Distal and Non-Distal Pairs

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Abstract

Our aim is to determine whether certain non-o-minimal expansions of o-minimal theories which are known to be NIP, are also distal. We observe that while tame pairs of o-minimal structures and the real field with a discrete multiplicative subgroup have distal theories, dense pairs of o-minimal structures and related examples do not.

Introduction

Over the last two decades, *NIP* (or *dependent*) theories, first introduced by Shelah, have attracted substantial interest. Properties of these theories have been studied in detail, and many examples of such theories have been constructed. Recently, Simon identified an important subclass of NIP theories called *distal theories*. The motivation behind this new notion is to single out NIP theories that can be considered purely unstable. O-minimal theories, the classical examples of unstable NIP theories, are distal. Here we examine some dependent expansions of o-minimal structures, to see if they are distal.

Let $\mathcal{A} = (A, <, \dots)$ be an o-minimal structure expanding an ordered group and let $B \subseteq A$. We consider theories of structures of the form $(\mathcal{A}, B)$ that satisfy one of the following conditions:

1. $\mathcal{A}$ is the real field and B is a cyclic multiplicative subgroup of $\mathbb{R}_{>0}$ (*discrete subgroup*),
2. $\mathcal{A}$ expands a real closed field, B is a proper elementary substructure such that there is a unique way to define a standard part map from A into B (*tame pairs*),
3. B is a proper elementary substructure of $\mathcal{A}$ dense in A (*dense pairs*),
4. $\mathcal{A}$ is the real field and B is a dense subgroup of the multiplicative group of $\mathbb{R}_{>0}$ with the Mann property (*dense subgroup*),
5. B is a dense, definably independent set (*independent set*).

The Distal Pairs

Suppose T is a $\mathcal{L}$-theory of an o-minimal group with $\mathcal{L}$ containing a constant symbol 1. Further assume that T admits universal axiomatization and has QE. Then let $T(\mathfrak{f})$ be an expansion by a function symbol $\mathfrak{f}$. Then suppose that the following conditions on $T(\mathfrak{f})$ hold:

(i) The theory $T(\mathfrak{f})$ has quantifier elimination.

(ii) For every $(\mathcal{C}, \mathfrak{f}) \models T(\mathfrak{f})$, $\mathcal{B} \preceq \mathcal{C}$ with $\mathfrak{f}(\mathcal{B}) \subseteq \mathcal{B}$ and every $c \in \mathcal{C}^k$, there are $l \in \mathbb{N}$ and $d \in \mathfrak{f}(\mathcal{B}\langle c\rangle)^l$ such that
$$\mathfrak{f}(\mathcal{B}\langle c\rangle) \subseteq \langle \mathfrak{f}(\mathcal{B}), d\rangle.$$

(iii) Let $m \geq n$ and let g, h be $\mathcal{L}$-terms of arities $m+k$ and $n+l$ respectively, $b_1 \in \mathbb{M}^k, b_2 \in \mathfrak{f}(\mathbb{M})^l$, $(a_i)_{i\in I}$ be an indiscernible sequence from $f(\mathbb{M})^n \times \mathbb{M}^{m-n}$ such that
   a. $I = I_1 + (c) + I_2$, where both I_1 and I_2 are infinite and without endpoints, and $(a_i)_{i\in I_1+I_2}$ is b_1b_2-indiscernible,
   b. $a_i = (a_{i,1}, \dots, a_{i,m})$ for each $i \in I$, and
   c. $\mathfrak{f}(g(a_i, b_1)) = h(a_{i,1}, \dots, a_{i,n}, b_2)$ for every $i \in I_1 + I_2$.

   Then $\mathfrak{f}(g(a_c, b_1)) = h(a_{c,1}, \dots, a_{c,n}, b_2)$.

Then $T(\mathfrak{f})$ is distal.

The Non-Distal Pairs

In this section we present sufficient conditions for non-distality of expansions of o-minimal theories by a single unary predicate, and give several examples of NIP theories satisfying these conditions. Let T be an o-minimal theory in a language $\mathcal{L}$ expanding that of ordered abelian groups, U a unary relation symbol not appearing in $\mathcal{L}$, and T_U an $\mathcal{L}(U) = \mathcal{L} \cup \{U\}$-theory expanding T. Let $\mathbb{M}$ be a monster model of T_U. We denote the interpretation of U in $\mathbb{M}$ by $U(\mathbb{M})$. We say that an $\mathcal{L}(U)$-definable subset X of $\mathbb{M}$ is *small* if there is no $\mathcal{L}$-definable (possibly with parameters) function $f : \mathbb{M}^m \to \mathbb{M}$ such that $f(X^m)$ contains an open interval in $\mathbb{M}$. When we say a set is dense in $\mathbb{M}$, we mean dense with respect to the usual order topology on $\mathbb{M}$.
Suppose the following conditions hold:

(i) $U(\mathbb{M})$ is small and dense in $\mathbb{M}$.

(ii) For $n \in \mathbb{N}$, $C \subseteq \mathbb{M}$, and $a, b \in \mathbb{M}^n$ both $\mathrm{dcl}_{\mathcal{L}}$-independent over $C \cup U(\mathbb{M})$,
$$tp_{\mathcal{L}}(a|C) = tp_{\mathcal{L}}(b|C) \Rightarrow tp_{\mathcal{L}(U)}(a|C) = tp_{\mathcal{L}(U)}(b|C).$$

Then T_U is not distal.

Main Result

The theories of expansions by discrete subgroups and tame pairs are distal. Those of dense pairs, dense subgroups, and independent sets are not.

Distality

We say T is *distal* if whenever $A \subseteq \mathbb{M}$, and $(a_i)_{i\in I}$ an indiscernible sequence from $\mathbb{M}^p$ such that

a. $I = I_1 + (c) + I_2$, and both I_1 and I_2 are infinite without endpoints,

b. $(a_i)_{i\in I_1+I_2}$ is A-indiscernible,

then $(a_i)_{i\in I}$ is A-indiscernible.
We will also use a localized notion of this, which is the natural analogue of the previous for a formula.

Proof Sketches

To show the positive result on the distality of certain pairs, we do a technical induction on the number of occurrences of the function symbol $\mathfrak{f}$.
Then to show the non-distality result, we construct an indiscernible sequence witnessing the non distality. The whole sequence will be indiscernible, while the sequence omitting the central element will have further indiscerniblility over a parameter transcendental over the predicate.

An Open Question

We observe the following interesting phenomenon: All examples of the above NIP theories that do not define a dense and codense set, are distal. However, all the examples that define a dense and codense set, are not distal. This not true in general. The expansion $(\mathbb{R}, <, \mathbb{Q})$ of the real line by a predicate for the set of rationals is dp-minimal and hence distal by work of Simon.

Follow Up

Proving a non-distality result is not a dead end. Distality is not preserved under reducts, but some of its consequences are. For example, results of Chernikov and Starchenko give certain combinatorial regularity to distal structures. This regularity is preserved under reducts. Thus it is an interesting question to check whether any of these non-distal examples admit a distal expansion. Another interesting question follows work of Simon, who shows that in any NIP theory, there is a natural decomposition of types into a "Stable" part, and a distal part.

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