Research Statement

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I work in mathematical logic. My main research area is model theory, in particular, o-minimality. In my Ph.D. thesis I analyzed groups definable in linear o-minimal structures. I also maintain an interest in modal logic.

1. Introduction

One of the central aspects of model theory is to study "definability" in mathematical structures. Given a first-order language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$, a subset X of M^n is called *definable (in $\mathcal{M}$ with parameters)* if there is a tuple $\bar{a} \in M^m$ and a formula $\phi(\bar{x}, \bar{a})$ from $\mathcal{L}$ such that $X = \{\bar{b} \in M^n : \phi(\bar{b}, \bar{a}) \text{ holds}\}$. Traditionally, the study of definability has been motivated by the following two facts:

- The definable sets in classical mathematical structures already constitute well-known collections of sets: the constructible sets if $\mathcal{M} = \langle \mathbb{C}, +, \cdot \rangle$, and the semi-algebraic sets if $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot \rangle$, for example.
- In much larger classes of mathematical structures the definable sets still present a nice or "tame" geometry.

The definition of an o-minimal structure isolates a finiteness property that holds for the semi-algebraic subsets of $\mathbb{R}$. A densely linearly ordered structure $\mathcal{M} = \langle M, <, \ldots \rangle$ is called *o-minimal* if every definable subset of M is a finite union of open intervals and points. The thrust of studying o-minimal structures is that the above finiteness property for the definable subsets of M results to finiteness properties for the definable subsets of M^n. In particular, every definable function $f : A \subseteq M^n \to M$ (that is, a function whose graph is a definable set) is piecewise continuous, and every definable subset of M^n is assigned a topological dimension. A standard reference for o-minimality is [vdD].

By a definable group we mean a group whose domain and operation are definable.

The main motivation for my work rises from the intuition that a group G definable in an o-minimal structure $\mathcal{M}$ should "resemble" a real Lie group. The foremost supporting evidence for this intuition was given in [Pi1], where it was shown that every such group G can be equipped with a (unique) "definable manifold" structure over $\mathcal{M}$ that makes it into a topological group. A series of definable analogues of classical theorems in the o-minimal context followed and the intuition was recently formalized in Pillay's conjecture stated below.

Let G be a group definable in an o-minimal structure $\mathcal{M}$. The group G is *definably compact* if for every definable continuous $\sigma : (a, b) \subseteq M \to G$, the limit $lim_{x \to a} \sigma(x)$ exists in G, taken with respect to the manifold topology of G. The structure $\mathcal{M}$ is *saturated* if every consistent set of $< |M|$ many formulas has a realization in M. A set $X \subseteq M^n$ is *type-definable* if it is the intersection of $< |M|$ many definable sets. A subgroup H of G has *bounded index* if $|G/H| < |M|$. If H has bounded index and $\pi : G \to G/H$ denotes the canonical homomorphism, then a set $A \subseteq G/H$ is *closed in the logic topology* if $\pi^{-1}(A) \subseteq G$ is type-definable.

Pillay's conjecture ([Pi2]). *Assume G is a definably compact group definable in a saturated o-minimal structure $\mathcal{M}$, and $\dim(G) = n$. Then there is a smallest type-definable subgroup G^{00} of G of bounded index such that G/G^{00}, when equipped with the logic topology, is a compact Lie group of dimension n.*

In [HPP] it is shown that Pillay's conjecture holds if $\mathcal{M}$ is an o-minimal expansion of an ordered field.

The next conjecture is intended to formalize the intuition that $\pi : G \to G/G^{00}$ is a kind of "standard part map". Let **Haar** denote the unique normalized Haar measure on G/G^{00}.

Compact domination conjecture ([HPP])**.** *Assume G satisfies the assumptions and the conclusion of Pillay's conjecture. Then, for all definable subsets $X \subseteq G$,*

$$\dim(X) < n \Rightarrow \mathbf{Haar}\big(\pi(X)\big) = 0.$$

In this case we say that G is *compactly dominated.*

In [HPP] it is shown that compact domination holds if $\dim(G) = 1$ or if G has "very good reduction".

In my thesis I analyzed groups definable in "linear" o-minimal structures. In this context, I first showed a definable analogue of the classical theorem that every connected abelian Lie group is Lie isomorphic to a direct sum of copies of the additive group $\mathbb{R}_a$ and the circle group S^1. I then applied my analysis and settled positively the above two conjectures for $\mathcal{M}$ a linear o-minimal structure.

2. Results

An o-minimal expansion $\mathcal{M} = \langle M, <, +, \dots \rangle$ of an ordered group is called *linear* ([LP]) if for every definable function $f : A \subseteq M^n \to M$, there is a partition of A into finitely many A_i, such that for each i, if $x, y, x + t, y + t \in A_i$, then

$$f(x + t) - f(x) = f(y + t) - f(y).$$

The main example of a linear o-minimal structure is that of an ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ over an ordered division ring D.

For the rest of this section, unless stated otherwise, $\mathcal{M} = \langle M, <, +, \dots \rangle$ denotes a linear o-minimal expansion of an ordered group, and definability is meant in $\mathcal{M}$ with parameters.

Let G be a definable group of dimension n. Assume G is definably connected, that is, it is not the disjoint union of any two open definable proper subsets. It is known that G is abelian. By Peterzil and Steinhorn, if G is not definably compact, then it has an one-dimensional torsion-free definable subgroup. By induction on the dimension of G, there are definable subgroups $\{0\} = G_0 < G_1 < \dots < G_r \leqslant G$, such that G/G_r is definably compact and G_{i+1}/G_i is an one-dimensional torsion-free group, for $i = 0, \dots, r-1$. By [EdEl1], the torsion-free subgroup G_r of G is definably isomorphic to $M^r = \langle M^r, +, 0 \rangle$. We call G/G_r the *compact part* of G. The dimension of G/G_r is then $s = n - r$.

A *lattice* L of rank $s \leq n$ is a subgroup of $M^n = \langle M^n, + \rangle$ generated by s linearly independent over $\mathbb{Z}$ elements of M^n. If $U \leqslant M^n$ is a subgroup of M^n and $L \leqslant U$ is a lattice, then U/L is called a *definable quotient group* if there is a definable complete set $S \subseteq U$ of representatives for U/L, such that the induced group structure $\langle S, +_S \rangle$ is definable. In this case, we identify U/L with $\langle S, +_S \rangle$.

Let $\{X_k : k < \omega\}$ be a collection of definable subsets of M^n. Assume that $U = \bigcup_{k<\omega} X_k$ is equipped with a binary map $\cdot$ so that $\langle U, \cdot \rangle$ is a group. Then U is called *a $\bigvee$-definable group* if, for all $i, j < \omega$, there is $k < \omega$, such that $X_i \cup X_j \subseteq X_k$ and the restriction of $\cdot$ to $X_i \times X_j$ is a definable function into M^n.

For $m, n \in \mathbb{Z} \setminus \{0\}$, and $x = (x_1, \dots, x_n) \in M$, let $\frac{m}{n}x$ be the unique $y = (y_1, \dots, y_n) \in M^n$ such that for all i, $mx_i = ny_i$. A set $X \subseteq M^n$ is called *convex* if for all $x, y \in X$ and $q \in \mathbb{Q} \cap [0, 1]$, $qa + (1 - q)b \in X$.

Theorem 1 ([El1]). *Let G be a definably connected group definable in a linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume the dimension of G is n, and the dimension of the compact part of G is s. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, +\rangle$ and a lattice $L \leqslant U$ of rank s.*

Theorem 1 has the following corollary.

Theorem 2 ([El1]). *Let G be a definably connected group definable in a saturated linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume the dimension of the compact part of G is s. Then there is a smallest type-definable subgroup G^{00} of G of bounded index such that G/G^{00}, when equipped with the logic topology, is a compact Lie group of dimension s.*

If G is in addition definably compact, then $s = n$ and Theorem 2 is Pillay's conjecture for $\mathcal{M}$ a saturated linear o-minimal expansion of an ordered group.

Using the analysis from the proof of Theorem 1, we show the following stronger version of compact domination.

Theorem 3 ([El2]). *Let G be a definably compact group definable in a saturated linear o-minimal expansion $\mathcal{M}$ of an ordered group. Then for all definable sets $A \subseteq G$ defined in* any *o-minimal expansion of $\mathcal{M}$,*

$$\dim(X) < n \Rightarrow \mathbf{Haar}\big(\pi(X)\big) = 0.$$

The *o-minimal fundamental group* $\pi_1(G)$ of G can be defined as in the classical case except that all paths and homotopies are taken to be definable.

Theorem 4 ([El1]). *Let G be a definably connected group definable in a linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume the dimension of the compact part of G is s. Then $\pi_1(G) \cong L \cong \mathbb{Z}^s$.*

The proofs of Theorems 1, 2, and 4 for G definably compact and $\mathcal{M}$ an ordered vector space over an ordered division ring appear in [ElSt].

Regarding Theorems 1 and 2 for G definably compact and $\mathcal{M}$ archimedean, an independent work was done also by A. Onshuus.

A different definition of the o-minimal fundamental group $\pi(G)$ was given by M. Edmundo for a $\bigvee$-definable group in any o-minimal structure, namely, $\pi(G)$ is the kernel of the "o-minimal universal covering homomorphism" $\widetilde{p} : \widetilde{G} \to G$ of G. We refer the reader to [EdEl2] for any terminology. We show there that the two o-minimal fundamental groups coincide.

Theorem 5 ([EdEl2]). *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and G a definably connected group definable in $\mathcal{M}$, such that $\pi_1(G)$ is countable. Then the o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \to G$ of G is a $\bigvee$-definable covering homomorphism, and $\pi_1(G) = \pi(G)$.*

3. Current and future work

3.1. **Definable quotient groups.** I am currently working on the following two projects. Let $\mathcal{M}$ be a linear o-minimal expansion of an ordered group.

Project. Given a lattice L of rank n generated by $v_1, \ldots, v_n \in M^n$, find necessary and sufficient conditions so that there is a convex $\bigvee$-definable subgroup U of M^n and U/L is a definably compact definable quotient group.

In particular, there is an example of an L that does not give rise to such a U/L. (See [El3, Claim 4.2].) The conditions that I am aiming to obtain are stated in the language of an archimedean valuation for the coordinates of $v_1, \ldots, v_n$. The analysis yields information about the possible shape of a definable complete set of representatives for U/L, and I aiming to use this information to also settle the following.

Project. Show that every group G definable in a linear o-minimal expansion $\mathcal{M}$ of an ordered group admits an affine embedding, that is, G is definably isomorphic to a definable group whose manifold topology is the subspace topology.

The affine embedding has been proved by Berarducci and Otero for $\mathcal{M}$ an o-minimal expansion of an ordered field.

3.2. **Possible generalizations of Theorems 1-3.** I am planning to work on the following.

Project. Show Pillay's conjecture in any o-minimal structure.

By Peterzil and Starchenko, the following trichotomy holds. If $\mathcal{M}$ is an o-minimal structure, then on a neighborhood around each point x of M^n the induced by $\mathcal{M}$ structure is either "trivial", or that of a vector space, or that of a real closed field. If G is a group definable in $\mathcal{M}$ and $x \in G$, the possible cases are the two latter ones. Since we know that Pillay's conjecture holds if $\mathcal{M}$ is globally a vector space or globally a real closed field, the conjecture should be possible to settle for any o-minimal structure $\mathcal{M}$, upon some "splitting". Such a splitting has already been practiced in [PePiS], and I am planning to adopt their argument here.

As far as Theorem 1 is concerned, let G be a group definable in an o-minimal expansion $\mathcal{M}$ of an ordered group. If $\mathcal{M}$ is not linear, then a field can be interpreted in $\mathcal{M}$, and it is easy to see that a straightforward generalization of Theorem 1 directly fails. One possible direction is to demand the critical features of G from the proof of Theorem 1. A *generic* set is a subset of G whose finitely many translations cover G.

Project. Let G be a definably connected group definable in an o-minimal expansion $\mathcal{M}$ of an ordered group. Assume the dimension of G is n, and the dimension of the compact part of G is s. Assume that (i) the compact part of G contains a generic convex set (such as a "parallelogram", [ElSt, Section 4, Step II]), (ii) G is definably locally isomorphic to $\langle M^n, + \rangle$, and (iii) the local isomorphism is uniform for all generic points of G ([ElSt, Corollary 4.4]). Then G satisfies the conclusion of Theorem 1, that is, G is definably isomorphic to a definable quotient group U/L, where $U \leqslant \langle M^n, + \rangle$ and $L \leqslant U$ is a lattice of rank s.

On the other hand, one could omit assumption (ii) from the last project and relax the conclusion so that U is not necessarily a subgroup of $\langle M^n, + \rangle$ but just any group (perhaps torsion-free, convex, or contractible). Moreover, instead of demanding the existence a "nicely shaped" generic subset of G in assumption (i), one could demand the existence of a "nicely measured" subset of G. By [HPP, Theorem 9.5], compact domination implies a "nice" measure on the collection of the definable subsets of G.

Project. Assuming G is compactly dominated, find reasonable conditions so that a version of the conclusion of Theorem 1 holds.

The contractibility conjecture from [B] is another possible assumption.

3.3. **Beyond o-minimal structures.** The resemblance between groups definable in o-minimal and ω-stable structures has intrigued many model-theorists. I am interested in extending my results, as well as other known results, to a more general unifying context.

Project. Find a suitable context where, given a definable group G, the Descending Chain Condition holds (a) for the definable subgroups of G of finite index, or even (b) for the type-definable subgroups of G of bounded index.

A natural such context is that of a geometric theory with the NIP ("Not the Independence Property"). The following is an additional property, common among o-minimal and strongly minimal theories, recently considered by J. Gagelman in his Ph.D. thesis: a pregeometric theory T is called *surgical* if for every definable set X and every definable equivalence relation E on X, at most finitely many E-classes have the same dimension as X.

Project. Show a version of Theorem 1 in a surgical modular geometric theory with the NIP.

In a class report at Notre Dame, I proved that every surgical modular pregeometric theory admits geometric elimination of imaginaries. In a different direction, I am curious to see if a similar result can be proved in more general settings, for example after replacing dimension by other notions of ranks.

3.4. **Zilber's trichotomy.** Zilber's trichotomy conjecture is open for a strongly minimal structure interpretable in an o-minimal structure. A special case was settled positively by Hasson and Kowalski. It has not been checked, however, if Hrushovski's construction of his counterexample to Zilber's general conjecture is possible to adopt here. I am planning, for example, to consider the following.

Project. Show that a non-trivial flat geometry inside an o-minimal structure does not interpret an infinite group.

3.5. **Possible application.** It is known that an algebraically closed valued field (ACVF) interprets both a field and an o-minimal group (that is, an ordered vector space over $\mathbb{Q}$). Groups definable in an ACVF are studied in [Hr]. My interest is to reduce several questions that arise in that context to my analysis of groups definable in linear o-minimal structures.

4. Modal logic

I would like to briefly discuss my interest in many-valued modal logics.

The language of propositional modal logic is that of classical propositional logic augmented by "modal operators". A model of such a language is build upon a Kripke-frame, that is, a directed graph, by defining a valuation of the sentences in each node that reflects the possible-world semantics of the modal operators. In the family of many-valued modal logics introduced in [F2], every edge of a Kripke-frame is labelled with an element of an underlying Heyting algebra $\mathcal{H}$ of truth values, and the definition of a valuation of the sentences into $\mathcal{H}$ is extended appropriately. In particular, if $\mathcal{H}$ is finite, the semantics corresponds to a multiple-expert semantics of interest to Knowledge Representation where every value from $\mathcal{H}$ defines a set of experts. In this equivalent semantics, the value of an edge (or of a sentence, respectively) represents the set of experts that believe the edge exists (or that the sentence is true, respectively).

My research activity in this area consists of transferring results from modal logic to many-valued modal logic. In my master thesis I generalized four classical model-theoretic constructions of modal logic, namely generated subframes, disjoint unions, bounded morphisms and canonical extensions, and proved truth-invariance results under these constructions. The truth-invariance results are relative to a value from $\mathcal{H}$ and they correspond to invariance of the epistemic consensus of a predefined group of experts. The work appears in [EK1].

The importance of the above four constructions stems partly from the Goldblatt-Thomason Theorem which states that a first-order definable class of Kripke frames is modally definable if and only if it is closed under these four constructions.

Project. Show an analogue of the Goldblatt-Thomason Theorem for many-valued modal logics.

A central notion of modal logic, and of its relation with computer science, is that of a "bisimulation" between two Kripke frames. We are currently developing its many-valued analogue in [EK2].

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