TAME EXPANSIONS OF o-MINIMAL STRUCTURES

I. HISTORY

- $(\mathbb{R}, 2^{\mathbb{Z}})$ – van den Dries ('85)
  (Axiomatization, quantifier elimination (near model completeness), structure of definable sets, d-minimal open core.)

- Dense pairs of o-minimal structures – vdDries ('98)
  $(\mathcal{M}, +, <, \dots)$ o-minimal group, $\mathcal{N} \precneqq \mathcal{M}$, N dense in M.
  (Same properties as above except o-minimal open core instead of d-minimal open core.)

- $(\mathbb{R}, G)$ $G \leq \mathbb{R}^{>0}$ multiplicative group of finite rank.
  $(\mathbb{R}, 2^{\mathbb{Q}})$, $(\mathbb{R}, 2^{\mathbb{Z}} 3^{\mathbb{Z}})$, ... vdDries-G. (2006)
  Same properties...

- More axiomatic approach – o-minimal + other assumptions
  Berenstein-Ealy-G. (2007)
  Observes imaginaries. TAME PAIRS

- $(\mathbb{R}, 2^{\mathbb{Q}} 3^{\mathbb{Q}}, 2^{\mathbb{Q}})$ G. (2008-Thesis)
  Asks: $(\mathbb{R}, 2^{\mathbb{Z}} 3^{\mathbb{Z}}, 2^{\mathbb{Z}})$? or equivalently $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$?

- $(\mathbb{R}$, power functions, $G)$ (G as above and doesn't interact much with the power functions)
  Hieronymi (2009)

- G.-Hieronymi (2011) – Similar setting to BEG & proves NIP.

- $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$ is no good Hieronymi (2011)

II. SETTING

T o-minimal theory extending the theory of ordered abelian groups, in a language L.

Definable means $L(P)$-definable

P a new predicate ; $L(P) = L \cup \{P\}$.

$\tilde{T}$ an $L(P)$-Theory extending T with the following properties holding for all $\tilde{\mathcal{M}} = (\mathcal{M}, P) \models \tilde{T}$:

(I) P is small: $\mathcal{M} \models T$, $X \subseteq M^m$ is large if there is an L-definable $f : (M^m)^n \to M$ s.t $f(X^n)$ has nonempty interior.

X is small if not large. (X is not necessarily def. in this def., but in the rest of the talk it will always be.)

- The definition is equivalent even if we allow $L(P)$-def. functions.
- When we assume P is small, the other small sets are exactly images of P under $L/L(P)$ - definable maps. They are also the same as P-internal sets.

(scl, sdim Here?)

(II) Near Model Completeness : Every definable $X \subseteq M^n$ is a boolean combination of sets of the form

$$\{x \in M^n : \exists z \in P^m \, \varphi(x,z)\}$$

where $\varphi(x,z)$ is an L-formula.

(III) O-Minimal Open Core : For every definable set $V \subseteq M^n$, its topological closure $cl(V)$ is L-definable.

OR definable open sets are L-definable.

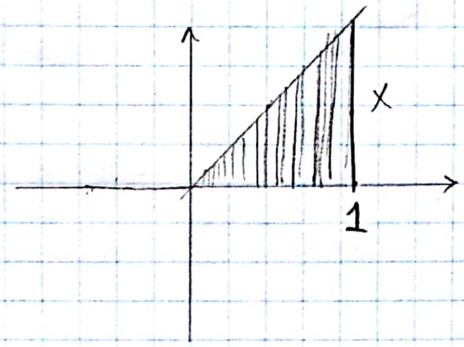
- It follows that P is dense in a finite union of intervals.
- All the examples above where the predicate is dense satisfy this condition.

III. RESULTS

Best result would be that every definable set is decomposed into sets of the form "small × L-definable" (up to definable bijections). However, this can't really be correct:

done in L-def

Example: $X = \bigcup_{\substack{a \in P^{>0} \\ 0<a<1}} \{a\} \times (0,a)$



$(0,a) \overset{\text{def. bij}}{\approx} (0,1)$

$t \longmapsto t/a$

$X \approx \{(x,y) : \underset{0<x<1}{x \in P},\ 0<y<1\} = (P \cap (0,1)) \times (0,1)$.

$(a,t) \longmapsto (a, t/a)$

BUT: $t \mapsto t/a$ might not be def.

Even when the theory extends RCF, we could construct examples.

The real problem is amorphousness.

So we come up with a new kind of "basic" sets: CONES.

First a stronger concept:

Def: $J \subseteq M^k$; define J being a supercone by recursion on k:

(i) $M^0 = \{*\}$ is a supercone.

(ii) $J \subseteq M^{n+1}$ is a supercone if the interior $U = cl(J)^{\circ}$ is an open cell, $\pi(J)$ is a supercone and for every $a \in \pi(J)$, J_a is contained in U_a and is co-small in it.

($\pi : M^{n+1} \to M^n$ projection onto first n coordinates.)

Def: $C \subseteq M^n$ is a k-cone if there are $S \subseteq M^m$ a small set, a uniform family $\mathcal{J} = (J_g)_{g \in S}$ of supercones in M^k and L-def. continuous $h : V \subseteq M^{m+k} \to M^n$ such that $\mathcal{J} \subseteq V$, $C = h(\mathcal{J})$ and for every $g \in S$, $h(g,-)$ is injective.

Note that the example above is a cone.

Theorem 1: Every definable set is a finite union of cones.

(Unfortunately, this is not a decomposition.)

large-dimension

Let's consider another example. Let $(\mathcal{M}, P)$ be a dense pair of o-minimal fields. Let $S = P + Pa$ for some $a \notin P$.

Define $f : M \longrightarrow M$

$$x \longmapsto \begin{cases} 0 & : \text{ if } x \notin S \\ r & : \text{ if } x = r + sa \in S \quad (r, s \in P) \end{cases}$$

Now $X = \Gamma(f)$ is a union of a small set $\{(x,r) : \exists s \in P \ \ x = r + sa\}$ and the set $\{(x,0) : x \notin P + Pa\}$ which is not L-definable but is co-small in M. (Both of them are cones.)

Also $\Gamma(f)$ is dense in M^2!

What can we say about the function f? It coincides with an L-def function on $M \setminus S$. We prove this happens in general:

<u>Theorem 2:</u> $f : X \subseteq M^n \longrightarrow M$ definable. There is a finite collection $\mathcal{C}$ of definable cones with $\bigcup_{C \in \mathcal{C}} C = X$ and f is (fiber) L-definable on each $C \in \mathcal{C}$.

<u>Idea of the ~~proofs~~ in 1-dimension</u> $(n = 1)$ ~

Let $X \subseteq M$ be defined by a formula $\varphi(x) : \exists \vec{y} \in P^m \ \psi(x, \vec{y})$ where $\psi(x, \vec{y})$ is an L-formula. (The general case is under control using condition (II).)

We need to find a partition $-\infty = a_0 \leq a_1 \leq \cdots \leq a_t \leq a_{t+1} = \infty$ of M such that for each $i = 0, \dots, t$ we have $X \cap (a_i, a_{i+1})$ is either small or co-small (in (a_i, a_{i+1})).

Let $S = \psi(\mathcal{M}) \subseteq M^{m+1}$. Then S is a union of L-cells. Also for a given $a \in P^m$, we may assume that S_a is either a point $\{s_a\}$ or an open interval $(f(a), g(a))$. (We may even choose f & g to be continuous functions)

Then X is a union of an image of P^m under an L-def. function, and an open set. By (III), this open set is a finite union of open intervals.

Theorem 3: Let $(G,*)$ be a group definable in $\tilde{\mathcal{M}}$. Suppose that G is of large dimension k. Then for every scl-generic $a \in G$, there is a $2k$-cone $C \subseteq G \times G$ containing (a,a) s.t. the operation $(x,y) \mapsto x * a^{-1} * y$ is given by an L-definable function on C.