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Definably Discrete Complete Expansions of Ordered Fields

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Abstract

Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ be a first order expansion of an ordered field. We say that $\mathcal{M}$ is definably discrete complete if every definable discrete subset of M has a maximum element. These expansions in which every unary definable subset is discrete or has interior, are definably (Dedekind) complete. Type complete (locally o-minimal) structures have been studied in [9], [4], and [8]. If $\mathcal{M}$ is definably complete and type complete, then every definable subset of M is discrete or has interior (see [8]). Here, we prove the converse of this result: If $\mathcal{M}$ is definably discrete complete and every definable subset of M is discrete or has interior, then $\mathcal{M}$ is type complete. Lastly, we suggest some problems.

Keywords: Definably Complete,Definably Discrete Complete,Type Complete,Locally o-Minimal.
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1 Introduction

Definable completeness is the first order version of Dedekind completeness in ordered structures. Structures having this property have some nice (first order versions of) topological and geometric properties inherited from the ordered real field. These structures have been studied during last decades (See, e.g., [7], [3], [4], and [8]).

Type complete (locally o-minimal) structures have been studied in [9], [4], and [8]. If $\mathcal{M}$ is definably complete and type complete, then every definable subset of M is discrete or has nonempty interior (see [8]). Here, we prove the converse of this result: If $\mathcal{M}$ is definably discrete complete and every definable subset of M is discrete or has nonempty interior, then $\mathcal{M}$ is type complete. This provides a criterion for type completeness of definably complete ordered fields.

Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \ldots)$ be a first order expansion of an ordered field.

Definition 1.1. The structure $\mathcal{M}$ is definably complete if every (nonempty) bounded definable subset of M has a least upper bound in M.

Note that by a definable subset of M, we mean a subset of M which is described by a first order formula in the language of the structure $\mathcal{M}$, with some parameters from M.

It is clear that definable completeness is a first order schema in the language $\mathcal{L} = \{<, +, \cdot, 0, 1, \ldots\}$, which we denote it by DC.

Definition 1.2. The structure $\mathcal{M}$ is said to be type complete if for every definable subset $A \subseteq M$ and every element $a \in M$, there exists $b > a$ such that $(a, b) \subset A$ or $(a, b) \subset (M \setminus A)$.

Type completeness which is a first order schema in the language $\mathcal{L} = \{<, +, \cdot, 0, 1, \ldots\}$, is denoted by TC. By $\mathcal{M} \vDash DC$ or $\mathcal{M} \vDash TC$, we mean the structure $\mathcal{M}$ satisfies definable completeness or type completeness, respectively.

Below, we give two others first order schemas that will be used.

DDC: Every (nonempty) bounded definable discrete subset of $\mathcal{M}$ has a maximum element.

ID: Every definable subset of $\mathcal{M}$ is discrete, or (otherwise) has (nonempty) interior.

Proposition 1.3. [8] *$\mathcal{M} \vDash DC{+}TC$ if and only if every definable subset of M is a disjoint union of some many open intervals and a bounded closed discrete definable subset of M.*

Theorem 1.4. [8] *If $\mathcal{M} \vDash DC + TC$, then $\mathcal{M} \vDash ID$. (But only the TC property is sufficient.)*

In this note, we prove that if $\mathcal{M} \vDash DC + ID$, then $\mathcal{M} \vDash TC$. Hence, ID and TC are equivalent on DC.

2 Main Results

Lemma 2.1. *If $\mathcal{M} \vDash ID$, then we have the following.*
i) The set of boundary points of any definable subset of M is discrete.
ii) Every definable discrete subset of M is closed and bounded.

Proof. i) Let $X \subseteq M$ be a definable set in $\mathcal{M}$. Then, the boundary of X, $bd(X)$, is a definable subset of M. Since the interior of $bd(X)$ is empty, then by the assumption, $bd(X)$ is discrete.

ii) Let X be a definable discrete subset of M. If $x \in M \setminus X$ is an accumulation point of X, then $X \cup \{x\}$ is a definable subset of M which can not be discrete. So, by the assumption, $X \cup \{x\}$ must have nonempty interior which is impossible. Hence, X is closed.

If X is not bounded, then the element $0 \in M$ will be an accumulation point of the definable discrete set $\{\frac{1}{x}|x \in X \vee x \neq 0\}$ which is impossible by (i). □

Proposition 2.2. *$\mathcal{M} \models DDC + ID$ if and only if $\mathcal{M} \models DC + ID$.*

Proof. The "if part" is clear. For the other direction, suppose that $\mathcal{M} \models DDC + ID$. Let X be a nonempty bounded definable subset of M. Then, by Lemma 2.1, $bd(X)$ is discrete. By the assumption, $bd(X)$ has a maximum element. This element is a least upper bound for X. □

Now, we can state our main result.

Theorem 2.3. *Let $\mathcal{M} \models DDC + ID$. Then, we have the following.*
i) $\mathcal{M} \models TC$.
ii) If every infinite definable subset of M has nonempty interior, then $\mathcal{M}$ is o-minimal, i.e. every definable subset of M is a finite union of open intervals and points.

Proof. i) By Theorem 1.4, it is sufficient to show that every definable subset of M is a disjoint union of intervals and a bounded closed discrete definable subset of M. Let $X \subseteq M$ be definable. Using Proposition 2.2, we have $\mathcal{M} \models DC$. Thus, by [2], the interior of X, X°, is a disjoint union of open intervals in M. Since $\mathcal{M} \models ID$, then the definable set $X \setminus X^\circ$ is discrete, so by Lemma 2.1, it is bounded and closed. As $X = X^\circ \cup (X \setminus X^\circ)$, we are done.

ii) Let $X \subseteq M$ be definable. Then, $X = \bigcup_{j \in J} I_j \cup D$ where I_j's are interval and D is a definable discrete set. But, by the assumption, D is finite. On the other hand, the set consisting of end points of the intervals I_j's is definable and discrete, so it is finite. Hence, every definable subset of M is a finite union of intervals and points. □

3 Some Problems about Grothendieck rings of some variants of o-minimal structures

Grothendieck ring for o-minimal structures is introduced in [6] and in [5] Grothendieck ring of o-minimal expansion of ordered abelian groups is specified.

Definition 3.1. We consider equivalence classes of definable sets that are definable isomorphic. these sets can be expand to a semiring and then a ring. The minimal expand ring that definable equivalence sets can be embed to it, nominated to Grothendieck ring. The Grothendieck ring of the theory of $\mathcal{M}$, denoted $\mathcal{K}_0$(M).

Problems

1.Investigate the Grothendieck ring for locally o-minimal expansions of an ordered field.
2.Investigate the Grothendieck ring of only discrete definable sets for locally o-minimal expansions of an ordered field.
3.If $\mathcal{M}$ be expand of an ordered group and its Grothendieck ring be ring of integer numbers, Is $\mathcal{M}$ o-minimal?

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