STRUCTURE THEOREMS IN TAME EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

PANTELIS E. ELEFTHERIOU, AYHAN GÜNAYDIN, AND PHILIPP HIERONYMI

Abstract. We study sets and groups definable in tame expansions of o-minimal structures. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of an o-minimal $\mathcal{L}$-structure $\mathcal{M}$ by a dense set P. We impose three tameness conditions on $\widetilde{\mathcal{M}}$ and prove a cone decomposition theorem for definable sets and functions in the realm of o-minimal semi-bounded structures. The proof involves induction on the notion of 'large dimension' for definable sets, an invariant which we herewith introduce and analyze. As a corollary, we obtain that (i) the large dimension of a definable set coincides with the combinatorial scl-dimension coming from a pregeometry given in Berenstein-Ealy-Günaydin [3], and (ii) the large dimension is invariant under definable bijections. We then illustrate how our results open the way to the study of groups definable in $\widetilde{\mathcal{M}}$, by proving that around scl-generic elements of a definable group, the group operation is given by an $\mathcal{L}$-definable map.

1. Introduction

Definable groups have been at the core of model theory for at least a period of three decades (see, for example, [5, 28, 34]) and have been crucially used in important applications of model theory to other areas of mathematics (such as in [23]). On the other hand, tame expansions of o-minimal structures have recently seen significant growth ([1, 3, 6, 8, 10, 12, 20, 26]) and are by now divided into two important classes of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. The missing ingredient for the study of definable groups in either category appears to be a cone decomposition theorem, inspired by an analogous theorem in the o-minimal semi-bounded setting ([13, 15, 29]), which was vitally used in the analysis of definable groups therein ([16]). In this paper, we prove this cone decomposition theorem for structures in the first category (Structure Theorem), and then apply it to the local study of definable groups. A relevant structure theorem for certain structures in the second category has been obtained in [37], benefiting largely by the presence of definable choice in that setting.

Let $\mathcal{M}$ be an o-minimal expansion of an ordered group with underlying language $\mathcal{L}$. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of $\mathcal{M}$ by a dense set P so that certain tameness

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conditions hold (those are listed in Section 2.1). For example, $\widetilde{\mathcal{M}}$ can be a dense pair ([10]), or P can be a dense independent set or multiplicative group with the Mann Property ([12]). To establish our structure theorem below, we introduce a new invariant for definable sets, the 'large dimension', which turns out to coincide with the combinatorial dimension coming from a pregeometry in [3]. These results are in the spirit of some standard and recent literature. In an o-minimal structure, a cell decomposition theorem is well-known ([11, 27]) and the associated 'topological dimension' equals the combinatorial dimension coming from the dcl-pregeometry ([32]). In a semi-bounded structure, cone decomposition theorems are known ([13, 15, 29]) and the associated 'long dimension' equals the dimension coming from the short closure pregeometry ([15]). In both settings, the equivalence of the two dimensions has proven extremely powerful in many occasions and in particular in the analysis of definable groups (see, for example, [15, 16, 17, 33]). Here, we apply the strategy from the semi-bounded setting to that of tame expansions of o-minimal structures and initiate the study of groups definable in $\widetilde{\mathcal{M}}$.

In Section 4 we introduce the notions of a *cone* and *large dimension* and in Section 5 we prove the following theorem.

Structure Theorem (5.1). *Let $X \subseteq M^n$ be an A-definable set. Then X is a finite union of A-definable cones. Furthermore, if $f : X \to M$ is an A-definable function, then there is a finite collection $\mathcal{C}$ of A-definable cones, whose union is X and such that f is fiber $\mathcal{L}$-definable with respect to each cone in $\mathcal{C}$.*

We then conclude that the large dimension is invariant under definable bijections (Corollary 5.3). The above Structure Theorem is a substantial improvement of the 'near-model completeness' results established in known cases (such as [1, 10, 12]) in that it achieves a decomposition of definable sets into *unions* (instead of boolean combinations) of cones, and it analyzes definable maps $f : M^k \to M$ for any k (instead of $k = 1$ that was handled in special cases).

In Section 6 we compare the large dimension of a definable set to the scl-dimension coming from [3]. In [3], the authors work under three similar tameness conditions on $\widetilde{\mathcal{M}}$ and prove that the *small closure* operator scl defines a pregeometry under further assumptions on $\mathcal{M}$ ([3, Corollary 77]). Here we observe that those further assumptions are in fact unnecessary (Corollary 6.4) and derive the equivalence of the two dimensions (Proposition 6.9), always.

In Section 7, we exploit this equivalence and set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$. Indeed, making use of desirable properties of 'scl-generic' elements (Fact 6.14), we are able to prove the following theorem.

Local theorem for definable groups (7.5). *Let $G = \langle G, * \rangle$ be a definable group of large dimension k. Then for every* scl*-generic element a in G, there is a $2k$-cone $C \subseteq G \times G$ containing (a, a) on which the operation*

$$(x, y) \mapsto x * a^{-1} * y$$

is given by an $\mathcal{L}$-definable map.

We note that an analogous local theorem for semi-bounded groups was proved in [15, Theorem 6.3] and was vitally used in the global analysis of semi-bounded groups in [16]. We expect that the present local theorem will be as crucial in forthcoming analysis of definable groups in $\widetilde{\mathcal{M}}$, and we list a series of open questions in the end of Section 7. The ultimate goal would be to understand definable groups in terms of $\mathcal{L}$-definable groups and small groups (Conjecture 7.7). Note that $\mathcal{L}$-definable groups have been exhaustively studied and are well-understood, some of the main results being proved in [7, 14, 16, 17, 24, 25, 30].

We next indicate some of the key aspects of this paper. Both the definition of the large dimension, as well as that of a cone, are based on the notion of a *supercone* given in Section 4, which in its turn is based on the notion of a *large* subset of M coming from [3] or [12]. Namely, a supercone J in M^n is defined, recursively on n, as a union of a special collection of large fibers over a supercone in M^{n-1}. The large dimension of a definable set X is then the maximum k such that a supercone from M^k can be *embedded* into X. The nature of this embedding is crucial: while the definition of the large dimension is given via a strong notion of embedding, proving its invariance under definable bijections in Corollary 5.3 requires an equivalent definition via a weaker notion of embedding. We establish that equivalence in Corollary 4.11.

Let us now describe the main idea behind the proof of the Structure Theorem in Section 5 that also explains the role of the large dimension in it and motivates all the preparatory work in Sections 3 and 4. The notion of a large/small set is defined in Section 2 and that of a k-cone in Section 4. Roughly speaking, a k-cone is a set of the form

$$h\left(\bigcup_{g\in S}\{g\}\times J_g\right),$$

where h is an $\mathcal{L}$-definable continuous map with each $h(g,-)$ being injective, $S\subseteq M^m$ is a small set, and $\{J_g\}_{g\in S}$ is a definable family of supercones in M^k. The proof of the Structure Theorem runs by induction on n. In the inductive step, let $X\subseteq M^{n+1}$. By the inductive hypothesis, we may assume that the projection $\pi(X)$ onto the first n coordinates is a k-cone, and by definability of smallness (Remark 3.4(a)), we may separate two cases. If all fibers of X above $\pi(X)$ are large, then we can simply follow the definition of a cone and conclude that X is a $k+1$-cone (using Lemma 4.3). If all fibers of X above $\pi(X)$ are small, then we first need to turn X into a small union of ($\mathcal{L}$-definable images of) subsets $J_g\subseteq\pi(X)$ as above. This is achieved using Lemma 3.7 and it is illustrated in Example 3.9. Unfortunately, the sets J_g we initially obtain are not yet in the desired form (that is, supercones), but we can make them into supercones after subtracting from them subsets of *smaller large dimension* (Statements (4) and (5) of the Structure Theorem 5.1). We can then write X as the union of a k-cone and a set of smaller large dimension, and conclude the proof of the Structure Theorem for definable sets (Statement (1)) by sub-induction on large dimension. The proof of the Structure Theorem for definable functions (Statement (2)) also makes use of Corollary 3.24, which is a statement about $\mathcal{L}$-definability of functions outside small sets.

In Section 5.3, we explore the optimality of our Structure Theorem. We prove that a stronger version where the notion of a cone is strengthened by requiring that h is injective on $\bigcup_{g\in S}\{g\}\times J_g$ is not possible. This is essentially due to the lack of definable choice in our setting (see, for example, [8, Section 5.5]). In Section 5.4, however, we isolate a key 'choice' property that implies a strengthened version of Lemma 3.7 (see Lemma 5.12), which in its turn guarantees a Strong Structure Theorem (5.13). This study suggests a new line of research where the behavior of $\mathcal{L}$-definable maps on small sets is pending to be explored. In Section 5.5, we include a list of open questions for subsequent work.

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2. The setting

Throughout this paper, we fix an o-minimal theory T expanding the theory of ordered abelian groups with a distinguished positive element 1. We also fix the language $\mathcal{L}$ of T and $\mathcal{L}(P)$ the language $\mathcal{L}$ augmented by a unary predicate symbol P. Let $\widetilde{T}$ be an $\mathcal{L}(P)$-theory expanding T. If $\mathcal{M} = \langle M, <, +, \dots \rangle \models T$, then $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of $\mathcal{M}$ that models $\widetilde{T}$. By 'A-definable' we mean 'definable in $\widetilde{\mathcal{M}}$ with parameters from A'. By '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{M}$ with parameters from A'. We omit the index A if we do not want to specify the parameters.

For a subset $X \subseteq M$, we write $\operatorname{dcl}(X)$ for the definable closure of X in $\mathcal{M}$, and $\operatorname{dcl}_{\mathcal{L}(P)}(X)$ for the definable closure of X in $\widetilde{\mathcal{M}}$. By the o-minimality of T, the operation that maps $X \subseteq M$ to $\operatorname{dcl}(X)$ is a pregeometry on M. We say that X is dcl-independent over A if for each proper subset $Y \subsetneq X$, $\operatorname{dcl}(Y \cup A) \neq \operatorname{dcl}(X \cup A)$.

The following definition is taken essentially from [12].

Definition 2.1. Let $X \subseteq M^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : M^{nm} \to M$ such that $f(X^m)$ contains an open interval in M. We call X *small* if it is not large.

Note that the complement of a small set is large (with a proof identical to that of [3, Lemma 20]). In Lemma 3.11 and Corollary 3.12 below we prove that smallness is equivalent to P-internality, in the usual sense of geometric stability theory.

If $X, Z \subseteq M^n$ are definable, we say that X is *small in* Z if $X \cap Z$ is small. We say that X is *co-small in* Z if $Z \setminus X$ is small.

2.1. Assumptions.

We assume that $\widetilde{T}$ satisfies the following three tameness conditions: for every model $\widetilde{\mathcal{M}} \models \widetilde{T}$,

(I) P is small.

(II) (Near model-completeness) Every A-definable set $X \subseteq M^n$ is a boolean combination of sets of the form
$$\{x \in M^n : \exists z \in P^m \varphi(x, z)\},$$
where $\varphi(x, y)$ is an $\mathcal{L}_A$-formula.

(III) (Open sets are $\mathcal{L}$-definable) For every parameter set A such that $A \backslash P$ is dcl-independent over P, and for every A-definable set $V \subset M^s$, its topological closure $cl(V) \subseteq M^s$ is $\mathcal{L}_A$-definable.

From now on, and unless stated otherwise, $\widetilde{T}$ satisfies Assumptions (I)-(III) and $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ is a sufficiently saturated model of $\widetilde{T}$.

Remark 2.2. Assumptions (I)-(III) are analogous to Assumptions (1)-(3) from [3, Theorem 3]. Here, however, we insist on having some control on the defining parameters. Moreover, an easy argument shows that under our assumptions, (3) from [3, Theorem 3] holds, but without the additional condition that the set S mentioned there is $\emptyset$-definable.

Notation-terminology. The topological closure of a set $X \subseteq M^n$ is denoted by $cl(X)$. If $X, Y \subseteq M$ and $b = (b_1, \dots, b_n)$, we sometimes write $X \cup b$ or Xb for $X \cup \{b_1, \dots, b_n\}$, and XY for $X \cup Y$. If $\varphi(x, y)$ is an $\mathcal{L}(P)$-formula and $a \in M^n$, then we write $\varphi(M^m, a)$ for

$$\{b \in M^m \ : \ \widetilde{\mathcal{M}} \models \varphi(b, a)\}.$$

Similarly, given any subset $X \subseteq M^m \times M^n$ and $a \in M^n$, we write X_a for

$$\{b \in M^m \ : \ (b, a) \in X\}.$$

For convenience, we sometimes write $f(t, X)$ for $f(\{t\} \times X)$. If $m \leq n$, then $\pi_m : M^n \to M^m$ denotes the projection onto the first m coordinates. By an open box in M^k, or a k-box, we mean a set $I_1 \times \cdots \times I_k \subseteq M^k$, where each $I_j \subseteq M$ is an open interval. By dimension of an $\mathcal{L}$-definable set we mean its usual o-minimal dimension, and the notions of dcl-independence, dcl-rank and dcl-generics are the usual notions attached to the dcl-pregeometry (see, for example, [33]). A family $\{J_g\}_{g\in S}$ of sets is called definable if $\bigcup_{g\in S}\{g\} \times J_g$ is definable, disjoint if every two elements of it are disjoint, and small if S is small. If for each $t \in T$, $\mathcal{J}_t = \{J_{g,t}\}_{g\in S_t}$ is a family of sets, we call $\{\mathcal{J}_t\}_{t\in T}$ definable if $\bigcup_{t\in T, g\in S_t}\{(g, t)\} \times J_{g,t}\}$ is definable.

Our examples are often given for structures over the reals (such as Example 4.9 and the counterexample in Section 5.3). But they can easily be adopted to the current, saturated setting, by moving to an elementary extension.

2.2. **Examples.**

Dense pairs. The first example we wish to consider is dense pairs of o-minimal structures. A *dense pair* $\langle \mathcal{M}, \mathcal{N} \rangle$ is a pair of models of T such that $\mathcal{N} \neq \mathcal{M}$, but $\mathcal{N}$ is dense in $\mathcal{M}$. Let $\widetilde{T} = T^d$ be the theory of dense pairs in the language $\mathcal{L}(P)$. By [10], T^d is complete and every model of T^d satisfies (I) and (II) ([10, Lemma 4.1] and [10, Theorem 1], respectively).

It is left to explain why (III) holds in dense pairs. Here we apply [6, Corollary 3.1]. Let A be a parameter set such that $A \setminus N$ is dcl-independent over N. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } N \cup A\}.$$

It is easy to see that D and A satisfy Assumptions (1) and (2) of [6, Corollary 3.1]. It is left to show that also the third assumption of that corollary holds. Towards that goal, recall the following notation from [10]. Given $\mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{Q} \models T$ with $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{Q}$ and $\mathcal{M} \subseteq \mathcal{O} \subseteq \mathcal{Q}$, we say that $\mathcal{N}$ and $\mathcal{O}$ are *free over* $\mathcal{M}$ (in $\mathcal{Q}$) if every subset $Y \subseteq \mathcal{N}$ that is dcl-independent over $\mathcal{M}$ is also dcl-independent over $\mathcal{O}$.

Proposition 2.3. *Let $a \in D$. Then the $\mathcal{L}(P)$-type over a over A is implied by the L-type over A and the fact that $a \in D$.*

Proof. Let $\langle \mathcal{M}, \mathcal{N} \rangle \models T^d$ be κ-saturated, where $\kappa > |T|$. Let Γ be the set of all isomorphisms $i : \langle \mathcal{M}_1, \mathcal{N}_1 \rangle \to \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ between substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$ such that $|M_1| < \kappa, |M_2| < \kappa$, $\mathcal{M}_1$ and $\mathcal{N}$ are free over $\mathcal{N}_1$ and $\mathcal{M}_2$ and $\mathcal{N}$ are free over

$\mathcal{N}_2$. By [10, Claim on p. 67], Γ has the back-and-forth property. Let $a, b \in D$ such that a and b satisfy the same $\mathcal{L}$-type over A. Then there is an $\mathcal{L}$-isomorphism

$$i : \operatorname{dcl}(a \cup A) \to \operatorname{dcl}(b \cup A).$$

Since both a and b are dcl-independent over $N \cup A$, the isomorphism expands to an isomorphisms

$$i : \big\langle \operatorname{dcl}(a \cup A), \operatorname{dcl}(A) \cap \mathcal{N} \big\rangle \to \big\langle \operatorname{dcl}(b \cup A), \operatorname{dcl}(A) \cap \mathcal{N} \big\rangle$$

of substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$. Since $a \cup (A \setminus N)$ is dcl-independent over N, $\operatorname{dcl}(a \cup A)$ and $\mathcal{N}$ are free over $\operatorname{dcl}(N) \cap \mathcal{N}$. By the same argument $\operatorname{dcl}(b \cup A)$ and $\mathcal{N}$ are free over $\operatorname{dcl}(A) \cap \mathcal{N}$. Hence $i \in \Gamma$. Since Γ is a back-and-forth system, a and b satisfy the same $\mathcal{L}(P)$-type over A. $\square$

Groups with the Mann property. Let Γ be a dense subgroup of $\mathbb{R}_{>0}$ that has the *Mann property*, that is for every $a_1, \dots, a_n \in \mathbb{Q}^\times$, there are finitely many $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $a_1\gamma_1 + \cdots + a_n\gamma_n = 1$ and $\sum_{i \in I} a_i\gamma_i \neq 0$ for every proper nonempty subset I of $\{1, \dots, n\}$. Every multiplicative subgroup of finite rank in $\mathbb{R}_{>0}$ has the Mann property, see [18].

Let $\mathcal{L}$ be the language of ordered rings augmented by a constant symbol for each $\gamma \in \Gamma$. Let T be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma} \rangle$ in that language and let $\widetilde{T} = T(\Gamma)$ be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma}, \Gamma \rangle$ in the language $\mathcal{L}(P)$. By [12, Theorem 7.5], every model of $T(\Gamma)$ satisfies (II). A proof that every model satisfies (I) is in [20, Proposition 3.5].

Again, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P \rangle \models T(\Gamma)$. Let A for every parameter set A such that $A \setminus P$ is dcl-independent over P. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } P \cup A\}.$$

One can check easily that assumptions (1) and (2) of [6, Corollary 3.1] follow from the o-minimality of T. Finally it is easy to see that almost the same proof as for Proposition 2.3, just using the back-and-forth system in the proof of Theorem 7.1 in [12] instead of [10, Claim on p. 67], shows that assumption (3) of [6, Corollary 3.1] is satisfied as well.

There are several other closely related examples. In [22] proper o-minimal expansions $\mathcal{R}$ of the real field and finite rank subgroups Γ of $\mathbb{R}_{>0}$ are constructed such that the structure $(\mathcal{R}, \Gamma)$ satisfies Assumptions (I)-(III). Indeed, the fact that these structures satisfy Assumptions (I) and (II) is immediate from results in [22]. Assumption (III) follows by the same argument as above. In [1, 20] certain expansions of the real field by subgroups of either the unit circle or an elliptic curve are studied. One can easily show using the above argument that these structures satisfy Assumptions (I)-(III) after adjusting their statements for the fact that P now lies in a 1-dimensional semialgebraic set in $\mathbb{R}^2$. Since no significant new argument is involved, we leave it to the reader to verify that our main results also hold in this slightly more general setting.

Independent sets. Let $\widetilde{T} = T^{\text{indep}}$ be an $\mathcal{L}(P)$-theory extending T by axioms stating that P is dense and dcl-independent. By [9], T^{indep} is complete and every model of T^{indep} satisfies (I) and (II) by [9, 2.1] and [9, 2.9], respectively. As usual, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P\rangle \models T^{\text{indep}}$. Let A be a parameter set such that $A \setminus P$ is dcl-independent over P. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } P \cup A\}.$$

From the o-minimality of T, assumptions (1) and (2) of [6, Corollary 3.1] follow easily as above. By [9, 2.12], assumption (3) of [6, Corollary 3.1] holds as well.

Non-examples.

(1) By Assumption (III), P must be dense in a finite union of open intervals and points. Indeed, the closure of P has to be $\mathcal{L}$-definable. Therefore, tame expansions of $\mathcal{M}$ by discrete sets, such as $\langle \mathbb{R}, 2^{\mathbb{Z}}\rangle$ do not belong to this setting.

(2) We do not know whether the theory of every expansion $\langle \mathcal{M}, P\rangle$ of an o-minimal structure $\mathcal{M}$ with o-minimal open core [8, 26] satisfies Assumptions (II) or (III). Assumption (I) does not hold in case P is a generic predicate.

(3) If $\widetilde{\mathcal{M}} = \langle M, <, +, \mathcal{P}\rangle$ is semi-bounded, that is, a pure ordered group expanded by the structure of a real closed field $\mathcal{P} = \langle P, \oplus, \otimes\rangle$ on some bounded open interval $P \subseteq M$, then Assumptions (II) and (III) hold by [15], but (I) does not.

Although all our known examples that satisfy Assumptions (I)-(III) have NIP (see [2, 21]), the following question stands open.

Question 2.4. *Do Assumptions (I)-(III) imply that $\widetilde{T}$ has NIP?*

2.3. **Basic facts for $\mathcal{L}$-definable and small sets.** We include some basic facts that will be used in the sequel.

Fact 2.5. *Let $f : X \subseteq M^m \to M^n$ be a finite-to-one $\mathcal{L}$-definable function. Then there is a finite partition $X = X_1 \cup \cdots \cup X_k$ into definable sets such that each $f_{\restriction X_i}$ is injective.*

Proof. Standard. $\square$

Fact 2.6. *Let $f : A \subseteq M^m \to M^n$ be an $\mathcal{L}$-definable function. Let*

$$X_f = \{a \in A : f^{-1}(f(a)) \textit{ is finite}\}.$$

Then $\dim f(A \setminus X_f) < \dim A$.

Proof. Let $R = f(A \setminus X_f)$. By definition of X_f, for every $r \in R$, $f^{-1}(r)$ has dimension > 0. Since $A \setminus X_f$ equals the disjoint union $\bigcup_{r\in R} f^{-1}(r)$, we have by standard properties of dimension:

$$\dim(A \setminus X_f) \geq \min_r \dim f^{-1}(r) + \dim R.$$

Hence, $\dim A \geq 1 + \dim R$ and $\dim R < \dim A$. $\square$

Fact 2.7. *Let $X \subseteq M^n$ be an $\mathcal{L}_{Aa}$-definable set of dimension n, and $b \in X$ a* dcl*-generic element of X over Aa. Then there are $A_1 \supseteq A$, such that*

$$\text{dcl-rank}(ab/A_1) = \text{dcl-rank}(ab/A),$$

and an $\mathcal{L}_{A_1}$-definable open box $V \subseteq X$ that contains b.

Proof. Standard. $\square$

Fact 2.8. *We have:*

(1) *If $X \subseteq M^m$ is co-small in $Z \subseteq M^m$ and I is open in Z, then $X \cap I$ is co-small in $Z \cap I$.*
(2) *If $J \subseteq M$ is a definable set and $f : M \to M^n$ is an $\mathcal{L}$-definable function which is injective on J, then f is finite-to-one on $cl(J)$.*

Proof. (1) is immediate from the definitions. For (2), $cl(J)$ is $\mathcal{L}$-definable (Assumption (III)) and J is dense in $cl(J)$. Hence, the statement then follows from the monotonicity theorem. □

A generalization of Fact 2.8(2) is given in Proposition 4.8(2) below.

3. Small sets

In this section we establish properties of small sets that will be important in the proof of the Structure Theorem. The two most crucial results are Lemma 3.7 and Corollary 3.24 below.

3.1. **Families of small sets and P-boundedness.** With the exception of Lemma 3.7 below, the results of this section were either established in [3] or are minor improvements of results in [3]. Since the assumptions in [3] differ from ours, we reprove the results here. Most of the proofs are direct adjustments from those in [3], but are included for the convenience of the reader. They often involve induction on formulas whose base step deals with a 'basic' set defined next.

Definition 3.1. A subset $X \subseteq M^n$ is called *basic over A* if it is of the form

$$\bigcup_{g \in P^m} \varphi(M, g),$$

for some $\mathcal{L}_A$-formula. We say X is *basic* if it is basic over some parameter set A.

Note that by Assumption (II) every definable set is a boolean combination of basic sets.

Lemma 3.2. *Let $p \in \mathbb{N}$. For $j = 1, \ldots, p$, let $\{S_{1,j,t}\}_{t\in M^l}, \{S_{2,j,t}\}_{t\in M^l}$ be A-definable families of subsets of P^n. Let $f_1, \ldots, f_p, h_1, \ldots, h_p : M^{n+l} \to M$ be $\mathcal{L}_A$-definable functions. Then there are A-definable families $\{Q_{j,t}\}_{t\in M^l}, \{R_{j,t}\}_{t\in M^l}$ of subsets of P^n, for $j = 1, \ldots, p$, such that for every $t \in M^l$,*

$$\bigcup_j f_j(S_{1,j,t}, t) \cap \bigcup_j h_j(S_{2,j,t}, t) = \bigcup_j f_j(Q_{j,t}, t),$$

$$\Big(M \setminus \bigcup_j f_j(S_{1,j,t}, t)\Big) \cup \bigcup_j h_j(S_{2,j,t}, t) = M \setminus \bigcup_j f_j(R_{j,t}, t)$$

Proof. Set

$$Q_{j,t} := \{g \in S_{1,j,t} : \bigvee_{i=1}^{p} \exists g' \in S_{2,i,t}\ h_i(g', t) = f_j(g, t)\}$$

and

$$R_{j,t} := \{g \in S_{1,j,t} : \bigwedge_{i=1}^{p} \forall g' \in S_{2,i,t}\ h_i(g', t) \neq f_j(g, t)\}.$$

□

Lemma 3.3. *Let $\{X_t\}_{t\in M^l}$ be an A-definable family of subsets of M. Then there are $m, n, p \in \mathbb{N}$ and for each $i = 1, \dots, m$ there are*

- *an A-definable family $\{S_{i,j,t}\}_{t\in M^l}$ of subsets of P^n, for each $j = 1, \dots, p$,*
- *$\mathcal{L}_A$-definable continuous functions $h_{i,1}, \dots, h_{i,p} : M^{n+l} \to M$,*
- *A-definable function $a_i : M^l \to M$,*

such that for $t \in M^l$,

(i) *$-\infty = a_0(t) \le a_1(t) \le \cdots \le a_m(t) = \infty$ is a decomposition of M, and*
(ii) *one of the following holds:*
 (a) *$[a_{i-1}(t), a_i(t)] \cap X_t = V_{i,t}$ or*
 (b) *$[a_{i-1}(t), a_i(t)] \cap X_t = (M \setminus V_{i,t}) \cap [a_{i-1}(t), a_i(t)]$,*
 where $V_{i,t} = \bigcup_j h_{i,j}(S_{i,j,t}, t)$.

Proof. First consider a definable family of basic sets, say $(\mathbb{D}_t)_{t\in M^l}$, that is a definable family of the form

$$\mathbb{D}_t = \bigcup_{g\in P^n} \varphi(M, g, t),$$

where $\varphi(x, y, z)$ is an $\mathcal{L}_A$-formula and $t \in M^l$. By cell decomposition, there are two finite sets J_1, J_2, $\mathcal{L}_A$-definable cells $(Y_{1,j})_{j\in J_1}$ and $(Y_{2,j})_{j\in J_2}$ and $\mathcal{L}_A$-definable continuous functions $(f_{1,j})_{j\in J_1}$, $(f_{2,j})_{j\in J_2}$ and $(f_{3,j})_{j\in J_2}$ such that

$$\mathbb{D}_t = \bigcup_{j\in J_1} f_{1,j}(Y_{1,j} \cap P^n, t) \cup \bigcup_{j\in J_2} \bigcup_{g\in Y_{2,j}\cap P^n} \big(f_{2,j}(g, t), f_{3,j}(g, t)\big).$$

Without loss of generality, we can assume that $|J_1| = |J_2|$. Set $p := |J_1|$ and assume that $J_1 = J_2 = \{1, \dots, p\}$. Set $U_t := \bigcup_{j\in J_2} \bigcup_{g\in Y_{2,j}\cap P^n} \big(f_{2,j}(g, t), f_{3,j}(g, t)\big)$. Note that U_t is open. By Assumption (II), U_t is a finite union of open intervals. Since finitely many intervals only have finitely many endpoints and U_t is At-definable, the endpoints of the intervals of U_t are At-definable. Let V_t be the topological closure of $\bigcup_{j\in J_1} f_{1,j}(Y_{1,j} \cap P^n, t)$. By Assumption (II) again, V_t is $\mathcal{L}$-definable. Hence it is a finite union of intervals and points. Since there are only finitely many endpoints and V_t is At-definable, these endpoints are At-definable. Hence we have a decomposition of M

$$-\infty = a_0(t) \le a_1(t) \le \cdots \le a_m(t) = \infty$$

such that either

- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t))$ or
- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t)) \cap \bigcup_{j\in J_1} f_{1,j}(Y_{1,j} \cap P^n, t)$.

In the first case set $S_{i,j,t} := \emptyset$ and set $h_{i,j}(x, y) = 0$ for all $(x, y) \in M^{n+l}$. In the second case set

$$S_{i,j,t} := \{g \in Y_{1,j} \cap P^n \ : \ f_{1,j}(g, t) \in (a_{i-1}(t), a_i(t))\},$$

and set $h_{i,j} = f_{1,j}$. By compactness, we can find an $m \in \mathbb{N}$ that works for every $t \in M^l$. Hence (i)-(ii) holds for $(\mathbb{D}_t)_{t\in M^n}$.

By Assumption (I) it is enough to check that if the statement of the Lemma holds for two definable $(X_t)_{t\in M^l}$ and $(Z_t)_{t\in M^l}$, then it also holds for $(M \setminus X_t)_{t\in M^l}$ and $(X_t \cup Z_t)_{t\in M^l}$. So suppose that the statement holds for $(X_t)_{t\in M^l}$ and $(Z_t)_{t\in M^l}$. It is immediate that the conclusion holds for $(M \setminus X_t)_{t\in M^l}$ as well. It is easy to check that Lemma 3.2 implies that the conclusion also holds for $(X_t \cup Z_t)_{t\in M^l}$. □

Remark 3.4. It is worth pointing out that in the previous lemma:

(a) the set
$$\{t \in M^n \ : \ X_t \cap [a_{i-1}(t), a_i(t)] \text{ is small } \}$$
is equal to
$$\{t \in M^n \ : \ X_t \cap [a_{i-1}(t), a_i(t)] = V_{i,t}\}$$
and hence A-definable. In particular, the set all $t \in M^n$ such that X_t is small is A-definable.

(b) the set of $(t, a_i(t))$ for which X_t is small in $(a_{i-1}(t), a_i(t))$ is A-definable.

We will make use of the following consequence of Lemma 3.3.

Corollary 3.5. *Let $\mathcal{X} = \{X_t\}_{t\in I}$ be an A-definable family of subsets of M, where each $X_t \subseteq M$ is small and $I \subseteq M^n$. Then there are $m \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_j : M^{m+n} \to M$ and A-definable families $\{S_{j,t}\}_{t\in I}$ of small sets $S_{j,t} \subseteq P^m$, $j = 1, \dots, p$, such that for every $t \in I$, $X_t = \bigcup_j h_j(S_{j,t}, t)$.*

The following example shows that in the last corollary the set $S_{j,t}$ has to depend on t.

Example 3.6. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and
$$X = \bigcup_{a\in M^{>0}} \{a\} \times X_a.$$
Let h_j and $S_{j,a}$ be as in Corollary 3.5, and assume towards a contradiction that all $S_{j,a}$'s equal some S_j. What follows shows that we cannot even relax h_j to be just $\mathcal{L}_A$-definable (without necessarily being continuous). So for every $a \in M^{>0}$,
$$(*) \qquad (0, a) \cap G = \bigcup_j h_j(S_j, a).$$
For every j and $p \in S_j$, let $z^j_p = \inf\{a \in M^{>0} : h_j(p, a) \in X_a\}$. By (*), for some j, $z^j_p = 0$. Fix that j. By o-minimality, $h_j(p, -)$ is eventually continuous and, hence, constant close to 0. That is, there is $a_p > 0$ and $c_p \in P$, such that for every $0 < a < a_p$, $h_j(p, a) = c_p$. Thus, if $0 < a < c_p$, we have $h_j(p, a) = c_p \notin (0, a) \cap P$, a contradiction.

We now derive a few corollaries of the above results. The next lemma shows how to turn a family $X = \{X_a\}_{a\in C}$ of small sets into a small family of subsets Z_g of C. This will be a crucial step in the proof of the Structure Theorem. There, we will further need to replace Z_{ig} by "supercones", which are defined in Section 4.

Lemma 3.7. *Let $X = \bigcup_{a\in C}\{a\} \times X_a$ be A-definable where each $X_a \subseteq M$ is small, non-empty, and $C \subseteq M^n$. Then there are $m, l \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_i : M^{m+n} \to M^{n+1}$, A-definable small sets $S_i \subseteq M^m$, $i = 1, \dots, l$, and families $\{Z_{ig}\}_{g\in S_i}$ with $Z_{ig} \subseteq C$, such that for*
$$X_i = h_i\left(\bigcup_{g\in S_i} \{g\} \times Z_{ig}\right)$$
we have

(1) *$X = X_1 \cup \dots \cup X_l$ is a disjoint union*

(2) *for every i and $g \in S_i$, $h_i(g, -) : M^n \to M^{n+1}$ is injective*

(3) *$C = \bigcup_{i,g} Z_{ig}$.*

Proof. We first observe that there are $m \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_i : M^{m+n} \to M$ and A-definable families Y_i of small sets $Y_{ia} \subseteq P^m$, $i = 1, \dots, p$, such that for every $a \in I$,

(1) $X_a = \bigcup_i h_i(Y_{ia}, a)$
(2) $\{h_i(Y_{ia}, a)\}_{i=1,\dots,p}$ are disjoint.

Indeed, this follows from Corollary 3.5; for (2), recursively replace Y_{ia}, $1 < i \leq p$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z, a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a \in C} \{a\} \times X_a = \bigcup_i \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a).$$

For every i, let $S_i = P^m$. For every i and $g \in S_i$, let

$$X_i = \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a),$$

which are also disjoint, and

$$Z_{ig} = \{a \in C : g \in Y_{ia}\}.$$

Since h_i, S_i and $\{Y_{ia}\}_{a \in C}$ are A-definable, so are X_i and $\{Z_{ig}\}_{g \in S_i}$. We have $C = \bigcup_{i,g} Z_{ig}$. Consider now the $\mathcal{L}_A$-definable continuous map $\hat{h}_i : M^{m+n} \to M^{n+1}$ with

$$\hat{h}_i(g, a) = (a, h_i(g, a)).$$

Then

$$X_i = \hat{h}_i \left(\bigcup_{g \in S_i} \{g\} \times Z_{ig} \right)$$

works. $\square$

Remark 3.8. As the last proof shows, in fact we obtain $S_i = P^m$. We decided, however, to keep the current formulation because the proof can then be adopted in similar situations (such as in Lemma 5.12 below). Had we kept the stronger formulation ($S_i = P^m$), what follows would result to a Structure Theorem 5.1 where in Definition 4.1 of a cone we could require $S \subseteq P^m$. However, we recover this information anyway, see Remark 4.2(e).

Let us illustrate Lemma 3.7 with an example.

Example 3.9. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and

$$X = \bigcup_{a \in M^{>0}} \{a\} \times X_a.$$

Then we can turn X into a small union of ($\mathcal{L}$-definable images of) large subsets of M, as follows. For every $g \in P$, let

$$J_g = \{a \in M : a > g\}.$$

Then

$$X = h \left(\bigcup_{g \in P} \{g\} \times J_g \right),$$

where $h : M^2 \to M^2$ switches the coordinates, $h(x, y) = (y, x)$. In this case, X is in fact seen to be 1-cone (according to Definition 4.1 below).

We now turn to examine better the notion of smallness.

Definition 3.10. A set $X \subseteq M^n$ is *P-bound* over A, if there is an $\mathcal{L}_A$-definable function $f : M^m \to M^n$ such that $X \subseteq f(P^m)$.

Lemma 3.11. *An A-definable set is small if and only if it is P-bound over A.*

Proof. Since P is small, it follows immediately that every P-bound set is small. For the other direction, observe first that, by Lemma 3.3, every A-definable small subset of M is P-bound over A. Now let $X \subseteq M^n$ be A-definable, and let $\pi_i : M^n \to M$ be the projection onto the i-coordinate. If X is small, so is $\pi_i(X)$ for $i = 1, \dots, n$. Since each A-definable small subset of M is P-bound over A, so is $\pi_i(X)$. Hence $\prod_{i=1}^n \pi_i(X)$ is P-bound over A and so is $X \subseteq \prod_{i=1}^n \pi_i(X)$. □

We show that in the definition of largeness and P-boundedness, we can replace $\mathcal{L}$-definability by definability. Recall from geometric stability theory that given two definable sets $X \subseteq M^n$ and $Y \subseteq M^k$, X is called *Y-internal* over A if there is an A-definable $f : M^{mk} \to M^n$ such that $X \subseteq f(Y^m)$.

Corollary 3.12. *Let X be a definable set.*

(1) *X is P-bound over A if and only if it is P-internal over A.*
(2) *X is large if and only if an open interval is X-internal.*

Proof. By Lemma 3.11, Definition 2.1 and Assumption (I), it is easy to see that (1) implies (2). For (1), let $F : M^k \to M^n$ be A-definable such that $X \subseteq h(P^k)$. Without loss of generality, we may assume that $A \setminus P$ is dcl-independent over P. For each $g \in P^n$, the singleton $\{F(g)\}$ equals its topological closure. Since $F(g)$ is definable over $A \cup g$ and $(A \cup g) \setminus P$ is dcl-independent over P, we get by Assumption (III) that $\{F(g)\}$ is $\mathcal{L}_{A\cup g}$-definable. Hence, by compactness, there are finitely many $\mathcal{L}_A$-functions $F_1, \dots, F_l$ such that for all $g \in P^k$, $F(g) = F_i(g)$ for some i. Hence

$$F(P^k) \subseteq \bigcup_i F_i(P^k).$$

However, the right hand side is P-bound over A, and hence so is $F(P^k)$. □

The following is then immediate.

Corollary 3.13. *Let $f : X \to M^n$ be a definable injective function. Then X is small if and only if $f(X)$ is small.*

A stronger version of the Corollary 3.13 is provided by the invariance result in Corollary 5.3 below. Here are two more corollaries of Lemma 3.11.

Corollary 3.14. *Let $Y \subseteq M^m$ be small and let $(X_t)_{t\in Y}$ be a definable family of small sets of M^n. Then $\bigcup_{t\in Y} X_t$ is small.*

Proof. By Lemma 3.11 and compactness, there is a definable family of $\mathcal{L}$-definable functions $(f_t)_{t\in Y}$ such that $X_t = f_t(M^k)$ for each $t \in Y$. Again by Lemma 3.11, there is also an $\mathcal{L}$-definable function $f : P^l \to M^m$ such that $Y \subseteq f(P^l)$. Set $h : M^{k+l} \to M^n$ be the function that takes (t, x) to $f_t(x)$. Then $\bigcup_{t\in Y} X_t \subseteq h(P^{k+l})$ and hence is P-bound. □

Corollary 3.15. *The union and cartesian product of finitely many small sets is small.*

Proof. Immediate from Lemma 3.11 and the definitions. □

3.2. **Definable functions outside small sets.** In this section we analyze the behavior of definable functions outside small, or rather *low*, sets.

Definition 3.16. We denote by $I_n(A) \subseteq M^n$ the set of all tuples $a = (a_1, \ldots, a_n) \in M^n$ that are dcl-independent over $P \cup A$.

Note that $I_n(A)$ is $\mathcal{L}(P)_A$-type definable. Indeed, $a \in I_n(A)$ if and only if for all $0 \leq i < n$, $m, l \in \mathbb{N}$ and $\mathcal{L}_A$-$(n+i)$-formula $\varphi(x, y)$, a satisfies:

$$\forall g \in P^n \,[\text{if } \varphi(g, a_1, \ldots, a_{i-1}, -) \text{ has } m \text{ realizations, then } \models \neg\varphi(g, a_1, \ldots, a_i)].$$

Lemma 3.17. *Let A be a parameter set such that $A \setminus P$ is* dcl*-independent over P, and let $\varphi(x, y)$ be an $\mathcal{L}(P)_A$-formula. Then there are $\mathcal{L}_A$-formulas $\psi_1(x, y), \ldots, \psi_m(x, y)$ such that for all $a \in I_n(A)$ there is $i \in \{1, \ldots, m\}$ with*

$$cl(\varphi(M^l, a)) = \psi_i(M^l, a).$$

Proof. Let $a = (a_1, \ldots, a_n) \in I_n(A)$. Since $a \in I_n(A)$, we have $A \cup \{a_1, \ldots, a_n\} \setminus P$ is dcl-independent over P. Since $I_n(A)$ is $\mathcal{L}(P)_A$-type definable, the statement of the Lemma follows from compactness and Assumption (III). □

In general, an $\mathcal{L}$-definable set X which is also A-definable need not be $\mathcal{L}_A$-definable. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, and $x, y \in M \setminus P$ such that there are unique $g, h \in P$ with $x = g + hy$. Then $\{g\}$ is $\mathcal{L}$-definable and $\{x, y\}$-definable, but in general not $\mathcal{L}_{\{x,y\}}$-definable. The following claim, however, implies, in particular, that every such X is always $\mathcal{L}_{A\cup P}$-definable.

Claim 3.18. *Let A be a parameter set. Let $X \subseteq M^n$ be an A-definable set. Then there is finite $B \subseteq A$ such that X is $B \cup P$-definable and B is* dcl*-independent over P. Hence, by Assumption (III), $cl(X)$ is $\mathcal{L}_{A\cup P}$-definable.*

Proof. Without loss of generality, we can assume that A is finite. Let $B = \{b_1, \ldots, b_k\} \subseteq A$ such that B is the maximal subset of A that is dcl-independent over P. Hence for every $a \in A \setminus B$ there is $g_a \in P^l$ and $h : M^{l+k} \to M$ $\mathcal{L}_\emptyset$-definable such that

$$h(g_a, b_1, \ldots, b_k) = a. \tag{1}$$

Set $H = \{g_a \,:\, a \in A \setminus B\}$. Since X is A-definable, by (1) it is $B \cup H$-definable. □

A positive answer to the following, currently open, question would give better control to the set of parameters (as indicated after Corollary 3.22 below).

Question 3.19. *For X as above, are there finite $B \subseteq A$ and $H \subseteq P \cap \mathrm{dcl}_{\mathcal{L}(P)}(A)$, such that X is $B \cup H$-definable and B is* dcl*-independent over P?*

Proposition 3.20. *Let A be a set of parameters and let $F : M^n \to M$ be A-definable. Then there are finitely many $\mathcal{L}_{A\cup P}$-definable functions $F_1, \ldots, F_m : M^n \to M$, such that for all $a \in I_n(A)$ there is $i \in \{1, \ldots, m\}$ such that*

$$F(a) = F_i(a)$$

Proof. By Claim 3.18, there is a finite $B \subseteq A$ such that B is dcl-independent over P and F is $B \cup P$-definable. Clearly, $(B \cup P) \setminus P$ is also dcl-independent over P. Let $\varphi(x, y)$ be an $\mathcal{L}(P)_{B\cup P}$-formula that defines the graph of F. Hence by Lemma 3.17 there are $\mathcal{L}_{B\cup P}$-formulas $\psi_1(x, y), \ldots, \psi_m(x, y)$ such that for all $a \in I_n(B \cup P)$ there is $i \in \{1, \ldots, m\}$ with

$$cl(\varphi(a, M)) = \psi_i(a, M).$$

Since $\varphi(a, M)$ is a single point, we have $\varphi(a, M) = \psi_i(a, M)$. Define $F_i : M^n \to M$ such that $F_i(a)$ is the unique $b \in M$ with $\psi_i(a, b)$ if such b exists, and 0 otherwise. Since ψ_i is an $\mathcal{L}_{B\cup P}$-formula, F_i is $\mathcal{L}_{B\cup P}$-definable. $\square$

Definition 3.21. We call $X \subseteq M^n$ *low over* B if there is $i \in \{1, \dots, n\}$ and $\mathcal{L}_B$-definable function $f : M^{n-1} \times M^l \to M$ such that

$$X = \{(a_1, ..., a_n) \in M^n \ : \ \exists g \in P^l \ f(a_{-i}, g) = a_i\},$$

where $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

Note that if a set $X \subseteq M$ is low, then it is small. A generalization of this statement is obtained in Lemma 4.19 below.

Corollary 3.22. *Let A be a set of parameters and let $F : M^n \to M$ be A-definable. Then there are finitely many sets $X_1, \dots, X_k \subseteq M^n$ low over A and finitely many $\mathcal{L}_{A\cup P}$-definable functions $F_1, \dots, F_m : M^n \to M$, such that for all $a \notin \bigcup_{i=1}^k X_i$, there is $i \in \{1, \dots, m\}$ such that*

$$F(a) = F_i(a)$$

Proof. Note that $a \notin I_n(A)$ if and only if there $i \in \{1, \dots, n\}$, an $\mathcal{L}_A$-definable function $f : M^{l+(n-1)} \to M$ and $g \in P^l$, such that $f(a_{-i}, g) = a_i$. Hence if $a \notin I_n(A)$ if and only if there is X low over A such that $a \in X$. The statement now follows from compactness and Proposition 3.20. $\square$

If the answer to Question 3.19 above is positive, then the same proof gives that the F_i's are all $\mathcal{L}_{A\cup H}$-definable, for some $H \subseteq P \cap \mathrm{dcl}_{\mathcal{L}(P)}(A)$. For $n = 1$ and $\widetilde{T} = T^d$ (dense pairs), this was achieved in [38, Theorem 5].

Since low subsets of M are small, we can easily get the following corollary of 3.22, known already for $\widetilde{T} = T^d$ by [10].

Corollary 3.23. *Let $f : M \to M$ be A-definable. Then f agrees off some small set with an $\mathcal{L}_{A\cup P}$-definable function $F : M \to M$.*

The Structure Theorem is intended, among others, to generalize the last corollary to arbitrary definable maps $f : X \subseteq M^n \to M$ (see also Corollary 4.20). For the moment, using compactness, we directly get the following uniform version of Corollary 3.22.

Corollary 3.24. *Let $f : S \times M^n \to M$ be an A-definable map. Then there are A-definable families $\{X_g^i\}_{g\in S}$ of low sets $X_g^i \subseteq M^n$, and $\{f_g^j : M^n \to M\}_{g\in S}$ of $\mathcal{L}_{A\cup P}$-definable maps, $j = 1, \dots, p$, such that for all $g \in S$ and $a \notin \bigcup_i X_g^i$, there is $j \in \{1, \dots, p\}$ such that*

$$f(g, a) = f_g^j(a).$$

4. Cones and large dimension

In this section, we introduce and analyze the two main objects of the paper, cones and large dimension.

4.1. **Cones.** We start with the definition of a cone.

Definition 4.1. We first define recursively the notion of a *supercone* $J \subseteq M^k$, $k \geq 0$.

- $M^0 = \{0\}$ is a supercone.
- A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : M^n \to M \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it.

A set $C \subseteq M^n$ is a *k-cone*, $k \geq 0$, if there are a small set $S \subseteq M^m$, a definable family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in M^k, and an $\mathcal{L}$-definable continuous function $h : M^{m+k} \to M^n$, such that

(1) $C = h\left(\bigcup_{g\in S}\{g\} \times J_g\right)$
(2) for every $g \in S$, $h(g, -) : M^k \to M^n$ is injective.

A *cone* is a k-cone for some k. We use the notation $C = h(S; \mathcal{J})$.

Now let $C = h(S; \mathcal{J})$ be a cone and $f : D \to M$ a definable function, $C \subseteq D$. We say that f is *fiber $\mathcal{L}_A$-definable with respect to C* if for every $g \in S$, f agrees on $h(g, J_g)$ with an $\mathcal{L}_{A\cup P}$-definable continuous function. We omit the index A if we do not want to specify the parameters.

Remark 4.2.

(a) The notion of being fiber $\mathcal{L}$-definable with respect to $h(S; \mathcal{J})$ depends on h, S and $\mathcal{J}$, as the following example shows. Let $C = h(S; \mathcal{J})$, where $h = id : M \to M$, $S = \{0, 1\}$, $J_0 = (-\infty, 0)$ and $J_1 = (0, \infty)$. Define $f : C \to M$ as $f(x) = i$ for $x \in J_i$, $i = 1, 2$. Then f is fiber $\mathcal{L}$-definable with respect to C. On the other hand, $J' = J_1 \cup J_2$ is a supercone, and f does not agree on J' with the restriction of a continuous function.

    Even if we omit the requirement of continuity in the definition of a fiber $\mathcal{L}$-definable function, we do not know if this notion becomes independent of h, S and $\mathcal{J}$.

(b) The 0-cones are exactly the small sets.

(c) The closure of a supercone $J \subseteq M^k$ is the closure of a k-cell that contains J. This can be proved by induction on k, since if $h_1, h_2 : M^k \to M$ are as in the definition of a supercone, we have

$$cl(J) = [h_1, h_2]_{\pi(cl(J))}.$$

    In particular, for all $t \in \pi(J)$,

$$cl(J)_t = [h_1(t), h_2(t)] = cl(J_t).$$

(d) If $C = h(S; \mathcal{J})$ is an A-definable cone, then h does not have to be $\mathcal{L}_A$-definable. We thus additionally mention this property when needed.

(e) We may replace S by a definable subset of P^l in the definition of a cone. Indeed, since S is P-bound, there is $\mathcal{L}$-definable $f : M^l \to M^m$ with $f(P^l) \supseteq S$, and we may thus replace S by $f^{-1}(S)$, and h by $H : M^{l+k} \to M^n$ with $H(x, y) = h(f(x), y)$. We decided, however, to keep the current definition because we can then adopt it in similar situations (such as Theorem 5.13 below). See also Remark 3.8.

The following two lemmas will be used in the proof of the Structure Theorem $(1)_n$, Cases I and II, respectively. In the first one, we prove that a suitable family of large subsets of M ranging over a k-cone gives rise to a $k+1$-cone.

Lemma 4.3. *Let $\{X_a\}_{a\in C}$ be an A-definable family of sets $X_a \subseteq M$, ranging over an A-definable k-cone $C = h(S;\mathcal{J}) \subseteq M^n$, where h is $\mathcal{L}_A$-definable. Assume that $h_1, h_2 : C \to M$ are fiber $\mathcal{L}$-definable with respect to C and such that for all $a \in C$, X_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it. Then $\bigcup_{a\in C}\{a\} \times X_a$ is an A-definable $k+1$-cone $\tau(S;\mathcal{J}')$, where τ is $\mathcal{L}_A$-definable.*

Proof. Assume $\mathcal{J} = \{J_g\}_{g\in S}$ is a family of supercones in M^k and $S \subseteq M^m$. Define and $\mathcal{L}_A$-definable continuous map $\tau : M^{m+k+1} \to M^{n+1}$ as follows

$$\tau(g, t, x) = (h(g, t), x)$$

and let

$$J'_g = \bigcup_{t\in J_g} \{t\} \times X_{h(g,t)}.$$

By the assumption on h_1 and h_2, $J'_g \subseteq M^{k+1}$ is a supercone. Moreover, each $\tau(g, -) : M^{k+1} \to M^{n+1}$ is injective, because so is $h(g, -) : M^k \to M^n$. It follows that

$$\bigcup_{a\in C} \{a\} \times X_a = \tau \left(\bigcup_{g\in S} \{g\} \times J'_g \right)$$

is a $k+1$-cone. □

In the second lemma, we prove that a small family of k-cones under a suitable map results again in a k-cone.

Lemma 4.4. *Let $\{Z_g\}_{g\in S}$, $S \subseteq M^m$ small, be an A-definable family of k-cones contained in M^n of the form*

$$Z_g = \sigma \left(\bigcup_{\gamma\in K} \{\gamma\} \times (Y_g)_\gamma \right),$$

where σ is some fixed $\mathcal{L}_A$-definable continuous map and $K \subseteq M^p$ is small. Let $h : M^{m+k} \to M^n$ be an $\mathcal{L}_A$-definable continuous map such that each $h(g, -) : M^k \to M^n$ is injective. Then $h\left(\bigcup_{g\in S}\{g\} \times Z_g\right)$ is also an A-definable k-cone $\tau(T;\mathcal{J})$, where τ is $\mathcal{L}_A$-definable.

Proof. We let $\tau : M^{m+p+k} \to M^n$ be defined by

$$\tau(g, \gamma, t) = h(g, \sigma(\gamma, t)).$$

It is then straightforward to check that

$$h \left(\bigcup_{g\in S} \{g\} \times Z_g \right) = \tau \left(\bigcup_{(g,\gamma)\in S\times K} \{(g, \gamma)\} \times (Y_g)_\gamma \right)$$

is the desired k-cone. □

We next prove a lemma that will be useful in the discussion of 'large dimension' in Section 4.3 below.

Lemma 4.5. *Let $J \subseteq M^n$ be a supercone and $\{X_s\}_{s\in S}$ a small definable family of subsets of M^n such that $J = \bigcup_{s\in S} X_s$. Then some X_s contains a supercone in M^n.*

Proof. By induction on n. If $n = 0$, it is obvious. If $n > 0$, let π denote the projection onto the first $n-1$ coordinates. For every $s \in S$, let

$$Y_s := \{t \in \pi(X_s) : \text{ the fiber } (X_s)_t \text{ is large}\}.$$

By Remark 3.4(a), $\{Y_s\}_{s\in S}$ is a definable family of sets. By Corollary 3.14, we have $\pi(J) = \bigcup_{s\in S} Y_s$. By Inductive Hypothesis, some Y_s contains a supercone K. Since for every $t \in K$, $(X_s)_t$ is large, Remark 3.4(b) implies that $X_s \supseteq \bigcup_{t\in K}\{t\} \times (X_s)_t$ contains a supercone in M^n. □

4.2. **$\mathcal{L}$-definable functions on supercones.** The goal of this section (Proposition 4.8(1) below) is to show that a supercone from M^m cannot be 'embedded' into M^n, for $n < m$. This will make meaningful the notion of 'large dimension' we introduce in Section 4.3. We start with a lemma.

Lemma 4.6. *Let $J \subseteq M^n$ be an A-definable supercone and $S \subseteq cl(J)$ an open A-definable cell. Then $S \cap J$ is an A-definable supercone with closure $cl(S)$.*

Proof. We work by induction on n. For $n = 0$ it is obvious. Assume we know the statement for subsets of M^k, $k < n$, and let $J \subseteq M^n$ be a supercone and $S \subseteq cl(J)$ be an open A-definable cell. Assume that $S = (f, g)_{\pi(S)}$. Since $\pi(S) \subseteq \pi(cl(J)) \subseteq cl(\pi(J))$, the inductive hypothesis gives that $\pi(S)\cap\pi(J)$ is an A-definable supercone $K \subseteq M^{n-1}$ with closure $cl(\pi(S))$. Let h_1, h_2 be as in the definition of a supercone for J. Since for every $t \in K$, J_t is co-small in $(h_1(t), h_2(t))$, we have that $(S\cap J)_t = (f(t), g(t)) \cap J_t$ is co-small in $(f(t), g(t))$. Hence $S \cap J = \bigcup_{t\in K}\{t\} \times (S \cap J)_t$ is a supercone with closure $cl(S)$. □

Before proving Proposition 4.8, we illustrate it with an example.

Example 4.7. Consider the function $f : M^2 \to M$ with $f(x_1, x_2) = x_1 + x_2$. Denote $J = M \setminus P$. We will show that $f_{\restriction J\times J}$ is not injective. The proof is inspired by an example in [3, page 5]. Assume towards a contradiction that $f_{\restriction J\times J}$ is injective. Pick any two distinct $t_0, t \in J$. Since $f_{\restriction J\times J}$ is injective, for every $b \in t_0 + J$, we have $b \notin t + J$. But $b \in t + M$, so $b \in t + P$. Since this holds for every $b \in t_0 + J$, we have that $t_0 + J \subseteq t + P$, which is a contradiction, since a large set cannot be contained in a small one.

Proposition 4.8. *Let $f : M^m \to M^n$ be an $\mathcal{L}$-definable function and $J \subseteq M^m$ a supercone such that $f_{\restriction J}$ is injective. Then*

(1) *$m \leq n$ and*
(2) *there is an $\mathcal{L}$-definable $X \subseteq cl(J)$ such that* $\dim(cl(J) \setminus X) < m$ *and $f_{\restriction X}$ is finite-to-one. Namely, $X = X_f \cap cl(J)$ from Fact 2.6.*

In particular, there is an open $\mathcal{L}$-definable $X \subseteq cl(J)$ such that $f_{\restriction X}$ is injective.

Proof. The last clause follows from Fact 2.5.

(1). Let $f : M^m \to M^n$ be an $\mathcal{L}$-definable function and $J \subseteq M^m$ a supercone such that $f_{\restriction J}$ is injective. Assume towards a contradiction that $m > n$. Without loss of generality, $n = m - 1$ (otherwise extend the range of f to M^n in an obvious way). So $J \subseteq M^{n+1}$. Let $J_1 = \pi(J)$ be the projection of J onto the first n coordinates, and $Y_1 = cl(J_1)$. For every $t \in J_1$, let J_t be the fiber of J above t, and $Y_t = cl(J_t)$. By Fact 2.8(2), for every $t \in J_1$, there is an open interval $X_t \subseteq Y_t$ on which $f(t, -)$ is injective. Together with Remark 4.2(c), the same is true for every $t \in Y_1$.

Claim 1. *There are open boxes $I_1 \subseteq Y_1$ and $I \subseteq M^n$, such that for every $t \in I_1$, $I \subseteq f(t, X_t)$.*

Proof of Claim 1. Since for every $t \in J_1$, $f(t, -)$ is injective on X_t, it follows that the dimension of the $\mathcal{L}$-definable set

$$Z = \bigcup_{t \in Y_1} \{t\} \times f(t, X_t)$$

is $n + 1$. By cell decomposition, there are open boxes $I_1 \subseteq Y_1$ and $I \subseteq M^n$ such that $I_1 \times I \subseteq Z$. In particular, for all $t \in I_1$, $I \subseteq f(t, X_t)$. $\square$

Let I_1, I be as in Claim 1, and pick any two distinct $t_0, t \in I_1 \cap J_1$. Since $f_{\restriction J}$ is injective, for any $b \in I \cap f(t_0, J_{t_0})$, we have $b \notin f(t, J_t)$. But $b \in f(t, X_t)$, so $b \in f(t, X_t \setminus J_t)$. Since this holds for every $b \in I \cap f(t_0, J_{t_0})$, we have that $I \cap f(t_0, J_{t_0}) \subseteq f(t, X_t \setminus J_t)$. Since J_{t_0} is large, so is $f(t_0, J_{t_0})$, by injectivity of $f(t_0, -)$ and Corollary 3.13.

Claim 2. *$I \cap f(t_0, J_{t_0})$ is large.*

Proof. Denote $f_{t_0}(-) = f(t_0, -)$. Since $I \subseteq f(t_0, X_{t_0}) \subseteq f(t_0, cl(J_{t_0}))$, we have $f_{t_0}^{-1}(I) \subseteq cl(J_{t_0})$. Let $I' \subseteq f_{t_0}^{-1}(I)$ be an open interval. By Lemma 4.6, $I' \cap J_{t_0}$ contains a supercone; in particular, it is large. Therefore, $f(t_0, I' \cap J_{t_0})$ is large, but the latter is contained in $I \cap f(t_0, J_{t_0})$. $\square$

We conclude that the large set $I \cap f(t_0, J_{t_0})$ is contained in the small set $f(t, X_t \setminus J_t)$, a contradiction.

(2). Denote

$$X_f = \{a \in cl(J) : f^{-1}(f(a)) \text{ is finite}\}.$$

We claim that $\dim(cl(J) \setminus X_f) < m$. Assume not. Let $I \subseteq cl(J) \setminus X_f$ be an open box. By Lemma 4.6, $I \cap J$ contains a supercone $K \subseteq M^m$. By Fact 2.6, $f(I)$ has dimension $l < m$. In particular, $f(I)$ is in definable bijection with a subset of M^l via the restriction of an $\mathcal{L}$-definable map $h : M^n \to M^l$. Consider now $g = h \circ f : M^m \to M^l$. Then g is $\mathcal{L}$-definable and injective on K. We have contradicted (1). $\square$

We show with an example that the assumption on J being a supercone (and not just satisfying $\dim(cl(J)) = m$) is necessary.

Example 4.9. This example is taken from [10]. Consider the expansion of the real field by the set of real algebraic numbers. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = r$ if $x = re + s$ for (necessarily unique) algebraic reals r and s, where e is the usual (transcendental) real number, while $h(x) = 0$ if x is not of this form. This function is definable in this setting. The usual projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$ is injective on $Graph(h)$ but of course not injective on any open subset of $cl(Graph(h)) = \mathbb{R}^2$.

The next definition and corollary will be useful when we discuss the notion of large dimension in Section 4.3.

Definition 4.10. Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable map, $J \subseteq M^k$ a supercone and $X \subseteq M^n$ a definable set. We say that

- f is a *strong embedding of J into X* if f is injective and $f(J) \subseteq X$.
- f is a *weak embedding of J into X* if $f_{\restriction J}$ is injective and $f(J) \subseteq X$.

Corollary 4.11. *Let $X \subseteq M^n$ be a definable set. The following are equivalent:*

(1) *there is a weak embedding of a supercone $J \subseteq M^k$ into X.*
(2) *there is a supercone $K \subseteq M^k$ and an $\mathcal{L}$-definable $f : M^k \to M^n$, injective on $cl(K)$, with $f(K) \subseteq X$.*
(3) *there is a strong embedding of a supercone $L \subseteq M^k$ into X.*

Proof. (3)⇒(1) is obvious.

(1)⇒(2). Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable map, injective on J, with $f(J) \subseteq X$. By Proposition 4.8, there is an open definable $S \subseteq cl(J)$ such that $f_{\restriction cl(S)}$ is injective. By Lemma 4.6, $J \cap S$ contains a supercone K.

(2)⇒(3). Let $S \subseteq cl(K)$ be open so that $f_{\restriction S}$ can be extended to an injective $\mathcal{L}$-definable map $F : M^k \to M^n$. By Lemma 4.6 again, $S \cap K$ contains a supercone L. □

4.3. **Large dimension.** We introduce an invariant for every definable set X which tends to measure 'how large' X is. This invariant will be used in the inductive proof of the Structure Theorem in Section 5.

Definition 4.12. Let $X \subseteq M^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. Equivalently, it is the maximum $k \in \mathbb{N}$ such that there is a strong embedding of a supercone $J \subseteq M^k$ into X. We also define the large dimension of the empty set to be $-\infty$. We denote the large dimension of X by $\operatorname{ldim}(X)$.

Clearly, the large dimension of a subset of M^n is bounded by n. In view of Corollary 4.11, the large dimension of X is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X. In Section 6, we will prove that the large dimension equals the 'scl-dimension' arising from a relevant pregeometry in [3]. Here we establish some of its basic properties. The first lemma is obvious.

Lemma 4.13. *For every definable $X, Y \subseteq M^n$, if $X \subseteq Y$, then* $\operatorname{ldim}(X) \leq \operatorname{ldim}(Y)$.

Lemma 4.14. *Let $\{Z_s\}_{s\in S}$ a small definable family of sets. Then*

$$\operatorname{ldim}\left(\bigcup_{s\in S} Z_s\right) = \max \operatorname{ldim} Z_s.$$

Proof. ($\leq$). Assume $f : M^n \to M^m$ is an $\mathcal{L}$-definable injective map, $J \subseteq M^n$ is a supercone, and $f(J) \subseteq \bigcup_{s\in S} Z_s$. We show that for some $s \in S$, $\operatorname{ldim}(Z_s) \geq n$. For every $s \in S$, let $X_s := f^{-1}(Z_s)$. Then $\{X_s \cap J\}_{s\in S}$ is a definable family of subsets of M^n that cover J, and by Lemma 4.5, one of them must contain a supercone $K \subseteq M^n$. Since $f(K) \subseteq Z_s$, we have that $\operatorname{ldim}(Z_s) \geq n$.

($\geq$). This is clear. □

In particular, we obtain the following standard property that holds for any good notion of dimension.

Corollary 4.15. *Let $X_1, \dots, X_k$ be definable sets. Then* $\operatorname{ldim}(X_1 \cup \cdots \cup X_l) = \max\{\operatorname{ldim}(X_1), \dots, \operatorname{ldim}(X_l)\}$.

Corollary 4.16. *If $C \subseteq M^n$ is a k-cone, then* $\operatorname{ldim}(C) = k$.

Proof. By Lemma 4.14 and the definition of a cone it suffices to show that every supercone in M^k has large dimension k. But this is clear. □

Lemma 4.17. *Let $X \subseteq M^{n+1}$ be a definable set, such that for every $t \in \pi(X)$ (the projection onto the first n coordinates), X_t is small. Then* $\operatorname{ldim}(X) = \operatorname{ldim}(\pi(X))$.

Proof. Let X_i, S_i, h_i and Z_{ig} be as in Lemma 3.7. In particular,

$$X_i = h_i \left(\bigcup_{g \in S_i} \{g\} \times Z_{ig} \right). \tag{2}$$

($\geq$). By Lemma 3.7(4), we have $\pi(X) = \bigcup_{ig} Z_{ig}$. By Lemma 4.14, for some i, g, we have $\operatorname{ldim}(Z_{ig}) = \operatorname{ldim}(\pi(X))$. By Equation (2) and Lemma 3.7(1), we obtain

$$\operatorname{ldim}(Z_{ig}) \leq \operatorname{ldim}(X_i) \leq \operatorname{ldim}(X).$$

($\leq$). By Corollary 4.15, $\operatorname{ldim}(X) = \max_i \operatorname{ldim}(X_i)$. By Equation (2), Lemma 3.7(3) and Lemma 4.14, for every i, $\operatorname{ldim}(X_i) = \max_g \operatorname{ldim}(Z_{ig})$. But $Z_{ig} \subseteq \pi(X)$, so $\operatorname{ldim}(X) \leq \operatorname{ldim}(\pi(X))$. □

Corollary 4.18. *Let $X \subseteq M^n$ be a definable set. Then $\operatorname{ldim}(X) = 0$ if and only if X is small.*

Proof. Right-to-left is immediate from the definitions of a small set and large dimension. For the left-to-right, we use induction on n. If $n = 1$, the statement is clear by Lemma 3.3. Assume we know the statement for all $l \leq n$ and let $X \subseteq M^{n+1}$.

Claim. *The projection of X onto any of its coordinates is small.*

Proof of Claim. Without loss of generality we may just prove that the projection $\pi(X)$ onto the first n coordinates is small. Since $\operatorname{ldim}(X) = 0$, using Lemma 3.3, we see that for every $t \in \pi(X)$, X_t is small. By Lemma 4.17, $\operatorname{ldim}(\pi(X)) = \operatorname{ldim}(X) = 0$. By Inductive Hypothesis, $\pi(X)$ is small. □

Since X is contained in the product of its coordinate projections, it is again small. □

Here is a lemma about low sets (Definition 3.21).

Lemma 4.19. *Let $X \subseteq M^n$ be a low definable set. Then* $\operatorname{ldim}(X) = n - 1$.

Proof. By Lemma 4.17. □

Corollary 4.20. *Let $f : M^k \to M^n$ be a definable function. Then f is given by an $\mathcal{L}$-definable function off a definable set of large dimension $< k$.*

Proof. By Corollaries 3.22 and 4.15 and Lemma 4.19. □

5. Structure theorem

We are now ready to prove the first main result of this paper, which consists of statements (1) and (2) below. The proof runs by simultaneous induction along with statements (3) - (5).

Theorem 5.1 (Structure Theorem)**.**

(1) *Let $X \subseteq M^n$ be an A-definable set. Then X is a finite union of A-definable cones $h(S; \mathcal{J})$, where h is $\mathcal{L}_A$-definable.*

(2) *Let $f : X \subseteq M^n \to M$ be an A-definable function. Then there is a finite collection $\mathcal{C}$ of A-definable cones of the form $h(S; \mathcal{J})$, where h is $\mathcal{L}_A$-definable, whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each $C \in \mathcal{C}$.*

(3) *Let $Y \subseteq M^{n+m}$ be a definable set and $\pi(Y)$ its projection onto the first n coordinates. Then*
$$\operatorname{ldim}(Y) \leq \operatorname{ldim}(\pi(Y)) + m.$$

(4) *Let $\{X_a\}_{a\in S}$ be an A-definable family of sets $X_a \subseteq M^n$. Then there are A-definable families $\{J^j_a\}_{a\in S}$, $j = 1, \dots, p$, where each $J^j_a \subseteq X_a$ is either empty or a supercone in M^n, with each $\{J^j_a\}_j$ disjoint, and such that for every $a \in S$,*
$$\operatorname{ldim}\left(X_a \setminus \bigcup_j J^j_a\right) < n.$$

(5) *Let $C \subseteq M^n$ be an A-definable l-cone, and $\{Z_t\}_t$ an A-definable family of subsets $Z_t \subseteq C$. Then there are A-definable families $\{Z^j_t\}_t$, $j = 1, \dots, p$, where each $Z^j_t \subseteq Z_t$ is either empty or an l-cone, such that for every t,*
$$\operatorname{ldim}\left(Z_t \setminus \bigcup_j Z^j_t\right) < l.$$
In fact, if $C = \sigma(\bigcup_{g\in S}\{g\} \times J_g)$, where σ is $\mathcal{L}_A$-definable, then
$$Z^j_t = \sigma\left(\bigcup_{g\in S}\{g\} \times (Y^j_t)_g\right),$$
where for every $g \in S$, $(Y^j_t)_g \subseteq J_g$ is either empty or a supercone in M^l.

Proof. We write $(1)_n$ - $(5)_n$ for the above statements and prove them by simultaneous induction on n. Statements $(1)_0$ - $(5)_0$ are trivial. We next prove that for $n > 0$, $(1)_l$ - $(5)_l$, $l < n$, imply $(1)_n$ - $(5)_n$.

$\mathbf{(1)_n} \Rightarrow \mathbf{(3)_n}$. Let $Y \subseteq M^{n+m}$ and $\pi(Y)$ its projection onto the first n coordinates. By $(1)_n$, $\pi(Y)$ is the finite union of cones $J_1, \dots, J_p$. Assume that J_i is a k_i-cone. By Lemma 4.3 (applied m times), $T_i = \bigcup_{t\in J_i}\{t\} \times M^m$ is a $k_i + m$-cone. By Corollary 4.16, $\operatorname{ldim}(T_i) = k_i + m$. Since Y is contained in $T_1 \cup \cdots \cup T_p$, it follows from Corollary 4.15 that $\operatorname{ldim}(Y) \leq \max_i\{k_i + m\} = \max_i\{k_i\} + m = \operatorname{ldim}(\pi(Y)) + m$.

$\mathbf{(3)_{n-1}}$ & $\mathbf{(4)_{n-1}} \Rightarrow \mathbf{(4)_n}$. Let π denote the projection onto the first $n-1$ coordinates. By Remark 3.4(b), there are finitely many pairs of A-definable functions

$h_1^i, h_2^i : S \times M^{n-1} \to M$, such that

$$X_a = \bigcup_i \bigcup_{t \in \pi(X_a)} \{t\} \times (X_{a,t}^i \cup F_{a,t}),$$

where each $X_{a,t}^i$ is co-small in $(h_1^i(a,t), h_2^i(a,t))$, with each $\{X_{a,t}^i\}_i$ disjoint, and each $F_{a,t}$ small. By Lemma 4.17, $\operatorname{ldim}\left(\bigcup_{t\in\pi(X_a)}\{t\}\times F_{a,t}\right) < n$. We may thus assume (using also Corollary 4.15) that

$$X_a = \bigcup_{t\in\pi(X_a)} \{t\} \times X_{a,t},$$

where each $X_{a,t}$ is co-small in $(h_1(a,t), h_2(a,t))$, for some A-definable $h_1, h_2 : S \times M^{n-1} \to M$. By $(4)_{n-1}$, there are A-definable families $\{K_a^j\}_{a\in S}$, where each $K_a^j \subseteq \pi(X_a)$ is either empty or a supercone in M^{n-1}, with each $\{K_a^j\}_j$ disjoint, such that

$$\operatorname{ldim}\left(\pi(X_a) \setminus \bigcup_j K_a^j\right) < n-1.$$

We can write

$$X_a = \left(\bigcup_j \bigcup_{t\in K_a^j} \{t\}\times X_{a,t}\right) \cup \left(\bigcup_{t\in \pi(X_a)\setminus\bigcup_j K_a^j} \{t\}\times X_{a,t}\right).$$

Each $\left\{J_a^j = \bigcup_{t\in K_a^j}\{t\}\times X_{a,t}\right\}_{a\in S}$ is an A-definable family, where $J_a^j \subseteq X_a$ is either empty (if K_a^j is empty) or a supercone in M^n (if K_a^j is a supercone in M^{n-1}), and each $\{J_a^j\}_j$ disjoint (since $\{K_a^j\}_j$ is). By $(3)_{n-1}$, and taking $m=1$,

$$\operatorname{ldim}\left(\bigcup_{t\in \pi(X_a)\setminus\bigcup_j K_a^j} \{t\}\times X_{a,t}\right) < n.$$

$\mathbf{(3)_{n-1}}$ & $\mathbf{(4)_{l\leq n}} \Rightarrow \mathbf{(5)_n}$. First, a claim.

Claim. *Let $l \leq n$ and $\{Y_t\}_{t\in T}$ be an A-definable family of subsets of $S\times M^l$, where S is small and A-definable. Then there are A-definable families $\{Y_t^j\}_{t\in T}$, with $Y_t^j \subseteq Y_t$, $j = 1,\dots,p$, such that for every $t\in T$ and $g\in S$, each fiber $(Y_t^j)_g \subseteq (Y_t)_g$ is either empty or a supercone in M^l, each $\{(Y_t^j)_g\}_j$ is disjoint, and*

$$\operatorname{ldim}\left((Y_t)_g \setminus \bigcup_j (Y_t^j)_g\right) < l.$$

Proof of Claim. Apply $(4)_l$ to the A-definable family of fibers $\{(Y_t)_g\}_{t\in T, g\in S}$ to get an A-definable family $\{X_{t,g}^j\}_{t\in T, g\in S}$, where each $X_{t,g}^j \subseteq (Y_t)_g$ is either empty or a supercone in M^l, with each $\{X_{t,g}^j\}_j$ disjoint, and such that for every t, $\operatorname{ldim}((Y_t)_g \setminus \bigcup_j X_{t,g}^j) < l$. Let

$$Y_t^j = \bigcup_{g\in S}\{g\}\times X_{t,g}^j.$$

Then $(Y_t^j)_g = X_{t,g}^j$ are as required. $\square$

We can now finish the proof of $(5)_n$. Let $C = \sigma(\bigcup_{g\in S}\{g\} \times J_g) \subseteq M^n$ be an A-definable l-cone, where σ is $\mathcal{L}_A$-definable, and $\{Z_t\}_{t\in T}$ an A-definable family of subsets of C. Then $l \le n$. For each $t \in T$, let

$$Y_t = \sigma^{-1}(Z_t) \subseteq \bigcup_{g\in S}\{g\} \times J_g \subseteq S \times M^l,$$

and let Y_t^j be as in the claim. We let

$$Z_t^j := \sigma(Y_t^j) = \sigma\left(\bigcup_{g\in S}\{g\} \times (Y_t^j)_g\right).$$

Since σ and each $\{Y_t^j\}_{t\in T}$ are A-definable, so is each $\{Z_t^j\}_{t\in T}$. Moreover, since each $(Y_t^j)_g \subseteq (Y_t)_g \subseteq J_g$ is either empty or a supercone in M^l, Z_t^j is either empty or an l-cone contained in Z_t, respectively. We have that

$$Z_t \setminus \bigcup_j Z_t^j = \sigma\left(\bigcup_{g\in S}\{g\} \times (Y_t)_g\right) \setminus \bigcup_j \sigma\left(\bigcup_{g\in S}\{g\} \times (Y_t^j)_g\right) =$$

$$\bigcup_{g\in S} \sigma\left(g, (Y_t)_g\right) \setminus \bigcup_{g\in S} \sigma\Big(g, \bigcup_j (Y_t^j)_g\Big) \subseteq$$

$$\bigcup_{g\in S}\left(\sigma\left(g, (Y_t)_g\right) \setminus \sigma\Big(g, \bigcup_j (Y_t^j)_g\Big)\right) =$$

$$= \bigcup_{g\in S} \sigma\Big(g, (Y_t)_g \setminus \bigcup_j (Y_t^j)_g\Big),$$

where the last equality is because each $\sigma(g, -)$ is injective. By Lemma 4.14 and since each $\sigma(g, -)$ is injective and $\mathcal{L}$-definable, the last set has large dimension equal to $\max_g \operatorname{ldim}\Big((Y_t)_g \setminus \bigcup_j (Y_t^j)_g\Big) < l$.

$\mathbf{(1)_{n-1}}$ & $\mathbf{(2)_{n-1}}$ & $\mathbf{(5)_{n-1}} \Rightarrow \mathbf{(1)_n}$. We prove $(1)_n$ by sub-induction on $\operatorname{ldim}(X)$. Let $X \subseteq M^n$ with $\operatorname{ldim}(X) = 0$. Then by Corollary 4.18, X is small, and hence a 0-cone. Assume now we know $(1)_n$ for $\operatorname{ldim}(X) < k$, and let $X \subseteq M^n$ be A-definable with $\operatorname{ldim}(X) = k$. By Remark 3.4(b), we may assume that there are A-definable $h_1, h_2 : M^{n-1} \to M$ such that for every $a \in \pi(X)$, X_a is contained in $(h_1(a), h_2(a))$, and it is either small in it for all $a \in \pi(X)$, or co-small in it for all $a \in \pi(X)$.

Case I. Assume that for every $a \in \pi(X)$, X_a is co-small in $(h_1(a), h_2(a))$. By cell decomposition and $(2)_{n-1}$, we may assume that $\pi(X)$ is a cone $h(S; \mathcal{J})$, where h is $\mathcal{L}_A$-definable, and h_1, h_2 are fiber $\mathcal{L}$-definable with respect to it. By Lemma 4.3, X is an A-definable $k+1$-cone.

Case II. Assume that for every $a \in \pi(X)$, X_a is small in $(h_1(a), h_2(a))$. By $(1)_{n-1}$, $\pi(X)$ is the finite union of A-definable cones $C_1, \ldots, C_r$. Then each

$$X_i = \bigcup_{a \in C_i} \{a\} \times X_a$$

is also A-definable. Assume C_i is an l-cone. By Lemma 4.17, and since $\operatorname{ldim}(X) = k$, we have $l \leq k$. If $l < k$, then, again by Lemma 4.17, $\operatorname{ldim}(X_i) = l < k$, and by Sub-Inductive Hypothesis, X_i is a finite union of A-definable cones. We may thus assume that $C = \pi(X) = \sigma(K; \mathcal{Y})$ is a k-cone, where σ is $\mathcal{L}_A$-definable, and $X = \bigcup_{a \in C} \{a\} \times X_a$ with each X_a small. By Lemma 3.7, we may further assume that

$$X = h\left(\bigcup_{g \in S} \{g\} \times Z_g\right),$$

for some $\mathcal{L}_A$-definable continuous $h : M^{m+k} \to M^n$, and A-definable small set $S \subseteq M^m$ and family $\{Z_g\}_{g \in S}$ with $Z_g \subseteq C$, such that

$$\text{for every } g \in S,\ h(g, -) : M^k \to M^n \text{ is injective.}$$

Since the family $\{Z_g\}_{g \in S}$ is A-definable and each $Z_g \subseteq C$, by $(5)_{n-1}$, there are finitely many A-definable families $\{Z_g^j\}_g$, where each $Z_g^j \subseteq Z_t$ is either empty or a k-cone, such that for every $g \in S$, $\operatorname{ldim}(Z_g \setminus \bigcup_j Z_g^j) < k$. In fact, if $C = \sigma\left(\bigcup_{\gamma \in K} \{\gamma\} \times Y_\gamma\right)$, $K \subseteq M^p$, then Z_g^j has form

$$Z_g^j = \sigma\left(\bigcup_{\gamma \in K} \{\gamma\} \times (Y_g^j)_\gamma\right),$$

where each $(Y_g^j)_\gamma \subseteq Y_\gamma$ is either empty or a supercone in M^l. Let $S_j \subseteq S$ be the set of all $g \in S$ for which $Z_g^j \neq \emptyset$. We have

$$X = h\left(\bigcup_j \bigcup_{g \in S_j} \{g\} \times Z_g^j\right) \cup h\left(\bigcup_{g \in S} \{g\} \times \left(Z_g \setminus \bigcup_j Z_g^j\right)\right).$$

Since each $h(g, -)$ is injective and $\mathcal{L}$-definable, it follows easily using Lemma 4.14 that

$$\operatorname{ldim} h\left(\bigcup_{g \in S} \{g\} \times \left(Z_g \setminus \bigcup_j Z_g^j\right)\right) = \max_g \operatorname{ldim}\left(Z_g \setminus \bigcup_j Z_g^j\right) < k.$$

Hence, by Sub-Inductive Hypothesis, $h\left(\bigcup_{g \in S} \{g\} \times (Z_g \setminus \bigcup_j Z_g^j)\right)$ is a finite union of cones. It remains to see that each $h\left(\bigcup_{g \in S_j} \{g\} \times Z_g^j\right)$ is a cone $\tau(T; \mathcal{J})$, where τ is $\mathcal{L}_A$-definable. But this is given by Lemma 4.4.

$\mathbf{(1)_n \,\&\, (3)_{n-1} \,\&\, (4)_n \Rightarrow (2)_n}$. First, a claim.

Claim. *Let $f : \bigcup_{g \in S} \{g\} \times J_g \to M$ be an A-definable function, where each $J_g \subseteq M^n$ is a supercone. Then there are A-definable families $\{Y_g^j\}_{g \in S}$ of supercones in M^n,* *and* $\{f^j_g : M^n \to M\}_{g\in S}$ *of* $\mathcal{L}_{A\cup P}$*-definable maps,* $j = 1, \dots, p$, *such that for every* $g \in S$,

- $\mathrm{ldim}(J_g \setminus \bigcup_j Y^j_g) < n$
- *for all* $j = 1, \dots, p$, f *agrees with* f^j_g *on* Y^j_g.

Proof of Claim. Let $X^i_g \subseteq M^n$ and f^j_g be as in Corollary 3.24. By Corollary 4.15 and Lemma 4.19, for each $g \in S$, $\mathrm{ldim}\left(\bigcup_i X^i_g\right) < n$. Let B^j_g be the subset of $J_g \setminus \bigcup_i X^i_g$ on which $f(g, -)$ agrees with f^j_g, so that $\mathrm{ldim}\left(J_g \setminus \bigcup_j B^j_g\right) < n$. Apply $(4)_n$ to get A-definable families of supercones or empty sets $Y^{j,k}_g \subseteq B^j_g$, such that $\mathrm{ldim}\left(B^j_g \setminus \bigcup_k Y^{j,k}_g\right) < n$. Since

$$J_g \setminus \bigcup_{j,k} Y^{j,k}_g \subseteq \left(J_g \setminus \bigcup_j B^j_g\right) \cup \bigcup_j \left(B^j_g \setminus \bigcup_k Y^{j,k}_g\right),$$

it follows that $\mathrm{ldim}\left(J_g \setminus \bigcup_{j,k} Y^{j,k}_g\right) < n$ and f agrees with f^j_g on each $Y^{j,k}_g$. $\square$

Now we can finish the proof of $(2)_n$, and thus of the Structure Theorem. Let $f : X \subseteq M^n \to M$ be an A-definable function. By $(1)_n$, we may assume $X \subseteq M^n$ is a k-cone, say $X = h(S; \mathcal{J})$, where h is $\mathcal{L}_A$-definable. By the last claim, there are A-definable families of supercones or empty sets $Y^j_g \subseteq M^n$, such that for all $g \in S$, $\mathrm{ldim}(J_g \setminus \bigcup_j Y^j_g) < n$ and $f \circ h$ agrees with some $\mathcal{L}_{A\cup P}$-definable function on each $\{g\} \times Y^j_g$. Therefore, f is fiber $\mathcal{L}_A$-definable with respect to each cone $h(S; \mathcal{Y}^j)$, where $\mathcal{Y}^j = \{Y^j_g\}_{g\in S}$.

Moreover, by Lemma 4.14, $\bigcup_{g\in S}\{g\} \times \left(J_g \setminus \bigcup_j Y^j_g\right)$ has large dimension less than n. Hence we can finish the proof by sub-induction on large dimension. $\square$

Remark 5.2. We do not know whether we can have disjoint unions in Structure Theorem (1) and (2). The problem is that in the above proof of $(3)_{n-1}$ & $(4)_{l\le n} \Rightarrow (5)_n$, we cannot deduce the disjointness of $\{Z^j_t\}_j$ from that of $\{(Y_t)^j_g\}_j$, because σ is not injective. Under one further assumption, however, we do obtain the disjointness; see Theorem 5.13 below.

5.1. **Invariance of large dimension.** We can now prove that the large dimension is invariant under definable bijections. Recall from Section 4.3 that that the large dimension of a definable set $X \subseteq M^n$ is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X.

Corollary 5.3 (Invariance of large dimension)**.** *Let* $f : X \to M^n$ *be a definable injective function. Then* $\mathrm{ldim}(X) = \mathrm{ldim} f(X)$.

Proof. Assume that $k \le \mathrm{ldim}(X)$. It suffices to show $k \le \mathrm{ldim} f(X)$. By the Structure Theorem, X is the union of finitely many cones such that f is fiber $\mathcal{L}$-definable with respect to each of them. By Corollary 4.15, one of them, say $h(S; \mathcal{J})$ must be a k-cone. Pick any $g \in S$. Then $f \circ h(g, -) : J \to M^n$ agrees with an $\mathcal{L}$-definable map on J and it is injective. Therefore, $k \le \mathrm{ldim} f(X)$. $\square$

5.2. **Uniform Structure Theorem.** Here we deduce a uniform version of the Structure Theorem. The proof uses a standard compactness argument, but we present it anyway for completeness. The main issue is to guarantee that certain notions involved in that of a cone are definable.

Lemma 5.4. *Being small is a definable notion (even for subsets of M^n).*

Proof. For $n = 1$, we proved it. For $n > 1$, X is small if and only if all its projections onto M are small, and hence the lemma. □

Let $\pi_m : M^n \to M^m$ denotes the projection onto the first m coordinates.

Definition 5.5. Let $n > 0$ and $X \subseteq M^n$ a definable set. A set of pairs of maps

$$\mathcal{H} = \{(h_1^i, h_2^i)\}_{i=0,\dots,n-1}$$

is a *skeleton for* X if, for every $0 \leq i < n$,

- $h_1^i < h_2^i : M^i \to M$ are $\mathcal{L}$-definable and continuous, and
- for every $t \in \pi_i(X)$, the fiber $\pi_{i+1}(X)_t$ is contained in $(h_1^i(t), h_2^i(t))$ and it is co-small in it.

So, a definable set $X \subseteq M^n$ is a supercone if and only if it has a (unique) skeleton. Moreover, if

$$\mathcal{H}_\alpha = \{((h_\alpha)_1^i, (h_\alpha)_2^i)\}_{i=0,\dots,n-1}$$

is an A-definable family of $\mathcal{L}$-definable and continuous maps, and $X_\alpha \subseteq M^n$ is an A-definable family of sets, then the set of α's such that $\mathcal{H}_\alpha$ is a skeleton for X_α is A-definable.

Theorem 5.6 (Uniform Structure Theorem)**.** *Let $\{X_t\}_{t\in I}$, be an A-definable family of subsets of M^n. Then there are k^j and A-definable families $\{Z_t^j\}_{t\in I}$, $j = 1, \dots, p$, of empty sets or k^j-cones, such that for all $t \in I$,*

$$X_t = Z_t^1 \cup \cdots \cup Z_t^p.$$

Moreover, for each $j = 1, \dots, p$, there are an $\mathcal{L}_A$-definable continuous family σ_t^j and A-definable families S_t^j and $\mathcal{J}_t^j$, such that Z_t^j is empty or a k^j-cone of the form

$$Z_t^j = \sigma_t^j(S_t^j; \mathcal{J}_t^j).$$

Furthermore, let $\{f_t : X_t \to M\}_{t\in I}$ be an A-definable family of maps. Then the above can be chosen so that, in addition, each f_t is fiber $\mathcal{L}_A$-definable with respect to each Z_t^j.

Proof. By Structure Theorem (1), every X_t is a finite union of At-definable cones $\sigma_t^j(S_t^j; \mathcal{J}_t^j)$, $j = 1, \dots, p_t$, where $S_t^j \subseteq M^{m_t^j}$ and $J_{t,x}^j \subseteq M^{k_t^j}$ is a supercone with skeleton

$$\mathcal{H}_{t,x}^j = \{((h_{t,x}^j)_1^i, (h_{t,x}^j)_2^i)\}_{i=0,\dots,n-1}$$

For each such j, let d be a tuple from A and

- $\varphi_t^j(x_1, x_2, y, t, d)$ be an $\mathcal{L}$-formula that defines $\sigma_t^j(x_1, x_2) = y$,
- $\chi_t^j(x_1, t, d)$ be a formula that defines $x_1 \in S_t^j$,
- $\psi_t^j(x_1, x_2, t, d)$ be a formula that defines $x_2 \in (J_t^j)_{x_1}$.

For each $t \in I$, let V_t be the set of all $\alpha \in I$ such that $y \in X_\alpha$ if and only if there is $j \in \{1, \dots, p_t\}$ such that

$$\varphi_t^j(x_1, x_2, y, \alpha, d) \ \ \& \ \ \chi_t^j(x_1, \alpha, d) \ \ \& \ \ \psi_t^j(x_1, x_2, \alpha, d)$$

holds and

- $\varphi_t^j(x_1, x_2, y, \alpha, d)$ defines a continuous map on $M^{m_t^j + k_t^j}$, and an injective map on $\{x_1\} \times M^{k_t^j}$,
- $\chi_t^j(M^{m_t^j}, \alpha, d)$ is small, and
- $\psi_t^j(M^{k_t^j}, x_2, \alpha, d)$ is a set with skeleton $\mathcal{H}_\alpha^j$.

Then each V_t is definable, and $I = \bigcup_t V_t$. By Compactness, I is a finite union of the V_t's, and the result easily follows.

The 'furthermore' statement can be proved similarly. □

We obtain immediately the following corollary.

Corollary 5.7. *Let $D \subseteq M^n \times M^m$ an A-definable set. Then D is a finite union of A-definable sets of the form*

$$\bigcup_{t \in C_1} \{t\} \times C_t,$$

where $C_1 \subseteq M^n$ is a cone and there is k such that each $C_t \subseteq M^m$ is a k-cone given by

$$C_t = \sigma_t(S_t, \mathcal{J}_t)$$

for some $\mathcal{L}_A$-definable continuous family of maps σ_t and A-definable families of small sets S_t and supercones $\mathcal{J}_t = \{J_{t,g}\}_{g \in S_t}$.

5.3. **Optimality of the Structure Theorem.** In this section, we prove that our Structure Theorem is in a certain sense optimal.

Definition 5.8. A *strong cone* is a cone $h(S; \mathcal{J})$ which, in addition to the properties of Definition 4.1, satisfies:

- $h : \bigcup_{g \in S} \{g\} \times J_g \to M^n$ is injective.

By *Strong Structure Theorem* we mean the Structure Theorem where cones are replaced everywhere by strong cones. Below we give a counterexample to the Strong Structure Theorem and in the next section we point out a 'choice' property that implies it.

Lemma 5.9. *Let $J \subseteq M^k$ be a supercone and $S \subseteq M^m$ small. Assume that $f : M^k \to M^m$ is an $\mathcal{L}$-definable continuous map that satisfies $f(J) \subseteq S$. Then $f_{\upharpoonright J}$ is constant.*

Proof. We work by induction on k. For $k = 0$, the statement is trivial. Now let $k > 1$ and assume we know the statement for all $J \subseteq M^{k'}$ with $k' < k$. Let $J \subseteq M^k$ and $f : M^k \to S$ be as in the statement with $f(J) \subseteq S$. For every $t \in \pi_1(J)$, by inductive hypothesis applied to $f(t, -) : M^{k-1} \to M^m$, there is unique $c_t \in S$ so that $f(\{t\} \times J_t) = \{c_t\}$. Since f is continuous, and by definition of a supercone, for every $t \in \pi_1(cl(J))$, there is also unique $c_t \in S$ so that $f(\{t\} \times cl(J)_t) = \{c_t\}$. We let $h : \pi_1(cl(J)) \to M^m$ be the map given by $t \mapsto c_t$. If f is not constant on J, there must be an interval $I \subseteq \pi_1(cl(J))$ on which h is injective. But $I \cap \pi_1(J) \subseteq M$ is a supercone (Lemma 4.6) and $h(I \cap \pi_1(J)) \subseteq S$, a contradiction. Therefore, f is constant on J. □

Lemma 5.10. *Let $n > 0$ and $J \subseteq M^n$ be a supercone. Then* $\mathrm{ldim}(cl(J) \setminus J) < \mathrm{ldim}(J)$.

Proof. By induction on n. For $n = 1$, it is clear from Lemma 3.3. By Remark 4.2(c) and since $\pi(cl(J)) \subseteq cl(\pi(J))$, we have $cl(J) \subseteq \bigcup_{t \in cl(\pi(J))} cl(J_t)$. Hence

$$cl(J) \setminus J \subseteq \left(\bigcup_{t \in \pi(J)} cl(J_t) \setminus J_t \right) \cap \left(\bigcup_{t \in cl(\pi(J)) \setminus \pi(J)} cl(J_t) \right),$$

where the first part has large dimension $= \mathrm{ldim}(\pi(J)) = n - 1 < n$ because of Lemma 4.17, and the second part has large dimension $\leq \mathrm{ldim}(cl(\pi(J)) \setminus \pi(J)) + 1 < (n-1)+1 = n$ because of Structure Theorem (3) and the inductive hypothesis. $\square$

Lemma 5.11. *Let $J_1, J_2 \subseteq M^k$ be two supercones and $h_1, h_2 : M^k \to M^n$ two $\mathcal{L}$-definable continuous injective maps. Then*

$$\dim\Big(h_1(cl(J_1)) \cap h_2(cl(J_2))\Big) = k \implies \mathrm{ldim}\Big(h_1(J_1) \cap h_2(J_2)\Big) = k.$$

Proof. Let

$$K_1 = h_1^{-1}\big(h_1(cl(J_1)) \cap h_2(cl(J_2))\big).$$

Then $K_1 \subseteq cl(J_1)$ and $\dim(K_1) = k$. By Lemma 4.6, $K_1 \cap J_1$ contains a supercone J. Now, since $J \subseteq K_1$, we have

$$h_2^{-1} h_1(J) \subseteq cl(J_1).$$

By Lemma 5.10 and Structure Theorem (4), $h_2^{-1}h_1(J) \cap J_2$ must contain a supercone L. We therefore have

$$h_2(L) \subseteq h_1(J) \cap h_2(J_2) \subseteq h_1(J_1) \cap h_2(J_2),$$

proving that $h_1(J_1) \cap h_2(J_2)$ has large dimension k. $\square$

Counterexample to the Strong Structure Theorem. Let $\mathcal{M} = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x_{\restriction[0,1]} \rangle$, $P = \mathrm{dcl}(\emptyset)$, and consider the dense pair $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$. It is easy to see that $P = \mathbb{Q}(\pi)^{rc}$. For $t \in M$, we denote by l_t the straight line of slope π that passes through $(t, 0)$. Define

$$U = \bigcup_{g \in P} l_g.$$

By an *endpart of* l_t, we mean $l_t \cap ([a, \infty) \times \mathbb{R})$, for some $a \in \mathbb{R}$. We prove that U is definable but not a finite union of strong cones.

Claim 1. *U is definable.*

Proof of Claim 1. For every $a \in M$, let $C_a = M \times [a, a+1)$ and $E_a \subseteq C_a \times C_a$ given by:

$$(x, y) E_a (x', y') \iff y' - y = \pi(x' - x) \text{ and } |x' - x| \leq 1.$$

Thus, if $(x, y) \in l_t \cap C_a$, then $[(x, y)]_{E_a}$ is the segment of l_t that lies in C_a. Define $p_a : C_a \to M^2$ via

$$p_a(x, y) = \text{ the midpoint of } [(x, y)]_{E_a},$$

and let

$$Y_a = p_a(C_a \cap P^2).$$

Clearly, for $t \in P$, we have $l_t \cap P^2 = \{(g, \pi(g-t)) : g \in G\}$, and for $t \notin P$, we have $l_t \cap P^2 = \emptyset$. We claim that

$$U = \bigcup_{a \in M} Y_a,$$

and hence U is definable.

($\subseteq$). Let $(x, y) \in l_t$, $t \in P$. We claim that $(x, y) \in p_a(C_a \cap P^2)$, for $a = y - \frac{1}{2}$. Indeed, (x, y) is the midpoint of $[(x, y)]_{E_a} = l_t \cap C_a$, and hence all we need is to find a point $(g_1, g_2) \in l_t \cap C_a \cap P^2$. Take any $g_2 \in [a, a+1) \cap P$ and let $g_1 = t + \frac{g_2}{\pi} \in P$. Then clearly $(g_1, g_2) \in l_t \cap C_a \cap P^2$ and hence $p_a(g_1, g_2) = (x, y)$.

($\supseteq$). Let $(x, y) = p_a(g_1, g_2) \in p_a(C_a \cap P^2)$. Then $y - g_2 = \pi(x - g_1)$. Hence, for $t = g_1 - \frac{g_2}{\pi}$, we have $(x, y) \in l_t$. $\square$

Claim 2. *U is not a finite union of strong cells.*

Proof of Claim 2. First we observe that $\mathrm{ldim}(U) = 1$. Indeed, U contains infinite $\mathcal{L}$-definable sets, so $\mathrm{ldim}(U) \geq 1$. It cannot be $\mathrm{ldim}(U) = 2$, by Lemma 4.17 and since each vertical fiber is small (it contains at most one element of each l_t, $t \in P$). Therefore $\mathrm{ldim}(U) = 1$.

Now assume, towards a contradiction, that U is a finite union of strong cones. Let $h(S; \mathcal{J})$ be one of them. In particular, h is injective on $\bigcup_{g \in S} \{g\} \times J_g$.

In the next two subclaims we make use of the expansion $\mathcal{M}' = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x \rangle$ of $\mathcal{M}$. Let $\mathcal{L}'$ be the underlying language of $\mathcal{M}'$. We still have $\mathrm{dcl}_{\mathcal{L}'}(0) = \mathbb{Q}(\pi)^{rc} = P$. Hence $\langle \mathcal{M}', P \rangle$ is still a dense pair.

Subclaim 1. *For every $g \in S$, $h(g, cl(J_g))$ must be contained in a unique l_t.*

Proof of Subclaim. Each of l_t and the family $\{l_t\}_{t \in M}$ is now $\mathcal{L}'$-definable. Consider the $\mathcal{L}'$-definable and continuous map $f : M \to M$ where

$$f(x) = t \;\Leftrightarrow\; h(g, x) \in l_t.$$

By Lemma 5.9 applied to $J = J_g$, $S = P$ and f, it follows that $h(g, J_g)$ must be contained in a unique l_t. By continuity of h, so does $h(g, cl(J_g))$. $\square$

Subclaim 2. *For every $t \in P$, there are only finitely many $g \in S$ such that $h(g, cl(J_g)) \subseteq l_t$.*

Proof of Subclaim. Assume, towards a contradiction, that for some $t \in P$ there are infinitely many $g \in S$ with $h(g, cl(J_g)) \subseteq l_t$. For each $g \in S$, denote by a_g the infimum of the projection of $h(g, cl(J_g))$ onto the first coordinate. By injectivity of h, for every two $g_1, g_2 \in S$, we have $h(g_1, J_{g_1}) \cap h(g_2, J_{g_2}) = \emptyset$. By Lemma 5.11, $h(g_1, cl(J_{g_1})) \cap h(g_2, cl(J_{g_2}))$ is finite (in fact, a singleton). Therefore, the set

$$\{a_g : g \in G \text{ and } h(g, cl(J_g)) \subseteq l_t\}$$

is an infinite discrete $\mathcal{L}'(P)$-definable subset of $\mathbb{R}$, a contradiction. $\square$

Since the subclaims hold for each of the finitely many strong cones, it turns out that for one of them, say $h(S; \mathcal{J})$, there is some $g \in S$ such that $h(g, cl(J_g))$ contains an endpart of l_0. So some endpart of l_0 is definable in $\widetilde{\mathcal{M}}$. But then its closure, which equals the endpart, is $\mathcal{L}$-definable. It follows easily that the full multiplication $x \mapsto \pi x$ is $\mathcal{L}$-definable, a contradiction. $\square$

5.4. **Future directions.** We now point out a key 'choice' property which guarantees the Strong Structure Theorem. Indeed, together with Corollary 3.5 it implies a strengthened version of Lemma 3.7 below, which is enough.

Choice Property: Let $h : Z \subseteq M^{n+k} \to M^l$ be an $\mathcal{L}_A$-definable continuous map and $S \subseteq M^n$ small. Then there are $p, m \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous map $h_i : M^{m+k} \to M^l$, $Y_i \subseteq M^m$ P-bound over A and A-definable families $X_i \subseteq M^{m+k}$ with $X_{ia} \subseteq Y_i$, $i = 1, \dots, p$, such that for every $g \in Y$ and $a \in \pi(Z)$,

(1) $h_i(-, a) : X_{ia} \to M^l$ is injective, and
(2) $h(S \cap Z_a, a) = \bigcup_i h_i(X_{ia}, a)$,

where $\pi(Z)$ denotes the projection of Z onto the last k coordinates.

Lemma 5.12. *If $\widetilde{M}$ satisfies the Choice Property, then Lemma 3.7 holds with the additional conclusion that each $h_i : \bigcup_{g \in S_i} \{g\} \times Z_{ig} \to M^{n+1}$ is injective.*

Proof. We first claim that there are $m, p \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_1, \dots, h_p : M^{m+n} \to M$, A-definable small sets $S_i \subseteq M^m$ and families $Y_i \subseteq M^{m+n}$ with $Y_{ia} \subseteq S_i$, such that for all $a \in I$,

(1) $h_i(-, a) : Y_{ia} \to M$ is injective
(2) $X_a = \bigcup_i h_i(Y_{ia}, a)$.
(3) $\{h_i(Y_{ia}, a)\}_{i=1,\dots,l}$ are disjoint.

Indeed, apply the Choice Property to each h_i from Corollary 3.5 to get (1) and (2). For (3), recursively replace Y_{ia}, $1 < i \leq l$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z, a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a \in C} \{a\} \times X_a = \bigcup_i \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a).$$

From this point on the argument continues identically with the corresponding part of Lemma 3.7, noting in the end that, by (1), each $h_i : \bigcup_{g \in S_i} \{g\} \times Z_{ig} \to M^{n+1}$ turns out to be injective. □

Theorem 5.13. *If $\widetilde{M}$ satisfies the Choice Property, then the Structure Theorem holds with cones replaced by strong cones. Moreover, the unions in Structure Theorem (1) and (2) are disjoint.*

Proof. The reader can check that Lemmas 4.3 and 4.4 hold with cones replaced everywhere by strong cones, with identical proofs. It is then a (rather lengthy) routine to check that the proof of the current statement is, again, identical with that of the Structure Theorem, with cones replaced everywhere by strong cones, and Lemma 3.7 replaced by Lemma 5.12.

For the moreover clause, and working by induction on large dimension, we need to prove that if D is a k-cone and C a definable set, both A-definable, then $D \setminus C$ is the disjoint union of finitely many k-cones and a set of large dimension $< k$, all A-definable. So let $D = h(S; \mathcal{J})$ and C be such. For every $g \in S$, write

$$X_g = J_g \setminus h(g, -)^{-1}(D \setminus C).$$

By Structure Theorem (4), there are A-definable families $\{Y_g^j\}_{g \in S}$, $j = 1, \dots, p$, where each $Y_g^j \subseteq X_g$ is either empty or a supercone in M^n, with each $\{Y_g^j\}_j$ disjoint, and such that for every $g \in S$,

$$\operatorname{ldim}\left(X_g \setminus \bigcup_j Y_g^j\right) < k.$$

For every j, let

$$D_j = h\left(\bigcup_{g \in S} \{g\} \times Y_g^j\right).$$

Since h is injective and each $\{(Y_t)_g^j\}_j$ is disjoint, so is $\{D_j\}_j$. Moreover, $E = h\left(\bigcup_{g \in S}\{g\} \times X_g \setminus \bigcup_j Y_g^j\right)$ has large dimension $< k$, by Lemma 4.14. Since $D \setminus C = D_1 \cup \cdots \cup D_p \cup E$, we are done. $\square$

5.5. **Open questions.** The counterexample to the Strong Structure Theorem relies on a somewhat unnatural condition on $\mathcal{M}$. It is thus plausible to ask the following question.

Question 5.14. *Under what assumptions on $\mathcal{M}$ does the Choice Property hold? For example, does it hold if $\mathcal{M}$ is a real closed field? If $\widetilde{\mathcal{M}}$ is a dense pair of real closed fields?*

In general, a supercone $J \subseteq M^n$ need not contain a product of supercones in M. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields and $J \subseteq M^2$ with

$$J = \bigcup_{a \in M} \{a\} \times (M \setminus aP).$$

Is it however true that J contains an image of such product under $\mathcal{L}$-definable map? More generally, we could ask the following question.

Question 5.15. *Would the Structure Theorem remain true if we defined:*

1. *supercones in M^k to be products $J_1 \times \cdots \times J_k$, where each J_i is a supercone in M?*
2. *k-cones to be of the form $h(S \times J)$? (That is, h and S are as before, but $J_g = J$ in Definition 4.1 is fixed.)*

6. Large dimension versus scl-dimension

In this section we use our Structure Theorem to establish the equality of the large dimension with the 'scl-dimension' arising from a relevant pregeometry in [3]. In Section 7 we use this equality to set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$.

We start by quoting [3, Definition 28], which was given independently from, and in complete analogy with, [15, Definition 5.2].

Definition 6.1. The *small closure* operator $\operatorname{scl} : \mathcal{P}(M) \to \mathcal{P}(M)$ is defined by:

$$a \in \operatorname{scl}(A) \Leftrightarrow a \text{ belongs to an } A\text{-definable small set.}$$

In [3] scl was shown to define a pregeometry under certain assumptions (in addition to their basic tameness conditions). We show that in the current context scl always defines a pregeometry. This follows from the first equality below, which is proved using only results from Section 3. In the interests of completeness, we also prove a second equality, using the Structure Theorem. Recall that $\operatorname{dcl}(A)$ denotes the usual definable closure of A in the o-minimal structure $\mathcal{M}$.

Lemma 6.2. $\mathrm{scl}(A) = \mathrm{dcl}(P \cup A) = \mathrm{dcl}_{\mathcal{L}(P)}(P \cup A)$.

Proof. $\mathrm{scl}(A) \subseteq \mathrm{dcl}(P \cup A)$. Let $b \in \mathrm{scl}(A)$. Then there are an $\mathcal{L}(P)$-formula $\varphi(x, y)$ and $a \in A^l$, such that $\varphi(\mathcal{M}, a)$ is small and contains b. Consider the $\emptyset$-definable family $\{\varphi(\mathcal{M}, t)\}_{t \in M^l}$. By Remark 3.4(a), the set I consisting of all $t \in M^l$ such that $\varphi(\mathcal{M}, t)$ is small is $\emptyset$-definable. Of course, I contains a. By Corollary 3.5, there is an $\mathcal{L}_\emptyset$-definable function $h : M^{m+l} \to M$ such that for all $t \in I$, $\varphi(\mathcal{M}, t) \subseteq h(P^m, t)$. Therefore $b \in h(P^m, a)$, and $b \in \mathrm{dcl}(P \cup A)$.

$\mathrm{scl}(A) \supseteq \mathrm{dcl}(P \cup A)$. Let $b \in \mathrm{dcl}(P \cup A)$. Then there is an $\mathcal{L}_\emptyset$-definable $h : M^{m+l} \to M$ and $a \subseteq A^l$ such that $b \in h(P^l, a)$. But the latter set is small, hence $b \in \mathrm{scl}(A)$.

$\mathrm{dcl}(P \cup A) = \mathrm{dcl}_{\mathcal{L}(P)}(P \cup A)$. It suffices to show $\mathrm{dcl}_{\mathcal{L}(P)}(A) \subseteq \mathrm{dcl}(P \cup A)$. Let $b = f(a)$, where $a \subseteq P \cup A$ and f is $\emptyset$-definable. By Structure Theorem, there is a $\emptyset$-definable cone $h(S; \mathcal{J})$, where h is $\mathcal{L}_\emptyset$-definable, containing a on which f is fiber $\mathcal{L}_\emptyset$-definable. Let $g \in S$ and $t \in J_g$ be so that $a = h(g, t)$. Since $h(g, -) : M^k \to M^n$ is $\mathcal{L}_g$-definable and injective, $t \in \mathrm{dcl}(P \cup A \cup S)$. Moreover, S is P-bound over $\emptyset$ (Lemma 3.11) and hence $t \in \mathrm{dcl}(A \cup P)$. Since $fh(g, -)$ agrees with an $\mathcal{L}_{A \cup P}$-definable map on J_g, it follows that

$$b = f(h(g, t)) \in \mathrm{dcl}(A \cup P).$$

□

Remark 6.3. In general $\mathrm{dcl}(P \cup A) \neq \mathrm{dcl}_{\mathcal{L}(P)}(A)$. For example, let $\langle \mathcal{M}, \mathcal{N} \rangle$ be a dense pair of real closed fields and let $\mathcal{N}_0$ be a real closed subfield of $\mathcal{N}$. Then $\mathrm{dcl}_{\mathcal{L}(P)}(\mathcal{N}_0) = \mathcal{N}_0$ by [10, Lemma 3.2].

The following corollary is then immediate.

Corollary 6.4. *The small closure operator* scl *defines a pregeometry.*

Definition 6.5. Let $A, B \subseteq M$. We say that B is scl-*independent over A* if for all $b \in B$, $b \notin \mathrm{scl}\big(A \cup (B \setminus \{b\})\big)$. A maximal scl-independent subset of B over A is called *a basis for B over A*.

By the Exchange property for scl, any two bases for B over A have the same cardinality. This allows us to define the *rank of B over A*:

$$\mathrm{rank}(B/A) = \text{ the cardinality of any basis of } B \text{ over } A.$$

In complete analogy with the corresponding fact for *acl* in a pregeometric theory, we can prove:

Lemma 6.6. *If p is a partial type over $A \subseteq M$ and $a \models p$ with* $\mathrm{rank}(a/A) = m$*, then for any set $B \supseteq A$ there is $a' \models p$ such that* $\mathrm{rank}(a'/B) \geq m$.

Proof. The proof of the analogous result for the rank coming from *acl* in a pregeometric theory is given, for example, in [19, page 315]. The proof of the present lemma is word-by-word the same with that one, after replacing an 'algebraic formula' by a 'formula defining a small set' in the definition of Φ^m_B ([19, Definition 2.2]) and the notion of 'algebraic independence' by that of 'scl-independence' we have here. □

It follows that the corresponding dimension of partial types and definable sets is well-defined and independent of the choice of the parameter set.

Definition 6.7. Let p be a partial type over $A \subset M$. The scl-*dimension of* p is defined as follows:

$$\text{scl-dim}(p) = \max\{\text{rank}(\bar{a}/A) : \bar{a} \subset M \text{ and } \bar{a} \models p\}.$$

Let X be a definable set. Then the scl-*dimension of* X, denoted by scl-dim(X) is the dimension of its defining formula.

We next prove the equivalence of the scl-dimension and large dimension of a definable set. First, by a standard routine, using the saturation of $\widetilde{\mathcal{M}}$, we observe the following fact about supercones.

Fact 6.8. *Let $J \subseteq M^k$ be an A-definable supercone. Then J contains a tuple of rank k over A.*

Proposition 6.9. *For every definable $X \subseteq M^n$.*

$$\text{ldim}(X) = \text{scl-dim}(X).$$

Proof. We may assume that X is $\emptyset$-definable.

$\leq$. Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable injective function and $J \subseteq M^k$ a supercone, such that $f(J) \subseteq X$. Suppose both f and J are defined over A. We need to show that $f(J)$ contains a tuple b with rank$(b/\emptyset) \geq k$. By Fact 6.8, J contains a tuple a of rank k over A. Let $b = f(a)$. Since f is injective, we have $a \in \text{dcl}(Ab)$ and $b \in \text{dcl}(Aa)$. In particular, $a \in scl(Ab)$ and $b \in \text{scl}(Aa)$. So a and b have the same rank over A. Hence,

$$\text{rank}(b/\emptyset) \geq \text{rank}(b/A) = \text{rank}(a/A) = k.$$

$\geq$. Let $b \in X$ be a tuple of rank k. By the Structure Theorem, b is contained in some l-cone $C \subseteq X$. We prove that $l \geq k$. Let $C = h(S; \mathcal{J})$, where $\mathcal{J}$ is a family of supercones in M^l. Suppose $b = h(g, a)$, for some $g \in S$ and $a \in J_g$. Since $h(g, -)$ is $\mathcal{L}_g$-definable and injective, we have $a \in \text{dcl}(gb)$ and $b \in \text{dcl}(ga)$. In particular, $a \in \text{scl}(gb)$ and $b \in \text{scl}(ga)$. Hence a and b have the same rank over g. But $a \in J \subseteq M^l$ and, hence,

$$k = \text{rank}(b/g) = \text{rank}(a/g) \leq l.$$

□

We next record several properties of the rank and large dimension, for future reference. By dcl-rank we denote the usual rank associated to dcl.

Lemma 6.10. *For every $a \in M$ and $A \subseteq M$, we have*

(1) $\text{scl}(A \cup P) = \text{scl}(A)$
(2) $\text{rank}(a/AP) = \text{rank}(a/A) = \text{dcl-rank}(a/AP)$.

Proof. Immediate from Lemma 6.2 and the definitions. □

Lemma 6.11. *Let $X, Y, X_1, \ldots, X_k$ be definable sets. Then:*

(1) $\text{ldim}(X) \leq \dim(cl(X))$. *Hence, if X is $\mathcal{L}$-definable,* $\text{ldim}X = \dim X$.
(2) $X \subseteq Y \subseteq M^n \Rightarrow \text{ldim}(X) \leq \text{ldim}(Y) \leq n$.
(3) *X is small if and only if* $\text{ldim}(X) = 0$.
(4) *If C is a k-cone, then* $\text{ldim}(C) = k$.
(5) $\text{ldim}(X_1 \cup \cdots \cup X_l) = \max\{\text{ldim}(X_1), \ldots, \text{ldim}(X_l)\}$.
(6) $\text{ldim}(X \times Y) = \text{ldim}(X) + \text{ldim}(Y)$.

Proof. (1). Assume X is A-definable and let $a \in X$ with $\text{rank}(a/A) = \text{ldim}(X)$. Since $a \in cl(X)$, we have

$$\text{ldim}(X) = \text{rank}(a/A) = \text{dcl-rank}(a/A \cup P) \leq \text{dcl-rank}(a/A) \leq \dim cl(X).$$

Now, if X is $\mathcal{L}$-definable, $\text{ldim}(X) \leq \dim cl(X) = \dim X$. On the other hand, if $\dim X = k$, one can $\mathcal{L}$-definably embed a k-box in X which of course is a k-cone. (2)-(5) were proved in Section 4, and (6) is by virtue of scl defining a pregeometry. □

Lemma 6.12. *Let $X \subseteq M^{m+n}$ be an A-definable set and let $\pi_m(X)$ be its projection onto the first m coordinates. Then*

1. *For every $k \in \mathbb{N}$, the set of all $t \in \pi_m(X)$ such that $\text{ldim}(X_t) = k$ is A-definable.*
2. *Assume that for every $t \in \pi_m(X)$, $\text{ldim}(X_t) = k$. Then*
$$\text{ldim}(X) = \text{ldim}(\pi_m(X)) + k.$$

Proof. (1). By induction on n. For $n = 0$, it is trivial. Now let $X \subseteq M^{m+n+1}$ and let $\pi_{m+n}(X)$ denote the projection onto the first $m + n$ coordinates. For $i = 0, 1$, let

$$Y^i = \{s \in \pi_{m+n}(X) : \text{ldim}(X_s) = i\}.$$

By Remark 3.4(a), each Y^i is A-definable. It clearly suffices to prove that the set of all $t \in \pi_m(X)$ such that $\text{ldim}(X_t) \geq k$ is A-definable. We observe that for every $t \in \pi_m(X)$,

$$(*) \qquad \text{ldim}(X_t) \geq k \iff \text{ldim}(Y^0)_t \geq k \text{ or } \text{ldim}(Y^1)_t \geq k - 1,$$

by Corollary 4.15 and, for left-to-right, Lemma 4.17 and Structure Theorem (3), and, for right-to-left, Lemmas 4.17 and 4.3. By Inductive Hypothesis, we are done.

(2). The proof is standard using the properties of a rank coming from a pregeometry. To see $\leq$, take any $a \in X$ and denote $a = (a_1, a_2) \in M^m \times M^n$. By the additivity property of the rank,

$$\text{rank}(a/A) = \text{rank}(a_2/Aa_1) + \text{rank}(a_1/A) \leq \text{ldim}(X_{a_2}) + \text{ldim}(\pi_m(X)) = k + \text{ldim}(\pi_m(X)).$$

To see $\geq$, take $a_1 \in \pi_m(X)$ with $\text{rank}(a_1/A) = \text{ldim}(\pi_m(X))$ and $a_2 \in X_{a_1}$ with $\text{rank}(a_2/Aa_1) = \text{ldim}(X_{a_2}) = k$. Again, by additivity,

$$\text{rank}(a/A) = \text{rank}(a_2/Aa_1) + \text{rank}(a_1/A) = k + \text{ldim}(\pi_m(X)).$$

□

6.1. scl**-generics.** For a treatment of the classical notion of dcl-generic elements, see, for example, [33]. Here we introduce the corresponding notion for scl.

Definition 6.13. Let $X \subseteq M^n$ be an $A \cup P$-definable set, and let $a \in X$. We say that a is a scl-*generic element of X over A* if it does not belong to any A-definable set of large dimension $< \text{ldim}(X)$. If $A = \emptyset$, we call a a scl-*generic element of X*.

By saturation, scl-*generic elements always exist.* More precisely, every $A \cup P$-definable set X contains an scl-generic element over A. Indeed, by Compactness and Lemma 6.11(5), the collection of all formulas which express that x belongs to X but not to any A-definable set of large dimension $< \text{ldim}(X)$ is consistent.

Two scl-generics are called *independent* if one (each) of them is scl-generic over the other. The facts that scl defines a pregeometry and that the scl-dim agrees with ldim imply:

Fact 6.14. *Let $G = \langle G, *\rangle$ be a definable group. If $a, b \in G$ are independent* scl*-generics, then so are a and $a * b^{-1}$.*

Note that none of the notions 'dcl-generic element' and 'scl-generic element' implies the other, but, by Lemma 6.10, if X is $A\cup P$-definable and $a \in X$, we have:

a is scl-generic over $A \cup P \Leftrightarrow a$ is scl-generic over $A \Leftrightarrow a$ is dcl-generic over $A \cup P$.

7. Definable groups

In this section we obtain our main application of the Structure Theorem. We fix a $\emptyset$-definable group $G = \langle G, *, 0_G\rangle$ with $G \subseteq M^n$ and $\mathrm{ldim}(G) = k$ and prove a local theorem for G: around scl-generic elements the group operation is given by an $\mathcal{L}$-definable map.

A convention on terminology. When we refer to a set $K = h(J) \subseteq M^n$ as an $A \cup P$-definable k-cone, we mean that $J \subseteq M^k$ is an A-definable supercone, and $h : M^k \to M^n$ is an $\mathcal{L}_{A\cup P}$-definable continuous and injective map. This convention is at no conflict with the definition of a cone, because every such $h(J)$ can be written as an $A \cup P$-definable cone $h'(S; \mathcal{J})$, where $S = \{g\}$ is the parameter set that defines h, $h'(g, -) = h(-)$ and $J_g = J$. Moreover, if an element a is contained in an A-definable k-cone $h'(S; \mathcal{J})$, then, since S is P-bound, it is clear that a is contained in an $A \cup P$-definable k-cone $h(J)$, as above.

Lemma 7.1. *Let $\{E_t = \tau_t(J_t)\}_{t\in E_1}$ be a definable family of l-cones, for some $\mathcal{L}_\emptyset$-definable continuous family τ_t, and a k-cone E_1. Then*

$$E = \bigcup_{t\in E_1} \{t\} \times E_t$$

is a $k + l$-cone.

Proof. Assume $E_1 = h(S; \mathcal{I})$, $S \subseteq M^m$, $E_1 \subseteq M^p$ and $E_t \subseteq M^n$. Define

$$\hat{h} : M^{m+k+l} \to M^{p+n} : (g, x, y) \mapsto (h(g, x), \tau_{h(g,x)}(y))$$

and, for every $g \in S$,

$$J_g = \bigcup_{x\in I_g} \{x\} \times J_{h(g,x)}.$$

The reader can verify that

$$E = \hat{h}\left(\bigcup_{g\in S} \{g\} \times J_g\right)$$

is a $k + l$-cone, as required. $\square$

Lemma 7.2. *Let $X \subseteq M^n$ be a $\emptyset$-definable set of large dimension k, (a, b) an* scl*-generic element of $X \times X$, and $D \subseteq X \times X$ a $\emptyset$-definable $2k$-cone containing (a, b). Then there are finite $A \subseteq M$ and $A\cup P$-definable family of k-cones $\{E_t = \tau_t(J_t)\}_{t\in E_1}$, where E_1 is a k-cone containing a, such that $b \in \bigcap_{t\in E_1} cl(E_t)$,*

$$(a, b) \in \bigcup_{t\in E_1} \{t\} \times E_t \subseteq D,$$

and (a, b) is scl*-generic of $X \times X$ over A.*

Proof. By Corollary 5.7, and since (a, b) is scl-generic of $X \times X$, it is contained in a $\emptyset$-definable set of the form

$$\bigcup_{t \in C_1} \{t\} \times C_t \subseteq D,$$

where $C_1 \subseteq X$ is a cone and there is l such that each $C_t \subseteq X$ is an l-cone given by

$$C_t = \sigma_t(S_t, \mathcal{J}_t)$$

for some $\mathcal{L}_\emptyset$-definable continuous family of maps σ_t and $\emptyset$-definable families of small sets S_t and supercones $\mathcal{J}_t = \{J_{t,g}\}_{g \in S_t}$. Since $b \in C_a \subseteq X$ and b is an scl-generic element of X over a, C_a must be a k-cone, and hence $l = k$. Fix some $g \in S_a \subseteq \mathrm{dcl}(P)$ such that $b \in \sigma_a(g, J_{a,g})$, and let $\hat{b} \in J_{a,g}$ with $b = \sigma_a(g, \hat{b})$. Then $b \in \mathrm{dcl}(Pa\hat{b})$ and $\hat{b} \in \mathrm{dcl}(Pab)$, hence

$$\text{dcl-rank}(a\hat{b}/P) = \text{dcl-rank}(ab/P) = 2k.$$

By Fact 2.7, there is $A \subseteq M$, such that

$$\text{dcl-rank}(a\hat{b}/AP) = \text{dcl-rank}(a\hat{b}/P)$$

and an $\mathcal{L}_{A \cup P}$-definable $\hat{B} \subseteq cl(J_{a,g})$ that contains $\hat{b}$. So, on the one hand, we have

$$\mathrm{rank}(a/AP) + \mathrm{rank}(\hat{b}/APa) = \mathrm{rank}(a\hat{b}/AP) = 2k$$

and hence $\mathrm{rank}(a/AP) = k$. On the other hand, if we let $B = \sigma_a(g, \hat{B})$, we have

$$b \in B \subseteq \sigma_a(g, cl(J_{a,g})).$$

Now consider the following $\mathcal{L}_{A \cup P}$-definable property $Q(t)$:

$$B \subseteq \sigma_t(g, cl(J_{t,g})).$$

Since a satisfies $Q(t)$ and is scl-generic of C_1 over AP, by the Structure Theorem, $Q(t)$ must hold on a $A \cup P$-definable cone $E_1 \subseteq C_1$ that contains a. Since

$$b \in B \subseteq \bigcap_{t \in E_1} \sigma_t(g, cl(J_{t,g})),$$

we have that the family $\{E_t = \tau_t(J_t) = \sigma_t(g, J_{t,g})\}_{t \in E_1}$ is as required. □

Remark 7.3. In general, there are no $\{E_t\}_{t \in E_1}$ as above so that $b \in \bigcap_{t \in E_1} E_t$. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, $X = M$

$$D = \bigcup_{c \in M} \{c\} \times (M \setminus cP),$$

and (a, b) any in M^2.

Corollary 7.4. *Let $X \subseteq M^n$ be a $\emptyset$-definable set of large dimension k. Let (a, b) be an* scl*-generic element of $X \times X$ and $f : X \times X \to X$ a $\emptyset$-definable function. Then there are finite $A \subseteq M$ and $A \cup P$-definable family of k-cones $\{E_t = \tau_t(J_t)\}_{t \in E_1}$, where E_1 is a k-cone containing a, such that $b \in \bigcap_{t \in E_1} cl(E_t)$, f agrees with an $\mathcal{L}_P$-definable continuous map on*

$$E = \bigcup_{t \in E_1} \{t\} \times E_t,$$

and (a, b) is an scl*-generic element of E over A.*

Proof. By the Structure Theorem, there is a $\emptyset$-definable $2k$-cone $D \subseteq G \times G$ that contains (a, b) and such that f agrees with an $\mathcal{L}_G$-definable continuous map on D. The statement then follows from Lemma 7.2. $\square$

We are now ready to prove the local theorem for definable groups.

Theorem 7.5 (Local theorem for definable groups)**.** *Let a be an* scl*-generic element of G. Then there is a $2k$-cone $C \subseteq G \times G$ containing (a, a), and an $\mathcal{L}$-definable map $F : M^n \times M^n \to M^n$, such that for every $(x, y) \in C$,*

$$x * a^{-1} * y = F(x, y).$$

Proof. Let $a_1 \in G$ be scl-generic over a, and let $a_2 = a_1^{-1} * a$. By Fact 6.14, a, a_1, a_2 are pairwise independent. By the Structure Theorem, for $i = 1, 2$, there is a Pa_i-definable k-cone $C_i = h_i(J_i) \subseteq G$ containing a, and $\mathcal{L}_{Pa_i}$-definable continuous $f_i : M^n \to M^n$ such that for every $x \in C_1$,

$$x * a_2^{-1} = f_2(x)$$

and for every $y \in C_2$,

$$a_1^{-1} * y = f_1(y).$$

Observe that $f_2(a) = a_1$ and $f_1(a) = a_2$.

We now look at the independent scl-generic elements a_1 and a_2. By Corollary 7.4, there are finite $A \subseteq M$ and $A \cup P$-definable family of k-cones $\{E_t = \tau_t(J_t)\}_{t \in E_1}$ in G, where $E_1 \subseteq G$ is a k-cone containing a_1 and $a_2 \in \bigcap_{t \in E_1} cl(E_t)$, such that $*$ agrees with an $\mathcal{L}_P$-definable continuous map $f : M^n \times M^n \to M^n$ on

$$E = \bigcup_{t \in E_1} \{t\} \times E_t,$$

and (a_1, a_2) is an scl-generic element of E over A. Observe that this implies (a, a_i) is also scl-generic of $G \times G$ over A. Moreover, since a_2 is dcl-generic of G over $A \cup P$, there is an $\mathcal{L}_{A \cup P}$-definable B of dimension k with

$$a_2 \in B \subseteq \bigcap_{t \in E_1} cl(E_t).$$

Claim. *For every $t \in E_1$, $f_1^{-1}(E_t) \cap h_1(J_1)$ has large dimension k.*

Proof of Claim. Denote $F_t = f_1^{-1} \tau_t$. Since a belongs to the $\mathcal{L}_{PAa_1}$-definable set $f_1^{-1}(B) \cap h_1(cl(J_1))$ and it is scl-generic over Aa_1, the set

$$f_1^{-1}(B) \cap h_1(cl(J_1)) \subseteq f_1^{-1}(cl(\tau_t(J_t)) \cap h_1(cl(J_1))$$

has dimension k. This implies that $F_t(cl(J_t)) \cap h_1(cl(J_1))$ has dimension k. By Lemma 5.11, $f_1^{-1}(E_t) \cap h_1(J_1) = F_t(J_t) \cap h_1(J_1)$ has large dimension k. $\square$

Now, since a belongs to the APa_2-definable set $f_2^{-1}(E_1) \cap h_2(J_2)$ and it is scl-generic over Aa_2, it must also belong to an APa_2-definable k-cone

$$C_1 \subseteq f_2^{-1}(E_1) \cap h_2(J_2).$$

For every $t \in C_1$, we let

$$D_t = f^{-1}(E_{f_2(t)}) \cap h_1(J_1).$$

By the Claim, $\text{ldim}(D_t) = k$. Since every $D_t \subseteq h_1(J_1)$, by Structure Theorem (5), we can find a definable family of k-cones

$$C_t = h_1(Y_t) \subseteq D_t, \; t \in C_1,$$

where $Y_t \subseteq J_1$ is a supercone in M^k, and $a \in C_a$. By Lemma 7.1, the set

$$C = \bigcup_{t \in C_1} \{t\} \times C_t$$

is a $2k$-cone. Moreover, it contains (a, a). We can now conclude as follows. For every $(x, y) \in C$,

$$x * a^{-1} * y = (x * a^{-1} * a_1) * (a_1^{-1} * y) = f_2(x) * f_1(y) = f(f_2(x), f_1(y)).$$

Set

$$F(x, y) = f(f_2(x), f_1(y)) : M^n \times M^n \to M^n.$$

□

Remark 7.6. (1) The above proof actually shows that there are a definable family of k-cones $\{C_t\}_{t \in C_1}$ in G, where $C_1 \subseteq G$ is a k-cone, such that

$$(a, a) \in C = \bigcup_{t \in C_1} \{t\} \times C_t,$$

and an $\mathcal{L}$-definable map $F : M^n \times M^n \to M^n$, such that for every $(x, y) \in C$,

$$x * a^{-1} * y = F(x, y).$$

(2) We observe that we cannot always have $C = C' \times C''$, where C', C'' are k-cones containing a. For example, consider the group $\mathcal{H} = \langle H = [0, 1), +\mathit{mod}\, 1\rangle$ in the real field, and let $T = \mathbb{Q}^{rc} \cap H$. Now let $g : H \to M$ be the translation $x \mapsto 2 + x$ on T, and identity elsewhere. Let G be the induced group on $(H \setminus T) \cup g(T)$. Clearly, G is definable in $\widetilde{M} = \langle \mathbb{R}, \mathbb{Q}^{rc}\rangle$, and it is easy to verify that the above observation holds for every $a \in G$. Of course, the conclusion of Theorem 7.5 holds for every $a \in H \setminus T$, by letting $C_1 = H \setminus T$, $C_t = H \setminus (T \cup (t + T))$ and $f = +\mathit{mod}\, 1$. Moreover, we can achieve $C = C' \times C'$, but only up to definable isomorphism. It is reasonable to ask whether that is always true, and we include some relevant (in fact, stronger) questions at the end of this section.

We expect that the above local theorem will play a crucial role in forthcoming analysis of groups definable in $\widetilde{M}$. The ultimate goal would be to understand definable groups in terms of $\mathcal{L}$-definable groups and small groups. Motivated by the successful analysis of semi-bounded groups in [16] and the more recent [4], we conjecture the following statement.

Conjecture 7.7. *Let $\langle G, *\rangle$ be a definable group. Then there is a short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{U} \arrow[r] \arrow[d, "\tau"] & K \arrow[r] & 0 \\ & & G & & \end{tikzcd}$$

where

- *$\mathcal{U}$ is $\bigvee$-definable*
- *$\mathcal{B}$ is $\bigvee$-definable in $\mathcal{L}$ with* $\dim(B) = \operatorname{ldim}(G)$.
- *K is definable and small*
- *$\tau : \mathcal{U} \to G$ is a surjective group homomorphism and*
- *all maps involved are $\bigvee$-definable.*

The conjecture is in a certain sense optimal: we next produce an example of a definable group G which is *not* a direct product of an $\mathcal{L}$-definable group by a small group. Using known examples of $\mathcal{L}$-definable groups B from [31, 36], which are not direct products of one-dimensional subgroups, it would be easy to provide such an G by restricting some of the one-dimensional subgroups of the universal cover of B to the subgroup P (say, in a dense pair). Our example below, however, is not constructed in this way, as it is *not* a subgroup of the examples in [31, 36].

Example 7.8. Let $\mathcal{M}$ be a non-archimedean real closed field, and $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle \models \widetilde{T}$. Let $G = \langle P \times [0,1), \oplus, 0\rangle$, where $x \oplus y = x + y \mod (1,1)$; that is,

$$x \oplus y = \begin{cases} x+y, & \text{if } x+y \in P \times [0,1) \\ x+y-(1,1), & \text{otherwise} \end{cases}$$

Then G is clearly not small. But it cannot contain any non-trivial $\mathcal{L}$-definable subgroup. Indeed, by o-minimality, every $\mathcal{L}$-definable subset of $P \times [0,1)$ must be contained in a finite union of fibers $\{g\} \times [0,1)$, $g \in P$. On the other hand, an $\mathcal{L}$-definable subgroup of G is a topological group containing some $\mathcal{L}$-definable neighborhood of 0 and, thus, also every fiber $\{n\} \times [0,1)$, $n \in \mathbb{Z}$.

The reader can verify that for $\mathcal{B} = Fin(M)$, $K = P$, $\mathcal{U} = \mathcal{B} \times K$ and $\tau(x,y) = (x,y) \mod (1,1)$, we obtain the diagram of Conjecture 7.7. Also, the set $V = \bigcup_{a=-1,0} \{a\} \times [0,1)$ is an $\mathcal{L}$-definable local subgroup of G, as in Theorem 7.5.

Finally, observe that G is a subgroup of the $\mathcal{L}$-definable group B, which *is* the direct product $B = S \times \langle M, +\rangle$, where S has domain $\{(x,x) : 0 \leq x < 1\}$ and operation $(x,y) \mapsto x+y \mod (1,1)$.

We finish with some open questions which we expect our local theorem to have an impact on.

Question 7.9. *Does G, up to definable isomorphism, contain an $\mathcal{L}$-definable local subgroup (in the sense of* [35, §23 (D)]*) whose dimension equals* $\operatorname{ldim} G$*?*

Question 7.10. *Assume* $\operatorname{ldim} G = \dim cl(G)$. *Is G, up to definable isomorphism, $\mathcal{L}$-definable?*

If Conjecture 7.7 is true, it would be nice to know what the small groups are.

Question 7.11. *Is every small definable group/set definably isomorphic to a group/set definable in the induced structure on P?*

Question 7.12. *Is G, up to definable isomorphism, a subgroup of an $\mathcal{L}$-definable group (whose dimension might be bigger than* $\operatorname{ldim}(G)$*)?*

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Department of Mathematics and Statistics, University of Konstanz, Box 216, 78457 Konstanz, Germany
E-mail address: panteleimon.eleftheriou@uni-konstanz.de

Department of Mathematics, Mimar Sinan Güzel Sanatlar Üniversitesi, Bomonti, Istanbul, Turkey
E-mail address: ayhan.gunaydin@msgsu.edu.tr

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: phierony@illinois.edu