A two cardinals theorem for small sets

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Introduction

Fix a monster model $\mathfrak{C}$. Give a classification of definable subsets of $\mathfrak{C}^n$ into “small” or “not small”.
Under some condition on the notion of “smallness”, we show that there is an elementary substructure of $\mathfrak{C}$ whose definable subsets are small iff they are countable.

The main tool is a theorem of J. Keisler on logic with the quantifier “there exist uncountably many”.

Contents

Small sets

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Small sets

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Examples

$\mathfrak{C}$	**Small**
any structure	finite
d-minimal structure	finite union of discrete sets
dense elementary pair of o-minimal structures	A Cartesian product does not cover $\mathfrak{C}$
structure with a dimension function	0-dimensional
superstable structure of infinite U-rank	finite rank
model of arithmetic	bounded
Unrestrained DC structure	definably enumerable

Small axioms

Let $\mathcal{S}$ be the family of small sets.

Reasonable notions of smallness satisfy the following properties:

- **S1** Every small set is a definable set (with parameters);
- **S2** singletons are small, $\mathfrak{C}$ is not small;
- **S3** $\mathcal{S}$ is an ideal: finite union and intersections of small sets are small, definable subset of a small sets are small;
- **S4** cartesian products of small sets are small;
- **S5** $\mathcal{S}$ is invariant under automorphisms;
- **S6** small unions of small sets are small: if $(X_i)_{i \in I}$ is a definable family, each X_i is small, and I is small, then $\bigcup_{i \in I} X_i$ is small.

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In many situations, the following also holds:

- **S7** smallness is a definable property: if $(X_i)_{i \in I}$ is a definable family, then $\left\{ i \in I : X_i \text{ is small} \right\}$ is also definable, with the same parameters as the family.

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Example

If small = finite, then (S7) holds iff $\mathfrak{C}$ eliminates the quantifier $\exists^\infty$.

More generally, (S7) is equivalent to the fact that $\mathfrak{C}$ eliminates the quantifier “there exists a large set of”.

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Remark

Axiom (S5) (invariance under automorphisms) follows from (S7).

Two cardinals

Examples

Lemma

If $\mathfrak{C}$ is d-minimal, there exists an uncountable model M such that all definable discrete subsets of M are countable.

Proof.

Let M be the Cauchy completion of a countable model. □

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Lemma

If $\mathfrak{C}$ is a dense pair of o-minimal structures, there exists an uncountable model $(B; A)$ such that the smaller structure A is countable.

Proof.

Let A be a countable o-minimal structure, and B be its Cauchy completion. □

Two cardinals theorem

From now on, we assume that we have a smallness notion satisfying the previous axioms, and the language is countable.

Theorem (Two cardinals)

There exists $M \prec \mathfrak{C}$ of cardinality $\aleph_1$ such that:
for every $X \subseteq M^n$ definable (with parameters in M),
X is small iff X is countable.

(we say that $X \subseteq M$ is small iff $X(\mathfrak{C})$ is small).

Small extension

Let $A \preceq B \preceq \mathfrak{C}$. B is a small extension of A if, for every X subset of A^n definable with parameters in A, if X is small, then $X(B) = X(A)$.
B is a proper small extension of A if, for every X as above, X is small iff $X(B) = X(A)$.

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Theorem (Small extension)

Every countable model A has a countable proper small extension.

The two cardinal theorem follows by applying the above theorem $\aleph_1$-many times:

Theorem (Small Extensions + Two Cardinals)

Let A be a countable model. There exists $B \geq A$ such that, for every $X \subseteq B^n$ definable in B,

1. *if X is large iff $|X| = \aleph_1$*
2. *if X is small and A-definable, then $X = X(A)$.*

Examples

1. Let $\mathfrak{C}$ be a model of arithmetic, and "small" = "bounded". A small extension is an end-extension. It is well-known that every countable model of arithmetic has proper end-extensions.
2. If small = finite, every elementary extension of A is a small extension.
3. If $\mathfrak{C}$ is d-minimal, an elementary extension B of A is small if does not enlarge any discrete definable subsets, and it is a proper small extension if $B \neq A$.

The main tool

ANNALS OF MATHEMATICAL LOGIC – NORTH-HOLLAND, AMSTERDAM (1970)

LOGIC WITH THE QUANTIFIER “THERE EXIST UNCOUNTABLY MANY”

H. Jerome KEISLER

University of Wisconsin

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This paper concerns the language L(Q) which is formed by adding, to the first order predicate logic L with identity, the additional quantifier (Qx) with the interpretation “there are uncountably many x”. The language L(Q) was first studied by Mostowski [23] who proposed the problem of finding an analogue of the Gödel

Keisler studied first-order logic with the added quantifier Qx = “there exists uncountably many x”.
The main result is that the logic of the quantifier is determined by the Standard Axioms, similar to the one I gave for the notion of smallness:

> In this paper we shall prove that the completeness theorem for L(Q) holds with the following very simple set of axiom schemes:
>
> $$\neg(Qx)(x \equiv y \vee x \equiv z)\ ;$$
>
> $$(\forall x)(\varphi \to \psi) \to ((Qx)\varphi \to (Qx)\psi)\ ;$$
>
> $$(Qx)\,\varphi(x) \leftrightarrow (Qy)\,\varphi(y)\ ;$$
>
> $$(Qy)(\exists x)\varphi \to (\exists x)(Qy)\varphi \vee (Qx)(\exists y)\varphi\ .$$
>
> Precisely these axioms were considered by Craig and Fuhrken in 1962, and they conjectured that the completeness theorem for L(Q) is true with these axioms. We prove their conjecture in §2.

Weak models and standard models

A weak model (M, q) for the $L(Q)$ logic is given by a first-order L-structure M, together with a family q of subsets of M.
The notion of satisfaction of an $L(Q)$ formula for such a model is the “obvious” one.
A weak model is a standard model iff q is the family of uncountable subsets of M.

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It is easy to see that $(\mathfrak{C}; \{\text{large subsets of } \mathfrak{C}\})$ satisfies the standard axioms (the main point is that $\mathfrak{C}$ eliminates the quantifier Q), and therefore it has a standard model.

Omitting types

The proof of the Completeness Theorem and of the existence of proper small extensions is based on some version of the following:

Theorem (Keisler: Omitting types)

Let Γ be a consistent set of $L(Q)$-sentences and Σ be a set of $L(Q)$-formulae in the variable x.
Assume that, for every $L(Q)$-formula $\phi(x)$, if $\exists x \phi(x)$ is consistent with Γ, then there exists $\sigma \in \Sigma$ such that $\exists x(\phi \mathbin{\&} \neg\sigma)$ is also consistent with Γ.
Then, Γ has a countable model omitting Σ.

Open problems

1. Two Cardinals Theorem without (S7).
2. A Two Cardinals Theorem for uncountable cardinals.
3. Given $A \preceq B$, we say that A is dense in B if A intersects every large subset of B.
    1. Give "natural" conditions for the existence of $B \succeq A$ such that B is a proper small extension of A and A is dense in B (that is, A intersects every large subset of B, small subsets of A are unchanged, while large subsets of A are enlarged).
    2. Give conditions for the existence of a completion B of $A \prec \mathfrak{C}$. (i.e., A is dense in B, and for every B' such that A is dense in B', B embeds in B' over A).