The Jacobian Property in T-convex fields

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Outline

1. T-convex fields
2. The multidimensional Jacobian property (JP)
3. JP in power bounded T-convex fields
4. Comments on the proof

T-convexity

- T an o-minimal expansion of the theory of real closed fields in a language $L \supseteq L_{rings}$.

Definition

For $R \models T$, a subring $V \subseteq R$ is a T-**convex subring** of R if: (1) V is a proper convex subring and (2) $f(V) \subseteq V$ for all 0-definable continuos functions $f : R \longrightarrow R$.

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- Convexity of $V \Rightarrow V$ is a valuation ring.
- (R, V) is called a T-**convex field**, considered as a valued field.

Facts

- (R, V) is Henselian of equicharacteristic 0, in fact: the residue field $\overline{R}$ models T.
- (R, V) is a weakly o-minimal structure.
- T power bounded $\Rightarrow \Gamma$ o-minimal.
- $T_{convex} :=$ the theory of all pairs (R, V) with $R \vDash T$ and $V \subseteq R$ a T-convex subring in the language $L \cup \{V\}$ is complete and weakly o-minimal.

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Definition (Auxiliary sorts)

$\mathrm{RV}^{\mathrm{eq}}$ is the union of all the imaginary sorts defined from $\mathrm{RV}^{(n)}$ $(n \geq 1)$.

- $\overline{R}$ and Γ are auxiliary sorts.
- $L_{\mathrm{RV}} = L+$ sorts in RV^{eq} and the canonical maps from the $RV^{(n)}$'s into them.

Jacobian property

Definition (Jacobian property)

Let X be a subset of R^n and $f : X \longrightarrow R$ a function. We say that f has the **Jacobian property** on X if either f is constant on X or there exists $z \in R^n \setminus \{0\}$ such that for all $x, x' \in X$,

$$v(f(x) - f(x') - \langle z, x - x' \rangle) > v(x - x') + v(z).$$

The case $n = 1$ and its positive characteristic variant are used in Motivic Integration *à la Cluckers-Loeser*.

Jacobian Property in R

Desired Result

Let $A \subseteq R \cup \mathrm{RV}^{eq}$ and $f : R^n \longrightarrow R$ be an A-definable function. Then there are an auxiliary sort $S \subseteq \mathrm{RV}^{eq}$ and an A-definable function $\chi : R^n \longrightarrow S$ such that f has the Jacobian property on every fiber $\chi^{-1}(q)$ of dimension n.

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Theorem

Assume that T is power bounded. The desired result holds for R.

Importance: Consequences

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Consequence 2

Applications to local geometry of definable sets:

- *inducing stratifications on tangent cones in R^n,*
- *classical stratifications for L-definable subsets of $\mathbb{R}^n$ and their tangent cones.*

Comments on the proof

Consequence 1 is actually employed in one proposed way of proving the theorem.

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$n = 0, 1$: follow, for instance, from work of Y. Yin.
Suppose JP holds for any A-definable function $R^d \longrightarrow R$, $d < n$.
For $f : R^n \longrightarrow R$ A-definable:

- (van den Dries-Lewenberg) There is a finite partition of R^n into convex definable sets X_i such that $f|_{X_i}$ is o-minimally definable (i.e. L-definable).
- $\mathrm{Jac}\,(f)$ exists on X_i.
- Further partitioning: There is an A-definable $\rho : R^n \longrightarrow S$ such that $\mathrm{rv}(\mathrm{Jac}\,(x))$ is constant on every fiber of ρ.

- Induction hypothesis $\Rightarrow$ there is a "nice" A-definable stratification for ρ.
- The stratification gives us an A-definable map $\chi : R^n \longrightarrow \mathrm{RV}^{eq}$ refining ρ.
- From properties of χ, a long but not difficult calculation shows that f has the Jacobian property on any fiber of χ of dimension n.

$\square$

THANKS!