SEMI-BOUNDED GROUPS

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ABSTRACT. We present structure theorems for groups definable in o-minimal structures. We are specifically interested in semibounded o-minimal structures; those are structures which lie strictly between an ordered vector space $\mathcal{V}$ and a real closed field $\mathcal{R}$. Jointly with Y. Peterzil [EP], we prove that every group definable in a semi-bounded o-minimal structure 'splits' into a group definable in $\mathcal{V}$ and a group definable in $\mathcal{R}$.

1. Introduction

A group is *definable* in a structure $\mathcal{M}$ if both its domain and the graph of its multiplication are definable in $\mathcal{M}$. Classical examples include algebraic groups, which are definable in algebraically closed fields, and compact real Lie groups, which are definable in o-minimal expansions of the real field. Definable groups have always been at the core of model theory, largely because of their prominent role in important applications of the subject, such as Hrushovski's proof of the function field Modell-Lang conjecture in all characteristics [Hr].

Let us provide some definitions. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We call $X \subseteq M^n$ *definable* (in $\mathcal{M}$) if there is a formula $\phi(\bar{x}, \bar{y}) \in \mathcal{L}$ and parameters $\bar{a} \in M^k$, such that

$$X = \{\bar{b} \in M^n : \mathcal{M} \models \phi(\bar{b}, \bar{a})\}.$$

Example 1.1. The circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is definable in $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$, but not in $\langle \mathbb{R}, <, +, 0\rangle$.

A function $f : A \subseteq M^m \to M^n$ is *definable* if its graph $\Gamma(f) \subseteq M^m \times M^n$ is a definable set. A group $G = \langle G, *, e_G\rangle$ is *definable* if $G \subseteq M^n$ and $* : M^{2n} \to M^n$ are definable.

Example 1.2. The circle S^1 with complex multiplication is a group definable $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$.

From a model-theoretic point of view, the interesting structures are those whose class of definable sets satisfy nice properties. Stable structures and o-minimal structures are such. Here we are concerned with the latter.

Definition 1.3. A densely linearly ordered structure $\mathcal{M} = \langle M, <, \dots\rangle$ is called *o-minimal* (order-minimal) if every definable subset of M is a finite union of open intervals (a, b), $a, b \in M \cup \{\pm\infty\}$, and points.

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The definable objects in an o-minimal structure have nice topological and geometric properties. First of all, on M we have the $<$-topology, and on each M^n we have the product topology. In this topology, every definable function $f : M^n \to M$ is piecewise continuous, and every definable subset $X \subseteq M^n$ is a finite union of definably homeomorphic images of open boxes, and points (a standard reference is [Dries]). We will make use of the following invariant, which can be defined for every definable $X \subseteq M^n$:

$$\begin{aligned}\dim(X) = \; &\max\{k : \; X \text{ contains a } k\text{-box } I^k \text{ up to definable} \\ &\text{bijection, where } I \text{ is an open interval in } M.\}\end{aligned}$$

This notion of dimension has nice properties, such as $\dim(X_1 \cup \cdots \cup X_k) = \max_i\{\dim(X_i)\}$.

We are interested in groups definable in o-minimal structures. Two main examples of o-minimal structures are:

(1) $\mathcal{R} = \langle R, <, +, \cdot \rangle$, a real closed field (that is, a model of the real field).
(2) $\mathcal{R}_{vect} = \langle R, <, +, \{x \mapsto rx\}_{r\in R}\rangle$, an ordered vector space over R.

Sets and groups definable in $\mathcal{R}$ are called *semialgebraic*, and those definable in $\mathcal{R}_{vect}$ are called *semilinear*. A natural question is the following:

Question 1. (van den Dries, 80's): Let $\mathcal{R}$ and $\mathcal{R}_{vect}$ be as above. Is there a structure $\mathcal{N}$ whose class of definable sets lies strictly between the classes of semilinear and semialgebraic sets?

Answer. Yes. Consider the structure $\mathcal{N} = \langle \mathcal{R}_{vect}, B\rangle$ where B denotes a semialgebraic set which is not semilinear (such as S^1). It is proved in [PSS] that $\mathcal{N}$ does not define all semialgebraic sets.

Remark 1.4. (1) By [MPP, Pet], if $R = \mathbb{R}$, then $\mathcal{N}$ is the unique structure whose class of definable sets lies strictly between that of $\mathcal{R}_{vect}$ and $\mathcal{R}$.

(2) By Trichotomy Theorem in [PeSt], there is an open interval $I \subseteq N$ and a definable real closed field with domain I.

(3) One could ask Question 1 in other settings as well. The question whether there is a structure strictly between the complex vector space and the complex field is a simple version of the famous Zilber Dichotomy Conjecture, work around which has given rise to the most striking applications of model theory, such as the aforementioned theorem by Hrushovski. The answer there is negative.

The second remark motivates the following definition.

Definition 1.5. An o-minimal structure $\mathcal{M}$ is called *semibounded* if there is a definable real closed field with bounded domain, and there is no definable real closed field with unbounded domain.

Groups definable in an o-minimal structure attracted a lot of attention after the following theorem was established:

Theorem 1.6. [Pi] *A group definable in an o-minimal structure admits a definable manifold topology that makes it into a topological group.*

Thus, every definable group has its own group topology, and every topological notion about a definable group we mention below is with respect to this group topology.

Definition 1.7. A definable group G is called *definably connected* if it contains no proper definable clopen subset. It is called *definably compact* if for every definable $\sigma : (a, b) \to G$ the limit $\lim_{x\to b^-} \sigma(x)$ exists (in G).

2. Examples and results

We now turn to see definable groups in the three examples we considered: $\mathcal{R}_{vect}$, real closed fields, semibounded structures.

Example 2.1. Let $\mathcal{R}_{vect} = \langle R, <, +, 0, \{x \mapsto rx\}_{r\in R}\rangle$. Let $G_a = \langle [0, a), \oplus, 0\rangle$ be the definable group given by:

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y < a \\ x + y - a & \text{if } x + y \geq a \end{cases}$$

That is, $\oplus = + \mod \mathbb{Z}a$, but the way it is defined above witnesses that $\oplus$ is definable in $\mathcal{R}_{vect}$. If we let $U_a = \bigcup_n [-na, na]$ be the subgroup of $\langle R, +\rangle$ generated by $[-a, a]$, then we have:

$$G_a \cong U_a/\mathbb{Z}a$$

In general,

Theorem 2.2. [ElSt] *Let G be a definably compact, definably connected group definable in R_{vect} with $\dim(G) = n$. Then there are*

- *an open subgroup $U \leqslant \langle R^n, +\rangle$ generated by a definable set*
- *a subgroup $L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n \leqslant U$, where $a_1, \ldots, a_n$ are $\mathbb{Z}$-independent,*

such that

$$G \cong U/L.$$

An L as above is called a *lattice of rank n*.

The proof uses the following lemma.

Local Lemma. *There are $c \in G$ and $V \subseteq G$ such that $\dim V = \dim G$, $c \in V$, and*

$$\forall x, y \in V,\ \ x \ominus c \oplus y = x - c + y$$

Example 2.3. Let $\langle R, <, +, \cdot, 0, 1\rangle$ be a real closed field. We cannot expect that every definable group is quotient U/L of a *subgroup* U of $\langle R^n, +\rangle$ by a lattice. For example, let $G^\times_b = \langle [1, b), \otimes, 1\rangle$ be the definable group given by:

$$x \oplus y = \begin{cases} xy, & \text{if } xy < b \\ xy/b & \text{if } xy \geq b \end{cases}$$

This group cannot be *definably* isomorphic to G_a since such an isomorphism would require the existence of an exponential function. Of course,

$$G_b^{\times} \cong \langle R^{>0}, \cdot \rangle / b^{\mathbb{Z}},$$

but it is unclear what could be a higher dimensional analogue of this isomorphism.

Example 2.4. Let $\mathcal{N} = \langle \mathcal{R}_{vect}, B \rangle$ be a semibounded structure. We again cannot expect that every definable group is quotient of a subgroup of $\langle R^n, + \rangle$ by a lattice. For example, $G_b^{\times}$ is definable in $\mathcal{N}$.

Question 2. (Peterzil 2009): Can groups definable in a semibounded structure be viewed as quotients U/L where U *contains* a subgroup of $\langle R^n, + \rangle$ rather than being itself one?.

For example, the group $G_a \times G_b^{\times}$ is isomorphic U/L where $U = U_a \times G_b^{\times}$. In order to answer Question 2, we need to define an invariant that counts how semilinear our group G is. Let us call an interval $I \subseteq M$ *short* if there is a real closed field definable on it, and *long* otherwise.

Definition 2.5. [El] Let $X \subseteq R^n$ be a definable in $\mathcal{N}$. Then the *linear dimension* of X is

$$\begin{aligned} \operatorname{ldim}(X) = \ & \max\{k : \ X \text{ contains an } k\text{-box } I^k, \text{ up to definable} \\ & \text{bijection, where } I \text{ is long}\}. \end{aligned}$$

Theorem 2.6. [EP] *Let G be a definably compact, definably connected group definable in a semibounded o-minimal structure $\mathcal{N}$ with $\dim(G) = n$ and $\operatorname{ldim}(G) = k$. Then there are*

- *an open subgroup $H \leqslant \langle R^k, + \rangle$ generated by a definable set*
- *a definable group K with $\operatorname{ldim}(K) = 0$,*
- *a lattice $L = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_k \leqslant U$, where $a_1, \ldots, a_k$ are $\mathbb{Z}$-independent.*

such that, if U is the group extension of K by H, then

$$G \cong U/L.$$

The proof uses the following lemma.

Local Lemma. *There are $c \in G$ and $V \subseteq G$ such that $\operatorname{ldim} V = \operatorname{ldim} G$, $c \in V$, and*

$$\forall x, y \in V, \ \ x \ominus c \oplus y = x - c + y$$

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