Nash groups

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Konstanz, July 2016

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- We are not a priori imposing any topological structure on G, or making any assumptions about continuity of the group operation (but see later).
- But we usually assume G to be semialgebraically connected in the sense of having no proper semialgebraic subgroup of finite index.
- A typical example is the interval $[0, 1)$ equipped with addition modulo 1, which is a semialgebraic group in the above sense, but not a topological group in the naive sense.

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- This can be seen model theoretically via elimination of maginaries in $(\mathbb{R}, +, \times)$. But we will see later on a more interesting reason.
- In any case we want to classify semialgebraic groups in terms of real algebraic groups and various constructions.

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- Firstly if U is an open semialgebraic subset of $\mathbb{R}^n$, then a function $f : U \to \mathbb{R}$ is said to be Nash if f is both analytic and semialgebraic, which essentially means that there is a polynomial $P(\bar{x}, y)$ such that $P(\bar{x}, f(\bar{x})) = 0$ for all $\bar{x} \in U$. (e.g. $f(x) = +\sqrt{x}$ on $(0, \infty)$).

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- Semialgebraic functions are piecewise Nash.

Nash manifolds II

- An (n-dimensional) Nash manifold is a Hausdorff topological space X with a covering by *finitely many* open sets V_i, each homeomorphic via some f_i to an open semialgebraic subset U_i of $\mathbb{R}^n$ such that the transition maps $f_i \circ f_j^{-1}$ between the open semialgebraic sets $f_j(V_i \cap V_j)$ and $f_i(V_i \cap V_j)$ are Nash (i.e each coordinate is Nash).

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- There is a natural notion of a Nash map between Nash manifolds, and a Nash group is a Nash manifold X with group operation given by a Nash map $X \times X \to X$ (i.e. a group object in the category of Nash manifolds).

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- $\mathbb{R}/\mathbb{Z}$ has naturally the structure of a Nash manifold (in fact group): $[0, 1]$ with $0, 1$ identified, and addition modulo 1 but it is non affine.
- In general little is known about arbitrary (not necessarily affine) Nash manifolds but there is a chance to classify the Nash groups.

Nash groups I

We mention some early results:

Theorem 0.1

(P 1986) Any semialgebraic group G can be semialgebraically equipped with the structure of a Nash group (in particular a Lie group) unique up to Nash isomorphism. The category of semialgebraic groups coincides with that of Nash groups.

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Theorem 0.2

(Hrushovski-P, 1994, with model theory proof) Any (even locally) Nash group G is Nash, locally isomorphic, to a real algebraic group H in the sense that there is Nash isomorphism between open semialgebraic neighbourhoods of the identity of G and of H.

Nash groups II

- Earlier, in the late 80's Madden-Stanton had noted the one-dimensional case of Theorem 0.2 and used it to classify one-dimensional Nash groups (included in our main result, to be stated later). In particular there are many distinct incarnations of the 1-dimensional compact Lie group S^1 as Nash groups.

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- We used Theorem 0.2 to prove the following (Hrushovski-P, 1994, with corrected proof 2012):

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Note this already takes us out of the category of algebraic groups: the proper finite covers of $SL(2, \mathbb{R})$ are neither real algebraic nor linear.

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Corollary 0.5

The locally Nash groups are, up to local Nash isomorphism precisely the quotients of universal covers of real algebraic groups by discrete subgroups.

Nash groups IV

The following easy result of Shiota is useful and yields some quick conclusions:

Theorem 0.6

Let X and Y be Nash manifolds with Y affine and let $f : X \to Y$ be a locally Nash map. Then f is a Nash map.

Corollary 0.7

- *(i) Any simply connected Nash group is a finite cover of a real algebraic group.*
- *(ii) Any torsion-free Nash group is a real algebraic group.*

Proof. ??

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- (ii) Classify the resulting Nash groups up to Nash isomorphism.
- We will answer question (i) in the commutative case, and give some examples showing some subtleties around (ii) compared to the one-dimensional case.

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- This passes to universal covers: in particular we have $0 \to L \to \tilde{G} \to_\pi \tilde{C} \to 0$.

Result

We have, with above notation the solution to (i) in the commutative case:

Theorem 0.8

Let G be a commutative real algebraic group, and let Γ be a discrete subgroup of $\tilde{G}$ (maybe trivial). Then $\tilde{G}/\Gamma$ has a (compatible) structure of Nash group just if $\tilde{C}/\pi(\Gamma)$ is compact.

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- Some ingredients in the proof are:
- The (obvious) fact that a compact locally Nash group is Nash.
- The universal cover of either $SO(2,\mathbb{R})$ or a real abelian variety does not have the compatible structure of a Nash group (a point that seems to have been overlooked by Madden-Stanton).
- $\tilde{G}/\Gamma$ has structure of Nash (definable) group if and only if $\tilde{G}$ has a "definable fundamental region" for Γ.

Fundamental region I

- A fundamental region Ω of $\tilde{G}$ for Γ is simply a neighbourhood Ω of 0 in $\tilde{G}$ such that every coset of Γ in $\tilde{G}$ meets Ω and both Ω and $\Omega + \Omega$ contain only finitely many elements of Γ.

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- The choice of a definable fundamental region in $\tilde{G}$ for Γ determines a Nash or definable group structure on $\tilde{G}/\Gamma$ (in the obvious way; explain).

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- Such a fundamental region is *definable* if Ω is a definable subset of the locally definable $\tilde{G}$ and the restriction of $+$ to Ω is definable.
- The choice of a definable fundamental region in $\tilde{G}$ for Γ determines a Nash or definable group structure on $\tilde{G}/\Gamma$ (in the obvious way; explain).
- The interesting new phenomena, compared with the one-dimensional case, is that different choices of definable fundamental region may give non isomorphic Nash group structures on the same locally Nash group $\tilde{G}/\Gamma$.

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- For the second choice it is the other way round, so they cannot be Nash isomorphic.