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Relative Quantifier Elimination for the Module of Continuous Semi-Algebraic Functions on a Real Closed Field

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Set-up and statement of the main theorem

We fix the following notation:

(i) R is a real closed field,

(ii) $M_R := C_{\text{s.a.}}(R,R)$ is the ordered module (over itself) of continuous semi-algebraic functions $R \longrightarrow R$ under pointwise addition,

(iii) L_R is the lattice of closed and semi-algebraic subsets of R,

(iv) $\widehat{R}$ is the real closure of the ordered field extension $R(t)$ of R, where t is a positive infinitesimal with respect to R, and

(v) $V_R :=$ conv. hull$_{\widehat{R}}(R)$ $(= R + \mathfrak{m}_{V_R}$ as subsets of $\widehat{R})$.

Lemma 1. *V_R is isomorphic to the ring of germs of continuous semi-algebraic functions $R \longrightarrow R$ at r^+ and at r^- for all $r \in R$, i.e. $V_R \cong C_{\text{s.a.}}(R,R)/I_{r^+}$ where*

$$I_{r^+} := \{f \in C_{\text{s.a.}}(R,R) \mid \exists \epsilon > 0 \text{ such that } f|_{[r,r+\epsilon]} = 0\},$$

and $V_R \cong C_{\text{s.a.}}(R,R)/I_{r^-}$, where

$$I_{r^-} := \{f \in C_{\text{s.a.}}(R,R) \mid \exists \epsilon > 0 \text{ such that } f|_{[r-\epsilon,r]} = 0\}.$$

Proof sketch. For $V_R \cong C_{\text{s.a.}}(R,R)/I_{r^+}$, consider the composite ring homomorphism

$$\begin{array}{ccccc} C_{\text{s.a.}}(R,R) & \longrightarrow & C_{\text{s.a.}}(\widehat{R},\widehat{R}) & \longrightarrow & \widehat{R} \\ f & \longmapsto & f_{\widehat{R}} & \longmapsto & f_{\widehat{R}}(r+t). \end{array}$$

$\square$

Definition 2. We let $\mathscr{M}_R$ be the 3-sorted module structure consisting of:

(i) a sort for M_R interpreted as a $\{+,-,0,\leq,(\lambda_\alpha)_{\alpha\in M_R}\}$-structure, where each λ_α is a unary function $M_R \longrightarrow M_R$ interpreted as multiplication by $\alpha \in M_R$;

(ii) a sort for L_R interpreted as a $\{\cup,\cap,\subseteq,\top,\bot\}$-structure;

(iii) a sort for V_R interpreted as a $\{+,\cdot,-,0,1,\leq\}$-structure;

(iv) a unary function $P : M_R \longrightarrow L_R$ interpreted as

$$P(f) = \{f \geq 0\} := \{r \in R \mid f(r) \geq 0\};$$

(v) for each $r \in R$, unary functions $\pi_{r^+}, \pi_{r^-} : M_R \longrightarrow V_R$ interpreted as

$$\pi_{r^+}(f) := f_{\widehat{R}}(r+t) \text{ and } \pi_{r^-}(f) := f_{\widehat{R}}(r-t).$$

We write $\mathscr{L}^{\text{m.s.}/R}$ for the 3-sorted language of $\mathscr{M}_R$.

$$\begin{tikzcd} & L_R \\ M_R \arrow[ur, "P"] \arrow[dr, bend left, "(\pi_{r^-})_{r\in R}"] \arrow[dr, bend right, "(\pi_{r^+})_{r\in R}"'] & \\ & V_R \end{tikzcd}$$

Figure 1. The 3-sorted module structure $\mathscr{M}_R$.

Theorem 3. *Every $\mathscr{L}^{\text{m.s.}/R}$-formula is equivalent modulo $\mathscr{M}_R$ to a formula with no quantifiers over the module sort. In particular, every $\mathscr{L}^{\text{m.s.}/R}$-formula is equivalent modulo $\mathscr{M}_R$ to a formula of the form*

$$\exists \xi_1 \ldots \xi_m \exists X_1 \ldots X_n \left[\gamma \,\&\, \bigwedge_{i=1}^{m} \xi_i = P(s_i) \,\&\, \bigwedge_{j=1}^{n} X_j = \pi_j(t_j) \right], \qquad (*)$$

where

- *$\xi_1,\ldots,\xi_m$ are lattice variables,*
- *$X_1,\ldots,X_n$ are germ variables,*
- *γ is a Boolean combination of lattice and germ formulas (with parameters),*
- *s_i, t_j are module terms, and*
- *π_j is either $\pi_{r_j^+}$ or $\pi_{r_j^-}$ for some $r_j \in R$.*

Moreover, if R is a recursive real closed field, every $\mathscr{L}^{\text{m.s.}/R}$-formula is effectively equivalent modulo $\mathscr{M}_R$ to a formula of the form $(*)$.

Corollary 4. *Let R be a recursive real closed field. Then the module M_R is decidable.*

Proof sketch. By Theorem 3, if the lattice L_R and the partially ordered ring V_R are decidable, then so is $\mathscr{M}_R$, and decidability of M_R then follows; that L_R is decidable follows from decidability of the monadic second-order theory of the infinite binary tree with two successor functions (see [3] and [5]), and decidability of the real closed valuation ring V_R follows from [2]. $\square$

Connection with lattice-ordered groups and reduction step

We follow some ideas from the proof in [1] of effective quantifier elimination for divisible lattice-ordered groups $G \hookrightarrow N^X$ (where X is a set and N is a totally ordered Abelian group) satisfying a patching condition, relative to the sublattice of $\mathcal{P}(X)$ given by the G-zero sets

$$P(f) = \{f \geq 0\} := \{x \in X \mid f(x) \geq 0\}, \; f \in G \hookrightarrow N^X.$$

In particular, the underlying group of the ordered module M_R is a divisible lattice-ordered group of functions $R \longrightarrow R$ which satisfies the patching condition due to the semi-algebraic Tietze extension theorem (see [4]); therefore, in the 2-sorted structure

$$(C_{\text{s.a.}}(R,R); +,-,0,\leq) \xrightarrow{P} (L_R; \cup,\cap,\subseteq,\top,\bot)$$

every formula is equivalent to a formula with no quantifiers over the group sort.

Using the fact that the set consisting of the lattice and germ sorts is closed in $\mathscr{L}^{\text{m.s.}/R}$ (in the sense of [6]) and after the appropriate simplification arguments, eliminating module quantifiers in $\mathscr{L}^{\text{m.s.}/R}$-formulas modulo $\mathscr{M}_R$ reduces to eliminating module quantifiers in formulas of the form

$$\exists h \in M_R : \; A \subseteq \{\lambda_\alpha(h) \geq f\} \,\&\, B \subseteq \{\lambda_\alpha(h) \leq g\}, \qquad (\dagger)$$

where $A, B \in L_R$, $f, g \in M_R$ and $\alpha \geq 0$.

Elimination step as existence of local solutions

Using the fact that for all $\alpha \in M_R$ with $\alpha \geq 0$ and all $f, g \in M_R$ we have $\{f \geq g\} = \{\lambda_\alpha(f) \geq \lambda_\alpha(g)\}$, if $\alpha > 0$ then $(\dagger)$ is equivalent to a formula with no module quantifiers by [1], so the main obstruction to eliminate module quantifiers is at points $r \in R$ with $\alpha(r) = 0$. In particular, by an iterated Tietze extension argument, eliminating the quantifier in $(\dagger)$ is equivalent to eliminating the quantifier in $(\dagger)$ locally around the boundary points of $\{\alpha = 0\}$:

Proposition 5. *Let $A, B \in L_R$, $f, g, \alpha \in M_R$, and suppose $\alpha \geq 0$. Let $\{a_1,\ldots,a_n\}$ be the boundary points of $\{\alpha = 0\}$. The following are equivalent:*

(i) *$(\dagger)$ holds in $\mathscr{M}_R$.*

(ii)(a) *$A \cap \{\alpha = 0\} \subseteq \{0 \geq f\}$, $B \cap \{\alpha = 0\} \subseteq \{0 \leq g\}$, $A \cap B \subseteq \{f \leq g\}$, and*

(b) *there is some $\epsilon > 0$ such that $(\dagger)$ is solvable in $Z_0 := \bigcup_{i=1}^n \overline{B}_\epsilon(a_i)$; i.e. there is some $h_0 \in M$ such that $Z_0 \cap A \subseteq \{\lambda_\alpha(h_0) \geq f\}$ and $Z_0 \cap B \subseteq \{\lambda_\alpha(h_0) \leq g\}$.*

Translating existence of local solutions to the germ sort

We make use of the ring of germs V_R to turn item (ii) (b) in Proposition 5 into a first-order statement for the module structure $\mathscr{M}_R$; the key ingredient to do so is Lemma 6 below, whose proof relies crucially in o-minimality of the field R and the fact that elements of V_R are R-bounded in $\widehat{R}$.

Lemma 6. *Let $f, g, \alpha \in M_R$ and suppose that $\alpha \geq 0$. Let also $r \in R$. The following are equivalent:*

(i) *$\mathscr{M}_R \models \exists Y \in V_R \, (\pi_{r^+}(\alpha)Y \geq \pi_{r^+}(f))$.*

(ii) *There exists a continuous semi-algebraic function $h : R \longrightarrow R$ and $\epsilon > 0$ such that*

$$[r, r+\epsilon] \subseteq \{\lambda_\alpha(h) \geq f\}.$$

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