Applications of o-minimality to some problems in Diophantine Geometry

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Some Bibliography

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Survey papers

T. Scanlon, *A proof of the André–Oort conjecture via mathematical logic [after Pila, Wilkie and Zannier]*, Sèminaire BOURBAKI Avril 2011 63ème année, 2010–2011, no 1037.

T. Scanlon, *Counting special points: Logic, diophantine geometry, and transcendence theory*, Bull. AMS (N.S.) 49 (2012), no. 1, 51 – 71.

An exercise

Setting

Let $\mathcal{C} =$ the family of all cosets of $\mathbb{C}$-linear subspaces of $\mathbb{C}^n$.

Let $\mathcal{S} \subseteq \mathcal{C}$ all cosets $H + b$, such that H has a basis in $\mathbb{Q}^n$ and $b \in \mathbb{Q}^n$. Call these "**special varieties**".

Let $\mathcal{S}_0 =$ all 0-dimensional $X \in \mathcal{S}$ (note: $\mathcal{S}_0 = \mathbb{Q}^n$). Call these "**special points**".

Problem

If $X \in \mathcal{C}$ and the special points are (Zariski) dense in X (i.e. $X \cap \mathcal{S}_0$ dense in X) then X is special (i.e. $X \in \mathcal{S}$).

Solution

An exercise. Also, find a "quantitative" assumption* on $X \cap \mathbb{Q}^n$ which ensures that X is special (e.g. $\#$ of points of height ... is ...).

A general problem scheme

Setting

$\mathcal{C}$ = an underlying family of sets

$\mathcal{S} \subseteq \mathcal{C}$ a marked collection of so-called “special” $\mathcal{C}$-sets

$\mathcal{S}_0$ = a set of so-called “special” points, often these are the $\mathcal{S}$-sets of dimension zero.

The problem scheme

Start with an ambient $\mathcal{S}$-set V and consider an arbitrary $\mathcal{C}$-set $X \subseteq V$. Assume that X has “many” special points ($X \cap \mathcal{S}_0$ is Zariski dense in V).
Show that X contains a special set of positive dimension. Under additional assumptions, show that X itself is a special set.

The Pila-Wilkie results (viewed in this scheme)

Fix $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, ... \rangle$ an o-minimal expansion of the real field.

$\mathcal{C} =$ the family of all definable sets in $\mathcal{M}$.

$\mathcal{S} =$ The family of connected semi-algebraic sets (defined over $\mathbb{Q}$).

$\mathcal{S}_0 =$ points in $(\mathbb{Q}^{alg})^n \cap \mathbb{R}^n$.

The Pila-Wilkie theorem(s)

Assume that $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{M}$. If $X \cap (\mathbb{Q}^{alg})^n$ is *large** then X contains a connected infinite semi-algebraic set defined over $\mathbb{Q}$. More precisely, if one removes **all** infinite connected semi-algebraic subsets of X then a *small** number of $\mathbb{Q}^{alg}$-points remains.

$X \cap (\mathbb{Q}^{alg})^n$ is **large*** if exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

$$limsup_T \frac{|\{\bar{q} \in X \cap (\mathbb{Q}_k^{alg})^n \,:\, height_k(\bar{q}) \leqslant T\}|}{T^\epsilon} = \infty.$$

From now on-the algebraic general problem scheme

The algebraic presentation

$\mathcal{C} =$ a family of complex algebraic (irreducible) varieties, (quasi) affine or projective.

$\mathcal{S} =$ a specified subfamily of "special" varieties.

$\mathcal{S}_0 =$ 0-dimensional $\mathcal{S}$-sets: special points.

$V =$ an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ an irreducible complex algebraic subvariety (so $X \in \mathcal{C}$)

Assumption

The special points ($X \cap \mathcal{S}_0$) are Zariski dense in X.

Goal

The variety X is special ($X \in \mathcal{S}$).

A test case-the multiplicative group (algebraic torus)

The algebraic side

Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X$ an irreducible algebraic variety$\}$.

$\mathcal{S} = \{p * A : A$ a conn. algebraic subgrp of $\mathbb{G}_m^n$ & p a torsion point$\}$.

$\mathcal{S}_0 =$ Torsion points in $(\mathbb{G}_m)^n$

Goal-a theorem of Laurent-1984)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = p * A$ for some connected $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.
Namely,
If $X \in \mathcal{C}$ and $X \cap \mathcal{S}_0$ is Zariski dense in X then $X \in \mathcal{S}$.

The Pila-Zannier strategy-vague description

First note: The field $\langle\mathbb{C},+,\cdot\rangle$ is definable in $\mathbb{R}$, (via $\mathbb{C}\sim\mathbb{R}^2$). Hence, every complex algebraic variety is definable in the o-minimal structure $\bar{\mathbb{R}}=\langle\mathbb{R},<,+,\cdot\rangle$.

But the strategy will force us to move to a different o-minimal structure:

An analytic presentation of the algebraic problem

As we'll see, in all cases there is a natural analytic covering map, call it $\Theta:\widetilde{V}\to V$, from an open set $\widetilde{V}\subseteq\mathbb{C}^n$ onto the algebraic variety V.

The idea-vague description

1. Using Θ, translate the algebraic problem from V to a problem about sets in $\widetilde{V}$, definable in some o-minimal expansion $\mathcal{M}$ of $\bar{\mathbb{R}}$.
2. Apply a Pila-Wilkie theorem in $\mathcal{M}$.
3. Use it to come back to V and conclude the result there.

Back to $(\mathbb{C}^*)^n$: the analytic presentation

- We have $V = (\mathbb{C}^*)^n = \mathbb{G}_m^n$.
- Take $\widetilde{V} = \mathbb{C}^n$ and $\Theta : \mathbb{C}^n \to (\mathbb{C}^*)^n$ defined by

$$\Theta(\bar{z}) = exp(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n}).$$

- So $\Theta : (\mathbb{C}^n, +) \to ((\mathbb{G}_m)^n, *)$ is a holomorphic group homomorphism.
- Let $\Gamma := Ker(\Theta) = (2\pi i\mathbb{Z})^n$. So, Θ is Γ-invariant, namely:

$$\forall \gamma = (\gamma_1, ..., \gamma_n) \in \Gamma \quad \Theta((z_1 + \gamma_1, ..., z_n + \gamma_n) = \Theta(z_1, ..., z_n).$$

“Special analytic” points and varieties

$\widetilde{\text{special}}$ points

Call $\bar{z} \in \mathbb{C}^n$ a $\widetilde{\text{special}}$ point **if** $\Theta(\bar{z})$ is a torsion element (namely if $\Theta(\bar{z})$ is a special point). Let $\widetilde{\mathbb{S}_0} =$ all $\widetilde{\text{special}}$ points.

exp is a homomorphism, so $\Theta(\bar{z})$ is a torsion point of order k iff $k\bar{z} \in \Gamma$.

So, $\widetilde{\mathbb{S}_0} := \{\bar{z} \in \mathbb{C}^n : \exists k \; k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

$\widetilde{\text{special}}$ varieties

An irreducible **analytic** $Y \subseteq \mathbb{C}^n$ is $\widetilde{\text{special}}$ if $\Theta(Y) = p * A$, where A is a connected algebraic subgroup of $(\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$. Namely, if $\Theta(Y) \in \mathbb{S}$.

So, $Y = \bar{q} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$, and $\bar{q} \in (2\pi i\mathbb{Q})^n$.

The multiplicative group-“weakly special”

Weakly special varieties

$Y \subseteq \mathbb{C}^n$ is called **weakly special** if $Y = \bar{z} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$ (but $\bar{z}$ arbitrary).
And $\Theta(Y) = \Theta(H) * \Theta(\bar{z}) \subseteq (\mathbb{C}^*)^n$ is called **a weakly special** subvariety. It is (an arbitrary) coset of a conn. algebraic subgroup of $(\mathbb{C}^*)^n$

Note

If $Y \subseteq \mathbb{C}^n$ is weakly special then both Y and $\Theta(Y)$ are algebraic varieties (although $\Theta = exp$ is a transcendental map).

A key observation

If $Y \subseteq \mathbb{C}^n$ and $\Theta(Y) \subseteq (\mathbb{C}^*)^n$ are both algebraic varieties then necessarily Y is weakly special. (will prove it later in certain settings)

Multiplicative group continues

Analytic presentation of Laurent's theorem

Assume that $Y \subseteq \mathbb{C}^n$ is an irreducible **analytic** variety, and $X = \Theta(Y) \subseteq (\mathbb{C}^*)^n$ is an algebraic variety, on which the torsion points are Zariski dense. Then Y is special, namely $Y = \bar{q} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$, and $\bar{q} \in (2\pi i\mathbb{Q})^n$.

The Pila-Zannier method

- Using the fact that $X \subseteq \mathbb{C}^n$ has many torsion points we shall conclude that $Y \subseteq (\mathbb{C}^*)^n$ has large*-number of "$2\pi i\mathbb{Q}^n$-points".
- Using Pila-Wilkie for Y, we shall conclude that Y contains an infinite semialgebraic subset of Y, and then also and algebraic subset A.
- Using ideas such as **the key observation** we conclude that A is special, so $\Theta(A)$ a special subvariety of X.
- With slightly more work, $\Theta(A) = X$.

The multiplicative case I. the (non)definability of Θ

We have $\Theta : \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $\Theta(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n})$.

The difficulty

Because $\Gamma = ker\Theta \subseteq \mathbb{C}^n$ is infinite and discrete, the map Θ, as well as $\Theta^{-1}(X)$ **cannot be definable in any o-minimal structure.**

We thus need to "truncate" Θ:

Fundamental sets

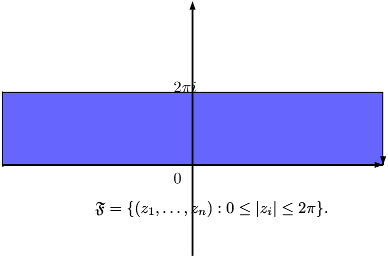
A **fundamental set** for Θ, is a set $\mathfrak{F} \subseteq \mathbb{C}^n$, such that (1) $\mathfrak{F} + \Gamma = \mathbb{C}^n$ ($\Rightarrow\ \Theta(\mathfrak{F}) = (\mathbb{C}^*)^n$.)

and (2) Only finitely many Γ-translates of $\mathfrak{F}$ intersect $Cl(\mathfrak{F})$ (a technical requirement).

- Here is one:

$$\mathfrak{F} = \{\bar{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : 0 \leqslant |Im(z_i)| \leqslant \pi\}.$$

The multiplicative case: the definability of $\Theta \restriction \mathfrak{F}$



$\mathfrak{F} = \{(z_1, \ldots, z_n) : 0 \leq |z_i| \leq 2\pi\}.$

$\Theta \restriction \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$:

We have $e^z = e^{x+iy} = e^x(cosy + i\, siny)$.

The map e^x is definable in $\mathbb{R}_{exp}$; the maps $cos, sin \restriction [0, 2\pi]$ are definable in $\mathbb{R}_{an}$, hence:

The map $e^z \restriction \{0 \leqslant Im(z) \leqslant 2\pi\}$ is definable in $\mathbb{R}_{an,exp}$.

It follows that $\Theta \restriction \mathfrak{F} = exp \restriction \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.
In particular, $\Theta^{-1}(X) \cap \mathfrak{F}$ is definable.

The multiplicative case II. From infinite to “large”

We assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is infinite.

Claim The set $\Theta^{-1}(X) \cap 2\pi i \mathbb{Q}^n \cap \mathfrak{F}$ is large*:

Proof X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.

• Since $X \cap Tor(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \ldots$ and elements $g_i \in X$, with $ord(g_i) = m_i$.

• Assume $g = (g_1, \ldots, g_n) \in X \subseteq (\mathbb{C}^*)^n$, and $ord(g) = m$. Then each g_j is an d_j-th primitive root of unity, with $d_j | m$ and $l.c.m(d_j) = m$.
• (By some basic Galois theory) $[\mathbb{Q}(g) : \mathbb{Q}] = \phi(m)$, where $\phi(m) = \#\{k \leqslant m : (k, m) = 1\}$ is the Euler totient function.
• So g has $\phi(m)$ conjugates over $\mathbb{Q}$, all lying in X.

• So, for every m_i, X contains at least $\phi(m_i)$-many torsion elements of order m_i.

Proof of Claim continues

Fact about Euler's function $\lim_m \phi(m)/m^{1/2} = \infty$.

- Hence, $\lim_i \frac{|\{g \in X : ord(g) = m_i\}|}{m_i^{1/2}} = \infty$.

- Let's move to the analytic side: recall that $\Theta(\bar{z})$ a torsion element of order m iff $\bar{z} = 2\pi i(r_1/s_1, \ldots, r_n/s_n)$, with $s_1, \ldots, s_n \mid m$.

- Because $\Theta^{-1}(X)$ is Γ-invariant, we can find such $\bar{z}$ in $\Theta^{-1}(X) \cap \mathfrak{F}$,
- hence each $r_i < 1$, so $ht(\bar{z}) \leqslant m$.
- So for each m_i there are $\phi(m_i)$-many $\bar{z} \in \Theta^{-1}(X) \cap \mathfrak{F}$ of $ht \leqslant m_i$.

Corollary

The following set is large*

$$\{\bar{q} = (q_1, \ldots, q_n) \in \mathbb{Q}^n : 2\pi i \bar{q} \in \Theta^{-1}(X) \cap \mathfrak{F}\}$$

□

The multiplicative case III from algebraic to (weakly) special

The Pila-Wilkie input

- The analytic set $\Theta^{-1}(X) \subseteq \mathbb{C}^n$ contains an infinite semi algebraic set S.
- The Zariski closure of S is a complex algebraic subset of $\Theta^{-1}(X)$, of positive dimension.
- Take a maximal such irreducible algebraic set $A \subseteq \Theta^{-1}(X)$.
- Note that $\Theta(A)$ is contained in the algebraic set X.
- We are in the realm of "key observation".

Goal: A **is weakly** $\widetilde{\textbf{special}}$ $=$ a coset of a linear s.space of $\mathbb{C}^n$ over $\mathbb{Q}$.

A proof of III using the classical Ax-Lindemann theorem

A-L Theorem

If $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ and $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$ then $tr.deg(\mathbb{C}(e^{\xi_1}, \ldots, e^{\xi_n})/\mathbb{C}) = m$.

Proof using A-L

- Take $H \subseteq \mathbb{C}^n$ a minimal subspace $/\mathbb{Q}$ with $A \subseteq H + p$ for $p \in \mathbb{C}^n$. Let $m = \dim H$.
- We have $\Theta(A) \subseteq \Theta(H) \star \Theta(p)$, and $\Theta(H) \leqslant (\mathbb{C}^*)^n$ algebraic.
- If $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ coordinate functions then $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$, so by Ax $tr.deg(\Theta(\bar{\xi}))/\mathbb{C}) = m = \dim(\Theta(H) * \Theta(p))$.
- Hence, $\Theta(A)$ is Zariski dense in $\Theta(H) * \Theta(p)$. So, $\Theta(H) * \Theta(p) \subseteq X$ (otherwise, $\Theta(\bar{z}) \in (\Theta(H) * \Theta(p)) \cap X$ has smaller dimension).
- Hence, $H + p \subseteq \Theta^{-1}(X)$, and recall $A \subseteq H + p$.
- By maximality, $A = H + p$, so A is weakly special. $\square$

Summary of proof in the multiplicative case

• We started with $X \subseteq (\mathbb{G}^m)^n$ such that $Tor(\mathbb{G}_m)^n \cap X$ is Zariski dense in X.

• Using number theory we concluded that $\Theta^{-1}(X)$ contains large*-many rational points.

• Using Pila-Wilkie, we concluded that $\Theta^{-1}(X)$ contains a nontrivial complex algebraic set A. Furthermore we can choose it so $A \cap \widetilde{\mathbb{S}}_0$ is nonempty. Take such A maximal.

• By Ax, A is weakly $\widetilde{\text{special}}$, hence $\widetilde{\text{special}}$ ($A \cap \widetilde{\mathbb{S}}_0 \neq \varnothing$).

• It follows that X contains a nontrivial special set $\Theta(A)$.

• By using the full strength of Pila-Wilkie we could show that X **is** actually special.

The general Pila-Zannier method

Recall the general problem scheme:
$\mathcal{C} =$ a family of complex algebraic (irreducible) varieties, (quasi) affine or projective.

$\mathcal{S} =$ a specified subfamily of "special" varieties.

$\mathcal{S}_0 =$ 0-dimensional $\mathcal{S}$-sets: special points.

$V =$ an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ an irreducible complex algebraic subvariety (so $X \in \mathcal{C}$)

Assumption

The special points ($X \cap \mathcal{S}_0$) are Zariski dense in X.

Goal

The variety X is special ($X \in \mathcal{S}$) (or at least contains a special variety).

An analytic presentation

An analytic covering map

In all our settings we have $\widetilde{V} =$ a (semi-algebraic) open subset of $\mathbb{C}^n$ (with $n = \dim V$). And $\Theta : \widetilde{V} \to V$ a **holomorphic**, **transcendental,** surjection.

General strategy

Instead of V and $X \subseteq V$ consider $\widetilde{V}$ and the complex analytic subvariety $\Theta^{-1}(X) \subseteq \widetilde{V}$.

Caution

In general, Θ and $\Theta^{-1}(X)$ are not definable in any "tame" structure. We will need to "truncate" it.

The analytic presentation: additional data

An underlying group action

We have $G =$ a real algebraic group acting semi-algebraically and transitively on $\widetilde{V}$. In some cases $\widetilde{V} = G$.

Γ= an infinite discrete subgroup of G (not necessarily normal).

The map $\Theta : \widetilde{V} \to V$ is Γ-invariant. Namely, $\Theta(x) = \Theta(y)$ if and only if $\Gamma x = \Gamma y$.

So, V can be identified with $\Gamma\backslash\widetilde{V}$.

If $X \subseteq V$ is a complex algebraic subvariety then $\Theta^{-1}(X) = \widetilde{X}$ is a Γ-invariant **analytic subvariety** of $\widetilde{V}$.

In general, $\widetilde{X}$ might have infinitely many connected components.

Special and $\widetilde{\text{special}}$ varieties and points

From special to $\widetilde{\textit{special}}$

An irreducible analytic subvariety $Y \subseteq \widetilde{V}$ **is called a $\widetilde{\textbf{special}}$ variety** if $\Theta(Y)$ is a special subvariety of V. In particular, $\Theta(Y)$ is algebraic (**!**).

A point $z \in \widetilde{V}$ is $\widetilde{\textbf{special}}$ if $\Theta(z)$ is a special point. Namely $\Theta(z) \in \mathcal{S}_0$.

Fact (an alternative definition): $\widetilde{\text{special}}$ varieties as orbits

An irreducible complex analytic variety $\widetilde{X} \subseteq \widetilde{V}$ is $\widetilde{\text{special}}$ iff
(i) $\Theta(\widetilde{X})$ is an algebraic subvariety of V.
(ii) There exists a real algebraic subgroup $H \subseteq G$ such that $\widetilde{X}$ is an orbit of H. In case $\widetilde{V} = G$ it means that $\widetilde{X}$ is a coset. (Note: it follows in either case that $\widetilde{X}$ is real algebraic).
(iii) $\widetilde{X} \cap \widetilde{S}_0 \neq \varnothing$.

- If only (i) and (ii) hold then $\widetilde{X}$ is called **weakly** $\widetilde{\textbf{special}}$.

The ingredients for the Pila-Zannier method

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\mathcal{S}_0 \subseteq V$ the set of special points.

I. Definability requirements (from algebraic to o-minimal)

One needs to establish the existence of a semialgebraic fundamental set $\mathfrak{F} \subseteq \widetilde{V}$ for Γ and the definability of $\Theta \restriction \mathfrak{F}$ in some o-minimal structure $\mathcal{M}$. In all examples, $\mathcal{M}$ is $\mathbb{R}_{an,exp}$.

For $X \subseteq V$ algebraic, let $\widetilde{X} \subseteq \widetilde{V}$ be an irreducible analytic component of $\Theta^{-1}(X)$. Note that $\widetilde{X} \cap \mathfrak{F} = (\Theta \restriction \mathfrak{F})^{-1}(X)$ is definable in $\mathcal{M}$.

II. Number theory goal

- The set $\widetilde{\mathcal{S}}_0 = \Theta^{-1}(\mathcal{S}_0)$ is contained in $\mathbb{Q}_k^{alg}$ for some k.

► If $X \cap \mathcal{S}_0$ is Zariski dense in X then $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*. This is "the lower bound".

The ingredients of the Pila-Zannier method (cont)

The Pila-Wilkie input

- Assume that we established that $\widetilde{\mathcal{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*.
- By PW, There exists a connected semi-algebraic nontrivial curve $C \subseteq \widetilde{X} \cap \mathfrak{F}$.
- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C. It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \widetilde{V}) \subseteq \widetilde{X}$.
- So $\widetilde{X}$ **contains a complex algebraic curve** (relative to the open semialgebraic $\widetilde{V}$).

The Pila-Zannier method: The punch-line!

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\widetilde{X} \subseteq \widetilde{V}$ a component of $\Theta^{-1}(X)$.

Ingredient III, the "Ax-Lindemann" goal

Assume that $\widetilde{A}$ is a maximal irreducible algebraic (relative to $\widetilde{V}$) subset of $\widetilde{X}$.

Then $\widetilde{A}$ is a weakly $\widetilde{\text{special}}$ variety. Namely,

(i) $\widetilde{A}$ is an orbit of a real algebraic subgroup of G (defined over $\mathbb{Q}$).

(ii) $\Theta(\widetilde{A})$ is an algebraic subvariety of V.

Summary of the Pila-Zannier method

We have $X \subseteq V$, $\Theta : \widetilde{V} \to V$ and $X \cap \mathcal{S}_0$ Zariski dense in X.

I. Definability

$\Theta \restriction \mathfrak{F}$ is definable in an o-minimal structure.

II. Number Theory

The set $\widetilde{S_0} \cap (\Theta^{-1}(X) \cap \mathfrak{F})$ is large*.

Application of the Pila-Wilkie Theorem.

III. Ax-Lindemann

If $\underbrace{\widetilde{A} \subseteq \Theta^{-1}(X)}$ is maximal irreducible algebraic then $\overbrace{\text{it is weakly}}$ special. (So, if in addition $\widetilde{A} \cap \widetilde{\mathcal{S}_0} \neq \varnothing$ then $\widetilde{A}$ is special).

We conclude: X contains a special variety $\Theta(\widetilde{A})$.

Another application of the Pila-Zannier method: The Manin-Mumford conjecture

Background: Abelian varieties

- An (complex) **abelian variety** is a smooth projective algebraic variety $V \subseteq \mathbb{P}^n(\mathbb{C})$, together with an algebraic binary operation $*$, which makes $\langle V, * \rangle$ an algebraic group.
- In complex dimension one, these are non-singular elliptic curves: $y^3 = x^2 + ax + b$ (the affine equation).
- In higher dimensions, explicit formulas are complicated.

Important properties of abelian varieties

- The group $(V, *)$ is (indeed) abelian, written as $(V, +)$ from now on.
- (over $\mathbb{C}$) The group $(V, +)$ admits the structure of a complex Lie group. Since $\mathbb{P}^n(\mathbb{C})$ is compact that group is compact.
- As a real Lie group, $(V, +)$ is isomorphic to a direct product of $(S^1)^{2m}$, where $m =$ complex dimension of V.
- For every $m \in \mathbb{N}$, the m-torsion subgroup of V is $(\mathbb{Z}/m\mathbb{Z})^{2n}$.

The Manin-Mumford conjecture

The setting

$V =$ an abelian variety in $\mathbb{P}^n(\mathbb{C})$, written additively $(V, +)$.

$\mathcal{C} =$ all irreducible algebraic subvarieties of V.

$\mathcal{S} =$ all cosets of the form $A + p$, where $p \in Tor(V)$ and A a connected algebraic subgroup (i.e. abelian subvariety) of V.

$\mathcal{S}_0 = Tor(V)$ the torsion elements of the group $(V, +)$.

The Manin-Mumford conjecture (Raynaud's Theorem, 1983)

Assume that V is a complex abelian variety defined over a number field, and $X \subseteq V$ an irreducible algebraic subvariety. If $X \cap Tor(V)$ is Zariski dense in V then $X = A + p$ as above.

The analytic presentation

- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \to V$.
- $\Gamma := Ker(\Theta)$ is a $2n$-lattice. I.e., $\Gamma = \Sigma_{i=1}^{2n} \mathbb{Z}\omega_i$, where $\omega_1, \ldots, \omega_{2n}$ are linearly independent over $\mathbb{R}$.
(Note: While every $2n$-lattice gives rise to a complex torus, it might not give rise, if $n > 1$, to an **projective** complex torus, i.e. abelian variety.)
- **special points** $= \Theta^{-1}(Tor(V)) = \mathbb{Q}\Gamma = \Sigma_{i=1}^{2n} \mathbb{Q}\omega_i$.
- **special varieties** are cosets of the form $\bar{z} + H$, where H a complex linear subspace defined over $\mathbb{Q}$ and $\bar{z} \in \mathbb{Q}\Gamma$.
- **Weakly special varieties** are arbitrary cosets of such H.

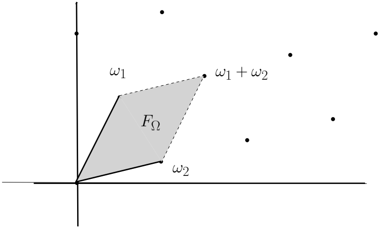
(weakly) special varieties as orbits

The weakly special varieties are exactly those orbits (i.e., cosets) of real subgroups of $(\mathbb{C}^n, +)$ which project onto algebraic subvarieties of V.

The Pila-Zannier method for Manin-Mumford

I. The fundamental set and definability of $\Theta \restriction \mathfrak{F}$

Consider the compact semilinear parallelogram
$\mathfrak{F} = \{\sum_{i=1}^{2n} t_i\omega_i : 0 \leqslant t_i \leqslant 1\}$. Then:
(i) $\Gamma + \mathfrak{F} = \mathbb{C}^n$.
(ii) The set $\{\gamma \in \Gamma : (\gamma + \mathfrak{F}) \cap \mathfrak{F} \neq \varnothing\}$ is finite.
$\mathfrak{F}$ is a fundamental set for Θ.



Since Θ is analytic and $\mathfrak{F}$ compact, $\Theta \restriction \mathfrak{F}$ is definable in the o-minimal $\mathbb{R}_{an}$.

Pila-Zannier for Manin-Mumford (cont)

II. Number Theory (on the algebraic side)

- V is an abelian variety defined over a number field F.
- $X \subseteq V$ is irreducible algebraic, with $X \cap Tor(V)$ Zariski dense in X.
- So, X is also defined over a number field $k \supseteq F$.

Number theoretic input (Masser)

There exists $\rho = \rho(V) > 0$ and a constant c, such that for every $p \in V$, if $ord(p) = T$ then $[F(p) : \mathbb{Q}] \geqslant cT^{\rho}$.

By conjugating $X \cap Tor(V)$ over k we conclude: if $\epsilon < \rho(V)$ then

$$limsup_T \frac{|\{P \in X : ord(P) \leqslant T\}|}{T^{\epsilon}} = \infty.$$

Conclusion: on the analytic side

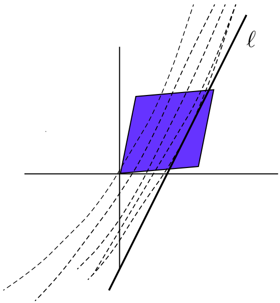
The set $\{(q_1, \ldots, q_{2n}) \in \mathbb{Q}^{2n} : \Sigma_{j=1}^{2n} q_j \omega_i \in \Theta^{-1}(X) \cap \mathfrak{F}\}$ is **large***.

III. Ax-Lindemann: an o-minimal argument

The Pila-Wilkie input

The analytic variety $\Theta^{-1}(X)$ contains an unbounded semialgebraic curve σ.

By the o-minimality of σ, when we translate it into $\mathfrak{F}$ by elements of Γ we get (inside $\widetilde{X}$) curves which are more and more "linear". Since $\widetilde{X} \cap \mathfrak{F}$ is compact, at the limit we get an affine line $\ell \subseteq \widetilde{X}$.



Finishing the proof of MM

On the analytic side

We saw that $\Theta^{-1}(X)$ contains a real affine line $\ell \subseteq \mathbb{C}^n$.

Back to the algebraic side

The variety $X \subseteq V$ contains a coset of a subgroup $\Theta(\ell)$.

The Zariski closure of $\Theta(\ell)$ is a coset of an algebraic subgroup of V, that is contained in X.

Hence, X contains a (weakly) special variety $z + A$, for $A \leqslant X$.

By using the full strength of Pila-Wilkie, we can show that X itself is a special variety.
END of the proof of Manin-Mumford.

Andre-Oort setting

The general analytic setting for Shimura varieties (simplified)

- $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.
- $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This is our $\widetilde{V}$.
- $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$, and consider the quotient $V = \Gamma \backslash \widetilde{V}$.

The Baily-Borel Theorem

There exists a holomorphic embedding $\Theta : \Gamma \backslash \widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

$Im(\Theta) = V$ is ***a Shimura variety*** (a non-specialist viewpoint).

André-Oort for $\mathbb{C}^n$: Preliminaries

We start with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.

The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ (transitively) as follows:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in \mathbb{H}$ then $A \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

Connection to elliptic curves

$\mathbb{H}$ is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$.

$E_{\tau_1} \cong E_{\tau_2} \Leftrightarrow \tau_1, \tau_2$ are in the same $SL(2, \mathbb{Z})$-orbit. So, $SL(2, \mathbb{Z})\backslash\mathbb{H}$ is the moduli space of elliptic curves.

The J-invariant

There exists a holomorphic, transcendental surjection $J : \mathbb{H} \to \mathbb{C}$ such that $J(\tau_1) = J(\tau_2) \Leftrightarrow SL(2, \mathbb{Z})\tau_1 = SL(2, \mathbb{Z})\tau_2$.

André-Oort for $\mathbb{C}^n$

We now begin on the analytic side

- $\widetilde{V} = \mathbb{H}^n$.
- $G(\mathbb{R}) = SL(2,\mathbb{R})^n$ acting on $\mathbb{H}^n$ in coordinates.
- The action is transitive so $\mathbb{H}^n = G(\mathbb{R})/stab_G(\bar{z})$ for any $\bar{z} \in \mathbb{H}^n$.
- Since $O(2,\mathbb{R})^n = stab(i,\dots,i)$, we have $\mathbb{H}^n = SL(2,\mathbb{R})^n/O(2,\mathbb{R})^n$ (namely, $K = O(2,\mathbb{R})^n$).

Note: $\widetilde{V}$ is not a group anymore. It is a semialgebraic homogenous space.

- Let $\Gamma = SL(2,\mathbb{Z})^n$ and $\Theta := (J,\dots,J) : \mathbb{H}^n \to \mathbb{C}^n$. Θ is a Γ-invariant surjection.

On the algebraic side

We let $V = \mathbb{C}^n \sim \Gamma\backslash\mathbb{H}^n$, via Θ.

Special varieties and points

Again, the definition begins on the analytic side.

Definition of special points: The set $\widetilde{\mathbb{S}}_0$

$(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is **special**, if for every i, the elliptic curve E_{τ_i} has complex multiplication ($End(E_\tau) \neq \mathbb{Z}$).
Equivalently, τ_i belongs to an imaginary quadratic extension of $\mathbb{Q}$.
(abstract definition of special points in Shimura varieties-omitted here).

Definition of special varieties

Recall: An irreducible analytic variety $Y \subseteq \mathbb{H}^n$ is **special** if
(i) Y is an orbit of a real algebraic group $H \leqslant SL(2, \mathbb{R})^n$.
(ii) $\Theta(Y) \subseteq \mathbb{C}^n$ is an algebraic variety.
(iii) $Y \cap \widetilde{\mathbb{S}}_0 \neq \varnothing$.

Special varieties and points in $V = \mathbb{C}^n$

The image under Θ of a $\widetilde{\text{special}}$ point is **special** in $\mathbb{C}^n$. $\mathbb{S}_0 := \Theta(\widetilde{\mathbb{S}}_0)$.

The Image under Θ of $\widetilde{\text{special}}$ variety is **special** in $\mathbb{C}^n$.

Examples of special varieties

• $\widetilde{X} = \{\tau\} \times \mathbb{H}^{n-1}$, with $\tau \in \widetilde{\mathbb{S}}_0$; it is an orbit of $H = \{1\} \times SL(2, \mathbb{R})^{n-1}$.
$\Theta(\widetilde{X}) = \{p\} \times \mathbb{C}^{n-1}$ is a special variety.

• $\widetilde{X} = \{(\tau, N\tau) : \tau \in \mathbb{H}\} \times \mathbb{H}^{n-2}$, for some $N \in \mathbb{N}$. It is an orbit of $H_1 \times SL(2, \mathbb{R})^{n-2}$, with $H_1 = \{(g, hgh^{-1}) : g \in SL(2, \mathbb{R})\}$ and

$$h = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$$

$\Theta(\widetilde{X}) = Z(\Phi_N) \times \mathbb{C}^{n-2}$ where Φ_N is the zero set of a modular polynomial.

Moonen's work

Every special variety in $\mathbb{C}^n$ is obtained from the above examples by permutation of variables and cartesian products.

The statement of theorem

The André-Oort Conjecture for $\mathbb{C}^n$ (a theorem of Pila)

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety and $X \cap \mathcal{S}_0$ is Zariski dense in X then X is special.

Notice that by the nature of the definitions, we immediately have an analytic presentation of the problem:

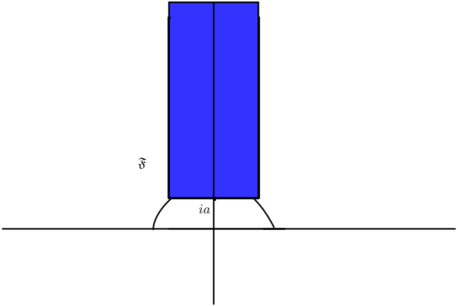
- We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$ given by the J function in each coordinate.
- We have notions of $\widetilde{\text{special}}$ points and varieties in $\mathbb{H}^n$.

The Pila Zannier method: I. The fundamental set

By the basic theory of elliptic curves, the following is a fundamental set for $SL(2,\mathbb{Z})$ (for every $0 < a < \sqrt{3}/2$):

$$\mathfrak{F} = \{z \in \mathbb{H}; -1/2 \leqslant Re(z) \leqslant 1/2 \,\&\, Im(z) > a\}.$$

So $\mathfrak{F}^n$ is a fundamental set for $SL(2,\mathbb{Z})^n$.

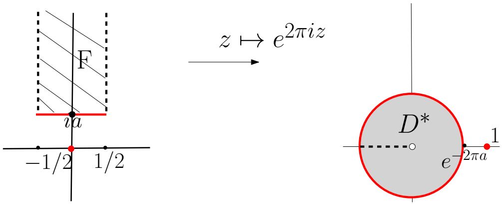


Pila-zanner method I: Definability of $J \restriction \mathfrak{F}$

Theorem

The restriction of J to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

Proof Consider first the map $z \mapsto e^{2\pi iz}$. It sends $\mathfrak{F}$ onto a punctured disc D^*. The "point" $Im(z) = \infty$ is sent to 0.



The proof continues

Because J is $\mathbb{Z}$-periodic ($z \mapsto z+1 \ \in SL(2,\mathbb{Z})$) it factors through $e^{2\pi iz}$.

$$\begin{tikzcd}
\mathfrak{F} \arrow[d, "e^{2\pi iz}"'] \arrow[dr, "J"] & \\
D^* \arrow[r, "\widetilde{J}"] & \mathbb{P}(\mathbb{C})
\end{tikzcd}$$

As before, the restriction of

$$e^{2\pi iz} = e^{2\pi i(x+iy)} = e^{-2\pi y}(cosx + isinx)$$

to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

It is known that as $Im(z) \to +\infty$, $J(z) \to +\infty$. Hence, $\widetilde{J}(q)$ tends to ∞ as $q \to 0$ in D^*, so $\widetilde{J}$ is meromorphic on the punctured disc. Hence, $\widetilde{J}$ is definable in $\mathbb{R}_{an}$.

It follows that $J|\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$. $\square$

II. Number Theory

We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$, and $X \subseteq \mathbb{C}^n$ algebraic, with $X \cap \mathcal{S}_0$ Zariski dense in X. We use $\mathfrak{F}$ for the fundamental set for Θ $(= \mathfrak{F}^n)$.

On the analytic side

Let $\widetilde{X} \subseteq \mathbb{H}^n$ be an irreducible **analytic** component of $\Theta^{-1}(X)$.

We already saw that if $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is special then each τ_i is imaginary quadratic.

Using a theorem of Siegel on imaginary quadratic fields, Pila proves:

Largness of special points

The set $\widetilde{\mathcal{S}}_0 \cap \widetilde{X} \cap \mathfrak{F}$ is large*.

III. The Ax-Lindemann statement

The Pila-Wilkie input

$\widetilde{X}$ contains an algebraic set of positive dimension (relative to $\mathbb{H}^n$). Let A be maximal irreducible such set.

Goal

A is weakly $\widetilde{\text{special}}$. Namely
(i) it is the orbit of a real algebraic subgroup of $SL(2, \mathbb{R})^n$, and
(ii) $\Theta(A)$ is algebraic.

Ax-Lindemann for $\mathbb{H}^n$ (third type of proof)

We have $\widetilde{X} \subseteq \mathbb{H}^n$ an analytic irreducible component of $\Theta^{-1}(X)$ and $A \subseteq \widetilde{X}$ is a maximal, relatively algebraic subset, of positive dimension. Namely, there exists an algebraic $\bar{A} \subseteq \mathbb{C}^n$ such that $A = \bar{A} \cap \mathbb{H}^n$.

Write $G := SL(2, \mathbb{R})^n$, and $\Gamma = SL(2, \mathbb{Z})^n$.

Without loss of generality $\dim(A \cap \mathfrak{F}) = \dim A$ (if not, replace $\widetilde{X}$ and A by $\gamma\widetilde{X}$ and γA, for some $\gamma \in \Gamma$).

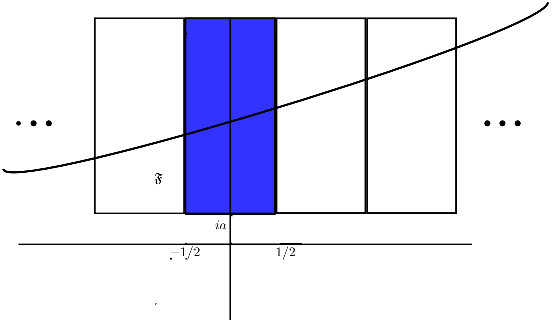
Fact A is not contained in finitely many Γ-translates of $\mathfrak{F}$.

WHY?

Otherwise $A \subseteq \bigcup_{i=1}^{k} \gamma_i \mathfrak{F}$. Because the real part of $\mathfrak{F}$ is bounded, it follows that $Re(z)$ is bounded for $z \in \bar{A} \cap \mathbb{H}^n$. This would imply (?) that A must be compact. But a compact complex analytic subset of $\mathbb{H}^n$ is finite. Contradiction.

Proof of A-L (cont)

We showed that A is not contained in finitely many Γ-translates of $\mathfrak{F}$



Hence, there are infinitely many $\gamma \in \Gamma$ such that $\dim(\gamma A \cap \mathfrak{F}) = \dim A$.

A-L continues

Let $G(A) = \{g \in G : \dim(gA \cap (\Theta^{-1}(X) \cap \mathfrak{F})) = \dim A\}$.

- As we showed, $\Gamma \cap G(A)$ is infinite.
- By analyticity of $\Theta^{-1}(X)$ and irreducibility of A, if $g \in G(A)$ then $gA \subseteq \Theta^{-1}(X)$.
- The set $G(A)$ is definable in $\mathbb{R}_{an,exp}$.

A counting Lemma (proof omitted)

The $\{\gamma \in SL(2,\mathbb{Z})^n : \gamma \in G(A)$ is large*.

A second use of Pila-Wilkie

By PW, $G(A)$ contains a semi-algebraic connected curve σ.

End of proof of A-L

We have $G(A) = \{g \in G : \dim(gA \cap (\Theta^{-1}(X) \cap \mathfrak{F})) = \dim A\}$.

A s semi-algebraic curve $\sigma \subseteq G(A)$. Without loss $e \in \sigma$.

So, $\sigma \cdot A \subseteq \Theta^{-1}(X)$ is a semi-algebraic set containing (a translate of) A. **By the maximality of A, $\sigma \cdot A = A$, hence the group $Stab_G(A)$ is infinite.**

Consider the real algebraic group $Stab_G(A) \subseteq G$. It is thus infinite and contains infinitely many Γ points (by a finer use of Pila-Wilkie).

Let H be the Zariski closure of $G(A) \cap \Gamma$. It is a real algebraic group defined over $\mathbb{Q}$ which stabilizes A. Using induction and decomposition of Shimura varieties, one can show that A is an orbit of H and that $\Theta(A)$ is algebraic, hence A is weakly special.
It follows that X contains a special variety. End of Pila's proof.

Further work around Pila-Zannier

André Oort for $\mathcal{A}_g$ for $g = 2$ (Pila Tsimerman)

Theorem The André- Oort conjecture holds for $\mathcal{A}_2$, the moduli space of abelian surfaces.

I. Definability: P-Starchenko.

II. Number Theory: Uses results of Tsimerman.

III. A-L: using strongly the low dimension of $\mathcal{A}_2$ ($\dim \mathcal{A}_2 = 3$).

Status of General André-Oort

Recent work of Klingler, Yafaev and Ullmo (2013)

I. The restriction of the Baily-Borel embedding of Shimura varieties to Siegel fundamental sets is definable in $\mathbb{R}_{an,exp}$ (!).
III. Ax-Lindemann holds for arbitrary Shimura varieties.

what is missing?

The number Theory part