G-structures and other dense/codense expansions

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Workshop on Tame Expansions of O-minimal Structures
Konstanz, October 1-4, 2018

Outline

- geometric theories
- dense/codense expansions
- axiomatization
- Q-independent sets and back-and-forth
- basic properties of dense/codense expansions
- "Q-bases"
- effect on acl
- effect on forking and SU-rank (SU-rank 1 case)
- effect on one-basedness (SU-rank 1 case)
- effect on NIP-like conditions
- application: linearity
- separating "geometry" and "random noise"

Geometric theories

Definition

A first order theory T is called **geometric** if

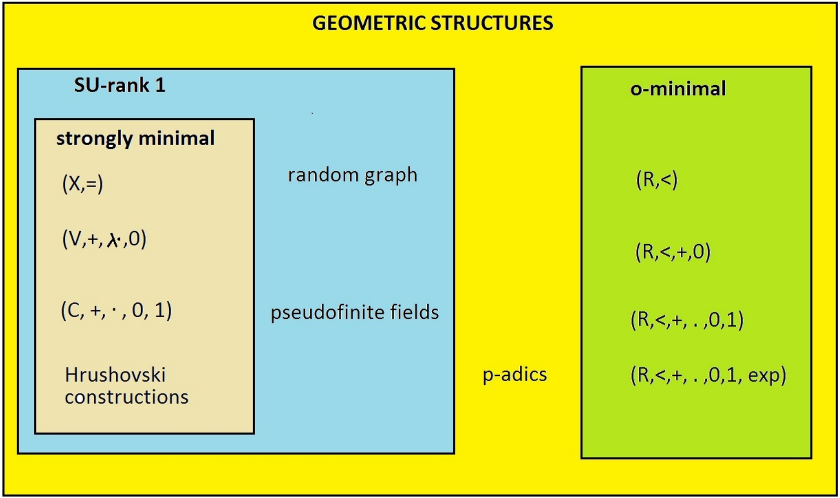
- in any model of T, *acl* satisfies the exchange property
- T eliminates quantifier $\exists^\infty$
  (i.e. for any $\phi(x, \bar{y})$ there is $n \in \omega$ such that whenever $|\phi(M, \bar{a})| > n$, $\phi(M, \bar{a})$ is infinite)

By **geometric structures** we mean models of geometric theories.

In any geometric structure, acl induces a pregeometry, with the natural notion of independence (denoted $\perp$) and dimension.

We can also define $\dim(\phi(\bar{x}, \bar{b})) = \max\{\dim(\bar{a}/\bar{b}) \mid \models \phi(\bar{a}, \bar{b})\}$.

A “map” of geometric structures



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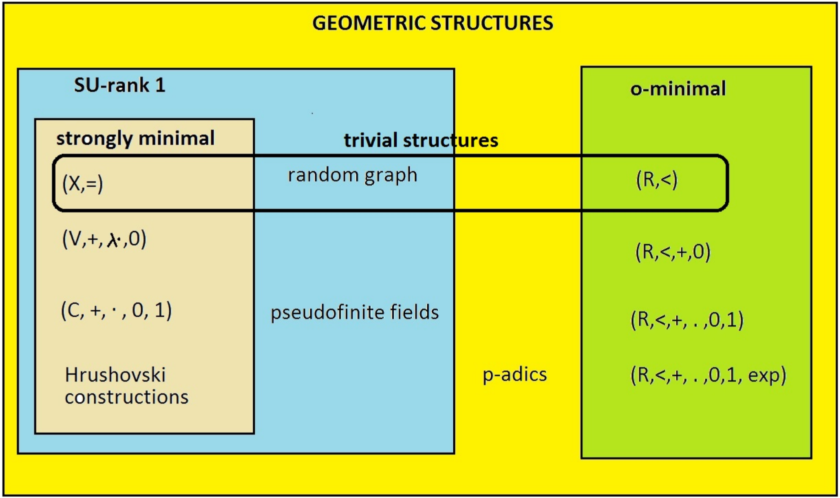

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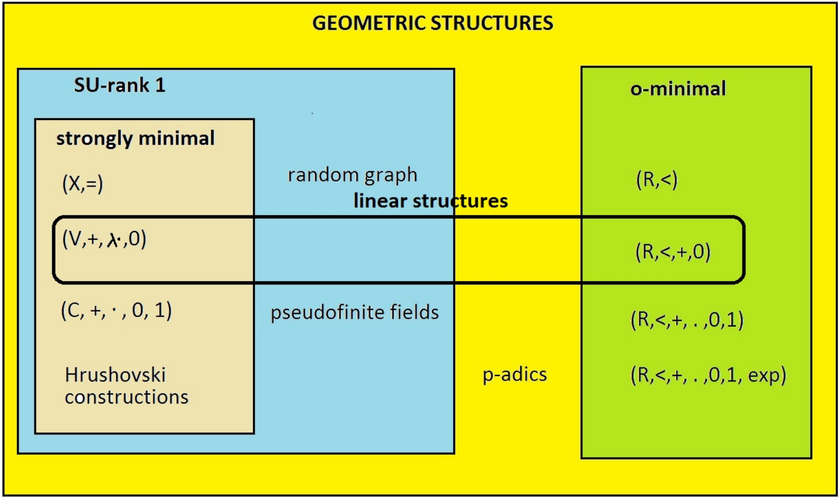

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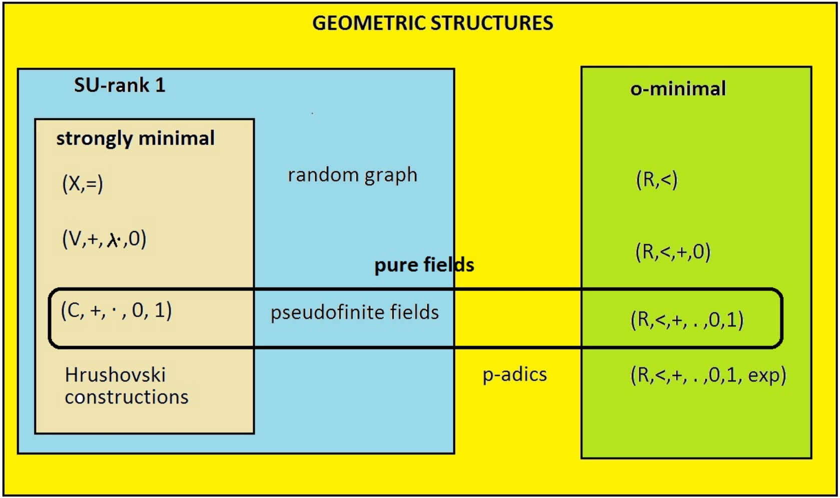
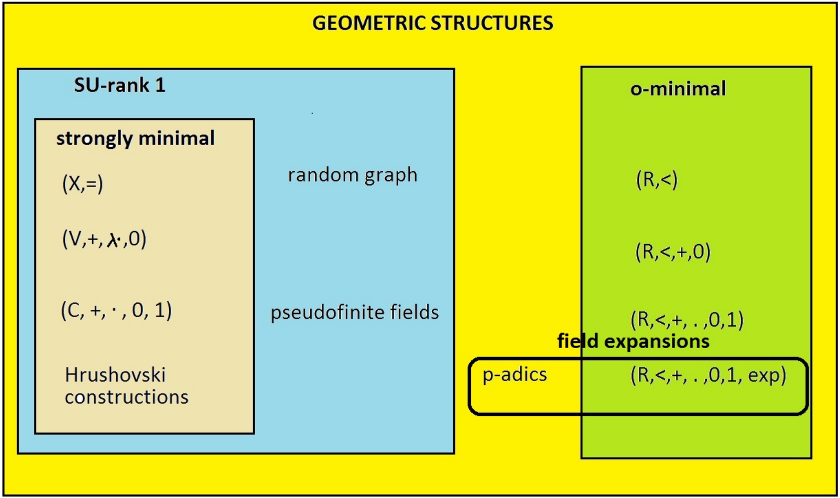


A “map” of geometric structures



Unary expansions

Add a new unary predicate symbol Q to the language $L = L(T)$ and consider models (M, Q) in the expanded language $L_Q = L \cup \{Q\}$, where $M \models T$.

Unary expansions: examples

- elementary pairs
  - belles paires (Poizat 1983) - stable case
  - dense pairs (van den Dries 1998) - o-minimal case
  - generic/lovely pairs (V. 2001; Ben Yaacov, Pillay, V. 2003) - supersimple SU-rank 1 and simple cases
  - lovely pairs of geometric structures (Berenstein, V., 2010)
- generic predicate (Chatzidakis, Pillay 1998)
- indiscernible sequence (Baldwin, Benedikt 2000)
- multiplicative subgroup of a field (Gunaydin, van de Dries 2005)
- independent dense subsets of o-minimal structures (Dolich, Miller, Steinhorn 2016)
- independent dense subsets of geometric structures (Berenstein, V. 2016)

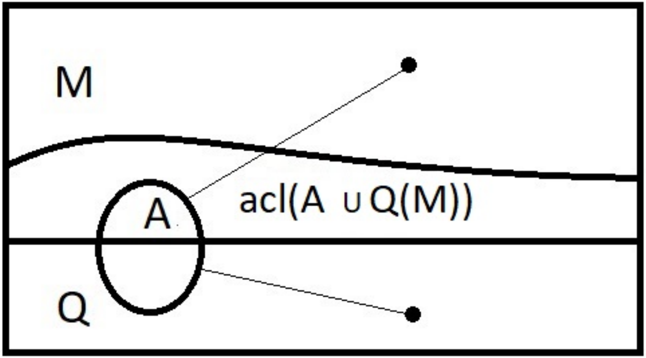
Dense/codense subsets

Definition

Let T be a geometric theory, $M \models T$.
A unary expansion (M, Q) of M is **dense/codense**, if any nonalgebraic 1-type $p(x, A)$ (in T) over a finite-dimensional $A \subset M$ has realizations in

- $Q(M)$ ("density" property)
- $M \setminus \operatorname{acl}(A \cup Q(M))$ ("codensity" or extension property).

Dense/codense in the geometric setting



Lovely pairs vs. H-structures

Definition

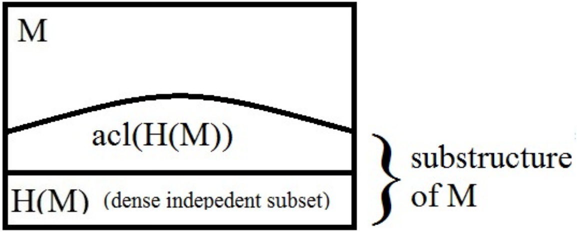
Given a geometric T, $M \models T$ and a dense/codense expansion (M, Q) of M,

- if $Q(M)$ is algebraically closed, we call (M, Q) a **lovely pair** (in this case $Q(M) \preceq M$);
- if $Q(M)$ is algebraically independent, we call (M, Q) an **H-structure**

Notation: we use (M, P) for lovely pairs and (M, H) for H-structures.

Lovely pairs vs. H-structures

In fact, for any H-structure (M, H),
$(M, \mathrm{acl}(H(M)))$ is a lovely pair (in particular, $\mathrm{acl}(H(M)) \preceq M$).



The field case: G-structures

We can consider an "intermediate" construction between lovely pairs and H-structures:

- Let T be the theory of an algebraically closed or real closed field $(K, +, \cdot, 0, 1)$, and let (K, H) be its H-structure.
- Let $G(K) =$ the multiplicative group generated by $H(K)$.
- (K, G) is dense-codense (we call it a G-structure), and $H(K) \subset G(K) \subset \mathrm{acl}(H(K))$.
- $G(K)$ is a free abelian group.
- $G(K)$ is linearly independent in K.
  This is an example of a group with the Mann property.
  Expansions of fields with multiplicative subgroups with the Mann property were studied by van den Dries and Gunaydin.

Axiomatizing density/codensity

Let T be a geometric theory (assume QE for convenience). Then a sufficiently saturated model of the following axioms in the langauge $L_Q = L \cup \{Q\}$ is a dense/codense expansion of T:

- T
- **density:** for any L-formula $\phi(x, \bar{y})$,

$$\forall \bar{y} \ (\exists^{\infty} x \ \phi(x, \bar{y}) \to \exists x \in Q \ \phi(x, \bar{y}))$$

- **extension (codensity):** for any L-formulas $\phi(x, \bar{y})$ and $\psi(x, \bar{y}, \bar{w})$ where ψ witnesses $x \in \operatorname{acl}(\bar{y}, \bar{w})$,

$$\forall \bar{y} \ (\exists^{\infty} x \ \phi(x, \bar{y}) \to \exists x \ (\phi(x, \bar{y}) \wedge \forall \bar{w} \in Q \ \neg\psi(x, \bar{y}, \bar{w})))$$

Theories T_P, T_H

Theories T_P and T_H can be obtained by adding axioms saying

- $\mathrm{acl}(P(M)) = P(M)$ (for T_P)
- $H(M)$ is acl-independent (for T_H)

Sufficiently saturated models of T_P / T_H are again lovely pairs/ H-structures.

Note: extension (codensity) axioms are not needed for T_H if T is "strongly non-trivial".

Theory T_G

What does a sufficiently saturated model (K^*, G) of $Th(K, +, \cdot, 0, 1, G)$ look like, for a G-structure generated by an H-structure?

Note that (K^*, G) is no longer generated by an H-structure: $G(K^*)$ will have divisible elements.

But $G(K^*)$ is still linearly independent, dense and codense.

Theory T_G

From the work of Gunaydin and van den Dries on fields with multiplicative subgroups having the Mann property, one gets that $Th(K, +, \cdot, 0, 1, G)$ can be axiomatized as follows:

For ACF:

- K is an algebraically closed field (of fixed characteristic)
- $G(K)$ is a subgroup of $K^\times$
- $G(K)$ is linearly independent over $\mathbb{Q}$ and satisfies the theory of free abelian groups (of infinite rank)

Theory T_G

For RCF:

- K is a real closed field
- $G(K)$ is a subgroup of $K^{>0}$
- $G(K)$ is dense in $K^{>0}$
- for any $n > 1$, $G^{[n]}$ (subgroup of nth powers) has infinite index in G
- $G(K)$ is linearly independent over $\mathbb{Q}$

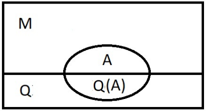
Theory T_G

Thus, in both cases we get a complete theory T^G.

From now on, by a G-structure we will mean a sufficiently saturated model of T^G.

Q-independence

For any unary expansion (M, Q) of a geometric structure M and a subset $A \subset M$, we say that A is Q-**independent**, if $A \mathop{\smile\!\!\!\!\!\!|}_{Q(A)} Q(M)$, i.e. for any finite $\bar{a}$ in A
$\dim(\bar{a}/Q(M)) = \dim(\bar{a}/Q(A))$.



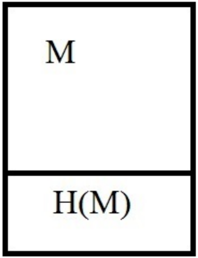
Note: Any $A \subset M$ can be extended to an Q-independent set by adding a subset of $Q(M)$.

Existence of H-structures (lovely pairs, G-structures)

Any structure $M \models T$ with an independent subset $H(M)$ can be extended to an H-structure (N, H) of T in such a way that M is H-independent in (N, H).

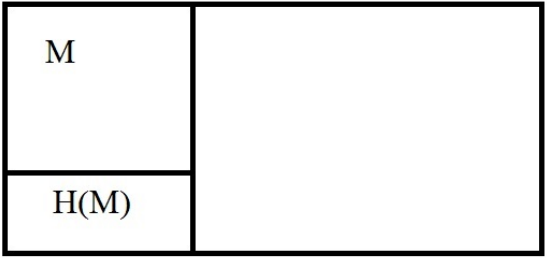
Existence of H-structures (lovely pairs, G-structures)

Given (M, H) ...



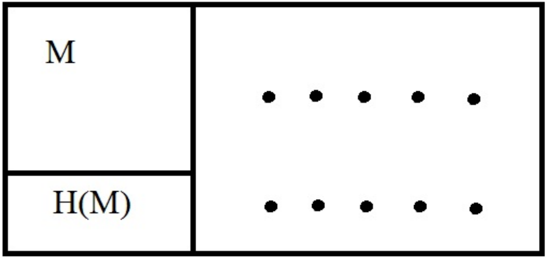
Existence of H-structures (lovely pairs, G-structures)

Given (M, H), take a saturated extension of M:



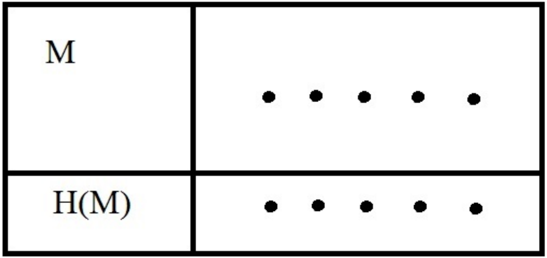
Existence of H-structures (lovely pairs, G-structures)

Choose two independent sets of independent realizations of all non-algebraic 1-types over M:



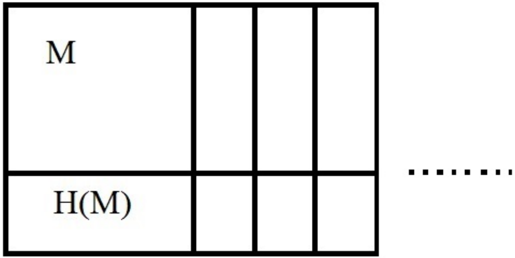
Existence of H-structures (lovely pairs, G-structures)

Include one of them in H:



Existence of H-structures (lovely pairs, G-structures)

Iterate ω times:



Existence of H-structures (lovely pairs, G-structures)

Take the union:

QE for Q-independent tuples

Given any two H-structures of T, (M, H) and (N, H), and H-independent tuples $\bar{a} \in M$, $\bar{b} \in N$, we have

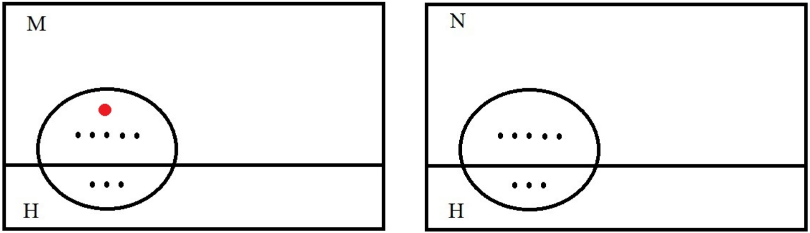
$$\mathrm{tp}(\bar{a}, H(\bar{a})) = \mathrm{tp}(\bar{b}, H(\bar{b})) \Rightarrow \mathrm{tp}_H(\bar{a}) = \mathrm{tp}_H(\bar{b}).$$

Same is true for P-independent tuples in lovely pairs.

In the case of G-structures, we need to add equality of (ordered) group types of G-parts of the tuples.

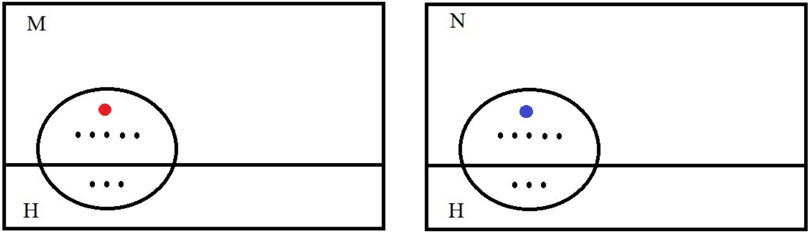
Back-and-forth for H-independent tuples

Given $c \in \operatorname{acl}(\bar{a})$ …



Back-and-forth for H-independent tuples

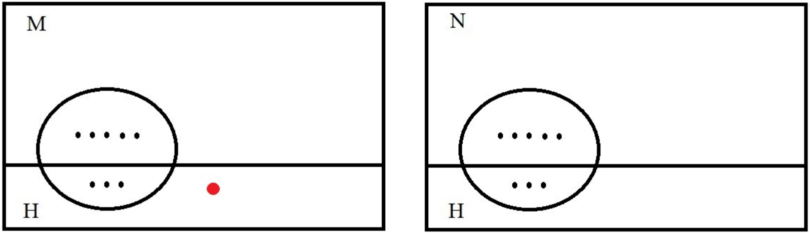
... find $d \in \mathrm{acl}(\bar{b})$, $\bar{b}d \equiv_L \bar{a}c$, with $c \in H \iff d \in H$.



Tuples are still H-independent.

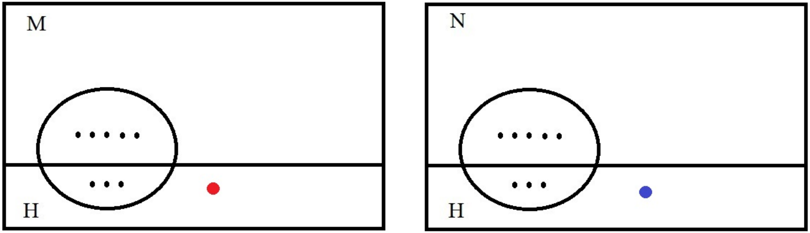
Back-and-forth for H-independent tuples

Given $c \in H(M)$ non-algebraic over $\bar{a}$ ...



Back-and-forth for H-independent tuples

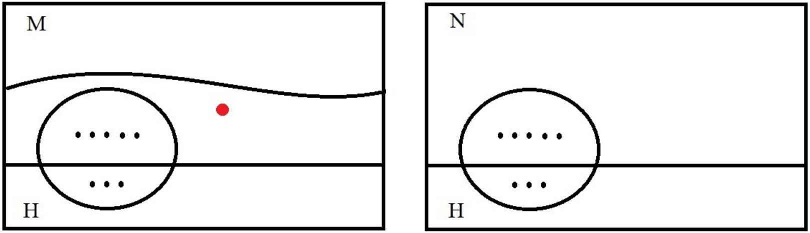
... find $d \in H(N)$ such that $\bar{b}d \equiv_L \bar{a}c$ (using density).



Tuples are still H-independent.

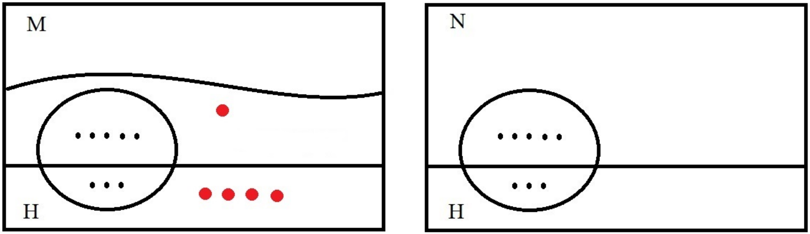
Back-and-forth for H-independent tuples

Given $c \in \mathrm{acl}(\bar{a}H(M)) \setminus H(M)$ ...



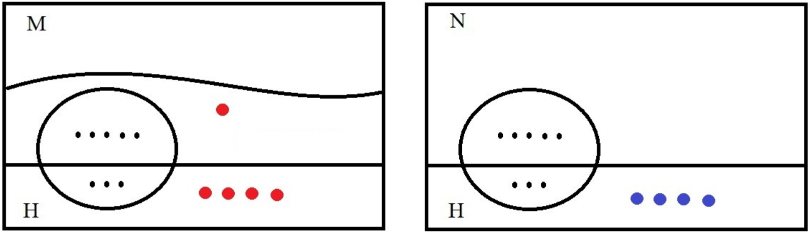
Back-and-forth for H-independent tuples

... find $\bar{h} \in H(M)$ such that $c \in \mathrm{acl}(\bar{a}\bar{h})$ ....



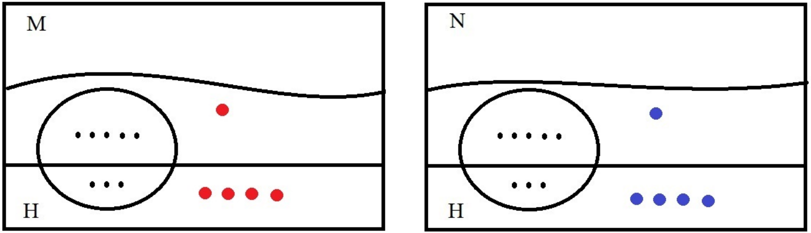
Back-and-forth for H-independent tuples

... then find $\bar{h}' \in H(N)$ such that $\bar{b}\bar{h}' \equiv_L \bar{a}\bar{h}$...



Back-and-forth for H-independent tuples

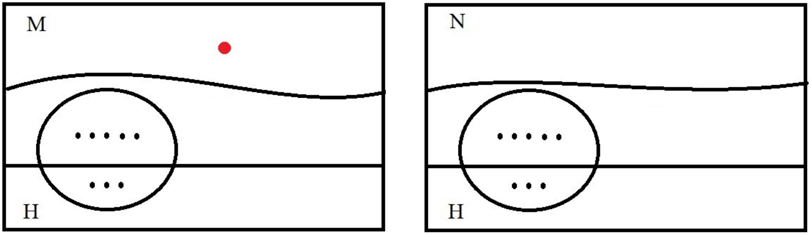
… then take $d \in N$ such that $\bar{b}\bar{h}'d \equiv_L \bar{a}\bar{h}c$.



$d \in \mathrm{acl}(\bar{b}\bar{h}') \Rightarrow d \notin H(N)$. Tuples are still H-independent.

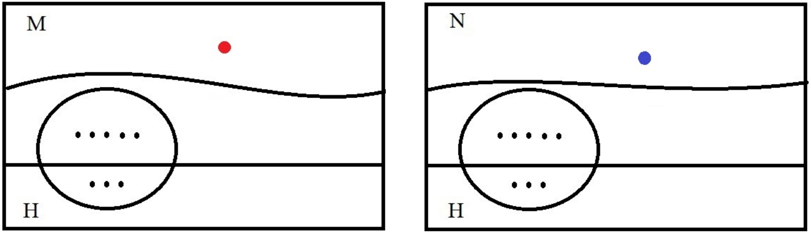
Back-and-forth for H-independent tuples

Given $c \in M \setminus \operatorname{acl}(\bar{a}H(M))$ ....



Back-and-forth for H-independent tuples

... find $d \in N \setminus \mathrm{acl}(\bar{b}H(N))$ with $\bar{b}d \equiv_L \bar{a}c$ (using extension).



Tuples are still H-independent.

Theory T_Q

As a consequence of QE for Q-independent tuples, we get:

- all lovely pairs / H-structures / G-structures are elementarily equivalent
- this gives rise to the complete theories
  - T_P (lovely pairs)
  - T_H (H-structures)
  - or (in the cases of ACF or RCF) T_G (G-structures)
- each of T_Q has an explicit axiomatization

Quantifier elimination

T_Q has QE down to boolean combination of formulas of the form $\exists \bar{y} \in Q\ \phi(\bar{x}, \bar{y})$, where ϕ is an L-formula.

Small and Large

We work in a sufficiently saturated $(M, Q) \models T_Q$.

- **Small closure** of $A \subset M$ is given by $\mathrm{scl}(A) = \mathrm{acl}(A \cup Q(M))$.
- An L_Q-definable subset X of M is **small** if $X \subset \mathrm{scl}(\bar{a})$ for a finite tuple $\bar{a} \in M$.
- Otherwise, we call X **large**.
- For any L_Q-definable set $X \subset M$, there is an L-definable set $Y \subset M$ such that $X \Delta Y$ is small.

Some properties of T_Q (T_P[1], T_H[2], T_G)

- When passing from T to T_Q the following properties are preserved:
  - stability (superstability, except for T_G)
  - simplicity (supersimplicity, except for T_G)
  - NIP
- In SU-rank 1 case:
  - one gets a reasonable description of forking in T_Q (in terms of forking over Q and forking of "Q-bases")
  - The SU-rank of T_Q reflects the "geometric complexity" of T.

[1] A. Berenstein, E. Vassiliev, On lovely pairs of geometric structures, Ann. Pure Appl. Logic, 161 (7), 2010, 866-878

[2] A. Berenstein, E. Vassiliev, Geometric structures with a dense independent subset, Selecta Mathematica - N.S., 22(1), 2016, 191-225

“Q-bases”

Suppose $C \subset M$ is Q-independent, $\bar{a} \in M$ a tuple.

We can split $\bar{a}$ into $\bar{a}'$ and $\bar{a}''$ where $\bar{a}$ is independent over $C \cup Q(M)$ and $\bar{a}'' \in \mathrm{acl}(\bar{a}'C\ Q(M))$.

We can find a finite $\bar{b} \in Q(M)$ such that $\bar{a}'' \in \mathrm{acl}(\bar{a}'\bar{b}C)$. Thus, $\bar{a}\bar{b}C$ is Q-independent.

Question: can we choose a minimal such $\bar{b}$ “canonically”?

“Q-bases”

H-structures: There is a unique minimal $\bar{b} \in H(M)$, call it the **H-basis** of $\bar{a}$ over C: $\bar{b} = HB(\bar{a}/C)$.

G-structures: There is no unique minimal $\bar{b} \in G(K)$, but all minimal $\bar{b}$ are interdefinable, in the group language, over $G(C)$. We call $\mathrm{dcl}_{gr}(\bar{b})$ the **G-basis** of $\bar{a}$ over C, $\mathbf{GB}(\bar{a}/C)$.

Lovely pairs: if M is supersimple of SU-rank 1 with EI/WEI/ GEI (e.g. ACF) we can take $Cb(\bar{a}/C)$.

What happens to acl?

Three closure operators in (M, Q): acl, acl_Q and scl.

Clearly, $\mathrm{acl}(A) \subset \mathrm{acl}_Q(A)$.

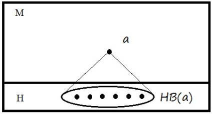
Any Q-independent acl-closed set is acl_Q closed. Thus, we have:

- $\mathrm{acl}_Q(A) \subset \mathrm{acl}(A \cup Q(M)) = \mathrm{scl}(A)$
- $\mathrm{acl}_Q(A) = \bigcap\{B | A \subset B,\ \ B = \mathrm{acl}(B) \text{ and is } Q - \text{independent}\}$

Question: when $\mathrm{acl}_Q = \mathrm{acl}$?

What happens to acl: H-structures

- $HB(\bar{a}) \in \mathrm{acl}_H(\bar{a})$
- $\mathrm{acl}_H(\bar{a}) = \mathrm{acl}(\bar{a}HB(\bar{a}))$
- acl_H-closed sets are always H-independent.
- If $a \in \mathrm{acl}(H(M))$, then a is interalgebraic with $HB(a)$ (in the sense of acl_H).



- $\mathrm{acl}_H = \mathrm{acl} \iff \mathrm{acl}$ is disintegrated, i.e. $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$.

What happens to acl: G-structures

Similar to H-structures:

$\mathrm{acl}_G(\bar{a}) = \mathrm{acl}(\bar{a}\mathbf{GB}(\bar{a}))$

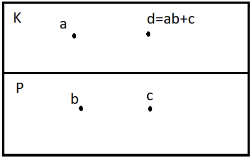
acl_G-closed sets are G-independent.

What happens to acl: lovely pair case

- In a pair (V, P) of vector spaces, any acl-closed set (subspace of V) is P-independent (by modularity), hence, $\mathrm{acl}_P = \mathrm{acl}$.

- In a pair (K, P) of algebraically closed fields, $\mathrm{acl}_P \neq \mathrm{acl}$:

  Take $a, b, c \in K$ algebraically independent, so that $b, c \in P(K)$ and $a \notin P(K)$. Let $d = ab + c$. Then $\mathrm{acl}_P(a, d) = \mathrm{acl}(a, b, c) \neq \mathrm{acl}(a, d)$.



- $\mathrm{acl} = \mathrm{acl}_P$ iff T is **linear**... More on this later in the talk.

Forking in (M, Q) in supersimple SU-rank 1 case

Let $C \subset B \subset M$, $\bar{a} \in M$.

Then $\bar{a} \downarrow^{Q}_{B} C \iff$

$\bar{a} \downarrow_{B\ Q(M)} C$ and $Q-\text{base of } \bar{a}\ C \downarrow_{Q-\text{base of}\, C} Q-\text{base of } B$.

Lovely pairs: $\bar{a} \downarrow^{P}_{B} C \iff$

$\bar{a} \downarrow_{B\ P(M)} C$ and $Cb(\bar{a}\ C/P(M)) \downarrow_{Cb(C/P(M))} Cb(B/P(M))$.

H-structures: (B, C H-independent) $\bar{a} \downarrow^{H}_{B} C \iff$

$\bar{a} \downarrow_{B\ H(M)} C$ and $HB(\bar{a}\ C) \downarrow_{HB(C)} HB(B)$

G-structures: (B, C G-independent) $\bar{a} \downarrow^{Q}_{B} C \iff$

$\bar{a} \downarrow_{B\ Q(M)} C$ and $GB(\bar{a}\ C) \downarrow_{GB(C)} GB(B)$

Properties of T_H: SU-rank 1 case

Description of forking in T_H for T supersimple of SU-rank 1:

Let $C \subset D$ be acl_H-closed, then $\mathrm{tp}(a/D)$ forks over C iff

- $a \in D \setminus C$ (becoming algebraic), or
- $a \in \mathrm{acl}(D \cup H) \setminus \mathrm{acl}(C \cup H)$ (becoming small), or
- $HB(a/D) \subsetneq HB(a/C)$ (reduction of H-basis)

It follows that $H(x)$ has SU-rank 1.

Properties of T_H: SU-rank 1 case

SU-rank of 1-types in T_H:
Suppose $C = \mathrm{acl}_H(C)$, a a single element.

- T trivial:
  - $SU(a/C) = 0 \iff a \in C$
  - $a \in \mathrm{scl}(C) \setminus C \Rightarrow SU(a/C) = 1$
  - $a \notin \mathrm{acl}(C) \Rightarrow SU(a/C) = 1$
- T nontrivial:
  - $SU(a/C) = 0 \iff a \in C$
  - $a \in \mathrm{scl}(C) \setminus C \Rightarrow SU(a/C) = |HB(a/C)|$
  - $a \notin \mathrm{scl}(C) \Rightarrow SU(a/C) = \omega$ (unless a is a "trivial" element)

Properties of T_H: SU-rank 1 case

Proposition

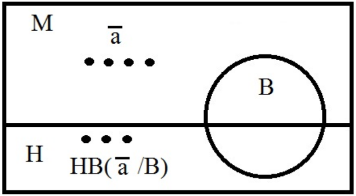
Let T be supersimple of SU-rank 1. Then T_H is supersimple

$$\text{and has } \mathrm{SU}-\text{rank } = \begin{cases} 1, & T \text{ is trivial} \\ \omega, & T \text{ nontrivial} \end{cases}$$

Properties of T_H: SU-rank 1 case

Canonical bases in T_H:

- T SU-rank 1, $B = \mathrm{acl}_H(B) \Rightarrow Cb_H(\bar{a}/B)$ is interalgebraic with $Cb(\bar{a}HB(\bar{a}/B)/B)$.



- In particular, we have geometric elimination of imaginaries in $(T_H)^{eq}$ down to T^{eq}.

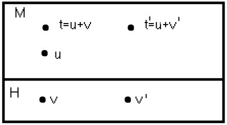
Properties of T_H: SU-rank 1 case

Recall: A theory is 1-based if $Cb(\bar{a}/B) \in \mathrm{acl}^{eq}(\bar{a})$.

1-basedness is not preserved when passing to T_H.

Properties of T_H: SU-rank 1 case

- Let T be the theory of infinite vector spaces over $\mathbb{F}_2$. Let (V, H) be an H-structure of T. Take $v \in H(V)$, $u \in V \setminus H(V)$, $t = u + v$.
- Then $Cb_H(t/u)$ is interalgebraic with $Cb(tHB(t/u)/u) = Cb(tv/u) = u$. However, $u \notin \mathrm{acl}_H(t) = span(t) = \{0, t\}$.



- But two independent realizations of $\mathrm{tp}_H(t/u)$ are enough: $u \in \mathrm{acl}_H(t, t')$. Thus T_H is "2-based" (true in general).

Properties of T_H: SU-rank 1 case

A recent result of Carmona:

- for $n \geq 2$: T n-ample $\iff$ T_H n-ample
- in particular:
  T 1-based (not 1-ample) $\Rightarrow$ T_H is CM-trivial (not 2-ample)

Properties of T_P: SU-rank 1 case

Proposition (V., 2001)

Let T be supersimple of SU-rank 1. Then T_P is supersimple

$$\text{and has SU}-\text{rank} \;=\; \begin{cases} 1, & T \text{ is trivial} \\ 2, & T \text{ one}-\text{based, nontrivial} \\ \omega, & T \text{ non}-\text{one}-\text{based} \end{cases}$$

(generalizing Buechler's 1991 result for s.m. structures)

If T is one-based then so is T_P (Ben Yaacov, Pillay, V. 2003).

Preservation of NIP in dense/codense expansions

- Berenstein, Dolich, Onshuus (2011):
  T is (strongly) dependent $\Rightarrow$ T_P is (strongly) dependent

- T is (strongly) dependent $\Rightarrow$ T_H is (strongly) dependent
  Idea of the proof of (T NIP $\Rightarrow$ T_H NIP):
  - Since T_H has QE down to H-bounded formulas, by a Chernikov-Simon's result, it suffices to show NIP over H;
  - Suppose T_H has IP over H: there is an L_H-formula $\phi(\bar{x}, \bar{y})$, $\bar{a} \in M$ and an indiscernible sequence $(\bar{b}_i : i < \omega)$ in $H(M)$ such that
  $$\models \phi(\bar{a}, \bar{b}_i) \iff i \text{ is even.}$$
  - We can replace $\phi(\bar{a}, H(M)^n)$ with $\psi(\bar{a}', H(M)^n)$ where ψ is an L-formula. Then ψ witnesses IP in T, a contradiction.

- RCF_G is dependent but not strongly dependent

Application of lovely pairs: a notion of linearity

What does it mean to be a **linear** geometric theory?

Linearity is well-defined and understood in:

- Strongly minimal theories: linearity = local modularity
- Supersimple theories of SU-rank 1: linearity = one-basedness (weaker than local modularity)
- o-minimal theories: CF-property, non-interpretability of an infinite field

Comparing strongly minimal and o-minimal settings

strongly minimal:

- linearity = local modularity
- linearity $\iff$ 1-basedness ($\bar{a} \equiv_B \bar{a}', \bar{a} \downarrow_B \bar{a}' \Rightarrow \bar{a} \downarrow_{\bar{a}'} B$)
- linearity $\iff$ no interpretable pseudoplane
- linear+nontrivial $\Rightarrow$ interpretability of infinite vector spaces
- non-local modularity $\not\Rightarrow$ intepretability of an infinite field

o-minimal:

- linearity = no interpretable infinite field
- local modularity $\Rightarrow$ linearity
- linearity $\not\Rightarrow$ local modularity
- linear+nontrivial $\Rightarrow$ interpretability of infinite vector spaces

In both settings, linearity $\iff$ any normal definable families of "plane curves" has dimension ≤ 1.

More on linearity: families of curves

Plane curve (with parameters $\bar{a}$) in a geometric structure $\mathcal{M}$ is a definable one-dimensional subset of M^2:

$C_{\bar{a}} = \{(x, y) | \mathcal{M} \models \phi(x, y, a_1, \ldots, a_n)\}$ and for any $(u, v) \in C_{\bar{a}}$, $\dim(uv/\bar{a}) \leq 1$.

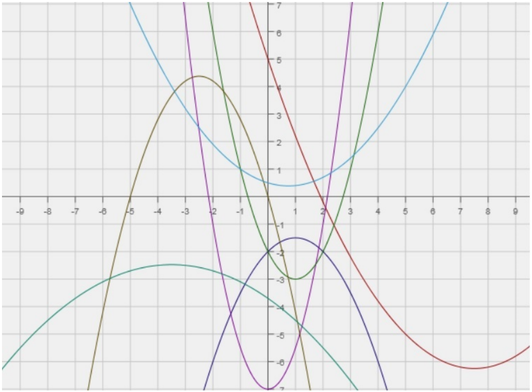
As we vary the parameters $\bar{a}$ over a definable subset of M^n we get a **definable family of plane curves**.

Nonlinear example

For example, $y = ax^2 + bx + c$, where $a \neq 0$, is a definable family of plane curves in $(\mathbb{R}, +, \cdot, 0, 1, <)$.

Since we are using 3 parameters (which can be chosen algebraically independent), this family has dimension 3.

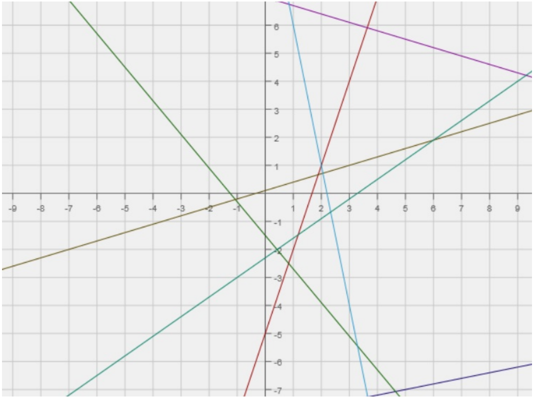
Nonlinear example



Different choices of (a, b, c) give different parabolas. Different parabolas can intersect in at most 2 different points. Such family is called normal: different curves have finite intersection.

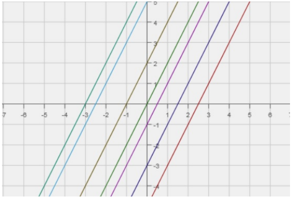
Another nonlinear example

A simpler example: $y = ax + b$, a normal family of dimension 2 (two-parameter family).



Linear example

In $(\mathbb{R}, +, 0, <)$ we can only form normal families of dimension ≤ 1 (one-parameter families): e.g. $y = (x + x) + a$.



Any two-parameter family that we can create, such as

$$x = a \vee y = b$$

or

$$y = (x + x) + a \vee y = (x + x) + b$$

will not be normal.

In search of general notion of linearity

A notion of linearity should:

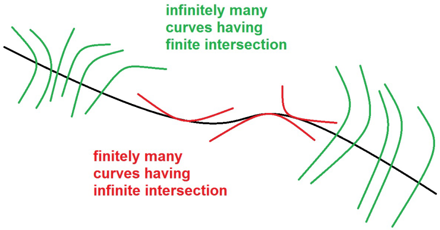
- have a definition in terms of combinatorial pregeometry (some form of modularity)
- have a definition in terms of definable sets (families of plane curves)
- be equivalent to non-(type)-definability of certain complicated structures and/or configurations (e.g. infinite fields, pseudoplanes, quasidesigns)
- be closed under reducts
- in the nontrivial case, imply certain connection with projective geometries over division rings, definability of infinite groups (vector spaces)
- should have a natural extension to non-geometric context (e.g. one-basedness in stable theories)

Main challenges in the general geometric case

- doing forking calculus without canonical bases (a strong tool in strongly minimal and SU-rank 1 cases)
- no definable topology (a strong tool in o-minimal or C-minimal cases)

Generic linearity

Call a definable family of plane curves **almost normal**, if each curve has infinite intersection with only finitely many other curves.



A geometric structure $\mathcal{M}$ is **generically linear**, if any almost normal family of plane curves in $\mathcal{M}$ has dimension ≤ 1.

In the strongly minimal and o-minimal cases: generic linearity $=$ linearity .

Theorem[3]

The following are equivalent for any geometric theory T:

1. T is **generically linear**
(any almost normal definable family of plane curves has dim ≤ 1)

2. T is **weakly locally modular**
(for any a, b, C such that $a \in \operatorname{acl}(bC)$, there exist $D \downarrow abC$ and $c \in \operatorname{acl}(CD)$ such that $a \in \operatorname{acl}(bcD)$)

3. T is **weakly 1-based**
(for any $\bar{a}, B$ there is $\bar{a}' \equiv_B a$ with $\bar{a} \downarrow_B \bar{a}'$ and $\bar{a} \downarrow_{\bar{a}'} \bar{B}$)

4. T has no complete type definable **almost quasidesign**.
(a pseudoplane-like configuration)

5. $\operatorname{acl}_P = \operatorname{acl}$ in any $(M, P) \models T_P$

6. scl is modular in any $(M, P) \models T_P$
($a \in \operatorname{scl}(bC) \Rightarrow$ there exists $c \in \operatorname{scl}(C)$ such that $a \in \operatorname{scl}(bc)$)

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[3] A. Berenstein, E. Vassiliev, Weakly one-based geometric theories, J. Symb. Logic, 77, No. 2, June 2012

Connection with “classical” lionearity

Generic linearity (weak local modularity, weak 1-basedness) is equivalent to

- local modularity, in the strongly minimal case
- one-basedness, in the SU-rank 1 case
- linearity (CF-property), in the o-minimal case
- linearity as defined by F. Maalouf, in the geometric C-minimal case

It is also closed under reducts.

Geometry of the small closure

Moreover, for a generically linear T we have:

- the geometry of $acl(- \cup P(M))$ is either trivial or splits in a disjoint union of projective geometries over division rings;
- the geometry of acl is a disjoint union of subgeometries of projective geometries over division rings;
- if T is ω-categorical (only one countable model), nontrivial and generically linear, then T_P interprets an infinite vector space over a finite field.

Structure induced on H: generic trivialization[4]

It turns out that the structure induced on $H(M)$ keeps the "random noise" while "forgetting" the geometry of M.

Given a sufficiently saturated H-structure (M, H), consider $H(M)$ together with traces of definable sets of M (without parameters).

Denote such structure by $H^*(M)$, and its theory by T^* (**generic trivialization** of T).

[4] A. Berenstein, E. Vassiliev, Generic trivializations of geometric theories, Math. Logic Q., 60, No. 4-5, 289-303 (2014)

Structure induced on H: generic trivialization

- T^* is a trivial ($\mathrm{acl}(A) = A$) geometric theory, with QE
- As T^* is a reduct of T_H and is trivial, we can expect T "nice" $\Rightarrow$ T^* "nice".
- More interestingly, we often have T^* "nice" $\Rightarrow$ T "nice".
- To show this we need $H^*(M)$ to somehow "control" M.
- Can be done by working in $\mathrm{acl}(H(M))$ which is a sufficiently saturated model of T.
- Easier when $\mathrm{acl} = \mathrm{dcl}$: any set definable over $\mathrm{dcl}(H(M))$ is also definable over $H(M)$.

Moving parameters into $H(M)$: when $\mathrm{acl} \neq \mathrm{dcl}$

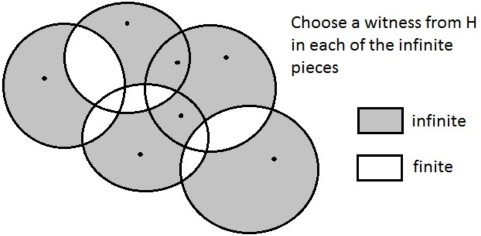
The main tool that allows to move parameters into $H(M)$:

Proposition

Let (M, H) be an H-structure of a geometric theory T. Let $D \subset M$ be a set L-definable over $\mathrm{acl}(H(M))$. Then there exists $D' \subset D$ L-definable over $H(M)$ such that $D \setminus D'$ is finite.

Idea of the Proof

- Let $D = \phi(M, \bar{a}, \bar{h})$, where $\bar{h} \in H(M)$ and $\bar{a} \in \mathrm{acl}(\bar{h})$, witnessed by an L-formula $\psi(\bar{y}, \bar{h})$.
- Consider all the conjugates of D over $\bar{h}$ (there are finitely many).
- Each conjugate is cut into disjoint pieces by boolean combinations with other conjugates.



- In each **infinite** disjoint piece pick an element of H, say c. Then the piece is L-definable over $\bar{h}c$ by $\forall \bar{y}(\psi(\bar{y}, \bar{h}) \to (\phi(x, \bar{y}, \bar{h}) \leftrightarrow \phi(c, \bar{y}, \bar{a})))$.

Properties of T^*: strongly minimal case

Proposition

T is strongly minimal $\iff$ T^* is strongly minimal (and, thus, is the theory of equality)

Proof:
$\Rightarrow$ clear
$\Leftarrow$ Suppose there is an infinite co-infinite $D \subset M$ definable over $\mathrm{acl}(H(M))$. Choose $D' \subset D$, definable over $H(M)$, with $D \setminus D'$ finite. Then $D' \cap H(M)$ is definable in $H^*(M)$ and is infinite and co-infinite.

Properties of T^*: SU-rank 1 case

Proposition

T is supersimple SU-rank 1 $\iff$ T^* is supersimple SU-rank 1

Proof:
$\Rightarrow$ follows from T_H being supersimple and $H(x)$ having SU-rank 1 in (M, H).
$\Leftarrow$ Assume T is not supersimple of SU-rank 1. Work over $\mathrm{acl}(H(M))$. Assume $\phi(x, \bar{a})$ is a non-algebraic formula that k-divides over $\emptyset$, witnessed by an indiscernible sequence $(\bar{a}_i : i < |T|^+)$.
For every $i < |T|^+$ we can find $\psi_i(x, \bar{h}_i)$ with $\bar{h}_i \in H(M)$ defining a co-finite subset of $\phi(M, \bar{a}_i)$. We may assume that $\psi_i = \psi$ are the same for each i and $\bar{h}_i$ form an indiscernible sequence. Then $\psi(x, \vec{h}_0)$ defines an infinite subset of $H^*(M)$ that k-divides over $\emptyset$ in T^*, a contradiction.

Properties of T^*: NIP case

Proposition

T is NIP $\iff$ T^* is NIP

Idea of the proof:
$\Rightarrow$ follows from T_H being NIP.
$\Leftarrow$ Suppose T has IP witnessed by $\phi(x, \bar{y})$ and an indiscernible sequence $I = (\bar{b}_i : i \in \omega)$ and $a \in M$ (non-algebraic over I) such that

$$\models \phi(a, \bar{b}_i) \iff i \text{ even.}$$

As in the SU-rank 1 case, we can "pull" I into $H(M)$.

Some questions

- If T is linear (i.e. weakly 1-based), is there a way to "recover" M from the geometry of scl and $H^*(M)$?
- For a geometric T, does T_H have elimination of $\exists^\infty$? (true for formulas $\phi(x, \bar{y})$ that imply $\bar{y} \in H$)
- Imaginaries in T_H?
  (Dolich, Miller, Steinhorn: EI holds in o-minimal case;
  we also have GEI in SU-rank 1 case)
- if T is nontrivial and linear, can we interpret an infinite group in T, or, at least, T_P?
- structure of weakly 1-based groups
- weak 1-basedness beyond geometric theories?
  (progress by Boxall, Bradley-Williams, Kestner, Omar Aziz, Penazzi, NDJFL 2013)

THANK YOU!