian product of finitely many small sets is small.*

Proof. Immediate from Lemma 3.11, Corollary 3.14 and the definitions. \square

In the case of dense pairs, we obtain the following interesting result.

Corollary 3.16. *Assume $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ is a dense pair. Then every \emptyset -definable small set $X \subseteq M^n$ is contained in P^n .*

Proof. By Corollary 3.11, there is an \mathcal{L}_{\emptyset} -definable $f : M^m \rightarrow M^n$, such that $X \subseteq f(P^m)$. By [12, Lemma 3.1], $X \subseteq P^n$. \square

3.2. Definable functions outside small sets. In this section we analyze the behavior of definable functions outside small, or rather *low*, sets. Note that Assumption (II) is not used in this section.

Definition 3.17. We denote by $I_n(A) \subseteq M^n$ the set of all tuples $a = (a_1, \dots, a_n) \in M^n$ that are dcl-independent over $P \cup A$.

Remark 3.18. (1) Note that $I_n(A)$ is $\mathcal{L}(P)_A$ -type definable. Indeed, $a \in I_n(A)$ if and only if for all $0 \leq i < n$, $m, l \in \mathbb{N}$ and \mathcal{L}_A -($l + i$)-formula $\varphi(x, y)$, a satisfies:

$$\forall g \in P^l [\text{if } \varphi(g, a_1, \dots, a_{i-1}, -) \text{ has } m \text{ realizations, then } \models \neg \varphi(g, a_1, \dots, a_i)].$$

(2) It is obvious that $I_n(A) = I_n(A \cup P)$ and $I_n(B) \supseteq I_n(A)$ for $B \subseteq A$.

Lemma 3.19. *Let $A \subseteq M$ that $A \setminus P$ is dcl-independent over P , and let $\varphi(x, y, z)$ be an $\mathcal{L}(P)_A$ -formula. Then there are \mathcal{L}_A -formulas $\psi_1(x, y, z), \dots, \psi_k(x, y, z)$ such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with*

$$cl(\varphi(a, b, M^l)) = \psi_i(a, b, M^l).$$

Proof. Let $a = (a_1, \dots, a_m) \in I_m(A)$ and $b = (b_1, \dots, b_n) \in P^n$. It follows that

$$(A \cup \{a_1, \dots, a_m\} \cup \{b_1, \dots, b_n\}) \setminus P$$

is dcl-independent over P . Since $I_m(A)$ is $\mathcal{L}(P)_A$ -type definable and P is definable, the statement of the lemma follows from compactness and Assumption (III). \square

Proposition 3.20. *Let $F : M^m \times M^n \rightarrow M$ be A -definable. Then there are $\mathcal{L}_{A \cup P}$ -definable continuous functions $F_i : Z_i \subseteq M^m \times M^n \rightarrow M$, $i = 1, \dots, k$, such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $(a, b) \in Z_i$ and*

$$F(a, b) = F_i(a, b).$$

Moreover, if A is dcl-independent over P , then the F_i 's can be chosen to be \mathcal{L}_A -definable.

Proof. By Lemma 2.5, there is a finite $B \subseteq A$ such that B is dcl-independent over P and F is $B \cup P$ -definable. So $(B \cup P) \setminus P$ is also dcl-independent over P . Let $\varphi(x, y, z)$ be an $\mathcal{L}(P)_{B \cup P}$ -formula that defines the graph of F . Hence by Lemma 3.19 there are $\mathcal{L}_{B \cup P}$ -formulas $\psi_1(x, y, z), \dots, \psi_k(x, y, z)$ such that for all $a \in I_m(B)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with

$$cl(\varphi(a, b, M)) = \psi_i(a, b, M).$$

Since $\varphi(a, b, M)$ is a single point, we have $\varphi(a, b, M) = \psi_i(a, b, M)$. Define $F_i : M^{m+n} \rightarrow M$ such that $F_i(a, b)$ is the unique $c \in M$ with $\psi_i(a, b, c)$ if such c exists, and 0 otherwise. Since ψ_i is an $\mathcal{L}_{B \cup P}$ -formula, F_i is $\mathcal{L}_{B \cup P}$ -definable. Thus we have $\mathcal{L}_{A \cup P}$ -definable functions $F_1, \dots, F_k : M^{m+n} \rightarrow M$, such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ such that $F(a, b) = F_i(a, b)$. Using cell decomposition in o-minimal structures, we can find an $\mathcal{L}_{A \cup P}$ -cell decomposition C_1, \dots, C_l of M^{m+n} such that each F_i is continuous on each C_j . The conclusion of the lemma now holds with the kl -many functions of the form $F_i|_{C_j}$, where $i = 1, \dots, k$ and $j = 1, \dots, l$.

For the ‘moreover’ clause, if $A \setminus P$ is dcl-independent over P , we need not replace A by $B \cup P$ in the above proof, which then shows that no further parameters from P are needed. \square

Corollary 3.21. *Let $F : P^n \rightarrow M$ be A -definable. Then there are $t \in \mathbb{N}$, \mathcal{L}_A -definable continuous functions $F_i : Z_i \subseteq M^{t+n} \rightarrow M$ with Z_i a cell, $i = 1, \dots, k$, and $u \in P^t$, such that for all $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $(u, b) \in Z_i$ and*

$$F(b) = F_i(u, b).$$

Proof. By Proposition 3.20 there are $\mathcal{L}_{A \cup P}$ -definable continuous function $H_i : Y_i \subseteq M^n \rightarrow M$, $i = 1, \dots, k$, such that for every $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $b \in Y_i$ and $F(b) = H_i(b)$. Now take $u \in P^t$ such that each H_i is \mathcal{L}_{A_u} -definable. For $i = 1, \dots, k$, pick an \mathcal{L}_A -definable function $F_i : Z_i \subseteq M^{t+n} \rightarrow M$ such that $(Z_i)_u = Y_i$ and $F_i(u, b) = H_i(b)$ for each $b \in Y_i$. By applying cell decomposition to M^{t+n} , we may further assume that each F_i is continuous and Z_i is a cell. \square

A slightly weaker version of Corollary 3.21 is known for dense pairs [12, Theorem 3(3)].

Definition 3.22. We call $X \subseteq M^n$, $n > 0$, *low over B* if there is $i \in \{1, \dots, n\}$ and \mathcal{L}_B -definable function $f : M^{n-1} \times M^l \rightarrow M$ such that

$$X = \{(a_1, \dots, a_n) \in M^n : \exists g \in P^l f(a_{-i}, g) = a_i\},$$

where $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

Note that if a set $X \subseteq M$ is low, then it is small and co-dense in M . Generalizations of this statement are obtained in Lemmas 4.14 and 4.31 below.

Corollary 3.23. *Let $F : M^n \rightarrow M$ be A -definable. Then there are $k, m, t \in \mathbb{N}$ and*

- *sets $X_j \subseteq M^n$ low over A , $j = 1, \dots, k$,*
- *\mathcal{L}_A -definable continuous functions $F_i : Z_i \subseteq M^{t+n} \rightarrow M$, $i = 1, \dots, m$,*
- *$u \in P^t$,*

such that for every $a \in M^n \setminus \bigcup_{j=1}^k X_j$, there is $i \in \{1, \dots, m\}$ with $a \in Z_i$ and

$$F(a) = F_i(u, a).$$

Proof. Note that $a \notin I_n(A)$ if and only if there are $i \in \{1, \dots, n\}$, an \mathcal{L}_A -definable function $f : M^{l+(n-1)} \rightarrow M$ and $g \in P^l$, such that $f(a_{-i}, g) = a_i$. Hence $a \notin I_n(A)$ if and only if there is X low over A such that $a \in X$. By compactness and Proposition 3.20, there are $k, m \in \mathbb{N}$ and

- $\mathcal{L}_{A \cup P}$ -definable functions $H_i : Y_i \subseteq M^n \rightarrow M$, $i = 1, \dots, m$
- sets $X_j \in M^n$ low over A , $j = 1, \dots, k$,

such that for every $a \in M^n \setminus (\bigcup_{j=1}^k X_j)$ there is $i \in \{1, \dots, m\}$ with $a \in Y_i$ and $F(a) = H_i(a)$. Now take $u \in P^t$ such that each H_i is \mathcal{L}_{Au} -definable, and continue as in the proof of Corollary 3.21. \square

Remark 3.24. It is natural to ask whether the extra parameter $u \in P^t$ in Corollary 3.23 can be chosen to be in $\text{dcl}_{\mathcal{L}(P)}(A)$. When the answer to Question 2.6 is positive, then the same proof gives that u is $\mathcal{L}_{A \cup H}$ -definable, for some $H \subseteq P \cap \text{dcl}_{\mathcal{L}(P)}(A)$. So in particular, this holds when $\tilde{T} = T^{\text{indep}}$ (independent set). When $\tilde{T} = T^d$ (dense pairs), we do not know the answer.

Remark 3.25. If $A \setminus P$ is dcl-independent over P , then using the ‘moreover’ clause of Proposition 3.20, we can see that in Corollaries 3.21 and 3.23, we obtain $t = 0$ and u be the empty tuple.

Since low subsets of M are small, we can easily get the following corollary of 3.23. This corollary is already known for $\tilde{T} = T^d$ by [12], with the aforementioned control in parameters also established in [45, Lemma 5]. We omit its proof since it is in fact a special case of Theorem 5.7(2) below.

Corollary 3.26. *Let $f : M \rightarrow M$ be A -definable. Then f agrees off some small set with an $\mathcal{L}_{A \cup P}$ -definable function $F : M \rightarrow M$.*

The Structure Theorem below is intended, among others, to generalize this corollary to arbitrary definable maps $f : X \subseteq M^n \rightarrow M$ (see Theorem 5.7(2)). For the moment, using compactness, we directly get the following uniform version of Corollary 3.23.

Corollary 3.27. *Let $f : Z \times M^n \subseteq M^{m+n} \rightarrow M$ be an A -definable map. Then there are $p, t \in \mathbb{N}$ and for each $i = 1, \dots, p$ there are*

- *an A -definable family $\{X_z^i\}_{z \in Z}$ of low subsets of M^n ,*
- *an \mathcal{L}_A -definable continuous function $f_i : Z_i \subseteq M^m \times P^t \times M^n \rightarrow M$,*

such that for all $z \in Z$ there is $u \in P^t$ such that for all $a \in M^n \setminus \bigcup_i X_z^i$, there is $i \in \{1, \dots, p\}$ with

$$f(z, a) = f_i(z, u, a).$$

Proof. The corollary follows easily from compactness and Corollary 3.23. \square

4. CONES AND LARGE DIMENSION

In this section, we introduce and analyze the two main objects of the paper, cones and large dimension.

4.1. Cones. As mentioned in the introduction, the notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both notions, supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way. The definitions appear to be quite technical in the beginning, but as it turns out they are in fact optimal in several ways (see Section 5.2, Question 5.14 and [18]).

Definition 4.1 (Supercones). We define recursively the notion of a *supercone* $J \subseteq M^k$, $k \geq 0$, as follows:

- $M^0 = \{0\}$ is a supercone.
- A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are \mathcal{L} -definable continuous $h_1, h_2 : M^n \rightarrow M \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it.

Abusing terminology, we say that a supercone J is *A-definable* if J is an A -definable set and its closure is \mathcal{L}_A -definable.

Note that, for $k > 0$, the interior U of $cl(J)$ is an open cell, and for every $a \in \pi(J)$, J_a is contained in U_a and it is co-small in it.

We remind the reader that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g \in S}$ with $\bigcup_{g \in S} \{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 4.2 (Uniform families of supercones). Let $\mathcal{J} = \bigcup_{g \in S} \{g\} \times J_g \subseteq M^{m+k}$ be a definable family of supercones. We call \mathcal{J} *uniform* if there is a cell $V \subseteq M^{m+k}$ containing \mathcal{J} , such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for \mathcal{J} . Abusing terminology, we call a uniform family *A-definable*, if it is an A -definable family of sets and has an \mathcal{L}_A -definable shell.

A shell for \mathcal{J} need not be unique. It is, however, canonical in the sense of Lemma 4.9 below. Note also that if \mathcal{J} is uniform, then so is each projection $\pi_{m+j}(\mathcal{J})$.

Definition 4.3 (Cones). A set $C \subseteq M^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq M^m$, a uniform family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in M^k , and an \mathcal{L} -definable continuous function $h : V \subseteq M^{m+k} \rightarrow M^n$, where V is a shell for \mathcal{J} , such that

- (1) $C = h(\mathcal{J})$, and
- (2) for every $g \in S$, $h(g, -) : V_g \subseteq M^k \rightarrow M^n$ is injective.

A *cone* is a k -cone for some k . Abusing terminology, we call a cone $h(\mathcal{J})$ *A-definable* if h is \mathcal{L}_A -definable and \mathcal{J} is A -definable.

Definition 4.4 (Fiber \mathcal{L} -definable maps). Let $C = h(\mathcal{J}) \subseteq M^n$ be a k -cone with $\mathcal{J} \subseteq M^{m+k}$, and $f : D \rightarrow M$ a definable function with $C \subseteq D$. We say that f is *fiber \mathcal{L} -definable with respect to C* if there is an \mathcal{L} -definable continuous function $F : V \subseteq M^{m+k} \rightarrow M$, where V is a shell for \mathcal{J} , such that

- $(f \circ h)(x) = F(x)$, for all $x \in \mathcal{J}$.

We call f *fiber \mathcal{L}_A -definable with respect to C* if F is \mathcal{L}_A -definable.

Remark 4.5.

- (1) If $J \subseteq M^n$ is a supercone, then $\pi_m(J)$ is a supercone, and for every $t \in \pi_m(J)$, J_t is a supercone with closure $cl(J)_t$.
- (2) Let $\{X_t\}_{t \in Z}$ be an A -definable family of subsets of M^n , $\{U_t\}_{t \in Z}$ an \mathcal{L}_A -definable family of subsets of M^n , and $\{C_t\}_{t \in Z}$ an A -definable family of cones in M^n . Using Remark 3.4(a), it is not hard to see that the sets
 - $\{t \in Z : X_t \text{ is a supercone with closure } cl(U_t)\}$
 - $\{t \in Z : X_t \text{ is a cone}\}$
 are both A -definable.
- (3) The 0-cones are exactly the small sets. Low subsets of M^n (Definition 3.22) are $n - 1$ cones, but not every $(n - 1)$ -cone is low.
- (4) The terminology of f being fiber \mathcal{L}_A -definable with respect to $C = h(\mathcal{J})$ is justified by the fact that, in that case, for every $g \in \pi(\mathcal{J})$, f agrees on $h(g, J_g)$ with an \mathcal{L}_{A_g} -definable map; namely $F \circ h(g, -)^{-1}$. But we require further that the family of these \mathcal{L}_{A_g} -definable maps is actually \mathcal{L}_A -definable and continuous. We illustrate this last point with Example 4.6 below. The same example also shows that the notion of being fiber \mathcal{L} -definable depends on h and \mathcal{J} .
- (5) It is easy to see that if $C = h(\mathcal{J})$ is an A -definable k -cone and $f : C \rightarrow M$ fiber \mathcal{L}_A -definable with respect to C , then the graph of f is an A -definable k -cone. We will not make use of this fact.
- (6) The closure of an A -definable cone $h(\mathcal{J})$ is \mathcal{L}_A -definable. Indeed, if $h : V \subseteq M^{m+k} \rightarrow M^n$ is as in Definition 4.3, then it is easy to check that $cl(h(\mathcal{J})) = cl(h(cl(\mathcal{J}) \cap V))$.
- (7) We may replace S by a definable subset of P^l in the definition of a cone C . Indeed, let $h : V \subseteq M^{m+k} \rightarrow M^n$ be as in that definition. Since S is P -bound, there is an \mathcal{L} -definable $f : M^l \rightarrow M^m$ with $f(P^l) \supseteq S$. By partitioning C into finitely many cones, we may assume that for some cell $Z \subseteq M^l$, $f : Z \rightarrow \pi(V)$ is continuous and $S \subseteq f(Z \cap P^l)$. So we may replace S by $S' := f^{-1}(S) \cap Z \subseteq P^l$, and h by $H : \bigcup_{g \in Z} \{g\} \times V_{f(g)} \rightarrow M^n$ with $H(x, y) = h(f(x), y)$. We decided, however, to keep the current definition because we can then adopt it in similar situations (such as Theorem 5.12 below). See also Remark 3.8.

Example 4.6. Consider a dense pair $\langle \mathcal{M}, P \rangle$ of real closed fields and let $S = P + aP$ for some $a \notin P$. The following map is taken from [12]. Let $f : S \rightarrow M$ be the a -definable map given by $f(x) = r$, where $x = r + sa$ for some (unique) $r, s \in P$. Then, clearly, for every $x = r + sa \in S$, $f(x, -) : M^0 \rightarrow M$ agrees with the \mathcal{L}_r -definable map H_r map given by $H_r(x, -) = r$. However, the family of maps H_r is not \mathcal{L} -definable. Now re-write S as the a -definable cone

$$S \times M = h(P^2)$$

where $h(p, q) = p + aq$, and let $F : M^2 \rightarrow M$ be the projection onto the first coordinate. Then, for every $(p, q) \in P^2$, we have

$$f(p + aq) = p = F(p, q),$$

witnessing that f is fiber \mathcal{L}_a -definable with respect to $h(P^2)$.

We next observe several easy consequences of the definitions that will be used in the proof of the Structure Theorem. The first lemma draws a connection between cones and the dcl-rank over tuples over P . Further results of this sort will be explored in Section 6.

Lemma 4.7. *Let $a \in M^n$ and $A \subseteq M$. Then*

$$\text{dcl-rank}(a/AP) = \min\{k \in \mathbb{N} : a \text{ is contained in an } A\text{-definable } k\text{-cone}\}.$$

Proof. (\leq). This follows easily from the definition of a k -cone.

(\geq). Let $a = (a_1, \dots, a_n)$ and set $k = \text{dcl-rank}(a/AP)$. We will find an A -definable k -cone that contains a . Without loss of generality, we may assume that $\text{dcl-rank}((a_1, \dots, a_k)/AP) = k$. Hence there are an \mathcal{L}_A -definable $Z \subseteq M^{l+n}$ and $s \in P^l$ such that $(s, a) \in Z$ and $\dim Z_s = k$. By cell decomposition in o-minimal structures, there are an \mathcal{L}_A -definable cell $X \subseteq M^{l+k}$ and a continuous \mathcal{L}_A -definable function $h : X \rightarrow M^n$ such that

- $\{(x, h(x, y)) : (x, y) \in X\} \subseteq Z$
- $(s, a_1, \dots, a_k) \in X$ and $h(s, a_1, \dots, a_k) = a$,
- X_y is an open cell and $h(y, -)$ is injective for each $y \in \pi_l(X)$.

In particular, $(X_y)_{y \in \pi_l(X)}$ is a uniform family of supercones with closure $cl(X)$. Thus

$$h \left(\bigcup_{g \in P^l \cap \pi_l(X)} \{g\} \times X_g \right)$$

is a k -cone containing a . □

Lemma 4.8. *Let C be an A -definable 0-cone in M^n and $f : C \rightarrow M$ be A -definable. Then there is a finite collection \mathcal{C} of A -definable 0-cones whose union is C and such that f is fiber \mathcal{L} -definable with respect to each cone in \mathcal{C} .*

Proof. Let S be A -definable small and $h : Z \subseteq M^m \rightarrow M^n$ be \mathcal{L}_A -definable and continuous such that $h(S) = C$. We may assume that $S \subseteq P^l$, for some l . Indeed, since S is P -bound over A , one can easily see that S is a finite union of sets $\sigma(S')$, where $S' \subseteq P^l$ is A -definable and $\sigma : W \rightarrow M^m$ is an \mathcal{L}_A -definable map. So C is a finite union of 0-cones of the form $h \circ \sigma(S')$.

Now, by Corollary 3.21 there are $k, t \in \mathbb{N}$ and, for $i = 1, \dots, k$, an \mathcal{L}_A -definable continuous function $F_i : Z_i \subseteq P^t \times M^l \rightarrow M$ with Z_i a cell, and $s \in P^t$, such that for all $g \in S$ there is $i \in \{1, \dots, k\}$ with $(f \circ h)(g) = F_i(s, g)$. Now set

$$S_i := \{(s, g) \in P^t \times S : (s, g) \in Z_i, (f \circ h)(g) = F_i(s, g)\}.$$

Set $\tau : M^t \times Z \rightarrow M^n$ to map (x, y) to $h(y)$. Then $\tau(S_i)$ is an A -definable 0-cone and f is fiber \mathcal{L}_A -definable with respect to $\tau(S_i)$. Moreover, $C = \bigcup_{i=1}^k \tau(S_i)$. □

Our next goal is to prove Lemmas 4.10 and 4.12 below, which will be used in the proof of the Structure Theorem (1)_n, Cases I and II, respectively. First, a lemma about shells.

Lemma 4.9. *Let $\mathcal{J} \subseteq M^{m+k}$ be an A -definable uniform family of supercones with an \mathcal{L}_A -definable shell V . Assume that $Z \subseteq M^{m+k}$ is an \mathcal{L}_A -definable cell containing \mathcal{J} . Then there are disjoint A -definable uniform families of supercones $\mathcal{J}_1, \dots, \mathcal{J}_n$ such that*

$$\mathcal{J} = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_n,$$

and each \mathcal{J}_i has an \mathcal{L}_A -definable shell $V_i \subseteq V \cap Z$.

Proof. First observe that for every $g \in \pi(\mathcal{J})$, $V_g \subseteq Z_g$. Indeed, $cl(V_g) = cl(\mathcal{J}_g) \subseteq cl(Z_g)$. Since V_g is an open cell, and Z_g is a cell too, this implies that $V_g \subseteq Z_g$.

Now let

$$D = \{g \in M^k : (V \cap Z)_g \text{ is an open cell}\}.$$

This set is \mathcal{L}_A -definable. Moreover, since for every $g \in \pi(\mathcal{J})$, $(V \cap Z)_g = V_g$, we obtain $\pi(\mathcal{J}) \subseteq D$. Let

$$D = D_1 \cup \dots \cup D_n$$

be a partition of D into \mathcal{L}_A -definable cells, and, for each i ,

$$\mathcal{J}_i = \mathcal{J} \cap (D_i \times M^k)$$

and

$$Z_i = (V \cap Z) \cap (D_i \times M^k).$$

Since both V, Z are cells and $D_i \subseteq D$, it is not hard to see that each Z_i is a cell. It clearly also contains \mathcal{J}_i . Finally, for every $g \in D_i$ and $0 < j \leq k$, we have

$$cl(\pi_{m+j}(\mathcal{J}_i)_g) = cl(\pi_{m+j}(V)_g) = cl(\pi_{m+j}(V \cap Z)_g) = cl(\pi_{m+j}(Z_i)_g),$$

showing that Z_i is a shell for \mathcal{J}_i . \square

We now prove that a suitable family of large subsets of M ranging over a k -cone gives rise to a $k+1$ -cone.

Lemma 4.10. *Let $C \subseteq M^n$ be an A -definable k -cone, let $\{X_a\}_{a \in C}$ be an A -definable family of subsets of M . Assume that $h_1, h_2 : C \rightarrow M \cup \{\pm\infty\}$ are fiber \mathcal{L}_A -definable with respect to C , and such that for all $a \in C$, X_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it. Then $\bigcup_{a \in C} \{a\} \times X_a$ is a finite disjoint union of A -definable $k+1$ -cones.*

Proof. Suppose that $C = h(\mathcal{J})$ for some uniform family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in M^k with shell V and \mathcal{L}_A -definable continuous $h : U \subseteq M^{m+k} \rightarrow M^n$, where U is a cell containing \mathcal{J} . By the assumption on h_1 and h_2 , there are \mathcal{L}_A -definable continuous functions $H_1, H_2 : Z \subseteq M^{m+k} \rightarrow M$, where Z is a cell containing \mathcal{J} , such that for every $g \in S$ and $t \in J_g$,

$$H_1(g, t) = h_1(h(g, t)) \text{ and } H_2(g, t) = h_2(h(g, t)).$$

By Lemma 4.9, we may assume that $V \subseteq U \cap Z$. Now set

$$V' := \{(g, t, x) : (g, t) \in V, H_1(g, t) < x < H_2(g, t)\}$$

and

$$J'_g := \bigcup_{t \in J_g} \{t\} \times X_{h(g, t)}.$$

It is easy to check that $\{J'_g\}_{g \in S}$ is a uniform family of supercones in M^{k+1} with closure $cl(V')$. Let $\tau : V \times M \rightarrow M^{n+1}$ map $((g, t), x)$ to $(h(g, t), x)$. For each $g \in S$ the function $\tau(g, -)$ is injective, because so is $h(g, -)$. Thus

$$\bigcup_{a \in C} \{a\} \times X_a = \tau \left(\bigcup_{g \in S} \{g\} \times J'_g \right)$$

is a $k+1$ -cone. \square

The proof of the Structure Theorem will run in parallel with its own uniform version (see Theorem 5.1(3) below), which prompts the following definition.

Definition 4.11 (Uniform families of cones). Let $\mathcal{C} := \{C_t\}_{t \in X \subseteq M^m}$ be a definable family of k -cones in M^n . We call \mathcal{C} *uniform* if there are

- an \mathcal{L} -definable continuous function $h : Z \subseteq M^{m+l+k} \rightarrow M^n$,
- a definable family $\{S_t\}_{t \in X}$ of small subsets of M^l ,
- a uniform definable family of supercones $Y = \{Y_{t,g}\}_{t \in X, g \in S_t}$ in M^k

such that $Y \subseteq Z$ and

- (i) $h(t, g, -) : Z_{t,g} \subseteq M^k \rightarrow M^n$ is injective for each $g \in S_t$,
- (ii) $C_t = h\left(\{t\} \times \left(\bigcup_{g \in S_t} \{g\} \times Y_{t,g}\right)\right)$.

Abusing terminology, we call \mathcal{C} *A-definable* if it is an A -definable family of sets, h is \mathcal{L}_A -definable, and $\{S_t\}_{t \in X}$ and $\{Y_{t,g}\}_{t \in X, g \in S_t}$ are A -definable.

We now prove that the union of a small uniform family of k -cones under a suitable map results again in a k -cone.

Lemma 4.12. *Let $\{C_t\}_{t \in K}$ be an A -definable uniform family of k -cones in M^n , with $K \subseteq M^m$ small, and let $\tau : W \subseteq M^{m+n} \rightarrow M^p$ be an \mathcal{L}_A -definable continuous map such that for each $t \in K$*

- $\{t\} \times C_t \subseteq W$,
- $\tau(t, -) : M^n \rightarrow M^p$ is injective.

Then $\tau\left(\bigcup_{t \in K} \{t\} \times C_t\right)$ is an A -definable k -cone in M^p .

Proof. Let $h : Z \subseteq M^{m+l+k} \rightarrow M^n$ be an \mathcal{L}_A -definable continuous function, $\{S_t\}_{t \in K}$ an A -definable family of small subsets of M^l , and $\{Y_{t,g}\}_{t \in K, g \in S_t}$ an A -definable family of supercones that witness that $\{C_t\}_{t \in K}$ is a uniform family of k -cones. Let $\sigma : Z \subseteq M^{m+l+k} \rightarrow M^p$ be defined by $\sigma(t, g, a) := \tau(t, h(t, g, a))$. We see directly that $\sigma(t, g, -)$ is injective, since $\tau(t, -)$ and $h(t, g, -)$ are injective. Note also that σ is \mathcal{L}_A -definable and continuous, since both h and τ are. Set

$$S := \bigcup_{t \in K} \{t\} \times S_t.$$

It is then straightforward to check that

$$\tau\left(\bigcup_{t \in K} \{t\} \times C_t\right) = \sigma\left(\bigcup_{(t,g) \in S} \{(t,g)\} \times Y_{t,g}\right)$$

is the desired k -cone. \square

The following lemma will be used in the last step of the proof of the Structure Theorem, $(1)_n \Rightarrow (3)_n$. It follows easily from Definition 4.2 and the next observations. Let $X \subseteq M^{m+n}$ be a set. Then for every $0 < j \leq n$ and $g \in \pi_m(X)$, we have

$$\pi_{m+j}(X)_g = \pi_j(X_g).$$

Let $X, Y \subseteq M^n$ and $0 < j \leq n$. Then

$$cl(X) = cl(Y) \Rightarrow cl(\pi_j(X)) = cl(\pi_j(Y)).$$

Indeed, $\pi_j(X) \subseteq \pi_j(cl(Y)) \subseteq cl(\pi_j(Y))$.

Lemma 4.13. *Let $U \subseteq M^{m+l+k}$ be an A -definable cell. Let*

$$\mathcal{K} = \{J_{t,g}\}_{t \in Y, g \in S_t}$$

be an A -definable family of supercones $J_{t,g} \subseteq M^k$, where $Y \subseteq M^m$ and $S_t \subseteq M^l$. Assume that for every $0 < j \leq k$, $t \in Y$ and $g \in S_t$,

$$(1) \quad cl(J_{t,g}) = cl(U_{t,g}).$$

Then U is a shell for \mathcal{K} . In particular, \mathcal{K} is an A -definable uniform family of supercones.

Proof. For every $t \in Y$ and $g \in S_t$, we have

$$\begin{aligned} cl(\mathcal{J}_{t,g}) = cl(U_{t,g}) &\Rightarrow cl(\pi_j(\mathcal{J}_{t,g})) = cl(\pi_j(U_{t,g})) \Rightarrow \\ &\Rightarrow cl(\pi_{m+l+j}(\mathcal{J})_{t,g}) = cl(\pi_{m+l+j}(U)_{t,g}), \end{aligned}$$

as required. \square

We finally include two lemmas that will be useful in the discussion of ‘large dimension’ in Section 4.3 below.

Lemma 4.14. *Let $J \subseteq M^n$, $n > 0$, be a supercone and $X \subseteq M^n$ a low set. Then $J \setminus X$ contains a supercone.*

Proof. Easy, following the definitions, by induction on n . \square

Lemma 4.15. *Let $J \subseteq M^n$ be a supercone and $\{X_s\}_{s \in S}$ a small definable family of subsets of M^n such that $J = \bigcup_{s \in S} X_s$. Then some X_s contains a supercone in M^n .*

Proof. By induction on n . If $n = 0$, it is obvious. If $n > 0$, for every $s \in S$, let

$$Y_s := \{t \in \pi(X_s) : \text{the fiber } (X_s)_t \text{ is large}\}.$$

By Remark 3.4(a), $\{Y_s\}_{s \in S}$ is a definable family of sets. By Corollary 3.14, we have $\pi(J) = \bigcup_{s \in S} Y_s$. By Inductive Hypothesis, some Y_s contains a supercone K . Since for every $t \in K$, $(X_s)_t$ is large, Remark 3.4(b) provides us with definable functions $h_1, h_2 : M^{n-1} \rightarrow M \cup \{\pm\infty\}$ such that for every $t \in \pi(X_s)$, $(X_s)_t$ is co-small in $(h_1(t), h_2(t))$. By Corollary 3.23, there are finitely many low sets in M^{n-1} off whose union h_1, h_2 are both \mathcal{L} -definable and continuous. Hence, by repeated use of Lemma 4.14, we obtain a supercone K' contained in K on which h_1, h_2 are both \mathcal{L} -definable. Therefore, the set

$$\bigcup_{t \in K'} \{t\} \times (X_s)_t \cap (h_1(t), h_2(t))$$

is a supercone contained in X_s . \square

4.2. \mathcal{L} -definable functions on supercones. The goal of this section (Proposition 4.19(1) below) is to show that a supercone from M^m cannot be ‘embedded’ into M^n , for $n < m$. This will make meaningful the notion of ‘large dimension’ we introduce in Section 4.3.

Lemma 4.16. *Let $J \subseteq M^n$ be an A -definable supercone and $S \subseteq cl(J)$ an open \mathcal{L}_A -definable cell. Then $S \cap J$ is an A -definable supercone with closure $cl(S)$.*

Proof. We work by induction on n . For $n = 0$ it is obvious. Assume we know the statement for subsets of M^k , $k < n$, and let $J \subseteq M^n$ be a supercone and $S \subseteq cl(J)$ be an open \mathcal{L}_A -definable cell. Since $\pi(S) \subseteq \pi(cl(J)) \subseteq cl(\pi(J))$, the inductive hypothesis gives that $\pi(S) \cap \pi(J)$ is an A -definable supercone $K \subseteq M^{n-1}$ with closure $cl(\pi(S))$. Since for every $t \in K$, J_t is co-small in $cl(J)_t$, we have that $(S \cap J)_t = S_t \cap J_t$ is co-small in S_t . Hence $S \cap J = \bigcup_{t \in K} \{t\} \times (S \cap J)_t$ is a supercone with closure $cl(S)$. \square

Lemma 4.17. *Let $K \subseteq M^{n+1}$ be a supercone. Then $cl(K) \setminus K$ is a finite union of sets of the form*

$$\bigcup_{g \in P^m} h(g, Z_g),$$

where $Z \subseteq P^m \times M^n$ is definable, $h : M^{m+n} \rightarrow M^{n+1}$ is \mathcal{L} -definable and each $h(g, -) : Z_g \rightarrow M^{n+1}$ is injective.

Proof. By induction on n . Denote $U = cl(K)$. For $n = 0$, this is clear since $U \setminus K$ is a small set and can be written as $h(P^m)$ with h as above. Now assume we know the statement for $k < n$, let $K \subseteq M^{n+1}$ be as above. We have:

$$(2) \quad U \setminus K = \left(\bigcup_{t \in \pi(K)} \{t\} \times (U_t \setminus K_t) \right) \cup \left(\bigcup_{t \in \pi(U) \setminus \pi(K)} \{t\} \times U_t \right).$$

By inductive hypothesis the second part is a finite union of sets of the form

$$T = \bigcup_{t \in X} \{t\} \times U_t,$$

where $X = \bigcup_{g \in P^m} h(g, Z_g)$, for suitable h . Observe that then

$$T = \bigcup_{g \in P^m} h'(g, W_g),$$

where $W_g = \bigcup_{v \in Z_g} \{v\} \times U_{h(g,v)}$ and $h'(g, v, u) = (h(g, v), u)$, as required.

The first part of the union in (2) is of the right form, as it follows immediately by applying Lemma 3.7. \square

Before proving Proposition 4.19, we illustrate it with an example.

Example 4.18. Consider the function $f : M^2 \rightarrow M$ with $f(x_1, x_2) = x_1 + x_2$. Let $J_1 = M \setminus P$ and for all $t \in J_1$, $J_t = J_1 \cap (t, \infty)$. Let $J = \bigcup_{t \in J_1} \{t\} \times J_t$. We will show that $f|_J$ is not injective. The proof is inspired by an example in [3, page 5]. Assume towards a contradiction that $f|_J$ is injective. Pick any two distinct $t_0 > t \in J$. Since $f|_J$ is injective, for every $b \in t_0 + J_{t_0}$, we have $b \notin t + J_t$. But $b \in t + cl(J_t)$, so $b \in t + P$. Since this holds for every $b \in t_0 + J_{t_0}$, we have that $t_0 + J_{t_0} \subseteq t + P$, which is a contradiction, since a large set cannot be contained in a small one.

Proposition 4.19. *Let $f : M^m \rightarrow M^n$ be an \mathcal{L} -definable function and $J \subseteq M^m$ a supercone, such that $f|_J$ is injective. Then*

- (1) $m \leq n$.
- (2) *there is an \mathcal{L} -definable $X \subseteq cl(J)$ such that $\dim(cl(J) \setminus X) < m$ and $f|_X$ is finite-to-one. Namely, $X = X_f \cap cl(J)$, with notation from Fact 2.9.*
- (3) *If $K \subseteq M^n$ is another supercone and $f : cl(J) \rightarrow cl(K)$ is injective, then $f(J) \cap K \neq \emptyset$.*

In particular, by (2), there is an open \mathcal{L} -definable $X \subseteq cl(J)$ such that $f|_X$ is injective.

Proof. The last clause follows from Fact 2.8.

We write $(1)_m - (3)_m$ for the above statements, and prove them simultaneously by induction on m . Statement $(1)_1$ is clear. Let $m \geq 1$.

$(1)_m \Rightarrow (2)_m$. Denote

$$X_f = \{a \in cl(J) : f^{-1}(f(a)) \text{ is finite}\}.$$

We claim that $\dim(\text{cl}(J) \setminus X_f) < m$. Assume not. Let $I \subseteq \text{cl}(J) \setminus X_f$ be an open box. By Lemma 4.16, $I \cap J$ contains a supercone $K \subseteq M^m$. By Fact 2.9, $f(I)$ has dimension $l < m$. In particular, $f(I)$ is in definable bijection with a subset of M^l via the restriction of an \mathcal{L} -definable map $h : M^n \rightarrow M^l$. Consider now $g = h \circ f : M^m \rightarrow M^l$. Then g is \mathcal{L} -definable and injective on K . We have contradicted $(1)_m$.

(1)_m \Rightarrow (3)_m. Let $K \subseteq M^m$ be a supercone and assume that $f : \text{cl}(J) \rightarrow \text{cl}(K)$ is injective. Suppose now for a contradiction that $f(J) \subseteq \text{cl}(K) \setminus K$. By Lemma 4.17 and Corollary 3.15, $\text{cl}(K) \setminus K$ is contained in the union of a small definable family of sets each of the form $h(g, Z_g)$ (for finitely many h 's), with each $Z_g \subseteq M^{m-1}$ and each $h(g, -) : Z_g \rightarrow M^m$ being \mathcal{L} -definable and injective. In particular, J is the union of a small definable family of sets of the form $f^{-1}h(g, Z_g) \cap J$. By Lemma 4.15, one of those sets must contain a supercone $L \subseteq M^n$. By Lemma 4.16, $T := \text{interior of } \text{cl}(L) \cap J$ is a supercone in M^m . But then the map $F = h(g, -)^{-1} \circ f : \text{cl}(T) \rightarrow M^{m-1}$ is an \mathcal{L} -definable map that is injective on T , contradicting $(1)_m$.

(2)_m & (3)_m \Rightarrow (1)_{m+1}. Let $f : M^{m+1} \rightarrow M^n$ be an \mathcal{L} -definable function and $J \subseteq M^{m+1}$ a supercone with closure V such that $f|_J$ is injective. Assume towards a contradiction that $m \geq n$. Let $J_1 = \pi_1(J)$ be the projection of J onto the first coordinate, and $V_1 = \pi_1(V)$. By $(2)_m$, for every $t \in J_1$, there is an open box $X_t \subseteq Y_t$ on which $f(t, -)$ is injective. By cell decomposition in o-minimal structures, and since J_1 is dense in V_1 , there is an open cell $U \subseteq V$, such that for every $t \in \pi_1(U)$, $f(t, -)$ is injective on U_t . By Lemma 4.16, $U \cap J$ is a supercone with closure $\text{cl}(U)$. We may thus replace J by $U \cap J$, and V by $\text{cl}(U)$, and assume from now on that for every $t \in V_1$, $f(t, -)$ is injective on V_t .

Claim 1. *There is an open interval $I_1 \subseteq V_1$ and an open box $I \subseteq M^n$, such that for every $t \in I_1$, $I \subseteq f(t, V_t)$.*

Proof of Claim 1. Since for every $t \in V_1$, $f(t, -)$ is injective on V_t , it follows that the dimension of the \mathcal{L} -definable set

$$Z = \bigcup_{t \in V_1} \{t\} \times f(t, V_t)$$

is $n + 1$. By cell decomposition, there is an open interval $I_1 \subseteq V_1$ and an open box $I \subseteq M^n$ such that $I_1 \times I \subseteq Z$. In particular, for all $t \in I_1$, $I \subseteq f(t, V_t)$. \square

By Claim 1, we can pick two distinct $t_0, t \in J_1$ such that

$$I \subseteq f(t_0, V_{t_0}) \cap f(t, V_t)$$

has dimension n . Since $f|_J$ is injective, for any $b \in I \cap f(t_0, J_{t_0})$, we have $b \notin f(t, J_t)$, and hence $b \in f(t, V_t \setminus J_t)$. Since this holds for every $b \in I \cap f(t_0, J_{t_0})$, we have that

$$I \cap f(t_0, J_{t_0}) \subseteq f(t, V_t \setminus J_t).$$

Claim 2. *There is a supercone $T \subseteq V_{t_0}$ such that $f(t_0, T) \subseteq I \cap f(t_0, J_{t_0})$.*

Proof. Denote $f_{t_0}(-) = f(t_0, -)$. So f_{t_0} is injective on V_{t_0} . Since $I \subseteq f(t_0, V_{t_0})$, we have $f_{t_0}^{-1}(I) \subseteq V_{t_0}$. Let $I' \subseteq f_{t_0}^{-1}(I)$ be an open cell. By Lemma 4.16, $T := I' \cap J_{t_0}$ is a supercone, as required. \square

We conclude that the map $f(t, -)^{-1} \circ f(t_0, -) : V_{t_0} \rightarrow V_t$ is an injective \mathcal{L} -definable map that maps T into $V_t \setminus J_t$, contradicting $(3)_m$. \square

We show with an example that the assumption on J being a supercone (and not just satisfying $\dim(\text{cl}(J)) = m$) is necessary.

Example 4.20. Let f be the function from Example 4.6. The usual projection map $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is injective on $\text{Graph}(f)$ but of course not injective on any open subset of $\text{cl}(\text{Graph}(f)) = \mathbb{R}^2$.

The next definition and corollary will be useful when we discuss the notion of large dimension in Section 4.3.

Definition 4.21. Let $f : M^k \rightarrow M^n$ be an \mathcal{L} -definable map, $J \subseteq M^k$ a supercone and $X \subseteq M^n$ a definable set. We say that

- f is a *strong embedding of J into X* if f is injective and $f(J) \subseteq X$.
- f is a *weak embedding of J into X* if $f|_J$ is injective and $f(J) \subseteq X$.

Corollary 4.22. Let $X \subseteq M^n$ be a definable set. The following are equivalent:

- (1) there is a weak embedding of a supercone $J \subseteq M^k$ into X .
- (2) there is a supercone $K \subseteq M^k$ and an \mathcal{L} -definable $f : M^k \rightarrow M^n$, injective on $\text{cl}(K)$, with $f(K) \subseteq X$.
- (3) there is a strong embedding of a supercone $L \subseteq M^k$ into X .

Proof. (3) \Rightarrow (1) is obvious.

(1) \Rightarrow (2). Let $f : M^k \rightarrow M^n$ be an \mathcal{L} -definable map, injective on J , with $f(J) \subseteq X$. By Proposition 4.19, there is an open definable $S \subseteq \text{cl}(J)$ such that $f|_{\text{cl}(S)}$ is injective. By Lemma 4.16, $J \cap S$ contains a supercone K .

(2) \Rightarrow (3). Let $S \subseteq \text{cl}(K)$ be open so that $f|_S$ can be extended to an injective \mathcal{L} -definable map $F : M^k \rightarrow M^n$. By Lemma 4.16 again, $S \cap K$ contains a supercone L . \square

4.3. Large dimension. We introduce an invariant for every definable set X which tends to measure ‘how large’ X is. This invariant will be used in the inductive proof of the Structure Theorem in Section 5.

Definition 4.23. Let $X \subseteq M^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k -cone. Equivalently, it is the maximum $k \in \mathbb{N}$ such that there is a strong embedding of a supercone $J \subseteq M^k$ into X . We also define the large dimension of the empty set to be $-\infty$. We denote the large dimension of X by $\text{ldim}(X)$.

Clearly, the large dimension of a subset of M^n is bounded by n . In view of Corollary 4.22, the large dimension of X is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X . In Section 6, we will prove that the large dimension equals the ‘scl-dimension’ arising from a relevant pregeometry in [3]. Here we establish some of its basic properties. The first lemma is obvious.

Lemma 4.24. For every definable $X, Y \subseteq M^n$, if $X \subseteq Y$, then $\text{ldim}(X) \leq \text{ldim}(Y)$.

Lemma 4.25. Let $\{Z_s\}_{s \in S}$ be a small definable family of sets. Then

$$\text{ldim}\left(\bigcup_{s \in S} Z_s\right) = \max \text{ldim} Z_s.$$

Proof. (\leq). Assume $f : M^n \rightarrow M^n$ is an \mathcal{L} -definable injective map, $J \subseteq M^n$ is a supercone, and $f(J) \subseteq \bigcup_{s \in S} Z_s$. We show that for some $s \in S$, $\text{ldim}(Z_s) \geq n$. For

every $s \in S$, let $X_s := f^{-1}(Z_s)$. Then $\{X_s \cap J\}_{s \in S}$ is a definable family of subsets of M^n that cover J , and by Lemma 4.15, one of them must contain a supercone $K \subseteq M^n$. Since $f(K) \subseteq Z_s$, we have that $\text{ldim}(Z_s) \geq n$.

(\geq). This is clear. \square

In particular, we obtain the following standard property that holds for any good notion of dimension.

Corollary 4.26. *Let X_1, \dots, X_l be definable sets. Then*

$$\text{ldim}(X_1 \cup \dots \cup X_l) = \max\{\text{ldim}(X_1), \dots, \text{ldim}(X_l)\}.$$

About supercones and cones we have:

Corollary 4.27. *If $C \subseteq M^n$ is a k -cone, then $\text{ldim}(C) = k$.*

Proof. By Lemma 4.25 and the definition of a cone it suffices to show that every supercone in M^k has large dimension k . But this is clear. \square

Corollary 4.28. *Let $n > 0$ and $J \subseteq M^n$ be a supercone. Then $\text{ldim}(\text{cl}(J) \setminus J) < n$.*

Proof. Immediate from Proposition 4.19(3) and the definitions. \square

Lemma 4.29. *Let $X \subseteq M^{n+1}$ be a definable set, such that for every $t \in \pi(X)$, X_t is small. Then $\text{ldim}(X) = \text{ldim}(\pi(X))$.*

Proof. Let U_i, S_i, h_i and Z_{ig} be as in Lemma 3.7. In particular,

$$(3) \quad U_i = h_i \left(\bigcup_{g \in S_i} \{g\} \times Z_{ig} \right).$$

(\geq). By Lemma 3.7(3), we have $\pi(X) = \bigcup_{i,g} Z_{ig}$. By Lemma 4.25, for some i, g , we have $\text{ldim}(Z_{ig}) = \text{ldim}(\pi(X))$. By Equation (3) and Lemma 3.7(1), we obtain

$$\text{ldim}(Z_{ig}) \leq \text{ldim}(U_i) \leq \text{ldim}(X).$$

(\leq). By Corollary 4.26, $\text{ldim}(X) = \max_i \text{ldim}(U_i)$. By Equation (3), Lemma 3.7(2) and Lemma 4.25, for every i , $\text{ldim}(U_i) = \max_g \text{ldim}(Z_{ig})$. But $Z_{ig} \subseteq \pi(X)$, so $\text{ldim}(X) \leq \text{ldim}(\pi(X))$. \square

Corollary 4.30. *Let $X \subseteq M^n$ be a definable set. Then $\text{ldim}(X) = 0$ if and only if X is small.*

Proof. Right-to-left is immediate from the definitions of a small set and large dimension. For the left-to-right, we use induction on n . If $n = 1$, the statement is clear by Lemma 3.3. Assume we know the statement for all $l \leq n$ and let $X \subseteq M^{n+1}$.

Claim. *The projection of X onto any of its coordinates is small.*

Proof of Claim. Without loss of generality we may just prove that the projection $\pi(X)$ onto the first n coordinates is small. Since $\text{ldim}(X) = 0$, using Lemma 3.3, we see that for every $t \in \pi(X)$, X_t is small. By Lemma 4.29, $\text{ldim}(\pi(X)) = \text{ldim}(X) = 0$. By Inductive Hypothesis, $\pi(X)$ is small. \square

Since X is contained in the product of its coordinate projections, it is again small. \square

In Definition 3.22, we introduced low sets. We are now able to determine their large dimension.

Lemma 4.31. *Let $X \subseteq M^n$ be a low definable set. Then $\text{ldim}(X) = n - 1$.*

Proof. By Remark 4.5(3) and Corollary 4.27. \square

Remark 4.32. We observe that the converse of Lemma 4.31 does not hold, even if we allow finite unions of low definable sets. For example, let $X := (M \setminus P) \times P$. One can see that X is a 1-cone. Suppose X is the finite union of low sets. Then the image of X under at least one of the coordinate projections has interior. But the images of X under the two coordinate projections are $M \setminus P$ and P . Neither of these two sets has nonempty interior.

5. STRUCTURE THEOREM

We are now ready to prove the main result of this paper, which consists of statements (1) and (2) below. The proof runs by simultaneous induction along with statement (3). The latter is a uniform version of (1).

Theorem 5.1 (Structure Theorem).

- (1) *Let $X \subseteq M^n$ be an A -definable set. Then X is a finite union of A -definable cones.*
- (2) *Let $f : X \subseteq M^n \rightarrow M$ be an A -definable function. Then there is a finite collection \mathcal{C} of A -definable cones whose union is X and such that f is fiber \mathcal{L}_A -definable with respect to each $C \in \mathcal{C}$.*
- (3) *Let $\{X_t\}_{t \in M^m}$ be an A -definable family of subsets of M^n . Then there is $p \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$,*
 - an A -definable subset $Y_i \subseteq M^m$,*
 - $k_i \in \mathbb{N}$,*
 - an A -definable uniform family of k_i -cones $\{C_t^i\}_{t \in Y_i}$,**such that for all $t \in M^m$*

$$X_t = \bigcup \{C_t^i : t \in Y_i\}.$$

Proof. We write $(1)_n - (3)_n$ for the above statements. We will now show by induction on n that $(1)_n - (3)_n$ hold. Statements $(1)_0 - (3)_0$ are trivial. Suppose now that $n > 0$ and $(1)_l - (3)_l$ hold for every $l < n$. It is left to show $(1)_n - (3)_n$.

(1)_n. Let $X \subseteq M^n$. By Remark 3.4(b), we may assume that there are A -definable $h_1, h_2 : M^{n-1} \rightarrow M \cup \{\pm\infty\}$ such that for every $a \in \pi(X)$, X_a is contained in $(h_1(a), h_2(a))$, and it is either small in it for all $a \in \pi(X)$, or co-small in it for all $a \in \pi(X)$. We handle the two cases separately.

Case I: For every $a \in \pi(X)$, X_a is co-small in $(h_1(a), h_2(a))$.

By $(2)_{n-1}$, we may assume that $\pi(X)$ is an A -definable cone, such that h_1, h_2 are fiber \mathcal{L}_A -definable with respect to it. By Lemma 4.10, X is a finite union of A -definable cones.

Case II: For every $a \in \pi(X)$, X_a is small in $(h_1(a), h_2(a))$.

By Lemma 3.7, we may assume that there are an \mathcal{L}_A -definable continuous function $h : Y \subseteq M^{m+n-1} \rightarrow M^n$, and A -definable small set $S \subseteq M^m$, and an A -definable family $\{Z_g\}_{g \in S}$ with $Z_g \subseteq \pi(X)$ such that

- $X = h\left(\bigcup_{g \in S} \{g\} \times Z_g\right)$, and
- for every $g \in S$, $h(g, -) : M^{n-1} \rightarrow M^n$ is injective.

By $(3)_{n-1}$, there is $p \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$,

- an A -definable subset $Y_i \subseteq S$,
- $k_i \in \mathbb{N}$,
- an A -definable uniform family of k_i -cones $\{C_g^i\}_{g \in Y_i}$,

such that for all $g \in S$,

$$Z_g = \bigcup \{C_g^j : g \in Y_j\}.$$

By Lemma 4.12, we have that for each $j \in \{1, \dots, p\}$,

$$h \left(\bigcup_{g \in Y_j} \{g\} \times C_g^j \right)$$

is an A -definable k_j -cone. Thus X is a finite union of A -definable cones.

$(1)_n \Rightarrow (2)_n$. Let $f : X \subseteq M^n \rightarrow M$ be an A -definable function. We prove $(2)_n$ by sub-induction on $\text{ldim}(X)$. Suppose first that $\text{ldim}(X) = 0$. By $(1)_n$ we can assume that X is a 0-cone. By Lemma 4.8 we can find a finite collection \mathcal{C} of A -definable cones whose union is X and such that f is fiber \mathcal{L}_A -definable with respect to each $C \in \mathcal{C}$. So we can now assume that $\text{ldim}(X) = k > 0$ and $(2)_n$ holds for all definable functions whose domain has $\text{ldim} < k$. By $(1)_n$, we may assume $X \subseteq M^n$ is an A -definable k -cone, say $X = h(\mathcal{J})$. Let $S = \pi(\mathcal{J})$. We now apply Corollary 3.27 to $f \circ h : \bigcup_{g \in S} \{g\} \times J_g \rightarrow M$ to get $p, t \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$

- an A -definable family $\{X_g \subseteq M^k\}_{g \in S}$ with $\text{ldim}(X_g) < k$,
- an \mathcal{L}_A -definable continuous function $f^i : Z_i \subseteq M^{l+t+k} \rightarrow M$

such that for every $g \in S$ there is $u \in P^t$ such that

(A) for all $a \in J_g \setminus X_g$ there is $i \in \{1, \dots, p\}$ such that $(f \circ h)(g, a) = f^i(g, u, a)$.

We denote the set of all pairs $(g, u) \in S \times P^t$ that satisfy (A) by K . For each $i \in \{1, \dots, p\}$ we define for $(g, u) \in K$,

$$B_{g,u}^i = \{a \in J_g \setminus X_g : (f \circ h)(g, a) = f^i(g, u, a)\}.$$

Note that for $g \in S$,

$$\bigcup_{u \in K_g} \left(J_g \setminus \bigcup_i B_{g,u}^i \right) \subseteq X_g.$$

Therefore

$$\text{ldim} \left(\bigcup_{u \in K_g} \left(J_g \setminus \bigcup_i B_{g,u}^i \right) \right) < k.$$

Since $h(g, -)$ is injective on J_g and \mathcal{L} -definable,

$$\text{ldim} h \left(\{g\} \times \bigcup_{u \in K_g} (J_g \setminus \bigcup_i B_{g,u}^i) \right) < k.$$

By Lemma 4.25

$$\text{ldim} h \left(\bigcup_{g \in S} \{g\} \times \left(\bigcup_{u \in K_g} (J_g \setminus \bigcup_i B_{g,u}^i) \right) \right) < k.$$

By sub-induction hypothesis, it is only left to show that the restriction of f to each

$$h \left(\bigcup_{g \in S} \{g\} \times \bigcup_{u \in K_g} B_{g,u}^i \right)$$

satisfies the conclusion of (2)_n. Let $i \in \{1, \dots, p\}$. Let $h' : M^{s+t+k} \rightarrow M^n$ map (g, u, a) to $h(g, a)$. Then

$$h \left(\bigcup_{g \in S} \{g\} \times \bigcup_{u \in K_g} B_{g,u}^i \right) = h' \left(\bigcup_{(g,u) \in K} \{(g,u)\} \times B_{g,u}^i \right).$$

By (3)_{n-1}, there is $q \in \mathbb{N}$ such that for every $j \in \{1, \dots, q\}$ there are an A -definable subset K^j of K , $k_j \in \{0, \dots, n\}$ and an A -definable uniform family of k_j -cones $\{Y_{g,u}^j\}_{(g,u) \in K^j}$ such that for each $(g, u) \in K$

$$B_{g,u}^i = \bigcup \{Y_{g,u}^j : (g, u) \in K^j\}.$$

By Lemma 4.12, we have that for each $j \in \{1, \dots, q\}$

$$h' \left(\bigcup_{(g,u) \in K^j} \{(g,u)\} \times (Y_{g,u}^j) \right)$$

is an A -definable k_j -cone $h'(\mathcal{Y}^j)$, where \mathcal{Y}^j denotes the inside family. Since

$$(f \circ h')(g, u, -) = (f \circ h)(g, -) = f^j(g, u, -)$$

on $Y_{g,u}^j$, we have that f is fiber \mathcal{L}_A -definable with respect to $h'(\mathcal{Y}^j)$.

(1)_n \Rightarrow (3)_n. This is by a standard (but lengthy) compactness argument, which we include for completeness. Let $\{X_t\}_{t \in M^m}$ be an A -definable family of subsets of M^n . Suppose that (3)_n fails. Then for every finite collection $\{C_t^1\}_{t \in Y_1}, \dots, \{C_t^p\}_{t \in Y_p}$ of A -definable uniform families of cones, there are $t \in M^m$ and $z \in M^n$ such that

$$z \in X_t \setminus \left(\bigcup_{i=1}^p C_t^i \right).$$

Since $\widetilde{\mathcal{M}}$ is sufficiently saturated, there is $x \in M^m$ and $z \in X_x$ such that for every A -definable uniform family of cones $\{C_t\}_{t \in Y}$ either $x \notin Y$ or $z \notin C_x$. For the rest of the proof, we fix this x and z . By (1)_n there is an Ax -definable k -cone $E \subseteq X_x$ with $z \in E$. This is not yet a contradiction, because we do not have a uniform family of cones such that E is one element of this family. Let $k' = \text{dcl-rank}(z/AxP)$. By Lemma 4.7, there is an Ax -definable k' -cone E' such that $z \in E'$. By (1)_n, there is an Ax -definable cone $F \subseteq E \cap E'$ such that $z \in F$. By Lemma 4.7, F is a k' -cone. Therefore we can assume that $F = E$ and $k = k'$. It is left to show that there is an A -definable uniform family of $\{C_t\}_{t \in Y}$ such that

- (I) $C_t \subseteq X_t$ for each $t \in Y$,
- (II) $x \in Y$ and $E = C_x$.

Let $\mathcal{J} = \{J_g\}_{g \in S}$ be an Ax -definable uniform family of supercones in M^k , and $h : Z \subseteq M^{l+k} \rightarrow M^n$ an \mathcal{L}_{Ax} -definable map, such that $E = h(\mathcal{J})$. Fix an $s \in S$ and $y \in J_s$ such that $h(s, y) = z$.

Pick an \mathcal{L}_A -definable function $h' : Z' \subseteq M^{m+l+k} \rightarrow M^n$ such that $h'(x, -, -) = h$. Thus in particular, $Z'_x = Z$. Let $U \subseteq M^{m+l+k}$ be an \mathcal{L}_A -definable cell such that h' is continuous on U and $(x, s, y) \in U$. Since $\text{dcl-rank}(z/AxP) = k$ we have that $\dim U_{x,s} = k$. By Lemma 4.16, $J_s \cap U_{x,s}$ is a supercone with closure $\text{cl}(U_{x,s}) = \text{cl}(U_x)_s$. We now take

- an A -definable family $\{S_t\}_{t \in M^m}$ of small subsets of M^l ,
- an A -definable family of $\{J'_{t,g}\}_{t \in M^m, g \in S_t}$ of subsets of M^k ,

such that $S_x = S$ and $J'_{x,g} = J_g$ for all $g \in S$. Note that we make no further claims about the objects just defined, in particular we do not claim that they directly give rise to a family of cones satisfying (I) and (II). Let

$$S'_t := \{g \in S_t : J'_{t,g} \cap U_{t,g} \text{ is a supercone with closure } \text{cl}(U_{t,g})\}.$$

By Remark 4.5(2), $(S'_t)_{t \in Y'}$ is an A -definable family. Let $Y' \subseteq M^m$ be the set of all $t \in Y$ such that $S'_t \neq \emptyset$ and

$$h' \left(t, \bigcup_{g \in S'_t} \{g\} \times (J'_{t,g} \cap U_{t,g}) \right) \subseteq X_t.$$

This set is A -definable. It is not hard to check that $s \in S'_x$ and hence $x \in Y'$. Denote

$$\mathcal{K} = \{J'_{t,g} \cap U_{t,g}\}_{t \in Y', g \in S'_t}.$$

By Lemma 4.13, \mathcal{K} is an A -definable uniform family of supercones and

$$\left\{ h' \left(t, \bigcup_{g \in S'_t} \{g\} \times (J'_{t,g} \cap U_{t,g}) \right) \right\}_{t \in Y'}$$

is an A -definable uniform family of k -cones satisfying (I) and (II). \square

Remark 5.2.

- (1) The proof of the Structure Theorem uses our standing assumption that $\widetilde{\mathcal{M}}$ is sufficiently saturated. However, by Remark 4.5(2), the Structure Theorem holds for any $\widetilde{\mathcal{M}} \models \widetilde{T}$.
- (2) Using a standard compactness argument, the reader can verify that the following uniform version of (2) easily follows (from (2)): let $\{X_t\}_{t \in M^m}$ be an A -definable family of subsets of M^n and $\{f_t : X_t \rightarrow M\}_{t \in M^m}$ an A -definable family of maps. Then the conclusion of (3) holds with every f_t being fiber \mathcal{L}_{At} -definable with respect to C_t^i .
- (3) We do not know whether we can have disjointness of the cones in the Structure Theorem. However, under one additional assumption, we do obtain it; see Theorem 5.12 below.

5.1. Corollaries of the Structure Theorem. We collect a few important corollaries of the Structure Theorem. The main result we are aiming for is Theorem 5.7, a generalization of Corollary 3.26. We start with showing the invariance of the large dimension under definable bijections. Recall from Section 4.3 that the large dimension of a definable set $X \subseteq M^n$ is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X .

Corollary 5.3 (Invariance of large dimension). *Let $f : X \rightarrow M^n$ be a definable injective function. Then $\text{ldim}(X) = \text{ldim} f(X)$.*

Proof. Assume that $k \leq \text{ldim}(X)$. It suffices to show $k \leq \text{ldim}f(X)$. By the Structure Theorem, X is the union of finitely many cones such that f is fiber \mathcal{L} -definable with respect to each of them. By Corollary 4.26, one of them, say $h(\mathcal{J})$ must be a k -cone. Pick any $g \in \pi(\mathcal{J})$. Then $(f \circ h)(g, -) : J \rightarrow M^n$ agrees with an \mathcal{L} -definable map on J and it is injective. Therefore, $k \leq \text{ldim}f(X)$. \square

The following is an easy consequence of Structure Theorem (3).

Corollary 5.4. *Let $D \subseteq M^m \times M^n$ be an A -definable set. Then D is a finite union of A -definable sets of the form*

$$\bigcup_{t \in \Gamma} \{t\} \times C_t,$$

where $\Gamma \subseteq M^m$ is an A -definable cone and there is k such that $\{C_t\}_{t \in \Gamma}$ is an A -definable uniform family of k -cones in M^n .

Proof. Left to the reader. \square

We now establish certain desirable properties of large dimension.

Corollary 5.5. *Let $X \subseteq M^{m+n}$ be an A -definable set and let $\pi_m(X)$ be its projection onto the first m coordinates. Then*

- (1) *For every $k \in \mathbb{N}$, the set of all $t \in \pi_m(X)$ such that $\text{ldim}(X_t) = k$ is A -definable.*
- (2) *Assume that for every $t \in \pi_m(X)$, $\text{ldim}(X_t) = k$. Then*

$$\text{ldim}(X) = \text{ldim}(\pi_m(X)) + k.$$

Proof. We observe that by [11, Proposition 1.4], we only need to prove both statements for $n = 1$. Statement (1) is then immediate by Lemma 3.3 and Remark 3.4(a).

(2). For $k = 0$, this is by Lemma 4.29. For $k = 1$, assume that $\text{ldim}(\pi_m(X)) = l$. By Structure Theorem (1), $\pi_m(X)$ is the finite union of cones J_1, \dots, J_p . Assume that J_i is a k_i -cone. By Lemma 4.10, $T_i = J_i \times M$ is a finite union of $k_i + 1$ -cones, and by Corollary 4.27, each of them has large dimension $k_i + 1$. Since X is contained in $T_1 \cup \dots \cup T_p$, it follows from Corollary 4.26 that $\text{ldim}(X) \leq \max_i k_i + 1 = l + 1$.

On the other hand, let C be an l -cone contained in $\pi_m(X)$. By Remark 3.4(b), there are definable $h_1, h_2 : M^m \rightarrow M$ such that for every $t \in \pi_m(X)$, X_t is co-small in $(h_1(t), h_2(t))$. By Structure Theorem (2), $\pi_m(X)$ contains an l -cone C' on which $h_1, h_2 : M^m \rightarrow M$ are both fiber \mathcal{L} -definable. By Lemma 4.10, it follows that X contains an $l + 1$ -cone. \square

Lemma 5.6. *Let $J_1, J_2 \subseteq M^k$ be two supercones and $h_1 : Z_1 \rightarrow M^n$, $h_2 : Z_2 \rightarrow M^n$ two \mathcal{L} -definable continuous injective maps, where Z_i is the interior of $\text{cl}(J_i)$, $i = 1, 2$. Then*

$$\dim(h_1(Z_1) \cap h_2(Z_2)) = k \implies \text{ldim}(h_1(J_1) \cap h_2(J_2)) = k.$$

Proof. Let

$$K_1 = h_1^{-1}(h_1(Z_1) \cap h_2(Z_2)).$$

Then $K_1 \subseteq Z_1$ and $\dim(K_1) = k$. By Lemma 4.16, $K_1 \cap J_1$ contains a supercone J . Since $J \subseteq K_1$, we have

$$h_2^{-1}h_1(J) \subseteq Z_2.$$

Observe that $h_2^{-1}h_1(J)$ has large dimension k and it is contained in the union of $Z_2 \setminus J_2$ and J_2 . By Corollary 4.28, $Z_2 \setminus J_2$ has large dimension $< k$. Hence

$$\text{ldim}(h_2^{-1}h_1(J) \cap J_2) = k.$$

Then $\text{ldim}(h_2(h_2^{-1}h_1(J) \cap J_2)) = k$. We observe

$$h_2(h_2^{-1}h_1(J) \cap J_2) \subseteq h_1(J) \cap h_2(J_2) \subseteq h_1(J_1) \cap h_2(J_2),$$

proving that $h_1(J_1) \cap h_2(J_2)$ has large dimension k . \square

We are now ready to prove the main result of this section. Statement (2) below is a higher dimensional analogue of Corollary 3.26. To our knowledge, it has not been known even in the special case of dense pairs of o-minimal structures.

Theorem 5.7.

- (1) *Let $X \subseteq M^n$ be A -definable. Then there are disjoint AP -definable supercones $J_1, \dots, J_p \subseteq X$ such that*

$$\text{ldim}\left(X \setminus \bigcup_{i=1}^p J_i\right) < n.$$

- (2) *Every A -definable map $f : M^n \rightarrow M$ is given by an \mathcal{L}_{AP} -definable map $F : M^n \rightarrow M$ off an AP -definable set of large dimension $< n$.*

Moreover, if $A \setminus P$ is dcl-independent over P , then in both statements the parameters from P can be omitted.

Proof. We again denote the above two statements by $(1)_n$ and $(2)_n$, and proceed by simultaneous induction on n . For $n = 0$, they are both trivial. Suppose now that $n > 0$ and $(1)_l$ and $(2)_l$ hold for every $l < n$. It is left to show $(1)_n$ and $(2)_n$.

(1)_n: Let $X \subseteq M^n$ and $\pi : M^n \rightarrow M^{n-1}$ be the usual projection onto the first $n - 1$ coordinates. By Corollary 5.5, the set

$$\{t \in X : \text{ldim}(X_{\pi(t)}) = 0\}$$

is A -definable and has $\text{ldim} < n$. Therefore, we can reduce to the case that $\dim X_a = 1$ for all $a \in \pi(X)$. By Remark 3.4(b), we may further assume that there are A -definable functions $h_1, h_2 : M^{n-1} \rightarrow M \cup \{\pm\infty\}$ such that for every $a \in \pi(X)$, X_a is co-small and contained in $(h_1(a), h_2(a))$. By $(2)_{n-1}$ there are \mathcal{L}_{AP} -definable functions $H_1, H_2 : M^{n-1} \rightarrow M$ and an AP -definable set $Z \subseteq M^{n-1}$ such that $\text{ldim}(Z) < n - 1$ and $H_1 = h_1$ and $H_2 = h_2$ on $M^{n-1} \setminus Z$. By $(1)_{n-1}$ there are disjoint AP -definable supercones J_1, \dots, J_p of M^{n-1} such that $J_i \subseteq \pi(X) \setminus Z$,

$$(*) \quad \text{ldim}\left((\pi(X) \setminus Z) \setminus \bigcup_{i=1}^p J_i\right) < n - 1.$$

By Lemma 4.16 and cell decomposition in o-minimal structures, we can assume that h_1, h_2 are continuous on the interior of each $cl(J_i)$. Then each $K_i := \bigcup_{t \in J_i} \{t\} \times X_t$ is an AP -definable supercone. It follows immediately from Corollary 5.5 and $(*)$ that $\text{ldim}(X \setminus \bigcup_{i=1}^p K_i) < n$, and that K_1, \dots, K_p are disjoint.

(1)_n \Rightarrow (2)_n: Let $f : M^n \rightarrow M$ be A -definable. By Corollaries 3.23 and 4.31, there are $m \in \mathbb{N}$ and

- an A -definable set $Z \subseteq M^n$ with $\text{ldim}(Z) < n$,
- $\mathcal{L}_{A \cup P}$ -definable functions $f_i : Z_i \rightarrow M$ for $i = 1, \dots, m$,

such that for each $a \in M^n \setminus Z$ there is $i \in \{1, \dots, m\}$ such that $a \in Z_i$ and $f(a) = f_i(a)$. Set

$$X_i := \{a \in M^n : f(a) = f_i(a) \wedge f(a) \neq f_j(a) \text{ for } j < i\}.$$

Note that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\text{ldim}(M^n \setminus \bigcup_{i=1}^m X_i) < n$. By $(1)_n$, for each $i = 1, \dots, m$, there are AP -definable supercones $J_{ik} \subseteq X_i$, $k = 1, \dots, p_i$, such that $\text{ldim}(X_i \setminus \bigcup_{k=1}^{p_i} J_{ik}) < n$. Note that $J_{ik} \cap J_{jl} = \emptyset$ for $i, j \in \{1, \dots, m\}$ with $i \neq j$ and $k = 1, \dots, p_i$, $l = 1, \dots, p_j$. Denote by V_{ik} the interior of $\text{cl}(J_{ik})$. By Lemma 5.6, for such i, j, k and l , $V_{ik} \cap V_{jl}$ has dimension $< n$, and hence, since V_{ik} and V_{jl} are open, empty. Thus define $F : M^n \rightarrow M$ to map $x \in V_{ik}$ to $f_i(a)$ and $x \notin \bigcup_i \bigcup_k V_{ik}$ to 0. Note that this function is well-defined and \mathcal{L}_{AP} -definable, since all f_i and V_{ik} are. Moreover, F agrees with f outside a set of large dimension $< n$; namely $X \setminus \bigcup_{i=1}^m \bigcup_{k=1}^{p_i} J_{ik}$.

The ‘moreover’ clause follows from the above proof and Remark 3.25. \square

We expect that Theorem 5.7 will find many applications in the future, and illustrate one here in the case of dense pairs. Namely, we answer the following question from Dolich-Miller-Steinhorn [9, page 702]: in dense pairs, is the graph of every \emptyset -definable unary map nowhere dense? This property is known to fail if we allow parameters, as the example in Introduction shows. In [8] the above authors isolate this property and examine it in the context of structures with o-minimal open core.

Proposition 5.8. *Let $\widetilde{M} = \langle M, P \rangle$ be a dense pair. Then the graph of every \emptyset -definable map $f : X \subseteq M \rightarrow M$ is nowhere dense.*

Proof. By Theorem 5.7, f agrees off a \emptyset -definable small set $S \subseteq X$ with an \mathcal{L}_\emptyset -definable function F . Clearly, the graph of $f|_{X \setminus S}$ is nowhere dense. We therefore only need to prove that the graph of $f|_S$ is nowhere dense. By Lemma 3.16, $S \subseteq P$. By [12, Lemma 3.1], $f(S) \subseteq P$. By [12, Theorem 3(3)], f is piecewise given by \mathcal{L} -definable functions, and hence its graph is nowhere dense. \square

5.2. Optimality of the Structure Theorem. In this section, we prove that our Structure Theorem is in a certain sense optimal.

Definition 5.9. A *strong cone* is a cone $h(\mathcal{J})$ which, in addition to the properties of Definition 4.3, satisfies:

- $h : \mathcal{J} \rightarrow M^n$ is injective.

By *Strong Structure Theorem* we mean the Structure Theorem where cones are replaced everywhere by strong cones. Below we give a counterexample to the Strong Structure Theorem and in the next section we point out a ‘choice property’ that implies it. We will need the following lemma.

Lemma 5.10. *Let $J \subseteq M^n$ be a supercone and $S \subseteq M^m$ small. Assume that $f : Z \subseteq M^n \rightarrow M^m$ is an \mathcal{L} -definable continuous map with $J \subseteq Z$ that satisfies $f(J) \subseteq S$. Then $f|_J$ is constant.*

Proof. We work by induction on n . For $n = 0$, the statement is trivial. Now let $n > 1$ and assume we know the statement for all $J \subseteq M^k$ with $k < n$. Let $J \subseteq M^n$ and $f : Z \rightarrow S$ be as in the statement with $f(J) \subseteq S$. For every $t \in \pi_1(J)$, by inductive hypothesis applied to $f(t, -) : Z_t \rightarrow M^m$, there is unique $c_t \in S$ so that $f(\{t\} \times J_t) = \{c_t\}$. Since f is continuous, and by definition of a supercone, for every $t \in \pi_1(Z)$, there is also unique $c_t \in S$ so that $f(\{t\} \times Z_t) = \{c_t\}$. We let $h : \pi_1(Z) \rightarrow M^m$ be the map given by $t \mapsto c_t$. If f is not constant on J , there must be an interval $I \subseteq \pi_1(Z)$ on which h is injective. But $I \cap \pi_1(J) \subseteq M$ is a supercone by Lemma 4.16, and $h(I \cap \pi_1(J)) \subseteq S$, a contradiction. Therefore, f is constant on J . \square

Counterexample to the Strong Structure Theorem. We consider two closely related o-minimal structures: $\mathcal{M} = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x \upharpoonright_{[0,1]} \rangle$ and its expansion $\mathcal{M}' = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x \rangle$. It is well-known that \mathcal{M} does not define unrestricted multiplication by π and that the theory of \mathcal{M}' is the theory of ordered $\mathbb{Q}(\pi)$ -vector spaces. We denote the language of \mathcal{M} by \mathcal{L} and the language of \mathcal{M}' by \mathcal{L}' .

We now set $P := \text{dcl}_{\mathcal{L}'}(\emptyset)$. We first observe that $P = \mathbb{Q}(\pi) = \text{dcl}_{\mathcal{L}'}(\emptyset)$. Indeed, since π is \mathcal{L}_{\emptyset} -definable, it is easy to see that $\mathbb{Q}(\pi) \subseteq P$. Note that $\mathbb{Q}(\pi)$ is a $\mathbb{Q}(\pi)$ -vector space and therefore a model of the theory of \mathcal{M}' . Thus $\text{dcl}_{\mathcal{L}'}(\emptyset) \subseteq \mathbb{Q}(\pi)$.

Since $P = \mathbb{Q}(\pi) = \text{dcl}_{\mathcal{L}'}(\emptyset)$, $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ is a dense pair of models of the theory of \mathcal{M} and $\langle \mathcal{M}', P \rangle$ is a dense pair of models of the theory of \mathcal{M}' . We will now show that the Strong Structure Theorem fails in $\widetilde{\mathcal{M}}$. Being able to work in the two different dense pairs will be crucial. In the following, whenever we say a set is definable without referring to a particular language, we mean definable in $\widetilde{\mathcal{M}}$.

For $t \in M$, we denote by l_t the straight line of slope π that passes through $(t, 0)$. Define

$$U = \bigcup_{g \in P} l_g.$$

We will prove that U is definable but not a finite union of strong cones. By an *endpart* of l_t , we mean $l_t \cap ([a, \infty) \times \mathbb{R})$, for some $a \in \mathbb{R}$.

Claim 1. *U is definable.*

Proof of Claim 1. For every $a \in M$, let $C_a = M \times [a, a+1)$ and $E_a \subseteq C_a \times C_a$ given by:

$$(x, y)E_a(x', y') \Leftrightarrow y' - y = \pi(x' - x) \text{ and } |x' - x| \leq 1.$$

Thus, if $(x, y) \in l_t \cap C_a$, then $[(x, y)]_{E_a}$ is the segment of l_t that lies in C_a . Define $p_a : C_a \rightarrow M^2$ via

$$p_a(x, y) = \text{the midpoint of } [(x, y)]_{E_a},$$

and let

$$Y_a = p_a(C_a \cap P^2).$$

Clearly, for $t \in P$, we have $l_t \cap P^2 = \{(g, \pi(g - t)) : g \in P\}$, and for $t \notin P$, we have $l_t \cap P^2 = \emptyset$. We claim that

$$U = \bigcup_{a \in M} Y_a,$$

and hence U is definable.

(\subseteq). Let $(x, y) \in l_t$, $t \in P$. We claim that $(x, y) \in p_a(C_a \cap P^2)$, for $a = y - \frac{1}{2}$. Indeed, (x, y) is the midpoint of $[(x, y)]_{E_a} = l_t \cap C_a$, and hence all we need is to find a point $(g_1, g_2) \in l_t \cap C_a \cap P^2$. Take any $g_2 \in [a, a+1) \cap P$ and let $g_1 = t + \frac{g_2}{\pi} \in P$. Then clearly $(g_1, g_2) \in l_t \cap C_a \cap P^2$ and hence $p_a(g_1, g_2) = (x, y)$.

(\supseteq). Let $(x, y) = p_a(g_1, g_2) \in p_a(C_a \cap P^2)$. Then $y - g_2 = \pi(x - g_1)$. Hence, for $t = g_1 - \frac{g_2}{\pi}$, we have $(x, y) \in l_t$. \square

Claim 2. *U is not a finite union of strong cones.*

Proof of Claim 2. First we observe that $\text{ldim}(U) = 1$. Indeed, U contains infinite \mathcal{L} -definable sets, so $\text{ldim}(U) \geq 1$. It cannot be $\text{ldim}(U) = 2$, by Lemma 4.29 and since each vertical fiber is small (it contains at most one element of each l_t , $t \in P$). Therefore $\text{ldim}(U) = 1$.

Now assume, towards a contradiction, that U is a finite union of strong cones. Let $h(\mathcal{J})$ be one of them, where $\mathcal{J} = \bigcup_{g \in S} \{g\} \times J_g$, and $h : Z \rightarrow M^2$. In particular, h is injective on \mathcal{J} . In the next two subclaims we make use of the expansion \mathcal{M}' of \mathcal{M} and the dense pair $\langle \mathcal{M}', P \rangle$.

Subclaim 1. *For every $g \in S$, $h(g, Z_g)$ must be contained in a unique l_t .*

Proof of Subclaim. Each of l_t and the family $\{l_t\}_{t \in M}$ is now \mathcal{L}' -definable. Consider the \mathcal{L}' -definable and continuous map $f : Z_g \rightarrow M$ where

$$f(x) = t \Leftrightarrow h(g, x) \in l_t.$$

By Lemma 5.10 applied to $J = J_g$, $S = P$ and f , it follows that $h(g, J_g)$ must be contained in a unique l_t . By continuity of h , so does $h(g, Z_g)$. \square

Subclaim 2. *For every $t \in P$, there are only finitely many $g \in S$ such that $h(g, Z_g) \subseteq l_t$.*

Proof of Subclaim. Assume, towards a contradiction, that for some $t \in P$ there are infinitely many $g \in S$ with $h(g, Z_g) \subseteq l_t$. For each $g \in S$, denote by a_g the infimum of the projection of $h(g, Z_g)$ onto the first coordinate. By injectivity of h , for every two $g_1, g_2 \in S$, we have $h(g_1, J_{g_1}) \cap h(g_2, J_{g_2}) = \emptyset$. By Lemma 5.6, $h(g_1, Z_{g_1}) \cap h(g_2, Z_{g_2})$ is finite (in fact, a singleton). Therefore, the set

$$\{a_g : g \in S \text{ and } h(g, Z_g) \subseteq l_t\}$$

is an infinite discrete $\mathcal{L}'(P)$ -definable subset of \mathbb{R} , a contradiction. \square

Since the subclaims hold for each of the finitely many strong cones, it turns out that for one of them, say $h(\mathcal{J})$, there is some $g \in \pi(\mathcal{J})$ such that $h(g, Z_g)$ contains an endpart of l_0 . So some endpart of l_0 is definable in $\widetilde{\mathcal{M}}$. But then its closure, which equals that endpart, is \mathcal{L} -definable. It follows easily that the full multiplication $x \mapsto \pi x$ is \mathcal{L} -definable, a contradiction. \square

5.3. Future directions. We now point out a key ‘choice property’ which guarantees the Strong Structure Theorem. Indeed, together with Corollary 3.5 it implies a strengthened version of Lemma 3.7 below, which is enough.

Choice Property: Let $h : Z \subseteq M^{n+k} \rightarrow M^l$ be an \mathcal{L}_A -definable continuous map and $S \subseteq M^n$ A -definable and small. Then there are $p, m \in \mathbb{N}$, \mathcal{L}_A -definable continuous maps $h_i : Z_i \subseteq M^{m+k} \rightarrow M^l$, $Y_i \subseteq M^m$ A -definable and small, and A -definable families $X_i \subseteq M^{m+k}$ with $X_{ia} \subseteq Y_i$, $i = 1, \dots, p$, such that for every $a \in \pi(Z)$,

- (1) $h_i(-, a) : X_{ia} \rightarrow M^l$ is injective, and
- (2) $h(S \cap Z_a, a) = \bigcup_i h_i(X_{ia}, a)$,

where $\pi(Z)$ denotes the projection of Z onto the last k coordinates.

Lemma 5.11. *If \widetilde{M} satisfies the Choice Property, then Lemma 3.7 holds with the additional conclusion that each $h_i : Z_i \rightarrow M^{n+1}$ is injective.*

Proof. We first claim that there are $m, p \in \mathbb{N}$, and for each $i = 1, \dots, p$, an \mathcal{L}_A -definable continuous function $h_i : Z_i \subseteq M^{m+n} \rightarrow M$, an A -definable small set $S_i \subseteq M^m$ and an A -definable family $Y_i \subseteq S_i \times C$, such that for all $a \in I$,

- (1) $h_i(-, a) : Y_{ia} \rightarrow M$ is injective,
- (2) $X_a = \bigcup_i h_i(Y_{ia}, a)$,
- (3) $\{h_i(Y_{ia}, a)\}_{i=1, \dots, p}$ are disjoint.

Indeed, apply the Choice Property to each h_i from Corollary 3.5 to get (1) and (2). For (3), recursively replace Y_{ia} , $1 < i \leq l$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z, a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a \in C} \{a\} \times X_a = \bigcup_i \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a).$$

From this point on the argument continues identically with the corresponding part of Lemma 3.7, noting in the end that, by (1), each \hat{h}_i turns out to be injective. \square

Theorem 5.12. *If \widetilde{M} satisfies the Choice Property, then the Structure Theorem holds with cones replaced by strong cones. Moreover, the unions of cones in Structure Theorem are disjoint.*

Proof. The reader can check that Lemmas 4.10 and 4.12 hold with cones replaced everywhere by strong cones, with identical proofs. Moreover, the Choice Property for $k = 0$ implies that every 0-cone is a finite union of strong 0-cones, and hence it is easy to obtain Lemma 4.8 with strong 0-cones in place of 0-cones, as well. It is then a (rather lengthy) routine to check that the proof of the current statement is, again, identical with that of the Structure Theorem, with cones replaced everywhere by strong cones and with the further condition that the unions of cones can be taken to be disjoint. In the proof, Lemma 3.7 has to be replaced by Lemma 5.11 in order to get strong cones and not just cones. The injectivity of the h_i 's in Lemma 5.11 guarantees the disjointness of the cones. We leave the details to the reader. \square

The counterexample to the Strong Structure Theorem relies on a somewhat unnatural condition on \mathcal{M} . In [20], we establish the Choice Property for a collection of structures $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$, such as when \mathcal{M} is a real closed field, or when P is a dense independent set. More generally, we can ask the following question.

Question 5.13. *Under what assumptions on \mathcal{M} or $\widetilde{\mathcal{M}}$ does the Choice Property hold?*

There are other ways in which one could try to improve the Structure Theorem. In general, a supercone $J \subseteq M^n$ does not contain a product of supercones in M . For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields and $J \subseteq M^2$ with

$$J = \bigcup_{a \in M} \{a\} \times (M \setminus aP).$$

It is natural to ask whether J contains an image of such product under \mathcal{L} -definable map. More generally, one could ask the following question.

Question 5.14. *Would the Structure Theorem remain true if we defined:*

- (1) *supercones in M^k to be products $J_1 \times \dots \times J_k$, where each J_i is a supercone in M ?*
- (2) *k -cones to be of the form $h(S \times J)$? (That is, h and S are as before, but $J_g = J$ in Definition 4.3 is fixed.)*

In subsequent work [18], we refute both questions, showing that our definitions and Structure Theorem are optimal in yet another way.

6. LARGE DIMENSION VERSUS scl-DIMENSION

In this section we use our Structure Theorem to establish the equality of the large dimension with the ‘scl-dimension’ arising from a relevant pregeometry in [3]. In Section 7 we use this equality to set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$.

We start by quoting [3, Definition 28], which was given independently from, and in complete analogy with, [17, Definition 5.2].

Definition 6.1. The *small closure* operator $\text{scl} : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ is defined by:

$$a \in \text{scl}(A) \Leftrightarrow a \text{ belongs to an } A\text{-definable small set.}$$

In [3] scl was shown to define a pregeometry under certain assumptions (in addition to their basic tameness conditions). We show that in the current context scl always defines a pregeometry. This follows from the first equality below, which is proved using only results from Section 3. In the interests of completeness, we also prove a second equality, using the Structure Theorem. Recall that $\text{dcl}(A)$ denotes the usual definable closure of A in the o-minimal structure \mathcal{M} .

Lemma 6.2. $\text{scl}(A) = \text{dcl}(P \cup A) = \text{dcl}_{\mathcal{L}(P)}(P \cup A)$.

Proof. $\text{scl}(A) \subseteq \text{dcl}(P \cup A)$. Let $b \in \text{scl}(A)$. Then there are an $\mathcal{L}(P)$ -formula $\varphi(x, y)$ and $a \in A^l$, such that $\varphi(\mathcal{M}, a)$ is small and contains b . Consider the \emptyset -definable family $\{\varphi(\mathcal{M}, t)\}_{t \in M^l}$. By Remark 3.4(a), the set I consisting of all $t \in M^l$ such that $\varphi(\mathcal{M}, t)$ is small is \emptyset -definable. Of course, I contains a . By Corollary 3.5, there is an \mathcal{L}_\emptyset -definable function $h : M^{m+l} \rightarrow M$ such that for all $t \in I$, $\varphi(\mathcal{M}, t) \subseteq h(P^m, t)$. Therefore $b \in h(P^m, a)$, and $b \in \text{dcl}(P \cup A)$.

$\text{scl}(A) \supseteq \text{dcl}(P \cup A)$. Let $b \in \text{dcl}(P \cup A)$. Then there is an \mathcal{L}_\emptyset -definable $h : M^{m+l} \rightarrow M$ and $a \subseteq A^l$ such that $b \in h(P^l, a)$. But the latter set is small, hence $b \in \text{scl}(A)$.

$\text{dcl}(P \cup A) = \text{dcl}_{\mathcal{L}(P)}(P \cup A)$. It suffices to show $\text{dcl}_{\mathcal{L}(P)}(P \cup A) \subseteq \text{dcl}(P \cup A)$. Let $b = f(a)$, where $a \subseteq P \cup A$ and f is \emptyset -definable. By Structure Theorem, there is a \emptyset -definable cone $h(\mathcal{J})$, where h is \mathcal{L}_\emptyset -definable, containing a on which f is fiber \mathcal{L}_\emptyset -definable. Denote $S = \pi(\mathcal{J})$. Let $g \in S$ and $t \in J_g$ be so that $a = h(g, t)$. Since $h(g, -) : M^k \rightarrow M^n$ is \mathcal{L}_g -definable and injective, $t \in \text{dcl}(P \cup A \cup S)$. Moreover, S is P -bound over \emptyset (Lemma 3.11) and hence $t \in \text{dcl}(A \cup P)$. Since $fh(g, -)$ agrees with an $\mathcal{L}_{A \cup P}$ -definable map on J_g , it follows that

$$b = f(h(g, t)) \in \text{dcl}(A \cup P).$$

□

Remark 6.3. In general $\text{dcl}(P \cup A) \neq \text{dcl}_{\mathcal{L}(P)}(A)$. For example, let $\langle \mathcal{M}, \mathcal{N} \rangle$ be a dense pair of real closed fields and let \mathcal{N}_0 be a real closed subfield of \mathcal{N} . Then $\text{dcl}_{\mathcal{L}(P)}(\mathcal{N}_0) = \mathcal{N}_0$ by [12, Lemma 3.2].

The following corollary is then immediate.

Corollary 6.4. *The small closure operator scl defines a pregeometry.*

Definition 6.5. Let $A, B \subseteq M$. We say that B is *scl-independent over A* if for all $b \in B$, $b \notin \text{scl}(A \cup (B \setminus \{b\}))$. A maximal scl-independent subset of B over A is called a *basis for B over A* .

By the Exchange property for scl , any two bases for B over A have the same cardinality. This allows us to define the *rank of B over A* :

$$\text{rank}(B/A) = \text{the cardinality of any basis of } B \text{ over } A.$$

In complete analogy with the corresponding fact for acl in a pregeometric theory, we can prove:

Lemma 6.6. *If p is a partial type over $A \subseteq M$ and $a \models p$ with $\text{rank}(a/A) = m$, then for any set $B \supseteq A$ there is $a' \models p$ such that $\text{rank}(a'/B) \geq m$.*

Proof. The proof of the analogous result for the rank coming from acl in a pregeometric theory is given, for example, in [24, page 315]. The proof of the present lemma is word-by-word the same with that one, after replacing an ‘algebraic formula’ by a ‘formula defining a small set’ in the definition of Φ_B^m ([24, Definition 2.2]) and the notion of ‘algebraic independence’ by that of ‘ scl -independence’ we have here. \square

It follows that the corresponding dimension of partial types and definable sets is well-defined and independent of the choice of the parameter set.

Definition 6.7. Let p be a partial type over $A \subset M$. The *scl -dimension of p* is defined as follows:

$$\text{scl-dim}(p) = \max\{\text{rank}(\bar{a}/A) : \bar{a} \subset M \text{ and } \bar{a} \models p\}.$$

Let X be a definable set. Then the *scl -dimension of X* , denoted by $\text{scl-dim}(X)$ is the dimension of its defining formula.

We next prove the equivalence of the scl -dimension and large dimension of a definable set. First, by a standard routine, using the saturation of \widehat{M} , we observe the following fact about supercones.

Fact 6.8. *Let $J \subseteq M^k$ be an A -definable supercone. Then J contains a tuple of rank k over A .*

Proposition 6.9. *For every definable $X \subseteq M^n$.*

$$\text{ldim}(X) = \text{scl-dim}(X).$$

Proof. We may assume that X is \emptyset -definable.

\leq . Let $f : M^k \rightarrow M^n$ be an \mathcal{L} -definable injective function and $J \subseteq M^k$ a supercone, such that $f(J) \subseteq X$. Suppose both f and J are defined over A . We need to show that $f(J)$ contains a tuple b with $\text{rank}(b/\emptyset) \geq k$. By Fact 6.8, J contains a tuple a of rank k over A . Let $b = f(a)$. Since f is injective, we have $a \in \text{dcl}(Ab)$ and $b \in \text{dcl}(Aa)$. In particular, $a \in \text{scl}(Ab)$ and $b \in \text{scl}(Aa)$. So a and b have the same rank over A . Hence,

$$\text{rank}(b/\emptyset) \geq \text{rank}(b/A) = \text{rank}(a/A) = k.$$

\geq . Let $b \in X$ be a tuple of rank k . By the Structure Theorem, b is contained in some l -cone $C \subseteq X$. We prove that $l \geq k$. Let $C = h(\mathcal{J})$, where \mathcal{J} is a uniform family of supercones in M^l . Suppose $b = h(g, a)$, for some $g \in \pi(\mathcal{J})$ and $a \in J_g$. Since $h(g, -)$ is \mathcal{L}_g -definable and injective, we have $a \in \text{dcl}(gb)$ and $b \in \text{dcl}(ga)$. In particular, $a \in \text{scl}(gb)$ and $b \in \text{scl}(ga)$. Hence a and b have the same rank over g . But $a \in J \subseteq M^l$ and, hence,

$$k = \text{rank}(b/g) = \text{rank}(a/g) \leq l.$$

\square

We next record several properties of the rank and large dimension, for future reference. By dcl-rank we denote the usual rank associated to dcl.

Lemma 6.10. *For every $a \in M$ and $A \subseteq M$, we have*

- (1) $\text{scl}(A \cup P) = \text{scl}(A)$
- (2) $\text{rank}(a/AP) = \text{rank}(a/A) = \text{dcl-rank}(a/AP)$.

Proof. Immediate from Lemma 6.2 and the definitions. \square

Lemma 6.11. *Let X, Y, X_1, \dots, X_k be definable sets. Then:*

- (1) $\text{ldim}(X) \leq \dim(\text{cl}(X))$. Hence, if X is \mathcal{L} -definable, $\text{ldim} X = \dim X$.
- (2) $X \subseteq Y \subseteq M^n \Rightarrow \text{ldim}(X) \leq \text{ldim}(Y) \leq n$.
- (3) X is small if and only if $\text{ldim}(X) = 0$.
- (4) If C is a k -cone, then $\text{ldim}(C) = k$.
- (5) $\text{ldim}(X_1 \cup \dots \cup X_l) = \max\{\text{ldim}(X_1), \dots, \text{ldim}(X_l)\}$.
- (6) $\text{ldim}(X \times Y) = \text{ldim}(X) + \text{ldim}(Y)$.

Proof. (1). Assume X is A -definable and let $a \in X$ with $\text{rank}(a/A) = \text{ldim}(X)$. Since $a \in \text{cl}(X)$, we have

$$\text{ldim}(X) = \text{rank}(a/A) = \text{dcl-rank}(a/A \cup P) \leq \text{dcl-rank}(a/A) \leq \dim \text{cl}(X).$$

Now, if X is \mathcal{L} -definable, $\text{ldim}(X) \leq \dim \text{cl}(X) = \dim X$. On the other hand, if $\dim X = k$, one can \mathcal{L} -definably embed a k -box in X which of course is a k -cone.

(2)-(5) were proved in Section 4, and (6) is by virtue of scl defining a pregeometry. \square

6.1. scl-generics. For a treatment of the classical notion of dcl-generic elements, see, for example, [39]. Here we introduce the corresponding notion for scl.

Definition 6.12. Let $X \subseteq M^n$ be an $A \cup P$ -definable set, and let $a \in X$. We say that a is a *scl-generic element of X over A* if it does not belong to any A -definable set of large dimension $< \text{ldim}(X)$. If $A = \emptyset$, we call a a *scl-generic element of X* .

By saturation, *scl-generic elements always exist*. More precisely, every $A \cup P$ -definable set X contains an scl-generic element over A . Indeed, by Compactness and Lemma 6.11(5), the collection of all formulas which express that x belongs to X but not to any A -definable set of large dimension $< \text{ldim}(X)$ is consistent.

Two scl-generics are called *independent* if one (each) of them is scl-generic over the other. The facts that scl defines a pregeometry and that the scl-dim agrees with ldim imply:

Fact 6.13. *Let $G = \langle G, * \rangle$ be a \emptyset -definable group. If $a, b \in G$ are independent scl-generics, then so are a and $a * b^{-1}$.*

Proof. We have

$$\text{rank}(b/a) = \text{rank}(a * b^{-1}/a).$$

So if b is scl-generic over a , then so is $a * b^{-1}$. \square

Note that none of the notions ‘dcl-generic element’ and ‘scl-generic element’ implies the other, but, by Lemma 6.10, if X is $A \cup P$ -definable and $a \in X$, we have:

$$a \text{ is scl-generic over } A \cup P \Leftrightarrow a \text{ is scl-generic over } A \Leftrightarrow a \text{ is dcl-generic over } A \cup P.$$

7. DEFINABLE GROUPS

In this section we obtain our main application of the Structure Theorem. We fix a \emptyset -definable group $G = \langle G, *, 0_G \rangle$ with $G \subseteq M^n$ and $\text{ldim}(G) = k$ and prove a local theorem for G : around scl-generic elements the group operation is given by an \mathcal{L} -definable map.

A convention on terminology. When we say that $h(J)$ is a k -cone, we mean that there is a k -cone $h'(\mathcal{J})$ and $g \in \pi(\mathcal{J})$, such that $J = J_g$ and $h(-) = h'(g, -)$. We call $h(J)$ $A \cup P$ -definable, if $h'(\mathcal{J})$ is A -definable. Likewise, when we say that $\mathcal{T} = \{\tau_t(J_t)\}_{t \in X}$ is a uniform family of k -cones, we mean that there is a uniform family $\mathcal{C} = \{C_t\}_{t \in X}$ of k -cones as in Definition 4.11 and $g \in \bigcap_t S_t$, such that for every $t \in X$, $J_t = Y_{t,g}$ and $\tau_t(-) = h(t, g, -)$. We call \mathcal{T} $A \cup P$ -definable if \mathcal{C} is A -definable. We write $\mathcal{T} = \{\tau(J_t)\}_{t \in X}$, if for all t, s , we have $\tau_t = \tau_s$.

Lemma 7.1. *Let $\{C_t = \tau_t(J_t)\}_{t \in \Gamma}$ be a uniform family of k -cones in M^n and $\Gamma \subseteq M^m$ a k' -cone. Then*

$$C = \bigcup_{t \in \Gamma} \{t\} \times C_t$$

is a $k' + k$ -cone.

Proof. Assume $\Gamma = \tau(\mathcal{I})$, where $\mathcal{I} = \bigcup_{s \in S} \{s\} \times I_s$ and $S \subseteq M^p$, and for every $t \in \Gamma$,

$$C_t = h(t, g, Y_{t,g}),$$

for some fixed $g \in \bigcap_t S_t$, and $h, \{Y_{t,g}\}_{t \in \Gamma}$ as in Definition 4.11. We define

$$h' : Z \subseteq M^{p+k'+k} \rightarrow M^{m+n} : (s, x, y) \mapsto (\tau(s, x), h(\tau(s, x), g, y)),$$

for a suitable Z , and, for every $s \in S$,

$$J_s = \bigcup_{x \in I_s} \{x\} \times Y_{\tau(s, x), g}.$$

The reader can verify that

$$C = h' \left(\bigcup_{s \in S} \{s\} \times J_s \right)$$

is a $k' + k$ -cone, as required. \square

Lemma 7.2. *Let $h(J)$ be a k -cone, and $\{D_t\}_{t \in \Gamma}$ a definable family of sets, such that for each $t \in \Gamma$, $\text{ldim}(D_t) = k$ and $D_t \subseteq h(J)$. Then there is a uniform definable family of k -cones $\{C_t = h(Y_t)\}_{t \in \Gamma}$ with $C_t \subseteq D_t$.*

Proof. This follows from a uniform version of Theorem 5.7(1), which can be proved easily via a standard compactness argument. Indeed, for every $t \in \Gamma$, let $X_t = h^{-1}(D_t) \subseteq J$. So $\text{ldim}(X_t) = k$. By the uniform Theorem 5.7(1), we can find a uniform family of supercones $Y_t \subseteq X_t$. Then $C_t = \{h(Y_t)\}_{t \in \Gamma}$ is as required. \square

Lemma 7.3. *Let $X \subseteq M^n$ be a \emptyset -definable set of large dimension k , (a, b) an scl-generic element of $X \times X$, and $D \subseteq X \times X$ a \emptyset -definable $2k$ -cone containing (a, b) . Then there is a P -definable uniform family of k -cones $\{E_t = \tau_t(J_t)\}_{t \in T}$, where T is a k -cone containing a , such that $b \in \bigcap_{t \in T} \text{cl}(E_t)$ and*

$$(a, b) \in \bigcup_{t \in T} \{t\} \times E_t \subseteq D.$$

Proof. By Corollary 5.4, and since (a, b) is scl-generic of $X \times X$, it is contained in a \emptyset -definable set of the form

$$\bigcup_{t \in \Gamma} \{t\} \times C_t \subseteq D,$$

where $\Gamma \subseteq X$ is a cone and there is l such that $\{C_t\}$ is an \emptyset -definable uniform family of l -cones contained in X . Write

$$C_t = h \left(\{t\} \times \left(\bigcup_{g \in S_t} \{g\} \times Y_{t,g} \right) \right),$$

as in Definition 4.11 where $h : Z \rightarrow M^n$. Since $a \in \Gamma \subseteq X$ and a is a scl-generic element of X , Γ must be a k -cone. Thus there is a supercone $J_0 \subseteq M^k$ and an \mathcal{L}_P -definable, continuous and injective map $f : U \subseteq M^k \rightarrow M^n$ such that $f(J_0) = \Gamma$. Let $\hat{a} \in M^k$ such that $f(\hat{a}) = a$. Because (a, b) is an scl-generic element of $X \times X$, \hat{a} is scl-generic over b . Since $b \in C_a \subseteq X$ and b is a scl-generic element of X over a , C_a must be a k -cone, and hence $l = k$. Fix $g \in S_a$ such that $b \in h(a, g, Y_{a,g})$. Because \hat{a} is scl-generic over b , there is an open box $B \subseteq M^k$ containing \hat{a} such that $b \in cl(h(f(x), g, Z_{f(x),g}))$ for every $x \in B$. By density of P we can assume that B is \mathcal{L}_P -definable. By Lemma 4.16, $J_0 \cap B$ is a supercone. Hence

$$(a, b) \in \bigcup_{t \in f(J_0 \cap B)} \{t\} \times h(t, g, Y_{t,g})$$

and $b \in \bigcap_{t \in f(J_0 \cap B)} cl(h(t, g, Y_{t,g}))$. Set $E_t = h(t, g, Y_{t,g})$. \square

Remark 7.4. In general, there is no $\{E_t\}_{t \in T}$ as above so that $b \in \bigcap_{t \in T} E_t$. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, $X = M$

$$D = \bigcup_{c \in M} \{c\} \times (M \setminus cP),$$

and (a, b) any element of D .

Corollary 7.5. *Let $X \subseteq M^n$ be a \emptyset -definable set of large dimension k . Let (a, b) be an scl-generic element of $X \times X$ and $f : X \times X \rightarrow X$ a \emptyset -definable function. Then there is a P -definable uniform family of k -cones $\{E_t = \tau_t(J_t)\}_{t \in T}$, where T is a k -cone containing a , such that $b \in \bigcap_{t \in T} cl(E_t)$ and f agrees with an \mathcal{L}_P -definable continuous map on*

$$E = \bigcup_{t \in T} \{t\} \times E_t.$$

Proof. By the Structure Theorem, there is a \emptyset -definable $2k$ -cone $D \subseteq G \times G$ that contains (a, b) and such that f agrees with an \mathcal{L}_P -definable continuous map on D . The statement then follows from Lemma 7.3. \square

We are now ready to prove the local theorem for definable groups.

Theorem 7.6 (Local theorem for definable groups). *Let a be an scl-generic element of G . Then there is a $2k$ -cone $C \subseteq G \times G$, whose closure contains (a, a) , and an \mathcal{L} -definable continuous map $F : Z \subseteq M^n \times M^n \rightarrow M^n$, such that for every $(x, y) \in C$,*

$$x * a^{-1} * y = F(x, y).$$

Moreover, F is a homeomorphism in each coordinate.

Proof. Let $a_1 \in G$ be scl-generic over a , and let $a_2 = a_1^{-1} * a$. By Fact 6.13, a, a_1, a_2 are pairwise independent. By the Structure Theorem, for $i = 1, 2$, there is a Pa_i -definable k -cone $C_i = h_i(J_i) \subseteq G$ containing a , and \mathcal{L}_{Pa_i} -definable continuous $f_i : Z_i \subseteq M^n \rightarrow M^n$ such that for every $x \in C_1$,

$$x * a_2^{-1} = f_2(x)$$

and for every $y \in C_2$,

$$a_1^{-1} * y = f_1(y).$$

Observe that $f_2(a) = a_1$ and $f_1(a) = a_2$.

We now look at the independent scl-generic elements a_1 and a_2 . By Corollary 7.5, there is a P -definable uniform family of k -cones $\{E_t = \tau_t(J_t)\}_{t \in T}$ in G , where $T \subseteq G$ is a k -cone containing a_1 and $a_2 \in \bigcap_{t \in T} cl(E_t)$, such that $*$ agrees with an \mathcal{L}_P -definable continuous map $f : Z \subseteq M^n \times M^n \rightarrow M^n$ on

$$E = \bigcup_{t \in T} \{t\} \times E_t.$$

Observe that (a, a_i) is also scl-generic of $G \times G$. Moreover, since a_2 is dcl-generic of G over P , there is an \mathcal{L}_P -definable B of dimension k with

$$a_2 \in B \subseteq \bigcap_{t \in T} cl(E_t).$$

Claim. *For every $t \in T$, $f_1^{-1}(E_t) \cap h_1(J_1)$ has large dimension k .*

Proof of Claim. Let $F_t = f_1^{-1}\tau_t$. Since a belongs to the \mathcal{L}_{Pa_1} -definable set $f_1^{-1}(B) \cap h_1(cl(J_1))$ and it is scl-generic over a_1 , the set

$$f_1^{-1}(B) \cap h_1(cl(J_1)) \subseteq f_1^{-1}(cl(\tau_t(J_t)) \cap h_1(cl(J_1)))$$

has dimension k . This implies that $F(cl(J_t) \cap h_1(cl(J_1)))$ has dimension k . By Lemma 5.6, $f_1^{-1}(E_t) \cap h_1(J_1) = F(J_t) \cap h_1(J_1)$ has large dimension k . \square

Now, since a belongs to the Pa_2 -definable set $f_2^{-1}(T) \cap h_2(J_2)$ and it is scl-generic over a_2 , it must also belong to a Pa_2 -definable k -cone

$$\Gamma \subseteq f_2^{-1}(T) \cap h_2(J_2).$$

For every $t \in \Gamma$, we let

$$D_t = f_1^{-1}(E_{f_2(t)}) \cap h_1(J_1).$$

By Claim, $\text{ldim}(D_t) = k$. Since every $D_t \subseteq h_1(J_1)$, by Lemma 7.2, we can find a uniform definable family of k -cones

$$C_t = h_1(Y_t) \subseteq D_t, \quad t \in \Gamma,$$

where $Y_t \subseteq J_1$ is a supercone in M^k , and $a \in \bigcap_{t \in \Gamma} C_t$. By Lemma 7.1, the set

$$C = \bigcup_{t \in \Gamma} \{t\} \times C_t$$

is a $2k$ -cone. We can now conclude as follows. For every $(x, y) \in C$,

$$x * a^{-1} * y = (x * a^{-1} * a_1) * (a_1^{-1} * y) = f_2(x) * f_1(y) = f(f_2(x), f_1(y)).$$

Set

$$F(x, y) = f(f_2(x), f_1(y)) : M^n \times M^n \rightarrow M^n.$$

For the “moreover” clause, we need to check that (a) each f_i can be chosen to be a homeomorphism, and (b) f can be chosen to be a homeomorphism in each coordinate.

The former fact follows from the scl-genericity of a over each a_i and the injectivity of each $x \mapsto x * a_i^{-1}$, and the latter fact from the scl-genericity of (a_1, a_2) and the injectivity of $*$ in each coordinate. \square

Remark 7.7. We observe that we cannot always have $C = C' \times C''$, where C', C'' are k -cones containing a . For example, consider the group $\mathcal{H} = \langle H = [0, 1], + \bmod 1 \rangle$ in the real field, and let $T = \mathbb{Q}^{rc} \cap H$. Now let $g : H \rightarrow M$ be the translation $x \mapsto 2 + x$ on T , and identity elsewhere. Let G be the induced group on $(H \setminus T) \cup g(T)$. Clearly, G is definable in $\widetilde{M} = \langle \mathbb{R}, \mathbb{Q}^{rc} \rangle$, and it is easy to verify that the above observation holds for every $a \in G$. Of course, the conclusion of Theorem 7.6 holds for every $a \in H \setminus T$, by letting $\Gamma = H \setminus T$, $C_t = H \setminus (T \cup (T - t))$ and $f = + \bmod 1$. Moreover, we can achieve $C = C' \times C''$, but only up to definable isomorphism. It is reasonable to ask whether that is always true, and we include some relevant (in fact, stronger) questions at the end of this section.

We expect that the above local theorem will play a crucial role in forthcoming analysis of groups definable in \widetilde{M} . The ultimate goal would be to understand definable groups in terms of \mathcal{L} -definable groups and small groups. Motivated by the successful analysis of semi-bounded groups in [21] and the more recent [4], we conjecture the following statement.

Conjecture 7.8. *Let $\langle G, * \rangle$ be a definable group. Then there is a short exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{U} & \longrightarrow & K \longrightarrow 0 \\ & & & & \downarrow \tau & & \\ & & & & G & & \end{array}$$

where

- \mathcal{U} is \mathbb{V} -definable
- \mathcal{B} is \mathbb{V} -definable in \mathcal{L} with $\dim(\mathcal{B}) = \text{l dim}(G)$.
- K is definable and small
- $\tau : \mathcal{U} \rightarrow G$ is a surjective group homomorphism and
- all maps involved are \mathbb{V} -definable.

The conjecture is in a certain sense optimal: we next produce an example of a definable group G which is *not* a direct product of an \mathcal{L} -definable group by a small group. Using known examples of \mathcal{L} -definable groups B from [36, 43], which are not direct products of one-dimensional subgroups, it would be easy to provide such an G by restricting some of the one-dimensional subgroups of the universal cover of B to the subgroup P (say, in a dense pair). Our example below, however, is not constructed in this way, as it is *not* a subgroup of the examples in [36, 43].

Example 7.9. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle \models T^d$. Let $G = \langle P \times [0, 1), \oplus, 0 \rangle$, where $x \oplus y = x + y \bmod (1, 1)$; that is,

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y \in P \times [0, 1) \\ x + y - (1, 1), & \text{otherwise} \end{cases}$$

Then G is clearly not small. But it cannot contain any non-trivial \mathcal{L} -definable subgroup. Indeed, by o-minimality, every \mathcal{L} -definable subset of $P \times [0, 1)$ must be contained in a finite union of fibers $\{g\} \times [0, 1)$, $g \in P$. On the other hand, an \mathcal{L} -definable

subgroup of G is a topological group containing some \mathcal{L} -definable neighborhood of 0 and, thus, also every fiber $\{n\} \times [0, 1)$, $n \in \mathbb{Z}$.

The reader can verify that for $\mathcal{B} = \text{Fin}(M)$, $K = P$, $\mathcal{U} = \mathcal{B} \times K$ and $\tau(x, y) = (x, y) \bmod (1, 1)$, we obtain the diagram of Conjecture 7.8.

Finally, observe that G is a subgroup of the \mathcal{L} -definable group B , which is the direct product $B = S \times \langle M, + \rangle$, where S has domain $\{(x, x) : 0 \leq x < 1\}$ and operation $(x, y) \mapsto x + y \bmod (1, 1)$.

We finish with some open questions which we expect our local theorem to have an impact on.

Question 7.10. *Does G , up to definable isomorphism, contain an \mathcal{L} -definable local subgroup (in the sense of [42, §23 (D)]) whose dimension equals $\text{ldim} G$?*

Question 7.11. *Assume $\text{ldim} G = \dim \text{cl}(G)$. Is G , up to definable isomorphism, \mathcal{L} -definable?*

If Conjecture 7.8 is true, it would be nice to know what the small groups are.

Question 7.12. *Is every small definable group/set definably isomorphic to a group/set definable in the induced structure on P ?*

Question 7.13. *Is G , up to definable isomorphism, a subgroup of an \mathcal{L} -definable group (whose dimension might be bigger than $\text{ldim}(G)$)?*

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