SMALL SETS IN MANN PAIRS

PANTELIS E. ELEFTHERIOU

Abstract. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ be an expansion of a real closed field $\mathcal{M}$ by a dense subgroup G of $\langle M^{>0}, \cdot\rangle$ that has the Mann property. We prove that the induced structure on G by $\mathcal{M}$ eliminates imaginaries. As a consequence, we obtain that every small set X definable in $\widetilde{\mathcal{M}}$ can be definably embedded into some G^l, uniformly in parameters. These results hold in a more general setting of expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ of an o-minimal structure by a dense set $P \subseteq M$, satisfying three tameness conditions.

1. Introduction

This note is a natural extension of the work in [6]. In that reference, expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ of an o-minimal structure $\mathcal{M}$ by a dense predicate $P \subseteq M$ were studied, and under three tameness conditions, it was shown that the induced structure P_{ind} on P by $\mathcal{M}$ eliminates imaginaries. The tameness conditions were verified for dense pairs of real closed fields, for expansions of $\mathcal{M}$ by an independent set P, and for expansions of a real closed field $\mathcal{M}$ by a dense divisible subgroup P of $\langle M^{>0}, \cdot\rangle$ with the Mann property (henceforth called *Mann pairs*). As pointed out in [6, Remark 4.10], without the divisibility assumption in Mann pairs, the third tameness condition no longer holds, and in [6, Question 4.11] it was asked whether in that case P_{ind} still eliminates imaginaries. In this note, we prove that it does. Namely, we replace the third tameness condition by a weaker one, which we verify for arbitrary Mann pairs, and prove that together with the other two tameness conditions it implies elimination of imaginaries for P_{ind}.

Let us fix our setting. Throughout this text, $\mathcal{M} = \langle M, <, +, 0, \ldots\rangle$ denotes an o-minimal expansion of an ordered group with a distinguished positive element 1. We denote by $\mathcal{L}$ its language, and by dcl the usual definable closure operator in $\mathcal{M}$. An '$\mathcal{L}$-definable' set is a set definable in $\mathcal{M}$ with parameters. We write '$\mathcal{L}_A$-definable' to specify that those parameters come from $A \subseteq M$. It is a well-known fact that $\mathcal{M}$ admits definable Skolem functions and eliminates imaginaries ([4, Chapter 6]).

Let $D, P \subseteq M$. The *D-induced structure on P by $\mathcal{M}$*, denoted by $P_{ind(D)}$, is a structure in the language

$$\mathcal{L}_{ind(D)} = \{R_{\phi(x)}(x) : \phi(x) \in \mathcal{L}_D\},$$

whose universe is P and, for every tuple $a \subseteq P$,

$$P_{ind(D)} \models R_\phi(a) \ \Leftrightarrow\ \mathcal{M} \models \phi(a).$$

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If $Q \subseteq P^n$, by a *trace on Q* we mean a set of the form $Y \cap Q$, where Y is $\mathcal{L}$-definable. We call $Y \cap P^n$ a *full trace.*

For the rest of this paper we fix some $P \subseteq M$ and denote $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$. We let $\mathcal{L}(P)$ denote the language of $\widetilde{\mathcal{M}}$; namely, the language $\mathcal{L}$ augmented by a unary predicate symbol P. We denote by $\operatorname{dcl}_{\mathcal{L}(P)}$ the definable closure operator in $\widetilde{\mathcal{M}}$. Unless stated otherwise, by '(A-)definable' we mean (A-)definable in $\widetilde{\mathcal{M}}$, where $A \subseteq M$. We use the letter D to denote an arbitrary, but not fixed, subset of M.

Tameness Conditions (for $\widetilde{\mathcal{M}}$ and D):

(OP) (Open definable sets are $\mathcal{L}$-definable.) For every set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^n$, its topological closure $\overline{V} \subseteq M^n$ is $\mathcal{L}_A$-definable.

$(\mathrm{dcl})_D$ Let $B, C \subseteq P$ and
$$A = \operatorname{dcl}(BD) \cap \operatorname{dcl}(CD) \cap P.$$
Then
$$\operatorname{dcl}(AD) = \operatorname{dcl}(BD) \cap \operatorname{dcl}(CD).$$

$(\mathrm{ind})_D$ Let $X \subseteq P^n$ be A-definable in $P_{ind(D)}$. Then there are $\mathcal{L}_{AD}$-definable sets $Y_1, \dots, Y_l \subseteq M^n$, and sets $Q_1, \dots, Q_l \subseteq P^k$, each $\emptyset$-definable in $P_{ind(D)}$, such that
$$X = \bigcup_i (Y_i \cap Q_i).$$

Conditions (OP) and $(\mathrm{dcl})_D$ are the same as in [6], and are already known to hold for Mann pairs ([6, Remark 4.11]. Condition $(\mathrm{ind})_D$ differs from the corresponding one in [6], in two ways: (a) X is now a *finite union* of traces (instead of a single trace), and (b) the traces are on *subsets* of P^n (instead of the whole P^n). These differences result in some non-trivial complications in the proof of our main theorem below, which are handled in Section 3. For now, let us state the main result.

Theorem 1.1. *Assume* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D. *Then* $P_{ind(D)}$ *eliminates imaginaries.*

Condition $(\mathrm{ind})_D$ is modeled after the current literature on Mann pairs, which we now explain. Assume $\mathcal{M} = \langle M, <, +, \cdot, 0, 1\rangle$ is a real closed field, and G a dense subgroup of $\langle M^{>0}, \cdot\rangle$. For every $a_1, \dots, a_r \in M$, a solution $(q_1, \dots, q_r)$ to the equation
$$a_1 x_1 + \dots + a_r x_r = 1$$
is called *non-degenerate* if for every non-empty $I \subseteq \{1, \dots, r\}$, $\sum_{i \in I} a_i q_i \neq 0$. We say that G has the *Mann property*, if for every $a_1, \dots, a_r \in M$, the above equation has only finitely many non-degenerate solutions $(q_1, \dots, q_r)$ in G^r.[1] Let us call such a pair $\langle \mathcal{M}, G\rangle$ a *Mann pair.* Examples of Mann pairs include all multiplicative subgroups of finite rank in $\mathbb{R}_{>0}$ ([8]), such as $2^{\mathbb{Q}}$ and $2^{\mathbb{Z}}3^{\mathbb{Z}}$. Van den Dries - Günaydin [5, Theorem 7.2] showed that in a Mann pair, where moreover G is divisible (such as $2^{\mathbb{Q}}$), every definable set $X \subseteq G^n$ is a full trace; in particular, $(\mathrm{ind})_D$ from [6] holds. Without the divisibility assumption, however, this is no longer true. Consider for example $G = 2^{\mathbb{Z}}3^{\mathbb{Z}}$ and let X be the subgroup of G consisting of all elements

[1]The original definition only involved equations with coefficients a_i in the prime field of $\mathcal{M}$, but, by [5, Proposition 5.6], the two definitions are equivalent.

divisible by 2. That is, $X = \{2^{2m}3^{2n} : m, n \in \mathbb{Z}\}$. This set is clearly dense and co-dense in $\mathbb{R}$, and hence cannot be a trace on any subset of G.

A substitute to [5, Theorem 7.2] was proved by Berenstein-Ealy-Günaydin [1], as follows. Consider, for every $d \in \mathbb{N}$, the set $G^{[d]}$ of all elements of G divisible by d,

$$G^{[d]} = \{x \in G : \exists y \in G,\, x = y^d\}.$$

Under the mild assumption that for every prime p, $G^{[p]}$ has finite index in G, [5, Theorem 7.5] provided a near model completeness result, which was then used in [1] to prove that every definable set $X \subseteq P^n$ is a finite union of traces on $\emptyset$-definable subsets of P^n (Fact 3.7 below). Together with the Lemma 3.6 below, we obtain the following result.

Corollary 1.2. *Assume $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G\rangle$ is a Mann pair, such that for every prime p, $G^{[p]}$ has finite index in G. Let $D \subseteq M$ be* dcl*-independent over P. Then* $(\mathrm{ind})_D$ *holds. In particular, $P_{ind(D)}$ eliminates imaginaries.*

Observe that Corollary 1.2 stands in contrast to the current literature, as it is known that in Mann pairs both existence of definable Skolem functions and elimination of imaginaries (for $\widetilde{\mathcal{M}}$) fail ([2]). Note also that the assumption of D being dcl-independent over P is necessary; namely, without it, $P_{ind(D)}$ need not eliminate imaginaries ([6, Example 5.1]).

Theorem 1.1 has the following important consequence. Recall from [3] that a set $X \subseteq M^n$ is called *P-bound over A* if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(P^m)$. The recent work in [7] provides an analysis for all definable sets in terms of '$\mathcal{L}$-definable-like' and P-bound sets. Using Theorem 1.1, we further reduce the study of P-bound sets to that of definable subsets of P^l.

Corollary 1.3. *Assume* (OP), $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for $\widetilde{\mathcal{M}}$ and every $D \subseteq M$ which is* dcl*-independent over P. Let $X \subseteq M^n$ be an A-definable set. If X is P-bound over A, then there is an $A \cup P$-definable injective map $\tau : X \to P^l$. If A is* dcl*-independent over P, then the extra parameters from P are not needed.*

Allowing parameters from P is standard practice when studying definability in this context; see for example also [7, Lemma 2.5 and Corollary 3.26].

2. Preliminaries

We assume familiarity with the basics of o-minimality and pregeometries, as can be found, for example, in [4] or [9]. Recall that $\mathcal{M} = \langle M, <, +, 0, \dots\rangle$ is our fixed o-minimal expansion of an ordered group with a distinguished positive element 1 and dcl denotes the usual definable closure operator. We denote the corresponding dimension by dim. The topological closure of a set $X \subseteq M^n$ is denoted by $\overline{X}$ and the frontier $\overline{X} \setminus X$ by $fr(X)$. If A, B are two sets, we often write AB for $A \cup B$. We denote by $\Gamma(f)$ the graph of a function f. If $T \subseteq M^m \times M^n$ and $x \in M^n$, we write T_x for

$$\{b \in M^m \ :\ (b, x) \in X\}.$$

2.1. **Elimination of imaginaries.** We recall that a structure $\mathcal{N}$ *eliminates imaginaries* if for every $\emptyset$-definable equivalence relation E on N^n, there is a $\emptyset$-definable map $f : N^n \to N^l$ such that for every $x, y \in N^n$,

$$E(x, y) \ \Leftrightarrow\ f(x) = f(y).$$

In the order setting, we have the following criterion (extracted from [9, Section 3]; for a proof see [6, Fact 2.2]).

Fact 2.1. *Let $\mathcal{N}$ be a sufficiently saturated structure with two distinct constants in its language. Suppose the following property holds.*

(*) *Let $B, C \subseteq N$ and $A = \mathrm{dcl}_{\mathcal{N}}(B) \cap \mathrm{dcl}_{\mathcal{N}}(C)$. If $X \subseteq N^n$ is B-definable and C-definable, then X is A-definable.*

Then $\mathcal{N}$ eliminates imaginaries.

2.2. **The induced structure.** Recall from the introduction that

$$P_{ind(D)} = \langle P, \{R \cap P^l : R \subseteq M^l \ \mathcal{L}_D\text{-definable }, l \in \mathbb{N}\}\rangle.$$

Note that for $A \subseteq P$,

- if $Q \subseteq P^n$ is A-definable in $P_{ind(D)}$, and $Y \subseteq M^n$ is $\mathcal{L}_{AD}$-definable, then $Q \cap Y$ is A-definable in $P_{ind(D)}$. Indeed, $Q \cap Y = Q \cap (Y \cap P^n)$.
- in general, if $Q \subseteq P^n$ is A-definable in $P_{ind(D)}$, then Q is AD-definable. The converse will be true in Mann pairs, by Lemma 3.6 and Fact 3.7 below.

Lemma 2.2. *Assume* $(\mathrm{ind})_D$. *Let $f : Z \subseteq P^n \to P$ be an A-definable map in $P_{ind(D)}$. Then there is an $\mathcal{L}_{AD}$-definable map $F : Z \subseteq M^n \to M^k$ that extends f.*

Proof. By $(\mathrm{ind})_D$, there are finitely many $\mathcal{L}_{AD}$-definable sets $T_1, \dots, T_l \subseteq M^{n+1}$ and $\emptyset$-definable sets $Q_1, \dots, Q_l \subseteq P^{n+1}$, such that $\Gamma(f) = \bigcup_i T_i \cap Q_i$. Fix i, and let f_i be the map whose graph equals $T_i \cap Q_i$. It clearly suffices to prove the lemma for f_i. By (OP) and o-minimality, each fiber $(Q_i)_x$ is dense in a finite union of open intervals and points, and hence each fiber $(T_i)_x$ is finite. Without loss of generality, we may assume that it is a singleton. The set

$$X_i = \{x \in \pi(T) : (T_i)_x \text{ is singleton}\}$$

is $\mathcal{L}_{AD}$-definable. So, $Q_i \subseteq X$. Now let

$$T_i' = \left(\bigcup_{x \in X_i} \{x\} \times (T_i)_x\right) \cup \{(x, 0) : x \in M^n \setminus X_i\}.$$

Then T_i' is $\mathcal{L}_{AD}$-definable, it is the graph of a function $F_i : Z_i' \subseteq M^n \to M$, and $\Gamma(f_i) = T_i' \cap Q_i$, as required. □

We denote the definable closure operator in $P_{ind(D)}$ by cl_D.

Corollary 2.3. *Assume* $(\mathrm{ind})_D$. *Then for every $A \subseteq P$, $\mathrm{cl}_D(A) = \mathrm{dcl}(AD) \cap P$.*

Proof. The inclusion $\supseteq$ is immediate from the definitions, whereas the inclusion $\subseteq$ is immediate from Lemma 2.2. □

By Corollary 2.3 and the fact that $\mathrm{dcl}(-D)$ defines a pregeometry in $\mathcal{M}$, it follows easily that, under $(\mathrm{ind})_D$, $cl_D(-)$ defines a pregeometry in $P_{ind(D)}$.

3. Proofs of the results

In this section we prove elimination of imaginaries for $P_{ind(D)}$ (Theorem 1.1) and deduce Corollary 1.3 from it. Our goal is to establish (*) from Fact 2.1 for $\mathcal{N} = P_{ind(D)}$ (Lemma 3.4 below). As in [6], the strategy is to reduce the proof of (*) to [9, Proposition 2.3], which is an assertion of (*) for $\mathcal{M}$. This reduction takes place in the proof of Lemma 3.4, and requires the key Lemma 3.3. The analogous key [6, Lemma 3.1] cannot help us in the current setting, because its assumptions are not met in the proof of Lemma 3.4. Moreover, the proof of Lemma 3.3 requires some new techniques.

First, some preliminary observations.

Fact 3.1. *Suppose* (OP) *holds. Then for every* $A \subseteq P$, $\mathrm{dcl}_{\mathcal{L}(P)}(A) = \mathrm{dcl}(A)$.

Proof. Take $x \in \mathrm{dcl}_{\mathcal{L}(P)}(A)$. That is, the set $\{x\}$ is A-definable in $\widetilde{\mathcal{M}}$. By (OP), we have that $\overline{\{x\}}$ is $\mathcal{L}_A$-definable. But $\overline{\{x\}} = \{x\}$. □

Lemma 3.2. *Let* $X \subseteq M^n$ *be an* $\mathcal{L}$*-definable set which is also* A*-definable, for some* $A \subseteq M$ *with* $A \setminus P$ dcl*-independent over* P*. Then* X *is* $\mathcal{L}_A$*-definable.*

Proof. We work by induction on $k = \dim X$. For $k = 0$, X is finite, and hence every element of it is in $\mathrm{dcl}_{\mathcal{L}(P)}(A)$. By Fact 3.1, it is in $\mathrm{dcl}(A)$. Now assume $k \geq 0$. By (OP), $\overline{X}$ is $\mathcal{L}_A$-definable. By o-minimality, $\dim fr(X) < k$. Since $fr(X) = \overline{X} \setminus X$ is both $\mathcal{L}$-definable and A-definable, by inductive hypothesis, it is $\mathcal{L}_A$-definable. So $X = \overline{X} \setminus fr(X)$ is $\mathcal{L}_A$-definable. □

Now, the key technical lemma.

Lemma 3.3. *Assume* (OP) *and* $(\mathrm{ind})_D$*, and that* D *is* dcl*-independent over* P*. Let* $B, C \subseteq P$*, and assume* $X \subseteq P^n$ *is* B*-definable and* C*-definable in* $P_{ind(D)}$*. Then there are* $W_1, \dots, W_l \subseteq M^n$*, each both* $\mathcal{L}_{BD}$*-definable and* $\mathcal{L}_{CD}$*-definable, and sets* $S_1, \dots, S_k \subseteq P^n$*, each* $\emptyset$*-definable in* $P_{ind(D)}$*, such that*

$$X = \bigcup_{i=1}^{l} S_i \cap W_i.$$

Proof. First note that X is both BD-definable and CD-definable in $\langle \mathcal{M}, P \rangle$. Since $B, C \subseteq P$, by (OP) it follows that $\overline{X}$ is $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable.

We perform induction on the dimension of $\overline{X}$. For $\dim \overline{X} = 0$, X is finite and equals $X = \overline{X} = \overline{X} \cap P^n$, as needed. Suppose now that $\dim X = k > 0$. By $(\mathrm{ind})_D$, there are $Y_i, Z_j \subseteq M^n$ and $Q_i, R_j \subseteq P^n$, $i = 1, \dots, l$, $j = 1, \dots, m$, such that each Y_i is $\mathcal{L}_{BD}$-definable, each Z_j is $\mathcal{L}_{CD}$-definable, and each of Q_i, R_j is $\emptyset$-definable in $P_{ind(D)}$, and moreover the following holds:

$$X = \bigcup_i Y_i \cap Q_i = \bigcup_j R_j \cap Z_j. \tag{1}$$

For every i, j, denote

$$T_{ij} = \{x \in \overline{X} : \text{ there is relatively open } V \subseteq \overline{X} \text{ around } x, \text{ with } V \cap (Q_i \cup R_k) \subseteq X\}.$$

It is immediate from the definition, that T_{ij} is relatively open in $\overline{X}$. By (OP), T is $\mathcal{L}$-definable. On the other hand, T_{ij} is BD-definable and CD-definable, because X is, and Q_i, R_j are D-definable. Hence, by Lemma 3.2, T_{ij} is $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable.

Let $T = \bigcup_{i,j} T_{ij}$. We claim that

$$\dim(\overline{X} \setminus T) < k.$$

Indeed, if not, there must be relatively open $U \subseteq \overline{X}$ with $U \cap T = \emptyset$. Since $\overline{X} \subseteq \bigcup_{i,j} \overline{Y_i \cap Z_j}$, we can find some i, j and relatively open $V \subseteq U$ which is also contained in $Y_i \cap Z_j$. Since $Y_i \cap Z_j \cap (Q_i \cup R_j) \subseteq X$, we obtain $V \cap (Q_i \cup R_j) \subseteq X$. But this contradicts $U \cap T = \emptyset$.

It is also clear from the definition of T that

$$X \cap T = \bigcup_{i,j} Q_i \cup R_j) \cap T.$$

Therefore, since Q_i and R_j are $\emptyset$-definable in $P_{ind(D)}$, $X \cap T$ is already in the desired form.

Now, since $\overline{X \setminus T} \subseteq \overline{\overline{X} \setminus T}$, we have $\dim \overline{X \setminus T} \leq \dim(\overline{X} \setminus T) < k$. But also $X \setminus T = X \cap (M^n \setminus T) \cap P^n$ is B-definable and C-definable in $P_{ind(D)}$, because X is. Therefore, by inductive hypothesis, $X \setminus T$ can be written in the desired form. Since

$$X = (X \cap T) \cup (X \setminus T),$$

we are done. □

Lemma 3.4. *Assume* (OP)*,* (dcl)$_D$ *and* (ind)$_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*. Let* $B, C \subseteq P$ *and* $A = cl_D(B) \cap cl_D(C)$*. If* $X \subseteq P^n$ *is* B*-definable and* C*-definable in* $P_{ind(D)}$*, then* X *is* A*-definable in* $P_{ind(D)}$*.*

Proof. Let $X \subseteq P^n$ be B-definable and C-definable in $P_{ind(D)}$. By Lemma 3.3, there are $W_1, \dots, W_l \subseteq M^n$, each both $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable, and $S_1, \dots, S_k \subseteq P^n$, each $\emptyset$-definable in $P_{ind(D)}$, such that

$$X = \bigcup_{i=1}^{l} S_i \cap W_i.$$

By [9, Proposition 2.3], each W_i is $\mathcal{L}$-definable over $\operatorname{dcl}(BD) \cap \operatorname{dcl}(CD)$. By (dcl)$_D$, W_i is $\mathcal{L}$-definable over $\operatorname{dcl}(BD) \cap \operatorname{dcl}(CD) \cap PD$. Hence X is definable over $\operatorname{dcl}(BD) \cap \operatorname{dcl}(CD) \cap P$ in $P_{ind(D)}$. But

$$\operatorname{dcl}(BD) \cap \operatorname{dcl}(CD) \cap P = cl_D(B) \cap cl_D(C) = A,$$

and hence X is A-definable in $P_{ind(D)}$. □

We can now conclude our results.

Proof of Theorem 1.1. By Fact 2.1 and Lemma 3.4. □

For the proof of Corollary 1.3, we additionally need the following lemma.

Lemma 3.5. *Assume* (ind)$_D$ *holds for* $\widetilde{\mathcal{M}}$ *and* D*. Let* $\mathcal{M}'$ *be the expansion of* $\mathcal{M}$ *with constants for all elements in* P*, and* $\widetilde{\mathcal{M}}' = \langle \mathcal{M}', P \rangle$*. Then* (ind)$_D$ *holds for* $\widetilde{\mathcal{M}}'$ *and* D*.*

Proof. Let $X \subseteq P^n$ be A-definable in the D-induced structure on P by $\mathcal{M}'$. It follows that X is AP-definable in $P_{ind(D)}$. Hence there are $\mathcal{L}_{APD}$-definable sets $Y_1, \dots, Y_l \subseteq M^n$, and P-definable $Q_1, \dots, Q_l \subseteq P^k$, such that

$$X = \bigcup_i (Y_i \cap Q_i).$$

Such Y_i's are $\mathcal{L}_{AD}$-definable in $\mathcal{M}$, and the Q_i's are $\emptyset$-definable, as required. □

Proof of Corollary 1.3. The case where A is dcl-independent over P is identical to that of [6, Theorem B]. The general case is identical to that of [6, Corollary 1.4], after replacing in [6, Lemma 3.4] the clause about $(\mathrm{ind})_D$ with Lemma 3.5 above. $\square$

3.1. **Mann pairs.** Here we turn to our targeted example of Mann pairs. We first prove a general lemma.

Lemma 3.6. *Assume* (OP) *and that D is* dcl-*independent over P. Let $A \subseteq P$. Suppose $X \subseteq P^n$ is an AD-definable set which is a finite union of traces on sets which are $\emptyset$-definable in $P_{ind(D)}$. Then there are $\mathcal{L}_{AD}$-definable sets $Y_1, \dots, Y_l \subseteq P^n$ and sets $Q_1, \dots, Q_l \subseteq P^n$, each $\emptyset$-definable in $P_{ind(D)}$, such that*

$$X = \bigcup_{i=1}^{l} (Y_i \cap Q_i).$$

Proof. The technique of the proof is similar to the one used in the proof of Lemma 3.3, and thus we are somewhat brief. Since X is AD-definable and $A \subseteq P$, by (OP) it follows that $\overline{X}$ is $\mathcal{L}_{AD}$-definable. We perform induction on the dimension of $\overline{X}$. For $\dim \overline{X} = 0$, X is finite and $X = \overline{X} = \overline{X} \cap P^n$, as needed. Suppose now that $\dim X = k > 0$. By assumption, there are $\mathcal{L}_{AD}$-definable sets $Z_1, \dots, Z_m \subseteq P^n$ and sets $R_1, \dots, R_l \subseteq P^n$, each $\emptyset$-definable in $P_{ind(D)}$, such that

$$X = \bigcup_{i=1}^{l} (Z_i \cap R_i).$$

Denote $R = \bigcup_i R_i$ and define

$$T = \{x \in \overline{X} : \text{ there is relatively open } V \subseteq \overline{X} \text{ around } x, \text{ with } V \cap R \subseteq X\}.$$

As in the proof of Lemma 3.3, T is relatively open in $\overline{X}$, and hence, by (OP), it is $\mathcal{L}$-definable. It is also AD-definable, because X is. Hence, by Lemma 3.2, T is $\mathcal{L}_{AD}$-definable. Moreover, it is easy to see that

$$\dim(\overline{X} \setminus T) < k,$$

and hence, by inductive hypothesis, the conclusion holds for $X \setminus T$. Also,

$$X \cap T = R \cap T,$$

and hence $X \cap T$ is already in the desired form. Since

$$X = (X \cap T) \cup (X \setminus T),$$

we are done. $\square$

The proof of Corollary 1.2 will be complete after we recall the fact below, which is extracted from [1]. First, observe that if $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G \rangle$ is a Mann pair, then for every $d \in \mathbb{N}$, $G^{[d]}$ is $\emptyset$-definable in $P_{ind(\emptyset)}$. Indeed, $G^{[d]}$ is the projection onto the first coordinate of the set $\{(x^d, x) : x \in M\} \cap G^2$.

Fact 3.7. *Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, G \rangle$ be a Mann pair, and $X \subseteq P^n$ a definable set. Then X is a finite union of traces on sets which are $\emptyset$-definable in $P_{ind(\emptyset)}$.*

Proof. Let X be such. By [1, Corollary 57], X is as a finite union of traces on sets of the form $g(G^{[d]})^n$, $d \in \mathbb{N}$. As pointed out in the proof of [1, Theorem 1], each such g can be chosen $\emptyset$-definable (in $\widetilde{\mathcal{M}}$). By Fact 3.1, $g \in \mathrm{dcl}(\emptyset)$. By the above observation, $g(G^{[d]})^n$ is $\emptyset$-definable in $P_{ind(\emptyset)}$. □

Proof of Corollary 1.2. By Lemma 3.6 and Fact 3.7, $(\mathrm{ind})_D$ holds. By [6], as explained in Remark 4.11 therein, (OP) and $(\mathrm{dcl})_D$ holds. By Theorem 1.1, we are done. □

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Department of Mathematics and Statistics, University of Konstanz, Box 216, 78457 Konstanz, Germany

E-mail address: panteleimon.eleftheriou@uni-konstanz.de