On expansions of the real field by complex subgroups

Erin Caulfield

University of Illinois at Urbana-Champaign

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Previous work

Expansions of the real field $\overline{\mathbb{R}}$ by subgroups of $\mathbb{C}^\times$ have been studied previously. Examples include:

- $(\overline{\mathbb{R}}, \Gamma)$, Γ an infinite finite rank subgroup of $\mathbb{S}^1$: Belegradek and Zilber, *The model theory of the field of reals with a subgroup of the unit circle* (2008)
- $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$: van den Dries, *The field of reals with a predicate for the powers of two* (1985)
- $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$: Günaydın, *Model theory of fields with multiplicative groups* (2008)

Motivation

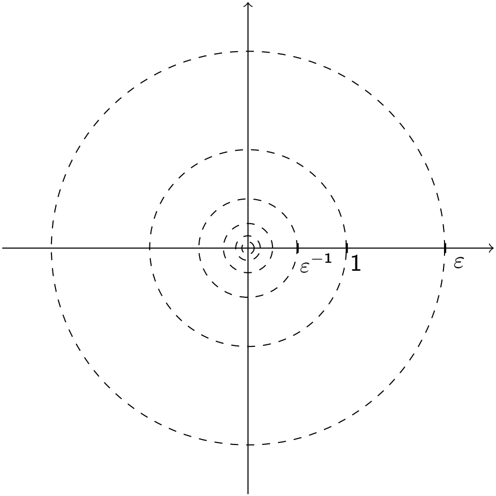
Theorem (Hieronymi, 2010)

Let S be an infinite cyclic subgroup of $(\mathbb{C}^\times, \cdot)$. Then exactly one of the following holds:

1. *$\mathbb{Z}$ is definable in $(\overline{\mathbb{R}}, S)$*
2. *$(\overline{\mathbb{R}}, S)$ is d-minimal*
3. *Every open definable set in $(\overline{\mathbb{R}}, S)$ is semialgebraic*

If S is a finite rank subgroup of $\mathbb{S}^1$, then $(\overline{\mathbb{R}}, S)$ satisfies (3). However, it was not known whether arbitrary finite rank subgroups of $\mathbb{C}^\times$ must satisfy one of (1)-(3).

In this talk, Γ will be a finite rank subgroup of $\mathbb{S}^1$ which is dense in $\mathbb{S}^1$ and Δ will be a subgroup of $\mathbb{R}$ of the form $\varepsilon^{\mathbb{Z}}$ for some $\varepsilon > 1$.



Theorem A

Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$ is a Boolean combination of sets of the form

$$\{x \in \mathbb{R}^m : \exists y \in (\Gamma\Delta)^n \text{ s.t. } (x,y) \in W\}$$

for some semialgebraic set $W \subseteq \mathbb{R}^{m+2n}$. Moreover, every open definable set in $(\overline{\mathbb{R}}, \Gamma\Delta)$ is definable in $(\overline{\mathbb{R}}, \Delta)$.

Let $\Gamma = (e^{i\varphi})^{\mathbb{Z}}$ for some $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ and let $\Delta = \varepsilon^{\mathbb{Z}}$. From Theorem A, it follows that $(\overline{\mathbb{R}}, \Gamma\Delta)$ does not satisfy any of (1)-(3).

$(\overline{\mathbb{R}}, \Gamma\Delta)$ does not define $\mathbb{Z}$

Let $X \subseteq \mathbb{R}$ be definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$. By Theorem A, X is a Boolean combination of sets $X_1, \ldots, X_k$, where for $i \in \{1, \ldots, k\}$,

$$\begin{aligned} X_i &= \{x \in \mathbb{R} : \exists y \in (\Gamma\Delta)^{n_i} \text{ s.t. } (x, y) \in W_i\} \\ &= \bigcup_{y \in (\Gamma\Delta)^{n_i}} \{x \in \mathbb{R} : (x, y) \in W_i\} \end{aligned}$$

for some $n_i \geq 1$ and semialgebraic set W_i. Each X_i is an F_σ set, and so X is Borel. But if $(\overline{\mathbb{R}}, \Gamma\Delta)$ defines $\mathbb{Z}$, then $(\overline{\mathbb{R}}, \Gamma\Delta)$ defines every projective subset of $\mathbb{R}$.

$(\overline{\mathbb{R}}, \Gamma\Delta)$ is not d-minimal

Let P be the projection of $\Gamma\Delta$ onto the real line. By density of $(e^{i\varphi})^{\mathbb{Z}}$ in $\mathbb{S}^1$, P is dense in $\mathbb{R}$. Since $\Gamma\Delta$ is countable, P is codense in $\mathbb{R}$. So $(\overline{\mathbb{R}}, \Gamma\Delta)$ is not d-minimal.

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Not every open set definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$ is semialgebraic

The complement of Δ in $\mathbb{R}^{>0}$ is open and definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$. Moreover,

$$\mathbb{R}^{>0} \setminus \Delta = \bigcup_{k\in\mathbb{Z}} (\varepsilon^k, \varepsilon^{k+1})$$

so by o-minimality of $\overline{\mathbb{R}}$, $\mathbb{R}^{>0} \setminus \Delta$ cannot be definable in $\overline{\mathbb{R}}$.

Since both Γ and Δ are definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$, we now consider $(\overline{\mathbb{R}}, \Gamma, \Delta)$.

The next theorem gives an axiomatization of

$$(\overline{\mathbb{R}}, \Gamma, \Delta, (\delta)_{\delta\in\Delta}, (\gamma)_{\gamma\in\Gamma}).$$

In this theorem, let K be a real closed field. Let G be a dense subgroup of $\mathbb{S}^1(K)$ with $\gamma \mapsto \gamma' : \Gamma \to G$ a group homomorphism. Let A be a subgroup of $K^{>0}$ with a group homomorphism $\delta \mapsto \delta' : \Delta \to A$ such that

1. ε' is the smallest element of A greater than 1, and
2. for every $k \in K^{>0}$, there is $a \in A$ such that $a \leq k < a\varepsilon'$.

Theorem B

$$(K, G, A, (\delta')_{\delta\in\Delta}, (\gamma')_{\gamma\in\Gamma}) \equiv (\overline{\mathbb{R}}, \Gamma, \Delta, (\delta)_{\delta\in\Delta}, (\gamma)_{\gamma\in\Gamma})$$

if and only if:

1. *for every $\gamma \in \Gamma$ and $n \in \mathbb{Z}^{>0}$, γ is an nth power in Γ if and only if γ' is an nth power in G;*
2. *for all primes p, $[p]\Gamma = [p]G$;*
3. *$(K, (\gamma'\delta')_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the orientation axioms for $\Gamma\Delta$;*
4. *$(K, GA, (\gamma'\delta')_{\gamma\in\Gamma,\delta\in\Delta})$ satisfies the Mann axioms for $\Gamma\Delta$;*
5. *all torsion points of G are in Γ.*

Theorem A is proved using Theorem B.

Orientation axioms

Definition

For $n \geq 1$, let $Q(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ and let $\gamma\delta := (\gamma_1\delta_1, \dots, \gamma_n\delta_n) \in (\Gamma\Delta)^n$. The orientation axiom for $\gamma\delta$ and Q is the sentence

$$Q(\mathrm{Re}(\gamma_1\delta_1), \dots, \mathrm{Re}(\gamma_n\delta_n)) > 0$$

if this holds in $\mathbb{R}$, and otherwise it is the sentence

$$Q(\mathrm{Re}(\gamma_1\delta_1), \dots, \mathrm{Re}(\gamma_n\delta_n)) \leq 0.$$

The set of orientation axioms for $\Gamma\Delta$ is the collection of these sentences for each $n \geq 1$, each $Q \in \mathbb{Z}[x_1, \dots, x_n]$, and each tuple $\gamma\delta \in (\Gamma\Delta)^n$.

The Mann property

Definition

Let K be a field of characteristic 0 and G a subgroup of $K^\times$. For $a_1, \ldots, a_n \in \mathbb{Q}$ ($n \geq 1$), a nondegenerate solution to

$$a_1x_1 + \ldots + a_nx_n = 1 \tag{$*$}$$

is a tuple $(g_1, \ldots, g_n) \in G^n$ such that

$$a_1g_1 + \ldots + a_ng_n = 1$$

and $\sum_{i \in I} a_ig_i \neq 0$ for each nonempty subset $I \subseteq \{1, \ldots, n\}$. G has the *Mann property* if every equation of the form $(*)$ has only finitely many nondegenerate solutions in G.

Definition

Let $\mathcal{L} := \mathcal{L}_{or}(P, V, \Gamma, \Delta)$ be the language of ordered rings, together with a binary relation symbol P, a unary relation symbol V, and names for each $\delta \in \Delta$ and $\gamma \in \Gamma$.

Theorem B

To prove Theorem B, we prove the following more general result.

Theorem B

Let

$$\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta}) \text{ and } \mathcal{N} := (L, H, B, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$$

be two models of the $\mathcal{L}$-theory T. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if

- $[p]G = [p]H$ *for all primes* p, *and*
- *for all* $\gamma \in \Gamma$, *and* $n \geq 1$, γ *is an nth power in* G *iff* γ *is an nth power in* H.

Definition

Let H, G be arbitrary groups with $H \leq G$. H is said to be *pure in* G if $H \cap G^{[m]} = H^{[m]}$ for all $m \geq 1$.

Definition

Let E, F be field extensions of a field k with $E, F \subseteq K$ for some field K. We say that E and F are *free over* k if any set $S \subseteq E$ which is algebraically dependent over F is algebraically dependent over k.

Let $\mathrm{Sub}(K, G, A)$ be the collection of $\mathcal{L}_{or}(P, V)$-structures (K', G', A') such that:

1. K' is a real closed subfield of K of cardinality less than κ
2. G' is a pure subgroup of G containing Γ
3. A' is a pure subgroup of A containing Δ
4. $K'(i)$ and $\mathbb{Q}(GA)$ are free over $\mathbb{Q}(G'A')$
5. For all $k' \in (K')^{>0}$, there is $a' \in A'$ such that $a' \leq k' < a'\varepsilon$.

Let $\mathcal{I}$ be the collection of isomorphisms between members of $\mathrm{Sub}(K, G, A)$ and $\mathrm{Sub}(L, H, B)$ that fix Δ and Γ pointwise.

Proof that $\mathcal{I}$ is a back-and-forth system

Let $a \in K \setminus K'$ and let $\iota \in \mathcal{I}$. We want to extend ι to $\iota' \in \mathcal{I}$ such that $a \in \mathrm{dom}(\iota')$. We prove this by considering four cases:

1. $a \in A$
2. $a \in \mathrm{Re}(G)$ or $a \in \mathrm{Im}(G)$
3. $a \in K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))$
4. $a \in K \setminus K'(\mathrm{Re}(GA) \cup \mathrm{Im}(GA))$

Theorem A: Quantifier reduction

Theorem A: Part I

Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$ is a Boolean combination of sets of the form

$$\{x \in \mathbb{R}^m : \exists y \in (\Gamma\Delta)^n \textit{ s.t. } (x, y) \in W\} \qquad (*)$$

for some semialgebraic set $W \subseteq \mathbb{R}^{m+2n}$.

Since Γ and Δ are definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$, we prove that every subset of $\mathbb{R}^m$ is definable in $(\overline{\mathbb{R}}, \Gamma, \Delta)$ is a Boolean combination of sets of the form $(*)$.

To prove this, we prove the following stronger theorem.

Theorem

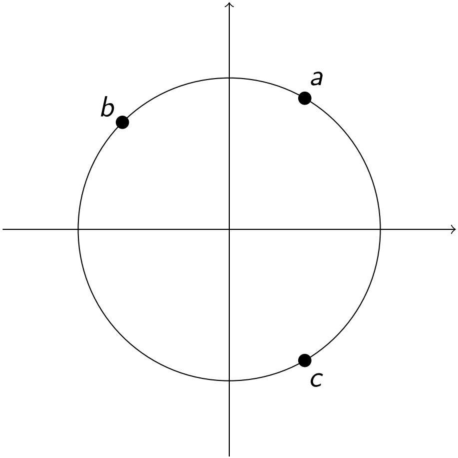
Let $\mathcal{M} := (K, G, A, (\gamma)_{\gamma\in\Gamma}, (\delta)_{\delta\in\Delta})$ *be a model of* T *such that* $[p]G$ *is finite for each prime* p. *Every subset of* K^m *definable in* M *is a Boolean combination of subsets of* K^m *defined in* $\mathcal{M}$ *by formulas of the form*

$$\exists y \exists z (V(y) \wedge P(z) \wedge \phi(x, y, z))$$

where $\phi(x, y, z)$ *is a quantifier free* $\mathcal{L}_{or}(K)$*-formula.*

Since Γ is a finite rank subgroup of $\mathbb{C}^\times$, $\left|\Gamma/\Gamma^{[n]}\right|$ is finite for each $n \geq 1$.

Next we define an orientation $\mathcal{O}$ on $\mathbb{S}^1(K)$. In this picture, $\mathcal{O}(a, b, c)$ holds.



Definition

Let $\mathcal{L}_{orm} = \{\mathcal{O}, 1, \cdot\}$. Let $\Gamma \subseteq G \subseteq \mathbb{S}^1(K)$. Let $x = (x_1, \ldots, x_n)$ and let $z = (z_{11}, z_{12}, \ldots, z_{n1}, z_{n2})$. For each $\mathcal{L}_{orm}(\Gamma)$-formula $\phi(x)$, there is an $\mathcal{L}_{or}(\Gamma)$-formula $\psi_\phi(z)$ such that for all $(a_1, \ldots, a_n) \in G^n$ with $a_i = (a_{i1}, a_{i2})$,

$$(G, \mathcal{O}, 1, \cdot) \models \phi(a_1, \ldots, a_n) \text{ if and only if}$$

$$(K, <, +, -, 0, 1, \cdot) \models \psi_\phi(a_{11}, a_{12}, \ldots, a_{n1}, a_{n2}).$$

Let

$$\Sigma_{orm}(\Gamma) = \{\psi_\phi : \phi \text{ an } \mathcal{L}_{orm}(\Gamma)\text{-formula}\}.$$

Definition

A *special* $\mathcal{L}_{or}(P, V)$-formula in x is a formula $\psi(x)$ of the form

$$\exists y \exists z (V(y) \wedge P(z) \wedge \theta^1_V(y) \wedge \theta^2_P(z) \wedge \phi(x, y, z)).$$

The following is the main lemma used in proving quantifier reduction.

Main lemma

Every $\mathcal{L}$-formula $\psi(x)$ is equivalent in T to a Boolean combination of special $\mathcal{L}$-formulas in x.

Future work

Currently we are studying expansions of $\overline{\mathbb{R}}$ by groups of the form $S := (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$, where $\varphi \notin 2\pi\mathbb{Q}$ and $a^{\mathbb{Z}} b^{\mathbb{Z}}$ is dense in $\mathbb{R}^{>0}$. Let $g : a^{\mathbb{Z}} b^{\mathbb{Z}} \to (e^{i\varphi})^{\mathbb{Z}}$ be defined by $g(a^n b^m) = e^{i\varphi n}$. We consider the structure

$$(\overline{\mathbb{R}}, b^{\mathbb{Z}}, a^{\mathbb{Z}} b^{\mathbb{Z}}, (e^{i\varphi})^{\mathbb{Z}}, g).$$

Question

Can we find a theory T and conditions for two models $\mathcal{M}, \mathcal{N}$ of T to be elementarily equivalent as in Theorem B?

Question

Let G be a finitely generated subgroup of $\mathbb{C}^\times$ and consider $(\overline{\mathbb{R}}, G)$. When can we analyze the definable sets of $(\overline{\mathbb{R}}, G)$ using methods similar to the ones used in Theorem A and Theorem B?