O-minimal geometry

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Contents

1. Axiom of o-minimality

1.1 Semialgebraic sets

Definition

A subset of $\mathbb{R}^n$ is called **semialgebraic** if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}$$

where $f, g_1, \ldots, g_k$ are real polynomials in n variables.

Example

The set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ is semialgebraic.

Remark

(1) *Let* $A, B \subset \mathbb{R}^n$ *be semialgebraic. Then* $A \cap B, A \cup B, \mathbb{R}^n \setminus A$ *are semialgebraic.*

(2) *Let* $A \subset \mathbb{R}^m, B \subset \mathbb{R}^n$ *be semialgebraic. Then* $A \times B$ *is semialgebraic.*

Theorem (Tarski)

Let $A \subset \mathbb{R}^{n+1}$ *be semialgebraic and let* $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *be the projection onto the first* n *coordinates. Then* $\pi(A)$ *is semialgebraic.*

1.2 Structures

Definition

A **structure on** $\mathbb{R}$ is a sequence $\mathcal{M} = (\mathcal{M}_n)_{n\in\mathbb{N}}$ such that the following holds for all $n, m \in \mathbb{N}$:

($\mathcal{S}1$) $\mathcal{M}_n$ consists of subsets of $\mathbb{R}^n$ such that $A \cap B, A \cup B, \mathbb{R}^n \setminus A \in \mathcal{M}_n$ for all $A, B \in \mathcal{M}_n$.

($\mathcal{S}2$) If $A \in \mathcal{M}_m, B \in \mathcal{M}_n$ then $A \times B \in \mathcal{M}_{m+n}$.

($\mathcal{S}3$) $\mathcal{M}_n$ contains the semialgebraic subsets of $\mathbb{R}^n$.

($\mathcal{S}4$) If $A \in \mathcal{M}_{n+1}$ then $\pi(A) \in \mathcal{M}_n$.

Definition

Let $\mathcal{M} = (\mathcal{M}_n)_{n\in\mathbb{N}}$ be a structure on $\mathbb{R}$.

(a) We say that $A \subset \mathbb{R}^n$ is definable (in $\mathcal{M}$) if $A \in \mathcal{M}_n$.

(b) We say that a function $f : A \to \mathbb{R}^m$ ($A \subset \mathbb{R}^n$) is **definable** if its graph is definable.

The above notion of a structure on $\mathbb{R}$ coincides with the logical notion of a structure with respect to a language

$$\{+,\cdot,-,0,1,\leq,\ldots\}$$

extending the language of ordered rings and the reals as universe. Definable sets/functions in the above setting are precisely the definable sets/functions (with parameters from $\mathbb{R}$) which are defined by formulas in the language.

Translation:

Sets	Formulas
$\cap$	$\wedge$
$\cup$	$\vee$
$\setminus$	$\neg$
π	$\exists$

Example

The formula $\exists y\ x = y^2$ defines the set $\{x \in \mathbb{R} \mid x \geq 0\}$.

Remark

Let $\mathcal{M}$ be a structure on $\mathbb{R}$.

(1) *Let A be definable. Then $\overline{A}, \mathring{A}$ and ∂A are definable.*

(2) *Let $f : A \to \mathbb{R}^m$ be definable. Then A and $f(A)$ are definable.*

1.3 O-minimal structures

Definition

A structure on $\mathbb{R}$ is called **o-minimal** if the following axiom holds:

($\mathcal{O}$) The definable subsets of $\mathbb{R}$ are the finite unions of intervals and points.

The axiom of o-minimality means that the unary definable sets have only finitely many connected components.

Examples

(1) The structure consisting of the semialgebraic sets (which is denoted by $\mathbb{R}$) is o-minimal.

(2) The structure $\mathbb{R}_{\exp}$ generated by the global exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is o-minimal.

2. Tameness

Fix an o-minimal structure $\mathcal{M}$ on $\mathbb{R}$. "Definable" means "definable in $\mathcal{M}$".

2.1 Cell decomposition

Theorem (Monotonicity)

Let $-\infty \leq a < b \leq \infty$ *and let* $f :]a,b[\to \mathbb{R}$ *be definable. Then there are points* $a_1 < \ldots < a_k$ *in* $]a,b[$ *such that on each subinterval* $]a_j, a_{j+1}[$ *with* $a_0 = a, a_{k+1} = b$*, the function is either constant, or strictly monotone and continuous.*

Proof:

Corollary

Let $f :]a, b[\to \mathbb{R}$ *be definable. Then* $\lim_{x \to b} f(x)$ *exists in* $\mathbb{R} \cup \{\pm\infty\}$.

Theorem

Let $f :]a, b[\to \mathbb{R}$ *be definable and let* $r \in \mathbb{N}$. *Then* f *is* C^r *up to a finite set.*

Definition (Cell)

A (definable) cell in $\mathbb{R}^n$ is recursively defined as follows:

$n = 1$: The cells in $\mathbb{R}$ are exactly the intervals and points.

$n \to n+1$: Assume that $X \subset \mathbb{R}^n$ is a cell and $f, g : X \to \mathbb{R}$ are continuous definable functions with $f(x) < g(x)$ for all $x \in X$. The the following sets are cells in $\mathbb{R}^{n+1}$ (with base X):

$$\Gamma(f) := \big\{(x,y) \in X \times \mathbb{R} \mid y = f(x)\big\},$$

$$(f,g) := \big\{(x,y) \in X \times \mathbb{R} \mid f(x) < y < g(x)\big\},$$

$$(f,\infty) := \big\{(x,y) \in X \times \mathbb{R} \mid y > f(x)\big\},$$

$$(-\infty,f) := \big\{(x,y) \in X \times \mathbb{R} \mid y < f(x)\big\},$$

$$(-\infty,\infty) := X \times \mathbb{R}.$$

Remark

(1) *There is a natural dimension attached to a cell.*
(2) *A cell is connected.*

Theorem (Cell decomposition)

(1) *Let $A \subset \mathbb{R}^n$ be a definable set. Then A can be partitioned into finitely many cells.*
(2) *Let $f : A \to \mathbb{R}$ be a definable function. Then A can be partitioned into finitely many cells such that on each cell f is continuous.*

Consequences:

- *There is a natural dimension attached to a definable set.*
- *A definable cell has only finitely many connected components. Each connected component is definable.*
- **Uniform finiteness:** *Let $A \subset \mathbb{R}^m \times \mathbb{R}^n$ be a definable family and assume that for each $t \in \mathbb{R}^m$ the set*

$$A_t := \{x \in \mathbb{R}^n \mid (t,x) \in A\}$$

  is finite. Then there is an $N \in \mathbb{N}$ such that $|A_t| \leq N$ for all $t \in \mathbb{R}^m$.
- **Curve selection:** *Let $A \subset \mathbb{R}^n$ be definable and let $x \in \overline{A} \setminus A$. Then there is a continuous definable curve $\gamma :]0,1[\to A$ such that $\lim_{t \to 0} \gamma(t) = x$.*

Remark

Let $A \subset \mathbb{R}^n$ be definable.

(1) $\overline{A}$ *is definable and* $\dim(\overline{A} \setminus A) < \dim(A)$.

(2) $\dim(A) = n$ *if and only if* $\mathring{A} \neq \emptyset$.

2.2 Smoothness

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be definable and let $r \in \mathbb{N}$. Then there is an open definable set $U \subset \mathbb{R}^n$ with $\dim(\mathbb{R}^n \setminus U) < n$ such that f is C^r on U.

Theorem (Inverse function)

Let $f : U \to \mathbb{R}^n$ be a definable C^1-map on a definable open set $U \subset \mathbb{R}^n$. Let $a \in U$ such that $Df(a)$ is invertible. Then there are definable open neighbourhoods $V \subset U$ of a and W of $f(a)$ such that $f : V \to W$ is a definable C^1-diffeomorphism.

2.3 Stratification

Let $r \in \mathbb{N}$.

Definition (Definable manifold)

Let $X \subset \mathbb{R}^n$. Then X is called a **definable (C^r-)manifold** if X is a submanifold of $\mathbb{R}^n$ and if X is a definable set.

Remark

Let X be a definable manifold. The dimension of X as a definable set coincides with the dimension of X as a submanifold of $\mathbb{R}^n$.

Definition (Nice definable manifold)

A k-dimensional C^r-definable manifold X is called **nice** if there are, after some linear change of coordinates, a definable open subset U of $\mathbb{R}^k$ and a definable C^r-map $f : U \to \mathbb{R}^{n-k}$ with bounded derivative such that X is the graph of f.

Theorem

Let $A \subset \mathbb{R}^n$ be definable set. Then A can be partitioned into finitely many nice definable manifolds, the so-called **strata**.

Corollary

Let $A \subset \mathbb{R}^n$ be a definable set of dimension k that is bounded. Then the k-dimensional volume of A is finite.

2.4 Triangulation

Definition (Simplex)

A **simplex** is of the form

$$(a_0, \ldots, a_k) := \left\{ \sum_{j=0}^{k} t_j a_j \;\middle|\; t_j > 0 \;\forall j, \sum_{j=0}^{k} t_j = 1 \right\},$$

where $a_0, \ldots, a_k \in \mathbb{R}^n$ are affine independent points. These points are called the **vertices** of the simplex.
The topological closure of the simplex is denoted by

$$[a_0, \ldots, a_k] := \left\{ \sum_{j=0}^{k} t_j a_j \;\middle|\; t_j \geq 0 \;\forall j, \sum_{j=0}^{k} t_j = 1 \right\}.$$

A **face** of a simplex $(a_0, \dots, a_k)$ is a simplex spanned by a nonempty subset of $\{a_0, \dots, a_k\}$.

Example

$[a_0, a_1]$ is the line segment between distinct points a_0 and a_1.
$[a_0, a_1, a_2]$ is the triangle spanned by points a_0, a_1, a_2 not on a line.

Definition (Complex)

A **complex** in $\mathbb{R}^n$ is a finite collection K of simplexes in $\mathbb{R}^n$, such that for all $\sigma_1, \sigma_2 \in K$ either

(a) $\overline{\sigma_1} \cap \overline{\sigma_2} = \emptyset$

or

(b) $\overline{\sigma_1} \cap \overline{\sigma_2} = \overline{\tau}$ for some common face τ of σ_1 and σ_2.

Definition (Polyhedron)

The union of simplexes of a complex is called the **polyhedron** spanned by the complex.

Theorem (Triangulation theorem)

Let $A \subset \mathbb{R}^n$ be definable. Then A is definably homeomorphic to a polyhedron spanned by a complex in $\mathbb{R}^n$.

2.4 Trivialization

A continuous map $f: X \to Y$ between topological spaces is often thought of as describing a "continuous" family of sets $\big(f^{-1}(y)\big)_{y \in Y}$ in X parametrized by the space Y.

From this viewpoint projections are the simplest:
Let $\pi : Y \times Z \to Y$ be the projection onto the first component.
Then $\pi^{-1}(y) = \{y\} \times Z$ for all $y \in Y$.

Definition (Trivial map)

A continuous map $f : X \to Y$ between topological spaces is **trivial** if there is a toplogical space Z and a homeomorphism $\varphi : X \to Y \times Z$ such that $f = \pi \circ \varphi$. The map $\varphi : X \to Y \times Z$ is then called a **trivialization** of f.
We also say that f is trivial over a subspace Y' of Y if $f|_{f^{-1}(Y')} : f^{-1}(Y') \to Y'$ is trivial.

Remark

Let $\varphi : X \to Y \times Z$ *be a trivialization of* $f : X \to Y$*. Then all fibers* $f^{-1}(y)$ *are homeomorphic to* Z*.*

Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ be definable and let $f : X \to Y$ be a definable continuous function.

Definition

We say that f is **definably trivial** if there is a definable set Z in some $\mathbb{R}^k$ and a definable homeomorphism $\varphi : X \to Y \times Z$ such that $f = \pi \circ \varphi$. The map $\varphi : X \to Y \times Z$ is then called a **definable trivialization** of f.
We also say that f is trivial over a definable subset Y' of Y if $f|_{f^{-1}(Y')} : f^{-1}(Y') \to Y'$ is definably trivial.

Theorem

Let $f : X \to Y$ be definable and continuous. Then Y can be partitioned into finitely many definable sets such that f is definably trivial over each of them.

Corollary

Let $A \subset \mathbb{R}^m \times \mathbb{R}^n$ be definable. Consider the definable family $(A_t)_{t \in \mathbb{R}^m}$ where

$$A_t := \{x \in \mathbb{R}^n \mid (t, x) \in A\}.$$

Then this family has only finitely many homoemorphism types.

2.4 Parametrization

Let $I :=]0,1[$.

Theorem (Parametrization theorem)

Let $A \subset \mathbb{R}^n$ be definable and bounded. Then for every $p \in \mathbb{N}$ there is a finite set Φ of definable C^p-maps $\varphi : I^{k_\varphi} \to \mathbb{R}^n$ with

$$A = \bigcup_{\varphi \in \Phi} \varphi(I^{k_\varphi})$$

such that $|D^\alpha \varphi|$ is bounded for every $\varphi \in \Phi$ and every $\alpha \in \mathbb{N}_0^{k_\varphi}$ with $||\alpha|| \leq p$.

Theorem (Reparametrization theorem)

Let $f : I^n \to \mathbb{R}^q$ be a definable map with bounded image. Then for every $p \in \mathbb{N}$ there is a finite set Φ of definable C^p-maps $\varphi : I^n \to \mathbb{R}^n$ with

$$I^n = \bigcup_{\varphi \in \Phi} \varphi(I^n)$$

such that the following holds for every $\varphi \in \Phi$ and every $\alpha \in \mathbb{N}_0^n$ with $||\alpha|| \leq p$:

(1) *$|D^\alpha \varphi|$ is bounded,*

(2) *$f \circ \varphi$ is C^p,*

(3) *$|D^\alpha (f \circ \varphi)|$ is bounded.*

3. The structure $\mathbb{R}_{\mathrm{an}}$

3.1 Semi- and subanalytic sets

Definition

Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ is called real analytic if for every $a \in U$ there is a convergent real power series $p(x)$ in n variables such that $f(x) = p(x - a)$ on a neighbourhood of a.

Definition

A subset A of $\mathbb{R}^n$ is called **semianalytic** if for every $a \in \mathbb{R}^n$ there is an open neighbourhood U of a such that $A \cap U$ is a finite union of sets of the form

$$\{x \in U \mid f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}$$

where $f, g_1, \ldots, g_k$ are real analytic on U.

Examples

(1) A semialgebraic set is semianalytic.

(2) The graphs of the exponential function and of the sine function are semianalytic.

Remarks

(1) *Let* $A, B \subset \mathbb{R}^n$ *be semianalytic. Then* $A \cap B, A \cup B, \mathbb{R}^n \setminus A$ *are semianalytic.*

(2) *Let* $A \subset \mathbb{R}^m, B \subset \mathbb{R}^n$ *be semianalytic. Then* $A \times B$ *is semianalytic.*

Definition

A subset A of $\mathbb{R}^n$ is called **subanalytic** if for every $a \in \mathbb{R}^n$ there is a neighbourhood U of a, some $p \in \mathbb{N}$ and a bounded semianalytic subset $B \subset \mathbb{R}^{n+p}$ such that $A \cap U = \pi_n(B)$ where $\pi_n : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the projection onto the first factor.

Remark

A semianalytic set is subanalytic. A subanalytic set is not necessarily semianalytic.

Remarks

(1) *Let* $A, B \subset \mathbb{R}^n$ *be subanalytic. Then* $A \cap B, A \cup B$ *are subanalytic.*

(2) *Let* $A \subset \mathbb{R}^m, B \subset \mathbb{R}^n$ *be subanalytic. Then* $A \times B$ *is subanalytic.*

(3) *Let* $A \subset \mathbb{R}^{n+1}$ *be subanalytic and bounded and let* $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ *be the projection onto the first n coordinates. Then* $\pi(A)$ *is subanalytic.*

Remark

The projection of a subanalytic set is not necessarily subanalytic.

Theorem (Gabrielov)

The complement of a subanalytic set is subanalytic.

Remark

The structure generated by the subanalytic sets is not o-minimal.

Remark

The map

$$\varphi_n : \mathbb{R}^n \to]-1,1[^n, x = (x_1, \ldots, x_n) \mapsto \Big(\frac{x_1}{\sqrt{x_1^2+1}}, \ldots, \frac{x_n}{\sqrt{x_n^2+1}},\Big),$$

is an analytic diffeomorphism that is semialgebraic.

Definition

A subset A of $\mathbb{R}^n$ is called **globally subanalytic** if $\varphi_n(A)$ is subanalytic.
A function $f : A \to \mathbb{R}^m$ is called globally subanalytic if its graph is a globally subanalytic set.

Examples

(1) A semialgebraic set is globally subanalytic.

(2) A bounded subanalytic set is globally subanalytic.

(3) The exponential function and the sine function are not globally subanalytic.

Definition

A function f is called **restricted analytic** if there is a real power series $p(x)$ that converges on an open neighbourhood of $[0,1]^n$ such that

$$f(x) = \begin{cases} p(x), & \text{if} \quad x \in [0,1]^n, \\ 0, & \text{else.} \end{cases}$$

Let $\mathcal{L}_{\mathrm{an}}$ be the language which extends the language $\{+,\cdot,-,0,1,\leq,\dots\}$ by function symbols for every restricted analytic function and let $\mathbb{R}_{\mathrm{an}}$ be the natural $\mathcal{L}_{\mathrm{an}}$-structure on $\mathbb{R}$.

Remark

Let $A \subset \mathbb{R}^n$.

(1) *Then A is definable in $\mathbb{R}_{\mathrm{an}}$ if and only if A is globally subanalytic.*

(2) *A function $f : A \to \mathbb{R}^m$ is definable in $\mathbb{R}_{\mathrm{an}}$ if and only if it is globally subanalytic.*

Theorem

The structure $\mathbb{R}_{\mathrm{an}}$ is o-minimal.

3.2 Proof of o-minimality

Thom's lemma

Let $f_1, \dots, f_k \in \mathbb{R}[T]$ be nonzero polynomials such that if $f_j' \neq 0$, then $f_j' \in \{f_1, \dots, f_k\}$. Let $\varepsilon : \{1, \dots, k\} \to \{-1, 0, 1\}$ (a "sign condition"), and set

$$A_\varepsilon := \Big\{t \in \mathbb{R} \mid \mathrm{sign}\big(f_j(t)\big) = \varepsilon(j), 1 \leq j \leq k\Big\}.$$

Then A_ε is empty, a point, or an interval.
If $A_\varepsilon \neq \emptyset$, then its closure is given by

$$\overline{A_\varepsilon} = \Big\{t \in \mathbb{R} \mid \mathrm{sign}\big(f_j(t)\big) \in \{\varepsilon(j), 0\}, 1 \leq j \leq k\Big\}.$$

If $A_\varepsilon = \emptyset$, then

$$\Big\{t \in \mathbb{R} \mid \mathrm{sign}\big(f_j(t)\big) \in \{\varepsilon(j), 0\}, 1 \leq j \leq k\Big\}$$

is empty or a point.

Let X be a nonempty topological space, E a ring of continuous real-valued functions $f : X \to \mathbb{R}$, the ring operations being pointwise addition and multiplication, with multiplicative identity the function on X that takes the constant value 1.

A subset A of X is called an E-set if A is a finite union of sets of the form

$$\{x \in X \mid f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}$$

where $f, g_1, \ldots, g_k \in E$.

Theorem

Let $f_1(T), \ldots, f_M(T) \in E[T]$. Then the list $f_1, \ldots, f_M$ can be augmented to a list $f_1, \ldots, f_N \in E[T]$ $(M \leq N)$, and X can be partitioned into finitely many E-sets $X_1, \ldots, X_k$ such that for each connected component C of each X_j there are continuous real-valued functions $\xi_{C,1} < \ldots < \xi_{C,m(C)}$ on C with the following properties:

(1) *Each function f_n has constant sign on each of the graphs $\Gamma(\xi_{C,j})$ and on each of the sets $(\xi_{C,j}, \xi_{C,j+1})$ (where $\xi_{C,0} \equiv -\infty$ and $\xi_{C,m(C)+1} \equiv +\infty$).*

(2) *Each of the sets $\Gamma(\xi_{C,j})$ and $(\xi_{C,j}, \xi_{C,j+1})$ is of the form*

$$\left\{(x,t) \in C \times \mathbb{R} \mid \mathrm{sign}\big(f_n(x,t)\big) = \varepsilon(n) \text{ for } 1 \leq n \leq N\right\}$$

for a suitable sign condition $\varepsilon : \{1, \ldots, N\} \to \{-1, 0, 1\}$.

Definition

The pair (X, E) has the **Łojasiewicz property** if each E-set has only finitely many connected components, and each component is also an E-set.

Corollary

If (X, E) *has the Łojasiewicz property then* $(X \times \mathbb{R}, E[T])$ *has also the Łojasiewicz property.*

By $\mathbb{R}\{x_1, \ldots, x_n\}$ we denote the ring of convergent real power series in the variables $x_1, \ldots, x_n$.

Definition

A convergent real power series $f \in \mathbb{R}\{x_1, \ldots, x_n\}$ is called **regular of order** d **in** x_n if

$$f(0, x_n) = a_d x_n^d + \text{ higher terms}$$

where $a_d \neq 0$.

Remark

Let $f \in \mathbb{R}\{x_1 \ldots, x_n\}$ be nonzero. Then there is a linear coordinate transformation φ of $\mathbb{R}^n$ such that $f \circ \varphi$ is regular in x_n.

Definition

A power series $p \in \mathbb{R}\{x_1, \ldots, x_{n-1}\}[x_n]$ of the form

$$p(x', x_n) = a_0(x') + a_1(x')x_n + \ldots a_{n-1}(x')x_n^{d-1} + x_n^d$$

is called a **Weierstraß polynomial of degree** d **in** x_n if

$$a_0(0) = \ldots = a_{d-1}(0) = 0.$$

Weierstraß preparation theorem

Let $f \in \mathbb{R}\{x_1, \ldots, x_n\}$ be regular of order d in x_n. Then there are uniquely determined $u \in \mathbb{R}\{x_1, \ldots, x_n\}$ with $u(0) \neq 0$ and a Weierstraß polynomial p of degree d in x_n such that $f = up$.

Gabrielov's theorem

Model theoretic formulation:

The $\mathcal{L}_{\mathrm{an}}$-theory of the $\mathcal{L}_{\mathrm{an}}$-structure $\mathbb{R}_{\mathrm{an}}$ is **model complete**; i.e. every definable set can be described by an existential formula.

3.3 Geometric properties

Definition

A **convergent real Puiseux series** is of the form $f(t^{1/p})$ where f is a convergent real power series in one variable and $p \in \mathbb{N}$.

Theorem

Let $f:]0,\infty[\to \mathbb{R}$ *be definable in* $\mathbb{R}_{\mathrm{an}}$. *Then there is a convergent real Puiseux series* g, *some* $r \in \mathbb{Q}$ *and some* $\varepsilon > 0$ *such that* $f(t) = t^r g(t)$ *on* $]0,\varepsilon[$.

Theorem (Lion-Rolin preparation)

Let $q \in \mathbb{N}_0$, let $A \subset \mathbb{R}^q \times \mathbb{R}$ and let $f : A \to \mathbb{R}, (x,t) \mapsto f(x,t)$, be definable in $\mathbb{R}_{\mathrm{an}}$.
Then there is a cell decomposition $\mathcal{C}$ of A such that the following holds. Let $C \in \mathcal{C}$ and let B denote the base of C. Assume that C is fat over the last variable t; i.e. C_x is a nonempty open interval (and not just a point) for every $x \in B$. Then the function $f|_C$ can be written as

$$f|_C(x,t) = a(x)|t-\theta(x)|^r u\big(x, t-\theta(x)\big)$$

where $r \in \mathbb{Q}$, the functions $a, \theta : B \to \mathbb{R}$ are definable and real analytic, $t \neq \theta(x)$ on C, and $u(x,t)$ is a so-called special unit on

$$C^\theta := \big\{(x, t-\theta(x)) \mid (x,t) \in C\big\};$$

i.e. u is of the following form:

$$u(x,t) = v\Big(b_1(x),\ldots,b_M(x),b_{M+1}(x)|t|^{1/q},b_{M+2}(x)|t|^{-1/q}\Big)$$

where

$$\begin{array}{rcl} \varphi : B\times\mathbb{R}\setminus\{0\} & \to & \mathbb{R}^{M+2}, \\ (x,t) & \mapsto & \Big(b_1(x),\ldots,b_M(x),b_{M+1}(x)|t|^{1/q},b_{M+2}(x)|t|^{-1/q}\Big) \end{array}$$

is a definable and real analytic function with $\varphi(C_\theta)\subset[-1,1]^{M+2}$ *and* v *is a real power series in* $M+2$ *variables that converges on an open neighbourhood of* $[-1,1]^{M+2}$ *and does not vanish there.*