Around and about real closed valued fields

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ordered valued field

Let $\mathcal{R}$ be a real closed field, consider the field of Laurent series $\mathcal{R}((t))$

$$\mathcal{R}((t)) = \{\sum_{i=N}^{\infty} a_i t^i : a_i \in \mathcal{R}, N \in \mathbb{Z}\}.$$

$\mathcal{R}((t))$ can be ordered in many different ways; let's consider $t > 0$, $t < r$ for any positive $r \in \mathcal{R}$. Then

$$t > t^2 > t^3 > \cdots > 0,$$

and in general,

$$\sum_{i=N}^{\infty} a_i t^i < \sum_{i=N}^{\infty} b_i t^i \iff a_N < b_N.$$

A picture

A picture
Another picture

valued field

A valuation is a function $v : K \to \Gamma \cup \{\infty\}$, where K is a field, Γ is an ordered abelian group, such that

$$\begin{aligned} v(xy) &= v(x) + v(y) \\ v(x+y) &\geq \min\{v(x), v(y)\} \\ v(x) = \infty &\leftrightarrow x = 0 \end{aligned}$$

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Valuation ring $\mathcal{O}_K = \{x \in K : v(x) \geq 0\}$
Maximal ideal $\mathfrak{m}_K = \{x \in K : v(x) > 0\}$
Residue field $k = \mathcal{O}_K/\mathfrak{m}_K$.

convexly valued ordered field

In the case of $K = \mathcal{R}((t))$,

$$v(\sum_{i=N}^{\infty} a_i t^i) = N,$$

$\mathcal{O}_K = \mathcal{R}[[t]]$ and the residue field is $\mathcal{R}$.

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With the given ordering, $\mathcal{O}_K$ is a *convex* subring: if $x, y \in \mathcal{O}_K$ and $x < z < y$ then $z \in \mathcal{O}_K$.

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More generally, a valuation v on a field is *convex* with respect to an ordering $<$ on the field if for all $0 < x < y$, $v(x) \geq v(y)$.

closure properties

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2) If $\Gamma_{\mathcal{R}}$ is divisible, $k_{\mathcal{R}}$ is real closed and $\mathcal{R}$ is henselian then $\mathcal{R}$ is real closed.

Analogous to algebraically closed valued fields.

AKE theorem

Ax-Kochen, Ersov, late 60's

Let K be a henselian valued field of characteristic 0. Then $\mathrm{Th}(K)$ in the language of valued fields is determined up to elementary equivalence by $\mathrm{Th}(\Gamma)$ in the language of ordered groups and $\mathrm{Th}(k)$ in the language of fields.

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Motto: a valued field is controlled by its value group and residue field.

Motivation: to what extent is this still true when further structure is added?
More generally: pursue analogies between algebraically closed valued fields and real closed valued fields.

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Mellor 2006

The theory RCVF has elimination of imaginaries in the sorted language $\mathcal{G}$ with sorts for the finitely generated $\mathcal{O}_K$-submodules and their torsors.

Again, analogous to algebraically closed valued fields.

some imaginaries in valued fields

The value group $K^\times/(\mathcal{O}_K \setminus \mathfrak{m}_K)$

aEb if and only if $a/b \in \mathcal{O}_k \setminus \mathfrak{m}_K$ if and only if $v(a) = v(b)$

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The RV sorts $K^\times/(1 + \mathfrak{m})$

aEb if and only if $a(1 + \mathfrak{m}) = b(1 + \mathfrak{m})$ if and only if $v(a - b) > v(b)$

a pure valued fields theorem

H.-Hrushovksi-Macpherson 2008

Let L, M, C be substructures of $\mathcal{U} \models \mathrm{ACVF}$. Assume that C is maximal and is a substructure of both L and M.

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Let L, M, C be substructures of $\mathcal{U} \models$ ACVF. Assume that C is maximal and is a substructure of both L and M.
Also assume that $\Gamma_L \cap \Gamma_M = \Gamma_c$ and that k_L is linearly independent from k_M over k_C.

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Then σ can be extended to an isomorphism of the field generated by L and M over C, which is the identity on M.

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Equivalently,

$$\mathrm{tp}(M/C\Gamma_L k_L) \vdash \mathrm{tp}(M/L).$$

an ordered valued fields theorem

Theorem (Ealy-H.-Maříková 2016)

The analogous statement with $\mathcal{U} \models \mathrm{RCVF}$.

outline of proof

Define $\sigma : C[L, M] \to C[L', M]$ by $\sigma(\sum_{i=1}^{n} \ell_i m_i) = \sum_{i=1}^{n} \sigma(\ell_i) m_i$.

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1) Show σ preserves the valuation (HHM, also Johnson).

Because C is maximal, WMA that the m_i are a *separated* basis for the finite-dimensional vector subspace of M that they generate over C such that in addition $v(\sum \ell_i m_i) = \min_i\{v(\ell_i) + v(m_i)\}$.

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As σ fixes Γ_M and is an isomorphism on Γ_L, the result follows.

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$$0 < \sigma(a) < m_{n+1} < a$$

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- it is not the case that $v(a - x) > v(a - \sigma(a))$,

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Now construct a pseudo-Cauchy sequence to find a closer such element x to a, giving a contradiction.

a stronger theorem

Theorem (Ealy-Haskell-Maříková 2016)

Assume L, M, C are elementary substructures of $\mathcal{U} \models \mathrm{RCVF}$, with C a common substructure of L and M, C maximal, $\Gamma_L \subseteq \Gamma_M$.
Assume that $\mathrm{kInt}^L_{C\Gamma_L}$ is algebraically independent from $\mathrm{kInt}^M_{C\Gamma_L}$ over $C\Gamma_L$.

the k-internal sorts

Given parameter set A, $kInt_A$ is the collection of sets, definable over A, that are *internal* to the residue field.

That is, $E \subset \mathrm{dcl}(k \cup a)$, where a is a finite tuple from A

Ex: $E = a(1 + \mathfrak{m})$.

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Given fields $C \subseteq L$, $S \subseteq \Gamma(L)$

$$\mathrm{kInt}^L_{CS} = \mathrm{kInt}_{CS} \cap \mathrm{acl}(Ck_L\{\mathrm{RV}_\gamma(L)\}_{\gamma \in S})$$

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Let $\sigma : L \to L'$ be an ordered valued field isomorphism fixing $\mathrm{kInt}^{M}_{C\Gamma_L}$.
Then σ extends to an ordered valued field isomorphism from $C(L, M)$ to $C(L'M)$ fixing M.

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Show that σ preserves the ordering: if not, then there is a change in an order relation on balls defined over M, which is a formula in the type of L over $\mathrm{kInt}^{M}_{C\Gamma_L}$.

perturbing the valuation

Choose $a_1, \ldots, a_r$ from L and $e_1, \ldots, e_r$ from M such that $v(a_i) = v(e_i)$ and $\{a_i\}$ is a $\mathbb{Q}$-basis for Γ_L over Γ_C.
Choose $b_1, \ldots b_s$ from L such that $\{\mathrm{res}(b_i)\}$ is a transcendence basis for k_L over k_C.
The assumption that $\mathrm{kInt}^L_{C\Gamma_L}$ is algebraically independent from $\mathrm{kInt}^M_{C\Gamma_L}$ over $C\Gamma_L$ is equivalent to saying the elements

$$\mathrm{res}(a_1/e_1), \ldots \mathrm{res}(a_r/e_r), \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s)$$

are algebraically independent over k_M.

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For each $0 \leq j \leq r-1$ choose a place

$$\begin{aligned} p^{(j)} : \mathrm{dcl}(k_M, &\mathrm{res}(b_1), \ldots, \mathrm{res}(b_s), \mathrm{res}(a_1/e_1), \ldots, \mathrm{res}(a_{j+1}/e_{j+1})) \to \\ &\mathrm{dcl}(k_M, \mathrm{res}(b_1), \ldots, \mathrm{res}(b_s), \mathrm{res}(a_1/e_1), \ldots, \mathrm{res}(a_j/e_j)) \end{aligned}$$

Let $p_{v'} : \mathrm{dcl}(C(L, M)) \to \mathrm{dcl}(k_M, k_L)$ be the composition. Let v' be a valuation associated to the place $p_{v'}$.
Then $\Gamma_{v'}(L) \cap \Gamma_{v'}(M) = \Gamma_{v'}(C)$ and $k_{v'}(L)$, $k_{v'}(M)$ are linearly disjoint over $k_{v'}(C)$.

σ preserves the ordering

Suppose not. Let $a = \sum_{i=1}^{n} \ell_i m_i > 0$ be a minimal counterexample. As before, WMA that the m_i form a separated basis for the space that they generate over C with respect to v'. From the construction of v', in fact, the basis is also separated over v and over L.
Hence we may assume that $v(\ell_i m_i) = 0$, so $v(m_i) \in \Gamma_L$.

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Thus $a > 0$ implies that $a + \sum_{i=1}^{n} \ell_i d_i > 0$ for any d_i with $v(d_i) > v(m_i) = -v(\ell_i)$.
In other words, the formula

$$x_1 B^{op}_{v(m_1)}(m_1) + \cdots + x_n B^{op}_{v(m_n)}(m_n) > 0$$

is a formula in the type of L over $\mathrm{kInt}^M_{C\Gamma_L}$. Since we assumed σ preserves this type, $\sigma(a) > 0$.

in the geometric sorts

As in previous theorems, let C, M be substructures of $\mathcal{U} \models$ RCVF, $A = \mathrm{dcl}(Ce)$, where e is a tuple of imaginaries in $\mathcal{G}$.

- Suppose $\Gamma_A \cap \Gamma_M = \Gamma_C$ and k_A and k_M are linearly independent over k_C. Then
$$\mathrm{tp}(A/Ck_M\Gamma_M) \vdash \mathrm{tp}(A/M).$$
- Suppose $\mathrm{kInt}^M_{C\Gamma_A}$ is independent from $\mathrm{kInt}^A_{C\Gamma_A}$. Then
$$\mathrm{tp}(A/C\Gamma_A\mathrm{kInt}^M_{C\Gamma_L}) \vdash \mathrm{tp}(A/M).$$

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$$\mathrm{tp}(A/C\Gamma_A\mathrm{kInt}^M_{C\Gamma_L}) \vdash \mathrm{tp}(A/M).$$

Proof: Find a resolution of A in the field sort with same value group and residue field. Then apply previous theorems.

further directions

Extend to a general T-convex theory.

How do functions behave on the interaction of L and M?
Resolutions still exist for the geometric sorts (provided the underlying o-minimal theory is power bounded). Are new sorts required to eliminate imaginaries?