ON WEAKLY O-MINIMAL NON-VALUATIONAL EXPANSIONS OF ORDERED GROUPS

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Abstract. Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a weakly o-minimal expansion of an ordered group. This paper has two parts. In the first part, we introduce the notion of $\mathcal{M}$ having no external limits and use it to prove that a large collection of non-valuational structures $\mathcal{M}$ do not admit definable Skolem functions. In the second part, we provide an alternative characterization of the canonical o-minimal completion from [11] and employ it to produce new examples of non-valuational structures.

1. Introduction

A structure $\mathcal{M} = \langle M, <, \dots \rangle$ is *weakly o-minimal* if $<$ is a dense linear order and every definable subset of M is a finite union of convex sets [2, 5]. Examples of weakly o-minimal structures are:

(a) $\mathcal{M} = \langle R, Fin(R) \rangle$, a non-archimedean real closed field R expanded by its natural valuation ring $Fin(R)$.

(b) $\mathcal{M} = \langle \mathbb{R}_{alg}, (0, \pi) \rangle$, the field of real algebraic numbers expanded by the convex set $(0, \pi)$.

These are the archetypical examples of two categories of weakly o-minimal structures that can be distinguished by their 'definable cuts'. A pair (C, D) of non-empty subsets of M is called a *cut in* $\mathcal{M}$ if $C < D$ and $C \cup D = M$. It is called a *definable cut* if C (and D) are definable. If $\mathcal{M} = \langle M, <, +, \dots \rangle$ expands an ordered group, then a definable cut (C, D) is called *non-valuational* if the infimum $\inf\{y - x : x \in C, y \in D\}$ exists in M (and must equal 0). Otherwise, it is called *valuational*. $\mathcal{M}$ is called *non-valuational* if every definable cut is non-valuational; otherwise, *valuational*. Example (a) above is valuational and (b) is non-valuational. We denote by $\bar{M}$ the set of all definable cuts (C, D) in M such that C has no maximum element. Then $\bar{M}$ has a natural order, where $(C, D) < (C', D')$ if and only if $C < C'$, which extends the order $<$ of M, where $a \in M$ is identified with $\big((-\infty, a), [a, +\infty)\big)$. In [5], a weak cell decomposition theorem was proved for every weakly o-minimal

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structure $\mathcal{M}$. In [5, 11], a strong monotonicity theorem and a strong cell decomposition theorem were shown in case $\mathcal{M}$ is non-valuational, exhibiting its resemblance to o-minimal structures. Moreover, it was shown that the property of being non-valuational is elementary [13].

Canonical examples of weakly o-minimal non-valuational structures are the 'o-minimal traces', obtained by considering dense pairs of o-minimal structures, as introduced in [3]. Let $\mathcal{N}$ be an o-minimal structure and $\mathcal{M}_1 \prec \mathcal{N}$ a dense elementary substructure. The theory of dense pairs is the theory of the structure $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ obtained by expanding $\mathcal{N}$ with a unary predicate for the universe M of $\mathcal{M}_1$. By a result of [1] the structure $\mathcal{M}$ induced from $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ on M is weakly o-minimal. Moreover, any definable set in $\mathcal{M}$ is of the form $M^n \cap S$ where $S \subseteq N^n$ is $\mathcal{N}$-definable, [3, Theorem 2]. Thus, any definable cut in $\mathcal{M}$ is of the form $\big((-\infty, a) \cap M, (a, +\infty)\big)$ for some $a \in N$. Since M is dense in N, we obtain that $\mathcal{M}$ is non-valuational.

In this paper, we study model theoretic and topological properties in the setting of weakly o-minimal non-valuational expansions $\mathcal{M} = \langle M, <, +, \dots \rangle$ of ordered groups. In Section 3, we introduce the notion of having no external limits (Definition 3.1) and use it to prove that a large collection of such $\mathcal{M}$ do not admit definable Skolem functions. We then explore different notions of topological compactness and illustrate their inadequacy in the current context, due to the main feature that the topology is definably totally disconnected. In Section 4, we turn to the canonical o-minimal completion $\bar{\mathcal{M}}$ of $\mathcal{M}$, introduced by Wencel [11], and propose that a model theoretic analysis of the pair $(\bar{\mathcal{M}}, \mathcal{M})$ may be fruitful for the the geometric study of definable sets in $\mathcal{M}$. To illustrate this proposal, we prove that the dimension of a definable set equals to the o-minimal dimension of its closure in $\bar{\mathcal{M}}$, and (re-)establish the basic properties of dimension, such as its invariance under definable injective functions. In Section 5, we give yet another application of appealing to the canonical o-minimal completion; namely, we employ it to produce new examples of structures $\mathcal{M}$, which, in particular, cannot be obtained as reducts o-minimal traces.

We now list the main results of the paper. In Section 2 we prove the following theorem.

Theorem 1. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Then there exists $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N and $\mathcal{N}$ realizes all irrational non-valuational cuts in $\mathcal{M}$.*

In Section 3, we introduce the notion of having no external limits and combine it with Theorem 1 to derive the next theorem, which generalizes a result of Shaw [9].

Theorem 2. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, $\mathcal{M}^*$ any proper expansion of $\mathcal{M}$ by non-valuational cuts. Then any reduct of $\mathcal{M}^*$ preserving the order and the group structure is weakly o-minimal and does not admit definable Skolem functions.*

In Section 4 we give a somewhat less technical description of the canonical o-minimal completion $\bar{\mathcal{M}}$ introduced in [11].

Proposition 1. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Consider the structure $\bar{\mathcal{M}}^*$ whose universe is $\bar{M}$ and whose atomic sets are*

$$\{\mathrm{Cl}_{\bar{M}}(S) : S \subseteq M^n \text{ (some } n\text{) } \mathcal{M}\text{-definable}\}.$$

Then $\bar{\mathcal{M}}^*$ and $\bar{\mathcal{M}}$ have the same definable sets.

Section 5 is dedicated to a collection of results that could be summarized as follows.

Theorem 3. *The class of o-minimal traces is not closed under taking reducts. There exists a weakly o-minimal non-valuational expansion of an ordered group which is not a reduct of an o-minimal trace.*

Our conclusion from the results of the present paper is that the realm of weakly o-minimal non-valuational expansions of ordered groups is wider than we first expected. It seems that a topological and geometric study of such structures cannot directly borrow ideas from the o-minimal context. Rather, one has to study the theory of the pairs $\langle \bar{\mathcal{M}}^*, \mathcal{M} \rangle$. The right abstract context for studying such pairs is yet to be found, as the context of dense pairs is not general enough.

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2. Non-valuational cuts

If (C, D) is a cut in an ordered structure $(\mathcal{M}, <, \cdots)$ we identify it with the (partial) type $\{x > C\} \cup \{x \leq D\}$. We recall that a cut in $\mathcal{M}$ is *rational* if it is realized in $\mathcal{M}$ (equivalently, if D has a minimum). As mentioned in the introduction, a cut (C, D) in an ordered group is defined to be non-valuational if

$$\inf\{d - c : c \in C, d \in D\} = 0. \tag{*}$$

As was pointed out by Kamensky, $(*)$ is, in fact, equivalent to the existence of the infimum, provided that the group is p-divisible for some p. Indeed, assume that (C, D) is such that

$$\inf\{d - c : c \in C, d \in D\} = \gamma > 0.$$

Take $d \in D, c \in C$ such that $d - d' < \frac{\gamma}{p}$ for all $d' \in D$ with $d' < d$ and similarly for $c' \in C$. Then $\frac{d-c}{p} \geq \frac{\gamma}{p}$, implying that $c + \frac{d-c}{p} \notin C$ so $c + \frac{d-c}{p} \in D$. But $d - c - \frac{d-c}{p} = \frac{(p-1)(d-c)}{p} \geq \frac{\gamma}{p}$, a contradiction.

Since weakly o-minimal groups are divisible ([5, Section 5]) in our setting the two notions are indeed equivalent.

In [6, Lemma 2.2] it is shown that if $\mathcal{M}$ is o-minimal, and a realizes an irrational cut over M then no $b \in \mathcal{M}(a) \setminus \mathcal{M}$ realizes a rational cut over $\mathcal{M}$. The following lemma is an analogue for non-valuational cuts. It follows from [12], but we provide a succinct proof.

Lemma 2.1. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Let (C, D) be an irrational non-valuational cut over M and $a \models (C, D)$ any realization. Then $\mathcal{M}(a)$ does not realize any valuational cuts over M.*

Proof. Let $b \in \mathcal{M}(a)$ be any element. Then there exists an $\mathcal{M}$-definable function f such that $b = f(a)$. By o-minimality and the fact that $a \notin M$ (because (C, D) is an irrational cut) there is an $\mathcal{M}$-definable interval I containing a such that f is continuous and strictly monotone or constant on I. If f is constant on I then $b \in M$ and there is nothing to prove. So we may assume without loss of generality that f is strictly increasing. Restricting I, if needed, we may also assume that I is closed and bounded. So f is uniformly continuous on I.

By [6, Lemma 2.2] the type $p \in S_1(M)$ of a positive infinitesimal element is not realized in $\mathcal{M}(a)$. It follows, since (C, D) is non-valuational, that for any $c \in C(\mathcal{M}(a))$, $d \in D(\mathcal{M}(a))$ and $0 < \delta \in \mathcal{M}(a)$ there are $c < c' < d' < d$ with $c' \in C(\mathcal{M})$ and $d' \in D(\mathcal{M})$ with $d' - c' < \delta$. Given any $0 < \epsilon \in \mathcal{M}(a)$ take $0 < \epsilon' < \epsilon$ with $\epsilon' \in \mathcal{M}$ and let $\delta \in M$ be such that $|f(x) - f(y)| < \epsilon'$ for any $x, y \in I$ with $|x - y| < \delta$. So

$$\inf\{f(d) - f(c) : c \in C(\mathcal{M}(a)), d \in D(\mathcal{M}(a))\} = 0.$$

Thus, b realizes a non-valuational cut over $\mathcal{M}$. Since $b \in \mathcal{M}(a)$ is arbitrary, this finishes the proof of the lemma. □

Our main theorem in this section is[1]:

Theorem 2.2. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Then there exists $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N and $\mathcal{N}$ realizes all irrational non-valuational cuts in $\mathcal{M}$.*

Proof. Let $\{(C_i, D_i)\}_{i \in I}$ be an enumeration of all the non-valuational cuts in $\mathcal{M}$, and $p \in S_1(M)$ the type of a positive infinitesimal. We construct $\mathcal{N}$ by induction as follows. For $i = 1$ let $\mathcal{M}_1 = \mathcal{M}$ if (C_1, D_1) is realized in M. Otherwise, let $\mathcal{M}_1 := \mathcal{M}(a_1)$ where $a_1 \models (C_1, D_1)$ is any realization. By [6, Lemma 2.2] p is not realized in $\mathcal{M}_1$.

Assume now that we have constructed $\mathcal{M}_i$ such that (C_j, D_j) is realized in $\mathcal{M}_i$ for all $j < i$ and such that p is not realized in $\mathcal{M}_i$. If i is a successor ordinal we let $\mathcal{M}_{i,0} = \mathcal{M}_{i-1}$ and if i is limit we let $\mathcal{M}_{i,0} := \bigcup_{j<i} \mathcal{M}_i$. If (C_i, D_i) is realized in $\mathcal{M}_{i,0}$ set $\mathcal{M}_i = \mathcal{M}_{i,0}$. Otherwise set $\mathcal{M}_i := \mathcal{M}_{i,0}(a_i)$ where $a_i \models (C_i, D_i)$ is any realization.

[1]We thank Y. Peterzil for pointing out this formulation of the theorem.

We prove that $\mathcal{M}_i$ does not realize p. If $\mathcal{M}_i = \mathcal{M}_{i+1}$ this follows from the induction hypothesis. So we assume that this is not the case. Thus $(C_i(\mathcal{M}_i), D_i(\mathcal{M}_i))$ defines a cut in $\mathcal{M}_i$. Observe that $(C_i(\mathcal{M}_i), D_i(\mathcal{M}_i))$ is still non-valuational. Indeed, if $\inf\{d_i - c_i : c_i \in C_i(\mathcal{M}_i), d_i \in D_i(\mathcal{M}_i)\}$ does not exist, then there are $c_i \in C_i(\mathcal{M}_i)$ and $d_i \in D_i(\mathcal{M}_i)$ such that $d_i - c_i \models p$, contradicting the inductive hypothesis. Thus, applying [6, Lemma 2.2] again the desired conclusion follows.

Let $\mathcal{N} := \bigcup_{i\in I} \mathcal{M}_i$. We claim that $\mathcal{N}$ has the desired properties. By construction $\mathcal{N}$ realizes all definable non-valuational cuts in $\mathcal{M}$. Now let $(a, b) \in N$ be any interval. Since $\mathcal{N}$ does not realize p there exists some $r \in M$ such that $0 < r < b - a$, so in order to show that $(a, b) \cap M \neq \emptyset$ it will suffice to show that a realizes a non-valuational cut, which is precisely the conclusion of the previous lemma. $\square$

The construction of the previous proof can be refined to give:

Corollary 2.3. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Let $\tilde{\mathcal{M}}$ be the expansion of $\mathcal{M}$ by unary predicates $\{C_i\}_{i\in I}$ interpreted as distinct irrational non-valuational cuts in M. Then there exists an elementary extension $\mathcal{M} \prec \mathcal{N}$ such that M is dense in N and $\tilde{\mathcal{M}}$ is precisely the structure induced on $\mathcal{M}$ by all externally definable subsets from $\mathcal{N}$.*

Proof. Construct $\mathcal{N}$ precisely as in the previous theorem, save that this time we only realize those cuts appearing in $\{(C_i, D_i)\}_{i\in I}$. Then $\mathcal{M} \prec \mathcal{N}$ and M is dense in N, so $(\mathcal{N}, \mathcal{M})$ is a dense pair. By [3, Theorem 2] the structure induced on $\mathcal{M}$ in the pair $(\mathcal{N}, \mathcal{M})$ is precisely the expansion of $\mathcal{M}$ by unary predicates for all cuts realized in $\mathcal{N}$. Thus, by construction, we get that $\tilde{M}$ is a reduct of the structure induced on $\mathcal{M}$ from $(\mathcal{N}, \mathcal{M})$. So it remains to show that any cut over M definable in $(\mathcal{N}, \mathcal{M})$ is definable in $\tilde{\mathcal{M}}$.

So let $a \in N$ be any element. We have to show the $(-\infty, a)\cap M$ is definable in $\tilde{\mathcal{M}}$. By construction there are $a_1, \dots, a_n$ realizing the cuts $C_{i_1}, \dots, C_{i_n}$ and an M-definable continuous function, f, such that $f(a_1, \dots, a_n) = a$. Choose $a_1, \dots, a_n$ and f so that n is minimal possible. For every $\eta \in \{-1, 1\}^n$ say that f is of type η at a point $\bar{c} \in N^n$ if

$$f_i(x_i) := f(c_1, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_n)$$

is strictly monotone at c_i and for all $1 \leq i \leq n$ and $f_i(x_i)$ is increasing at c_i if and only if $\eta(i) = 1$. By the minimality of n there is some $\eta \in \{-1, 1\}^n$ such that f is of type η at $(a_1, \dots, a_n)$. In particular, the set F_η of points $\bar{x}$ such that f is of type η at x is M-definable and open. Let $L_i := C_i$ if $\eta(i) = -1$ and $L_i := D_i$ otherwise, then $L := \prod_{i=1}^n L_i \cap F_\eta$ is open and definable in $\tilde{M}$, and for any $\bar{x} \in L$ we have that $f(\bar{x}) < f(a_1, \dots, a_n) = a$. Since M is dense in N and $L(N)$ is not empty also $L(M)$ is not empty. Thus

$$x \in (-\infty, a) \iff \exists y(y \in L \wedge x < f(y))$$

and the right hand side is $\tilde{M}$-definable. $\square$

3. No external limits

In this section we introduce the notion of having no external limits and prove Theorem 2 from the introduction.

Definition 3.1. A weakly o-minimal structure, $\mathcal{M}$ *has no external limits* if whenever (a, b) is an interval in M (with $a, b \in M \cup \{\pm\infty\}$) and $\gamma : (a, b) \to M$ a definable function such that either $\lim_{t\to a} \gamma(t)$ or $\lim_{t\to b} \gamma(t)$ exist in $\bar{M}$, then the limit is in M. Otherwise, we say that $\mathcal{M}$ *has external limits.*

By [11, Lemma 1.3] for every $\mathcal{M}$ and γ as above the limit $\lim_{t\to b} \gamma(t)$ exists in $\bar{M} \cup \{\pm\infty\}$. So $\mathcal{M}$ has no external limits precisely when the limits are always in $M \cup \{\pm\infty\}$.

Dense pairs provide us with the basic example of weakly o-minimal structures with no external limits.

Lemma 3.2. *Let $(\mathcal{N}, \mathcal{M})$ be a dense pair of o-minimal expansions of ordered groups. Let $\mathcal{M}^*$ be the induced structure on $\mathcal{M}$. Then $\mathcal{M}^*$ has no external limits.*

Proof. By [3, Theorem 3] every function $f : \mathcal{M} \to \mathcal{M}$ definable in $(\mathcal{N}, \mathcal{M})$ is piecewise given by $\mathcal{M}$-definable functions $f_1, \dots, f_k$ (some k). I.e., for every $x \in M$ there is $1 \le i \le k$ such that $f(x) = f_i(x)$. Let $\gamma : (a, b) \to M$ be a definable bounded function. By definition, γ is definable in $(\mathcal{N}, \mathcal{M})$ so γ is given, piecewise by $\gamma_1, \dots, \gamma_k$ with all γ_i $\mathcal{M}$-definable. As $\mathcal{M}^*$ is weakly o-minimal the set $x \in M$ such that $f(x) = f_i(x)$ is $\mathcal{M}^*$-definable, and therefore a finite union of definable convex sets. In particular, there is some $1 \le i \le k$ such that $f = f_i$ on a final segment $(a', b) \subseteq (a, b)$. Since $\mathcal{M}$ is o-minimal and γ is bounded $\lim_{t\to b} \gamma(t) \in M$. $\square$

As an application we get a generalization of a theorem of Shaw [9].

Theorem 3.3. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, $\mathcal{M}^*$ any proper expansion of $\mathcal{M}$ by non-valuational cuts. Then $\mathcal{M}^*$ is weakly o-minimal and does not admit definable Skolem functions.*

The theorem will follow immediately from Corollary 2.3 and Lemma 3.2 once we prove:

Lemma 3.4. *Let $\mathcal{M}$ be a weakly o-minimal expansion of an ordered group. Then $\mathcal{M}$ is o-minimal if and only if it has no external limits and admits definable Skolem functions.*

Proof. The left-to-right direction is well-known. For the right-to-left direction, let (C, D) be any definable cut in $\mathcal{M}$. Fix some $a \in M$ and consider the definable set

$$S := \{(x, y) \in M^2 : 0 < x < a, y \in C, x + y \notin C\}.$$

Since $\mathcal{M}$ admits definable Skolem functions, there is a definable function $f : (0, a) \to M$ such that $(x, f(x)) \in S$ for all $x \in (0, a)$. So $\lim_{t\to 0} f(x) =$

$\sup(C)$. Since $\mathcal{M}$ has no external limits, $\sup C \in M$ i.e., $\sup(C) = \inf(D)$ implying that (C, D) is a rational cut. Since (C, D) was arbitrary, it follows that all definable cuts in $\mathcal{M}$ are rational, so weak o-minimality implies that, in fact, $\mathcal{M}$ is o-minimal. □

We point out that both properties of having no external limits and of being non-valuational are preserved under ordered reducts (the latter requiring the preservation of the group structure as well). So we get:

Corollary 3.5. *Let $(\mathcal{N}, \mathcal{M})$ be a dense pair of o-minimal expansions of an ordered group. Let $\tilde{\mathcal{M}}$ be an any expansion of $\mathcal{M}$ by sets definable in $(\mathcal{N}, \mathcal{M})$. Then $\tilde{\mathcal{M}}$ does not admit definable Skolem functions.*

These observations combine with [10] to give:

Corollary 3.6. *Let $\mathcal{M}$ be a valuational weakly o-minimal expansion of an ordered group. Then $\mathcal{M}$ has external limits.*

Proof. Since weakly o-minimal ordered groups are divisible abelian ([5, Section 5]) the reduct $\mathcal{M}_+ := (M, <, +)$ is o-minimal. Let $\mathcal{M}_U := (M, <, +, U)$ where U is a unary predicate interpreted as a valuational cut definable in $\mathcal{M}$. Then $\mathcal{M}_U$ is an expansion of $\mathcal{M}_+$ by a valuational cut and by [10, Theorem 4.12] it has definable Skolem functions. By Lemma 3.4 $\mathcal{M}_U$ has external limits. Therefore so does $\mathcal{M}$, as it is an expansion of $\mathcal{M}_U$. □

We give an example of a weakly o-minimal structure (but not an expansion of an ordered group) which does have external limits.

Example 3.7. Let $\mathcal{R}$ be the structure obtained by appending two real closed fields one "on top" of the other. More precisely, the language is given by $(\leq, R_1, R_2, +_1, \cdot_1, +2, \cdot_2)$ and the theory of $\mathcal{R}$ is axiomatised by:

(1) R_1, R_2 are unary predicates such that $(\forall x)(R_1(x) \leftrightarrow \neg R_2(x))$ and $(\forall x, y)(R_1(x) \wedge R_2(y) \to x < y)$.
(2) $+_i, \cdot_i$ are ternary relations supported only on triples of elements in R_i. They are graphs of functions on their domains, and R_i is a real closed field with respect to these operations.
(3) $\leq$ is an order relation compatible with the field ordering of R_1 and R_2 together with (1) above.

It follows immediately from quantifier elimination for real closed fields, that the above theory is complete and has quantifier elimination (after adding constants for $0, 1$ in both fields, and relation symbols for the inverse function in both fields). Thus $\mathcal{R}$ is weakly o-minimal, and the only definable cut in $\mathcal{R}$ not realized in $\mathcal{R}$ is $(R_1(R), R_2(R))$. It follows that $\bar{R} = R \cup *$ where $*$ is a new point realizing the above cut.

However, $\mathcal{R}$ does have external limits. Take the function $x \mapsto x^{-1}$ in the field structure on R_1 on the interval $(0, 1)$. Clearly its limit, as $x \to 0^+$ is $*$.

Remark 3.8. In [11] Wencel introduces the notion of strong cell decomposition (see Definition 4.3 below) and proves that in the context of expansions of ordered groups this notion is equivalent to notion of being non-valuational. We point out that the previous example, though not expanding a group, has the strong cell decomposition.

Indeed, by quantifier elimination any definable subset of R^n is the disjoint union of a definable subset of R_1^n and a definable subset of R_2^n both definable in the language of ordered fields. Thus, by cell decomposition in R_1 and in R_2 (as stably embedded o-minimal sub-structures) any definable set in $\mathcal{R}^n$ is a union of cells in R_1^n and in R_2^n. Note that the iterative convex hull (see 4.3) of any cell $C \subseteq R_i^n$ is C itself. Thus, any such cell is a strong cell, and therefore $\mathcal{R}$ has strong cell decomposition.

Conjecture 3.9. *Let $\mathcal{M} = \langle M, <, +, \cdots \rangle$ be a weakly o-minimal non-valuational expansion of an ordered group. Then $\mathcal{M}$ has no external limits.*

3.1. **Notions of definable compactness.** Defining a notion of "definable compactness" in the weakly o-minimal setting is problematic, since the topology is definably totally disconnected, unless the structure is, in fact, o-minimal. In the present sub-section we explore several variants of definable compactness, showing they all fail to satisfy some of the basic properties of compact sets. Our examples are not pathological, they are simple and natural. This failure to find a satisfactory analogue of definable compactness in the present setting will lead us in the next section to suggest a different approach to the problem.

The direct analogue of the notion of definable compactness from the o-minimal setting would be the following.

Definition 3.10. Let $\mathcal{M}$ be an ordered structure. We say that a definable set $X \subseteq M^n$ is *curves closed* if for any definable curve $\gamma : (a, b) \to X$ (with $a, b \in M \cup \{\pm\infty\}$) the limit $\lim\limits_{t\to b} \gamma(t)$ exists and is an element of X.

Remark 3.11. If $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group, then in the definition of curves closed we may restrict to continuous definable functions.

We next give a few observations supporting the view that curves closedness captures some sort of compactness, as well as some examples demonstrating its shortcomings.

Lemma 3.12. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group with no external limits. Then any closed and bounded set is curves closed.*

Proof. Let $S \subseteq M^n$ be a definable closed and bounded set. Let $\gamma(a, b) \to S$ be a definable curve. We may assume, using the strong monotonicity theorem [5, Proposition 6.4] and passing to a final segment, that – writing $\gamma := (\gamma_1, \ldots, \gamma_n)$ – γ_i is monotone for all i. Since S is bounded, it follows that $\lim_{t\to b} \gamma(t)$ exists. As $\mathcal{M}$ has no external limits it follows that $\lim_{t\to b} \gamma(t) \in M^n$, and as S is closed $\lim_{t\to b} \gamma(t) \in S$. $\square$

Question 3.13. Is "non-valuational" really needed in the above?

Remark 3.14. In the above proof we have used Proposition 6.4 of [5]. From now on we will use this result freely without reference.

The converse of the previous lemma is, however, not true:

Example 3.15. Let $\mathcal{M}$ be the field of real algebraic numbers expanded by a predicate for the irrational cut at π. This is an expansion of a real closed field by a non-valuational cut, and therefore has no external limits. We will construct an unbounded $\mathcal{M}$-definable set which is curves closed.

Let $f(x) = \frac{1}{\pi - x}$ and $g(x) = \frac{1}{\pi - 2x}$. Let

$$S := \{(x, y) \in M^2 : 0 < x < \frac{\pi}{2}, f(x) < y < g(x)\}.$$

Then S is $\mathcal{M}$-definable. It is unbounded, and we will show that it is curves closed. Observe that $\Gamma_f \cap M^2 = \emptyset$ and $\Gamma_g \cap M^2 = \emptyset$, that $\{(\pi/2, y) : \frac{2}{\pi} \leq y\} \cap M^2 = 0$ and that $(0, \frac{1}{\pi}) \notin M^2$. It follows that S is closed in M^2. Thus, if we can show that there are no definable unbounded curves in S, the proof of the previous lemma will give the desired conclusion. So assume towards a contradiction that there is an unbounded $\mathcal{M}$-definable curve $\gamma : (a, b) \to S$. Let γ_1, γ_2 be the components of γ. Then, obviously, $\lim_{t\to b}(\gamma_1(t)) = \pi$. By the strong monotonicity theorem, there is a final segment $(c, b) \subseteq (a, b)$ such that $\gamma|(c, d)$ is the graph of an $\mathcal{M}$-definable function $f : (d, \pi) \to M$ with $\lim_{t\to\pi} f(t) = \infty$. We now conclude using the following general lemma.

Lemma 3.16. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group with no external limits. Suppose that $C \subseteq M$ is a definable bounded convex set. Assume that $f : C \to M$ is definable and that $\lim_{t\to\sup C} f(t) = \infty$ then $\sup C \in M$.*

Proof. By the strong monotonicity theorem, we may assume without loss of generality that f is strictly increasing and continuous on C. Fix some $a \in C$ and denote $b = f(a)$. Then $f^{-1} : (b, \infty) \to C$ is $\mathcal{M}$-definable and $\lim_{t\to\infty} f^{-1}(t) = \sup C$. Since $\mathcal{M}$ has no external limits, $\sup C \in M$. □

The above has another, not surprising, application:

Corollary 3.17. *If $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group with no external limits, then the image of any closed and bounded interval in M under a definable continuous function is closed and bounded. In particular, the continuous image of a closed and bounded interval cannot be the whole of M.*

Proof. We only have to show that if $[a, b] \subseteq M$ and $f : [a, b] \to M$ is definable and continuous then $F_{a,b} := f([a, b])$ is closed. So assume that $c \in \partial F_{a,b}$. By weak o-minimality, we may assume that there exists d such that $(c, d) \subseteq F_{a,b}$ (otherwise replace f with $-f$). Consider $f^{-1}((c, d))$. By weak o-minimality it is a finite union of definable convex subsets of $[a, b]$, restricting to a closed sub-interval, we may assume that $f^{-1}((c, d))$ is a definable convex set C.

By restricting further (changing d if needed) we may assume by the strong monotonicity theorem that f is strictly monotone or constant on C. If f is constant then it must be by continuity that $f(C) = c$. So we assume that f is, say, strictly decreasing on C. Then $\inf f(C) = c$ and $f^{-1} : (c, d) \to C$ has a limit e at c, and $e \in [a, b]$ implying $f(e) = c$. $\square$

The above does not, however, generalise to higher dimensions. The image of a closed and bounded set under a continuous function need not be closed:

Example 3.18. Let $\mathcal{M}$ and S be as in Example 3.15. Consider the projection $\pi_1 : S \to M$ onto the first coordinate. Clearly $\pi(S) = (0, \pi)$ which is neither curves-closed nor closed.

We finally consider two alternative ways for defining compactness in our setting which we claim also fail to capture the desired properties. Consider the algebro-geometric notion of a complete variety, adapted to the present setting. Say that an M-definable set X is *geometrically complete* if for any definable $\mathcal{M}$-space Y the projection $\pi : X \times Y \to Y$ is a closed map. Then if $\mathcal{M}$ is a strictly weakly o-minimal non-valuational expansion of a field no interval is geometrically complete. Indeed, consider the set S of example 3.15, then $S \subseteq [0, 4] \times \mathbb{R}$ and is closed, but $\pi_1(S) = (0, \pi)$ which is not closed. Since any two closed intervals in $\mathcal{M}$ are definably homeomorphic, this proves our claim.

In [4] yet another notion of definable compactness is introduced. A definable set X in a topological structure is *types closed* if every definable type p extending X has a limit in X, i.e., if there exists a point $a \in X$ such that for any definable open set $U \in p$ we have $a \in U$ (namely, viewed as an ultra-filter on the class of definable sets, p converges to a). In the context of o-minimal expansion of ordered groups Hrushovski and Loeser mention that this notion coincides with definable compactness. This follows readily from the main result of [7]. Indeed, let X be a definable closed and bounded set, and let p be a definable type extending X. By [7] if $a \models p$ is any realization then no irrational cut is realized in $\mathcal{M}(a)$. In particular $\mathrm{st}_M(a)$ – the standard part of a with respect to M – is defined (because X is bounded), and is readily seen to be a limit point of p (because p is definable and X is closed). However, it is easy to check that an interval in a weakly o-minimal structure is types closed if and only if it is o-minimal (with respect to the induced structure).

4. The canonical o-minimal completion

As we have seen in the previous sub-section, basic topological notions such as (definable) compactness and connectedness are impossible to define in the present context. As these problems arise due to the underlying topology being definably totally disintegrated, a natural solution to the problem would be to look for some sort of (canonical) completion, with a better behaved topology. In order to study the completion model theoretically, we have to equip it with some structure, making sure that this structure has a tractable interaction with the structure we started with. In the present section we suggest how to construct such a completion, and study its basic properties. This is only a first step in that direction, as it is not yet clear to us what the right completion of a definable space (or even a definable set) should be.

In this context, the present section is dedicated to the study of the canonical o-minimal completion $\bar{\mathcal{M}}$ of a weakly o-minimal nonvaluational structure, introduced by Wencel in [11]. Our long term goal is to study the pair $(\bar{\mathcal{M}}, M)$ (the expansion of the structrue $\bar{\mathcal{M}}$ by a unary predicate for the set M, the universe of $\mathcal{M}$). Here we develop some of its basic properties. In the next section we will give examples showing that this class of structures is strictly larger than the class of dense pairs.

We start with a simple characterization of the structure $\bar{\mathcal{M}}$.

Definition 4.1. Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Let $\bar{\mathcal{M}}^*$ be the structure whose universe is $\bar{M}$ and whose definable sets are

$$\{\mathrm{Cl}_{\bar{M}}(S) : S \subseteq M^n, \mathcal{M} - \text{definable}\}$$

Our main result (Proposition 4.8 below) is that $\bar{\mathcal{M}}^*$ and $\bar{\mathcal{M}}$ have the same definable sets. We proceed to define $\bar{\mathcal{M}}$ and establish some results for it.

Definition 4.2. Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group, $C \subseteq M^n$ an $\mathcal{M}$-definable set. A function $f : C \to \bar{M}$ is definable if the set $\{(x, y) : y < f(x)\}$ is $\mathcal{M}$-definable.

Definition 4.3. Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Define recursively:

(1) A 0-strong cell in $\mathcal{M}$ is a point in M. Its iterative convex hull in $\bar{M}$ is itself.
(2) A 1-strong-cell M is a definable convex set (s, t) with $s, t \in \bar{M}$. The iterative convex hull of (s, t) in $\bar{M}$ is $\{x \in \bar{M} : s < x < t\}$.
(3) Let $C \subseteq M^n$ be a strong cell. An $\mathcal{M}$-definable function $f : C \to \bar{M}$ is strongly continuous if it can be extended continuously to $\bar{C}$, the iterative convex hull of C.
(4) Let $\eta \in \{0, 1\}^n$. An $\eta^\frown 0$-strong-cell in M^{n+1} is a set

$$\{(x, f(x)) : x \in C\}$$

where $C \subseteq M^n$ is an η-strong-cell, and $f : C \to M$ is an $\mathcal{M}$-definable strongly continuous function. The iterative convex hull of C in M^{n+1} is $\{(x, \bar{f}(x)) : x \in \bar{C}\}$ where $\bar{f}$ is the (unique) continuous extension of f to $\bar{C}$.
(5) Let $\eta \in \{0, 1\}^n$. An $\eta^\frown 1$-strong-cell in M^{n+1} is a set

$$\{(x, y) : x \in C\, f(x) < y < g(x)\}$$

where $C \subseteq M^n$ is an η-strong-cell, and $f, g : C \to \bar{M}$ are $\mathcal{M}$-definable strongly continuous functions. The iterative convex hull of C in

M^{n+1} is
$$\{(x,y) : x \in \bar{C}, \bar{f}(x) < y < \bar{g}(x)\}$$
where $\bar{f}, \bar{g}$ are the (unique) continuous extensions of f, g to $\bar{C}$.

A weakly o-minimal structure $\mathcal{M}$ has the *strong cell decomposition property* if whenever $X_1, \dots, X_k$ are $\mathcal{M}$-definable subsets of M^n there exists a decomposition of M^n into strong cells partitioning each of the X_i.

Fact 4.4. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Then:*

(1) *$\mathcal{M}$ admits strong cell decomposition* [11, Theorem 2.15].
(2) *Let $\bar{\mathcal{M}}$ be the structure whose universe is $\bar{M}$ and whose atomic sets are*
$$\{\bar{C} : C \text{ a strong cell in } \mathcal{M}\}$$
where for an $\mathcal{M}$-definable strong cell C we let $\bar{C}$ be its iterative convex hull in $\bar{M}$. Then $\bar{M}$ is o-minimal [11, Section 3].
(3) *The structure induced from $\bar{\mathcal{M}}$ on M is $\mathcal{M}$. This follows from* [13] *Proposition 2.2, the discussion following it, and Proposition 2.13.*

The structure $\bar{\mathcal{M}}$ above is called the *canonical o-minimal completion of* $\mathcal{M}$. Before we proceed to prove that it has the same definable sets as $\bar{\mathcal{M}}^*$, we give some evidence supporting Wencel's claim for the canonicity of his construction.

Lemma 4.5. *Let $\mathcal{N}$ be an o-minimal structure, $M \subseteq N$ a dense subset. Assume that $\mathcal{N}'$ is an o-minimal structure with universe N and the same order relation as $\mathcal{N}$, and such that for any $\mathcal{N}$-definable set $S \subseteq N^n$ there exists an $\mathcal{N}'$-definable set S' such that $S \cap M^n = S' \cap M^n$. Then $\mathcal{N}'$ is an expansion of $\mathcal{N}$.*

Proof. By cell decomposition, in $\mathcal{N}$ it will suffice to show that any $\mathcal{N}$-definable open cell is also $\mathcal{N}'$-definable. So let C be such an open cell. Then $C = \operatorname{int}\operatorname{Cl}(C)$. In addition, because C is open and M is dense in N we also get $\operatorname{Cl}(C \cap M^n) = \operatorname{Cl}(C)$.

Let C' be $\mathcal{N}'$-definable such that $C' \cap M^n = C \cap M^n$. Since $\operatorname{Cl}(C) = \operatorname{Cl}(C \cap M^n)$ we get that $\operatorname{Cl}(C') \supseteq \operatorname{Cl}(C)$, implying that $C = \operatorname{int}\operatorname{Cl}(C) \subseteq \operatorname{int}\operatorname{Cl}(C')$.

On the other hand, if $\emptyset \neq \operatorname{int}\operatorname{Cl}(C') \setminus \operatorname{Cl}(C)$ then $M^n \cap (\operatorname{int}\operatorname{Cl}(C') \setminus \operatorname{Cl}(C)) \neq \emptyset$. By o-minimality of $\mathcal{N}'$ this implies that there exists a definable open box $B \subseteq C'$ such that $B \cap \operatorname{Cl}(C) = \emptyset$. This contradicts the assumption that $C \cap M^n = C' \cap M^n$.

It follows that $C = \operatorname{int}\operatorname{Cl}(C')$. Sinc the left hand side is $\mathcal{N}'$-definable the lemma is proved. ☐

Corollary 4.6. *Let $\mathcal{N}$, $\mathcal{N}'$ be o-minimal structures with universe N and the same underlying order. Assume that for some dense $M \subseteq N$, the trace of $\mathcal{N}$ on M and the trace of $\mathcal{N}'$ on M are the same. Then $\mathcal{N}$ and $\mathcal{N}'$ have the same definable sets.*

Specialising this last corollary to the case where $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group, and $N = \bar{M}$, we get:

Corollary 4.7. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Then $\bar{\mathcal{M}}$ is the unique – up to a change of signature – o-minimal structure on $\bar{M}$ whose trace on M is $\mathcal{M}$.*

We now return to the proof of Proposition 3 from Introduction.

Proposition 4.8. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Then $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}^*$ have the same definable sets.*

Proof. Since $\mathcal{M}$ admits strong cell decomposition, it will suffice to show that if $C \subseteq M$ is a strong cell then $\mathrm{Cl}_{\bar{M}}(C)$ is definable in $\bar{\mathcal{M}}$, and that $\bar{C}$ is definable in $\bar{\mathcal{M}}^*$.

The first implication is obvious. Indeed, it is easy to check that C is dense in $\bar{C}$. So $\mathrm{Cl}_{\bar{M}}(C) = \mathrm{Cl}_{\bar{M}}(\bar{C})$, and since $\bar{\mathcal{M}}$ is o-minimal the right hand side is $\bar{\mathcal{M}}$-definable.

The other implication is proved by induction. For 0-cells and for 1-cells, there is nothing to prove. So assume we have proved the result for all strong cells in M^n and we prove it for $(n+1)$-strong-cells in M^{n+1}. Let $C \subseteq M^n$ be a strong cell. Let $f : C \to M$ be an $\mathcal{M}$-definable strongly continuous function, and let D be Γ_f, the graph of f. Then $\bar{D} := \{(x, \bar{f}(x)) : x \in \bar{C}\}$. We have to show that $\bar{D}$ is $\bar{\mathcal{M}}^*$-definable. As f is strongly continuous we get that $\bar{D} = \mathrm{Cl}_{\bar{M}}(\Gamma_f) \cap (\bar{C} \times \bar{M})$, which is $\bar{\mathcal{M}}$-definable by the inductive hypothesis.

Now let $f, g : C \to \bar{M}$ be $\mathcal{M}$-definable strongly continuous functions with $f < g$. We have to show that the iterative convex hull of $D := (f, g)$ is $\bar{\mathcal{M}}^*$-definable. By definition

$$\bar{D} = \{(x, y) : x \in \bar{C}, \bar{f}(x) < y < \bar{f}(g)\}.$$

Since, by induction $\bar{C}$ is $\bar{\mathcal{M}}^*$-definable, it will suffice to show that $\bar{f}$ is $\bar{\mathcal{M}}^*$-definable. If f is the constant function $-\infty$, there is nothing to prove. So we assume this is not the case. By definition, the set

$$F := \{(x, y) : x \in C, y < f(x)\}$$

is $\mathcal{M}$-definable. For every $c \in \bar{C}$ let

$$s(c) := \sup\{y : (c, y) \in \mathrm{Cl}_{\bar{M}}(F)\}.$$

Since f is strongly continuous, $s(c)$ is well defined, and by definition it coincides with f on C. Since C is dense in $\bar{C}$ and $\bar{f}$ is the unique continuous extension of f to $\bar{C}$, necessarily $s = \bar{f}$, and as s is $\bar{\mathcal{M}}^*$-definable, we are done. □

We note that since the structure induced on M from $\bar{\mathcal{M}}^*$ is $\mathcal{M}$, we obtain the following.

Corollary 4.9. *The closed and bounded $\mathcal{M}$-definable subsets of M^n (any n) are precisely the traces of definably compact $\bar{\mathcal{M}}^*$-definable sets in M^n.*

4.1. **Definable functions.** We next use Proposition 4.8 to obtain some information about definable functions in the canonical o-minimal completion.

Definition 4.10. Let $S \subseteq M^n$ be a definable set. A definable function $f : S \to \bar{M}$ is strongly continuous if there exists an $\mathcal{M}^*$-definable set $S \subseteq B$ such that f extends continuously to B.

Note the above definition is weaker than Definition 4.3. E.g., if $\mathbb{R}^{\pi}_{\mathrm{alg}}$ is the extension of the field of real algebraic numbers by a predicate for the interval $(0, \pi)$ then the function $f(x) = \begin{cases} 1, & \text{if } x < \pi \\ 0 & \text{otherwise} \end{cases}$ is strongly continuous in the sense of the above definition, but not in that of Definition 4.3. With the present definition, however, we can show that any continuous definable function is strongly continuous on a large set.

Lemma 4.11. *Let $S \subseteq M^n$ be a definable set and $f : S \to \bar{M}$ a definable continuous function. Then there exists a definable set $S' \subseteq S$ such that f is strongly continuous on S' and* $\dim(\mathrm{Cl}_{\bar{\mathcal{M}}}(S) \setminus \mathrm{Cl}_{\bar{M}}(S')) < \dim \mathrm{Cl}_{\bar{M}}(S)$

Proof. Let $\Gamma := \mathrm{Cl}_{\bar{M}}\,\Gamma_f$. By Proposition 4.8 we know that Γ is $\mathcal{M}^*$-definable. Let $\Gamma_0 := \{x \in \Gamma : |\pi^{-1}(\pi(x)) \cap \Gamma| = 1\}$, where $\pi : M^{n+1} \to M^n$ is the projection onto the first n-coordinates. Let $S_0 := \pi(\Gamma_0)$. Let $\Gamma_1 := \Gamma \setminus \Gamma_0$ and $S_1 := \pi(\Gamma_1)$. Note that $\dim(S_1) < \dim(\mathrm{Cl}_{\bar{M}}(S))$. Otherwise there must be $s \in S_1 \cap S \neq \emptyset$ and $t_1, t_2 \in \bar{M}$ such that $(s, t_i) \in \Gamma_1$. Assume, without loss that $t_1 = f(s)$, implying that for all $\delta > 0$ there exists $s' \in B_\delta(s) \cap S$ such that $|f(s') - t_2| < \frac{1}{2}|t_1 - t_2|$, contradicting the continuity of f on S.

So let $S_2 := S \cap S_0$ and $\Gamma_2 := \pi^{-1}(S_2) \cap \Gamma$. Then Γ_2 is the graph of a function, $\bar{f} : S_2 \to \bar{M}$, and by o-minimality $\bar{f}$ is continuous on a large set. □

On the local level, the previous lemma becomes a little trickier. Consider the following example:

Example 4.12. Let $\mathbb{R}_{\mathrm{arg}} := (\mathbb{R}, \leq, +, \cdot, \mathrm{arg})$ where arg is the argument function on $\mathbb{R}^2 \setminus [0, \infty)$. Then $\mathbb{R}_{\mathrm{arg}}$ is o-minimal as a reduct of $\mathbb{R}_{\mathrm{an}}$. Let $\mathcal{R}$ be a countable elementary sub-structure with universe R, and $\alpha \in \mathbb{R} \setminus R$ with $\frac{1}{4} < \alpha < \frac{3}{4}$, and define

$$f(t) = \begin{cases} t, & \text{if } t \in [0, \frac{1}{4}] \\ \frac{1}{4}, & \text{if } t \in [\frac{1}{4}, \alpha) \\ 1, & \text{if } t \in (\alpha, \frac{3}{4}] \\ 4(1-t), & \text{if } t \in [\frac{3}{4}, 1] \end{cases}$$

Then f is a continuous function definable in $\mathcal{R}_\alpha$, the expansion of $\mathcal{R}$ by a predicate for $x < \alpha$. Note that $\mathcal{R}_\alpha$ is an o-minimal trace, whence weakly o-minimal and non-valuational. Finally, observe that $f(0) = f(1) = 0$ so the function $F(x, y) = (x^2 + y^2) f(\frac{\arg(x,y)}{2\pi})$ (with $F(0, 0) = 0$) is also definable and continuous. Clearly, F does not have a continuous extension to $\mathbb{R}^2$ (or even to $R(\alpha)^2$), but it does have a continuous extension $\bar{F}$ to $\mathbb{R}^2 \setminus \arg^{-1}(\alpha)$. Note, however, that $\bar{F}$ is continuous at the origin. Thus, $(0,0)$ is a point where F is continuous, but there is no (relatively) open neighbourhood of $(0,0)$ in $\mathbb{R}^2$ where F can be extended.

A local analysis of the proof of Lemma 4.11 shows that the above example is, in a way, the worst case scenario. To formulate the local version of Lemma 4.11 we need a couple of definitions:

Definition 4.13. (1) Let $\mathcal{M}$ be an o-minimal structure and $S \subseteq M^n$ a definable set. The dimension of S at $a \in M^n$ is

$$\dim_a(S) := \min\{\dim(U \cap S) : a \in U, U \text{ open}\}.$$

This is the dimension of the germ of S at a.

(2) Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Let $S \subseteq M^n$ and $f : S \to \bar{M}$ be definable with f continuous on S. Let $\operatorname{dom}_{\bar{M}} f := \operatorname{dom} \bar{f}$, where $\bar{f}$ is the continuous extension of f to $\bar{M}$ as provided by Lemma 4.11.

Corollary 4.14. *Let $S \subseteq M^n$ and $f : S \to M$ a definable continuous function then for any $a \in S$ the function f is strongly continuous at a, i.e., $a \in \operatorname{dom}_{\bar{M}} f$ and $\dim_a(\operatorname{dom}_{\bar{M}} f) = \dim_a(\operatorname{Cl}_{\bar{M}}(S))$. In fact, we even have $\dim_a(\operatorname{Cl}_{\bar{M}}(S) \setminus \operatorname{dom}_{\bar{M}} f) < \dim_a(\operatorname{Cl}_{\bar{M}}(S))$.*

Proof. Since f is continuous at a it is bounded in some neighbourhood of a, so we may assume without loss of generality that f is bounded on S. Consider, as above, $\Gamma := \operatorname{Cl}_{\bar{M}} \Gamma_f$ and $\Gamma_0 := \{x \in \Gamma : |\pi^{-1}(\pi(x)) \cap \Gamma| = 1\}$. Let $B_0 := \pi(\Gamma_0)$ where π is the projection onto the first n-coordinates. Then Γ_0 is the graph of a function, $\bar{f}$ with domain B_0. By curve selection in $\mathcal{M}^*$, if $\bar{f}$ is not continuous at some point $b \in B_0$ there are definable curves $\gamma_1, \gamma_2 \subseteq B$ such that $\lim \gamma_i(t) = b$ but $\lim(\bar{f}(\gamma_1(t))) \neq \lim(\bar{f}(\gamma_2(t)))$. Since f is bounded on S, it must also be that $\bar{f}$ is bounded on B_0. Thus, the above limits exist. But then $\lim(\bar{f}(\gamma_i(t))) \in \operatorname{Cl}_{\bar{M}} \Gamma_f$, contradicting the assumtion that $b_0 \in B_0$.

Since by continuity of f we must have $a \in B_0$, and – as we proved in the previous lemma – B_0 has small co-dimension, the corollary is proved. □

4.2. **dimension.** In the above results the notion of dimension in the canonical o-minimal completion of a weakly o-minimal non-valuational structure played an important role. In fact, to definable sets in such structures three natural notions of dimension can be associated:

Definition 4.15. Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group, $S \subseteq M^n$ a definable set:

(1) The topological dimension of S, $\dim_\tau(S)$, is the largest $r \leq n$ such that there exists a projection $\pi : M^n \to M^r$ such that $\pi(S)$ has non-empty interior.

(2) The combinatorial dimension of S, $\dim_{\mathrm{cl}}(S)$, is the dcl-dimension of S.
(3) The o-minimal dimension of S, $\dim_o(S)$, is $\dim(\mathrm{Cl}_{\bar{M}}(S))$.

We will need the following easy observation:

Lemma 4.16. *Let $\mathcal{M}$ be a weakly o-minimal structure, $S \subseteq M^n$ a definable open set. If $S' \subseteq S$ is a definable dense subset of S then $\dim_\tau(S) = \dim_\tau(S')$.*

Proof. If $\dim_\tau(S) = k$ then there is a projection $\pi : S \to M^k$ such that $\pi(S)$ has non-empty interior. It follows that $\pi(S')$ is dense in $\pi(S)$, and by decomposing $\pi(S)$ into cells, we are reduced to the case that S' is an open cell. Now if $U \subseteq M^n$ and $f : U \to M$ is a definable continuous function then, since the underlying order is dense, the graph of f is nowhere dense in M^{n+1}. So partition S' into strong cells. If no cell is open then each cell is, with respect to some choice of projection onto M^{n-1}, the graph of a definable function, whence nowhere dense in M^n. Since the finite union of nowhere dense sets is nowhere dense, this leads to a contradiction. □

As could be expected, these three notions of dimension are, essentially, the same:

Lemma 4.17. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group, $S \subseteq M^n$ a definable set. Then $\dim_o(S) = \dim_\tau(S) \geq \dim_{\mathrm{cl}}(S)$, and if $\mathcal{M}$ is ω_1-saturated, then equality holds throughout.*

Proof. First, we prove the equality of the o-minimal and topological dimensions. If $S \subseteq M^n$ is a definable open set then there exists a definable M-closed box $B \subseteq S$. Since M is dense in $\bar{M}$ we know that $\bar{B} := \mathrm{Cl}_{\bar{M}}(B) \subseteq \mathrm{Cl}_{\bar{M}}(S)$ and $\bar{B}$ is a box in $\bar{M}^n$. If follows that $n = \dim(\bar{M}) \leq \dim(\mathrm{Cl}_{\bar{M}}(S)) \leq n$, so equality holds. If follows that if S contains an open set then $\dim_\tau(S) = \dim_o(S) = n$. Now, if $S \subseteq M^n$ and $\dim_\tau(S) = r$ let $\pi : M^n \to M^r$ witness this. Then $r = \dim_\tau(S) = \dim_\tau(\pi(S)) = \dim_o(\pi(S)) \leq \dim_o(S)$. So it remains to prove the other inequality. So assume that $\dim_o(S) = r'$. Since in the o-minimal setting the o-minimal dimension is equal to the topological dimension there is some projection $\pi : \bar{M}^n \to \bar{M}^{r'}$ such that $\pi(\mathrm{Cl}_{\bar{M}}(S))$ contains an open box $\bar{B}$. Denote $\bar{S} := \mathrm{Cl}_{\bar{M}}(S)$ and let $B := \bar{B} \cap M^{r'}$. Let $b \in \bar{B}$ be any point and $c \in \pi^{-1}(b)$. Then, as $\bar{S} = \mathrm{Cl}_{\bar{M}}(S)$ for any $\bar{M}$-definable box $\bar{D} \ni c$ there exists $c' \in S \cap \bar{D}$, and choosing $\bar{D}$ small enough we conclude that $\pi(c') \in B$. Since B and b were arbitrary, it follows that $\pi(S) \cap \bar{B}$ is dense in $\bar{B}$ and therefore in B. By the previous lemma this implies that $\pi(S)$ contains an open set.

It remains to show that $\dim_\tau(S) \geq \dim_{\mathrm{cl}}(S)$ and that equality holds if $\mathcal{M}$ is saturated enough. Since for any definable sets S_1, S_2 we have that $\mathrm{Cl}_{\mathcal{M}}(S_1 \cup S_2) = \mathrm{Cl}_{\bar{\mathcal{M}}(S_1)} \cup \mathrm{Cl}_{\bar{\mathcal{M}}}(S_2)$ it follows form what we proved, and from the corresponding equality in the o-minimal setting that

$$(*) \qquad \dim_\tau(S_1 \cup S_2) = \max\{\dim_\tau(S_1), \dim_\tau(S_2)\}.$$

Thus, it will suffice to prove the claim under the assumption that S is a cell. In that case, it follows immediately from the definition that $\dim_\tau(S) \geq \dim_{\mathrm{cl}}(S)$.

From the definition we observe that in order to prove the remaining part of the claim it suffices to assume that S is open. In that case, however, the result follows from a standard compactness argument. □

The above results allow us to reduce the proofs of properties of the dimension in weakly o-minimal non-valuational expansions of an ordered group to the analogous properties in the o-minimal setting. We give two basic examples illustrating this:

Corollary 4.18. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Then the dimension function is preserved under definable bijections. I.e., if $S \subseteq M^n$ is definable and $f : S \to M^m$ is a definable bijection, then* $\dim(S) = \dim(f(S))$.

Proof. Partition S into strong cells such that $f|S$ is continuous. Restricting to a strong cell of maximal dimension, it thus suffices to prove the result under the assumption that f is continuous. By further restricting S we may assume that $f|S$ is strongly continuous (either use Lemma 4.11 or just the definition of strong cell decomposition with respect to the graph of f). Let B be $\mathcal{M}^*$-definable such that f extends continuously to B. Without loss of generality, we may assume that $S \subseteq B \subseteq \mathrm{Cl}_{\bar{M}}(S)$, so that B is dense in $\mathrm{Cl}_{\bar{M}}(S)$ and $\dim(B) = \dim(\mathrm{Cl}_{\bar{M}}(S))$. Then

$$\dim(B) = \dim(\mathrm{Cl}_{\bar{\mathcal{M}}}(S)) = \dim(\mathrm{Cl}_{\bar{\mathcal{M}}}(S) \cap B) = \dim(f(B))$$

where the last equality is true because o-minimal dimension is preserved under bijections. Now $f(B) \supseteq f(S)$, and $f(B)$ is $\mathcal{M}^*$-definable, so $\dim(f(B)) \geq \dim(\mathrm{Cl}_{\bar{M}}(f(S))$. The other inequality is obtained by replacing f with f^{-1}, and the result now follows. □

We now turn to the addition formula:

Lemma 4.19. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Let $S \subseteq M^n \times M^m$ be definable, π the standard projection on M^m. Assume that* $\dim(\pi^{-1}(a) \cap S) = k$ *for all $a \in T := \pi(S)$. Then* $\dim(S) = k + \dim(T)$.

Proof. Let S be as in the statement of the lemma, $S_0 := \mathrm{Cl}_{\bar{M}}(S)$ and T_0 its projection onto M^m. Then $\dim(S_0) = \dim(S)$ and as

$$\mathrm{Cl}_{\bar{M}}(T) = \mathrm{Cl}_{\bar{M}}(\pi(S)) \supseteq \pi(\mathrm{Cl}_{\bar{M}}(S)) = T_0 \supseteq T$$

also $\dim(T_0) = \dim(T)$. Note also that for any $a \in T$ we have that $\mathrm{Cl}_{\bar{M}}(S(a)) \subseteq S_0(a)$, where $S(a) := \{x \in M^n : (x, a) \in S\}$. Since dimension is definable in the o-minimal setting, $T_1 := \{a \in T_0 : \dim(S_0(a)) \geq k\}$ is $\mathcal{M}^*$-definable, and contains T. Thus $\dim(T_1) = \dim(T_0) = \dim(T)$.

By the addition formula in the o-minimal setting $\dim(S_0) \geq k + \dim(T_1)$, and by the previous equality $\dim(S) \geq k + \dim(T)$. This finishes the proof in case $k = n$. So we now assume that $k < n$. It follows that for all $a \in T$ there is some projection $\sigma_a : M^n \to M^k$ such that $\sigma(S(a))$ contains an open set. Since there are only finitely many projections there must be, by $(*)$ above, a projection $\sigma : M^n \to M^k$ such that $\dim(T_1) = \dim(T)$, where $T_1 := \{a \in T : \dim(\sigma(S(a)) = k\}$. Restricting ourselves to $\pi^{-1}(T_1) \cap S$, we may assume that $T_1 = T$. For simplicity, assume that σ is the projection onto the first k coordinates. It follows that for all $a \in T$, and $b := (b_1, \dots, b_n) \in S(a)$ there is a definable function $f_a(x, a) : M^k \to M^n$ such that $f_a(b_1, \dots, b_k, a) = (b_{k+1}, \dots, b_n)$. By compactness (alternatively, by strong cell decomposition), and reducing S further, we may assume that the function $f_a(x, y)$ does not depend on a.

Consider the set $S_1 \subseteq M^k \times M^m$ defined by

$$S_1 := \{(\sigma(b), a) : b \in S(a), a \in T\}.$$

By assumption $\dim(S_1(a)) = k$ for all $a \in T$. So by what we have already shown we know that $\dim(S_1) = \dim(T) + k$. But S is in definable bijection with S_1 given by $F(b, a) := (b, f(b, a), a)$, so by the previous corollary the result now follows. □

Lemma 4.20. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of a group, $S \subseteq M^n$ a definable set. Then* $\dim(\mathrm{Cl}_{\mathcal{M}}(S) \setminus S) < \dim S$.

Proof. Let $\bar{S} := \mathrm{Cl}_{\bar{M}}(S)$. By Proposition 2.3(1) of [13] there is a decomposition of $\bar{S}$ into basic cells (cf. for the definition). Let $\mathcal{P}$ be such a decomposition of $\bar{S}$. For each cell $C \in \mathcal{P}$ let $\partial C := \mathrm{Cl}_{\bar{M}}(C) \setminus C$. Denote $\partial \bar{S} := \bigcup_{C \in \mathcal{P}} \partial C$. Note that $\partial \bar{S}$ contains the boundary of S, but probably contains more points. Then $\dim_{\bar{\mathcal{M}}^*}(\partial \bar{S}) < \dim_{\mathcal{M}^*} \bar{S} = \dim_{\mathcal{M}} S$. It will suffice to show that $\mathrm{Cl}_{\mathcal{M}}(S) \setminus S \subseteq \partial \bar{S}$. By the discussion preceding [13, Proposition 2.3] (and, essentially, from the definition) $C \cap M^n$ is either empty or a strong cell. So $\mathcal{P}$ induces a decomposition $\mathcal{P}_M$ of $\mathrm{Cl}_{\mathcal{M}}(S)$ into refined strong cells. Now, if $p \in \mathrm{Cl}_{\mathcal{M}}(S) \setminus S$ then there is some $C \in \mathcal{P}$ such that $p \in \mathrm{Cl}_{\mathcal{M}}(C \cap M)$ and $p \notin C \cap M$. Thus, $p \in \partial_{\mathcal{M}}(C \cap M)$. Clearly, $p \notin C$, and $p \in \mathrm{Cl}_{\bar{M}}(C)$. So $p \in \partial C$. □

Remark 4.21. Since we have shown that dimension in $\mathcal{M}$ is the geometric dimension, and since $\mathcal{M}$ is weakly o-minimal, the proof of the analogous claim for o-minimal theories goes through essentially unaltered to the present context. However, the present proof is shorter.

Remark 4.22.
(1) The results of the present sub-section do not depend on the structure being an expansion of an ordered group, but only on the strong cell decomposition property.
(2) The results in the present sub-section are most probably all known. We include them for ease of reference and to demonstrate the usefulness of the canonical o-minimal completion as a tool in the study of weakly o-minimal non-valuational expansions of ordered groups.

5. O-minimal traces

Using the canonical o-minimal completion, we see that any weakly o-minimal expansion of an ordered group is obtained as the trace of an o-minimal structure on a dense subset. Of course, given an o-minimal structure $\mathcal{N}$ and a dense subset M, the structure induced on M can be wild (take $\mathbb{Q} \subseteq \mathbb{R}$, the latter equipped with the full field structure).

As already mentioned, one case where this construction does give a well behaved structure is in the case where M is the universe of an elementary sub-model of $\mathcal{N}$. One is then naturally led to ask whether this is the only way weakly o-minimal non-valuational structures are obtained. More precisely:

Definition 5.1. A weakly o-minimal expansion of an ordered group, $\mathcal{M}$ is an *o-minimal trace* if there is exists a dense pair of o-minimal structures $(\mathcal{N}, \mathcal{M}')$ such that $\mathcal{M}'$ has universe M and $\mathcal{M}$ is the structure induced on M from $(\mathcal{N}, \mathcal{M}')$.

In the present section we provide an example showing that ordered reducts of o-minimal traces need not themselves be o-minimal traces. We then provide an example of a weakly o-minimal non-valuational expansion of an ordered group that is not a reduct of an o-minimal trace.

Before we proceed to the example itself, one word concerning Definition 5.1 is in place. The theory of dense pairs, as developed in [3] requires that the o-minimal structure expands a group. Thus it seems natural to require in Definition 5.1 that the group structure on $\mathcal{M}$ be part of the language of $\mathcal{N}$. We point out that this assumption is harmless. As we have seen in Corollary 4.7 if $\mathcal{M}$ is a trace, then – up to a change of signature – $\bar{\mathcal{M}}$ is the only possible candidate for $\mathcal{N}$. Moreover, by Wencel's work, if $\mathcal{M}$ expands a group (field) then so does $\bar{\mathcal{M}}$. Of course, a priori, the group structure need not be part of the signature of $\mathcal{N}$. But there is a $\emptyset$-definable formula in $\varphi(x, y, \bar{z})$ and a parameter $\bar{a} \in N$ such that $\varphi(x, y, \bar{a})$ makes N into an ordered abelian group. Since $\mathcal{M} \prec \mathcal{N}$ one can also find such a parameter $\bar{a}' \in M$ in M, and there is no harm introducing $\varphi(x, y, \bar{a}')$ as an atomic function in the signature of $\mathcal{N}$.

In view of the above, we will henceforth assume that if $\mathcal{M}$ is an o-minimal trace then the group operation is part of the signature of the o-minimal structure witnessing it. Now we proceed to the construction of the example. Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_{\mathrm{alg}}$ the real closure of $\mathbb{Q}$. Then $(\mathbb{R}, \mathbb{R}_{\mathrm{alg}})$ is a dense pair. Let $\widetilde{\mathbb{R}_{\mathrm{alg}}}$ be the structure on $\mathbb{R}_{\mathrm{alg}}$ induced from $\mathbb{R}$, and $\mathbb{R}^{\pi}_{\mathrm{alg}}$ the reduct $(\mathbb{R}_{\mathrm{alg}}, \leq, +, \pi\cdot)$, i.e., the additive group of the field of real algebraic numbers equipped with an additional binary relation $y > \pi x$ (which, for convenience we denote as the unary function $x \mapsto \pi x$).

The structure $\mathbb{R}^{\pi}_{\mathrm{alg}}$ is a weakly o-minimal non-valuational expansion of an ordered group with no external limits, being the reduct of $\widetilde{\mathbb{R}_{\mathrm{alg}}}$, which has these properties by virtue of being an o-minimal trace. First we prove:

Lemma 5.2. *For all $\alpha \in \mathbb{Q}(\pi)$ the relation $\alpha x < y$ is $\emptyset$-definable in $\mathbb{R}^{\pi}_{\mathrm{alg}}$. Moreover, for $\alpha_1, \dots, \alpha_n \in \mathbb{Q}(\pi)$ the relation $\sum_{i=1}^{n} \alpha_i x_i < 0$ is $\emptyset$-definable.*

Proof. We will show that if $\alpha, \beta \in \mathbb{Q}(\pi)$ are definable in $\mathbb{R}^{\pi}_{\mathrm{alg}}$ then so is $\alpha + \beta$ and that π^n is definable for all $n \in \mathbb{Z}$. Since the domain of all of these functions is $\mathbb{R}_{\mathrm{alg}}$ and all these functions are strongly continuous, it will follow immediately from the definition of the canonical o-minimal completion that these functions are definable in the completion, which will finish the proof.

Indeed, if α, β are definable by S_α, S_β then $\alpha + \beta$ is defined by

$$(\exists z_1, z_2)(S_\alpha(x, z_1) \wedge S_\beta(x, z_2) \wedge y > z_1 + z_2).$$

If $x \mapsto \pi^n x$ is definable by $P_n(x, y)$ then $x \mapsto \pi^{n+1} x$ is defined by

$$(\exists z)(P_n(x, z) \wedge P_1(z, y)).$$

So to conclude the first part of the lemma it remains only to note that $P_{-1}(x, y)$ is given by $P_1(y, x)$.

For the second part of the lemma we will show that if $n > 1$ and $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then:

$$\sum_{i=1}^{n} \alpha_i x_i < 0 \iff (\forall x'_1, \dots x'_n) \left(\bigwedge_{i=1}^{n} x'_i < \alpha_i x \to \sum_{i=1^n} x'_i < 0 \right)$$

unless $x_i = 0$ for all i. The lef-to-right direction is clear, so we have to show the other implication. The assumption implies $\sum_{i=1}^{n} \alpha_i x_i \leq 0$, so we only have to check that equality cannot hold. First, observe that since the x_i are not all 0, we may assume that the x_i are linearly independent over $\mathbb{Q}$. Indeed, if $x_1 = \sum_{i=2}^{n} q_i x_i$ for $q_i \in \mathbb{Q}$ we get

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 \sum_{i=2}^{n} q_i x_i + \sum_{i=2}^{n} \alpha_i x_i = \sum_{n=2}^{n} (q_i \alpha_1 + \alpha_i) x_i$$

and $\{(q_i \alpha_1 + \alpha_i)\}_{i=2}^{n}$ are still independent over $\mathbb{Q}$. Now α_i are polynomials in π with rational co-efficients and x_i are real algebraic numbers, so we can write $\sum_{i=1}^{n} \alpha_i x_i = \sum_{i=0}^{k} \beta_i \pi^i = 0$ where β_i are $\mathbb{Q}$-linear combinations of $x_1, \dots, x_n$. So $\beta_i = 0$ for all i if and only if the α_i are all 0, which is impossible, since they are linearly independent. $\square$

Let $\overline{\mathbb{R}^{\pi}_{\mathrm{alg}}}$ be the set of definable cuts in $\mathbb{R}^{\pi}_{\mathrm{alg}}$. By [1] those are precisely the cuts defined by $\sum_{i=1}^{n} \alpha_i x_i$ for $\alpha_1, \dots, \alpha_n \in \mathbb{Q}(\pi)$ and $x_1, \dots, x_n \in \mathbb{R}_{\mathrm{alg}}$ (we do not claim, of course, that any two such cuts are distinct). The above lemma implies that the mapping $\alpha x \mapsto x \otimes \alpha$ extends to a bijection $\overline{\mathbb{R}^{\pi}_{\mathrm{alg}}} \to \mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ and this bijection is an isomorphism of ordered $\mathbb{Q}$-vector spaces (in fact, it is automatically an isomorphism of $\mathbb{Q}(\pi)$-vector spaces). Let T_0 be the theory asserting that for an ordered $\mathbb{Q}$-vector space R the mapping defined above (from $\overline{R}$ onto $R \otimes \mathbb{Q}(\pi)$) is an isomorphism of ordered $\mathbb{Q}(\pi)$-vector spaces.

Proposition 5.3. *The theory* $T_0 \models \mathrm{Th}(\mathbb{R}^\pi_{\mathrm{alg}})$ *has quantifier elimination in the language of ordered* $\mathbb{Q}$*-vector spaces expanded by predicates* $\sum_{i=1}^n \alpha_i x_i < 0$ *for all* $\alpha_1, \dots, \alpha_n$ *linearly independent over* $\mathbb{Q}$.

Proof. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be two saturated models of T_0 of the same cardinality. We will show that if $A_i \subseteq \mathcal{Q}_i$ are small divisible subgroups, and $f : A_1 \to A_2$ is a partial isomorphism (in the expanded langauge), then for every $a \in \mathbb{Q}_1$ the isomorphim f can be extended to a.

Since f is an isomorphism in the expanded language and $\mathcal{Q}_i \models T_0$ we can extend f to an isomorphism of $A_1 \otimes \mathbb{Q}(\pi)$ with $A_2 \otimes \mathbb{Q}(\pi)$. Identify a with $a \otimes 1$. By quantifier elimination in the theory of ordered divisible abeian groups and since $A_1 \otimes \mathbb{Q}(\pi)$ is a divisible abelian sub-group, $\mathrm{tp}(a/A_1 \otimes \mathbb{Q}(\pi))$ is determined by the cut it realises. Thus, it will suffice to show that the same cut over $A_2 \otimes \mathbb{Q}(\pi)$ is realised in $\mathcal{Q}_2 \otimes \mathbb{Q}(\pi)$. But since $\mathcal{Q}_2$ is a saturated model of T_0 we automatically get that $\mathcal{Q}_2 \otimes \mathbb{Q}(\pi)$ is saturated (for $\leq$), as required. □

As a corollary we get:

Corollary 5.4. *The canonical o-minimal completion of* $\mathbb{R}^\pi_{\mathrm{alg}}$ *is* $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ *as an ordered* $\mathbb{Q}(\pi)$*-vector space.*

Proof. We identify $\mathbb{R}_{\mathrm{alg}}$ with $\{r \otimes 1\}_{r \in \mathbb{R}_{\mathrm{alg}}}$. Since this mapping extends to an isomorphism of $\mathbb{Q}$-vector spaces (on $\overline{\mathbb{R}^\pi_{\mathrm{alg}}}$) it follows that $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ induces on the image of $\mathbb{R}_{\mathrm{alg}}$ a structure of an ordered $\mathbb{Q}$-vector space. Moreover, for all $\alpha_1, \dots, \alpha_i \in \mathbb{Q}(\pi)$ and $r_1. \dots r_n \in \mathbb{R}_{\mathrm{alg}}$ the cut $\sum_{i=1}^n \alpha_i r_i$ is externally definable in the image of $\mathbb{R}_{\mathrm{alg}}$ by the set $x < \sum_{i=1}^n r_i \otimes \alpha_i$. More generally, the relation $\sum_{i=1}^n \alpha_i x_i < 0$ is definable in the $\mathbb{Q}(\pi)$-vector space structure on $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$. Thus every atomically definable set of $\mathbb{R}^\pi_{\mathrm{alg}}$ is definable in the structure induced from $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$. Since, by the previous proposition, $\mathbb{R}^\pi_{\mathrm{alg}}$ has quantifier elimination, this implies that the structure induced on $\mathbb{R}_{\mathrm{alg}}$ from $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ expands $\mathbb{R}^\pi_{\mathrm{alg}}$. Since the theory of ordered $\mathbb{Q}$-vector spaces has quantifier elimination, we get that – in fact – the induced structure on $\mathbb{R}_{\mathrm{alg}}$ is precisely $\mathbb{R}^\pi_{\mathrm{alg}}$.

On the other hand, the completion of $\mathbb{R}^\pi_{\mathrm{alg}}$ is, as a set, $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ and as a structure it is an expansion of the $\mathbb{Q}(\pi)$-vector space structure on $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$, since the function $x \mapsto \pi x$ is a strongly continuous function on $\mathbb{R}_{\mathrm{alg}}$ (into $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$, and therefore its unique extension to $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ is part of the completion. By Corollary 4.7 and the previous pragraph, the conclusion follows. □

We will then show:

Lemma 5.5. *Let* S *be any set definable in* $\mathbb{R}^\pi_{\mathrm{alg}}$*, but not in* $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$*. Then the reduct of* $\mathbb{R}^\pi_{\mathrm{alg}}$ *given by* $(\mathbb{R}_{\mathrm{alg}}, \leq, +, S)$ *is not o-minimal.*

Proof. Any reduct $(\mathbb{R}_{\mathrm{alg}}, \leq, +, S)$ is a weakly o-minimal non-valuational expansion of an ordered group (as these properties are preserved under taking ordered reducts keeping the group structure). Therefore $(\mathbb{R}_{\mathrm{alg}}, \leq, +, S)$ has the strong cell decomposition. Thus, it will suffice to prove the lemma under the assumption that S is a refined strong cell.

The lemma is obvious in the case of 1-cells (and is vacuously true for 0-cells). Assume the lemma is true for strong cells in $(\mathbb{R}^{\pi}_{\mathrm{alg}})^n$, and we prove it for cells in $(\mathbb{R}^{\pi}_{\mathrm{alg}})^{n+1}$. So assume that S is a strong $(n+1)$-cell. The projection S_n on the first n-coordinates is a strong n-cell, and definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +, S)$. Thus, if S_n is not definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$, by induction $(\mathbb{R}_{\mathrm{alg}}, \leq, +, S)$ is not o-minimal. So we are reduced to the case where S_n is definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$. It will suffice to check the case where $S = \Gamma_f$ where $f : S_n \to \mathbb{R}$ is continuous and definable. If $f(S_n) \subseteq \mathbb{R} \setminus \mathbb{R}_{\mathrm{alg}}$ the structure is not o-minimal, so we may assume that $f(S_n) \subseteq \mathbb{R}_{\mathrm{alg}}$, and we have to show that in that case f is definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$.

Since f is strongly continuous $\bar{f}$, the iterative convex hull of f is continuous and definable in the canonical o-minimal completion, which by the previous lemma is $\mathbb{R}^{\pi}_{vs}$, implying that f is piecewise-linear. Let us assume, first, that f is linear. Then $f(v_1, \ldots, v_n) = \sum a_i v_i$ where $a_i \in \mathbb{Q}(\pi)$. We will show that the assumption that $f(S_n) \subseteq \mathbb{R}_{\mathrm{alg}}$ implies that $a_i \in \mathbb{Q}$ for all i. Assume not, then for a fixed $v = (v_1, \ldots, v_n) \in S_n$ we have that $f(v)$ is a polynomial in π with coefficients in $\mathbb{R}_{\mathrm{alg}}$. Since π is transcendental, the assumption that $f(v) \in \mathbb{R}_{\mathrm{alg}}$ implies that this is the zero polynomial. The coefficients of the polynomial $f(v)$ (as a polynomial in π) are linear combinations of $v_1, \ldots, v_n$ over $\mathbb{Q}$. Since S_n is open, we can choose $(v_1, \ldots, v_n)$ linearly independent over $\mathbb{Q}$ assuring that $f(v)$ is a non-zero polynomial in π, leading to a contradiction. Thus f is definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$. Now let $f_1, \ldots, f_k$ be the linear pieces of f. The sets $S_{n,i} := \{v \in S_n : f(v) = f_i(v)\}$ are thus definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$ for all $1 \leq i \leq k$. By the inductive assumption $S_{n,i}$ is definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$ for all i, implying that S is definable in $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$ as claimed. $\square$

In view of the discussion following Definition 5.1 these two lemmas combine to prove:

Theorem 5.6. $\mathbb{R}^{\pi}_{\mathrm{alg}}$ *is a weakly o-minimal non-valuational expansion of an ordered group which is not an o-minimal trace.*

Proof. By Corollary 4.7 and Lemma 5.4 if $\mathbb{R}^{\pi}_{\mathrm{alg}}$ is an o-minimal trace, it must be the trace of $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ as an ordered $\mathbb{Q}(\pi)$-vector space – possibly with a different signature. Thus, we are looking for an o-minimal reduct $\mathcal{R}$ of $\mathbb{R}^{\pi}_{\mathrm{alg}}$ with the properties that

- $\mathcal{R}$ expands $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$,
- The extension of $\mathcal{R}$ to $\mathbb{R}$ has the same definable sets as $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ (as a $\mathbb{Q}(\pi)$-vector space).

Since the extension of $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$ to $\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi)$ is $(\mathbb{R}_{\mathrm{alg}} \otimes \mathbb{Q}(\pi), \leq, +)$, a proper reduct of the $\mathbb{Q}(\pi)$-vecto space structure, it must be that $\mathcal{R}$ is a proper expansion of $(\mathbb{R}_{\mathrm{alg}}, \leq, +)$. But by Lemma 5.5 no such expansion is o-minimal. $\square$

Thus we obtain:

Corollary 5.7. *The class of o-minimal traces is not closed under taking ordered reducts.*

It is now natural to ask whether any weakly o-minimal non-valuational expansion of an ordered group is a reduct of an o-minimal trace. The following example shows that this, too, is not the case.

Let $\mathbb{Q}^{\pi}_{vs} := (\mathbb{Q}, \leq, +, \alpha\cdot)_{\alpha\in\mathbb{Q}(\pi)}$ where $\alpha\cdot$ is the binary relation $y > \alpha x$. Clearly, $\mathbb{Q}^{\pi}_{vs} \models T_0$, so by quantifier elimination $\mathbb{Q}^{\pi}_{vs} \equiv \mathbb{R}^{\pi}_{\mathrm{alg}}$. In particular, since $\mathbb{R}^{\pi}_{\mathrm{alg}}$ is weakly o-minimal and non-valuational, so is $\mathbb{Q}^{\pi}_{vs}$.

Theorem 5.8. *$\mathbb{Q}^{\pi}_{vs}$ is not a reduct of an o-minimal trace.*

Proof. Assume towards a contradiction that there exists an o-minimal structure $\mathcal{Q}$ with universe $\mathbb{Q}$, and $\mathcal{R} \succ \mathcal{Q}$ such that $(\mathcal{R}, \mathcal{Q})$ is a dense pair, and the structure induced on $\mathcal{Q}$ from $\mathcal{R}$ expands $\mathbb{Q}^{\pi}_{vs}$.

The desired conclusion now follows from [8] as follows. First, by Theorem A thereof, either $\mathcal{R}$ is linearly bounded or there is a definable binary operation $\cdot$ such that $(R, \leq, +, \cdot)$ is a real closed field. Since $\mathcal{Q} \prec \mathcal{R}$, in the latter case we would have a binary operation $\cdot_Q$ definable in $\mathcal{Q}$ making $(\mathbb{Q}, \leq, +, \cdot)$ into a real closed field, which is impossible, because $(\mathbb{Q}, +)$ is the standard addition on $\mathbb{Q}$, and therefore there exists at most one field structure (definable or not) expanding it, but $(\mathbb{Q}, +, \cdot)$ is not real closed.

So we are reduced to the linearly bounded case. By Theorem B of [8] every definable endomorphism of $(R, +)$ is $\emptyset$-definable. Thus, it will suffice to show that $x \mapsto \pi x$ is a definable endomorphism of $(R, +)$, since then it is $\emptyset$-definable, contradicting the assumption that $\mathcal{Q} \prec \mathcal{R}$. But this should now be obvious, since $x \mapsto \pi x$ is definable as a function from $\mathcal{Q}$ to $\overline{\mathcal{Q}}$ and strongly continuous, so by Corollary 4.7 $x \mapsto \pi x$ is a definable continuous function in $\mathcal{R}$, which is clearly an endomorphism of $(R, +)$. $\square$

The above proof actually shows more:

Corollary 5.9. *If $\mathcal{Q} \equiv \mathbb{Q}^{\pi}_{vs}$ then $\mathcal{Q}$ is not an o-minimal trace.*

However, we do have:

Example 5.10. $\mathbb{Q}^{\pi}_{vs} \equiv \mathbb{R}^{\pi}_{\mathrm{alg}}$. Thus, there exists a weakly o-minimal non-valuational expansion of an ordered group which is not a reduct of an o-minimal trace, whereas some elementarily equivalent structure is a reduct of an o-minimal trace.

Question 5.11. Is every weakly o-minimal non-valuational structure elementarily equivalent to a reduct of an o-minimal trace?

The above examples show that weakly o-minimal non-valuational expansion of ordered groups need not be o-minimal traces, or even reducts of o-minimal traces. It is natural, therefore, to ask, given such a structure, $\mathcal{M}$, whether it is necessarily the case that any $\mathcal{N} \equiv \mathcal{M}$ is not an o-minimal trace. We have partial answers to the above question. First we prove:

Lemma 5.12. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $\mathcal{M}$ is an o-minimal trace. Then so is any $\mathcal{N} \succ \mathcal{M}$.*

Proof. Let $\widetilde{\mathcal{M}}$ be an o-minimal structure on $\bar{M}$ witnessing that $\mathcal{M}$ is an o-minimal trace. By definition this means that M is the universe of an elementary sub-structure $\mathcal{M}_0 \prec \widetilde{\mathcal{M}}$ such that the structure induced on M from $\widetilde{\mathcal{M}}$ is $\mathcal{M}$. It follows that $\mathcal{M}$ has the same definable sets as $\mathcal{M}_0$ expanded by predicates for all $\mathcal{M}$-definable cuts.

Let $\mathcal{L}_0$ be the signature of $\mathcal{M}_0$, and let $\mathcal{N} \succ \mathcal{M}$. Then the reduct $\mathcal{N}_0 := \mathcal{N}|\mathcal{L}_0$ is o-minimal. Expand $\mathcal{N}$ by predicates for all $\mathcal{N}$-definable cuts. By Corollary 2.3 there exists an o-minimal $\overline{\mathcal{N}} \succ \mathcal{N}_0$ such that N is dense in $\bar{N}$ and the structure induced on $\mathcal{N}_0$ in the dense pair $(\overline{\mathcal{N}}, \mathcal{N}_0)$ is precisely the expansion of $\mathcal{N}_0$ by all cuts realized in $\overline{\mathcal{N}}$, i.e., by all $\mathcal{N}$-definable cuts. But since the theory of dense pairs is complete, and $\mathcal{N}_0 \equiv \mathcal{M}_0$, it follows that $(\overline{\mathcal{N}}, \mathcal{N}_0) \equiv (\widetilde{\mathcal{M}}, \mathcal{M}_0)$. In particular, $\mathcal{M}_0$ with the induced structure from $\widetilde{\mathcal{M}}$ is precisely $\mathcal{M}$, and is elementarily equivalent to $\mathcal{N}_0$ with the induced structure from $\overline{\mathcal{N}}$. Since $\mathcal{M}$ is also elementarily equivalent to $\mathcal{N}$, it follows that $\mathcal{N}_0$ expanded by all $\mathcal{N}$-definable cuts has the same definable sets as $\mathcal{N}$, so $\mathcal{N}$ is an o-minimal trace. □

The last lemma implies:

Corollary 5.13. *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $\mathcal{M}$ is an o-minimal trace. Then:*

- *Any $|M|^+$-saturated $\mathcal{N} \equiv \mathcal{M}$ is an o-minimal trace.*
- *If $\mathcal{M}$ is a prime model then any $\mathcal{N} \equiv \mathcal{M}$ is a o-minimal trace.*

Question 5.14. Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $\mathcal{M}$ is a reduct of an o-minimal trace. Is any $\mathcal{N} \succ \mathcal{M}$ a reduct of an o-minimal trace?

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