Topological large fields, their generic expansions to differential fields and transfer results.

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Outline of the talk

- dp-minimal fields with a generic derivation,
- Transfer results : elimination of imaginaries, continuous definable functions, open core,
- Applications to dense pairs,
- Further directions.

dp-minimal fields

Definition

A theory T is not *dp*-minimal if there is a model $\mathcal{M}$ of T, $a_{ij} \in M$ and uniformly unary definable sets $X_i,\ Y_j \subseteq M$, $i,\ j \in \mathbb{N}$, such that
$a_{ij} \in X_{i'} \leftrightarrow i = i'$,
$a_{ij} \in Y_{j'} \leftrightarrow j = j.'$

A structure is dp-minimal if its theory is.

Examples of dp-minimal fields

Let $\mathbf{K} := (K, +, \cdot, -, 0, 1)$.

• $(\mathbf{K}, <)$ an ordered real-closed field, ie a model of RCF [o-minimal theory]

• $(\mathbf{K}, v)$ a non-trivially valued algebraically closed field, respectively of ACVF [C-minimal theory],

• $(\mathbf{K}, v)$ a p-adically closed valued field of rank d, respectively of $p\mathrm{CF}_d$ [p-minimal theory]

• $(\mathbf{K}, <, v)$ an ordered valued real-closed field, respectively of RCVF [weakly o-minimal theory].

and more...

dp-minimal fields

Theorem (Johnson)

If $\mathcal{K} := (K, +, \cdot, 0, 1, \cdots)$ is an expansion of an infinite field with a dp-minimal theory but not strongly minimal, then $\mathcal{K}$ can be endowed with a non-discrete Hausdorff definable field topology, namely $\mathcal{K}$ has a uniformly definable basis of neighbourhoods of zero compatible with the field operations, etc $\cdots$

Moreover, Johnson shows that any definable subset of $\mathcal{K}$ has finite boundary and every infinite definable set has non-empty interior (so $\mathcal{K}$ eliminates $\exists^\infty$).
Furthermore, the topology on K is induced either by a non-trivial valuation or an absolute value.

Correspondences

Now for a dp-minimal field $\mathcal{K}$, we will describe a generalisation of a cell decomposition theorem due to L. Mathews (for certain topological fields).

Definition

Let $E,\ F$ be two definable subsets of K^n, then a correspondence f is a definable subset $graph(f)$ of $E \times F$ such that

$$0 < |\{y \in F : (x, y) \in graph(f)\}| < \infty, \text{forall } x \in E.$$

A correspondence f is an m-correspondence if for all $x \in E$, $|\{y \in F : (x, y) \in graph(f)\}| = m$.

dp-minimal fields-definable sets

Let X be a A-definable subset of $\mathcal{K}^n$ with A a subset of K, then:

Theorem (Simon-Walsberg)

There a finitely many A-definable subsets X_i with $X = \bigcup X_i$ such that X_i is the graph of a A-definable continuous m-correspondence $f : U_i \rightrightarrows K^{n-d}$, where U_i is a A-definable open subset of K^d, for some $0 \leq d \leq n$.

Conventions: if $d = 0$, $f : K^0 \rightrightarrows K^{n-d}$, then $graph(f)$ is identified with a finite set and
if $d = n$, $f : U \rightrightarrows K^0$, $graph(f)$ is identified with U (an open subset of K^n).

Note that when $\mathrm{acl} = \mathrm{dcl}$, we may replace "correspondence" by the graph of a definable function.

dp-minimal fields-dimension

Let $\mathcal{K}$ be a dp-minimal field and let X be a A-definable subset of $\mathcal{K}^n$. We have several notions of dimensions:

• the topological dimension:
let $X \subseteq K^n$, then $\dim(X) := max\{\ell :$ there is a projection $\pi : K^n \to K^\ell$ such that $\pi(X)$ has non-empty interior$\}$.

• the acl-dimension $(\mathrm{acl}-\mathrm{dim})$, defined as follows: $\mathrm{acl}-\mathrm{dim}(\bar{u}/A) := min\{\ell :$ there is a subtuple $\bar{d}$ of $\bar{u}$ of length ℓ such that $\bar{u} \in \mathrm{acl}(A, \bar{d})\}$. Then $\mathrm{acl}-\mathrm{dim}(X/A) := max\{\mathrm{acl}-\mathrm{dim}(\bar{u}/A) : \bar{u} \in X\}$.
Note that it is not assumed that acl has the exchange.

dp-minimal fields-dimension

Theorem (Simon-Walsberg)

Then $\dim(X) = \mathrm{acl} - \dim(X)(= dp - rank(X))$.

Let fr(X):=$\overline{X} \setminus X$, where $\overline{X}$ denotes the closure of X.

Theorem (Simon-Walsberg)

$\dim(fr(X)) < \dim(X)$.

dp-minimal fields with a generic derivation

From now on, $\mathcal{K} := (K, +, -, \cdot, 0, 1, \cdots)$ denote a dp-minimal field of characteristic 0 and assume that $\mathcal{K}$ is not strongly minimal. Furthermore, we will assume that:

• the language $\mathcal{L}$ is a relational expansion of the ring (field) language and every relation and its complement is the union of an algebraic set and an open subset.

• The theory T of $\mathcal{K}$ admits quantifier elimination in the language $\mathcal{L}$.

Examples

Let $\mathcal{L}$ be the language of fields. Let div be a binary relation.

1. Let $\mathcal{L}_< := \mathcal{L} \cup \{<\}$, then RCF admits quantifier elimination (Tarski),
2. Let $\mathcal{L}_{div} := \mathcal{L} \cup \{div\}$, then ACVF admits quantifier-elimination (Robinson).
3. Let $\mathcal{L}_{<,div} := \mathcal{L}_< \cup \{div\}$, then RCVF admits quantifier-elimination (Cherlin-Dickmann).
4. Let $\mathcal{L}_p := \mathcal{L} \cup \{div, c_1, \cdots, c_d, P_n;\ n \geq 1\}$, then $p\mathrm{CF}_d$ admits quantifier elimination in $\mathcal{L}_p$ (Macintyre, Prestel-Roquette).

In all the above cases, the relations and their complements satisfy the hypothesis to be the union of an open set with an algebraic set.

dp-minimal fields with a generic derivation-some notation:

We consider the *generic* expansion of $\mathcal{K}$ with a derivation δ, namely we put no a priori continuity assumptions on δ. Denote by $\mathcal{L}_\delta := \mathcal{L} \cup \{\delta\}$ and T_δ the $\mathcal{L}_\delta$-theory $T \cup \{\delta$ is a derivation $\}$.

For $a \in K$ and $m \geqslant 0$, we let

$\delta^m(a)$ denote the m^{th}-derivative of a, $m \geq 1$, with $\delta^0(a) = a$,

$$\mathrm{Jet}_m(a) = \bar{\delta}^m(a) = (\delta^0(a), \delta^1(a), \delta^2(a), \ldots, \delta^m(a)), \text{and}$$

for $X \subset K$, $\mathrm{Jet}_m(X) = \{\bar{\delta}^m(a) : a \in X\}$.

dp-minimal fields with a generic derivation-some notation:

By assumption on $\mathcal{L}$, any $\mathcal{L}_\delta$-term $t(x)$ with $x = (x_1, \ldots, x_n)$, is equivalent, modulo the theory of differential fields, to an $\mathcal{L}$-term $t^*(\bar{\delta}^{m_1}(x_1), \cdots, \bar{\delta}^{m_n}(x_n))$ for some $(m_1, \cdots, m_n) \in \mathbb{N}^n$.

So with any $\mathcal{L}_\delta$-quantifier-free formula $\varphi(x)$, we may associate an equivalent $\mathcal{L}_\delta$-formula $\varphi^*(\bar{\delta}^m(x))$, $m \in \mathbb{N}$, (modulo the theory of differential fields) where φ^* is a $\mathcal{L}$-quantifier-free formula which arises by uniformly replacing every occurrence of $\delta^m(x_i)$ by a new variable y_i^m in φ with the following choice for the order of variables $\varphi^*(y_1^0, \cdots, y_1^m, \cdots, y_n^0, \cdots, y_n^m)$. So we get

$$\varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^*(\bar{\delta}^m(x_1), \ldots, \bar{\delta}^m(x_n)).$$

Scheme (DL)

Let T as before, $\mathcal{K} \models T$ and $\chi(x, \bar{y})$ be an $\mathcal{L}$-formula such that for any $\bar{a} \subset K$, $\chi(K, \bar{a})$ is an open neighbourhood of 0 in K.
Set $T_\delta^* := T_\delta \cup (DL)$, where (DL) is the following list of axioms:

Let $\bar{a} := (\bar{a}_1, \cdots, \bar{a}_n)$ $\bar{a}_i \subset K$, $1 \leq i \leq n$, $n \geq 1$, set

$$W_{\bar{a}} := \chi(K, \bar{a}_1) \times \cdots \times \chi(K, \bar{a}_n).$$

$\mathcal{K}$ satisfies (DL) if for every $n \geq 1$, for every differential polynomial $f(X) \in K\{X\}$, with $f(X) = f^*(X, \delta(X), \ldots, \delta^n(X))$ and for any $\bar{a} \subset K$, we have:

$$\exists \bar{\alpha}\Big((f^*(\bar{\alpha}) = 0 \wedge s_f^*(\bar{\alpha}) \neq 0) \Rightarrow \Big(\exists z \big(f(z) = 0 \wedge s_f(z) \neq 0 \wedge (\bar{\delta}(z) - \bar{\alpha}) \in W_{\bar{a}}\big)\Big)\Big).$$

Axiomatisation of differential t-large e.c. topological fields of characteristic 0

Under the further hypothesis, called t-large–it adapts in this topological setting the property of largeness (Pop)-, the theory T_δ^* is consistent and axiomatizes the class of existentially closed models of T_δ. In this particular setting, we get:

Theorem (Guzy-P)

Let T be a dp-minimal theory of t-large $\mathcal{L}$-fields of characteristic 0, admitting quantifier elimination.
Then T_δ^* is the model-completion of T_δ and admits quantifier elimination.

The above theorem was stated for a larger class of topological fields.

Examples

- We obtain for the theory T_δ^*:

1. $\mathrm{CODF} = \mathrm{RCF}_\delta^*$ in case $T = \mathrm{RCF}$,
2. RCVF_δ^* in case $T = \mathrm{RCVF}$ (an expansion of CODF),
3. $p\mathrm{CF}_\delta^*$ in case $T = p\mathrm{CF}_d$,
4. $\mathrm{ACVF}_{0,0}{}_\delta^*$ in case $T = \mathrm{ACVF}_{0,0}$ (an expansion of DCF_0),

First properties (direct consequences of the axiomatisation)

Using the fact that T_δ^* admits q.e. (and the forgetful functor), one can observe:

- (Guzy-P.) If T is NIP, then T_δ^* is NIP.
- (Chernikov, 2015) If T is distal, then T_δ^* is distal.

Let $\mathcal{K} \models T_\delta^*$ and denote by C_K its subfield of constants.
Using the axiomatisation (respectively the geometrical axiomatisation), two observations:

- Then C_K is dense in K.
- (Brouette, Cousins, Pillay, P.–in case $\mathcal{L}$ is the language of rings–) Then $C_K \models T$.

So we get an elementary pair (K, C_K) of models of T.

Order of a definable set

Since T_δ^* admits quantifier elimination, every $\mathcal{L}_\delta$-definable set $X \subseteq K^n$ is of the form $\mathrm{Jet}_m^{-1}(Y)$ for some quantifier-free $\mathcal{L}$-definable set $Y \subseteq K^{(m+1)n}$.

Definition (Order)

Let $X \subseteq K^n$ be an $\mathcal{L}_\delta$-definable set. The *order of* X, denoted by $o(X)$, is the smallest integer m such that $X = \mathrm{Jet}_m^{-1}(Y)$ for some $\mathcal{L}$-definable set $Y \subseteq K^{(m+1)n}$.

Open core

Property $(\star)$: For any $X \subseteq K^n$ $\mathcal{L}_\delta$-definable non-empty subset, there is an integer $m \geq o(X)$ and an $\mathcal{L}$-definable set $Z \subseteq K^{(m+1)n}$ such that

1. $x \in X$ if and only if $\mathrm{Jet}_m(x) \in Z$ and
2. $\overline{Z} = \overline{\mathrm{Jet}_m(X)}$.

Note that equivalently in Property $(\star)$ one can require that $m = o(X)$.

Open core-continued

Lemma (C-P)

Property $(\star)$ is equivalent to: T^*_δ has $\mathcal{L}$-open core.

$(\Rightarrow)$ one shows that given an $\mathcal{L}_\delta$-definable set X, its closure $\overline{X}$ is $\mathcal{L}$-definable.

Claim: $\overline{X} = \overline{\pi(\overline{Z})}$, where Z has the property $(\star)$ and π is the projection sending each block of $(m+1)$ coordinates to its first coordinate.

Open core-continued

$(\Leftarrow)$ Conversely, if the theory T_δ^* has $\mathcal{L}$-open core, then:

take $Y \subset K^{(o(X)+1)n}$ be an $\mathcal{L}$-definable set such that $X = \mathrm{Jet}_{o(x)}^{-1}(Y)$.
Set $Z := Y \cap \overline{\mathrm{Jet}_{o(X)}(X)}$. Since $\overline{\mathrm{Jet}_{o(X)}(X)}$ is both closed and $\mathcal{L}_\delta$-definable, it is $\mathcal{L}$-definable (T_δ^* has open core).
So the set Z is $\mathcal{L}$-definable. Since

$$\mathrm{Jet}_{o(X)}(X) \subseteq Z \subseteq \overline{\mathrm{Jet}_{o(X)}(X)},$$

both properties (1) and (2) are easily shown.

Elimination of imaginaries

Let $\mathcal{G}$ be a collection of sorts of $\mathcal{L}^{eq}$. We let $\mathcal{L}^{\mathcal{G}}$ denote the restriction of $\mathcal{L}^{eq}$ to the home sort together with the new sorts in $\mathcal{G}$ and their respective quotient maps.

Theorem (C-P)

Suppose that T admits elimination of imaginaries in $\mathcal{L}^{\mathcal{G}}$. If T^*_δ has $\mathcal{L}$-open core, then T^*_δ admits elimination of imaginaries in $\mathcal{L}^{\mathcal{G}}_\delta$.

We follow an argument of Marcus Tressl to show EI on CODF.

Proof:
Fix a model K of T_δ^* and let $X \subseteq K^n$ be a non-empty $\mathcal{L}_\delta$-definable set.
We will show that X has an $\mathcal{L}_\delta$-code in $\mathcal{G}(K)$, namely, there is a tuple $e \in \mathcal{G}(K)$ such that for all $\sigma \in \mathrm{Aut}_{\mathcal{L}_\delta}(K)$

$$\sigma(X) = X \text{ if and only if } \sigma(e) = e.$$

Observation 1: every $\mathcal{L}$-definable set has an $\mathcal{L}_\delta$-code in $\mathcal{G}(K)$.

$$\widetilde{X} := \mathrm{Jet}_{o(X)}^{-1}(\overline{\mathrm{Jet}_{o(X)}(X)}).$$

By the open core assumption, $\overline{\mathrm{Jet}_{o(X)}(X)}$ is $\mathcal{L}$-definable.
We proceed by induction on $\dim(\overline{\mathrm{Jet}_{o(X)}(X)})$.

Elimination of imaginaries

Conditional to having $\mathcal{L}$-open core, we obtain the following corollaries:

- For $T = \mathrm{RCF}$, yet another proof that CODF admits elimination of imaginaries in the language of differential fields.

- For $T = \mathrm{ACVF}$, $T = \mathrm{RCVF}$ and $T = p\mathrm{CF}$, a proof that T_δ^* has elimination of imaginaries in the language $\mathcal{L}_\delta^{\mathcal{G}}$ where $\mathcal{G}$ corresponds to the so called *geometric sorts* (by the corresponding results of Haskell-Hrushovski-Macpherson, Mellor and Hrushovski-Martin-Rideau respectively).

Continuous definable functions and open core

Theorem (C-P)

Assume that $\mathrm{acl}_{\mathcal{L}} = \mathrm{dcl}_{\mathcal{L}}$ (finite Skolem functions) for models of T. Let $X \subseteq K^n$ be an $\mathcal{L}$-definable set and $f : X \to K$ be a continuous $\mathcal{L}_\delta$-definable function. Then f is $\mathcal{L}$-definable.

Corollary

CODF has $\mathcal{L}$-open core.

Continuous definable functions and open core

Let $\mathcal{L}_{\mathrm{RCVF}}$ and $\mathcal{L}_{p\mathrm{CF}_d}$ be the languages in which RCVF and $p\mathrm{CF}_d$ eliminate quantifiers respectively. Let $\mathcal{L}_\Gamma$ be the 2-sorted language

$$(K, \mathcal{L})$$

$$\begin{cases} (\Gamma \cup \{\infty\}, \mathcal{L}_{oag}) & \text{if } \mathcal{L} := \mathcal{L}_{\mathrm{RCVF}} \\ (\Gamma \cup \{\infty\}, \mathcal{L}_{\mathsf{Pres}}) & \text{if } \mathcal{L} := \mathcal{L}_p\mathrm{CF}_d \end{cases}$$

$$v \colon K \to \Gamma \cup \{\infty\}.$$

Let $\mathcal{L}_{\Gamma,\delta}$ be the extension of $\mathcal{L}_\Gamma$ in which we replace $\mathcal{L}$ by $\mathcal{L}_\delta$ in the valued field.

Theorem (C-P)

Let T be RCVF or $p\mathrm{CF}_d$. Let K be a model of T^*_δ. Then the $\mathcal{L}_{\Gamma,\delta}$ theory of K has quantifier elimination.

Continuous definable functions and open core

Corollary (C-P)

Let T be RCVF or pCF_d. Let K be a model of T^*_δ. Then every $\mathcal{L}_{\Gamma,\delta}$-definable subset $X \subseteq \Gamma \cup \{\infty\}$ is $\mathcal{L}_\Gamma$-definable.

Theorem (C-P)

Assume that $\mathrm{acl}_\mathcal{L} = \mathrm{dcl}_\mathcal{L}$ (finite Skolem functions) for models of T. Let $X \subseteq K^n$ be an $\mathcal{L}$-definable set and $f : X \to \Gamma \cup \{\infty\}$ be a continuous $\mathcal{L}_{\Gamma,\delta}$-definable function. Then f is $\mathcal{L}_\Gamma$-definable.

Corollary (C-P)

RCVF^*_δ and pCF^*_δ have $\mathcal{L}$-open core.

Applications to dense pairs

Let $\mathcal{L}^2 := \mathcal{L} \cup \{P\}$ where P is a new unary predicate P.
Let T^2 be the $\mathcal{L}^2$-theory of dense elementary pairs (K, F).

Recall that if K is a model of T^*_δ, then (K, C_K) is a model of T^2.

Assume from now that T is geometric.

Theorem (van den Dries/Berenstein-Vassiliev/Fornasiero/....)

The theory T^2 is complete.

Corollary (C-P)

Every model (K, F) of T^2 has an $\mathcal{L}^2$-elementary extension (K^*, F^*) such that K^* is a model of T^*_δ with constant field $C_{K^*} = F^*$.

Applications to dense pairs

Corollary (C-P)

Assume that T_δ has open core. Then T^2 has $\mathcal{L}$-open core.

Corollary (Boxall-Hieronymi/Fornasiero)

If T is RCF, RCVF or $p\mathrm{CF}$, then T^2 has $\mathcal{L}$-open core.

Applications to dense pairs

Theorem (Hieronymi, Nell)

Let T be an o-minimal theory extending the theory of ordered abelian groups. Then the theory T^2 is not distal.

Question: (Simon) Does T^2 admit a distal expansion?

Theorem (Nell)

Let F be an ordered field, A be an ordered F-vector space and B a dense subspace. Let $\mathcal{L}$ denote the language of ordered F-vector spaces. Then, the $\mathcal{L}^2$-theory of (A, B) has a distal expansion.

Applications to dense pairs

Theorem (Chernikov)

Assume that T is distal. Then T^*_δ is distal.

Let T^2_δ be the $\mathcal{L}^2_\delta$-theory extending T^*_δ by the axiom

$$\forall x(P(x) \leftrightarrow \delta(x) = 0),$$

i.e., P is interpreted as the constant field.

Corollary (C-P)

The theory T^2_δ is a distal expansion of T^2.
In particular, the theories of dense pairs of real-closed fields, dense pairs of p-adically closed fields and dense pairs of real closed valued fields admit a distal expansion.

Further directions

Let K be a model of T^*_δ.

- Show that if $f\colon X \rightrightarrows K$ is a continuous $\mathcal{L}_\delta$-definable correspondence and $X \subseteq K^n$ is $\mathcal{L}$-definable, then f is $\mathcal{L}$-definable.
- Develop the formalism of T^*_δ when T is a theory in a multi-sorted language. This might provide a way to deal with valued fields such as $\mathbb{C}((X))$ and $\mathbb{R}((X))$.
- Extend this formalism to some dp-minimal expansions of fields.

Thank you for your attention.