Definable groups

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Introduction

o-Minimal expansion of a real closed field

$$\mathcal{R} := \langle \mathrm{R}, <, +, \cdot, \cdots \rangle \qquad\qquad K := R(i)$$

Definable := definable (with parameters) in $\mathcal{R}$.
Definable group: A group G,

$$G \subseteq R^k \text{ \& graph } \Gamma(\cdot) \subseteq R^{3k} \text{ definable.}$$

Examples

- Algebraic subgroups of $GL(n, K)$.
- Semialgebraic groups: $(R^{>0}, \cdot)$;
- Groups definable in $\mathbb{R}_{exp}$, e.g.: $\left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{R} \right\}$

  (Peterzil-Pillay-Starchenko(2002)).

dim=1

- $[0,1) \subseteq \mathbb{R}$, addition $\mod 1$.
- $\mathbb{T} := \left\{a+bi \in \mathbb{C} : a^2+b^2=1\right\} \leq \mathbb{C}^*$.

$[0,1) \ncong \mathbb{T}$ semialgebraically.
(Mamino 2011): IF

$$\phi : [0,1) \to \mathbb{T}$$

is a nontrivial definable homomorphism. WMA ϕ is 0-definable & everywhere differentiable.

$$\lim_{x \to 1} \phi(x) = \phi(0) = 1$$

$$\phi'(x) = \phi(x)\phi'(0)$$

$\therefore \phi(x) = e^{x\phi'(0)}$ and $1 = \lim_{x\to 1} \phi(x) = e^{\phi'(0)}$.
$\therefore \phi'(0) = 2k\pi i$ for some $k \in \mathbb{Z}$, $k \neq 0$,
$\therefore \pi$ is 0-definable, CONTRADICTION.

Theorem (Pillay1988)

G d.group $\Longrightarrow$ *G d.manifold & multiplication and inversion continuous.*

Remarks

1. If $R = \mathbb{R}$, a definable group is a Lie group.
2. Not every Lie group is definable in an o-minimal expansion of the real field.

Topological group $\Longrightarrow$ regular space.
By *Robson's embedding theorem:*

$$G \hookrightarrow R^m.$$

$\therefore$ the topology of (the image of) G is induced by that of R^m.

Corollary

G d.group.

1. *$H \leq G$, H definable $\Longrightarrow$ H closed.*
2. *G infinite $\Rightarrow \exists H \leq G$, H definable infinite abelian.*
3. *IF $H \leq G$ definable THEN:*
$$H \text{ open} \iff [G : H] \text{ finite} \iff \dim H = \dim G.$$
4. *G^0: definably connected component of the identity.*
G^0 = smallest d.subgroup of G of finite index.

Definably connected group = d.group no proper d.subgroup of finite index.

Exercise 1. G d.group.

1. G has descending chain condition on d.subgroups (dcc).
2. For *any* $X \subseteq G$,

$$C_G(X) := \text{centralizer of } X \text{ in } G \leq G$$

   is definable.
3. If G is definably connected. Then,
   1. $\dim G = 1 \implies G$ commutative.
   2. Any definable action of G on a finite set is trivial.
   3. $G' := \langle \{xyx^{-1}y^{-1} : x, y \in G\} \rangle$ finite $\implies G$ is abelian.

Any subset X of a d.group G is contained in a smallest d.*subgroup*

$$h(X) := \text{definable hull of } X \leq G.$$

Lemma

G d.group.

1. *$X \subseteq G$ set of commuting elements $\Longrightarrow h(X) \leq G$ is abelian.*
2. *$X \trianglelefteq G \Longrightarrow h(X) \trianglelefteq G$.*

Proof.

1. $Z(C_G(X))$ is abelian d. and contains X
$\therefore h(X) \subseteq Z(C_G(X))$.
2. For any $g \in G$, $X^g = X$
$\therefore X \subseteq h(X) \bigcap h(X)^g$ $\qquad \therefore h(X) = h(X) \bigcap h(X)^g$
$\therefore h(X) \leq h(X)^g$, for any $g \in G$
$\therefore h(X) = h(X)^g$. $\square$

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Euler characteristic and torsion

$X \subseteq R^k$ definable set. Partition of X into cells: $X = \bigsqcup_{C \in \mathcal{D}_X} C$.

The (model theoretic) Euler characteristic of X is

$$E(X) := \sum_{C \in \mathcal{D}_X} (-1)^{dim(C)}.$$

Remark.

1. X finite $\Longrightarrow E(X) = card(X)$.
2. X triangulated: $E(X) = \#$vertices -$\#$edges + $\#$faces - $\cdots$
3. $R = \mathbb{R}$ and X is closed and bounded $\Longrightarrow E(X) = \chi(X)$.
   BUT $\chi((a,b)) = \chi(\text{point}) = 1 = E(\text{point}) \neq E((a,b)) = -1$.

Theorem

1. X and Y disjoint $\Longrightarrow E(X \cup Y) = E(X) + E(Y)$.
2. $E(X \times Y) = E(X)E(Y)$.
3. $f : X \to Y$ definible & $E(f^{-1}(y)) = m$, for each $y \in Y \Longrightarrow$
$$E(X) = E(\bigcup_{y \in Y} f^{-1}(y)) = E(Y) \cdot m.$$
4. $f : X \to Y$ definable bijection $\Longrightarrow E(X) = E(Y)$.

Example

$E(\mathbb{T}) = 0$: Write $\mathbb{T}$ as a union of two 0-cells and two 1-cells, then $E(\mathbb{T}) = 1 + 1 + (-1) + (-1)$.

Theorem (Strzebonski 1994)

G d.group.

1. *If* $K \leq H \leq G$ *definable then*
$$E(G) = E(H)E(G/H)$$
and
$$E(G/K) = E(G/H)E(H/K).$$
2. $p | E(G) \implies G$ *has an element of order* p*,* p *prime.*

Proof.

1. By definable choice $\exists f : G \to H \times (G/H)$ definable bijection.

Proof (cont.)

2 (If $p|E(G)$ then G has an element of order p). Action of $\mathbb{Z}/p\mathbb{Z}$ on

$$X = \{(a_0, \ldots, a_{p-1}) \in G^p : a_0 \cdot \cdots \cdot a_{p-1} = 1\}$$

by cyclic permutations. The orbit of $x \in X$ has either 1 element or p elements.

$$X = \bigcup_{|orb(x)|=1} orb(x) \cup \bigcup_{|orb(x)|=p} orb(x).$$

$X \to G^{p-1}$ definable bijection $\Longrightarrow E(X) = E(G)^{p-1}$ is divisible by p,

$$E(\bigcup_{|orb(x)|=p} orb(x)) = pE(\{orb(x) : |orb(x)| = p\}),$$

$\therefore$ p divides $E(\bigcup_{|orb(x)|=1} orb(x))$.
$\therefore$ $\exists a \neq 1$ S.T. $x = (a, \ldots, a) \in X$, I.E. $a^p = 1$. $\square$

Exercise 2

G d.group.

1. $E(G) = 0 \Longrightarrow G$ has elements of order p, for each prime p.
2. $E(G) = \pm 1 \Longleftrightarrow G$ is torsion-free.
3. G torsion-free $\Longrightarrow G$ definably connected.
4. Quotients of torsion-free d.groups are torsion-free.

Theorem (Strzebonski 1994)

G infinite d.group. Then,

1. $\nexists n \forall x \in \ |\langle x\rangle| \leq n$, *I.E. G does not have bounded exponent;*
2. *G abelian $\Longrightarrow$ the torsion subgroup $G[m]$ is finite, for each $m > 0$.*

Exercise 3.

G abelian d.connected group $\Longrightarrow$ divisible.

A definable group G is definably compact if it is closed and bounded.

Theorem (Peterzil-Steinhorn (1999))

G d.group.
G NOT d.compact $\Longrightarrow \exists H \leq G$ definable, $\dim H = 1$ & H torsion-free.

Theorem (Edmundo-O_(2004))

G d.connected d.compact abelian group $\Longrightarrow$ for each $m > 0$, the torsion subgroups

$$G[m] \cong (\mathbb{Z}/m\mathbb{Z})^{\dim G}.$$

Exercise 4.

1. G d.connected d.compact group $\implies E(G) = 0$,
   $\therefore G$ has p-torsion for each prime p.
2. G abelian d.group $\implies \exists$ d.subgroups

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n \leq G$$

   S.T.

   1. G/G_n is d.compact,
   2. G_{i+1}/G_i is a torsion-free one-dimensional group $(0 \leq i < n)$.

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The Lie algebra of a definable group

Definition

$m \geq 0$. Definable C^m-manifold of dimension n:

- definible set: M
- *definable C^m-atlas* on M:

$$\{(U_i, \varphi_i)\}_{i=1}^{s}$$

  $\varphi_i : U_i \to V_i$ definable bijection, $V_i \subseteq R^n$ open S.T. the transition maps are d.C^m-maps.
- identify two d.C^m-atlas on M if their union is a d.C^m-atlas on M.

Remark

G d.group $\Longrightarrow$ for each $m \geq 0$, G is a definable C^m-group.

Definition

M d.C^1-manifold, $a \in M$. Tangent space of M at a:

$$T_a(M) := \left\{ \overline{\alpha} \mid \alpha : [0,1] \to M, \alpha(0) = a, \alpha \text{ d.}C^1\text{-map} \right\},$$

$\overline{\alpha} = \overline{\beta} :\Leftrightarrow \alpha'(0) = \beta'(0)$.

$\varphi : U \to R^n$, $a \in U \subseteq M$ & identify $T_a(M)$ with R^m via $\overline{\alpha} \mapsto (\varphi \circ \alpha)'(a)$.

Definition

M, N d.C^m-manifolds, $f : M \to N$ d.C^m-map.
Differential of f at point $a \in M$:

$$d_a f : T_a M \to T_{f(a)} N : \overline{\alpha} \mapsto d_a f(\overline{\alpha}) := \overline{f \circ \alpha}.$$

Remark.

$\dim f(M) = rk(d_a f)$ for some $a \in M$.

Exercise 5.

$f : M \to N$ d.C^m-map.

1. If M is d.connected, $d_a f = 0$ for all $a \in M \iff f$ is constant.
2. f injective $\implies d_a f$ injective, for some $a \in M$.

Lemma

G & H d.C^m-groups , $f : G \to H$ d.homomorphism $\Longrightarrow$ f d.C^m-map.

Proof.

$g \in G$.

$$L_g : G \to G : h \mapsto gh$$

d.C^m-map.
(V, ψ) of G.
$\therefore \exists V_0 \subseteq V$ open definable, $\exists (U, \varphi)$ of H S.T. $f(V_0) \subseteq U$.
$\therefore \exists c \in V_0$ S.T. $\varphi \circ f \circ \psi^{-1}$ is d.C^m at $\psi(c)$

$$\begin{tikzcd} V_0 \arrow[r, "f"] & U \arrow[d, "\varphi"] \\ R^n \supseteq \psi(V_0) \arrow[u, "\psi^{-1}"] \arrow[r, "\varphi\circ f\circ\psi^{-1}"'] & \varphi(U) \subseteq R^n. \end{tikzcd}$$

$\therefore f$ is d.C^m at c. $\square$

Exercise 6.

G d.group $\Longrightarrow$ *unique* d.C^m-group structure.

Theorem

IF G d.connected, $f_1, f_2 : G \to G$ d. homomorphisms THEN

$$f_1 = f_2 \iff d_e f_1 = d_e f_2.$$

Proof.

$h \in G$,

$$f_i = L_{f_i(h)} \circ f_i \circ L_{h^{-1}} \qquad (i = 1, 2).$$

$$\therefore d_h f_i = d_e L_{f_i(h)} \circ d_e f_i \circ d_h L_{h^{-1}}.$$

$D := d_e f_1 = d_e f_2$.

$\therefore$ f_1 and f_2 solutions (neighbourhood of e) of differential equation

$$\left.\begin{array}{rc} d_x(\phi) = & F(x, \phi(x)) \\ \phi(e) = & e \end{array}\right\},$$

$F(x, y) := d_e L_y D d_x L_{x^{-1}}$. $\square$

Theorem (Peterzil-Pillay-Starchenko 2000)

IF G *d.group,* X *d.set,* $\alpha : G \curvearrowright X$ *d.transitive.*
THEN for every $m \geq 0$,

$$X \text{ and } G \text{ are d.} C^m\text{-manifolds}$$

S.T.

- G *d.*C^m*-group,*
- α *d.*C^m*-action.*

Exercise 7.

$H \leq G$ definable $\Longrightarrow$ for every $m \geq 0$, G/H is a d.C^m-manifold.

Lemma

$\alpha : G \curvearrowright X$ d.transitive C^m-action.
IF $x \in X$ and $G_x := \{g \in G : \alpha(g,x) = x\}$ THEN for each $g \in G$,

$$rk(d_g\alpha(-,x)) = \dim G - \dim G_x.$$

Proof.

$$\dim \alpha(G,x) = rk(d_h\alpha(-,x)), \text{ for some } h \in G.$$
$$\dim \alpha(G,x) = \dim X = \dim G/G_x = \dim G - \dim G_x.$$

$\therefore$ STP $rk(d_g\alpha(-,x))$ is constant on G.

$$\alpha(h,x) = \alpha(g, \alpha(g^{-1}h, x)), \text{ for all } h, g \in G,$$

$\therefore \alpha(-,x) = \alpha(g,-) \circ \alpha(-,x) \circ L_{g^{-1}}$, for all $g \in G$.

$$d_g\alpha(-,x) = d_x\alpha(g,-) \circ d_e\alpha(-,x) \circ d_g L_{g^{-1}}.$$

$d_x\alpha(g,-)$ and $d_g L_{g^{-1}}$ are invertible

$$\therefore rk(d_g\alpha(-,x)) = rk(d_e\alpha(-,x)), \text{ for all } g \in G.$$

$\square$

Corollary

G d.group.

1. *IF* $\alpha : G \curvearrowright X$ *d.transitive* C^m*-action.* $H \leq G$ *d.connected. THEN for each* $x \in X$,

$$H \subseteq G_x \iff T_e H \subseteq \ker d_e\alpha(-, x).$$

   $T_e G_x = \ker d_e\alpha(-, x)$.

2. *IF* $H_1, H_2 \leq G$ *d.connected THEN* $\boxed{H_1 = H_2 \iff T_e H_1 = T_e H_2}$

3. *IF* $f : G \to H$ *is a d.homomorphism and* $H_1 \leq H$ *definable THEN*

$$T_e f^{-1}(H_1) = (d_e f)^{-1}(T_e H_1).$$

$$\boxed{T_e \ker f = \ker d_e f}$$

4. $f \in Aut^{def}(G) \implies d_e f \in Aut(T_e(G))$.

Proof.

1. $[H \subseteq G_x \Rightarrow T_e H \subseteq \ker d_e\alpha(-,x)]$: $H \subseteq G_x \Rightarrow \alpha(-,x)$ const. on H.
   $\therefore d_e\alpha(-,x) = 0$ on $T_e H$.
   $[\Leftarrow]$ $\beta := \alpha_{|H\times X}$. $\beta : H \curvearrowright X$ d.C^m-action & $d_e\beta(-,x) = 0$
   $\therefore d_h\beta(-,x) = 0$ for all $h \in H$. $\therefore \beta(-,x)$ constant $\therefore H \subseteq G_x$.
   $[T_e G_x = \ker d_e\alpha(-,x)]$: $T_e G_x \subseteq \ker d_e\alpha(-,x)$.
   $\dim T_e G_x = \dim G_x = \dim G - rk(d_e\alpha(-,x)) =$
   $\dim G - (\dim T_e G - \dim \ker d_e\alpha(-,x))$.
2. $[H_1 \subseteq H_2 \iff T_e H_1 \subseteq T_e H_2]$: $G \curvearrowright G/H_2$. Apply (1) $x = eH_2$.
3. $[T_e f^{-1}(H_1) = (d_e f)^{-1}(T_e H_1)]$: $\alpha : H \curvearrowright H/H_1 \Rightarrow \beta : G \curvearrowright H/H_1$,
   $\beta(g, hH_1) := \alpha(f(g), hH_1)$. $x := eH_1$.
   $\therefore G_x = f^{-1}(H_1)$
   $\therefore T_e f^{-1}(H_1) = \ker d_e\beta(-,x)$ (by (1)).
   $\beta(-,x) = \alpha(-,x) \circ f \implies d_e\beta(-,x) = d_e\alpha(-,x) \circ d_e f$.
   $\therefore \ker d_e\beta(-,x) = (d_e f)^{-1}(\ker d_e\alpha(-,x))$.
4. $[f \in Aut^{def}(G) \implies d_e f \in Aut(T_e(G))]$: By (3).

□

Definition

$\mathbb{K}$ ch.0. A Lie algebra over $\mathbb{K}$ is a fin.dim. $\mathbb{K}$-vector space $\mathfrak{h}$ & bilinear map $[-,-] : \mathfrak{h}\times\mathfrak{h} \to \mathfrak{h}$ S.T.

1. $[x,x] = 0$;
2. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$.

$\mathfrak{h}$ anticommutative: $[y,x] = -[x,y]$; non associative, if $[-,-] \neq 0$.

Example

A an associative $\mathbb{K}$-algebra (e.g. $M(n,\mathbb{K})$) $\Longrightarrow \mathfrak{a} = (A,[-,-])$ with $[x,y] := xy - yx$ is a Lie algebra ($gl(n,\mathbb{K})$).

Morphism of Lie algebras: $f : \mathfrak{h}_1 \to \mathfrak{h}_2$ linear & $f([x,y]) = [fx, fy]$.
$Aut(\mathfrak{h}) := \{f : \mathfrak{h} \to \mathfrak{h} \mid f$ bijection & morphism of Lie algebras$\}$.

Definitions

$\mathfrak{h}$ Lie algebra. $\mathfrak{h}_1$, $\mathfrak{h}_2$ subspaces of $\mathfrak{h}$.

- $[\mathfrak{h}_1, \mathfrak{h}_2] := \langle \{[x, y] : x \in \mathfrak{h}_1, y \in \mathfrak{h}_2\} \rangle_{\mathbb{K}}$.
- $\mathfrak{h}_1$ Lie subalgebra of $\mathfrak{h}$: $[\mathfrak{h}_1, \mathfrak{h}_1] \subseteq \mathfrak{h}_1$.
- $\mathfrak{h}_1$ ideal of $\mathfrak{h}$: $[\mathfrak{h}_1, \mathfrak{h}] \subseteq \mathfrak{h}_1$.
- $\mathfrak{h}$ commutative: $[\mathfrak{h}, \mathfrak{h}] = 0$.
- $\mathfrak{h}$ semisimple: no nontrivial commutative ideals.
- $\mathfrak{h}$ simple: no nontrivial proper ideals.

Examples

1. $\dim \mathfrak{h} = 1 \implies \mathfrak{h}$ commutative.
2. $\dim \mathfrak{h} = 2 \implies \mathfrak{h} = \langle \{x, y\} \rangle_{\mathbb{K}}$ with $[x, y] := 0$ or $[x, y] := y$.
3. $sl(n, \mathbb{K}) := \{x \in gl(n, \mathbb{K}) : Tr(x) = 0\}$ semisimple.
4. $sl(n, \mathbb{K})$ ideal of $gl(n, \mathbb{K})$ and $[gl(n, \mathbb{K}), gl(n, \mathbb{K})] \subseteq sl(n, \mathbb{K})$.

G d.group. $g \in G$.

$$\boxed{Int(g) : G \to G : h \mapsto ghg^{-1}}$$

$Int(g) \in Aut^{def}(G)$

$$\boxed{\therefore Ad(g) := d_e Int(g) : T_e G \to T_e G}$$

$\therefore Ad(g) \in Aut(T_e G)$.

$$\boxed{\therefore Ad : G \to Aut(T_e G) : g \mapsto Ad(g)}$$

Ad d.homomorphism: $Ad(gh) = d_e Int(gh) = d_e(Int(g) \circ Int(h)) = d_e Int(g) d_e Int(h) = Ad(g)Ad(h)$.

$$\boxed{\therefore ad := d_e Ad : T_e G \to T_e(Aut(T_e G))}.$$

Identify $T_e(Aut(T_eG))$ with $End(T_eG)$ via

$$\overline{\alpha} \mapsto \alpha'(0) := \lim_{t\to 0} \frac{\alpha(t)-\alpha(0)}{t},$$

α takes values in $Aut(T_eG)$ and limit in $End(T_eG)$.

$$\boxed{\therefore ad : T_eG \to End(T_eG)}.$$

Definition

Lie algebra of a d.group G:

$$\mathfrak{g} := (T_eG, [-,-]) \qquad [x,y] := ad(x)(y).$$

Exercise 8.

G d.group.

1. $ad : \mathfrak{g} \to End(T_e G)$ is a morphism of Lie algebras.
2. $H \leq G$ definable $\Longrightarrow \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.
3. $n = \dim G \Longrightarrow G/Z(G) \cong H \leq GL(n, R)$ definably (identify $Aut(T_e(G))$ with $GL(n, R)$).

Definitions

$\mathfrak{g}$ Lie algebra, $\mathfrak{h}$ subspace of $\mathfrak{g}$

- Centre of $\mathfrak{g}$: $\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}$.
- Centralizer of $\mathfrak{h}$ in $\mathfrak{g}$: $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [x, \mathfrak{h}] = 0\}$.
- Normalizer of $\mathfrak{h}$ in $\mathfrak{g}$: $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$.

Exercise 9.

$\mathfrak{g}$ Lie algebra, $\mathfrak{h}$ subspace of $\mathfrak{g}$

1. $\mathfrak{z}(\mathfrak{g})$ ideal of $\mathfrak{g}$ and Lie algebra of $Z(G)$ is $\mathfrak{z}(\mathfrak{g})$.
2. $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ are subalgebras of $\mathfrak{g}$.

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.group. $\mathfrak{h}$ subspace of $\mathfrak{g}$.

1. $$f \in Aut^{def}(G) \Longrightarrow d_e f \in Aut(\mathfrak{g}).$$
2. $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ *is the Lie algebra of*
$$C_G(\mathfrak{h}) := \left\{g \in G : Ad(g)_{|\mathfrak{h}} = id_{\mathfrak{h}}\right\}.$$
3. $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ *is the Lie algebra of*
$$N_G(\mathfrak{h}) := \{g \in G : Ad(g)(\mathfrak{h}) \subseteq \mathfrak{h}\}.$$

Exercise 10

G d.connected group.

1. G is commutative $\iff \mathfrak{g}$ is commutative, and
2. if $H \leq G$ is a d.connected subgroup of G then,

$$H \text{ is normal in } G \iff \mathfrak{h} \text{ is an ideal of } \mathfrak{g}.$$

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Definably compact subgroups

G connected LIE GROUP $\Longrightarrow \exists K_1$ maximal compact subgroup (unique up to conjugation) S.T.

$$G \approx K_1 \times \mathbb{R}^m.$$

$G = SL(2, \mathbb{R}) \Longrightarrow K_1 = SO(2, \mathbb{R})$.

Example (Strzebonski 1994)

$$G := R \times [0,1) \qquad (a,t) * (b,s) := \begin{cases} (a+b, t+s \mod 1) & \text{if t+s<1} \\ (a+b+1, t+s \mod 1) & \text{o/w.} \end{cases}$$

$(G, *)$ 2-dim. d.group.
$G \not\approx R^2$, $E(G) = 0$.
G has no proper d.subgroups with $E(G) = 0$.
$\therefore$ G has NO d.compact subgroups.

Lemma

G d.group $\Longrightarrow \exists N \trianglelefteq G$ definable & torsion-free S.T.

$$\forall H \trianglelefteq G \text{ definable \& torsion-free}, H \subseteq N$$

Proof.

Let $N \trianglelefteq G$ definable and torsion-free of maximal dimension.
$H \subseteq N$, for every $H \trianglelefteq G$ definable and torsion-free:
$HN/N \overset{def}{\cong} H/(H \cap N)$ torsion-free $\quad \therefore E(HN/N) = \pm 1.$

$$\therefore E(HN) = E(N)E(HN/N) = \pm 1.$$

$N, HN \trianglelefteq G$ definable torsion-free, $N \subseteq HN$ and N of maximal dimension,

$$\therefore \dim N = \dim HN.$$

Both d.connected $\quad \therefore N = HN$ and $\quad \therefore H \subseteq N$. $\square$

Theorem (Conversano 2014)

G d.connected group. $N \trianglelefteq G$ the unique maximal torsion-free.

$$\overline{G} := G/N.$$

THEN

1. *$\exists K_1 \leq \overline{G}$ maximal d.compact;*
2. *K_1 is d.connected and unique up to conjugation;*
3. *$\exists H \leq \overline{G}$ definable torsion-free S.T.*

$$\overline{G} = K_1 H \qquad \& \qquad K_1 \cap H = 1.$$

Exercise 11

1. H is *maximal* torsion-free subgroup of $\overline{G}$.
2. Preimage of H in G is a maximal torsion-free d.subgroup of G.
3. G d.group $\Longrightarrow \exists H_1 \leq G$ maximal torsion-free definable,
   G NOT d.compact $\Longrightarrow H_1 \neq \{1\}$.

Theorem (Peterzil-Starchenko 2005)

IF G d.group, $\dim G = n$ *THEN*

$$G \text{ torsion-free} \iff G \text{ d.diffeomorphic to } R^n.$$

Theorem (Conversano 2014)

IF G d.connected group THEN

- $\exists K_1 \leq G/N$ *maximal d.compact*
- $$G \overset{def}{\approx} K_1 \times R^s,$$

  d.homeomorphism, $s = \dim G - \dim K_1$.

Definable-torus T of d.group G: $T \leq G$ d.connected d.compact abelian.
$SO(3, \mathbb{R})$: maximal tori $\cong SO(2, \mathbb{R})$.

Example (Peterzil-Steinhorn 1999)

T d.-torus of d.group $G \not\Rightarrow T \overset{def}{\cong} T_1 \times \cdots \times T_1$ & $\dim T_1 = 1$:

$$\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \leq \mathbb{R},$$

$\{v_1, \ldots, v_n\}$ generic (n^2 components algebraically independent over $\mathbb{Q}$).
WMA

$$G := \mathbb{R}^n / \Gamma$$

definable.

- $\forall w \in \mathbb{Q}^n \setminus \{0\}$, $\langle w + \Gamma \rangle \leq G$ is dense.
- For every $H \leq G$ definable $H \cap (\mathbb{Q}^n \setminus \{0\}) \neq \emptyset$.

$\therefore$ H is dense in G, H closed in $G \implies H = G$.

Theorem (Berarducci 2008)

IF

T d.-torus of a d.compact group G

THEN

- $H < G$ *definable* $\Longrightarrow E(T/H) = 0$
- $E(G/T) \neq 0 \Longrightarrow T$ *maximal d.-torus of* G.

Theorem (Berarducci 2008, Edmundo 2005)

IF

G d.connected d.compact group

THEN

- *for each T maximal d.-torus of G, $G = \bigcup_{g \in G} T^g$, and*
- T_1, T_2 *maximal d.-tori of* $G \Longrightarrow T_1 = T_2^g$, *for some* $g \in G$.

Exercise 12

G d.connected d.compact group.

1. $Z(G) = \bigcap \{T : T$ maximal d.-torus of $G\}$.
2. $G/Z(G)$ is centreless.

Definably simple and semisimple groups

Definition

G d.group. G semisimple $:\Longleftrightarrow$ NO $H \trianglelefteq G$, H infinite abelian.

Exercise 13

G d.group, G semisimple $\Longleftrightarrow$ NO $H \trianglelefteq G$, H infinite abelian *definable*.

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.connected group. THEN

1. *G semisimple group $\Longleftrightarrow$ $\mathfrak{g}$ semisimple Lie algebra.*
2. *IF G centreless THEN G is d.simple $\Longleftrightarrow$ $\mathfrak{g}$ simple Lie algebra.*

Theorem (Peterzil-Pillay-Starchenko 2000)

G d.connected semisimple centreless group $\dim G = n$*. THEN*

$$G \overset{def}{\cong} H^0 \leq GL(n, R),$$

H^0 semialgebraically connected component of an algebraic linear group H.

Proof.

$$G \overset{def}{\cong} G_1 := Ad(G) \leq Aut(\mathfrak{g}) \leq GL(n, R).$$

$$\dim G_1 = \dim G = \dim \mathfrak{g} \overset{(*)}{=} \dim Aut(\mathfrak{g}),$$

$(*)$ transfer from the reals.
$\therefore$ $G_1 \leq Aut(\mathfrak{g})$ finite index, $Aut(\mathfrak{g})$ algebraic group.
$\therefore$ G_1 semialgebraically connected component of $Aut(\mathfrak{g})$. □

Theorem (Peterzil-Pillay-Starchenko 2000-2002)

G d.connected group.

1. *G d.simple $\Longrightarrow G \overset{def}{\cong} H(R)^0$, H real algebraic group/$\mathbb{R}^{alg}$.*
2. *G semisimple $\Longrightarrow Z(G)$ is finite and*

$$G/Z(G) \overset{def}{\cong} H_1 \times \cdots \times H_s,$$

   H_i d.simple.

3. *G d.simple $\Longrightarrow G \equiv H$, H centreless simple Lie group.*

Remark.(Hrushovski-Peterzil-Pillay 2011)

G d. connected group $\not\Rightarrow G \equiv H$, Lie group:
$\langle R, <, +, \cdot, exp\rangle$ nonstandard model of the theory of $\mathbb{R}_{exp}$. $\alpha \in R$ infinite.

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : u, v, t \in R, t > 0 \right\} \not\equiv \text{Lie group.}$$

Definition

G d.group. Solvable radical of G

$$R(G) := \langle \bigcup \{H \trianglelefteq G : H \text{ solvable}\} \rangle$$

Lemma (Baro-Jaligot-O_ 2012)

G d.group $\Longrightarrow$ $R(G)$ definable & solvable,
$\therefore$ $G/R(G)$ semisimple.

Theorem (Peterzil-Starchenko 2000)

G d.connected d.compact group $\Longrightarrow$ $G/Z(G)$ semisimple.

Exercise 14

G d.connected d.compact solvable group $\Longrightarrow$ G abelian.

Commutator subgroup

G d.group $\not\Rightarrow$

$$[G, G] := \langle\{[x, y] : x, y \in G\}\rangle$$

definable.

Example (Conversano 2009)

$\exists G$ d.connected group, definable/$\mathbb{R}$ S.T. $[G, G]$ is NOT definable,

$$1 \to [0, 1) \to G \to PSL(2, \mathbb{R}) \to 1.$$

Theorem (Hrushovski-Peterzil-Pillay 2011)

G d.compact d.connected $\Longrightarrow [G, G]$ definable and semisimple &

$$G = Z^0(G)\,[G, G]\,,$$

$Z^0(G) \cap [G, G]$ finite.

This reduce many questions of d.compact groups to the commutative and semisimple cases.

Example (Mamino 2011)

$\exists G$ d.compact definable/$\mathbb{R}$ S.T. $[G : G]$ has NO definable semidirect complement in G:

$$SU_2 = \left\{a + b\boldsymbol{i} + c\boldsymbol{j} + d\,\boldsymbol{k} : a^2 + b^2 + c^2 + d^2 = 1\right\}.$$

$$G := ([0, 1) \times SU_2)/\Gamma,$$

$\Gamma = \{(0, 1), (1/2, -1)\}$. $\therefore\ G' \cong SU_2$.

Solvable groups

Theorem (Edmundo 2003)

G d.group.

$$G \text{ solvable} \Longrightarrow G/N \text{ is d.compact,}$$

$N \trianglelefteq G$ maximal normal torsion-free.

Theorem (Edmundo 2003)

G d.group.

$$G \text{ solvable d.connected} \Longrightarrow G' := [G, G] \text{ nilpotent.}$$

Lemma (Peterzil-Starchenko 2005)

G d.groups. G torsion-free d.group $\Longrightarrow \exists$ d.subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \triangleleft \cdots \triangleleft G_n = G$$

S.T. G_{i+1}/G_i torsion-free abelian d.group.
$\therefore$ G torsion-free $\Longrightarrow$ G is d.connected and solvable.

Proof.

G counterexample of minimal dimension.
G d.connected $\Longrightarrow \dim G > 1$.
G d.simple:
O/W $\exists H \triangleleft G$, H & G/H torsion-free $\therefore$ 2 d. normal series for H and G/H which induce corresponding series for G, CONTRADICTION.
G d.simple $\Longrightarrow G \equiv H$, H simple centreless Lie group, H has torsion, a contradiction. □

Theorem (Baro-Jaligot-O_ 2012)

G d. group. G solvable d.connected group $\Longrightarrow$ derived series & lower central series of G consist of d.groups. $\therefore$ G' is definable.

Definition

Commutator width (cm) of group G:

$$cm(G) := \min\left\{m : G' = \{[x_1, y_1] \cdot \cdots \cdot [x_m, y_m] : x_i, y_i \in G\}\right\},$$

if m exists, o/w $cm(G) := \infty$.

- G finite simple group $\Longrightarrow$ $cm(G) = 1$ (Ore conjecture, 2010).
- G d.compact d.simple $\Longrightarrow$ $cm(G) = 1$.

Question 1

G d.simple $\Longrightarrow$ $cm(G) = 1$?

Definition

G group. A maximal nilpotent $Q \leq G$ is Cartan subgroup of G IF $\forall H \trianglelefteq Q$, H finite index in $Q \Longrightarrow H$ finite index in $N_G(H)$.

G d.connected d.compact group $\Longrightarrow$

Cartan subgroup of G = maximal d.-torus T of G

- they are all conjugate
- d.connected
- $T^G = G$, T any maximal d.-torus.

Cartan subgroups of $SL(2, \mathbb{R})$

$G := SL(2, \mathbb{R})$. 2 Cartan subgroups, up to conjugacy:

$$Q_1 := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \neq 0 \right\} \quad \& \quad Q_2 := \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

- Q_1 NOT d.connected
- $X := Q_1^G \cup Q_2^G = \{A \in SL(2, \mathbb{R}) : Tr(A) \neq 2\} \cup \{\pm I\} \neq G$
- X dense in G.

Theorem (Baro-Jaligot-O_ 2014)

G solvable d.connected group. THEN

- *Cartan subgroups of G exist and are definable*
- $Q \leq G$ *Cartan* $\Longrightarrow Q$ *d.connected & selfnormalizing*
- $Q_1, Q_2 \leq G$ *Cartan* $\Longrightarrow Q_1 = Q_2^g$
- $Q \leq G$ *Cartan* $\Longrightarrow Q^G$ *dense in* G.

Definition

G torsion-free d.group. G definably completely solvable if $\exists$ d.series

$$\{1\} = G_0 \trianglelefteq G_1 \triangleleft \cdots \triangleleft G_n = G$$

S.T. G_{i+1}/G_i is one-dimensional.

Exercise 15

G torsion-free abelian d.group $\Longrightarrow$ G d.completely solvable.

G connected LIE GROUP. G split-solvable IF $\exists$ series

$$\{0\} = \mathfrak{g}_0 \trianglelefteq \mathfrak{g}_1 \triangleleft \cdots \triangleleft \mathfrak{g}_n = \mathfrak{g}$$

S.T. $\dim \mathfrak{g}_i = i$ $(0 \leq i \leq n)$.

Example

Solvable $\not\Rightarrow$ split-solvable: $\mathbb{R}^2 \rtimes_\varphi \mathbb{R}$ connected torsion-free,

$$\varphi : \mathbb{R} \to GL(2, \mathbb{R}) : t \mapsto \varphi(t) := \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

Theorem (Conversano-Onshuus-Starchenko 2016)

G d.group.

$$G \text{ torsion-free} \Longrightarrow G \text{ d.completely solvable.}$$

Question 2

Which Lie groups are Lie isomorphic to a definable group?

$$\left.\begin{array}{l}\text{(1) Compact}\\ \text{(2) Semisimple centreless}\end{array}\right\} : \text{ Lie isomorphic to semialgebraic groups.}$$

Theorem (Conversano-Onshuus-Starchenko 2016)

G connected torsion-free solvable LIE GROUP. Then,

$$G \text{ Lie isomophic to a d.group} \iff G \text{ is split-solvable.}$$

Theorem (Conversano-Onshuus-Starchenko 2016)

G solvable LIE GROUP. Then,
G Lie isomophic to d.group $\iff \exists H \trianglelefteq_{Lie} G$, H connected torsion-free split-solvable & G/H compact.

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