The cell decomposition theorem in d-minimal expansions of the real field

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Let $\mathfrak{R}$ be an expansion of the real field.

'Definable' means 'definable in $\mathfrak{R}$ possibly with parameters'.

Cells in $\mathbb{R}^n$ are defined inductively as follows:

- The only cell in $\mathbb{R}^0$ is $\mathbb{R}^0$.
- $C \subseteq \mathbb{R}^{n+1}$ is a cell in $\mathbb{R}^{n+1}$ if (1) C is a graph of a definable continuous function $D \to \mathbb{R}$ where D is a cell in $\mathbb{R}^n$; or (2) there are definable continuous functions $f, g\colon D \to \mathbb{R}$ such that D is a cell in $\mathbb{R}^n$, $f < g$ and $C = (f, g)$.

A cell decomposition of $\mathbb{R}^n$ is defined inductively as follows:

- The only cell decomposition of $\mathbb{R}^0$ is $\{\mathbb{R}^0\}$.
- $\mathcal{D}$ is a cell decomposition of $\mathbb{R}^{n+1}$ if $\mathcal{D}$ is a finite partition of $\mathbb{R}^{n+1}$ that satisfies the following:
  1. for each $S \in \mathcal{D}$, every connected component of S is a cell in $\mathbb{R}^{n+1}$;
  2. for each $S \in \mathcal{D}$, if X_1, X_2 are connected component of S, then $\pi X_1 = \pi X_2$ where $\pi \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection to the first n coordinates
  3. if $\pi \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection to the first n coordinates, then $\{\pi S : S \in \mathcal{D}\}$ is a cell decomposition.

Cell Decomposition Theorem in d-minimal expansions of the real field (T.)

If $\mathfrak{R}$ is d-minimal and $\mathcal{A}$ is a finite collection of definable subsets of $\mathbb{R}^n$, then there is a cell decomposition of $\mathbb{R}^n$ compatible with $\mathcal{A}$.

Dimension and full dimension

For $d \leq n$, let $\Pi(n, d)$ denote the set of all coordinate projections $\mathbb{R}^n \to \mathbb{R}^d$:

$$(x_1, \dots, x_n) \mapsto (x_{i_1}, \dots, x_{i_d})$$

where $1 \leq i_1 < \cdots < i_d \leq n$.
Let $S \subseteq \mathbb{R}^n$ be nonempty.

- $\dim S$ is the largest $d \in \mathbb{N}$ such that πS has interior for some $\pi \in \Pi(n, d)$;
- $\operatorname{fdim} S$ is the ordered pair (d, k) where $d = \dim S$ and k is the cardinality of the set $\{\pi \in \Pi(n, d) : \pi S \text{ has interior}\}$.

Let $\pi \in \Pi(n, d)$ and $S \subseteq \mathbb{R}^n$.

S is a π**-special submanifold** if S is definable and for every $y \in \pi S$, there is a box B about y such that π homeomorphically maps each connected component of $S \cap \pi^{-1}B$ onto B.
S is a **special submanifold** if S is a π-special submanifold for some $\pi \in \Pi(n, \dim S)$.

Decomposition Theorem

Suppose $\mathfrak{R}$ is a d-minimal expansion of the real field. Let $\mathcal{A}$ be a finite collection of definable subsets of $\mathbb{R}^n$. Then there is a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ into special submanifolds compatible with $\mathcal{A}$.

Let $S \subseteq \mathbb{R}^n$ be definable and $\pi \in \Pi(n, \dim S)$. We say S is π-**good** if

- πS is open;
- for every open box $B \subseteq \mathbb{R}^n$, $\pi(S \cap B)$ either has interior or is empty;
- $\operatorname{cl} S \cap \pi^{-1} x = \operatorname{cl}(S \cap \pi^{-1} x)$ and $\dim(S \cap \pi^{-1} x) = 0$ for every $x \in \pi S$.

Let $S \subseteq \mathbb{R}^n$, $d \leq n$ and $\pi \in \Pi(n, d)$.

For $a \in S$, $a \in \operatorname{reg}_\pi S$ iff there is a box B about x such that $\pi{\restriction}(B \cap S)$ homeomorphically maps $B \cap S$ onto an open subset of $\mathbb{R}^d$.

As corollary of the proof of Partition Lemma (C. Miller), we have

Lemma

If $\mathfrak{R}$ is d-minimal, $S \subseteq \mathbb{R}^n$ be definable and $\pi \in \Pi(n, \dim S)$ where πS has interior, then there is a definable, open, and dense $U \subseteq \mathbb{R}^{\dim S}$ such that $S \cap \pi^{-1}U$ is π-good.

Lemma

Suppose $\mathfrak{R}$ is d-minimal. Let $S \subseteq \mathbb{R}^n$ be definable and $\pi \in \Pi(n, \dim S)$ where πS has interior and $S \cap \pi^{-1}x$ is discrete for every $x \in \mathbb{R}^{\dim S}$. Then there is a definable, open, and dense $U \subseteq \mathbb{R}^{\dim S}$ such that $S \cap \pi^{-1}U = \mathrm{reg}_\pi(S \cap \pi^{-1}U)$.

Lemma

Let $S \subseteq \mathbb{R}^n$ be bounded and $\pi \in \Pi(n, d)$ be the projection on the first d coordinates. Suppose S is π-good, $S = \operatorname{reg}_\pi S$, πS is a finite disjoint union of simply-connected sets, and S_x is finite for every $x \in \pi S$. Then for every connected component X of S, πX is a connected component of πS and $\pi{\restriction}X : X \to \pi X$ is a homeomorphism.

$\mathcal{U}(0) = \{\mathbb{R}^0\}$

$\mathcal{U}(n+1)$ = the collection of all open definable $U \subseteq \mathbb{R}^{n+1}$ such that

- the projection πU on the first n coordinates is in $\mathcal{U}(n)$;
- if X is a connected component of U, then X is a cell and πX is a connected component of πU.

Let $0 \leq d \leq n$ and $\pi \in \Pi(n, d)$.

$\mathcal{M}(n, d, \pi)$ = the collection of all definable $M \subseteq \mathbb{R}^n$ for which there are $U_1, \ldots, U_m \in \mathcal{U}(d)$ such that

- $U_1, \ldots, U_m$ are pairwise disjoint;
- $\pi M = U_1 \cup \cdots \cup U_m$;
- for all $x \in \mathbb{R}^d$, $M \cap \pi^{-1}x$ is discrete;
- if X is a connected component of M, then πX is a connected component of πM and $\pi \upharpoonright X : X \to \pi X$ is a homeomorphism.

$\mathcal{M}(n,d) = \bigcup_{\pi \in \Pi(n,d)} \mathcal{M}(n,d,\pi)$.

$\mathcal{M}(n) = \bigcup_{0 \leq d \leq n} \mathcal{M}(n,d)$.

Assume $\mathfrak{R}$ is d-minimal.

Decomposition Theorem (T.)

(I$_n$) If $A \subseteq \mathbb{R}^n$ is definable and bounded, $\dim A < n$ and $\pi \in \Pi(n, \dim A)$, then there exist definable, open $U \subseteq \mathbb{R}^{\dim A}$ and $\mathcal{Q} \subseteq \mathcal{M}(n, \dim A, \pi)$ finite pairwise disjoint such that (1) U is dense in $\mathbb{R}^{\dim A}$, (2) $A \cap \pi^{-1}U = \bigcup \mathcal{Q}$ and (3) for every $Q \in \mathcal{Q}$, the projection under π of each connected component of Q is a connected component of U and $\operatorname{fr} Q \cap \pi^{-1}U$ is a finite union of elements in $\mathcal{Q}$.

(II$_n$) If $\mathcal{A}$ is a finite collection of definable and bounded subsets of $\mathbb{R}^n$, then there is a finite partition $\mathcal{P}$ of $\mathbb{R}^n$ by elements of $\mathcal{M}(n)$ such that $\mathcal{P}$ is compatible with $\mathcal{A}$, and for each $P \in \mathcal{P}$, $\operatorname{fr} P$ is a finite union of elements in $\mathcal{P}$.