Pairs of theories satisfying a Mordell-Lang condition

Alexi Block Gorman
Joint with Philipp Hieronymi and Elliot Kaplan

arXiv:1806.00030

October 3, 2018

Dense Pairs and Lovely Pairs

Definition

*Let $\mathcal{B}$ be an o-minimal $\mathcal{L}$-structure, and $\mathcal{A} \preccurlyeq \mathcal{B}$ a submodel such that $A \subsetneq B$ is dense. Then we call $(\mathcal{B}, \mathcal{A})$ a **dense pair**.*

In [van den Dries,1998], there is a partial Q.E. result for dense pairs; namely, that for $T^d := Th(\mathcal{B}, \mathcal{A})$ in the language $\mathcal{L}^2 := \mathcal{L} \cup \{U\}$ (where $A = U(B)$) the following holds:

Theorem 1 (van den Dries)

Each $\mathcal{L}^2$-formula $\psi(\vec{y})$ is equivalent in T^d to a boolean combination of formulas of the form

$$\exists \vec{x} U(\vec{x}) \wedge \phi(\vec{x}, \vec{y})$$

where $\phi(\vec{x}, \vec{y})$ is an $\mathcal{L}$-formula.

Lovely Pairs

This work inspired Alex Bernstein and Yevgeniy Vasilyev to define the following in [Berenstein and Vassilev, 2010]:

Definition

We say $(M, P(M))$ *is a* ***lovely pair of geometric structures*** *if* $M \models T$ *where* T *is a geometric theory,* $P(M)$ *is an algebraically closed subset of* M*, and the following properties are satisfied:*

- *(Coheir/density) non-algebraic unary types over finite dimensional* acl*-closed parameter sets are realized in the predicate*
- *(Extension/codensity) non-alg. unary types over fin.-dim.* acl*-closed* A *are realized outside of* $\mathrm{acl}(P(M) \cup A)$

$(\mathbb{R}, \mathbb{Q})$ as vector spaces

Let $K \subseteq \mathbb{R}$ be a subfield with $\mathbb{Q} \subsetneq K$. Consider the structure $(\mathbb{R}, \mathcal{Q}, <, 0, +, (\lambda_k)_{k \in K})$ where $\mathcal{Q}$ is a unary predicate that picks out $\mathbb{Q}$, and λ_k is a function symbol that abbreviates $x \mapsto kx$ for each $k \in K$.

$(\mathbb{R}, \mathbb{Q})$ as vector spaces

Let $K \subseteq \mathbb{R}$ be a subfield with $\mathbb{Q} \subsetneq K$. Consider the structure $(\mathbb{R}, \mathcal{Q}, <, 0, +, (\lambda_k)_{k \in K})$ where $\mathcal{Q}$ is a unary predicate that picks out $\mathbb{Q}$, and λ_k is a function symbol that abbreviates $x \mapsto kx$ for each $k \in K$.

Obs. 1: We can't use the dense pairs, lovely pairs, or H-structure frameworks, since $\mathcal{Q}$ is not a K-vector space and not acl-closed, nor is it acl-independent.

$(\mathbb{R}, \mathbb{Q})$ as vector spaces

Let $K \subseteq \mathbb{R}$ be a subfield with $\mathbb{Q} \subsetneq K$. Consider the structure $(\mathbb{R}, \mathcal{Q}, <, 0, +, (\lambda_k)_{k \in K})$ where $\mathcal{Q}$ is a unary predicate that picks out $\mathbb{Q}$, and λ_k is a function symbol that abbreviates $x \mapsto kx$ for each $k \in K$.

Obs. 1: We can't use the dense pairs, lovely pairs, or H-structure frameworks, since $\mathcal{Q}$ is not a K-vector space and not acl-closed, nor is it acl-independent.

Obs. 2: We can treat $\mathcal{Q}$ as a $\mathbb{Q}$-vector space, i.e. a model of a different theory in a sublanguage.

$(\mathbb{R}, \mathbb{Q})$ as vector spaces

Let $K \subseteq \mathbb{R}$ be a subfield with $\mathbb{Q} \subsetneq K$. Consider the structure $(\mathbb{R}, \mathcal{Q}, <, 0, +, (\lambda_k)_{k \in K})$ where $\mathcal{Q}$ is a unary predicate that picks out $\mathbb{Q}$, and λ_k is a function symbol that abbreviates $x \mapsto kx$ for each $k \in K$.

Obs. 1: We can't use the dense pairs, lovely pairs, or H-structure frameworks, since $\mathcal{Q}$ is not a K-vector space and not acl-closed, nor is it acl-independent.

Obs. 2: We can treat $\mathcal{Q}$ as a $\mathbb{Q}$-vector space, i.e. a model of a different theory in a sublanguage.

Theorem 2 (B. G. - Hieronymi - Kaplan)

The theory of pairs $(\mathcal{R}, \mathcal{Q})$ with $\mathcal{R}$ an ordered K-v.s. and $\mathcal{Q}$ and ordered $\mathbb{Q}$-v.s. is near-model complete, and decidable when $\dim(K|\mathbb{Q}) = \infty$ and K has a computable presentation and basis over $\mathbb{Q}$.

ML-Theories

Let T_β be a consistent geometric $\mathcal{L}_\beta$-theory and T_α a consistent $\mathcal{L}_\alpha$-theory. Let $\mathcal{L} := \mathcal{L}_\beta \cap \mathcal{L}_\alpha, \mathcal{L}^2 = \mathcal{L}_\beta \cup \mathcal{L}_\alpha \cup \{U\}$. An *ML*-theory is a $\mathcal{L}^2$-theory T with models $(\mathcal{B}, \mathcal{A})$ where $\mathcal{B} \models T_\beta$, $\mathcal{A} \models T_\alpha$, $A = U(B)$, and:

ML-Theories

Let T_β be a consistent geometric $\mathcal{L}_\beta$-theory and T_α a consistent $\mathcal{L}_\alpha$-theory. Let $\mathcal{L} := \mathcal{L}_\beta \cap \mathcal{L}_\alpha$, $\mathcal{L}^2 = \mathcal{L}_\beta \cup \mathcal{L}_\alpha \cup \{U\}$. An *ML*-theory is a $\mathcal{L}^2$-theory T with models $(\mathcal{B}, \mathcal{A})$ where $\mathcal{B} \models T_\beta$, $\mathcal{A} \models T_\alpha$, $A = U(B)$, and:

- (ML condition) For every algebraic $\mathcal{L}_\beta$-formula $\varphi(\vec{y}, x)$ and every $\vec{a} \in A^{|\vec{y}|}$, there is an $\mathcal{L}_\alpha$-formula $\psi(\vec{y}, x)$ such that $T \models U(x) \to (\varphi(\vec{a}, x) \leftrightarrow \psi(\vec{a}, x))$.

ML-Theories

Let T_β be a consistent geometric $\mathcal{L}_\beta$-theory and T_α a consistent $\mathcal{L}_\alpha$-theory. Let $\mathcal{L} := \mathcal{L}_\beta \cap \mathcal{L}_\alpha, \mathcal{L}^2 = \mathcal{L}_\beta \cup \mathcal{L}_\alpha \cup \{U\}$. An *ML*-theory is a $\mathcal{L}^2$-theory T with models $(\mathcal{B}, \mathcal{A})$ where $\mathcal{B} \models T_\beta$, $\mathcal{A} \models T_\alpha$, $A = U(B)$, and:

- (ML condition) For every algebraic $\mathcal{L}_\beta$-formula $\varphi(\vec{y}, x)$ and every $\vec{a} \in A^{|\vec{y}|}$, there is an $\mathcal{L}_\alpha$-formula $\psi(\vec{y}, x)$ such that $T \models U(x) \to (\varphi(\vec{a}, x) \leftrightarrow \psi(\vec{a}, x))$.
- (Algebraic Extension) technical condition, holds e.g. if $A = \mathrm{acl}_\mathcal{B}(A)$, A is $\mathrm{acl}_\mathcal{B}$-independent, or $\mathrm{acl}_\mathcal{B} = \mathrm{dcl}_\mathcal{B}$.

ML-Theories

Let T_β be a consistent geometric $\mathcal{L}_\beta$-theory and T_α a consistent $\mathcal{L}_\alpha$-theory. Let $\mathcal{L} := \mathcal{L}_\beta \cap \mathcal{L}_\alpha, \mathcal{L}^2 = \mathcal{L}_\beta \cup \mathcal{L}_\alpha \cup \{U\}$. An *ML*-theory is a $\mathcal{L}^2$-theory T with models $(\mathcal{B}, \mathcal{A})$ where $\mathcal{B} \models T_\beta$, $\mathcal{A} \models T_\alpha$, $A = U(B)$, and:

- (ML condition) For every algebraic $\mathcal{L}_\beta$-formula $\varphi(\vec{y}, x)$ and every $\vec{a} \in A^{|\vec{y}|}$, there is an $\mathcal{L}_\alpha$-formula $\psi(\vec{y}, x)$ such that $T \models U(x) \to (\varphi(\vec{a}, x) \leftrightarrow \psi(\vec{a}, x))$.
- (Algebraic Extension) technical condition, holds e.g. if $A = \mathrm{acl}_\mathcal{B}(A)$, A is $\mathrm{acl}_\mathcal{B}$-independent, or $\mathrm{acl}_\mathcal{B} = \mathrm{dcl}_\mathcal{B}$.
- For any κ-saturated $(\mathcal{B}, \mathcal{A}) \models T$, any small $C \subseteq B$, and any non-algebraic $\mathcal{L}_\beta(C)$-type $q(x)$:
  - (Density) if $p(x)$ is a $\mathcal{L}_\alpha(U(C))$-type s.t. $\mathrm{qf}(q|_{\mathcal{L}(U(C))}) = \mathrm{qf}(p|_{\mathcal{L}(U(C))})$, then $p(x) \cup q(x)$ is realized in A
  - (Codensity) there is $b \in B \setminus \mathrm{acl}_\mathcal{B}(A \cup C)$ realizing $q(x)$.

Algebraic Extension

Since we're among friends, here is the precise statement of the Algebraic Extension condition:

Definition

Let C be a finite subset of B with $C \downarrow_{U(C)} A$, and let $a \in \mathrm{acl}(C) \cap A$.
Set $q = \mathrm{tp}_{\mathcal{L}_\beta}(C)$ and set $p = \mathrm{tp}_{\mathcal{L}_\alpha}(U(C))$. Let $\varphi(x)$ isolate $\mathrm{tp}_{\mathcal{L}_\beta}(a|C)$ and let $\psi(x)$ isolate $\mathrm{tp}_{\mathcal{L}_\alpha}(a|U(C))$.
We say that T satisfies the ***Algebraic Extension condition*** *if for all models of T and for all C and a as above:*

$$T \cup q \cup p_U \models \exists x(U(x) \wedge \varphi(x) \wedge \psi_U(x)).$$

Again, this is immediate if A is $\mathrm{acl}_{\mathcal{B}}$-independent, if $A = \mathrm{acl}_{\mathcal{B}}(A)$, or $\mathrm{acl}_{\mathcal{B}} = \mathrm{dcl}_{\mathcal{B}}$.

Known Examples

To justify our choice of definitions, here are some of the known examples that are also ML-theories:

- dense pairs
- lovely pairs
- H-structures
- (K, G) where $K \models ACF$ or RCF, $G \leq K^\times$ is a Mann Group

Known Examples

To justify our choice of definitions, here are some of the known examples that are also ML-theories:

- dense pairs
- lovely pairs
- H-structures
- (K, G) where $K \models ACF$ or RCF, $G \leq K^\times$ is a Mann Group

The name “ML-theory” comes from this last example, as Ayhan Günaydin and Lou van den Dries showed that the Mann property is equivalent to the Mordell-Lang property for such pairs.

Main Theorem

These conditions hit the right level of generality to obtain:

Theorem 3 (B. G. - Hieronymi - Kaplan)

If T is an ML-theory, then each $\mathcal{L}^2$-formula $\theta(\vec{x})$ is equivalent in T to a boolean combination of formulas of the form

$$\exists \vec{y} U(\vec{y}) \wedge \psi(\vec{y}) \wedge \phi(\vec{x}, \vec{y})$$

where $\psi(\vec{y})$ is an $\mathcal{L}_\alpha$-formula and $\phi(\vec{x}, \vec{y})$ is an $\mathcal{L}_\beta$-formula.

Theorem 4 (B. G. - Hieronymi - Kaplan)

An ML-theory T is complete precisely if T_α and T_β are complete.

Open Core

For a structure $\mathcal{M}$ with a definable topology, we define the **open core of** $\mathcal{M}$ to be the structure $\mathcal{M}^\circ := (M; U \in \mathcal{U})$, where $\mathcal{U}$ is the collection of definable open subsets of M^n for all $n \in \mathbb{N}$.

Assume that every open set in the definable topology is empty or infinite, and that Assumption (I) from [Boxall and Hieronymi, 2012] is satisfied.

Theorem 5 (B. G. - Hieronymi - Kaplan)

If T_β is equipped with a definable topology satisfying the above assumptions, then $(\mathcal{B}, \mathcal{A})^\circ = \mathcal{B}^\circ$ for any $(\mathcal{B}, \mathcal{A}) \models T$.

PRC Fields

Let $\mathcal{L}_\alpha := \{0, 1, +, \cdot, -, <_1, <_2, \ldots, <_n\}$. An ***n*-ordered field** is an $\mathcal{L}_\alpha$-structure $\mathcal{K} = (K, \ldots)$ such that $(K, 0, 1, +, \cdot, -, <_i)$ is an ordered field for $i = 1, \ldots, n$.

PRC Fields

Let $\mathcal{L}_\alpha := \{0, 1, +, \cdot, -, <_1, <_2, \dots, <_n\}$. An ***n*-ordered field** is an $\mathcal{L}_\alpha$-structure $\mathcal{K} = (K, \dots)$ such that $(K, 0, 1, +, \cdot, -, <_i)$ is an ordered field for $i = 1, \dots, n$.

Say that $\mathcal{K}$ is a **pseudo real closed field** (PRCF) if $\mathcal{K}$ is an existentially closed n-ordered field. PRC fields form an elementary class, due to van den Dries. Let T_α be the theory of PRCF in the language $\mathcal{L}_\alpha$.

Let $\mathcal{L}_\beta = \{0, 1, +, \cdot, -, <_1\}$ and let T_β be the $\mathcal{L}_\beta$-theory of real closed fields. We let T_n^d be the $\mathcal{L}^2$-theory whose models $(\mathcal{R}, \mathcal{K})$ are such that $\mathcal{R} \models T_\beta$, K is dense in R, and $\mathcal{K} \models T_\alpha$.

RC-PRC Pairs

As any model $\mathcal{K} \models T_\alpha$ is dense in its $<_1$-real closure $\overline{\mathcal{K}}$, the pair $(\overline{\mathcal{K}}, \mathcal{K})$ is a model of T_n^d.

Theorem 6 (B. G. - Hieronymi - Kaplan)

T_n^d is an ML-theory, hence near model complete.

Corollary

Every open set definable in $(\mathcal{R}, \mathcal{K}) \models T_n^d$ is semialgebraic over $\mathcal{R}$.

As far as we know, this is the “smallest” example of a pair of fields which is known to be near model complete.

p-Adics with a Dense Independent Set

The following adapts the work of Philipp Hieronymi, Travis Nell, and Erik Walsberg from o-minimal to p-adic setting:

Let $\mathcal{L}_\beta = \{0, 1, +, \cdot, \mathcal{O}, P_2, P_3, \ldots\}$ and let T_β be the $\mathcal{L}_\beta$-theory $\mathrm{Th}(\mathbb{Q}_p)$ of p-adically closed fields.

For an arbitrary theory T'_α in a relational language $\mathcal{L}'_\alpha$, we define the theory T_α in the language $\mathcal{L}_\alpha := \mathcal{L}'_\alpha \cup \{E\}$ as follows:

- for any $\mathcal{A} \models T_\alpha$, E is an equivalence relation on A^2 with infinite classes
- the $\mathcal{L}'_\alpha$-structure on A is such that each relation $R \in \mathcal{L}'_\alpha$ is E-invarient, and $\mathcal{A}/E \models T'_\alpha$.

p-Adics with a Dense Independent Set

We consider the $\mathcal{L}^2$-theory T_e whose models $(\mathcal{Q}_p, \mathcal{A})$ satisfy:

- $\mathcal{Q}_p \models T_\beta$ and $\mathcal{A} \models T_\alpha$.
- A is dense and $\mathrm{acl}_{\mathcal{Q}_p}$-independent.
- Each equivalence class of E on A is dense in $\mathcal{Q}_p$.

Theorem 7 (B. G. - Hieronymi - Kaplan)

T_e is an ML-theory, hence near model complete.

Corollary

The p-adic open core of $(\mathcal{Q}_p, \mathcal{A}) \models T_e$ is $\mathcal{Q}_p^\circ$.

Neostability Theory and Future Work

We would like to be able to say more about the ML-theory T in the case that T_α and T_β are strongly dependent, NTP$_2$, and so on.

We have already shown that NIP is preserved using machinery adapted from [Chernikov and Simon, 2013]:

Theorem 8 (B. G. - Hieronymi - Kaplan)

If T_α and T_β are both NIP, then T is NIP.

Neostability Theory and Future Work

We would like to be able to say more about the ML-theory T in the case that T_α and T_β are strongly dependent, NTP$_2$, and so on.

We have already shown that NIP is preserved using machinery adapted from [Chernikov and Simon, 2013]:

Theorem 8 (B. G. - Hieronymi - Kaplan)

If T_α and T_β are both NIP, then T is NIP.

Thank you!

arXiv:1806.00030

References I

Alexander Berenstein and Evgueni Vassilev (2010)
On lovely pairs of geometric structures
Ann. Pure Appl. Logic 161(7), 866 – 878.

Gareth Boxall and Philipp Hieronymi (2012)
Expansions which introduce no new open sets
Journal of Symbolic Logic 77(1), 111 – 121.

Artem Chernikov and Pierre Simon (2013)
Externally definable sets and dependent pairs
Israel Journal of Mathematics 194(1), 409 – 425.

Lou van den Dries (1998)
Dense pairs of o-minimal structures
Fundamenta Mathematicae 157(1), 61 – 78.