SMALL SETS IN DENSE PAIRS

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Abstract. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of an o-minimal structure $\mathcal{M}$ by a dense set $P \subseteq M$, such that two tameness conditions hold. We prove that the induced structure on P by $\mathcal{M}$ eliminates imaginaries. As a corollary, we obtain that every small set X definable in $\widetilde{\mathcal{M}}$ can be definably embedded into some P^l, uniformly in parameters. We verify the tameness conditions in three examples: dense pairs, expansions of $\mathcal{M}$ by a dense independent set, and expansions of $\mathcal{M}$ by a multiplicative group with the Mann property.

1. Introduction

Elimination of imaginaries is a classical theme in model theory. If a structure eliminates imaginaries, then quotients of definable sets by definable equivalent relations can be treated as definable.

Definition 1.1. A structure $\mathcal{N}$ *eliminates imaginaries* if for every $\emptyset$-definable equivalence relation E on N^n, there is a $\emptyset$-definable map $f : X \to N^l$ such that for every $x, y \in X$,

$$E(x, y) \Leftrightarrow f(x) = f(y).$$

In particular, M^n/E is in bijection with the $\emptyset$-definable set $\{f(a) : a \in M^n\}$.

We fix throughout this paper an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group with a distinguished positive element 1. We denote by $\mathcal{L}$ its language, and by dcl the usual definable closure operator in $\mathcal{M}$. An '$\mathcal{L}$-definable' set is a set definable in $\mathcal{M}$ with parameters. We write '$\mathcal{L}_A$-definable' to specify that those parameters come from $A \subseteq M$. It is a well-known fact that $\mathcal{M}$ admits definable Skolem functions and eliminates imaginaries ([4, Chapter 6]).

Definition 1.2. Let $D, P \subseteq M$. The *D-induced structure on P by $\mathcal{M}$*, denoted by $P_{ind(D)}$, is a structure whose language is

$$\mathcal{L}_{ind(D)} = \{R_{\phi(x)}(x) : \phi(x) \in \mathcal{L}_D\}$$

and, for every tuple $a \subseteq P$,

$$P_{ind(D)} \models R_\phi(a) \Leftrightarrow \mathcal{M} \models \phi(a).$$

For an extensive account on $P_{ind(D)}$, see [12, Section 3.1.2]. Notice that $P_{ind(D)}$ is weakly o-minimal and, as such, it is not necessarily the same with the structure induced on P by the pair $\langle \mathcal{M}, P \rangle$ (see Remark 2.3 below).

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For the rest of the paper we fix some $P \subseteq M$ and denote $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$. Unless stated otherwise, by '(A-)definable' we mean (A-)definable in $\widetilde{\mathcal{M}}$, where $A \subseteq M$. We also let D denote a subset of M.

Consider the following two properties for $\widetilde{\mathcal{M}}$ and D:

(OP) (Open definable sets are $\mathcal{L}$-definable.) For every set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^n$, its topological closure $\overline{V} \subseteq M^n$ is $\mathcal{L}_A$-definable.

$(\mathrm{dcl})_D$ Let $B, C \subseteq P$ and

$$A = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P.$$

Then

$$\mathrm{dcl}(AD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).$$

Property (OP) is Assumption (III) from [7], and a justification for its terminology is provided in [7, Lemma 2.5]. Property $(\mathrm{dcl})_D$ is introduced here and is verified in Section 4 in several examples in case D is dcl-independent over P. The thrust of $(\mathrm{dcl})_D$ is that it concerns only definability in $\mathcal{M}$.

Our main result is the following:

Theorem A. *Suppose* (OP) *and* $(\mathrm{dcl})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*. Then* $P_{ind(D)}$ *eliminates imaginaries.*

Theorem A stands in contrast to the general intuition that in pairs with tame geometric behavior on the class of all definable sets, 'choice properties' generally fail. Such pairs have been extensively studied in recent years, and they include dense pairs [3], expansions of $\mathcal{M}$ by a dense independent set or by a multiplicative group with the Mann Property. It is known that a dense pair does not eliminate imaginaries, and neither it nor $P_{ind(D)}$ admits definable Skolem functions ([1, Section 5] and [6]). If P is a dense independent set, then $\widetilde{\mathcal{M}}$ eliminates imaginaries but does not admit definable Skolem functions ([2]).

In [7], the above and further examples were all put under a common perspective and a program was initiated for understanding their definable sets in terms of $\mathcal{L}$-definable sets and 'P-bound' sets. In particular, they were shown to satisfy (OP). An important application of Theorem A is (Theorem B below) that the study of P-bound sets can be further reduced to that of definable (in $\widetilde{\mathcal{M}}$) subsets of P^l. This reduction is the main motivation of the present work.

Definition 1.3 ([3])**.** A set $X \subseteq M^n$ is called *P-bound over A* if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(P^m)$.

In the aforementioned examples, P-boundness amounts to a precise topological notion of smallness ([7, Definition 2.1]), as well as to the classical notion of P-internality from geometric stability theory ([7, Corollary 3.12]). Our application of Theorem A is the following.

Theorem B. *Suppose* (OP) *and* $(\mathrm{dcl})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*, and let* $X \subseteq M^n$ *be a* D*-definable set. If* X *is* P*-bound over* D*, then there is a* D*-definable injective map* $\tau : X \to P^l$.

Notice that $\tau(X) \subseteq P^l$ is definable in $\widetilde{\mathcal{M}}$, but not necessarily definable in $P_{ind(D)}$, for any $D \subseteq M$.

In Section 4, we verify $(\mathrm{dcl})_D$ in our examples under the assumption that D is dcl-independent over P.

Proposition 1.4. *Suppose $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ is a dense pair, or an expansion of $\mathcal{M}$ by a dense independent set or an expansion of $\mathcal{M}$ by a multiplicative group with the Mann Property. Let $D \subseteq M$ be* dcl*-independent over P. Then $(dcl)_D$ holds, and hence $P_{ind(D)}$ eliminates imaginaries.*

We show in Example 5.1 that the assumption of D being dcl-independent over P is necessary; namely, without it, $P_{ind(D)}$ need not eliminate imaginaries. However, even without it, one still obtains the following corollary, which in particular applies to our examples.

Corollary 1.5. *Suppose (OP) and $(dcl)_D$ hold for $\widetilde{\mathcal{M}}$ and every $D \subseteq M$ which is* dcl*-independent over P. Let $X \subseteq M^n$ be an A-definable set. If X is P-bound over A, then there is an $A \cup P$-definable injective map $\tau : X \to P^l$.*

Allowing parameters from P is standard practice when studying definability in this context; see for example also [7, Lemma 2.5, Corollary 3.24]. We note that Corollary 1.5 settles affirmatively [7, Question 7.12] in our examples. The same question was asked to the author by E. Baro and A. Martin-Pizarro during the Summer School in Tame Geometry in Konstanz in 2016.

In Section 5, we deal with the optimality of our results. Besides the aforementioned Example 5.1, we prove in Corollary 5.3 that $(\mathrm{dcl})_D$ is necessary for $P_{ind(D)}$ to eliminate imaginaries. More precisely, we introduce a further property for D, called $(\mathrm{dcl}')_D$, and show that if $\widetilde{\mathcal{M}}$ has (OP) and D is dcl-independent over P, then

$$P_{ind(D)} \text{ eliminates imaginaries} \quad \Leftrightarrow \quad (\mathrm{dcl})_D \quad \Leftrightarrow \quad (\mathrm{dcl}')_D. \tag{1}$$

Finally, in Example 5.5 we observe that if we do not assume (OP), the above three properties need not hold. We do not know whether they hold, if we assume (OP).

Our proof is influenced by two previous accounts on elimination of imaginaries in ordered structures, [11] and [13], but diverges from both of them substantially. As noted in Fact 2.2 below, to prove that an ordered pregeometric structure $\mathcal{N}$ eliminates imaginaries, the following is enough:

(*) Let $B, C \subseteq N$ and $A = \mathrm{dcl}_{\mathcal{N}}(B) \cap \mathrm{dcl}_{\mathcal{N}}(C)$. If $X \subseteq N^n$ is B-definable and C-definable, then X is A-definable.

So our goal is to prove (*) for $\mathcal{N} = P_{ind(D)}$ (Lemma 3.3). There are two obstacles in adopting the existing accounts. First, Pillay [11] establishes (*) for an o-minimal $\mathcal{N}$ under the additional assumption that $A \preccurlyeq \mathcal{N}$, which need no longer be true here. Second, $P_{ind(D)}$ is only weakly o-minimal, and Wencel's proof [13] of elimination of imaginaries in certain weakly o-minimal structures (even under the assumption $A \preccurlyeq P_{ind(D)}$) unfortunately contains a serious gap (see Section 3.1). We are thus led to produce a new strategy for proving (*) that moreover works under our general assumptions (OP) and $(\mathrm{dcl})_D$.

Structure of the paper. In Section 2, we recall some basic facts about elimination of imaginaries and discuss the pregeometry in $P_{ind(D)}$. Section 3 contains the proofs Theorems A and B. We also point out the aforementioned gap in [13]. A major part of our work is in Section 4, where we prove (1) and verify $(\mathrm{dcl})_D$ in the three main examples: dense pairs, expansions of $\mathcal{M}$ by a dense independent set, and expansions by a multiplicative group with the Mann property. Finally, we provide examples to establish the optimality of our results.

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2. Preliminaries

We assume familiarity with the basics of o-minimality and pregeometries, as can be found, for example, in [4] or [11]. A tuple of elements is denoted just by one element, and we write $b \subseteq B$ if b is a tuple with coordinates from B. The set of realizations of a formula $\phi(x)$ in a structure $\mathcal{N}$ is denoted by $\phi(N^n)$, where x is an n-tuple. If $A \subseteq N$, we write $A \preccurlyeq \mathcal{N}$ to denote that A is an elementary substructure of $\mathcal{N}$ in the language of $\mathcal{N}$. If $A, B \subseteq N$, we often write AB for $A \cup B$. We denote by $\mathrm{Aut}(\mathcal{N})$ the group of automorphisms of $\mathcal{N}$.

Recall that $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$ is our fixed o-minimal expansion of an ordered group with a distinguished positive element 1. We denote the definable closure operator in $\mathcal{M}$ by dcl, and the corresponding rank by rk. The topological closure of a set $X \subseteq M^n$ is denoted by $\overline{X}$. If $X, Y \subseteq M^n$, we call X dense in Y, if $\overline{X \cap Y} = \overline{Y}$.

2.1. **Elimination of imaginaries.** The property of elimination of imaginaries (Definition 1.1) can be formulated in many ways. In Fact 2.1 we state one which will be useful for our purposes. Notice that Definition 1.1 is sometimes called *uniform* elimination of imaginaries, whereas elimination of imaginaries is reserved for the condition that every definable set X has a canonical base (see below). However, in the presence of two distinct constants in our $\mathcal{L}$, the two notions coincide ([10, Lemma 4.3]).

For the rest of this subsection, let $\mathcal{N}$ be a structure with two distinct constants in its language, and $X \subseteq N^n$ a definable set. We call $A \subseteq N$ a *defining set for X*, if X is A-definable. The following fact is noted in [11, Section 3].

Fact 2.1. *Assume $\mathcal{N}$ is an ordered structure. The $\mathcal{N}$ eliminates imaginaries if and only if every definable set has a smallest definably closed defining set.*

It is well-known that if C is the smallest definably closed defining set for X, then $C = \mathrm{dcl}(p)$, for some tuple $p \subseteq N$ satisfying: for every $\tau \in \mathrm{Aut}(\mathcal{N})$,

$$\tau(p) = p \quad \Leftrightarrow \quad \tau(X) = X.$$

Such a tuple p is called a *canonical base* for X. Clearly, if X is A-definable, then $p \subseteq \mathrm{dcl}(A)$.

The next fact can also be extracted from [11, Section 3].

Fact 2.2. *Assume $\mathcal{N}$ is an ordered pregeometric structure that satisfies (*) from the introduction. Then $\mathcal{N}$ eliminates imaginaries.*

Proof. Let $X \subseteq N^n$ be a definable set. We need to show that X has a smallest definably closed defining set. Assume X is B_0-definable, $B_0 \subseteq N$ is finite and $\mathrm{dcl}_{\mathcal{N}}$-independent, and $|B_0|$ is least possible. We claim that $B = \mathrm{dcl}_{\mathcal{N}}(B_0)$ is the smallest definably closed defining set for X. Suppose not. Then there is another set $C \subseteq N$ such that $B \not\subseteq \mathrm{dcl}_{\mathcal{N}}(C)$, and X is C-definable. Let $A = B \cap \mathrm{dcl}_{\mathcal{N}}(C)$.

By (*), X is A-definable. Moreover, $A \subsetneq B$ and A is $\mathrm{dcl}_{\mathcal{N}}$-closed. By the exchange property in pregeometric theories, $A = \mathrm{dcl}_{\mathcal{N}}(A_0)$, for some $\mathrm{dcl}_{\mathcal{N}}$-independent A_0 with size $|A_0| < |B_0|$. Then X is also A_0-definable, contradicting the choice of B_0. $\square$

2.2. **The induced structure.** Recall from the introduction that

$$P_{ind(D)} = \langle P, \{R \cap P^l : R \subseteq M^l \ \mathcal{L}_D\text{-definable }, l \in \mathbb{N}\}\rangle.$$

Note that for every $Z \subseteq M^n$ and $B \subseteq P$, the following are equivalent:

(1) Z is $\mathcal{L}_{BD}$-definable
(2) $Z \cap P^n$ is B-definable in $P_{ind(D)}$
(3) $Z \cap P^n$ is BD-definable in $P_{ind(D)}$.

Remark 2.3. $P_{ind(D)}$ is weakly o-minimal. Namely, every definable $X \subseteq P$ in $P_{ind(D)}$ is a finite union of convex subsets of P. Indeed, $X = Y \cap P$ for an $\mathcal{L}$-definable $Y \subseteq M$. This description is no longer true for a subset $X \subseteq P$ which is definable in $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$. For example, if $\mathcal{M} = \mathbb{R}$ and $P = 2^{\mathbb{Z}}3^{\mathbb{Z}}$, the set of elements in P which are divisible by 2 is not a finite union of convex subsets of P.[1]

Lemma 2.4. *Let $f : P^n \to P^k$ be an A-definable map in $P_{ind(D)}$. Then there is an $\mathcal{L}_D$-definable map $F : M^n \to M^k$ that extends f.*

Proof. By definition of $P_{ind(D)}$, there is an $\mathcal{L}_{AD}$-definable set $T \subseteq M^{n+k}$ such that $\Gamma(f) = T \cap P^{n+k}$. By o-minimality, the set

$$X = \{x \in \pi(T) : T_x \text{ is singleton}\}$$

is $\mathcal{L}_{AD}$-definable. Obviously, $P^n \subseteq X$. Now let

$$T' = \left(\bigcup_{x\in X} \{x\} \times T_x\right) \cup \{(x, 0) : x \in M^n \setminus X\}.$$

Then T' is $\mathcal{L}_{AD}$-definable, it is the graph of a function $F : M^n \to M^k$, and $\Gamma(f) = T' \cap P^{n+k}$, as required. $\square$

We denote the definable closure operator in $P_{ind(D)}$ by cl_D.

Corollary 2.5. *For every $A \subseteq P$, $\mathrm{cl}_D(A) = \mathrm{dcl}(AD) \cap P$.*

Proof. The inclusion $\supseteq$ is immediate from the definitions, whereas the inclusion $\subseteq$ is immediate from Lemma 2.4. $\square$

By Corollary 2.5 and the fact that $\mathrm{dcl}(-D)$ defines a pregeometry in $\mathcal{M}$, it follows easily that $cl_D(-)$ defines a pregeometry in $P_{ind(D)}$. This pregeometry need not satisfy usual properties known for the definable closure in o-minimal structures. For example, as pointed out in [6, Proposition 2.4], if $\widetilde{\mathcal{M}}$ is a dense pair, then there are $A \subseteq M$, such that $cl_D(A) \not\preccurlyeq P_{ind(D)}$. It is natural to ask the following:

Question 2.6. *Under what assumptions on $\widetilde{\mathcal{M}}$ and D, is it true that for all $A \subseteq M$, $cl_D(A) \preccurlyeq P_{ind(D)}$?*

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[1] I thank Philipp Hieronymi for first pointing out this example.

3. Proofs of Theorems A and B

Throughout this section we fix $D \subseteq M$, and assume that (OP) *and* $(\mathrm{dcl})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D.

We are now ready to prove the main theorems of this paper. We begin with a key lemma. Note that its proof only uses (OP) for parameter sets $A \subseteq P$. Right afterwards we present an example that illustrates the main points of the proof.

Lemma 3.1. *Let $B, C \subseteq P$, $X \subseteq P^n$, and $Y, Z \subseteq M^n$ such that Y is $\mathcal{L}_B$-definable, Z is $\mathcal{L}_C$-definable and*

$$X = P^n \cap Y = P^n \cap Z. \tag{2}$$

Then there is $W \subseteq M^n$, both $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable, such that

$$X = P^n \cap W.$$

Proof. We work by induction on $\dim(Y \cup Z)$. First note that, by (2), X is both B-definable and C-definable in $\langle \mathcal{M}, P\rangle$. Since $B, C \subseteq P$, by (OP) it follows that $\overline{X}$ is $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable.

If $\dim(Y \cup Z) = 0$, then X is finite, and hence $X = \overline{X}$. So X is both $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable, and can let $W = \overline{X}$.

Assume $\dim(Y \cup Z) = n > 0$. Let

$$K = \overline{(\overline{X} \setminus (Y \cup Z)) \cap P^n}.$$

Claim 1. *K is $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable.*

Proof of Claim 1. It suffices to prove that

$$(\overline{X} \setminus (Y \cup Z)) \cap P^n = (\overline{X} \setminus Y) \cap P^n = (\overline{X} \setminus Z) \cap P^n.$$

since the second (respectively, third) part is B-definable (respectively, C-definable) in $\langle \mathcal{M}, P\rangle$ and hence its closure $\mathcal{L}_B$-definable (respectively, $\mathcal{L}_C$-definable). We prove the first equality, the other being completely analogous. We only need to prove $\supseteq$. Let $x \in (\overline{X} \setminus Y) \cap P^n$. We claim that $x \notin Z$. Indeed, if $x \in Z$, then $x \in Z \cap P^n = X$, and hence $x \in X \setminus Y$, contradicting (2). $\square$

Claim 2. We have

$$\dim K < n.$$

Proof of Claim 2. By (2), we have $\overline{X} \subseteq \overline{Y \cup Z}$. Therefore,

$$\overline{X} \setminus (Y \cup Z) \subseteq \overline{(Y \cup Z)} \setminus (Y \cup Z)$$

has dimension $< n$, and hence so does $K \subseteq \overline{\overline{X} \setminus (Y \cup Z)}$. $\square$

Claim 3. We have

$$(\overline{X} \setminus K) \cap P^n \subseteq X.$$

Proof of Claim 3. Let $x \in (\overline{X} \setminus K) \cap P^n$. By (2) it suffices to show that $x \in Y \cup Z$. Assume not. Then $x \in (\overline{X} \setminus (Y \cup Z) \cap P^n \subseteq K$, a contradiction. $\square$

By Claim 3 and since $X \subseteq P^n$, we can write

$$X = (X \cap K) \cup (X \cap (\overline{X} \setminus K)) = (X \cap K) \cup (P^n \cap (\overline{X} \setminus K)). \tag{3}$$

We also have $X \cap K = P^n \cap Y \cap K = P^n \cap Z \cap K$. By Claim 1, $Y \cap K$ is $\mathcal{L}_B$-definable and $Z \cap K$ is $\mathcal{L}_C$-definable. By Claim 2, $\dim((Y \cap K) \cup (Z \cap K)) < n$. Hence, by inductive hypothesis, there is $W_1 \subseteq M^n$, both $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable, such that

$$X \cap K = P^n \cap W_1. \tag{4}$$

Let also $W_2 = \overline{X} \setminus K$. Again by Claim 1, W_2 is $\mathcal{L}_B$-definable and $\mathcal{L}_C$-definable. Therefore, for $W = W_1 \cup W_2$, (3) and (4) give

$$X = P^n \cap W,$$

as required. □

Example 3.2. The above proof can be illustrated as follows. Let $\widetilde{\mathcal{M}} = \langle \widetilde{\mathbb{R}}, P \rangle$ be a dense pair with $\widetilde{\mathbb{R}}$ the real field, l_β and l_γ two non-parallel lines in $\mathbb{R}^2$,

$$Y = \mathbb{R}^2 \setminus l_\beta \quad \text{and} \quad Z = \mathbb{R}^2 \setminus l_\gamma,$$

and X, K, W_1 and W_2 the sets defined in the above statement and proof. Apart from the intersection point $c \in l_\beta \cap l_\gamma$, the two lines cannot contain any other element of P^2. Indeed, such an element would belong to only one of $Y \cap P^2$ and $Z \cap P^2$, contradicting (2). There are two cases:

Case I. $c \in P^2$. In this case, $K = \{c\}$, $W_2 = \mathbb{R}^2 \setminus \{c\} = Y \cup Z$ and

$$X = (\mathbb{R}^2 \setminus \{c\}) \cap P^2.$$

Case II. $c \notin P^2$. In this case, $K = \emptyset$, $W_2 = \mathbb{R}^2$ and

$$X = \mathbb{R}^2 \cap P^2.$$

Note that in the second case, even though we also have $X = (Y \cup Z) \cap P^2$, the set $Y \cup Z$ is neither $\mathcal{L}_B$-definable nor $\mathcal{L}_C$-definable. In both cases, $W_1 = \emptyset$ and $X = (\overline{X} \setminus K) \cap P^2$. We leave it to the reader to construct examples on $\widetilde{\mathbb{R}}$ where the above cases actually occur.

Lemma 3.3. *Let $B, C \subseteq P$ and $A = cl_P(B) \cap cl_P(C)$. If $X \subseteq P^n$ is B-definable and C-definable in $P_{ind(D)}$, then X is A-definable in $P_{ind(D)}$.*

Proof. Let $X \subseteq P^n$ be B-definable and C-definable in $P_{ind(D)}$. So

$$X = P^n \cap Y = P^n \cap Z,$$

for some $\mathcal{L}_{BD}$-definable $Y \subseteq M^n$ and $\mathcal{L}_{CD}$-definable $Z \subseteq M^n$. By Lemma 3.1, there is $W \subseteq M^n$, both $\mathcal{L}_{BD}$-definable and $\mathcal{L}_{CD}$-definable, such that $X = P^n \cap W$. By [11, Proposition 2.3], W is $\mathcal{L}$- definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD)$. By $(\mathrm{dcl})_D$, W is $\mathcal{L}$-definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap PD$. Hence X is definable over $\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P$ in $P_{ind(D)}$. But

$$\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P = cl_P(B) \cap cl_P(C) = A,$$

and hence X is A-definable. □

We can now conclude Theorems A and B.

Proof of Theorem A. By Fact 2.2 and Lemma 3.3 □

Proof of Theorem B. Let $h : M^m \to M^n$ be an $\mathcal{L}_A$-definable map such that $X \subseteq h(P^m)$, and consider the following equivalence relation E on M^m:

$$xEy \Leftrightarrow h(x) = h(y).$$

Note that $E \cap (P^m \times P^m)$ is an equivalence relation on P^m, which is $\emptyset$-definable in $P_{ind(D)}$. Since $P_{ind(D)}$ eliminates imaginaries, there is a $\emptyset$-definable in $P_{ind(D)}$ map $f : P^m \to P^l$, for some l, such that for every $x, y \in P^m$,

$$f(x) = f(y) \Leftrightarrow xEy.$$

Define $\tau : X \to P^l$, given by $\tau(h(x)) = f(x)$. Then τ is well-defined and injective. Since

$$\tau(y) = z \ \Leftrightarrow \ \exists x \in P^m,\ h(x) = y \ \text{ and } \ f(x) = z.$$

it is also D-definable (in $\widetilde{\mathcal{M}}$). $\square$

Finally, we turn to the proof of Corollary 1.5, where the parameter set for X may not be dcl-independent over P. We need the following lemma.

Lemma 3.4. *Suppose* (OP) *and* $(\mathrm{dcl})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*. Let* $\mathcal{M}'$ *be the expansion of* $\mathcal{M}$ *with constants for all elements in* P*, and* $\widetilde{\mathcal{M}}' = \langle \mathcal{M}', P \rangle$*. Then* (OP) *and* $(\mathrm{dcl})_D$ *hold for* $\widetilde{\mathcal{M}}'$ *and* D*.*

Proof. For (OP), let $V \subseteq M$ be A-definable in $\widetilde{\mathcal{M}}'$. So V is AP-definable in $\widetilde{\mathcal{M}}$. If $A \setminus P$ is $\mathrm{dcl}_{\mathcal{M}'}$-independent over P, then $AP \setminus P = A \setminus P$ is dcl-independent over P, and hence it is dcl-independent over P. By (OP) for $\widetilde{\mathcal{M}}$, $\overline{V}$ is $\mathcal{L}_{AP}$-definable. Hence it is A-definable in $\mathcal{M}'$.

For $(\mathrm{dcl})_D$ for $\mathcal{M}'$, observe that if $B \subseteq P$, then

$$\mathrm{dcl}_{\mathcal{M}'}(BD) = \mathrm{dcl}(PD).$$

Now let $B, C \subseteq P$ and

$$A = \mathrm{dcl}_{\mathcal{M}'}(BD) \cap \mathrm{dcl}_{\mathcal{M}'}(CD) \cap P.$$

We have

$$\mathrm{dcl}_{\mathcal{M}'}(AD) = \mathrm{dcl}(PD) = \mathrm{dcl}_{\mathcal{M}'}(BD) \cap \mathrm{dcl}_{\mathcal{M}'}(CD),$$

as required. $\square$

Proof of Corollary 1.5. Let D be a maximal subset of A which is dcl-independent over P. Then $\mathrm{dcl}(A) \subseteq \mathrm{dcl}(DP)$. Let $\widetilde{\mathcal{M}}'$ be the expansion of $\widetilde{\mathcal{M}}$ as in Lemma 3.4. Hence (OP) and $(\mathrm{dcl})_D$ hold for $\widetilde{\mathcal{M}}'$ and D. Moreover, X is D-definable in $\widetilde{\mathcal{M}}'$. By Theorem B, there is an injective map $\tau : X \to P^l$, which is D-definable in $\widetilde{\mathcal{M}}'$. Hence τ is DP-definable in $\widetilde{\mathcal{M}}$, and thus also AP-definable. $\square$

Remark 3.5. In Example 5.1 below we show that the assumption of D being dcl-independent is necessary for $P_{ind(D)}$ to eliminate imaginaries. However, it is still possible to have A not dcl-independent over P, and yet, $P_{ind(A)}$ eliminate imaginaries. This is the case whenever there are $D \subseteq A$, dcl-independent over P, and $P_0 \subseteq P$, such that

$$\mathrm{dcl}(A) = \mathrm{dcl}(DP_0). \tag{5}$$

Indeed, if E is a $\emptyset$-definable equivalence relation in $P_{ind(A)}$, then it is also $\emptyset$-definable in $P_{ind(D)}$. Let f be as in Definition 1.1, $\emptyset$-definable in $P_{ind(D)}$. Then f is $\emptyset$-definable in $P_{ind(A)}$.

An example where assumption (5) holds, for A not dcl-independent over P, is when $\widetilde{\mathcal{M}}$ is a dense pair, $d \notin P$, $A = \operatorname{dcl}(dP)$ and $D = \{d\}$.

3.1. **On weakly o-minimal structures.** The reader may wonder why we do not directly apply or adopt elimination of imaginaries results known for weakly o-minimal structures. Wencel [13] claims that a weakly o-minimal structure $\mathcal{N}$ with 'strong cell decomposition' property (SCD), such that for every $A \subseteq N$, $\operatorname{dcl}_{\mathcal{N}}(A) \preccurlyeq \mathcal{N}$, eliminates imaginaries. One natural approach would be to assert those two assumptions for our $P_{ind(D)}$. As pointed out right before Question 2.6, the latter property fails for the case of dense pairs, yet it might be true in other settings. Even so, the proof in [13] appears to contain a serious gap. Namely, Theorem 6.3 in that reference is proved by imitating the proof of Pillay [11, Proposition 3.2], where at some point one needs to verify the following statement (as pointed out in our introduction).

(**) Let $B, C \subseteq N$ and $N_0 \preccurlyeq \mathcal{N}$ with $N_0 = \operatorname{dcl}_{\mathcal{N}}(B) \cap \operatorname{dcl}_{\mathcal{N}}(C)$. If $X \subseteq N^n$ is B-definable and C-definable, then X is N_0-definable.

To establish (**), the author uses Proposition 6.2, but it is unclear to us how to obtain its assumption, namely that every convex subset of $\mathcal{N}$ which is B-definable and C-definable is also N_0-definable. What one can extract from Wencel's account is the following fact, whose verification is left to the reader.

Fact 3.6 (Wencel [13])**.** *Let $\mathcal{N}$ be a weakly o-minimal structure with* (SCD)*, and assume that:*

(1) *for every $B, C \subseteq N$ with $N_0 = \operatorname{dcl}_{\mathcal{N}}(B) \cap \operatorname{dcl}_{\mathcal{N}}(C)$, every convex subset of $\mathcal{N}$ which is B-definable and C-definable is also N_0-definable.*
(2) *for every $A \subseteq N$, $\operatorname{dcl}_{\mathcal{N}}(A) \preccurlyeq \mathcal{N}$.*

Then $\mathcal{N}$ admits elimination of imaginaries.

4. $(\mathrm{DCL})_D$ IN MAIN EXAMPLES

In this section we establish the assumptions of Theorems A and B in three main examples: dense pairs, expansions of $\mathcal{M}$ by a dense independent set, and expansions of $\mathcal{M}$ by a multiplicative group with the Mann property. Property (OP) was shown to hold for them in [7, Section 2]. We thus show $(\operatorname{dcl})_D$. Our strategy is to introduce yet another property for $\widetilde{\mathcal{M}}$ and D, prove that together with (OP) it implies $(\operatorname{dcl})_D$ (Proposition 4.3), and then verify it in our examples.

Consider the following property for $\widetilde{\mathcal{M}}$ and D:

$(\operatorname{dcl}')_D$ For every $\alpha \in \operatorname{dcl}(PD)$, there is $q \subseteq P$, such that
$$\operatorname{dcl}(qD) = \operatorname{dcl}_{\mathcal{L}(P)}(\alpha D).$$

Remark 4.1. We could equally well impose $(\operatorname{dcl}')_D$ as the main assumption for this paper, instead of $(\operatorname{dcl})_D$. We chose, however, the latter because it involves only definability in $\mathcal{M}$.

In Proposition 5.4 below we give a complete picture of the main properties mentioned in this paper. For handling our examples in this section, we only need Proposition 4.3 below. First a very useful fact.

Fact 4.2. *Suppose* (OP) *holds, and D is* dcl*-independent over P. Then for every $A \subseteq P$, $\operatorname{dcl}_{\mathcal{L}(P)}(AD) = \operatorname{dcl}(AD)$.*

Proof. Take $x \in \mathrm{dcl}_{\mathcal{L}(P)}(AD)$. That is, the set $\{x\}$ is AD-definable in $\widetilde{\mathcal{M}}$. By (OP), since $AD \setminus P \subseteq D$ is dcl-independent over P, we have that $\overline{\{x\}}$ is $\mathcal{L}_{AD}$-definable. But $\overline{\{x\}} = \{x\}$. □

Proposition 4.3. *Suppose D is* dcl*-independent over P. Then:*

$$\text{(OP) } \textit{and } (\mathrm{dcl}')_D \Rightarrow (\mathrm{dcl})_D.$$

Proof. Let $B, C \subseteq P$ and

$$A = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P.$$

We will show that

$$\mathrm{dcl}(AD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).$$

($\subseteq$). This part follows immediately from properties of dcl. Indeed:

$$\begin{aligned}\mathrm{dcl}(AD) = \mathrm{dcl}(\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap PD) \subseteq \mathrm{dcl}(\mathrm{dcl}(BD) \cap \mathrm{dcl}(CD)) = \\ = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).\end{aligned}$$

($\supseteq$). Let $\alpha \in \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD)$. By $(\mathrm{dcl}')_D$, there is $q \subseteq P$ such that

$$\mathrm{dcl}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D).$$

Observe that

$$q \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(BD) \cap \mathrm{dcl}_{\mathcal{L}(P)}(CD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD),$$

by Fact 4.2. Hence $qD \subseteq \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap PD$ and hence

$$\alpha \in \mathrm{dcl}(qD) \subseteq \mathrm{dcl}(AD).$$

□

We now proceed to prove $(\mathrm{dcl})_D$ in our three examples.

In the rest of this section, we fix $D \subseteq M$ be dcl-independent over P.

4.1. **Dense pairs.** Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair. We will need the following proposition.

Proposition 4.4. *Let $f : X \subseteq M^k \to M$ be an $\mathcal{L}$-definable continuous function, such that X is $\mathcal{L}_P$-definable and P^{k+1} is dense in $\Gamma(f)$. Then f is $\mathcal{L}_P$-definable.*

The proof is based on [3, Theorem 3(3)], a version of which is stated below.

Definition 4.5. Let $f : A \to C$ and $f_i : B_i \to C$ be some maps, $i = 1, \dots, k$. We say that f *is piecewise given by* $f_1, \dots, f_n$ if $A \subseteq \bigcup_i B_i$ and for every $x \in A$, there is $i \in \{1, \dots, n\}$ with $x \in B_i$ and $f(x) = f_i(x)$.

We state the following version of [3, Theorem 3(3)]. The only difference is that here K is not required to equal P^k.

Fact 4.6. *If $f : K \subseteq P^k \to P$ is definable in $\widetilde{\mathcal{M}}$, then f is piecewise given by functions $f_i : P^k \to P$ definable in P.*

Proof. We can expand f to a map $\hat{f} : P^k \to P$ by letting $\hat{f}(x) = 0$ if $x \notin K$. Apply [3, Theorem 3(3)] to get that $\hat{f}$ is piecewise given by maps $f_1, \dots, f_n : P^k \to P$ definable in P. Then f is piecewise given by the same maps. □

Proof of Proposition 4.4. By continuity, it is sufficient to prove that $f_{\restriction X}$ is $\mathcal{L}_P$-definable for some $\mathcal{L}_P$-definable set Y with $\dim(X \setminus Y) < \dim(X)$. Let

$$K = \{x \in X \cap P^k : f(x) \in P\}.$$

By Fact 4.6, $f_{\restriction K}$ is piecewise given by maps $F_1, \dots, F_n : P^k \to P$ which are definable in P. Let $\phi_i(x, y)$ be the $\mathcal{L}_P$-formula that defines F_i,

$$X_i = \{x \in X : \text{ there is unique } y \text{ with } \phi_i(x, y)\},$$

and $f_i : X_i \to M$ the $\mathcal{L}_P$-definable map defined by $\phi_i(x, y)$. Take a cell decomposition $\mathcal{C}$ of M^{k+1} into $\mathcal{L}_P$-definable cells that partitions $\bigcup_i \Gamma(f_i) \cap (X \times M)$. So, for every $C \in \pi(\mathcal{C})$, each $f_{i\restriction C}$ is continuous, and for every $i, j \in \{1, \dots, n\}$, one of the following holds:

$$f_{i\restriction C} < f_{j\restriction C} \quad \text{or} \quad f_{i\restriction C} = f_{j\restriction C} \quad \text{or} \quad f_{i\restriction C} > f_{j\restriction C}.$$

Let $C \in \pi(\mathcal{C})$ be open. The restriction $f_{\restriction K}$ must agree on $K \cap C$ with some F_i. Since K is dense in X, and both $f_{\restriction C}$ and $f_{i\restriction C}$ are continuous, we obtain that $f_{\restriction C} = f_{i\restriction C}$. Let Y be the union of all those $C \in \mathcal{C}$ that are open in X. As a consequence, we obtain that $f_{\restriction Y}$ is $\mathcal{L}_P$-definable, and $\dim(X \setminus Y) < \dim X$, as needed. □

Remark 4.7. It is possible to prove Proposition 4.4 for any $\widetilde{\mathcal{M}}$ as fixed in this paper, by adopting arguments from [8, Lemmas 1.1 - 1.3]. However we make no use of this more general statement, and hence omit it.

Before going to the proof $(\mathrm{dcl}')_D$, we illustrate it with an example.

Example 4.8. Suppose that for $\alpha \in \mathrm{dcl}(bD)$, there is an $\mathcal{L}_D$-definable map $f : M^2 \to M$ with $\alpha = f(b)$, and such that $f^{-1}(\alpha)$ is the graph of an $\mathcal{L}$-definable continuous map $h : M \to M$. Then h is $\mathcal{L}_{\alpha D}$-definable.

Case I. $f^{-1}(\alpha) \cap P^2 = \{b\}$. Then $q = b$ works.
Case II. $f^{-1}(\alpha) \cap P^2$ is dense in $f^{-1}(\alpha)$. By Proposition 4.4, h is actually $\mathcal{L}_P$ definable. Then the canonical base for h in the sense of $\mathcal{M}$, $q \in \mathrm{dcl}(P) = P$, works.

Proof of $(\mathrm{dcl}')_D$**.** Let $\alpha \in \mathrm{dcl}(bD)$, where $b \in P^n$ is dcl-independent over D. If $\alpha \in \mathrm{dcl}(D)$, then the empty tuple $q = \emptyset$ works. Indeed, we have

$$\mathrm{dcl}_{\mathcal{L}(P)}(aD) = \mathrm{dcl}_{\mathcal{L}(P)}(D) = \mathrm{dcl}(D),$$

by Fact 4.2.

So let us suppose that $\alpha \notin \mathrm{dcl}(D)$. Write $b = (b_1, b_2)$ where $b_1 \in P^{n-1}$. (If $n = 1$, then b_1 is the empty tuple.) Since $\alpha \in \mathrm{dcl}(bD)$, and b is dcl-independent over D, there is $\mathcal{L}_D$-definable continuous map $f : X \subseteq M^n \to M$, such that

$$\alpha = f(b).$$

Let $S = f^{-1}(\alpha)$. Our goal is to find a tuple $q \subseteq P$, such that the conclusion of the statement holds. The first attempt would be to let q be the canonical base of S in the sense of $\mathcal{M}$. As shown below, such q satisfies $\mathrm{dcl}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D)$, but need not be in P. The idea in what follows is to replace S by the graph of a suitable $\mathcal{L}_P$-definable function $h : Z \subseteq M^k \to M$, whose canonical base we know is in $\mathrm{dcl}(P) = P$. We construct that h, by meeting the two assumptions of Proposition 4.4:

(a) the domain of h is $\mathcal{L}_P$-definable and
(b) P^{k+1} is dense in its graph.

Observe that

$$S = \{x \in M^n : f(x) = f(b)\}$$

is $\mathcal{L}_{bD}$-definable. Since $\alpha \notin \mathrm{dcl}(D)$, there is an $\mathcal{L}_b$-definable box around b, whose intersection with S has dimension $< n$. After restricting the domain of f to that box if necessary, we may assume that S has dimension $< n$. By restricting even further, and after permuting coordinates if necessary, we may assume that the closure $\overline{S}$ is the graph of an $\mathcal{L}_{bD}$-definable map $h_0 : Z_0 \subseteq M^{n-1} \to M$. (If $n = 1$, then $S = \{b_2\}$.)

Now let $S_1 = \overline{S \cap P^n}$. The set $S \cap P^n$ is bD-definable (in $\widetilde{\mathcal{M}}$). Since $bD \setminus P \subseteq D$ is dcl-independent over P, by (OP) we obtain that its closure S_1 is $\mathcal{L}_{bD}$-definable. Since $S_1 \subseteq \overline{S}$, it follows that S_1 is the graph of an $\mathcal{L}$-definable function $h_1 : Z_1 \subseteq M^{n-1} \to M$. Observe that P^n is dense in the graph of h_1 (so we have met (b)), but the domain of h_1 need not be $\mathcal{L}_P$-definable. Observe that S_1 is also αD-definable (but not necessarily $\mathcal{L}_{\alpha D}$-definable).

Let k be the 'local dimension' of Z_1 around b_1, that is, the minimum among all $\dim(Z_1 \cap T)$, where $T \subseteq M^{n-1}$ is a box containing b_1. It is then not hard to find a coordinate projection $\pi : M^n \to M^{k+1}$ such that $\pi(S_1)$ is the graph of a map $h : Z \subseteq M^k \to M$, with $\dim Z = k$, and for every $b \in Z_1$,

$$h(\pi(b)) = h_0(b).$$

Since S_1 is $\mathcal{L}_{bD}$-definable and αD-definable, so is h. Moreover, by restricting the domain of π around b if necessary, we may assume that $\pi_{\restriction S_1}$ is injective. Observe that Z contains an open box containing $\pi(b_1)$ that is $\mathcal{L}_{b_1}$-definable. We may replace Z by that box, and hence $h : Z \subseteq M^k \to M$ has $\mathcal{L}_P$-definable domain. Moreover, since P^n is dense in S_1, it follows that P^{k+1} is dense in $\Gamma(h)$. We have thus met (a) and (b).

We can now conclude as follows. By Proposition 4.4, h is $\mathcal{L}_P$-definable. Let $q \subseteq P$ be a canonical base for h in the sense of $\mathcal{M}$. We prove

$$\mathrm{dcl}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D).$$

($\subseteq$) holds, since q is the canonical base for h, and h is αD-definable. For ($\supseteq$), observe that α is defined by the formula

$$\exists x \in \Gamma(h), f(\pi^{-1}(x)) = \alpha,$$

and hence $\alpha \in \mathrm{dcl}(qD)$. By Fact 4.2, it follows that

$$\mathrm{dcl}_{\mathcal{L}(P)}(\alpha D) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(qD) = \mathrm{dcl}(qD),$$

as required. □

Remark 4.9. The above proof actually shows:

Suppose that for $\widetilde{\mathcal{M}}$ has (OP) *and $D \subseteq M$ is* dcl*-independent, such that:*

(1) *the statement of Proposition 4.4 holds, and*
(2) *every $\mathcal{L}$-definable set has a canonical base in P.*

Then $(\mathrm{dcl}')_D$ *holds.*

By Remark 4.7, we know (1). However, we do not know any examples other than dense pairs that satisfy (OP) and (2). It is natural to ask whether $(\mathrm{dcl}')_D$ implies $P \preccurlyeq \mathcal{M}$.

4.2. **Expansions of $\mathcal{M}$ by a dense independent set.** Assume that $P \subseteq M$ is a dense and dcl-independent set.

Proof of $(\mathrm{dcl}')_D$**.** Let $\alpha \in \mathrm{dcl}(bD)$, with $b \in P^n$ and n least possible. In particular, b is dcl-independent over D. We prove that $q = b$ works. There is an $\mathcal{L}_D$-definable map $f : M^n \to M$ with $f(b) = \alpha$. By [2, 1.6], there is an $\mathcal{L}_D$-decomposition of $M^n = \bigcup_i C_i$, such that

- each $f_{\restriction C_i}$ is regular, and
- each open C_i is strongly regular.

Since b is dcl-independent over D, it must belong to an open, and hence regular, C_i. If $f_{\restriction C_i}$ is constant in some coordinate, say the last one, then $\alpha \in \mathrm{dcl}(b_1 pD)$, contradicting the assumption on n. So assume f is strongly regular. By [2, 1.8], $f^{-1}(\alpha) \cap P^n$ is finite, so

$$b \in \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D).$$

Hence $\mathrm{dcl}_{\mathcal{L}(P)}(bD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D)$, and

$$\mathrm{dcl}(bD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D),$$

by Fact 4.2, as required. □

4.3. **Expansions of $\mathcal{M}$ by a multiplicative group with the Mann property.** Let $\mathcal{M} = \langle M, <, +, \cdot, 0, 1\rangle$ be a real closed field. Let G be a dense subgroup of $\langle M^{>0}, \cdot\rangle$. For every $a_1, \dots, a_r \in M$, we call a solution $(q_1, \dots, q_r)$ to the equation

$$a_1 x_1 + \dots + a_r x_r = 1,$$

non-degenerate if for every non-empty $I \subseteq \{1, \dots, r\}$, $\sum_{i \in I} a_i q_i \neq 0$. We say that G has the *Mann property*, if for every $a_1, \dots, a_r \in M$, the above equation has only finitely many non-degenerate solutions $(q_1, \dots, q_r)$ in G^r. Observe that the original definition only involved equations with coefficients a_i from the prime field of $\mathcal{M}$. However, by [5, Proposition 5.6], the two definitions are equivalent.

We now assume that P is a dense subgroup of $\langle M^{>0}, \cdot\rangle$ with the Mann property, and work in $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$.

Proof of $(\mathrm{dcl}')_D$**.** Let $\alpha \in \mathrm{dcl}(PD)$. Denote $K = \mathrm{dcl}(D)$. Then there is a polynomial $Q(x, y) \in K[x, y]$, where x is an n-tuple, and $b_1, \dots, b_n \in P$, such that

$$Q(b_1, \dots, b_n, \alpha) = 0. \tag{6}$$

Indeed, let $f : M^n \to M$ be an $\mathcal{L}_D$-definable map and $b \in P^n$ such that $\alpha = f(b)$. By quantifier elimination for real closed fields, the graph of f is a finite union of sets of the form

$$\{(x, y) \in M^{n+1} : p(x, y) = 0, q_1(x, y) > 0, \dots, q_s(x, y) > 0\},$$

where $p, q_1, \dots, q_s \in K[x, y]$. In particular, there is a polynomial $Q(x, y) \in K[x, y]$, such that for every $x \in R^n$,

$$Q(x, f(x)) = 0.$$

Now, let $b = (b_1, \dots, b_n)$ and, for $i = (i_1, \dots, i_n) \in \mathbb{N}^n$, denote

$$b^i = b_1^{i_1} \cdot \dots \cdot b_n^{i_n}.$$

Let us also write

$$Q(b_1, \dots, b_n, \alpha) = d_1 b^{j_1} \alpha^{k_1} + \dots + d_r b^{j_r} \alpha^{k_r},$$

for some $d_m \in \mathrm{dcl}(D)$ and suitable indices j_m and k_m, $m = 1, \ldots, r$. We may assume that no sub-sum of the above expression is 0, or else replace $Q(b_1, \ldots, b_n, \alpha)$ by that sub-sum. Now, divide equation (6) by $d_r b^{j_r} \alpha^{k_r}$ to obtain an equation

$$q_1 a_1 + \cdots + q_{r-1} a_{r-1} = 1, \tag{7}$$

where

$$q_m = \frac{b^{j_m}}{b^{j_r}} \in P$$

and

$$a_m = -\frac{d_m}{d_r} \alpha^{k_m - k_r} \in \mathrm{dcl}(\alpha D),$$

for $m = 1, \ldots, r-1$. Let $q = (q_1, \ldots, q_{r-1})$. Equation (7) still has the property that no sub-sum of the expression on the left is 0. In other words, q is a non-degenerate solution to

$$x_1 a_1 + \cdots + x_{r-1} a_{r-1} = 1.$$

By Mann property, the last equation has only finitely many non-degenerate solutions in P, and, since being a non-degenerate solution to that equation is an $\mathcal{L}(P)_{\alpha D}$-definable property, we obtain that each $q_m \in \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D)$. Hence $q \in \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D)$.

Moreover, multiplying (7) by α^{k_r}, we obtain that α is solution to a polynomial equation with coefficients in $\mathrm{dcl}(qD)$. Hence $\alpha \in \mathrm{dcl}(qD)$. By Fact 4.2, it follows that

$$\mathrm{dcl}_{\mathcal{L}(P)}(\alpha D) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(qD) = \mathrm{dcl}(qD),$$

as required. □

5. Optimality of the assumptions

In this section, we establish that our results are optimal, in three ways:

- (Example 5.1). If D is not dcl-independent over P, then $P_{ind(D)}$ need not eliminate imaginaries.
- (Proposition 5.4). Suppose $\widetilde{\mathcal{M}}$ has (OP) and D is dcl-independent over P. Then:

$$P_{ind(D)} \text{ eliminates imaginaries} \quad \Leftrightarrow \quad (\mathrm{dcl})_D \quad \Leftrightarrow \quad (\mathrm{dcl}')_D.$$

- (Example 5.5). If we do not assume (OP), the above three properties need not hold.

Example 5.1. We give an example of $\widetilde{\mathcal{M}}$ and D where D is not dcl-independent over P and $P_{ind(D)}$ does not eliminate imaginaries. Let $\mathcal{M}$ be any o-minimal expansion of a real closed field, and $P \subseteq M$ any set such that there are $b_1, b_2, c_1, c_2 \in P$ and $e \in M$, with

(1) $\{b_1, b_2, c_1, c_2, e\}$ dcl-independent, and
(2) $e \notin \mathrm{dcl}(P)$.

(Such an $\mathcal{M}$ could be a dense pair, Mann pair, or expansion of $\mathcal{M}$ by a dense independent set – we will not use any further properties than the above two.) Let $d \in M$ be defined by

$$b_1 + b_2 d = c_1 + c_2 e,$$

and $D = \{d, e\}$. Clearly, D is not dcl-independent over P. By (2), $d \notin \mathrm{dcl}(P)$. Moreover, none of b_1, b_2 is in $\mathrm{dcl}(c_1, c_2, d, e)$, since otherwise both would be in it and hence $\mathrm{rk}(b_1, b_2, c_1, c_2, d, e) = 4$, a contradicting (1).

Observe also that (b_1, b_2) is the unique solution in P^2 to the equation

$$x_1 + x_2 d = c_1 + c_2 e.$$

Indeed, if $(e_1, e_2) \in P^2$ was another one, then $b_1 + b_2 d = e_1 + e_2 d$, yielding $d \in \mathrm{dcl}(P)$, a contradiction. So the set

$$X = \{b_1, b_2\} = \{(x_1, x_2) \in P^2 : x_1 + x_2 d = c_1 + c_2 e\}$$

is both $\{b_1, b_2\}$-definable and $\{c_1, c_2\}$-definable in $P_{ind(D)}$. We claim that X cannot have a smallest cl_P-closed defining set A. Indeed, such a set would have to be contained in $cl_P(b_1, b_2) \cap cl_P(c_1, c_2)$. However,

$$b_i \notin \mathrm{dcl}(c_1, c_2, d, e) \cap P = cl_P(c_1, c_2).$$

Hence $b_i \notin A = cl_P(A)$, which is a contradiction, since $X = \{b_1, b_2\}$ is A-definable.

Our next goal is Proposition 5.4 below. First, a general statement.

Lemma 5.2. *Suppose $P_{ind(D)}$ eliminates imaginaries. Then for every $\alpha \in \mathrm{dcl}(PD)$, there is $q \subseteq P$, such that*

$$\mathrm{dcl}_{\mathcal{L}(P)}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D).$$

Proof. Let $\alpha \in \mathrm{dcl}(PD)$. So there are $b \subseteq P^n$ and an $\mathcal{L}_D$-definable map $f : M^n \to M$, such that $\alpha = f(b)$. The set

$$X = f^{-1}(\alpha) \cap P^n = \{x \in P^n : f(x) = f(b)\}$$

is b-definable in $P_{ind(D)}$. Let $q \subseteq P$ be a canonical base for it in the sense of $P_{ind(D)}$. Now let $\tau \in Aut(\widetilde{\mathcal{M}})$. Observe that X is also αD-definable (in $\widetilde{\mathcal{M}}$). Hence we have

$$\tau_{\restriction \alpha D} = id_{\alpha D} \ \Leftrightarrow\ \tau(X) = X \ \Leftrightarrow\ \tau(q) = q,$$

showing that

$$\mathrm{dcl}_{\mathcal{L}(P)}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(\alpha D).$$

□

Corollary 5.3. *Suppose D is* dcl*-independent over P. Then:*

$$\text{(OP) } \textit{and } P_{ind(D)} \textit{ eliminates imaginaries} \Rightarrow (\mathrm{dcl}')_D.$$

Proof. Let $q \subseteq P$ be as in Lemma 5.2. By Fact 4.2, we have $\mathrm{dcl}(qD) = \mathrm{dcl}_{\mathcal{L}(P)}(qD)$. By Lemma 5.2, the result follows. □

Proposition 5.4. *Suppose $\widetilde{\mathcal{M}}$ has* (OP) *and D is* dcl*-independent over P. Then*

$$P_{ind(D)} \textit{ eliminates imaginaries } \ \Leftrightarrow\ (\mathrm{dcl})_D \ \Leftrightarrow\ (\mathrm{dcl}')_D.$$

Proof. By Theorem A, Proposition 4.3 and Corollary 5.3. □

We finally show that if we do not assume (OP), the above three properties need not hold. We do not know whether (OP) implies them.

Example 5.5. Let $\mathcal{M}$ be our fixed o-minimal expansion of an ordered group, p_1, p_2 two dcl-independent elements, $P = \{p_1\}$ and $\alpha = p_1 + p_2$. Then $(\mathrm{dcl}')_\emptyset$ fails for this α, since there is no $q \in P$ such that $\alpha \in \mathrm{dcl}(q)$. Of course, (OP) also fails for $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$. Indeed, $\{p_2\}$ is dcl-independent over P, but $\alpha \in \mathrm{dcl}_{\mathcal{L}(P)}(p_2) \setminus \mathrm{dcl}(p_2)$, and hence, by Fact 4.2, (OP) fails.

References

[1] A. Dolich, C. Miller, C. Steinhorn, *Structures having o-minimal open core*, Trans. AMS 362 (2010), 1371-1411.

[2] A. Dolich, C. Miller, C. Steinhorn, *Expansions of o-minimal structures by dense independent sets*, APAL 167 (2016), 684-706.

[3] L. van den Dries, *Dense pairs of o-minimal structures*, Fundamenta Mathematicae 157 (1988), 61-78.

[4] L. van den Dries, Tame topology and o-minimal structures, Cambridge University Press, Cambridge, 1998.

[5] L. van den Dries, A. Günaydın, *The fields of real and complex numbers with a small multiplicative group*, Proc. London Math. Soc. 93 (2006), 43-81.

[6] P. Eleftheriou, A. Hasson, and G. Keren, *On definable Skolem functions in weakly o-minimal non-valuational structures*, preprint (2016).

[7] P. Eleftheriou, A. Günaydin, and P. Hieronymi, *Structure theorems in tame expansions of o-minimal structures by dense sets*, preprint (2015).

[8] A. Hasson, A. Onshuus, *Embedded o-minimal structures*, Bull. London Math. Soc 42 (2010), 64-74.

[9] W. Hodges, A shorter model theory, Cambridge University Press, 1997.

[10] D. Marker, *Introduction to the Model Theory of Fields*, Chapter 1 in Model Theory of Fields, 1–37, Springer-Verlag, Berlin, 1996.

[11] A. Pillay, *Some remarks on definable equivalence relations in o-minimal structures*, J. Symb. Logic, Volume 51, Number 3 (1986), 709-714.

[12] P. Simon, A guide to NIP theories, Lecture Notes in Logic, Cambridge University Press, 1st edition, 2015.

[13] R. Wencel, *On the strong cell decomposition property for weakly o-minimal structures*, Math. Log. Quart. 59 (2013), 452–470.

[14] R. Wencel, *Weakly o-minimal nonvaluational structures*, Ann. Pure Appl. Logic 154 (2013), 139–162.

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