

Definable groups

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Introduction

o-Minimal expansion of a real closed field

$$\mathcal{R} := \langle \mathbb{R}, <, +, \cdot, \dots \rangle \qquad K := R(i)$$

Definable := definable (with parameters) in \mathcal{R} .

Definable group: A group G ,

$G \subseteq R^k$ & graph $\Gamma(\cdot) \subseteq R^{3k}$ definable.

Examples

- Algebraic subgroups of $GL(n, K)$.
- Semialgebraic groups: $(R^{>0}, \cdot)$;
- Groups definable in \mathbb{R}_{exp} , e.g.: $\left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{R} \right\}$
(Peterzil-Pillay-Starchenko(2002)).

dim=1

- $[0, 1) \subseteq \mathbb{R}$, addition mod 1.
- $\mathbb{T} := \{a + bi \in \mathbb{C} : a^2 + b^2 = 1\} \leq \mathbb{C}^*$.

$[0, 1) \not\cong \mathbb{T}$ semialgebraically.

(Mamino 2011): IF

$$\phi : [0, 1) \rightarrow \mathbb{T}$$

is a nontrivial definable homomorphism. WMA ϕ is 0-definable & everywhere differentiable.

$$\lim_{x \rightarrow 1} \phi(x) = \phi(0) = 1$$

$$\phi'(x) = \phi(x)\phi'(0)$$

$$\therefore \phi(x) = e^{x\phi'(0)} \text{ and } 1 = \lim_{x \rightarrow 1} \phi(x) = e^{\phi'(0)}.$$

$$\therefore \phi'(0) = 2k\pi i \text{ for some } k \in \mathbb{Z}, k \neq 0,$$

$$\therefore \pi \text{ is 0-definable, CONTRADICTION.}$$

Theorem (Pillay1988)

G d.group $\implies G$ d.manifold & multiplication and inversion continuous.

Remarks

- 1 If $R = \mathbb{R}$, a definable group is a Lie group.
- 2 Not every Lie group is definable in an o-minimal expansion of the real field.

Topological group \implies regular space.

By *Robson's embedding theorem*:

$$G \hookrightarrow R^m.$$

\therefore the topology of (the image of) G is induced by that of R^m .

Corollary

G d.group.

- ① $H \leq G$, H definable $\implies H$ closed.
- ② G infinite $\implies \exists H \leq G$, H definable infinite abelian.
- ③ IF $H \leq G$ definable THEN:
 H open $\iff [G : H]$ finite $\iff \dim H = \dim G$.
- ④ G^0 : definably connected component of the identity.
 G^0 = smallest d.subgroup of G of finite index.

Definably connected group = d.group no proper d.subgroup of finite index.

Exercise 1. G d.group.

- 1 G has descending chain condition on d.subgroups (dcc).
- 2 For any $X \subseteq G$,

$$C_G(X) := \text{centralizer of } X \text{ in } G \leq G$$

is definable.

- 3 If G is definably connected. Then,
 - 1 $\dim G = 1 \implies G$ commutative.
 - 2 Any definable action of G on a finite set is trivial.
 - 3 $G' := \langle \{xyx^{-1}y^{-1} : x, y \in G\} \rangle$ finite $\implies G$ is abelian.

Any subset X of a d.group G is contained in a smallest d.subgroup

$$h(X) := \text{definable hull of } X \leq G.$$

Lemma

G d.group.

- ① $X \subseteq G$ set of commuting elements $\implies h(X) \leq G$ is abelian.
- ② $X \trianglelefteq G \implies h(X) \trianglelefteq G$.

Proof.

1. $Z(C_G(X))$ is abelian d. and contains X

$$\therefore h(X) \subseteq Z(C_G(X)).$$

2. For any $g \in G$, $X^g = X$

$$\therefore X \subseteq h(X) \cap h(X)^g \quad \therefore h(X) = h(X) \cap h(X)^g$$

$$\therefore h(X) \leq h(X)^g, \text{ for any } g \in G$$

$$\therefore h(X) = h(X)^g.$$



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Euler characteristic and torsion

$X \subseteq R^k$ definable set. Partition of X into cells: $X = \bigsqcup_{C \in \mathcal{D}_X} C$.

The (model theoretic) **Euler characteristic** of X is

$$E(X) := \sum_{C \in \mathcal{D}_X} (-1)^{\dim(C)}.$$

Remark.

- ❶ X finite $\implies E(X) = \text{card}(X)$.
- ❷ X triangulated: $E(X) = \#\text{vertices} - \#\text{edges} + \#\text{faces} - \dots$
- ❸ $R = \mathbb{R}$ and X is closed and bounded $\implies E(X) = \chi(X)$.
BUT $\chi((a, b)) = \chi(\text{point}) = 1 = E(\text{point}) \neq E((a, b)) = -1$.

Theorem

- ① X and Y disjoint $\implies E(X \cup Y) = E(X) + E(Y)$.
- ② $E(X \times Y) = E(X)E(Y)$.
- ③ $f : X \rightarrow Y$ definable & $E(f^{-1}(y)) = m$, for each $y \in Y \implies$

$$E(X) = E\left(\bigcup_{y \in Y} f^{-1}(y)\right) = E(Y) \cdot m.$$

- ④ $f : X \rightarrow Y$ definable bijection $\implies E(X) = E(Y)$.

Example

$E(\mathbb{T}) = 0$: Write \mathbb{T} as a union of two 0-cells and two 1-cells, then
 $E(\mathbb{T}) = 1 + 1 + (-1) + (-1)$.

Theorem (Strzebonski 1994)

G d.group.

- ① If $K \leq H \leq G$ definable then

$$E(G) = E(H)E(G/H)$$

and

$$E(G/K) = E(G/H)E(H/K).$$

- ② $p|E(G) \implies G$ has an element of order p , p prime.

Proof.

1. By definable choice $\exists f : G \rightarrow H \times (G/H)$ definable bijection.

Proof (cont.)

2 (If $p|E(G)$ then G has an element of order p). Action of $\mathbb{Z}/p\mathbb{Z}$ on

$$X = \{(a_0, \dots, a_{p-1}) \in G^p : a_0 \cdots a_{p-1} = 1\}$$

by cyclic permutations. The orbit of $x \in X$ has either 1 element or p elements.

$$X = \bigcup_{|orb(x)|=1} orb(x) \cup \bigcup_{|orb(x)|=p} orb(x).$$

$X \rightarrow G^{p-1}$ definable bijection $\implies E(X) = E(G)^{p-1}$ is divisible by p ,

$$E\left(\bigcup_{|orb(x)|=p} orb(x)\right) = pE(\{orb(x) : |orb(x)| = p\}),$$

$\therefore p$ divides $E(\bigcup_{|orb(x)|=1} orb(x))$.

$\therefore \exists a \neq 1$ S.T. $x = (a, \dots, a) \in X$, I.E. $a^p = 1$.



Exercise 2

G d.group.

- 1 $E(G) = 0 \implies G$ has elements of order p , for each prime p .
- 2 $E(G) = \pm 1 \iff G$ is torsion-free.
- 3 G torsion-free $\implies G$ definably connected.
- 4 Quotients of torsion-free d.groups are torsion-free.

Theorem (Strzebonski 1994)

G infinite d.group. Then,

- 1 $\nexists n \forall x \in G \quad |\langle x \rangle| \leq n$, I.E. *G does not have bounded exponent;*
- 2 *G abelian \implies the torsion subgroup $G[m]$ is finite, for each $m > 0$.*

Exercise 3.

G abelian d.connected group \implies divisible.

A definable group G is **definably compact** if it is closed and bounded.

Theorem (Peterzil-Steinhorn (1999))

G *d.group*.

G *NOT d.compact* $\implies \exists H \leq G$ definable, $\dim H = 1$ & H torsion-free.

Theorem (Edmundo-O_ (2004))

G *d.connected d.compact abelian group* \implies for each $m > 0$, the torsion subgroups

$$G[m] \cong (\mathbb{Z}/m\mathbb{Z})^{\dim G}.$$

Exercise 4.

- ① G d.connected d.compact group $\implies E(G) = 0$,
 $\therefore G$ has p -torsion for each prime p .
- ② G abelian d.group $\implies \exists$ d.subgroups

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n \leq G$$

S.T.

- ① G/G_n is d.compact,
- ② G_{i+1}/G_i is a torsion-free one-dimensional group ($0 \leq i < n$).

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The Lie algebra of a definable group

Definition

$m \geq 0$. Definable C^m -manifold of dimension n :

- definable set: M
- *definable C^m -atlas* on M :

$$\{(U_i, \varphi_i)\}_{i=1}^s$$

$\varphi_i : U_i \rightarrow V_i$ definable bijection, $V_i \subseteq \mathbb{R}^n$ open S.T. the transition maps are d. C^m -maps.

- identify two d. C^m -atlas on M if their union is a d. C^m -atlas on M .

Remark

G d.group \implies for each $m \geq 0$, G is a definable C^m -group.

Definition

M d. C^1 -manifold, $a \in M$. Tangent space of M at a :

$$T_a(M) := \{ \overline{\alpha} \mid \alpha : [0, 1] \rightarrow M, \alpha(0) = a, \alpha \text{ d.}C^1\text{-map} \},$$

$$\overline{\alpha} = \overline{\beta} :\Leftrightarrow \alpha'(0) = \beta'(0).$$

$\varphi : U \rightarrow R^n$, $a \in U \subseteq M$ & identify $T_a(M)$ with R^m via $\overline{\alpha} \mapsto (\varphi \circ \alpha)'(a)$.

Definition

M, N d. C^m -manifolds, $f : M \rightarrow N$ d. C^m -map.

Differential of f at point $a \in M$:

$$d_a f : T_a M \rightarrow T_{f(a)} N : \overline{\alpha} \mapsto d_a f(\overline{\alpha}) := \overline{f \circ \alpha}.$$

Remark.

$\dim f(M) = \text{rk}(d_a f)$ for some $a \in M$.

Exercise 5.

$f : M \rightarrow N$ d. C^m -map.

- ❶ If M is d.connected, $d_a f = 0$ for all $a \in M \iff f$ is constant.
- ❷ f injective $\implies d_a f$ injective, for some $a \in M$.

Lemma

G & H d. C^m -groups, $f : G \rightarrow H$ d.homomorphism $\implies f$ d. C^m -map.

Proof.

$g \in G$.

$$L_g : G \rightarrow G : h \mapsto gh$$

d. C^m -map.

(V, ψ) of G .

$\therefore \exists V_0 \subseteq V$ open definable, $\exists (U, \varphi)$ of H S.T. $f(V_0) \subseteq U$.

$\therefore \exists c \in V_0$ S.T. $\varphi \circ f \circ \psi^{-1}$ is d. C^m at $\psi(c)$

$$\begin{array}{ccc} V_0 & \xrightarrow{f} & U \\ \psi^{-1} \uparrow & & \downarrow \varphi \\ R^n \supseteq \psi(V_0) & \xrightarrow{\varphi \circ f \circ \psi^{-1}} & \varphi(U) \subseteq R^n. \end{array}$$

$\therefore f$ is d. C^m at c .



Exercise 6.

G d.group \implies *unique* d. C^m -group structure.

Theorem

IF G d.connected, $f_1, f_2 : G \rightarrow G$ d. homomorphisms THEN

$$f_1 = f_2 \iff d_e f_1 = d_e f_2.$$

Proof.

$h \in G$,

$$f_i = L_{f_i(h)} \circ f_i \circ L_{h^{-1}} \quad (i = 1, 2).$$

$$\therefore d_h f_i = d_e L_{f_i(h)} \circ d_e f_i \circ d_h L_{h^{-1}}.$$

$$D := d_e f_1 = d_e f_2.$$

$\therefore f_1$ and f_2 solutions (neighbourhood of e) of differential equation

$$\left. \begin{aligned} d_x(\phi) &= F(x, \phi(x)) \\ \phi(e) &= e \end{aligned} \right\},$$

$$F(x, y) := d_e L_y D d_x L_{x^{-1}}.$$



Theorem (Peterzil-Pillay-Starchenko 2000)

IF G d.group, X d.set, $\alpha : G \curvearrowright X$ d.transitive.

THEN for every $m \geq 0$,

X and G are $d.C^m$ -manifolds

S.T.

- G $d.C^m$ -group,
- α $d.C^m$ -action.

Exercise 7.

$H \leq G$ definable \implies for every $m \geq 0$, G/H is a $d.C^m$ -manifold.

Lemma

$\alpha : G \curvearrowright X$ d.transitive C^m -action.

IF $x \in X$ and $G_x := \{g \in G : \alpha(g, x) = x\}$ THEN for each $g \in G$,
$$\text{rk}(d_g \alpha(-, x)) = \dim G - \dim G_x.$$

Proof.

$\dim \alpha(G, x) = \text{rk}(d_h \alpha(-, x))$, for some $h \in G$.

$\dim \alpha(G, x) = \dim X = \dim G / G_x = \dim G - \dim G_x$.

\therefore STP $\text{rk}(d_g \alpha(-, x))$ is constant on G .

$\alpha(h, x) = \alpha(g, \alpha(g^{-1}h, x))$, for all $h, g \in G$,

$\therefore \alpha(-, x) = \alpha(g, -) \circ \alpha(-, x) \circ L_{g^{-1}}$, for all $g \in G$.

$$d_g \alpha(-, x) = d_x \alpha(g, -) \circ d_e \alpha(-, x) \circ d_g L_{g^{-1}}.$$

$d_x \alpha(g, -)$ and $d_g L_{g^{-1}}$ are invertible

$\therefore \text{rk}(d_g \alpha(-, x)) = \text{rk}(d_e \alpha(-, x))$, for all $g \in G$.



Corollary

G d.group.

- ① IF $\alpha : G \curvearrowright X$ d.transitive C^m -action. $H \leq G$ d.connected.
THEN for each $x \in X$,

$$H \subseteq G_x \iff T_e H \subseteq \ker d_e \alpha(-, x).$$

$$T_e G_x = \ker d_e \alpha(-, x).$$

- ② IF $H_1, H_2 \leq G$ d.connected THEN $H_1 = H_2 \iff T_e H_1 = T_e H_2$
- ③ IF $f : G \rightarrow H$ is a d.homomorphism and $H_1 \leq H$ definable THEN

$$T_e f^{-1}(H_1) = (d_e f)^{-1}(T_e H_1).$$

$$T_e \ker f = \ker d_e f$$

- ④ $f \in \text{Aut}^{\text{def}}(G) \implies d_e f \in \text{Aut}(T_e(G)).$

Proof.

- ① $[H \subseteq G_x \Rightarrow T_e H \subseteq \ker d_e \alpha(-, x)]: H \subseteq G_x \Rightarrow \alpha(-, x)$ const. on H .
 $\therefore d_e \alpha(-, x) = 0$ on $T_e H$.
 $[\Leftarrow] \beta := \alpha|_{H \times X}$. $\beta : H \curvearrowright X$ d. C^m -action & $d_e \beta(-, x) = 0$
 $\therefore d_h \beta(-, x) = 0$ for all $h \in H$. $\therefore \beta(-, x)$ constant $\therefore H \subseteq G_x$.
 $[T_e G_x = \ker d_e \alpha(-, x)]: T_e G_x \subseteq \ker d_e \alpha(-, x)$.
 $\dim T_e G_x = \dim G_x = \dim G - \text{rk}(d_e \alpha(-, x)) =$
 $\dim G - (\dim T_e G - \dim \ker d_e \alpha(-, x))$.
- ② $[H_1 \subseteq H_2 \iff T_e H_1 \subseteq T_e H_2]: G \curvearrowright G/H_2$. Apply (1) $x = eH_2$.
- ③ $[T_e f^{-1}(H_1) = (d_e f)^{-1}(T_e H_1)]: \alpha : H \curvearrowright H/H_1 \Rightarrow \beta : G \curvearrowright H/H_1$,
 $\beta(g, hH_1) := \alpha(f(g), hH_1)$. $x := eH_1$.
 $\therefore G_x = f^{-1}(H_1)$
 $\therefore T_e f^{-1}(H_1) = \ker d_e \beta(-, x)$ (by (1)).
 $\beta(-, x) = \alpha(-, x) \circ f \implies d_e \beta(-, x) = d_e \alpha(-, x) \circ d_e f$.
 $\therefore \ker d_e \beta(-, x) = (d_e f)^{-1}(\ker d_e \alpha(-, x))$.
- ④ $[f \in \text{Aut}^{\text{def}}(G) \implies d_e f \in \text{Aut}(T_e(G))]:$ By (3).



Definition

\mathbb{K} ch.0. A **Lie algebra** over \mathbb{K} is a fin.dim. \mathbb{K} -vector space \mathfrak{h} & bilinear map $[-, -] : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$ S.T.

- ❶ $[x, x] = 0$;
- ❷ $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

\mathfrak{h} anticommutative: $[y, x] = -[x, y]$; non associative, if $[-, -] \neq 0$.

Example

A an associative \mathbb{K} -algebra (e.g. $M(n, \mathbb{K})$) $\implies \mathfrak{a} = (A, [-, -])$ with $[x, y] := xy - yx$ is a Lie algebra ($\mathfrak{gl}(n, \mathbb{K})$).

Morphism of Lie algebras: $f : \mathfrak{h}_1 \rightarrow \mathfrak{h}_2$ linear & $f([x, y]) = [fx, fy]$.
 $\text{Aut}(\mathfrak{h}) := \{f : \mathfrak{h} \rightarrow \mathfrak{h} \mid f \text{ bijection \& morphism of Lie algebras}\}.$

Definitions

\mathfrak{h} Lie algebra. $\mathfrak{h}_1, \mathfrak{h}_2$ subspaces of \mathfrak{h} .

- $[\mathfrak{h}_1, \mathfrak{h}_2] := \langle \{[x, y] : x \in \mathfrak{h}_1, y \in \mathfrak{h}_2\} \rangle_{\mathbb{K}}$.
- \mathfrak{h}_1 Lie subalgebra of \mathfrak{h} : $[\mathfrak{h}_1, \mathfrak{h}_1] \subseteq \mathfrak{h}_1$.
- \mathfrak{h}_1 ideal of \mathfrak{h} : $[\mathfrak{h}_1, \mathfrak{h}] \subseteq \mathfrak{h}_1$.
- \mathfrak{h} commutative: $[\mathfrak{h}, \mathfrak{h}] = 0$.
- \mathfrak{h} semisimple: no nontrivial commutative ideals.
- \mathfrak{h} simple: no nontrivial proper ideals.

Examples

- ① $\dim \mathfrak{h} = 1 \implies \mathfrak{h}$ commutative.
- ② $\dim \mathfrak{h} = 2 \implies \mathfrak{h} = \langle \{x, y\} \rangle_{\mathbb{K}}$ with $[x, y] := 0$ or $[x, y] := y$.
- ③ $sl(n, \mathbb{K}) := \{x \in gl(n, \mathbb{K}) : Tr(x) = 0\}$ semisimple.
- ④ $sl(n, \mathbb{K})$ ideal of $gl(n, \mathbb{K})$ and $[gl(n, \mathbb{K}), gl(n, \mathbb{K})] \subseteq sl(n, \mathbb{K})$.

G d.group. $g \in G$.

$$\text{Int}(g) : G \rightarrow G : h \mapsto ghg^{-1}$$

$$\text{Int}(g) \in \text{Aut}^{\text{def}}(G)$$

$$\therefore \text{Ad}(g) := d_e \text{Int}(g) : T_e G \rightarrow T_e G$$

$$\therefore \text{Ad}(g) \in \text{Aut}(T_e G).$$

$$\therefore \text{Ad} : G \rightarrow \text{Aut}(T_e G) : g \mapsto \text{Ad}(g)$$

$$\text{Ad d.homomorphism: } \text{Ad}(gh) = d_e \text{Int}(gh) = d_e(\text{Int}(g) \circ \text{Int}(h)) = d_e \text{Int}(g) d_e \text{Int}(h) = \text{Ad}(g) \text{Ad}(h).$$

$$\therefore \text{ad} := d_e \text{Ad} : T_e G \rightarrow T_e(\text{Aut}(T_e G)).$$

Identify $T_e(\text{Aut}(T_e G))$ with $\text{End}(T_e G)$ via

$$\bar{\alpha} \mapsto \alpha'(0) := \lim_{t \rightarrow 0} \frac{\alpha(t) - \alpha(0)}{t},$$

α takes values in $\text{Aut}(T_e G)$ and limit in $\text{End}(T_e G)$.

$$\boxed{\therefore ad : T_e G \rightarrow \text{End}(T_e G)}.$$

Definition

Lie algebra of a d.group G :

$$\mathfrak{g} := (T_e G, [-, -]) \quad [x, y] := ad(x)(y).$$

Exercise 8.

G d.group.

- 1 $ad : \mathfrak{g} \rightarrow \text{End}(T_e G)$ is a morphism of Lie algebras.
- 2 $H \leq G$ definable $\implies \mathfrak{h}$ is a Lie subalgebra of \mathfrak{g} .
- 3 $n = \dim G \implies G/Z(G) \cong H \leq GL(n, R)$ definably (identify $\text{Aut}(T_e(G))$ with $GL(n, R)$).

Definitions

\mathfrak{g} Lie algebra, \mathfrak{h} subspace of \mathfrak{g}

- **Centre of \mathfrak{g} :** $\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}$.
- **Centralizer of \mathfrak{h} in \mathfrak{g} :** $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\}$.
- **Normalizer of \mathfrak{h} in \mathfrak{g} :** $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$.

Exercise 9.

\mathfrak{g} Lie algebra, \mathfrak{h} subspace of \mathfrak{g}

- 1 $\mathfrak{z}(\mathfrak{g})$ ideal of \mathfrak{g} and Lie algebra of $Z(G)$ is $\mathfrak{z}(\mathfrak{g})$.
- 2 $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ are subalgebras of \mathfrak{g} .

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.group. \mathfrak{h} subspace of \mathfrak{g} .

①

$$f \in \operatorname{Aut}^{\operatorname{def}}(G) \implies d_e f \in \operatorname{Aut}(\mathfrak{g}).$$

② $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ is the Lie algebra of

$$C_G(\mathfrak{h}) := \{g \in G : \operatorname{Ad}(g)|_{\mathfrak{h}} = \operatorname{id}_{\mathfrak{h}}\}.$$

③ $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ is the Lie algebra of

$$N_G(\mathfrak{h}) := \{g \in G : \operatorname{Ad}(g)(\mathfrak{h}) \subseteq \mathfrak{h}\}.$$

Exercise 10

G d.connected group.

- ① G is commutative $\iff \mathfrak{g}$ is commutative, and
- ② if $H \leq G$ is a d.connected subgroup of G then,

H is normal in $G \iff \mathfrak{h}$ is an ideal of \mathfrak{g} .

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Definably compact subgroups

G connected LIE GROUP $\implies \exists K_1$ maximal compact subgroup
(unique up to conjugation) S.T.

$$G \approx K_1 \times \mathbb{R}^m.$$

$$G = SL(2, \mathbb{R}) \implies K_1 = SO(2, \mathbb{R}).$$

Example (Strzebonski 1994)

There are semialgebraic groups over \mathbb{R} which are not definably homeomorphic to any \mathbb{R}^m but they do not have a definably compact subgroup.

(See example 5.3 in Strzebonski, Adam W. Euler characteristic in semialgebraic and other o-minimal groups. J. Pure Appl. Algebra 96 (1994).)

Lemma

G d.group $\implies \exists N \trianglelefteq G$ definable & torsion-free S.T.

$$\forall H \trianglelefteq G \text{ definable \& torsion-free, } H \subseteq N$$

Proof.

Let $N \trianglelefteq G$ definable and torsion-free of maximal dimension.

$H \subseteq N$, for every $H \trianglelefteq G$ definable and torsion-free:

$$HN/N \stackrel{\text{def}}{\cong} H/(H \cap N) \text{ torsion-free} \quad \therefore E(HN/N) = \pm 1.$$

$$\therefore E(HN) = E(N)E(HN/N) = \pm 1.$$

$N, HN \trianglelefteq G$ definable torsion-free, $N \subseteq HN$ and N of maximal dimension,

$$\therefore \dim N = \dim HN.$$

Both d.connected $\therefore N = HN$ and $\therefore H \subseteq N$.



Theorem (Conversano 2014)

G d.connected group. $N \trianglelefteq G$ the unique maximal torsion-free.

$$\overline{G} := G/N.$$

THEN

- ① $\exists K_1 \leq \overline{G}$ maximal d.compact;
- ② K_1 is d.connected and unique up to conjugation;
- ③ $\exists H \leq \overline{G}$ definable torsion-free S.T.

$$\overline{G} = K_1 H \quad \& \quad K_1 \cap H = 1.$$

Exercise 11

- ① H is maximal torsion-free subgroup of \overline{G} .
- ② Preimage of H in G is a maximal torsion-free d.subgroup of G .
- ③ G d.group $\implies \exists H_1 \leq G$ maximal torsion-free definable,
 G NOT d.compact $\implies H_1 \neq \{1\}$.

Theorem (Peterzil-Starchenko 2005)

IF G d.group, $\dim G = n$ THEN

$$G \text{ torsion-free} \iff G \text{ d.diffeomorphic to } R^n.$$

Theorem (Conversano 2014)

IF G d.connected group THEN

- $\exists K_1 \leq G/N$ maximal d.compact
-

$$G \stackrel{\text{def}}{\approx} K_1 \times R^s,$$

d.homeomorphism, $s = \dim G - \dim K_1$.

Definable-torus T of d.group G : $T \leq G$ d.connected d.compact abelian.
 $SO(3, \mathbb{R})$: maximal tori $\cong SO(2, \mathbb{R})$.

Example (Peterzil-Steinhorn 1999)

T d.-torus of d.group $G \not\stackrel{def}{\cong} T_1 \times \cdots \times T_1$ & $\dim T_1 = 1$.
(See example 5.2 in Peterzil, Ya'acov; Steinhorn, Charles Definable compactness and definable subgroups of o-minimal groups. J. London Math. Soc. (2) 59 (1999), no. 3, 769–786.)

Theorem (Berarducci 2008)

IF

T d.-torus of a d.compact group G

THEN

- *$H < G$ definable $\implies E(T/H) = 0$*
- *$E(G/T) \neq 0 \implies T$ maximal d.-torus of G .*

Theorem (Berarducci 2008, Edmundo 2005)

IF

G d.connected d.compact group

THEN

- *for each T maximal d.-torus of G , $G = \bigcup_{g \in G} T^g$, and*
- *T_1, T_2 maximal d.-tori of $G \implies T_1 = T_2^g$, for some $g \in G$.*

Exercise 12

G d.connected d.compact group.

- 1 $Z(G) = \bigcap \{ T : T \text{ maximal d.-torus of } G \}.$
- 2 $G/Z(G)$ is centreless.

Definably simple and semisimple groups

Definition

G d.group. G **semisimple** \iff NO $H \trianglelefteq G$, H infinite abelian.

Exercise 13

G d.group, G semisimple \iff NO $H \trianglelefteq G$, H infinite abelian *definable*.

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.connected group. THEN

- 1 G semisimple group $\iff \mathfrak{g}$ semisimple Lie algebra.
- 2 IF G centreless THEN G is d.simple $\iff \mathfrak{g}$ simple Lie algebra.

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.connected semisimple centreless group $\dim G = n$. THEN

$$G \stackrel{\text{def}}{\cong} H^0 \leq GL(n, R),$$

H^0 semialgebraically connected component of an algebraic linear group H .

Proof.

$$G \stackrel{\text{def}}{\cong} G_1 := \text{Ad}(G) \leq \text{Aut}(\mathfrak{g}) \leq GL(n, R).$$

$$\dim G_1 = \dim G = \dim \mathfrak{g} \stackrel{(*)}{=} \dim \text{Aut}(\mathfrak{g}),$$

(*) transfer from the reals.

$\therefore G_1 \leq \text{Aut}(\mathfrak{g})$ finite index, $\text{Aut}(\mathfrak{g})$ algebraic group.

$\therefore G_1$ semialgebraically connected component of $\text{Aut}(\mathfrak{g})$. □

Theorem (Peterzil-Pillay-Starchenko 2000-2002)

G d.connected group.

- ① G d.simple $\implies G \stackrel{\text{def}}{\cong} H(R)^0$, H real algebraic group/ \mathbb{R}^{alg} .
- ② G semisimple $\implies Z(G)$ is finite and

$$G/Z(G) \stackrel{\text{def}}{\cong} H_1 \times \cdots \times H_s,$$

H_i d.simple.

- ③ G d.simple $\implies G \equiv H$, H centreless simple Lie group.

Remark.(Hrushovski-Peterzil-Pillay 2011)

G d. connected group $\not\equiv G \equiv H$, Lie group:

$\langle R, <, +, \cdot, \exp \rangle$ nonstandard model of the theory of \mathbb{R}_{exp} . $\alpha \in R$ infinite.

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : u, v, t \in R, t > 0 \right\} \not\equiv \text{Lie group.}$$

Definition

G d.group. Solvable radical of G

$$R(G) := \langle \bigcup \{H \trianglelefteq G : H \text{ solvable}\} \rangle$$

Lemma (Baro-Jaligot-O_ 2012)

G d.group $\implies R(G)$ definable & solvable,
 $\therefore G/R(G)$ semisimple.

Theorem (Peterzil-Starchenko 2000)

G d.connected d.compact group $\implies G/Z(G)$ semisimple.

Exercise 14

G d.connected d.compact solvable group $\implies G$ abelian.

Commutator subgroup

G d.group \nRightarrow

$$[G, G] := \langle \{[x, y] : x, y \in G\} \rangle$$

definable.

Example (Conversano 2009)

$\exists G$ d.connected group, definable/ \mathbb{R} S.T. $[G, G]$ is NOT definable,

$$1 \rightarrow [0, 1) \rightarrow G \rightarrow PSL(2, \mathbb{R}) \rightarrow 1.$$

Theorem (Hrushovski-Peterzil-Pillay 2011)

G d.compact d.connected $\implies [G, G]$ definable and semisimple &

$$G = Z^0(G)[G, G],$$

$Z^0(G) \cap [G, G]$ finite.

This reduce many questions of d.compact groups to the commutative and semisimple cases.

Example (Mamino 2011)

$\exists G$ d.compact definable/ \mathbb{R} S.T. $[G : G]$ has NO definable semidirect complement in G :

$$SU_2 = \{a + bi + cj + dk : a^2 + b^2 + c^2 + d^2 = 1\}.$$

$$G := ([0, 1) \times SU_2)/\Gamma,$$

$$\Gamma = \{(0, 1), (1/2, -1)\}. \therefore G' \cong SU_2.$$

Solvable groups

Theorem (Edmundo 2003)

G *d.group*.

G solvable $\implies G/N$ is *d.compact*,

$N \trianglelefteq G$ maximal normal torsion-free.

Theorem (Edmundo 2003)

G *d.group*.

G solvable *d.connected* $\implies G' := [G, G]$ nilpotent.

Lemma (Peterzil-Starchenko 2005)

G d.groups. G torsion-free d.group $\implies \exists$ d.subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \triangleleft \cdots \triangleleft G_n = G$$

S.T. G_{i+1}/G_i torsion-free abelian d.group.

$\therefore G$ torsion-free $\implies G$ is d.connected and solvable.

Proof.

G counterexample of minimal dimension.

G d.connected $\implies \dim G > 1$.

G d.simple:

O/W $\exists H \triangleleft G$, H & G/H torsion-free \therefore 2 d. normal series for H and G/H which induce corresponding series for G , CONTRADICTION.

G d.simple $\implies G \cong H$, H simple centreless Lie group, H has torsion, a contradiction. □

Theorem (Baro-Jaligot-O_ 2012)

G d. group. G solvable d.connected group \implies derived series & lower central series of G consist of d.groups. $\therefore G'$ is definable.

Definition

Commutator width (cm) of group G :

$$cm(G) := \min \{ m : G' = \{ [x_1, y_1] \cdots [x_m, y_m] : x_i, y_i \in G \} \},$$

if m exists, o/w $cm(G) := \infty$.

- G finite simple group $\implies cm(G) = 1$ (Ore conjecture, 2010).
- G d.compact d.simple $\implies cm(G) = 1$.

Question 1

G d.simple $\implies cm(G) = 1$?

Definition

G group. A maximal nilpotent $Q \leq G$ is **Cartan subgroup** of G IF $\forall H \trianglelefteq Q$, H finite index in $Q \implies H$ finite index in $N_G(H)$.

G d.connected d.compact group \implies

Cartan subgroup of G = maximal d.-torus T of G

- they are all conjugate
- d.connected
- $T^G = G$, T any maximal d.-torus.

Cartan subgroups of $SL(2, \mathbb{R})$

$G := SL(2, \mathbb{R})$. 2 Cartan subgroups, up to conjugacy:

$$Q_1 := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \neq 0 \right\} \quad \& \quad Q_2 := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

- Q_1 NOT d.connected
- $X := Q_1^G \cup Q_2^G = \{A \in SL(2, \mathbb{R}) : \text{Tr}(A) \neq 2\} \cup \{\pm I\} \neq G$
- X dense in G .

Theorem (Baro-Jaligot-O_ 2014)

G solvable d.connected group. THEN

- *Cartan subgroups of G exist and are definable*
- *$Q \leq G$ Cartan $\implies Q$ d.connected & selfnormalizing*
- *$Q_1, Q_2 \leq G$ Cartan $\implies Q_1 = Q_2^G$*
- *$Q \leq G$ Cartan $\implies Q^G$ dense in G.*

Definition

G torsion-free d.group. G **definably completely solvable** if \exists d.series

$$\{1\} = G_0 \trianglelefteq G_1 \triangleleft \cdots \triangleleft G_n = G$$

S.T. G_{i+1}/G_i is one-dimensional.

Exercise 15

G torsion-free abelian d.group $\implies G$ d.completely solvable.

G connected LIE GROUP. G **split-solvable** IF \exists series

$$\{0\} = \mathfrak{g}_0 \trianglelefteq \mathfrak{g}_1 \triangleleft \cdots \triangleleft \mathfrak{g}_n = \mathfrak{g}$$

S.T. $\dim \mathfrak{g}_i = i$ ($0 \leq i \leq n$).

Example

Solvable \nRightarrow split-solvable: $\mathbb{R}^2 \rtimes_{\varphi} \mathbb{R}$ connected torsion-free,

$$\varphi : \mathbb{R} \rightarrow GL(2, \mathbb{R}) : t \mapsto \varphi(t) := \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

Theorem (Conversano-Onshuus-Starchenko 2016)

G *d.group*.

G *torsion-free* $\implies G$ *d.completely solvable*.

Question 2

Which Lie groups are Lie isomorphic to a definable group?

- (1) Compact
(2) Semisimple centreless
- } : Lie isomorphic to semialgebraic groups.

Theorem (Conversano-Onshuus-Starchenko 2016)

*G connected torsion-free solvable LIE GROUP. Then,
 G Lie isomorphic to a d.group $\iff G$ is split-solvable.*

Theorem (Conversano-Onshuus-Starchenko 2016)

*G solvable LIE GROUP. Then,
 G Lie isomorphic to d.group $\iff \exists H \trianglelefteq_{\text{Lie}} G$, H connected torsion-free split-solvable & G/H compact.*



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