Simply-connected locally Nash groups

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Summer School in Tame Geometry,
Konstanz, 2016

Outline

1. What is a locally Nash group?
2. Why do we study locally Nash groups?
3. A description of simply-connected two-dimensional abelian locally Nash groups

Locally Nash Groups (LNGs)

Definition

Let G be a (real) Lie group with an analytic atlas $\mathcal{A}$. Then G is a LNG if $\exists (U, \varphi) \in \mathcal{A}$, $1 \in U$, for which $\{(gU, \varphi_g)\}_{g \in G}$ is an analytic atlas

$$\varphi_g : gU \to \varphi(U) : gu \mapsto \varphi(u)$$

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$\therefore$ for (U,φ), $\exists V, V_g \subset U$ $(g\in G)$, $1\in U$, s.t.

$$(\mathcal{C}_1)\quad \varphi(V)\times\varphi(V) \longrightarrow \varphi(U) : (x,y) \mapsto \varphi(\varphi^{-1}(x)\varphi^{-1}(y))$$

$$(\mathcal{C}_2)\quad \varphi(V_g) \longrightarrow \varphi(U) : x \mapsto \varphi(g^{-1}(\varphi^{-1}(x))^{-1}g)$$

are Nash maps.

Definition

Let $\big(G, (U, \varphi)\big)$ and $\big(H, (V, \psi)\big)$ be LNGs.

A homomorphism (isomorphism) of Lie groups $f : G \to H$ is a locally Nash homomorphism (resp. isomorphism) if $\exists W \subset U \cap f^{-1}(V)$, $1 \in W$, s.t.

$$\varphi(W) \to \psi(V) : x \mapsto \psi \circ f \circ \varphi^{-1}(x)$$

is a Nash map.

Example

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3. $\big(\mathbb{R}, (\mathbb{R}, id)\big) \quad \overset{id}{\simeq} \quad \big(\mathbb{R}, ((-1, 1), id)\big)$.
4. The direct product of LNGs is a LNG.
5. The universal covering of a LNG is a LNG.
6. The quotient of a LNG by a discrete subgroup is a LNG.
7. Every algebraic group has a natural structure of LNG.

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Theorem (Hrushovski-Pillay [HP94])

The LNGs are, up to local Nash isomorphism, precisely the quotient of universal coverings of algebraic groups by discrete subgroups.

IF $\left(\mathbb{R}^n, (U, \varphi)\right)$ is a LNG THEN

$$\varphi(V) \times \varphi(V) \longrightarrow \varphi(U) : (x, y) \mapsto \varphi(\varphi^{-1}(x) + \varphi^{-1}(y))$$

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Algebraic Addition Theorem (AAT)

Let U be an open neighborhood of 0. An analytic map $\varphi : U \to \mathbb{R}^n$ admits an AAT if $\varphi_1(u + v), \ldots, \varphi_n(u + v)$ are algebraic over

$$\mathbb{R}(\varphi_1(u), \ldots, \varphi_n(u), \varphi_1(v), \ldots, \varphi_n(v)).$$

This definition generalizes to holomorphic and meromorphic maps.

A natural problem in Tame Geometry is to find a classification of semialgebraic groups over the real field, *i.e.* of groups definable over

$$\mathcal{R} :=< \mathbb{R}; <, +, \cdot, 0, 1 > .$$

In [Pil88], it is shown that these groups admit an additional analytic structure and hence they can be viewed as Nash groups (and conversely, every Nash group is a semialgebraic group).

The one-dimensional classification of Nash groups was done by Madden and Stanton in [MS92] by getting first a classification of simply-connected one-dimensional LNG.

Notation

IF $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a real meromorphic map admitting an AAT s.t. $\exists U \subset \mathbb{R}^n$, $0 \in U$, $\exists k \in \mathbb{R}^n$ s.t.

$$\psi : U \to \mathbb{R}^n : x \mapsto \varphi(x+k)$$

is a real analytic diffeomorphism THEN $\big(\mathbb{R}^n, (U, \psi)\big)$ is a LNG.
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The Weierstrass $\wp_\Lambda$-function $\wp_\Lambda : \mathbb{C} \to \mathbb{C}$ is a meromorphic function admitting an AAT which is not analytic at 0.
An adecuate translation of $\wp_\Omega$ is an analytic diffeomorphism at 0, $\therefore$ $(\mathbb{R}, \wp_\Lambda)$ is a LNG.

Theorem (Madden-Stanton [MS92])

Classification of simply-connected one-dimensional LNGs:

- $(\mathbb{R}, id)$,
- $(\mathbb{R}, exp)$,
- $(\mathbb{R}, \sin)$,
- $(\mathbb{R}, \wp_\Lambda)$ *for some lattice* $\Lambda < (\mathbb{C}, +)$.

None of the above LNGs is locally Nash isomorphic to one of a different class. The fourth case subdivides in different cases.

Simply-connected two-dimensional abelian LNGs

Theorem (Baro-dV-Otero [BdVO15])

1. *Extension: Every simply-connected two-dimensional abelian LNG is locally Nash isomorphic to one of the form* $(\mathbb{R}^2, \varphi)$ *for some real meromorphic map* $\varphi := (\varphi_1, \varphi_2) : \mathbb{C}^2 \to \mathbb{C}^2$ *admitting an AAT.*
2. *Painlevé families: Both* φ_1 *and* φ_2 *are algebraic over a field of a unique member of the six following families:*

$(\varphi_1, \varphi_2) : \mathbb{C}^2 \to \mathbb{C}^2$. Both φ_1, φ_2 are algebraic over one and only one of:

Painlevé families

$$\mathcal{P}_1 := \left\{\mathbb{C}\big((u, v) \circ \alpha\big) : \alpha \in GL_2(\mathbb{C})\right\}$$

$$\mathcal{P}_2 := \left\{\mathbb{C}\big((u, e^v) \circ \alpha\big) : \alpha \in GL_2(\mathbb{C})\right\}$$

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$\mathcal{P}_4 := \left\{\mathbb{C}\big((\wp_\Omega(u), v - a\zeta_\Omega(u)) \circ \alpha\big) : a \in \{0, 1\}; \Omega < (\mathbb{C}, +); \alpha \in GL_2(\mathbb{C})\right\}$

$\mathcal{P}_5 := \left\{\mathbb{C}\left(\left(\wp_\Omega(u), \frac{\sigma_\Omega(u-a)}{\sigma_\Omega(u)} e^v\right) \circ \alpha\right) : a \in \mathbb{C}; \Omega < (\mathbb{C}, +); \alpha \in GL_2(\mathbb{C})\right\}$

$\mathcal{P}_6$: All the possible fields of abelian functions (corresponding to lattices of $\mathbb{C}^2$) of transcendence degree 2 over $\mathbb{C}$.

Proof: Extension

Proposition (Baro-dV-Otero [BdVO15])

IF $(\mathbb{R}^n, (U, \varphi))$ *is a LNG THEN* $\exists \psi := (\psi_1, \dots, \psi_n) : \mathbb{C}^n \to \mathbb{C}^n$ *a real meromorphic map admitting an AAT s.t.*

1. $(\mathbb{R}^n, (U, \varphi)) = (\mathbb{R}^n, \psi)$,
2. $\exists \psi_0 : \mathbb{C}^n \to \mathbb{C}$ *a real meromorphic map algebraic over* $\mathbb{R}(\psi_1, \dots, \psi_n)$ *s.t. for each* $f \in \mathbb{R}(\psi_0, \psi_1, \dots, \psi_n)$

$$f(u+v) \in \mathbb{R}\big(\psi_0(u), \psi_1(u) \dots, \psi_n(u), \psi_0(v), \psi_1(v) \dots, \psi_n(v)\big).$$

$\therefore$ the classification of locally Nash structures of $\mathbb{R}^2$ reduces to the study of real meromorphic maps $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ that admit an AAT.

Proof: Painlevé families

Theorem (Painlevé [Pai03])

IF $\varphi_1, \varphi_2 : \mathbb{C}^2 \to \mathbb{C}$ *are functionally independent meromorphic functions s.t.* $\varphi := (\varphi_1, \varphi_2)$ *admits an AAT THEN* $\exists i \in \{1, \ldots, 6\}$ *s.t.* φ_1 *and* φ_2 *are algebraic over one of the fields of the family* $\mathcal{P}_i$*.*

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