DEFINABLE QUOTIENTS OF LOCALLY DEFINABLE GROUPS

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Abstract. We analyze groups definable in o-minimal structures as a combination of semi-linear groups and groups definable in o-minimal expansions of real closed fields. The analysis goes through their locally definable covers. We thus investigate locally definable abelian groups $\mathcal{U}$ in various settings and examine conditions under which the quotient of $\mathcal{U}$ by a discrete subgroup might be definable. This turns out to be related to the existence of the type-definable subgroup $\mathcal{U}^{00}$ and to the divisibility of $\mathcal{U}$.

As a corollary of the above analysis, we prove the Compact Domination Conjecture in o-minimal expansions of ordered groups.

1. Introduction

This paper is divided into two, mostly independent, parts. In the first part we analyze definable groups in the standard setting of an o-minimal expansion of an ordered group, in terms of semi-linear groups and groups definable in o-minimal expansions of real closed fields. At a certain stage of the argument we are required to examine another problem, which leads us to the second part. Here we consider $\bigvee$-definable abelian groups in various settings and ask when such groups have definable quotients.

Part I

Let $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ be an o-minimal expansion of an ordered group. When $\mathcal{M}$ expands a real closed field (with $+$ not necessarily one of the field operations) there is strong compatibility of definable sets with the field structure. For example, each definable function is piecewise differentiable with respect to the field structure. Other powerful tools, such as the triangulation theorem, are available as well ([4]). At the other end, when $\mathcal{M}$ is a linear structure, such as a reduct of an ordered vector space over an ordered division ring, then every definable set is *semi-linear*.

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There is a third possibility, where $\mathcal{M}$ is not linear, so by the Trichotomy Theorem for o-minimal structures (see [26]), there is a definable real closed field R on some interval in M, and yet the underlying domain of R is necessarily a bounded interval and not the whole of M. Such a structure is called *semi-bounded* (and non-linear), and definable sets in this case turn out to be a combination of semi-linear sets and sets definable in o-minimal expansions of fields (see [5], [23], [13]). An important example is the expansion of the ordered vector space $\langle \mathbb{R}; <, +, x \mapsto ax \rangle_{a\in\mathbb{R}}$ by all bounded semialgebraic sets.

The ultimate goal of this project is to examine definable groups in o-minimal expansions of ordered groups, and to reduce the analysis of definable groups to semi-linear groups and groups which are definable in o-minimal expansions of real closed fields. Therefore, most of our work is intended for a semi-bounded structure which is non-linear. Such a reduction was proposed in Conjecture 2 from [23] and a first step towards it was carried out in [13].

Let us recall some definitions.

Short sets and long dimension. Let $\mathcal{M}$ be a sufficiently saturated o-minimal expansion of an ordered group. Following [23], we call an element $a \in M$ *short* if either $a = 0$ or the interval $(0, a)$ supports a definable real closed field; otherwise a is called *tall*. An element of M^n is called *short* if all its coordinates are short. An interval (a, b) is called *short* if $b - a$ is short, and otherwise it is called *long*. As is shown in [5], $\mathcal{M}$ is semi-bounded if and only if all unbounded rays $(a, +\infty)$ are long. However, a semi-bounded $\mathcal{M}$ also has bounded intervals which are long. A definable set $X \subseteq M^n$ is called *short* if it is in definable bijection with a subset of I^n for some short interval I. The image of a short set under a definable map is short.

Following [13], we say that *the long dimension* of a definable $X \subseteq M^n$, $\mathrm{lgdim}(X)$, is the maximum k such that X contains a definable homeomorphic image of I^k, for some long interval I. The results in [13] show that every definable subset of M^n can be decomposed into "long cones" and as a result it follows that a definable $X \subseteq M^n$ is short if and only if $\mathrm{lgdim}(X) = 0$. We call X *strongly long* if $\mathrm{lgdim}(X) = \dim(X)$; this is for example the case with a cartesian product of long intervals. Note that all these notions are invariant under definable bijections.

Roughly speaking, strongly long sets and short sets are "orthogonal" to each other. The idea is that the structure which $\mathcal{M}$ induces on short sets comes from an o-minimal expansion of a real closed field, while the structure induced on strongly long sets is closely related to the semi-linear structure. More precisely, implicit in [13] one can find: if $p(x)$ is a complete n-dimensional type over A such that every n-dimensional formula in $p(x)$ is strongly long then its semi-linear formulas determine the type.

As we will see in examples (see Section 5), the analysis of definable groups forces us to use the language of $\bigvee$-definable groups, so we recall some definitions.

$\bigvee$-definable and locally definable sets. Let $\mathcal{M}$ be a sufficiently saturated, not necessarily o-minimal, structure. A $\bigvee$*-definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n for some fixed n (where $|I|$ is small relative to the saturation of $\mathcal{M}$) and for every $i, j \in I$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). Following [6], we say that $\langle \mathcal{U}, \cdot \rangle$ is *locally definable* if $|I|$ is countable. We are mostly interested here in *definably generated* groups, namely $\bigvee$-definable groups which are generated as a group by a definable subset. These groups are locally definable. An important example of such groups is the universal cover of a definable group (see [10]).

A map $\phi : \mathcal{U} \to \mathcal{H}$ between $\bigvee$-definable (locally definable) groups is called $\bigvee$*-definable (locally definable)* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $graph(\phi) \cap (X \times Y)$ is a definable set.

In an o-minimal expansion of an ordered group, a $\bigvee$-definable group $\mathcal{U}$ is called *short* if $\mathcal{U}$ is given as a bounded union of definable short sets. If $\mathcal{U} = \bigcup_{i \in I} X_i$ then we let $\operatorname{lgdim}(\mathcal{U}) = max_i(\operatorname{lgdim}(X_i))$. We say that $\mathcal{U}$ is *strongly long* if $\dim(\mathcal{U}) = \operatorname{lgdim}(\mathcal{U})$.

We also need the following definitions.

Definition 1.1. (1) (see [6]) For a $\bigvee$-definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a bounded union of definable sets).

(2) (see [1]) In an o-minimal structure, a $\bigvee$-definable group $\mathcal{U}$ is called *connected* if there is no $\bigvee$-definable compatible subset $\emptyset \subsetneqq \mathcal{V} \subsetneqq \mathcal{U}$ which is both closed and open with respect to the group topology (see Section 2.1 below).

Remark 1.2. It is easy to see that, in an o-minimal structure, if a $\bigvee$-definable group $\mathcal{U}$ is generated by a definably connected set which contains the identity, then it is connected.

We are now ready to state the main results of this paper.

We assume in the rest of Part I that $\mathcal{M} = \langle M, <, +, \cdots \rangle$ is an o-minimal expansion of an ordered group.

Note that in the special case where $\mathcal{M}$ expands a real closed field, the results below become trivial (since in this case all definable sets are short), and in the case where $\mathcal{M}$ is semi-linear, they reduce to the main theorem from [15] (since in this case every definable short set is finite).

1.1. **The universal cover of a definably compact group.** We first note (see [23, Lemma 7.1]) that every definably compact group in a semi-bounded structure is necessarily bounded; namely, it is contained in some cartesian product of bounded intervals.

Theorem 1.3. *Let G be a definably compact, definably connected group of long dimension k and let $\widehat{F}:\widehat{G}\to G$ be the universal cover of G. Then there exist an open subgroup $\widehat{\mathcal{H}}\subseteq\langle M^k,+\rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i:\widehat{\mathcal{H}}\to\widehat{G}$, with $i(\widehat{\mathcal{H}})$ central in G, such that $\mathcal{U}=\widehat{G}/i(\widehat{\mathcal{H}})$ is generated by a short definable set. Namely, we have the following exact sequence with locally definable maps i, π and $\widehat{F}$:*

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] & \widehat{G} \arrow[r, "\pi"] \arrow[d, "\widehat{F}"] & \mathcal{U} \arrow[r] & 1 \\ & & G & & \end{tikzcd}$$

If we let $\mathcal{H}=\widehat{F}(i(\widehat{\mathcal{H}}))$, then $\mathcal{H}$ is the largest connected, strongly long, locally definable subgroup of G, namely it contains every other such group.

When G is abelian, it follows from [8] and [23] that $\ker(\widehat{F})\simeq\mathbb{Z}^{\dim G}$. Note that since $\mathcal{U}$ is generated by a definable short set, there is a definable real closed field R such that $\mathcal{U}$ is $\bigvee$-definable in an o-minimal expansion of R. One immediate corollary of the above theorem is that every definably compact group G which is strongly long is semi-linear, because in this case $\mathcal{H}=G$.

1.2. **Covers by extensions of definable short groups.** In the next result we want to replace the locally definable group $\mathcal{U}$ from Theorem 1.3 by a definable short group $\overline{K}$. Roughly speaking, it says that G is close to being an extension of a short definable group by a semi-linear group, and the distance from being such a group is measured by the kernel of the map $\overline{F}$ below.

Theorem 1.4. *Let G be a definably compact, definably connected group of long dimension k. Then G has a locally definable cover $\overline{F}:\overline{G}\to G$ with the following properties: there is an open subgroup $\widehat{\mathcal{H}}\subseteq\langle M^k,+\rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i:\widehat{\mathcal{H}}\to\overline{G}$, with $i(\widehat{\mathcal{H}})$ central in $\overline{G}$, such that $\overline{K}=\widehat{G}/i(\widehat{\mathcal{H}})$ is a definably compact* **definable** *short group. Namely, we have the following exact sequence with locally definable maps i, π and $\overline{F}$:*

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] & \overline{G} \arrow[r, "\pi"] \arrow[d, "\overline{F}"] & \overline{K} \arrow[r] & 1 \\ & & G & & \end{tikzcd}$$

If we take $\mathcal{H} \subseteq G$ as in Theorem 1.3, then there is also a locally definable, central extension G' of $\overline{K}$ by $\mathcal{H}$, with a locally definable homomorphism from G' onto G.

When G is abelian so is $\overline{G}$ and $\ker(\overline{F}) \simeq \mathbb{Z}^k + F$, for a finite group F.

Part II

1.3. **$\bigvee$-definable abelian groups and their definable quotients.** In order to pass from the locally group $\mathcal{U}$ in Theorem 1.3 to the definable group $\overline{K}$ in Theorem 1.4 we need to examine general $\bigvee$-definable abelian groups and their possible definable quotients. In Theorem 7.9 we prove the following.

Theorem 1.5. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group in an o-minimal expansion of an ordered group, with $\mathcal{U}$ generated by a definably compact definable set. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definably compact definable group.

The analysis in this part of the paper goes through new results about the existence of the type-definable group $\mathcal{U}^{00}$, for a $\bigvee$-definable abelian group $\mathcal{U}$. Recall ([19, Section 7]) that for a definable, or $\bigvee$-definable group $\mathcal{U}$, we write $\mathcal{U}^{00}$ for the smallest, if such exists, type-definable subgroup of $\mathcal{U}$ of bounded index (in particular we require that $\mathcal{U}^{00}$ is contained in a definable subset of $\mathcal{U}$). From now on we use the expression "$\mathcal{U}^{00}$ *exists*" to mean that "there exists a smallest type-definable subgroup of $\mathcal{U}$ of bounded index, which we denote by $\mathcal{U}^{00}$". (By "a type definable subgroup of $\mathcal{U}$ of bounded index" we mean a type definable subgroup $\mathcal{H}$ of $\mathcal{U}$ such that there are no new cosets of $\mathcal{H}$ in $\mathcal{U}$ in elementary extensions of $\mathcal{M}$).

When $\mathcal{U}$ is a definable group in an NIP structure, then $\mathcal{U}^{00}$ exists (see Shelah's result in [19, Proposition 6.1]). When $\mathcal{U}$ is a $\bigvee$-definable group in an NIP structure, then $\mathcal{U}^{00}$ may not always exist. However, if we assume that *some* type-definable subgroup of bounded index exists, then there is a smallest one (see [19, Proposition 7.4]). Recall that a definable $X \subseteq \mathcal{U}$ is called *generic* if boundedly many translates of X cover $\mathcal{U}$. In Theorem 6.10 we prove:

Theorem 1.6. *Let $\mathcal{U}$ be an abelian $\bigvee$-definable group in an NIP structure. If the definable non-generic sets in $\mathcal{U}$ form an ideal and $\mathcal{U}$ contains at least one definable generic set, then $\mathcal{U}^{00}$ exists.*

We also prove (see Corollary 7.8):

Theorem 1.7. *Assume that $\mathcal{U}$ is a connected abelian $\bigvee$-definable group in an o-minimal expansion of an ordered group. If $\mathcal{U}$ is generated by a definable, definably compact subset and if $\mathcal{U}$ contains at least one definable generic set, then*

(i) $\mathcal{U}^{00}$ *exists.*
(ii) $\mathcal{U}/\mathcal{U}^{00}$*, equipped with the logic topology (see Section 7) is isomorphic to* $\mathbb{R}^k \times \mathbb{T}^r$*, with* $k + r = \dim \mathcal{U}$ *and* $\mathbb{T}$ *the circle group.*
(iii) There is a definable, definably compact G*, with* $\dim G = \dim \mathcal{U}$*, and a* $\bigvee$*-definable surjective homomorphism* $\phi : \mathcal{U} \to G$*.*

In the same section we discuss the following conjecture, which we do not know to be true even when $\mathcal{U}$ is a subgroup of a definable group:

Conjecture 1.8. *Let* $\mathcal{U}$ *be a connected abelian* $\bigvee$*-definable group in an o-minimal structure which is definably generated. Then*
(i) $\mathcal{U}$ *contains a definable generic set.*
(ii) $\mathcal{U}$ *is divisible.*

We also show that it is sufficient to prove (i) under restricted conditions, in order to deduce the full conjecture.

1.4. **Compact Domination.** The relationship between a definable group G and the compact Lie group G/G^{00} has been the topic of quite a few papers. In [12], [18], [20] the related so-called Compact Domination Conjecture was solved for semi-linear groups and for groups definable in expansions of real closed fields. Using the above analysis we can complete the proof of the conjecture for groups definable in arbitrary o-minimal expansions of ordered groups.

Theorem 1.9. *Let* G *be a definably compact, definably connected group. Let* $\pi : G \to G/G^{00}$ *denote the canonical homomorphism. Then,* G *is compactly dominated by* G/G^{00}*. That is, for every definable set* $X \subseteq G$*,*

$$\dim(X) < n \Rightarrow \mathbf{Haar}\big(\pi(X)\big) = 0. \tag{1}$$

1.5. **Notation.** Let us finish this section with a couple of notational remarks. Given a group $\langle G, + \rangle$ and a set $X \subseteq G$, we denote, for every $n \in \mathbb{N}$,

$$X(n) = \overbrace{(X - X) + \cdots + (X - X)}^{n-\text{times}}$$

We assume familiarity with the notion of definable compactness. Whenever we write that a set is definably compact, or definably connected, we assume in particular that it is definable.

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2. Preliminaries: $\bigvee$-definable groups, extensions of abelian groups, pushout and pullback

From now until the end of Section 4, we assume that $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of an ordered group.

However, the only use of this assumption is to guarantee the existence of definable Skolem functions, which allows us to replace every definable quotient by a definable set. Any structure in which this is true will be just as good, or, if we are willing to work in $\mathcal{M}^{eq}$ then any structure will work.

2.1. **$\bigvee$-definable groups and compatible subgroups.** If $\mathcal{U} \subseteq M^n$ is a $\bigvee$-definable group then, by [2, Theorem 4.8], it can be endowed with a manifold-like topology τ, making it into a topological group. Namely, there is a bounded collection $\{U_i : i \in I\}$ of definable subsets of $\mathcal{U}$, whose union equals $\mathcal{U}$, such that each U_i is in definable bijection with an open subset of M^k ($k = \dim\mathcal{U}$), and the transition maps are continuous. The group operation and group inverse are continuous with respect to this induced topology. The topology τ is determined by the ambient topology of M^n in the sense that at every generic point of $\mathcal{U}$ the two topologies coincide.

From now on, whenever we refer to a topology on G, it is τ we are considering.

As mentioned earlier, for a $\bigvee$-definable group $\mathcal{U}$, we say that a subset $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset* if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set. Clearly then, the only compatible $\bigvee$-definable subgroups of a definable group are the definable ones. Note that if $\phi : \mathcal{U} \to \mathcal{V}$ is a $\bigvee$-definable homomorphism between $\bigvee$-definable groups then $\ker(\phi)$ is a compatible $\bigvee$-definable normal subgroup of $\mathcal{U}$. Compatible subgroups are used in order to obtain $\bigvee$-definable quotients, but for that we need to restrict ourselves to locally definable groups.

Together with [6, Theorem 4.2], we have:

Fact 2.1. *If $\mathcal{U}$ is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup, $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a $\bigvee$-definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \to \mathcal{V}$ whose kernel is $\mathcal{H}$.*

2.2. **Pushouts and definability.** *In the following three subsections, all groups are assumed to be abelian and all arrows represent group homomorphisms.*

Several steps of the proof require us to construct extensions of abelian groups with certain maps attached to them. All constructions are standard in the classical theory of abelian groups but because we are concerned here with definability issues we review the basic notions (see [16] for the classical treatment). The proofs of these basic results are given in the appendix. Although we chose to present the constructions below in the more common language of pushouts and pullbacks, it is also possible to carry them out in the less canonical (but possibly more constructive) language of sections and cocycles.

Definition 2.2. Given homomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \\ C & \end{tikzcd}$$

the triple (D, γ, δ) (or just D) is called *a pushout* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma"] \\ C \arrow[r, "\delta"] & D \end{tikzcd}$$

and for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma'"] \\ C \arrow[r, "\delta'"] & D' \end{tikzcd} \tag{2}$$

there is a unique $\phi : D \to D'$ such that $\phi\gamma = \gamma'$ and $\pi\delta = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D \to D'$ as required then we say that *the pushout is (locally) definable.*

Proposition 2.3. *Assume that we are given the following diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \\ C & \end{tikzcd}$$

(i) Let (D, γ, δ) be a pushout. Then

$$\ker(\gamma) = \alpha(\ker(\beta)).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data is definable. Then there exists a definable pushout (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data is locally definable and $\alpha(A)$ is a compatible subgroup of B. Then there exists a locally definable pushout (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that α is injective. If we let $E = B/\alpha(A)$ and $\pi : B \to E$ the projection map then there is a locally definable surjection $\pi' : D \to E$ such that the diagram below commutes and both sequences are exact. In particular, $\ker(\pi') = \delta(C)$ *is a compatible subgroup of D.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\pi} & E & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle id_E} & & \\ 0 & \longrightarrow & C & \xrightarrow{\delta} & D & \xrightarrow{\pi'} & E & \longrightarrow & 0 \end{array}$$

We also need the following general fact, for which we could not find a reference (see appendix for proof):

Lemma 2.4. *Assume that we are given the following commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} \\ C & \xrightarrow{\delta} & D \\ \Big\downarrow{\scriptstyle\eta} & & \Big\downarrow{\scriptstyle\mu} \\ E & \xrightarrow{\xi} & F \end{array} \tag{3}$$

with D the pushout of B and C over A (via $\alpha, \beta, \gamma, \delta$), and F the pushout of B and E over A (via α, $\eta\beta$, $\mu\gamma$ and ξ). Then F is also the pushout of E and D over C (via η, δ, μ, ξ).

2.3. **Pullbacks and definability.**

Definition 2.5. Given homomorphisms

$$\begin{array}{ccc} & & B \\ & & \Big\downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

the triple (D, γ, δ) (or just D) is called *a pullback* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$\begin{array}{ccc} D & \xrightarrow{\gamma} & B \\ \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

and for every commutative diagram

$$
\begin{tikzcd}
D' \arrow[r, "\gamma'"] \arrow[d, "\delta'"'] & B \arrow[d, "\alpha"] \\
C \arrow[r, "\beta"'] & A
\end{tikzcd}
\tag{4}
$$

there is a unique $\phi : D' \to D$ such that $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D' \to D$ as required then we say that *the pullback is (locally) definable.*

Proposition 2.6. *Assume that we are given the following diagram*

$$
\begin{tikzcd}
 & B \arrow[d, "\alpha"] \\
C \arrow[r, "\beta"'] & A
\end{tikzcd}
$$

(i) Let (D, γ, δ) be a pullback. Then

$$\gamma(\ker(\delta)) = \ker(\alpha).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data is definable. Then there exists a unique definable pullback (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data is locally definable. Then there exists a locally definable pullback (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that β is surjective. Let $G = \ker(\gamma)$ and $H = \ker(\beta)$. Then G, H are locally definable and compatible in D and C, respectively. Moreover, there is a locally definable isomorphism $j : G \to H$ such that the following diagram commutes and both sequences are exact.

$$
\begin{tikzcd}
0 \arrow[r] & G \arrow[r, "id_G"] \arrow[d, "j"'] & D \arrow[r, "\gamma"] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H \arrow[r, "id_H"] & C \arrow[r, "\beta"] & A \arrow[r] & 0
\end{tikzcd}
$$

2.4. **Additional lemmas.**

Lemma 2.7. *Assume that the sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

is exact and that we have a surjective homomorphism $\mu : C \to D$. Let $A' = \ker(\mu\pi) \subseteq B$. Then the following diagram commutes and both sequences are *exact. If all data is locally definable then so is A' and the associated maps.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "i"'] & B \arrow[r, "\pi"] \arrow[d, "id"] & C \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & A' \arrow[r, "id"] & B \arrow[r, "\mu\pi"] & D \arrow[r] & 0
\end{tikzcd}$$

Proof. This is trivial. □

Lemma 2.8. *Assume that we have surjective homomorphisms $F : B \to G$ and $F' : B \to G'$ with $\ker(F') \subseteq \ker(F)$. Then there is a canonical surjective homomorphism $h : G' \to G$, given by $h(g') = g$ if and only if there exists $b \in B$ with $F'(b) = g'$ and $F(b) = g$. The kernel of h equals $F'(\ker(F))$ and if all data is locally definable then so is h.*

Proof. Algebraically, this is just the fact that if $B_1 \subseteq B_2 \subseteq B$ then there is a canonical homomorphism $h : B/B_1 \to B/B_2$, whose kernel is B_2/B_1.

As for definability, assume that B, G, G', and F, F' are $\bigvee$-definable, and take definable sets $X \subseteq G$ and $X' \subseteq G'$. We want to show that the intersection of *graph*(h) with $X' \times X$ is definable. Since F', F are $\bigvee$-definable and surjective, there exists a definable $Y \subseteq B$ such that $F'(Y) \supset X'$ and $F(Y) \supseteq X$. Now, for every $g' \in X'$ there exists $b \in Y$ such that $F'(b) = g'$, and we have $h(g') = F(b)$. Thus, the intersection of *graph*(h) with $X' \times X$ is definable. □

Remark 2.9. All statements from Proposition 2.3 to Lemma 2.8 hold under the more general assumption that $\mathcal{M}$ is any sufficiently saturated structure (not necessarily o-minimal) which has definable Skolem functions. This is because the definability issues in the statements are all handled based on Fact 2.1, which can be proved for such a more general $\mathcal{M}$.

3. The universal cover of G

3.1. **The local subgroup $H \subseteq G$ and the generic set $B+H$.** We assume that $\langle G, + \rangle$ is a definable abelian group. Recall that $X \subseteq G$ is *generic* if finitely many group translates of X cover G. Using terminology from [22], a definable set $X \subseteq G$ is called *G-linear* if for every $g, h \in X$ there is an open neighborhood U of 0 (here and below, we always refer to the group topology of G), such that $(g - X) \cap U = (h - X) \cap U$. Clearly, every open subset of a definable subgroup of G is a G-linear set. More generally, every group translate of such a set is also G-linear. As is shown in [22], if a G-linear subset contains 0 then it contains an infinitesimal subgroup of G. When the group G is $\langle M^n, + \rangle$ a G-linear subset is also called *affine*. We call a definable G-linear subset $X \subseteq G$ *a local subgroup of G* if it is definably connected and $0 \in X$.

The G-linear set $G_0 \subseteq G$ and the H-linear set $H_0 \subseteq H$ are *definably isomorphic* if there exists a definable bijection $\phi : G_0 \to H_0$ such that for every $g, h, k \in G_0$, $g - h + k \in G_0$ if and only if $\phi(g) - \phi(h) + \phi(k) \in H_0$, in which case we have $\phi(g - h + k) = \phi(g) - \phi(h) + \phi(k)$. If $\phi : G_0 \to H_0$ is an isomorphism of local subgroups of G and H, respectively, and $\phi(0_G) = 0_H \in H_0$, then one can show that for all $g, k \in G_0$, if $g + k \in G_0$ then $\phi(g) + \phi(h) \in H_0$ and we have $\phi(g + h) = \phi(g) + \phi(h)$.

Two G-linear sets $X, Y \subseteq G$ are called *G-equivalent* if for every $x \in X$ and $y \in Y$, there are open neighborhoods $V \ni x$ and $W \ni y$ such that $(V \cap X) - x = (W \cap Y) - y$. Clearly, two G-equivalent sets have the same dimension (because they are locally the same up to group translation).

Our starting point is Proposition 5.4 from [13], which comes out of the analysis of definable sets in semi-bounded structures.

We recall that *a k-long cone in M^n* is a set of the form

$$C = \left\{ b + \sum_{i=1}^{k} \lambda_i(t_i) : b \in B,\, t_i \in J_i \right\},$$

where B is a short cell, each $J_i = (0, a_i)$ is a long interval and $\lambda_1, \ldots, \lambda_k$ are M-independent partial linear maps from J_i into M^n (that is, for all $t_1, \ldots, t_k \in M$, $\lambda_1(t_1) + \cdots + \lambda_k(t_k) = 0$ implies $t_1 = \cdots = t_k = 0$). It is required further that for each $x \in C$ there are unique b and λ_i's with $x = b + \sum_{i=1}^{k} \lambda_i(t_i)$. For such a C, we let

$$\langle C\rangle^+ = \left\{ \sum_{i=1}^{k} v_i t_i : t_i \in J_i \right\} \quad \text{and} \quad \langle C\rangle = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in \pm J_i \cup \{0\} \right\}.$$

A *long cone* is a k-long cone for some k.

Below we use $\oplus$ and $\ominus$ for group addition and subtraction in G and use $+$ and $-$ for the group operations in $\mathcal{M}$.

Fact 3.1. [13] *Let $\langle G, \oplus\rangle$ be a definably compact abelian group of long dimension k. Then G contains a definable, generic, k-long cone C on which the group topology of G agrees with the o-minimal topology. Furthermore, for every $a \in C$ there exists an open neighborhood $V \subseteq G$ of a such that for all $x, y \in V \cap (a + \langle C\rangle)$,*

$$x \ominus a \oplus y = x - a + y.$$

Our goal is to prove:

Proposition 3.2. *Let $\langle G, \oplus\rangle$ be a definably compact, definably connected abelian group. Then there exists a definably connected, k-dimensional strongly long local subgroup set $H \subseteq G$ and a definable short set $B \subseteq G$, $\dim(B) = \dim(G) - k$, satisfying:*

(1) $\langle H, \oplus\rangle$ is definably isomorphic, as a local group, to $\langle H', +\rangle$, where $H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k) \subseteq M^k$, with each $e_i > 0$ tall in M.
(2) The set $H \oplus B = \{h \oplus b : h \in H\ b \in B\}$ is generic in G.

We first need the following.

Lemma 3.3. *Let $\{X_b : b \in B\}$ be a definable family of subsets of M^n, each of long dimension $\leq \ell$, and B a short set. If $S = \bigcup_{b\in B} X_b$ then* $\mathrm{lgdim}(S) \leq \ell$.

Proof. First consider the set $\bar{S} = \{(b,x) \in B\times S : x \in X_b\}$. By [13, Corollary 3.4], the set $\bar{S}$ has long dimension $\leq \ell$. But clearly S is the image of $\bar{S}$ under a definable map. It now follows from [13, Corollary 2.25] that $\mathrm{lgdim}(S) \leq \mathrm{lgdim}(\bar{S}) \leq \ell$. □

Proof of Proposition 3.2. We start with the generic k-long cone $C \subseteq G$ given by Fact 3.1. We assume that the group topology (denoted τ-topology) on C agrees with the ambient M^n-topology (denoted "affine topology").

Let us write $C = B + \langle C\rangle^+$. For every $a \in B$, we let $C_a = a + \langle C\rangle^+$. Each set $\langle C_a, +\rangle$ is an affine subset of $\langle M^n, +\rangle$ which is definably isomorphic to $\langle C\rangle$ and hence in particular, definably connected (with respect to both topologies). The property of C given to us by Fact 3.1 implies that each $\langle C_a, \oplus\rangle$ is locally a G-linear subset of G and that the identity map gives an isomorphism with $\langle C_a, +\rangle$. Because C_a is definably connected it follows from [22, Lemma 2.4] that it is actually a G-linear (not only locally) subset of G. Moreover, for every $a, b \in C$, the sets $\langle C_a, \oplus\rangle$ and $\langle C_b, \oplus\rangle$ are G-equivalent.

Note that if $a \in B$ then $a \notin C_a$. We mend it by fixing a point $c_0 \in \langle C\rangle^+$ and defining $\alpha : B \to C$ by $\alpha(b) = b + c_0$. The map α is continuous with respect to the affine topology and for every $b \in B$ we have $\alpha(b) \in C_b$. For each $b \in B$ we now let $H_b = C_b \ominus \alpha(b)$, and write

$$C = \bigcup_{b\in B} C_b = \bigcup_{b\in B} \alpha(b) \oplus H_b.$$

We therefore have: For every $b \in B$, the set H_b is a local subgroup of G which is definably isomorphic to the $\langle C\rangle$. In particular, all the H_b's are definably isomorphic to each other, as local subgroups.

Our goal is to find a single strongly long, G-linear set H, contained in all H_b's, such that $\alpha(B) \oplus H$ is still generic in G.

Claim 3.4. *For every $a, b \in B$,* $\mathrm{lgdim}(H_a \Delta H_b) < k$.

Proof of Claim We start by considering the set $\overline{H} = \bigcup_{b\in B} H_b \subseteq G$. Since all the H_b's are definably isomorphic to each other, the union $\overline{H}$ is also a local subgroup of G, and G-equivalent to $\langle C\rangle$ (see [22, Lemma 2.13]). Hence, its dimension equals k. We therefore have $\dim(H_b) = \dim(\overline{H}) = k$.

Assume, towards contradiction, that the claim fails. Then without loss of generality, $\mathrm{lgdim}(H_a \setminus H_b) = k$. It follows from [13] that $H_a \setminus H_b$ contains the image, call it A, of a definable continuous embedding $f : (0,d)^k \to A \subseteq H_a \setminus H_b$, for some tall element $d > 0$. Furthermore, since $\dim(\overline{H}) = k$, we may choose A to be relatively open in $\overline{H}$ and such that the affine topology and τ-topology coincide on A. We now fix a "middle point"

$$a_0 = f\left(\frac{d}{2}, \frac{d}{2}, \ldots, \frac{d}{2}\right) \in A \subseteq H_a.$$

Notice
(5)
If $S \subseteq A$ is a definably connected short set and $a_0 \in S$, then $Cl(S) \subseteq A$.

Our goal is to show that as we move continuously from a to b within the short set B, we obtain a definable short path β in A which starts at a_0 and eventually goes out of A. This will yield a contradiction to (5).

We first observe that we may assume that B is a short interval $[a, b] \subseteq M$. Indeed, this is done by considering, instead of B, the image of a short interval $[0, s]$ under some definable continuous $\gamma : [0, s] \to B$ with $\gamma(0) = a$ and $\gamma(s) = b$.

Since $a_0 \in H_a$, it is of the form $a_0 = (a + \sum_i \lambda_i(t_i)) \ominus \alpha(a)$. Fix $c_0 = \sum_i \lambda_i(t_i) \in \langle C\rangle^+$ and consider the map $\beta : [a, b] \to \overline{H}$, defined by $\beta(t) = (t + c_0) \ominus \alpha(t)$. We have $\beta(a) = a_0 \in A$ and $\beta(b) \in H_b \setminus A$.

We claim that β is continuous with respect to the affine topology on $[a, b]$ and the group topology on $\overline{H}$. Indeed, let $V \subseteq \overline{H}$ be a τ-open set. Then, the set W of all $(x, y) \in C \times C$ such that $x \ominus y \in \overline{H}$, is open in $C \times C$ with respect to the group topology τ. However, the group topology on $C \times C$ is just the product of the topology on C, so, by Fact 3.1 it coincides with the affine topology. The map $a \mapsto (a + c_0, \alpha(a))$ is a continuous map from $[a, b]$ into $C \times C$, with respect to the affine topology, so the pre-image of W under this map is open in $[a, b]$. This is exactly $\beta^{-1}(V)$.

By continuity, the set $\beta([a, b]) \subseteq \overline{H}$ is definably connected with respect to the group topology. Moreover, we have $\beta(a) = a_0 \in A$ and $\beta(b) \in \overline{H} \setminus A$. Because A is open in $\overline{H}$ the set $\beta^{-1}(A)$ is open in $[a, b]$, definable and contains a. Because $\beta(b) \notin A$, we can find $a_1 < b$ such that $\beta([a, a_1)) \subseteq A$ and $\lim_{t\to a_1} \beta(t) \notin A$. By (5), this is impossible. We therefore showed that $\text{lgdim}(H_a \Delta H_b) < k$. This ends the proof of the claim. $\square$

We now let $\widetilde{H} = \bigcap_{b\in B} H_b$.

Claim 3.5. *(i) For every* $b \in B$, $\text{lgdim}(H_b \setminus \widetilde{H}) < k$.

(ii) The set $\alpha(B) \oplus \widetilde{H}$ *is a subset of* C *and* $\text{lgdim}(C \setminus (\alpha(B) \oplus \widetilde{H})) < k$.

(iii) The set $\alpha(B) \oplus \widetilde{H}$ *is generic in* G.

Proof of Claim (i) The set $H_b \setminus \widetilde{H}$ equals the union $\bigcup_{a\in B}(H_b \setminus H_a)$. By the last claim, each set in the union has long dimension smaller than k. By Lemma 3.3, $\text{lgdim} \bigcup_{a\in B}(H_b \setminus H_a) < k$.

(ii) Since $\widetilde{H} \subseteq H_b = C_b \ominus \alpha(b)$, we have for every $b \in B$,

$$\alpha(b) \oplus \widetilde{H} \subseteq \alpha(b) \oplus H_b = C_b \subseteq C.$$

We also have

$$C \setminus (\alpha(B) \oplus \widetilde{H}) \subseteq \bigcup_{b\in B} \alpha(b) \oplus (H_b \setminus \widetilde{H}).$$

Because each set in the union has long dimension smaller than k and because the union is taken over a short set, it follows again from Lemma 3.3, that $\mathrm{lgdim}(C \setminus (\alpha(B) \oplus \widetilde{H})) < k$.

(iii) Since C was generic in G the fact that $\alpha(B) \oplus \widetilde{H}$ is generic follows from (ii) and [13, Lemma 4.14]. End of proof of claim. $\square$

We now return to the proof of Proposition 3.2 and let $B' = \alpha(B)$. We have a strongly long definable local subgroup $\widetilde{H}$ of G, which is definably isomorphic to an affine subset of $\langle M^k, +\rangle$ and furthermore $B' \oplus \widetilde{H}$ is generic. By [13], $\widetilde{H}$ contains finitely many strongly long cones $H_1, \dots, H_m$ of dimension k, such that $\mathrm{lgdim}(\widetilde{H} \setminus \bigcup_i H_i) < k$. It follows that at least one of the sets of the form $B' \oplus H_i$ is generic in G. The set H_i is an affine subset of M^n (being a strongly long cone) and also a G-linear subset of G (being an open subset of the local subgroup $\widetilde{H}$). As an affine set, H_i is definably isomorphic to a product of long intervals call it $H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k)$. By translating H_i in G, we can further assume that it contains 0_G, hence it is a local subgroup of G. We call it H. This ends the proof of Proposition 3.2. $\square$

For $H' \subseteq M^k$ and $H \subseteq G$ as in Proposition 3.2, let $f : H' \to H$ be the acclaimed isomorphism of local groups. We let $\mathcal{H} = \langle H \rangle$ be the subgroup of G generated by H. Since H is a local abelian subgroup of G of dimension k, $\mathcal{H}$ is a locally definable abelian subgroup of G of dimension k (see [22, Lemma 2.18]. One can show that the universal cover of $\mathcal{H}$ is a locally definable subgroup $\widehat{\mathcal{H}}$ of $\langle M^k, +\rangle$. Indeed, let $\widehat{\mathcal{H}} = \langle H' \rangle$ be the subgroup of $\langle M^k, +\rangle$ generated by H'. Then we can extend f to a map $\widehat{f} : \widehat{\mathcal{H}} \to \mathcal{H}$ with, for every $x_1, \dots, x_l \in H'$,

$$\widehat{f}(x_1 + \cdots + x_l) = f(x_1) \oplus f(x_2) \oplus \cdots \oplus f(x_l)$$

is a $\bigvee$-definable covering map for $\mathcal{H}$. (The fact that $\widehat{f}$ is well-defined is provided by the same argument as for [15, Lemma 4.27]). For simplicity, we still use f for the map $\widehat{f}$. Since $\widehat{\mathcal{H}}$ is torsion-free, it is the universal cover of $\mathcal{H}$.

3.2. **Proof of Theorem 1.3.** We proceed with the same notation. Namely, $\langle G, \oplus\rangle$ is a definable abelian group, $H \subseteq G$ the definable strongly long set and B the definable short set from Proposition 3.2, $\mathcal{H} = \langle H \rangle$ and $f : \widehat{\mathcal{H}} \to \mathcal{H}$ the covering map from an open subgroup of $\langle M^k, +\rangle$ onto $\mathcal{H}$. The pre-image of H under f is the set $H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k)$.

We let $\mathcal{H}_0'$ be the subset of M^k that consists of all short elements (by this we mean all elements of M^k all of whose coordinates are short). By [23, Lemma 3.4], $\langle \mathcal{H}_0', +\rangle$ is a subgroup of $\langle M^k, +\rangle$ and moreover, it is a subset of H'. It follows that $\mathcal{H}_0 = f(\mathcal{H}_0')$ is a subgroup of $\mathcal{H}$ which is isomorphic to $\mathcal{H}_0'$ (note that $\mathcal{H}_0$ is a locally definable set, but not, in general, a definable one).

In order to simplify the notation, we will write $+$ for the group operation of G. In few cases we will also use $+$ for the usual operation on M^k, and this will be clear from the context.

We define $\mathcal{B} = \bigcup_{n\in\mathbb{N}} B(n)$, where the notation $B(n)$ is given in Section 1.5. Since each $B(n)$ is a short definable set, $\mathcal{B}$ is a short locally definable subgroup of G.

Claim 3.6. $\mathcal{H} + \mathcal{B} = G$.

Proof. By Proposition 3.2, the set $H + B$ is a generic subset of G and is contained in $\mathcal{H} + \mathcal{B}$ (we use here the fact that $B \subseteq \mathcal{B}$ since $0 \in B$). Since G is definably connected we have $\mathcal{H} + \mathcal{B} = G$. $\square$

The following claim is crucial to the rest of the analysis.

Claim 3.7. *The locally definable group $\mathcal{H}_0 \cap \mathcal{B}$ compatible in $\mathcal{B}$.*

Proof. Let $X \subseteq \mathcal{B}$ be a definable set. The set $\mathcal{B}$ is a bounded union of short sets, and X is contained in one of these so must also be short. We prove that, in general, the intersection of any definable short $X \subseteq G$ with $\mathcal{H}_0$ is definable.

Since $\mathcal{H}_0 \subseteq H$ we may assume that X is a subset of H. Let us consider $X' = f^{-1}(X) \subseteq M^k$. The set X' is a finite union of definably connected short subsets of M^k. If one of these components intersects $\mathcal{H}_0'$ non-trivially then it must be entirely contained in $\mathcal{H}_0$ (since it is a short set). Hence, $X' \cap \mathcal{H}_0'$ is a finite union of components of X' and therefore definable. Its image under f is the definable set $X \cap \mathcal{H}_0$. $\square$

Step 1 By Claim 3.7 and Fact 2.1, the quotient $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$ is a locally definable group and hence we obtain the following short exact sequence of locally definable groups:

$$0 \longrightarrow \mathcal{H}_0 \cap \mathcal{B} \xrightarrow{i_0} \mathcal{B} \xrightarrow{\pi_{\mathcal{B}}} \mathcal{K} \longrightarrow 0 \tag{6}$$

Claim 3.8. $\dim H + \dim \mathcal{K} = \dim G$.

Proof. Because $\mathcal{H} + \mathcal{B} = G$, we have

$$\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H} \cap \mathcal{B}) = \dim G.$$

Indeed, this is true for definable groups, and can be proved similarly here by considering a sufficiently small neighborhood of 0 in the locally definable group $\mathcal{H} \cap \mathcal{B}$.

But $\mathcal{H}_0$ is open in $\mathcal{H}$ and therefore $\dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim(\mathcal{H} \cap \mathcal{B})$, so we also have $\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim G$. Because $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$, we have $\dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim \mathcal{K}$. We can now conclude $\dim \mathcal{H} + \dim \mathcal{K} = \dim G$. $\square$

Step 2. Since $\mathcal{H}_0 \cap \mathcal{B}$ embeds into $\mathcal{H}$ and $\mathcal{H}_0 \cap \mathcal{B}$ is a compatible subgroup of $\mathcal{B}$, we can apply Lemma 2.3 and obtain a locally definable group D (the pushout of $\mathcal{H}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$) with the following diagram commuting

(7)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_0 \cap \mathcal{B} \arrow[r, "i_0"] \arrow[d, "id"'] & \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\gamma"] & \mathcal{K} \arrow[r] \arrow[d, "id_{\mathcal{K}}"] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r, "j"] & D \arrow[r, "\pi_D"] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

The maps γ and j are injective. Note that since $\mathcal{H}$ and $\mathcal{B}$ are subgroups of G, we also have a commutative diagram (with all maps being inclusions)

(8)
$$\begin{tikzcd}
\mathcal{H}_0 \cap \mathcal{B} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{H} \arrow[r] & G
\end{tikzcd}$$

It follows from the definition of pushouts that there exists a locally definable map $\phi : D \to G$ such that $\phi\gamma : \mathcal{B} \to G$ and $\phi j : \mathcal{H} \to G$ are the inclusion maps. The restriction of ϕ to $j(\mathcal{H})$ is therefore injective and furthermore, the set $\phi(D)$ contains $\mathcal{H} + \mathcal{B}$ and hence, by Claim 3.6, ϕ is surjective on G.

Step 3 Consider now the universal cover $f : \widehat{\mathcal{H}} \to \mathcal{H}$ where $\widehat{\mathcal{H}}$ is identified with an open subgroup of $\langle M^k, + \rangle$ as before. As we saw, the group $\widehat{\mathcal{H}}$ has a subgroup $\mathcal{H}_0'$ which is isomorphic via f to $\mathcal{H}_0$. Hence, there is a locally definable embedding $\beta : \mathcal{H}_0 \cap \mathcal{B} \to \widehat{\mathcal{H}}$ such that $f\beta = id_{\mathcal{H}_0 \cap \mathcal{B}}$. Our goal is to use this embedding in order to interpolate an exact sequence between the two sequences in (7) (see (11) below).

We let $\widehat{D}$ be the pushout of $\widehat{\mathcal{H}}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$. Namely, we have

(9)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_0 \cap \mathcal{B} \arrow[r, "i_0"] \arrow[d, "\beta"'] & \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\gamma''"] & \mathcal{K} \arrow[r] \arrow[d, "id_{\mathcal{K}}"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "\widehat{\delta}"] & \widehat{D} \arrow[r, "\pi_{\widehat{D}}"] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

Step 4 Next, we consider the diagram

(10)
$$\begin{tikzcd}
\mathcal{H}_0 \cap \mathcal{B} \arrow[r, "i_0"] \arrow[d, "\beta"'] & \mathcal{B} \arrow[d, "\gamma"] \\
\widehat{\mathcal{H}} \arrow[r, "jf"] & D
\end{tikzcd}$$

Since $f\beta = id$, it follows from (7) that the above diagram commutes. Since $\widehat{D}$ was a pushout, there exists a locally definable $\gamma' : \widehat{D} \to D$ such that $\gamma'\gamma'' = \gamma$ and $\gamma'\widehat{\delta} = jf$.

Putting the above together with (7) and (9), we obtain

(11)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}_0 \cap \mathcal{B} \arrow[r, "i_0"] \arrow[d, "\beta"'] & \mathcal{B} \arrow[r, "\pi_{\mathcal{B}}"] \arrow[d, "\gamma''"] & \mathcal{K} \arrow[r] \arrow[d, "id_{\mathcal{K}}"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "\widehat{\delta}"] \arrow[d, "f"'] & \widehat{D} \arrow[r, "\pi_{\widehat{D}}"] \arrow[d, "\gamma'"] & \mathcal{K} \arrow[r] \arrow[d, "id_{\mathcal{K}}"] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r, "j"] & D \arrow[r, "\pi_D"] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

Note that in order to conclude that the above diagram commutes, we still need to verify that the bottom right square commutes, namely, $(id_{\mathcal{K}})\pi_{\widehat{D}} = (\pi_D)\gamma'$.

We now apply Lemma 2.4 and conclude that the group D is the pushout of $\mathcal{H}$ and $\widehat{D}$ over $\widehat{\mathcal{H}}$. As a corollary we conclude, by Lemma 2.3 (and the fact that f is surjective),

(12) $$(i)\ \pi_{\widehat{D}} = (\pi_D)\gamma' \ (ii)\ \ker(\gamma') = \widehat{\delta}(\ker f)\ (iii)\ \gamma' \text{ is surjective.}$$

In particular, (11) commutes.

If we now return to the surjective $\phi : D \to G$ and compose it with γ', we obtain a surjection $\phi\gamma' : \widehat{D} \to G$.

Let us summarize what we have so far:

(13)
$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "\widehat{\delta}"] & \widehat{D} \arrow[r, "\pi_{\widehat{D}}"] \arrow[d, "\phi\gamma'"] & \mathcal{K} \arrow[r] & 0 \\
 & & G & &
\end{tikzcd}$$

Step 5 Let $\mu : \mathcal{U} \to \mathcal{K}$ of $\mathcal{K}$ be the universal cover of $\mathcal{K}$, (see [10, Theorem 3.11] for its existence and its local definability) and apply the pullback construction from Proposition 2.6 to $\mathcal{U}$, $\mathcal{K}$ and $\widehat{D}$.

We obtain a $\bigvee$-definable group $\widehat{G}$ (the pullback of $\mathcal{U}$ and $\widehat{D}$ over $\mathcal{K}$), with associated $\bigvee$-definable maps such that the following sequences are exact and commute (since the kernels of $\pi_{\widehat{G}}$ and $\pi_{\widehat{G}}$ are isomorphic we identify them both with $\widehat{\mathcal{H}}$ and assume that the map between them is the identity). By Proposition 2.6, we also have

(14) $$\pi_{\widehat{G}}(\ker(\eta)) = \ker(\mu).$$

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] \arrow[d, "id"'] & \widehat{G} \arrow[r, "\pi_{\widehat{G}}"] \arrow[d, "\eta"] & \mathcal{U} \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "\widehat{\delta}"] & \widehat{D} \arrow[r, "\pi_{\widehat{D}}"] & \mathcal{K} \arrow[r] & 0
\end{tikzcd}
\tag{15}
$$

Because μ is surjective, so is η, so we obtain a surjective homomorphism $\widehat{F} := \phi\gamma'\eta : \widehat{G} \to G$. It can be inferred from what we have so far that $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$.

Note that $\dim \widehat{G} = \dim \mathcal{U} + \dim \widehat{\mathcal{H}}$ and, since $\mathcal{U}$ is the universal cover of $\mathcal{K}$, $\dim \mathcal{U} = \dim \mathcal{K}$. By Claim 3.8, we have $\dim \widehat{G} = \dim G$. Since $\mathcal{U}$ and $\widehat{\mathcal{H}}$ are torsion-free then so is $\widehat{G}$ and therefore $\widehat{F} : \widehat{G} \to G$ is isomorphic to the universal cover of G.

We therefore obtain

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] & \widehat{G} \arrow[r, "\pi_{\widehat{G}}"] \arrow[d, "\widehat{F}"] & \mathcal{U} \arrow[r] & 0 \\
 & & G & &
\end{tikzcd}
\tag{16}
$$

This ends the proof of the first part Theorem 1.3 for an abelian definably connected, definably compact G.

Assume now that G is an arbitrary definably compact, definably connected group. By [20, Corollary 6.4], the group G is the almost direct product of the definably connected groups $Z(G)^0$ and $[G, G]$, and $[G, G]$ is a semisimple group. The group G is then the homomorphic image of the direct sum $A \oplus S$ with A abelian, S semi-simple, both definably compact, and the kernel of this homomorphism is finite. We may therefore assume that $G = A \oplus S$. By [20, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short. It follows that $\mathrm{lgdim}(G) = \mathrm{lgdim}(A)$. By the abelian case, we obtain the following for the universal cover $\widehat{A}$ of A.

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r] & \widehat{A} \arrow[r] \arrow[d, "\widehat{F}"] & \mathcal{U} \arrow[r] & 0 \\
 & & A & &
\end{tikzcd}
$$

Next, we consider $p : \widehat{S} \to S$ the universal cover of S. By taking the direct product we obtain:

$$\begin{tikzcd} 0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] & \widehat{A} \oplus \widehat{S} \arrow[r, "\pi"] \arrow[d, "\hat{F}\cdot p"] & \mathcal{U} \oplus \widehat{S} \arrow[r] & 0 \\ & & G = A \oplus S & & \end{tikzcd} \tag{17}$$

In order to finish the proof of Theorem 1.3 we need to see:

Lemma 3.9. *The group $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$ contains every connected, locally definable, strongly long subgroup of G.*

Proof. We first prove the analogous result for the universal cover $\widehat{G}$ of G, namely we prove that $i(\widehat{\mathcal{H}})$ contains every connected, locally definable, strongly long subgroup of $\widehat{G}$. For simplicity, we assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Assume that $\mathcal{V} \subseteq \widehat{G}$ is a connected, locally definable subgroup with $\dim(\mathcal{V}) = \mathrm{lgdim}(\mathcal{V}) = \ell$. Because $\mathrm{lgdim}(\widehat{G}) = k$ we must have $\ell \leq k$. We will show that the group $\mathcal{V} \cap \widehat{\mathcal{H}}$ has bounded index in $\mathcal{V}$, so by connectedness the two must be equal.

Because $\mathcal{U}$ is short, there exists at least one $u \in \mathcal{U}$ such that $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$. Since $\mathcal{V}$ is a group we can use translation in $\mathcal{V}$ to show that *for every $u \in \pi(\mathcal{V})$*, we must have $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$. In particular, $\mathrm{lgdim}(\mathcal{V} \cap \widehat{\mathcal{H}}) = \ell$.

Write $\mathcal{V} = \bigcup_i V_i$ a countable union of definable sets which we may assume to be all strongly long of dimension ℓ. For every V_i, consider the definable projection $\pi(V_i) \subseteq \mathcal{U}$. By Lemma 10.1 (proved in the appendix), the set F_i of all $u \in \pi(V_i)$ such that $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$ is definable, so because $\dim(V_i) = \ell$, this set must be finite.

If we let $F = \bigcup_i F_i$ then for every $u \in \pi(\mathcal{V}) \setminus F$, $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) < \ell$, which, as we saw above, implies that $F = \pi(\mathcal{V})$. Because $|F| \leq \aleph_0$ it follows that the index of $\mathcal{V} \cap \widehat{\mathcal{H}}$ in $\mathcal{V}$ is at most $\aleph_0$. We thus can conclude that $\mathcal{V} \cap \widehat{\mathcal{H}} = \mathcal{V}$, so $\mathcal{V} \subseteq \widehat{\mathcal{H}}$.

Assume now that $\mathcal{V} \subseteq G$ is a connected, locally definable, strongly long subgroup of G and let $\widehat{\mathcal{V}} \subseteq \widehat{G}$ be the pre-image of $\mathcal{V}$ under $\widehat{F}$. The group $\widehat{\mathcal{V}}$ is strongly long and locally definable, and the connected component of the identity (see [1, Proposition 1]), call it $\widehat{\mathcal{V}}^0$, is still strongly long (since it has the same dimension and long dimension as $\widehat{\mathcal{V}}$). By what we just saw, $\widehat{\mathcal{V}}^0$ is contained in $\widehat{\mathcal{H}}$ and hence $\widehat{F}(\mathcal{V}^0)$ is a $\bigvee$-definable subgroup of $\mathcal{H} \cap \mathcal{V}$, which has bounded index in $\mathcal{V}$. Because $\mathcal{V}$ is connected it follows $\widehat{F}(\mathcal{V}^0) = \mathcal{V} \subseteq \mathcal{H}$. □

This ends the proof of Theorem 1.3.

4. Replacing the locally definable group $\mathcal{U}$ with a definable group

We now proceed to prove Theorem 1.4. The goal is to replace the locally definable group $\mathcal{U}$ in (16) with *a definable* short group. We assume first that G is abelian. We refer to the notation of (15).

Step 1 Let $\Lambda = \ker(\widehat{F})$ and let $\Lambda_1 = \pi_{\widehat{G}}(\Lambda) \subseteq \mathcal{U}$. *For simplicity, we write π for $\pi_{\widehat{G}}$.*

In Part II of the paper we will establish, independently Theorem 1.5, which we now recall:

Theorem 1.5 *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact definable set. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definably compact definable group.

Let us see that $\mathcal{U}$, the universal cover of $\mathcal{K}$ from (15) satisfies the above assumptions. The group $\widehat{G}$ is the universal cover of G. Because G is definably connected and definably compact, we can find a definable, definably connected, definably compact $X \subseteq \widehat{G}$ which contains the identity and such that $\widehat{F}(X) = G$. Indeed, to see that X can be chosen definably compact, note that if not then by [6, Lemma 5.1 and Theorem 5.2], $\widehat{G}$ has a definable, 1-dimensional, torsion-free subgroup H. However, the image of H in G must be a definable 1-dimensional torsion-free subgroup of G, contradicting the fact that G is definably compact. Thus we can find a definably compact X with $X + \ker(\widehat{F}) = \widehat{G}$. By [10, Theorem 1.4, Corollary 1.5], $\ker(\widehat{F})$ is isomorphic to the fundamental group of G, $\pi_1^{def}(G)$, which is finitely generated. It follows that Λ_1 is finitely generated, $\mathcal{U} = \pi_{\widehat{G}}(X) + \Lambda_1$, and $\pi_{\widehat{G}}(X)$ is definably compact and definably connected. Since X generates $\widehat{G}$, the set $\pi_{\widehat{G}}(X)$ generates $\mathcal{U}$. By Remark 1.2, $\mathcal{U}$ is connected.

We can now apply Theorem 1.5 and conclude that there is a definably compact group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\ker(\widehat{\mu}) = \Lambda_0 \subseteq \Lambda_1$.

Our goal is to prove: *There are locally definable extensions $\overline{G}$ and G' of $\overline{K}$, by the group $\widehat{\mathcal{H}}$ and $\mathcal{H}$, respectively, and surjective homomorphisms from $\overline{G}$ and G' onto G.*

First, by Lemma 2.7, we have a locally definable group $\widehat{\mathcal{H}}' = \ker(\widehat{\mu}\pi_{\widehat{G}}) = \pi_{\widehat{G}}^{-1}(\Lambda_0) \subseteq \widehat{G}$ such that (we write i for the identity on $\widehat{\mathcal{H}}$ on the top left) the diagram commutes and the following sequences are exact.

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] \arrow[d, "i"'] & \widehat{G} \arrow[r, "\pi_{\widehat{G}}"] \arrow[d, "id"] & \mathcal{U} \arrow[r] \arrow[d, "\widehat{\mu}"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}}' \arrow[r, "id"] & \widehat{G} \arrow[r, "\hat{\mu}\pi_{\widehat{G}}"] & \overline{K} \arrow[r] & 0
\end{tikzcd} \tag{18}$$

Because $ker(\widehat{\mu}) \subseteq \pi_{\widehat{G}}(\Lambda)$, the group $\widehat{\mathcal{H}}'$ is contained in the group $i(\widehat{\mathcal{H}})+\Lambda$. Since $i(\widehat{\mathcal{H}})$ is a divisible subgroup of $\widehat{\mathcal{H}}'$, there exists a subgroup $\Lambda' \subseteq \Lambda$ such that $\widehat{\mathcal{H}}'$ equals the direct sum of $i(\widehat{\mathcal{H}})$ and Λ'. Because $\ker(\pi_{\widehat{G}}) = i(\widehat{\mathcal{H}})$, the group Λ' is isomorphic, via π, to Λ_0, so Λ' is finitely generated. We now have a group homomorphism $p : \widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}$, given via the identification of $\widehat{\mathcal{H}}'$ with $i(\widehat{\mathcal{H}}) \oplus \Lambda'$. Namely, $p(i(h) + \lambda) = h$.

We claim that p is a locally definable map. Indeed, $\widehat{\mathcal{H}}'$ is the union of sets of the form $i(H_i) + F_i$, where H_i is definable and F_i is a finite subset of Λ'. Because the sum of $\widehat{\mathcal{H}}$ and Λ' is direct, each element g of $i(H_i) + F_i$ has a unique representation as $g = i(h) + f$, for $h \in H_i$ and $f \in F_i$. Therefore the restriction of p to $i(H_i)+F_i$ is definable. It follows that p is locally definable.

Step 2. We apply Proposition 2.3 to the diagram

$$\begin{tikzcd}
\widehat{\mathcal{H}}' \arrow[r, "id"] \arrow[d, "p"'] & \widehat{G} \\
\widehat{\mathcal{H}} &
\end{tikzcd}$$

and obtain a locally definable pushout $\overline{G}$, such that the following diagram commutes and the sequences are exact:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}}' \arrow[r, "id"] \arrow[d, "p"'] & \widehat{G} \arrow[r, "\hat{\mu}\pi_{\widehat{G}}"] \arrow[d, "\widehat{\alpha}"] & \overline{K} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i_1"] & \overline{G} \arrow[r, "\pi_{\overline{G}}"] & \overline{K} \arrow[r] & 0
\end{tikzcd}$$

Because p is surjective the map $\widehat{\alpha} : \widehat{G} \to \overline{G}$ is also surjective. Moreover, by Lemma 2.3, the kernel of $\widehat{\alpha}$ equals $\ker p = \Lambda'$ so is contained in $\Lambda = \ker(\widehat{F})$.

Step 3. We now have surjective maps $\widehat{F} : \widehat{G} \to G$ and $\widehat{\alpha} : \widehat{G} \to \overline{G}$, both $\bigvee$-definable with $\ker(\widehat{\alpha}) \subseteq \ker(\widehat{F})$. By Lemma 2.8 we have a $\bigvee$-definable surjective $\overline{F} : \overline{G} \to G$, with $\ker(\overline{F}) = \widehat{\alpha}(\ker(\widehat{F}))$. We therefore obtained the following diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i_1"] & \overline{G} \arrow[r, "\pi_{\overline{G}}"] \arrow[d, "\overline{F}"] & \overline{K} \arrow[r] & 0 \\
 & & G & &
\end{tikzcd}
\tag{19}
$$

Finally, let us calculate $\ker(\overline{F})$: Recall that Λ' is isomorphic to Λ_0 the kernel of the universal covering map $\widehat{\mu} : \mathcal{U} \to \overline{K}$. Because $\overline{K}$ is a short definably compact group, it follows from [11] that $\ker(\widehat{\mu}) = \pi_1^{def}(\overline{K}) = \mathbb{Z}^d$, where $\pi_1^{def}(\overline{K})$ is the o-minimal fundamental group of $\overline{K}$ and

$$d = \dim(\overline{K}) = \dim(\mathcal{U}) = \dim(G) - k,$$

for $k = \text{lgdim}(G)$. The map $\widehat{F} : \widehat{G} \to G$ is the universal covering map of G and therefore, as shown in [10, Theorem 1.4, Corollary 1.5], $\ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^\ell$, for some ℓ. Furthermore, for every $m \in \mathbb{N}$, the group of m-torsion points $G[m]$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\ell$. By [23, Theorem 7.6], $G[m] = (\mathbb{Z}/m\mathbb{Z})^{\dim(G)}$, hence we can conclude

$$\Lambda = \ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^{\dim G}.$$

We now have $\ker(\overline{F}) = \widehat{\alpha}(\Lambda) \simeq \Lambda/\Lambda'$, with $\Lambda \simeq \mathbb{Z}^{\dim(G)}$ and $\Lambda' \simeq \mathbb{Z}^{\dim(G)-k}$. Hence, $\ker(\overline{F})$ is isomorphic to the direct sum of $\mathbb{Z}^k$ and a finite group, as required.

Question Can $\overline{K}$ be chosen so that $\ker(\overline{F}) \simeq \mathbb{Z}^k$?

Next, consider $\mathcal{H} \subseteq G$ as in Theorem 1.3. We want to see that we can obtain a similar diagram to (19), with $\mathcal{H}$ instead of $\widehat{\mathcal{H}}$. For simplicity, assume that i_1 is the identity. First notice that by the last clause of Theorem 1.3, we must have $\overline{F}(\widehat{\mathcal{H}}) \subseteq \mathcal{H}$. However, using exactly the same proof as in Lemma 3.9, we can show that that $\overline{F}(\widehat{\mathcal{H}})$ is also the largest connected strongly long, locally definable, subgroup of G, hence it equals $\mathcal{H}$. We therefore have

$$
\begin{tikzcd}
\widehat{\mathcal{H}} \arrow[r, "i_1"] \arrow[d, "\overline{F}\restriction\widehat{\mathcal{H}}"'] & \overline{G} \\
\mathcal{H} &
\end{tikzcd}
$$

We can now obtain G', the pushout of $\overline{G}$ and $\mathcal{H}$ over $\widehat{\mathcal{H}}$:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i_1"] \arrow[d, "\overline{F}\restriction\widehat{\mathcal{H}}"'] & \overline{G} \arrow[r, "\pi_{\overline{G}}"] \arrow[d, "\alpha'"] & \overline{K} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r, "i'"] & G' \arrow[r, "\pi_{G'}"] & \overline{K} \arrow[r] & 0
\end{tikzcd}$$

Clearly, $\ker(\overline{F} \restriction \widehat{\mathcal{H}}) \subseteq \ker(\overline{F})$, so by Proposition 2.3, $\ker(\alpha') = i(ker\overline{F} \restriction H) \subseteq \ker \overline{F}$. By Lemma 2.8, we have a homomorphism from G' onto G as we want. We therefore have:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H} \arrow[r, "i'"] & G' \arrow[r, "\pi_{\overline{G}}"] \arrow[d, "h'"] & \overline{K} \arrow[r] & 0 \\
& & G & &
\end{tikzcd} \tag{20}$$

This ends the switch from (19) to (20), and with that the proof of Theorem 1.4 in the case that G is abelian. In order to conclude the same result for arbitrary definably compact, definably connected G, we repeat the same arguments as in the last part of the proof of Theorem 1.3. $\square$

4.1. **Special cases.** As was pointed out earlier, we use Theorem 1.5 to guarantee that there is a definable group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\Lambda_0 := \ker(\widehat{\mu})$ a subgroup of $\pi_{\widehat{G}}(\ker \widehat{F})$ (see notation of Theorem 1.3). In certain simple cases we can see why this is true directly:

Assume G is abelian. Let $\mathcal{K}$ and $\mathcal{H}$ be as in Section 3.2. Namely, $\mathcal{K}$ is the group obtained as the quotient of the locally definable subgroup $\mathcal{B}$ of G by the compatible subgroup $\mathcal{H}_0 \cap \mathcal{B}$, and $\mathcal{H}$ is the largest locally definable, connected strongly long subgroup of G.

(1) **Assume that $\mathcal{K}$ is definable**.

In this case we take $\Lambda_0 = \ker(\mu)$, where $\mu : \mathcal{U} \to \mathcal{K}$. Obviously, $\mathcal{U}/\Lambda_0$ is definable, so we need only to see that $\Lambda_0 \subseteq \pi_{\widehat{G}}(\ker \widehat{F})$. Let $u \in \ker(\mu)$. By (14), $u = \pi_{\widehat{G}}(v)$, for some $v \in \ker(\eta)$. But then $\widehat{F}(v) = \phi\gamma'\eta(v) = 0$.

(2) **Assume that $\mathcal{H}$ is definable**.

We denote by $\overline{K}$ the definable group $G/\mathcal{H}$. From Theorem 1.3 and its proof we obtain the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{H}} \arrow[r, "i"] \arrow[d, "f"'] & \widehat{G} \arrow[r, "\pi_{\widehat{G}}"] \arrow[d, "\widehat{F}"'] & \mathcal{U} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r, "id"] & G \arrow[r, "\pi_G"] & \overline{K} \arrow[r] & 0
\end{tikzcd}$$

But now there is a natural map $\mu : \mathcal{U} \to \overline{K}$ which makes the above diagram commute, and it is easy to verify by construction that $\ker(\mu) \subseteq \pi_{\widehat{G}}(\ker(\widehat{F}))$. We now take $\Lambda_0 = \ker(\mu)$.

5. Examples of non-extensions

In this section we provide examples that motivate the statements of Theorem 1.3 and 1.4. More specifically, we give examples of definable groups which cannot themselves be written as extensions of short (locally) definable groups by strongly long (locally) definable subgroups. This is what forces us to move our analysis to the level of universal covers.

In the following examples, we fix $\mathcal{M} = \langle M, +, <, 0, R\rangle$ to be an expansion of an ordered divisible abelian group by a real closed field R, whose domain is a bounded interval $(0, a) \subseteq M$. In particular, $\mathcal{M}$ is semi-bounded, o-minimal, and $(0, a)$ is short. Let also $b \in M$ be any long positive element. In the first two examples, we define semi-linear groups which have the same domain $[0, a) \times [0, b)$ but different operations.

Example 5.1. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, +\rangle$ generated by the vectors $(a, 0)$ and (v_1, b), and let $G = \langle [0, a) \times [0, b), \star, 0\rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

By [15, Claim 2.7(ii)], G is definable.

Let us see what the various groups of Theorems 1.3 and 1.4 are in this case.

We let $\widehat{G}$ be the subgroup of M^2 generated by $[0, a] \times [0, b]$. The group $\widehat{G}$ is torsion-free and it is easy to see that there is a locally definable covering map $\widehat{F} : \widehat{G} \to G$. Hence, $\widehat{G}$ is the universal cover of G. The group $\widehat{\mathcal{H}} = \{0\} \times \bigcup_n(-nb, nb)$, is a locally definable compatible subgroup of $\widehat{G}$ and the quotient $\widehat{G}/\widehat{\mathcal{H}}$ is isomorphic to the short group $\bigcup_n(-na, na)$.

We have $\mathrm{lgdim}(\widehat{\mathcal{H}}) = \dim(\widehat{\mathcal{H}}) = 1$, so $\widehat{\mathcal{H}}$ is strongly long. As in the proof of Proposition 3.9, the group $\widehat{\mathcal{H}}$ is the largest strongly long, connected, locally definable subgroup of $\widehat{\mathcal{H}}$.

Now, we let $\mathcal{H} = \widehat{F}(\widehat{H})$. This is the subgroup of G generated by the set $H = \{0\} \times [0, b)$ and we can describe it explicitly. Let $S \subseteq [0, a)$ be the set containing all elements of the form $n(a - v_1) \mod a$. By the choice of v_1, the set S has to be infinite. By the definition of the operation $\star$, it is easy to see that

$$\mathcal{H} = \bigcup_{s \in S} \{s\} \times [0, b),$$

which is not definable (so in particular not compatible in G). This shows the need in Theorem 1.3 to work with the universal cover of G rather than with G itself. Note that $\widehat{F}$ restricted to $\widehat{H}$ is an isomorphism onto $\mathcal{H}$.

In fact, G does not contain any infinite strongly long definable subgroup. Indeed, if it did, then its connected component must be contained in $\mathcal{H}$ and therefore pre-image of this component under $\widehat{F} \restriction \widehat{\mathcal{H}}$ would be a proper definable subgroup of $\widehat{\mathcal{H}}$ and, thus, of $\langle M, +\rangle$, a contradiction.

Now consider the subgroup $K = \langle [0, a) \times \{0\}, \star, 0\rangle$ of G and let $\widehat{K}$ be its universal cover. We can write

$$G = \mathcal{H} \star K.$$

Of course $\mathcal{H} \cap K$ is infinite, so this is not a direct sum. However, the universal cover $\widehat{G}$ of G is a direct sum

$$\widehat{G} = \widehat{H} \oplus \widehat{K},$$

whereas, if we let

$$\overline{G} = \widehat{\mathcal{H}} \oplus K,$$

then we can define a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}^k$.

Example 5.2. Pick any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, +\rangle$ which is generated by the vectors (a, u_2) and $(0, b)$, and let again $G = \langle [0, a) \times [0, b), \star, 0\rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

Here we observe that $H = \{0\} \times [0, b)$ itself is the largest strongly long locally definable subgroup of G and, hence, G is itself an extension of a short definable group by H. However, H does not have a definable complement in G; namely, G cannot be written as a direct sum of H with some definable subgroup of it. The proof of this goes back to [29]. See also [25].

The universal cover $\widehat{\mathcal{H}}$ of $\mathcal{H}$ is again the subgroup of M^2 generated by H. Let $\mathcal{K}$ be the subgroup of G generated by $K = [0, a) \times \{0\}$, and $\widehat{K}$ its universal cover. Then we can write

$$G = H \star \mathcal{K},$$

where again $H \cap \mathcal{K}$ is not finite, so this is not a direct sum. The universal cover $\widehat{G}$ of G is again a direct sum

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K}.$$

If we let $\overline{K} = \langle [0, a) \times \{0\}, \star_K, 0\rangle$ be the group with operation $\star_K = +$ mod a, and

$$\overline{G} = \widehat{\mathcal{H}} \oplus \overline{K},$$

then we can define a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}^k$.

We finally give an example for Theorems 1.3 and 1.4 of a definable group G which contains no infinite proper definable subgroup.

Example 5.3. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent, and any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, +\rangle$ which is generated by the vectors (a, u_2) and (v_1, b). We define the group G with domain

$$\big([0,a)\times[0,b-u_2)\big)\cup\big([v_1,a)\times[b-u_2,b)\big),$$

and group operation again

$$x\star y=z \Leftrightarrow x+y-z\in L.$$

It is not too hard to verify that the above is indeed a definable group - this will appear in a subsequent paper ([14]).

In this case, G does not contain any infinite proper definable subgroup. This again originates in [29]. We let $\mathcal{H}$ the subgroup of G generated by $H=\{0\}\times[0,b-u_2)$, and $\widehat{\mathcal{H}}$ its universal cover. We also let $\mathcal{K}$ be the subgroup of G generated by $K=[0,a)\times\{0\}$, and $\widehat{K}$ its universal cover. Then we have:

$$G=\mathcal{H}\star\mathcal{K},$$

with $\mathcal{H}\cap\mathcal{K}$ infinite, and

$$\widehat{G}=\widehat{\mathcal{H}}\oplus\widehat{K}.$$

Finally, if we let $\overline{K}=\langle[0,a)\times\{0\},\star_K,0\rangle$ be the group with operation $\star_K = +$ mod a, and

$$\overline{G}=\widehat{\mathcal{H}}\oplus\overline{K},$$

then we can define a surjective homomorphism $\overline{F}:\overline{G}\to G$ with $\ker\overline{F}\simeq\mathbb{Z}^k$.

Question In all the above examples, the universal cover $\widehat{G}$ was the direct sum of the groups $\widehat{\mathcal{H}}$ and $\mathcal{U}$ (rather then just an extension of $\mathcal{U}$ by $\widehat{\mathcal{H}}$). Can $\widehat{G}$ always be realized as a direct sum of $\widehat{H}$ and $\mathcal{U}$?

6. $\bigvee$-definable groups and type-definable subgroups of bounded index

Our goal in this and in the next section is to prove Theorem 1.5. The results in these sections are totally independent from all that we established thus far.

In this section, we assume that $\mathcal{M}$ is a sufficiently saturated, not necessarily o-minimal, structure.

By *a small subset of M* we mean a subset $A\subseteq M$ such that every type over A is realized in M. Every small definable set is therefore finite.

6.1. Definable quotients of $\bigvee$-definable groups.

Definition 6.1. Given a $\bigvee$-definable group $\mathcal{U}$ and $\Lambda_0\subseteq\mathcal{U}$ a normal subgroup, we say that $\mathcal{U}/\Lambda_0$ is *definable (interpretable)* if there is a definable (interpretable) group $\overline{K}$ and a surjective $\bigvee$-definable homomorphism $\mu:\mathcal{U}\to\overline{K}$ whose kernel is Λ_0.

Here is a criterion for definability:

Lemma 6.2. *Let $\mathcal{M}$ be an arbitrary κ-saturated structure. Let $\langle \mathcal{U}, +\rangle$ be a $\bigvee$-definable abelian group and Λ_0 a small subgroup of $\mathcal{U}$. Then the following are equivalent:*

1. *The quotient $\mathcal{U}/\Lambda_0$ is interpretable in $\mathcal{M}$.*
2. *There is a definable $X \subseteq \mathcal{U}$ such that (1) $X + \Lambda_0 = \mathcal{U}$ and (2) for every definable $Y \subseteq \mathcal{U}$, $Y \cap \Lambda_0$ is finite.*
3. *There is a definable $X \subseteq \mathcal{U}$ such that (1) $X + \Lambda_0 = \mathcal{U}$ and (2) $X \cap \Lambda_0$ is finite.*

Proof. ($1 \Rightarrow 2$). We assume that there is a $\bigvee$-definable surjective $\mu : \mathcal{U} \to \overline{K}$ with kernel Λ_0, and $\overline{K}$ definable. By saturation, there is a definable subset $X \subseteq \mathcal{U}$ such that $\mu(X) = \overline{K}$ and hence $X +_{\mathcal{U}} \Lambda_0 = \mathcal{U}$. Given any definable $Y \subseteq \mathcal{U}$, the restriction of μ to Y is definable and thus the small set $\ker(\mu_{\restriction Y}) = Y \cap \Lambda_0$ is definable and, hence, finite.

($2 \Rightarrow 3$). This is obvious.

($3 \Rightarrow 1$) We claim first that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \Lambda_0$ is finite. Indeed, since $Y \subseteq \Lambda_0 + X$, by saturation there is a finite $F \subseteq \Lambda_0$ such that $Y \subseteq F + X$. We assume that $X \cap \Lambda_0$ is finite, and since F is a finite subset of Λ_0 it follows that $(F + X) \cap \Lambda_0$ is finite which clearly implies $Y \cap \Lambda_0$ finite.

Fix a finite $F_1 = (X - X) \cap \Lambda_0$ and $F_2 = (X + X - X) \cap \Lambda_0$.

We now define on X an equivalence relation $x \sim y$ if $x - y \in \Lambda_0$ if and only if $x - y \in F_1$. This is a definable relation since F_1 is finite. We can also define a group operation on the equivalence classes: $[x] + [y] = [z]$ if and only if $x + y - z \in \Lambda_0$ iff $x + y - z \in F_2$. The group we get, call it $\overline{K}$, is clearly isomorphic to $\mathcal{U}/\Lambda_0$, and we have a $\bigvee$-definable homomorphism from $\mathcal{U}$ onto $\overline{K}$, whose kernel is Λ_0. □

Note that in the case that $\mathcal{M}$ is an o-minimal structure, then by [6, Corollary 8.1], the group $\mathcal{U}$ has strong definable choice for definable families of subsets of $\mathcal{U}$. Namely, for every definable family of subsets of $\mathcal{U}$, $\{X_t : t \in T\}$, there is a definable function $f : T \to \bigcup X_t$ such that for every $t \in T$, $f(t) \in X_t$ and if $X_{t_1} = X_{t_2}$ then $f(t_1) = f(t_2)$. In particular, every interpretable quotient of $\mathcal{U}$ is definably isomorphic to a definable group.

6.2. **Subgroups of bounded index of $\bigvee$-definable groups.** Let $\mathcal{U}$ be a $\bigvee$-definable group in an o-minimal structure. It is not always true that $\mathcal{U}$ has some type-definable subgroup of bounded index. For example, consider a sufficiently saturated ordered divisible abelian group $\langle G, <, +\rangle$ and in it take an infinite increasing sequence of incomparable elements $0 < a_1 < a_2 < \cdots$ (that is, for every $n \in \mathbb{N}$, we have $na_i < a_{i+1}$). The subgroup $\bigcup_i(-a_i, a_i)$ of G is a $\bigvee$-definable group which does not have any type-definable subgroup of bounded index. However, as is shown in [19] (see Proposition 6.1 and Proposition 7.4), if $\mathcal{U}$ does have some type-definable subgroup of bounded index then it has a smallest one; namely $\mathcal{U}^{00}$ exists.

Our goal here is to show, under various assumptions on $\mathcal{U}$, that the non-generic definable sets give rise to type-definable subgroups of bounded index.

As is shown in [24], using Dolich's results, if G is a definably compact, abelian group in an o-minimal expansion of a real closed field then the non-generic definable sets form an ideal. Later, it was pointed out in [15] and [23, Section 8] that the same proof works in expansions of groups. We start by re-proving an analogue of the result for $\bigvee$-definable groups (see Lemma 6.7 below). We first define the corresponding notion of genericity and prove some basic facts about it.

Definition 6.3. Let $\mathcal{U}$ be a $\bigvee$-definable group. A definable $X \subseteq \mathcal{U}$ is called *left-generic* if there is a small subset $A \subseteq \mathcal{U}$ such that $\mathcal{U} = \bigcup_{g \in A} gX$. We similarly define *right-generic*. The set X is called *generic* if it is both left-generic and right-generic.

It is easy to see that a definable $X \subseteq \mathcal{U}$ is generic if and only if for every definable $Y \subseteq \mathcal{U}$, there are finitely many translates of X which cover Y.

Fact 6.4. *(1) If $\mathcal{U}$ is a $\bigvee$-definable group, then every $\bigvee$-definable subgroup of bounded index is a compatible subgroup. In particular, if $X \subseteq \mathcal{U}$ is a definable left-generic set, then the subgroup generated by X is a compatible subgroup.*

(2) Assume that $\mathcal{U}$ is a $\bigvee$-definable group in an o-minimal structure. If $\mathcal{U}$ is connected and $X \subseteq \mathcal{U}$ is a left-generic set, then X generates $\mathcal{U}$.

Proof. (1) Assume that $\mathcal{V}$ is a $\bigvee$-definable subgroup of bounded index. We need to see that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \mathcal{V}$ is definable. Since $\mathcal{V}$ has bounded index in $\mathcal{U}$ its complement in $\mathcal{U}$ is also a bounded union of definable sets, hence a $\bigvee$-definable set. But then $Y \cap \mathcal{V}$ and $Y \setminus \mathcal{V}$ are both $\bigvee$-definable sets, so by compactness $Y \cap \mathcal{V}$ must be definable.

(2) Assume now that $\mathcal{U}$ is a $\bigvee$-definable connected group in an o-minimal structure and $X \subseteq \mathcal{U}$ is a left-generic set. By (1), the group $\mathcal{V}$ generated by X is compatible, of bounded index. But then $\dim \mathcal{V} = \dim \mathcal{U}$, so by [1, Proposition 1], $\mathcal{V} = \mathcal{U}$. □

Fact 6.5. *Let $\langle \mathcal{U}, + \rangle$ be an abelian, definably generated group. If $X \subseteq \mathcal{U}$ is a definable set then X is generic if and only if there exists a finitely generated (in particular countable) group $\Gamma \leqslant \mathcal{U}$ such that $\mathcal{U} = X + \Gamma$.*

Proof. Clearly, if Γ exists then X is generic. For the converse, assume that $\mathcal{U}$ is generated by the definable set $Y \subseteq \mathcal{U}$, with $0 \in Y$. Because X is generic in $\mathcal{U}$, there is a finite set $F \subseteq \mathcal{U}$ such that the sets $-Y$, Y and $X + X$ are all contained in $X + F$.

Let $Y(n)$ be as in the notation from Section 1.5. If we now let Γ be the group generated by F, then $\mathcal{U} = \bigcup_n Y(n) = X + \Gamma$. □

Let us now point out why we can obtain Dolich's result in our setting.

Fact 6.6. *Let $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ be a small elementary submodel. If $\mathcal{U}$ is a $\bigvee$-definable group and $X_t \subseteq \mathcal{U}$ is a t-definable, definably compact set such that $X_t \cap M_0 = \emptyset$, then there are $t_1, \ldots, t_k$, all of the same type as t over M_0 such that $X_{t_1} \cap \cdots \cap X_{t_k} = \emptyset$.*

Proof. We need to translate the problem from the group topology to the M^n-topology. As we already noted it is shown in [2] that $\mathcal{U}$ can be covered by a fixed collection of definable open sets $\bigcup_i U_i$ such that each U_i is definably homeomorphic to an open subset of M^n. We will assume that $\mathcal{M}_0$ contains the parameters defining the V_i's (this is sufficient for the argument below). By logical compactness, X_t is contained in finitely many V_i's, say $V_1, \cdots, V_m$. Now, by definable compactness, we can replace each of the V_i's by an open set W_i such that $Cl(W_i) \subseteq V_i$ and X_t is still contained in $W_1, \ldots, W_m$. Each $X(i) = X_t \cap Cl(W_i)$ is definably compact and we finish the proof as in [15, Lemma 3.10]. □

For a $\bigvee$-definable group $\mathcal{U}$, we call a definable $X \subseteq \mathcal{U}$ *relatively definably compact* if the closure of X in $\mathcal{U}$ is definably compact. Clearly, X is relatively definably compact if and only if it is contained in some definably compact subset of $\mathcal{U}$.

Lemma 6.7. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Assume that $\mathcal{U}$ is a $\bigvee$-definable abelian group, and $X, Y \subseteq \mathcal{U}$ are definable, with X relatively definably compact. If X and Y are non-generic, then $X \cup Y$ is still non-generic.*

Proof. This is just a small variation on the work in [24]. Because commutativity plays only a minor role we use multiplicative notation for possible future use.

We may assume that $\mathcal{U}$ contains a definable generic set (otherwise, the conclusion is trivial).

We need to prove that if $X \subseteq \mathcal{U}$ is definable, relatively definably compact and non-generic, and if $Z \supseteq X$ is definable and generic then $Z \setminus X$ is generic.

Fix $\mathcal{M}_0$ over which all sets are definable. Without loss of generality, X is definably compact (since the closure of a non-generic set is non-generic).

We first prove the result for Z of the form $W \cdot W$, when W is generic. Since X is not generic, no finitely many translates of X cover W (because W is generic). It follows from compactness that there is $g \in W$ such that $g \notin \bigcup_{h \in M_0} hX$. Changing roles, there is $g \in W$ such that $Xg^{-1} \cap M_0 = \emptyset$. We now apply Dolich's result (Fact 6.6) to the definably compact Xg^{-1}. It follows that there are $g_1, \ldots, g_r$, all realizing the same type as g, so in particular all are in W, such that $Xg_1^{-1} \cap \cdots \cap Xg_r^{-1} = \emptyset$. This in turn implies that $\bigcup_{i=1}^r (W \setminus Xg_i^{-1}) = W$. For each $i = 1, \ldots, r$ we have

$$W \setminus Xg_i^{-1} = (Wg_i \setminus X)g_i^{-1} \subseteq (WW \setminus X)g_i^{-1}.$$

Therefore, it follows that W is contained in the finite union $\bigcup_{i=1}^r (WW \setminus X)g_i^{-1}$ and since W is generic it follows that $WW \setminus X$ is generic, as needed.

We now consider an arbitrary definable generic set $Z \subseteq \mathcal{U}$, with $X \subseteq Z$ non-generic. Because Z is generic, finitely many translates of Z cover $Z \cdot Z$. Namely, $Z \cdot Z \subseteq \bigcup_{i=1}^{t} h_i Z$. If $X' = \bigcup_{i=1}^{t} h_i X$ then X' is still non-generic (and relatively definably compact), so by the case we have just proved, $ZZ \setminus X'$ is generic. However this set difference is contained in

$$\bigcup_{i=1}^{t} h_i Z \setminus \bigcup_{i=1}^{t} h_i X \subseteq \bigcup_{i=1}^{t} h_i (Z \setminus X),$$

hence this right-most union is generic. It follows that $Z \setminus X$ is generic. $\square$

Corollary 6.8. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Assume that $\mathcal{U}$ is a $\bigvee$-definable abelian group which contains a definable generic set and is generated by a definably compact set. Then the definable non-generic subsets of $\mathcal{U}$ form an ideal.*

Proof. Every definable subset of $\mathcal{U}$ must be relatively definably compact, because it is contained in some definably compact set. Then apply Lemma 6.7. $\square$

We next show how the conclusion of the last corollary can be used to guarantee the existence of $\mathcal{U}^{00}$. We recall a definition [24]:

Definition 6.9. Given a $\bigvee$-definable group $\mathcal{U}$ and a definable set $X \subseteq \mathcal{U}$,

$$Stab_{ng}(X) = \{g \in \mathcal{U} : gX \Delta X \text{ is non-generic in } \mathcal{U}\}.$$

Theorem 6.10. *Let $\mathcal{U}$ be an abelian $\bigvee$-definable group in an NIP structure $\mathcal{M}$. Assume that the non-generic definable subsets of $\mathcal{U}$ form an ideal and that $\mathcal{U}$ contains some definable generic set. Then for any definable generic set X, the set $Stab_{ng}(X)$ is a type-definable group and has bounded index in $\mathcal{U}$. In particular, $\mathcal{U}^{00}$ exists.*

Proof. The fact the definable non-generic sets form an ideal implies that for every definable set X, the set $Stab_{ng}(X)$ is a subgroup. We assume that $X \subseteq \mathcal{U}$ is a definable generic set and show that this group is type-definable.

First note that for every $g \in \mathcal{U}$, if $gX \Delta X$ is non-generic, then in particular $gX \cap X \neq \emptyset$ and therefore $g \in XX^{-1}$. It follows that $Stab_{ng}(X)$ is contained in XX^{-1}.

Next, note that a subset of $\mathcal{U}$ is generic if and only if finitely many translates of it cover X (since X itself is generic). Now, for every n, we consider the statement in g: "n many translates of $gX \Delta X$ do not cover X". Here again we note that for $h(gX\Delta X) \cap X$ to be non-empty we must have $h \in XX^{-1} \cup X(gX)^{-1}$. Hence, it is sufficient to write the first-order formula saying that for every $h_1, \ldots, h_n \in XX^{-1} \cup X(gX)^{-1}$, $X \nsubseteq \bigcup_{i=1}^{n} h_i(gX\Delta X)$. The union of all these formulas together with the formula for XX^{-1} is the type which defines $Stab_{ng}(X)$.

It remains to see that this group has bounded index in $\mathcal{U}$. This is a similar argument to the proof of [19, Corollary 3.4] but in that paper the amenability of definable groups and, as a result, the fact that every generic set has positive measure, played an important role. Since a generic subset of a $\bigvee$-definable group may require infinitely many translates to cover the group, we cannot a-priori conclude that it has positive measure, even if the group is amenable. Assume then towards contradiction that $Stab_{ng}(X)$ had unbounded index. Then we can find a sequence of indiscernibles over $\mathcal{M}_0$, $g_1, \dots, g_n, \dots \in \mathcal{U}$ which are all in different cosets of $Stab_{ng}(X)$. In particular, it means that $g_i X \Delta g_j X$ is generic, for $i \neq j$.

Consider now the sequence $X_i = g_{2i} X \Delta g_{2i+1} X$, $i \in \mathbb{N}$. By NIP, there is a k, such that the sequence $\{X_i : i \in \mathbb{N}\}$ is k-inconsistent.

Consider now the type $tp(g_i/M_0)$ and find some M_0-definable set W containing g_i. Because of indiscernibility, all g_i's are in W. It follows that all the $g_i X$, and therefore also all X_i, are contained in WX. Because each X_i is generic, finitely many translates of X_i cover WX. By indiscernibility, there is some ℓ such that for every i there are ℓ-many translates of X_i which cover WX.

We then have countably many set $X_i \subseteq WX$, such that on one hand the intersection of every k of them is empty and on the other hand there is some ℓ such that for each i, ℓ-many translates of X_i cover WX. To obtain a contradiction it is sufficient to prove the following lemma (it is here that we need to find an alternative argument to the measure theoretic one):

Lemma 6.11. *Let G be an arbitrary abelian group, $A \subseteq G$ an arbitrary subset. For every k and ℓ there is a fixed number $N = N(k, \ell)$ such that there are at most N subsets of A with the property that each covers A with ℓ-many translates and every k of them have empty intersection.*

Proof. We are going to use the following fact about abelian groups, taken from [21] (see problems 7 and 16 on p. 82):

Fact 6.12. *For every abelian group G, and for every set $A \subseteq G$ and m, it is not possible to find $A_1, \dots A_{m+1} \subseteq A$ pairwise disjoint such that each A_i covers A by m-many translates.*

Returning to the proof of the lemma, we are going to show that $N = k\ell$ works. Assume for contradiction that there are $k\ell + 1$ subsets $X_1, \dots, X_{kl+1}$ of A, each covering A by ℓ-many translates, with an empty intersection of every k of them. We work with the group $G' = G \times C_k$, where $C_k = \{0, \dots, k-1\}$ is the cyclic group. For $i = 1, \dots, k\ell + 1$, we define $Y_i \subseteq G'$ as follows: For $x \in G$ and $n \in \mathbb{N}$, we have $(x, n) \in Y_i$ if and only if $x \in X_i$ and n is the maximum number such that for some distinct $i_1, \dots, i_n < i$, we have $x \in X_{i_1} \cap \dots \cap X_{i_n} \cap X_i$. Notice that even though i might be larger than k, because of our assumption that every k sets among the X_i's intersect trivially, the maximum n we pick is indeed at most $k - 1$. Note also that the projection of each Y_i on the first coordinate is X_i.

We claim that the Y_i's are pairwise disjoint. Indeed, if $x \in X_i \cap X_j$ and $i < j$ then by the definition of the sets, if $(x, n) \in Y_i$ and $(x, n') \in Y_j$ then $n < n'$, so $Y_i \cap Y_j = \emptyset$.

Now, let $A' = A \times C_k$. We claim that each Y_i covers A' by $k\ell$-many translates. Indeed, if $A \subseteq \bigcup_{j=1}^{\ell} g_{ij} \cdot X_i$ then

$$A' \subseteq \bigcup_{p \in C_k} \bigcup_{j=1}^{\ell} (g_{ij}, p) \cdot Y_i.$$

We therefore found $N+1$ pairwise disjoint subsets of A', each covering A' in N translates, contradicting Fact 6.12. $\square$

With that we finish the proof of Theorem 6.10. $\square$

Remark 6.13. The last theorem implies that for a $\bigvee$-definable abelian group $\langle \mathcal{U}, + \rangle$ in an NIP structure, if the non-generic definable sets form an ideal, then $\mathcal{U}^{00}$ exists if and only if $\mathcal{U}$ contains a definable generic set (we have just proved the right-to-left direction. The converse is immediate since every definable set containing $\mathcal{U}^{00}$ is generic).

7. Divisibility, genericity and definable quotients

In this section, $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of an ordered group.

Proposition 7.1. *If $\mathcal{U}$ is an infinite $\bigvee$-definable group of positive dimension, then it has unbounded exponent. In particular, for every n, the subgroup of n torsion points, $\mathcal{U}[n]$, is bounded.*

Proof. By the Trichotomy Theorem ([26]), there exists a neighborhood of the identity which is in definable bijection with an open subset of R^n for some real closed field R or of V^n for some ordered vector space V (we use here the definability of a group operation near the identity of $\mathcal{U}$).

In the linear case, the group operation of $\mathcal{U}$ is locally isomorphic near $e_{\mathcal{U}}$ to $+$ near $0 \in M^n$ (see [15, Proposition 4.1 and Corollary 4.4] for a similar argument). Clearly then the map $x \mapsto kx$ is non-constant.

Assume then that we are in the field case. Namely, we assume that some definable neighborhood V of e is definably homeomorphic to an open subset of R^n, with e identified with $0 \in \mathbb{R}^n$, and that a real closed field is definable in V. The following argument was suggested by S. Starchenko. If $M(x, y) = xy$ is the group product of elements near e, then it is R-differentiable and its differential at (e, e) is $x+y$. It follows that the differential of the map $x \mapsto x^n$ is nx. Therefore, for every n, the map $x \mapsto x^n$ is not the constant map.

As for the last clause, note that $\mathcal{U}[n]$ is a $\bigvee$-definable subgroup of $\mathcal{U}$ of exponent n, hence must have dimension zero, so its intersection with every definable subset of $\mathcal{U}$ is finite. It follows that $\mathcal{U}[n]$ is bounded. $\square$

Remark 7.2. Although we did not write down the details, we believe that the above result is actually true without any assumptions on the ambient o-minimal $\mathcal{M}$. This can be seen by expressing a neighborhood of $e_{\mathcal{U}}$ as a direct product of neighborhoods, in cartesian powers of orthogonal real closed fields and ordered vector spaces.

Assume that $\mathcal{U} = \bigcup_{i\in I} X_i$ and that $\mathcal{U}^{00}$ exists. Given the projection $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$, we define the *logic topology* on $\mathcal{U}/\mathcal{U}^{00}$ by: $F \subseteq \mathcal{U}/\mathcal{U}^{00}$ is closed if and only if for every $i \in I$, $\pi^{-1}(F) \cap X_i$ is type-definable. We first prove a general lemma.

Lemma 7.3. *Let $\mathcal{U}$ be a locally definable group for which $\mathcal{U}^{00}$ exists and let $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$ be the projection map. If $K_0 \subseteq \mathcal{U}/\mathcal{U}^{00}$ is a compact set, then $\pi^{-1}(K_0)$ is contained in a definable subset of $\mathcal{U}$.*

Proof. We write $\mathcal{U} = \bigcup_{n\in\mathbb{N}} X_n$, and we assume that the union is increasing. If the result fails then there is a sequence $k_n \to \infty$ and $x_n \in X_{k_n} \setminus X_{k_n-1}$ such that $\pi(x_n) \in K_0$. Since K_0 is compact we may assume that the sequence $\pi(x_n)$ converges to some $a \in K_0$. The set $\pi^{-1}(a)$ is a coset of $\mathcal{U}^{00}$ and therefore contained in some definable set $Z \subseteq \mathcal{U}$. Since a can be realized as the intersection of countably many open sets, there is, by logical compactness, some open neighborhood $V \ni a$ in $\mathcal{U}/\mathcal{U}^{00}$ such that $\pi^{-1}(V) \subseteq Z$. But then, the whole tail of the sequence $\{\pi(x_n)\}$ belongs to V and therefore the tail of $\{x_n\}$ is contained in Z, contradicting our assumption on the sequence. $\square$

Theorem 7.4. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact set. If $\mathcal{U}^{00}$ exists, then*

1. *(1) $\mathcal{U}^{00} \neq \mathcal{U}$.*
2. *(2) The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times K$, for some compact Lie group K. (Later we will see that $K \simeq \mathbb{T}^r$, with $k + r = \dim \mathcal{U}$).*
3. *(3) $\mathcal{U}$ and $\mathcal{U}^{00}$ are divisible.*
4. *(4) $\mathcal{U}^{00}$ is torsion-free.*

Proof. (1) By [19, Lemma 7.5], the group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is a locally compact group. If $\mathcal{U}$ is a definable group then it is necessarily definably compact and therefore, by [15], [19], [23], the topological dimension of $\mathcal{U}/\mathcal{U}^{00}$ equals the o-minimal dimension of $\mathcal{U}$. In particular, $\mathcal{U}^{00} \neq \mathcal{U}$. If $\mathcal{U}$ is not definable then, since by definition $\mathcal{U}^{00}$ is contained in a definable set, we must have $\mathcal{U}^{00} \neq \mathcal{U}$.

(2) Let us denote the group $\mathcal{U}/\mathcal{U}^{00}$ by L. By [3, Lemma 2.6] (applied to $\mathcal{U}$ instead of G there), the image of every definable, definably connected subset of $\mathcal{U}$ under π is a connected subset of L. As in the proof of Theorem 2.9 in [3], the group L is locally connected, and since $\mathcal{U}$ is connected, the group L must actually be connected. We next see that this is a Lie group.

Since $\mathcal{U}$ is generated by a definable set, say $X \subseteq \mathcal{U}$, its image $\pi(\mathcal{U}) = L$ is generated by $\pi(X)$ which is a compact set ($\pi(X)$ is a quotient of X by a type-definable equivalence relation with bounded quotient, see [28]). Hence, the group L is so-called compactly generated. By [17, Theorem 7.57], the group L is then isomorphic, as a topological group, to a direct product $\mathbb{R}^k \times K$, for some compact abelian group K. This proves (2).

In what follows, we use $+$ for the group operation of $\mathcal{U}$ and write $\mathcal{U}$ as an increasing countable union $\bigcup_{k=1}^{\infty} X(k)$ (with $X(k)$ as in the notation from Section 1.5).

(3) Let us see that $\mathcal{U}$ is divisible. Given $n \in \mathbb{N}$, consider the map $z \mapsto nz : \mathcal{U} \to \mathcal{U}$. For a subset Z of $\mathcal{U}$, let nZ denote the image of Z under this map. The kernel of this map is $\mathcal{U}[n]$. By Proposition 7.1, $\mathcal{U}[n]$ must have dimension 0, and therefore, $\dim(n\mathcal{U}) = \dim(\mathcal{U})$.

Since $\mathcal{U}$ is connected, by [1, Proposition 1] it is sufficient to show that for every n, the group $n\mathcal{U}$ is a compatible subgroup of $\mathcal{U}$, namely that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap n\mathcal{U}$ is definable.

We claim that $Y \cap n\mathcal{U}$ is contained in $nX(j)$ for some j. Assume towards a contradiction that this fails. Then for every j there exists $x_j \in \mathcal{U}$ such that $nx_j \in Y \setminus nX(j)$. Hence, $x_j \notin X(j)$ and therefore there is a sequence $k_j \to \infty$ such that $x_j \in X(k_j) \setminus X(k_j - 1)$ and $nx_j \in Y$. Consider the projection $\pi(Y)$ and $\pi(x_j)$ in L. Because Y is definable the set $\pi(Y)$ is compact.

By Lemma 7.3, because the sequence $\{x_j\}$ is not contained in any definable subset of $\mathcal{U}$, its image $\{\pi(x_j)\}$ is not contained in any compact subset of L. At the same time, $n\pi(x_j)$ is contained in the compact set $\pi(Y)$. However, since L is isomorphic to $\mathbb{R}^k \times K$, for a compact group K, the map $x \mapsto nx$ is a proper map on L and hence this is impossible. We therefore showed that

$$Y \cap n\mathcal{U} \subseteq nX(j) \subseteq n\mathcal{U},$$

and so $Y \cap n\mathcal{U} = Y \cap nX(j)$ which is a definable set. We can conclude that the group $n\mathcal{U}$ is a compatible subgroup of $\mathcal{U}$, of the same dimension and therefore $n\mathcal{U} = \mathcal{U}$. It follows that $\mathcal{U}$ is divisible.

Let us see that $\mathcal{U}^{00}$ is also divisible. Indeed, consider the map $x \mapsto nx$ from $\mathcal{U}$ onto $\mathcal{U}$. It sends $\mathcal{U}^{00}$ onto the group $n\mathcal{U}^{00}$ and therefore $[\mathcal{U} : \mathcal{U}^{00}] \leq [\mathcal{U} : n\mathcal{U}^{00}]$. Since $\mathcal{U}^{00}$ is the smallest type-definable subgroup of bounded index we must have $n\mathcal{U}^{00} = \mathcal{U}^{00}$, so $\mathcal{U}^{00}$ is divisible.

(4) This is a repetition of an argument from [24]. Because $\mathcal{U}^{00}$ exists there is a definable generic set $X \subseteq \mathcal{U}$ which we now fix. By Theorem 6.10, the group $Stab_{ng}(X)$ contains $\mathcal{U}^{00}$, so it is sufficient to prove that for every n, there is a definable $Y \subseteq \mathcal{U}$ such that $Stab_{ng}(Y) \cap \mathcal{U}[n] = \{0\}$. We do that as follows. Because $\mathcal{U}$ is divisible, the $\bigvee$-definable map $h \mapsto nh$ is surjective. By compactness, there exists a definable $Y_1 \subseteq \mathcal{U}$ which maps onto X. However, since $\mathcal{U}[n]$ is compatible and has dimension zero, every element of X has only finitely many pre-images in Y_1. By definable choice, we can find a definable $Y \subseteq Y_1$ such that the map $h \mapsto nh$ induces a bijection from Y onto X. The set Y is generic in $\mathcal{U}$ as well (since its image is generic and the kernel of the map has dimension zero) and for every $g \in \mathcal{U}[n]$ we have $(g + Y) \cap Y = \emptyset$. Hence, the only element of $\mathcal{U}[n]$ which belongs to $Stab_{ng}(Y)$ is 0. It follows that $\mathcal{U}^{00}$ is torsion-free. $\square$

As a corollary, we can formulate the following criterion for recognizing $\mathcal{U}^{00}$, generalizing a result from [3] and [19]:

Proposition 7.5. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact set. Assume that $H \leqslant \mathcal{U}$ is type-definable of bounded index. Then $H = \mathcal{U}^{00}$ if and only if H is torsion-free.*

In particular, if $\mathcal{U}$ is torsion-free then $\mathcal{U}^{00}$, if it exists, is the only type-definable subgroup of bounded index.

Proof. Since H is type-definable of bounded index, by [19, Proposition 7.4] $\mathcal{U}^{00}$ exists.

If $H = \mathcal{U}^{00}$, then by Theorem 7.4 it is torsion-free.

For the converse, assume that $H \leqslant \mathcal{U}$ is torsion-free. We let $L = \mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology. Because $\mathcal{U}^{00} \leqslant H$, the map $\pi : \mathcal{U} \to L$ sends the type-definable group H onto a compact subgroup of L. If $\pi(H)$ is non-trivial (namely, $H \neq \mathcal{U}^{00}$) then $\pi(H)$ has torsion. However, $\ker(\pi) = \mathcal{U}^{00}$ is divisible (see Theorem 7.4) and therefore H has torsion. Contradiction. □

Theorem 7.6. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact set. Then the following are equivalent.*

(1) $\mathcal{U}$ contains a definable generic set.
(2) $\mathcal{U}^{00}$ exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for some $k \in \mathbb{N}$ and a compact Lie group K.
(3) There exists a definable, definably compact group G and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$ with $\ker(\phi) \simeq \mathbb{Z}^{k'}$, for some $k' \in \mathbb{N}$.

Assume now that the above hold. If k is as in (2), and $\phi : \mathcal{U} \to G$ and k' are as in (3), then $k = k'$.

Proof. (1) ⇔ (2). By Corollary 6.8, Remark 6.13 and Theorem 7.4.

(2) ⇒ (3). Let $L = \mathbb{R}^k \times K$ and $\pi_{\mathcal{U}} : \mathcal{U} \to L$ be the projection map (whose kernel is $\mathcal{U}^{00}$).

We now fix generators $z_1, \ldots, z_k \in \mathbb{R}^k$ for $\mathbb{Z}^k$, and find $u_1, \ldots, u_k \in \mathcal{U}$ with $\pi_{\mathcal{U}}(u_i) = (z_i, 0)$. If we let $\Gamma \leqslant \mathcal{U}$ be the subgroup generated by $u_1, \ldots, u_k$ then $\pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Note that since $z_1, \ldots, z_k$ are $\mathbb{Z}$-independent, the restriction of $\pi_{\mathcal{U}}$ to Γ is injective, namely $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

By Lemma 7.3, there is a definable $X \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq X$. It follows from [3, Lemma 1.7] that the set $\pi_{\mathcal{U}}(X)$ contains not only K but also an open neighborhood of K. But then, there is an m such that $m\pi_{\mathcal{U}}(X) + \mathbb{Z}^k = L$. This implies that $\pi_{\mathcal{U}}(mX + \Gamma) = L$ and hence $mX + \mathcal{U}^{00} + \Gamma \subseteq mX + X + \Gamma = \mathcal{U}$. We let $Y = mX + X$ and then $Y + \Gamma = \mathcal{U}$.

We claim that $Y \cap \Gamma$ is finite. Indeed, if $Y \cap \Gamma$ were infinite then, since $\pi_{\mathcal{U}}$ is injective on Γ, the set $\pi_{\mathcal{U}}(Y) \cap \mathbb{Z}^k$ is infinite, contradicting the compactness of $\pi_{\mathcal{U}}(Y)$. We can now apply Lemma 6.2 and conclude that there is a definable group G and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$ whose kernel is Γ. Because $\mathcal{U}$ is generated by a definably compact set it follows that G must be definably compact.

(3) $\Rightarrow$ (1). By logical compactness, there is a definable $X \subseteq \mathcal{U}$ such that $\phi(X) = G$. But then $X + \ker(\phi) = \mathcal{U}$, and since $\ker(\phi) = \mathbb{Z}^{k'}$ is small, X is generic in $\mathcal{U}$.

Assume now that the conditions hold, k is as in (2), and $\phi : \mathcal{U} \to G$ and k' are as in (3). We will prove that $k = k'$. Consider the map $\pi_U : U \to \mathbb{R}^k \times K$ and let Γ be the image of $\ker(\phi)$ under π_U.

We first claim that $k \leq k'$. Let $X \subseteq \mathcal{U}$ be so that $\phi(X) = G$. Then $X + \ker(\phi) = G$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y and Γ' be the projections of $\pi_{\mathcal{U}}(X)$ and Γ, respectively, into $\mathbb{R}^k$. We have $Y + \Gamma' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_{k'}$ be the generators of $\ker(\phi)$ and let $v_1, \ldots, v_{k'} \in \mathbb{R}^k$ be their images in Γ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_{k'}$ then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that $k \leq k'$.

Now let us prove that $k' \leq k$. Note first that $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, we must have $\ker(\phi) \cap \mathcal{U}^{00} \subseteq \ker(\phi) \cap X$ finite. However, by Theorem 7.4, the group $\mathcal{U}^{00}$ is torsion-free, hence $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$.

It follows that $\Gamma = \pi_{\mathcal{U}}(\ker \phi)$ is of rank k'. It is also discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of 0 and no other elements of Γ.

Now, since K is compact, no element of Γ can be in K and therefore the projection of Γ onto $\Gamma' \subseteq \mathbb{R}^k$ is an isomorphism. Furthermore, Γ' is also discrete, which implies that $k' \leq k$. $\square$

We conjecture that the above three conditions hold always:

Conjecture A. *If $\mathcal{U}$ is a connected abelian $\bigvee$-definable group, generated by a definably compact set, then $\mathcal{U}$ contains a definable generic set.*

Theorem 7.7. *Let $\mathcal{U}$ be a connected $\bigvee$-definable group. Let G be a definably compact group and $\phi : \mathcal{U} \to G$ a surjective $\bigvee$-definable homomorphism with $\ker(\phi) \simeq \mathbb{Z}^k$.*

Then, $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$ and $\phi(\mathcal{U}^{00}) = G^{00}$. Furthermore there is a topological covering map $\phi' : \mathcal{U}/\mathcal{U}^{00} \to G/G^{00}$, with respect to the logic topologies, such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "\phi"] \arrow[d, "\pi_{\mathcal{U}}"'] & G \arrow[d, "\pi_G"] \\
\mathcal{U}/\mathcal{U}^{00} \arrow[r, "\phi'"] & G/G^{00}
\end{tikzcd} \tag{21}$$

The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times \mathbb{T}^r$, for $\mathbb{T}$ the circle group and $k+r = \dim(\mathcal{U})$. If $\mathcal{U}$ is torsion-free, then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim \mathcal{U}}$.

Proof. Let $\Gamma = \ker(\phi)$. We first claim that $\Gamma \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, we must have $\Gamma \cap \mathcal{U}^{00} \subseteq \Gamma \cap X$ finite. However, by Theorem 7.4, the group $\mathcal{U}^{00}$ is torsion-free, hence $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

We claim that $\phi(\mathcal{U}^{00}) = G^{00}$. First note that since $\mathcal{U}^{00}$ has bounded index in $\mathcal{U}$ and ϕ is surjective, the group $\phi(\mathcal{U}^{00})$ has bounded index in G. Because $\Gamma \cap \mathcal{U}^{00} = \{0\}$ the restriction of ϕ to $\mathcal{U}^{00}$ is injective and hence $\phi(\mathcal{U}^{00})$ is torsion-free. By [3], we must have $\phi(\mathcal{U}^{00}) = G^{00}$.

By the work in [15], [19] and [23], we have

$$G/G^{00} \simeq \mathbb{T}^{\dim(G)} = \mathbb{T}^{\dim(\mathcal{U})}.$$

We now consider $\pi_G : G \to G/G^{00}$ and define $\phi' : L \to G/G^{00}$ as follows: For $u \in \mathcal{U}$, let $\phi'(\pi_{\mathcal{U}}(u)) = \pi_G(\phi(u))$. Since $\phi(\mathcal{U}^{00}) = G^{00}$ this map is a well-defined homomorphism which makes the above diagram commute. It is left to see that ϕ' is a covering map.

It follows from what we established thus far that $\ker(\phi') = \pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Since $\phi' : L \to G/G^{00}$ is a surjective homomorphism of Lie groups it is sufficient to check that it is continuous. If $W \subseteq G/G^{00}$ is open then $V = \pi_G^{-1}(W)$ is a $\bigvee$-definable subset of G and hence $\phi^{-1}(V)$ is a $\bigvee$-definable subset of $\mathcal{U}$ (because $\ker \phi$ is a small group). By commutation, this last set equals $\pi_{\mathcal{U}}^{-1}(\phi'^{-1}(W))$ and therefore $\phi'^{-1}(W)$ is open in L.

By Theorem 7.6, $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for a compact Lie group K. We now have a covering map $\phi' : \mathbb{R}^k \times K \to G/G^{00} = \mathbb{T}^{k+r}$, with $k + r = \dim(G) = \dim(\mathcal{U})$ and $\ker(\phi') = \mathbb{Z}^k \subseteq \mathbb{R}^k$. It follows that $K \simeq \mathbb{T}^r$. $\square$

By Theorems 7.6 and 7.7, we obtain:

Corollary 7.8. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact definable set. Assume that $\mathcal{U}$ contains at least one definable generic set. Then*
(i) $\mathcal{U}^{00}$ exists.
(ii) $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, with $k + r = \dim \mathcal{U}$.
(iii) There is a definable, definably compact group G, with $\dim G = \dim \mathcal{U}$, and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$.

Finally, we get Theorem 1.5, which took us on this excursion.

Theorem 7.9. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is generated by a definably compact definable set. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definably compact definable group.

Proof. Since $X + \Lambda = \mathcal{U}$, X is generic. We can apply Theorems 7.6 and 7.7 and obtain a definably compact definable group G and a $\bigvee$-definable homomorphism $\phi : \mathcal{U} \to G$ such that the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "\phi"] \arrow[d, "\pi_{\mathcal{U}}"'] & G \arrow[d, "\pi_G"] \\
\mathcal{U}/\mathcal{U}^{00} \arrow[r, "\phi'"] & G/G^{00}
\end{tikzcd}
\tag{22}
$$

We have $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, where $k + r = \dim\mathcal{U}$ and $\Gamma = \ker(\phi) \simeq \mathbb{Z}^k$. Because $\mathcal{U}^{00} \cap \Gamma = \{0\}$ (see proof of Theorem 7.7), we may assume that $\pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. We now consider $\Delta = \pi_{\mathcal{U}}(\Lambda) \subseteq \mathbb{R}^k \times \mathbb{T}^r$ and let $\Delta' \subseteq \mathbb{R}^k$ be the projection of Δ into $\mathbb{R}^k$. Since $X + \Lambda = \mathcal{U}$, we have $\pi_{\mathcal{U}}(X) + \Delta = \mathbb{R}^k \times \mathbb{T}^r$. Hence, if Y is the projection of $\pi_{\mathcal{U}}(X)$ into $\mathbb{R}^k$ then we have $Y + \Delta' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \dots, \lambda_m$ be generators of Λ and let $v_1, \dots, v_m \in \mathbb{R}^k$ be their images in Δ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \dots, v_m$ then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that among $v_1, \dots, v_m$ there are elements $v_{i_1}, \dots, v_{i_k}$ which are $\mathbb{R}$-independent. It follows that $\lambda_{i_1}, \dots, \lambda_{i_k} \in \Delta$ are $\mathbb{Z}$-independent. If we let Λ' be the group generated by $\lambda_{i_1}, \dots, \lambda_{i_k}$ then we immediately see that the restriction of $\pi_{\mathcal{U}}$ to Λ' is injective. We claim that $\mathcal{U}/\Lambda'$ is definable.

First, let us see that for every definable $Z \subseteq \mathcal{U}$, the set $Z \cap \Lambda'$ is finite. Indeed, $\pi_{\mathcal{U}}(Z)$ is a compact subset of $\mathbb{R}^k \times \mathbb{T}^r$ and hence $\pi_{\mathcal{U}}(Z) \cap (\mathbb{Z}^k \times \{0\})$ is finite. Because $\pi_{\mathcal{U}}|\Lambda'$ is injective it follows that $Z \cap \Lambda'$ is also finite.

We can now take a compact set $K \subseteq \mathbb{R}^k \times \mathbb{T}^r$ such that $K + \mathbb{Z}^k = \mathbb{R}^k \times \mathbb{T}^r$. It follows that $\pi_{\mathcal{U}}^{-1}(K) + \Lambda' = \mathcal{U}$. By Lemma 7.3, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq Z$. We now have $Z + \Lambda' = \mathcal{U}$ and $Z \cap \Lambda'$ finite. By Lemma 6.2, $\mathcal{U}/\Lambda'$ is definable. □

We end this section with a second conjecture.

Conjecture B. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Then*
(i) $\mathcal{U}$ contains a definable generic set.
(ii) $\mathcal{U}$ is divisible.

Although we cannot prove the above conjecture, we can reduce it to proving (i) under additional assumptions.

Claim 7.10. *Conjecture A implies Conjecture B.*

Proof. We assume that Conjecture A is true.

Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Let $\mathcal{V}$ be the universal cover of $\mathcal{U}$ (see [10]). Because $\mathcal{U}$ is the homomorphic image of $\mathcal{V}$ under a $\bigvee$-definable homomorphism whose kernel is a set of dimension 0, it is sufficient to prove that $\mathcal{V}$ contains a generic set and that $\mathcal{V}$ is divisible.

The group $\mathcal{V}$ is connected, torsion-free and generated by a definable set $X \subseteq \mathcal{V}$. We work by induction on $\dim(\mathcal{V})$.

Let Y be the closure of X with respect to the group topology of $\mathcal{V}$.

Case 1 The set Y is definably compact.

Since $\mathcal{V}$ is generated by Y, then by our standing assumption we may conclude that $\mathcal{V}$ contains a definable generic set. By Theorems 7.6 and 7.4, $\mathcal{V}$ is divisible.

Case 2 The set Y is not definably compact.

In this case, we can apply [6, Theorem 5.2] and obtain a definable 1-dimensional, definably connected, divisible, torsion-free subgroup of $\mathcal{V}$, call it H. Clearly, H is a compatible subgroup of $\mathcal{V}$, hence the group $\mathcal{V}/H$ is $\bigvee$-definable, connected ([6, Corollary 4.8]), torsion-free and definably generated (by the image of X under the projection map). We have $\dim(\mathcal{V}/H) < \dim \mathcal{V}$, so by induction, the conjecture holds for $\mathcal{V}/H$, hence it is divisible and contains a definable generic set Z. Because H is divisible as well, it follows that $\mathcal{V}$ is divisible. It is easy to see that the pre-image of Z in $\mathcal{V}$ is a definable generic subset of $\mathcal{V}$. $\square$

Finally, although we know that $\mathcal{U}$ needs to be definably generated in order to guarantee (i) (by Fact 6.4(2)), we do not know if the same is true for (ii).

Conjecture C. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group. Then $\mathcal{U}$ is divisible.*

8. Compact Domination

We assume that $\mathcal{M}$ is a sufficiently saturated semi-bounded o-minimal expansion of an ordered group, G is a definably connected, definably compact, abelian definable group and $\pi : G \to G/G^{00}$ is the natural projection. We equip G/G^{00} with the Haar measure, denoted by $m(Z)$, and prove: for every definable $X \subseteq G$, the set of $h \in G/G^{00}$ for which $\pi^{-1}(h) \cap X \neq \emptyset$ and $\pi^{-1}(h) \cap (G \setminus X) \neq \emptyset$ has measure zero. As is pointed out in [19], it is sufficient to prove that

(23) for every definable $X \subseteq G$, if $\dim X < \dim G$, then $m(\pi X) = 0$.

We say then that G (and π) *satisfy Compact Domination.* We split the argument into two cases:

I. G is abelian.

If we consider the universal covering map $\phi : \widehat{G} \to G$ then, using the commutative diagram in Theorem 7.7 (and the fact that $\ker \phi'$ has measure zero), it is sufficient to prove (23) for the universal cover $\widehat{G}$.

Recall by Theorem 1.3 the sequence:

$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{i} \widehat{G} \xrightarrow{f} \mathcal{U} \longrightarrow 0$$

with $\widehat{\mathcal{H}}$ an open subgroup of $\langle M^k, +\rangle$, $\mathrm{lgdim}(\widehat{\mathcal{H}}) = k = \mathrm{lgdim}(\widehat{G})$ and $\mathcal{U}$ a short $\bigvee$-definable group of dimension n. Note that $\widehat{G}$ contains a definable generic set (for example, by Theorem 7.6), and hence so does $\mathcal{U}$. By Theorem 7.6 again, $\mathcal{U}$ has a definable, definably compact quotient K. By [18], the group K, with its map onto K/K^{00} satisfies Compact Domination, and therefore $\pi_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$ also satisfies Compact Domination.

We now consider $\widehat{\mathcal{H}}$ and claim that

(24) the set of all short elements in M^k is contained in $\widehat{\mathcal{H}}^{00}$.

Indeed, recall from Section 3.1 that $\widehat{\mathcal{H}}$ is generated by a subset $H' \subseteq M^k$,

$$H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k),$$

with each $e_i > 0$ tall in M. We define, for each $n \in \mathbb{N}$, $H_i = \frac{1}{n}H'$, and claim that

$$\widehat{\mathcal{H}}^{00} = \bigcap_n H_n.$$

Indeed, $\widehat{\mathcal{H}}^{00}$ is a torsion-free subgroup of $\widehat{\mathcal{H}}$. Moreover, each H_n is generic in $\widehat{\mathcal{H}}$ because we have $\widehat{\mathcal{H}} = H_n + \mathbb{Q}e_1 + \cdots + \mathbb{Q}e_k$. It follows that $\widehat{\mathcal{H}}^{00}$ has bounded index in $\widehat{\mathcal{H}}$, and thus Proposition 7.5 gives $\widehat{\mathcal{H}}^{00} = \bigcap_n H_n$. Finally, since each e_i is tall, we obtain (24).

We now claim that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}}) = i(\widehat{\mathcal{H}}^{00})$. This follows from the fact that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}})$ has bounded index in $i(\widehat{\mathcal{H}})$ and it is torsion-free (Proposition 7.5). Next, we claim that $f(\widehat{G}^{00}) = \mathcal{U}^{00}$. Since $f(\widehat{G}^{00})$ has bounded index it must contain $\mathcal{U}^{00}$. Because $\widehat{G}^{00}$ is torsion-free and $\ker(f) = i(\widehat{\mathcal{H}}^{00}) = i(\widehat{\mathcal{H}}) \cap \widehat{G}^{00}$ is divisible (Theorem 7.4), it follows that $f(\widehat{G}^{00})$ is torsion-free so must equal $\mathcal{U}^{00}$. We therefore have the following commutative diagram of exact sequences:

(25)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} & \xrightarrow{f} & \mathcal{U} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \pi_{\widehat{\mathcal{H}}}} & & \downarrow{\scriptstyle \pi_{\widehat{G}}} & & \downarrow{\scriptstyle \pi_{\mathcal{U}}} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00} & \xrightarrow{\widehat{i}} & \widehat{G}/\widehat{G}^{00} & \xrightarrow{\widehat{f}} & \mathcal{U}/\mathcal{U}^{00} & \longrightarrow & 0
\end{array}$$

As in the proof of Theorem 7.7, the map $\widehat{f}$ is continuous.

Assume now that $X \subseteq \widehat{G}$ is a definable set of dimension smaller than $\dim \widehat{G}$.

Case 1 $\dim f(X) < \dim \mathcal{U}$.

In this case, by Compact Domination for $\mathcal{U}$, we have $m(\pi_{\mathcal{U}}(f(X))) = 0$, and therefore, by the commutation of the above diagram we must have $m(\pi_{\widehat{G}}(X)) = 0$.

Most of the work goes towards the proof of the second case. For simplicity, let us assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Case 2. $\dim f(X) = \dim \mathcal{U}$.

The idea of this argument is as follows: We identify, locally, $\widehat{G}$ as the cartesian product $\widehat{\mathcal{H}} \times U$ and $\widehat{G}/\widehat{G}^{00}$ with $\mathbb{R}^k \times \mathbb{R}^n$. We then show that in our case the projection of X into $\widehat{\mathcal{H}}$ has long dimension smaller than k and as a result the image of this projection in $\mathbb{R}^k$ has measure 0. This will be sufficient to conclude that $m(\pi_{\widehat{G}}(X)) = 0$.

Let $W \subseteq \widehat{G}$ be a definable generic set. Since W is definable there is a definable partial section $s : f(W) \to \widehat{G}$, namely, $sf(u) = u$ for all $u \in f(W)$. The map s is piecewise continuous (with respect to the τ-topologies of $\hat{G}$ and $\mathcal{U}$) and therefore W has a definable subset $W' \subseteq W$, still generic in $\widehat{G}$ such that s is continuous on $f(W')$. Moreover, using Compact Domination for $\mathcal{U}$, it follows from [19, Claim 3, p.590] that the set $f(W')$ contains a coset of $\mathcal{U}^{00}$ so we may assume after translation that $W' = W$, $f(W)$ contains $\mathcal{U}^{00}$ and $s : f(W) \to \widehat{G}$ is continuous. Because W is generic, finitely many translates of W cover X. It is sufficient to prove the result for the intersection of X with each such translate, therefore we may assume that $X \subseteq W$.

We let U denote the definable set $f(W) \subseteq \mathcal{U}$. Since U is a short set, the associated cocycle $\sigma(x, y) = s(x) + s(y) - s(x + y)$ is a map from $U \times U$ into a short subset of $\widehat{\mathcal{H}}$. By choosing $s(0_{\mathcal{U}}) = 0_{\widehat{G}}$, we have $\sigma(0, 0) = 0$. Hence, by (24) and since s is continuous, $\sigma(f(W) \times f(W))$ is contained in $\widehat{\mathcal{H}}^{00}$. The map $(h, u) \mapsto h + s(u)$ gives a $\bigvee$-definable bijection between the $\bigvee$-definable sets $\widehat{\mathcal{H}} \times U$ and $f^{-1}(U)$. Its inverse is $g \mapsto (g - sf(g), f(g))$.

Assume that $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00} \simeq \mathbb{R}^k$ and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^n$. We now define $h : f^{-1}(U) \to \mathbb{R}^k \times \mathbb{R}^n$ by

$$h(g) = (\pi_{\widehat{\mathcal{H}}}(g - sf(g)), \pi_{\mathcal{U}}(f(g))).$$

We claim that h is a partial group homomorphism, namely, for all $g_1, g_2 \in f^{-1}(U)$, if $g_1 + g_2 \in f^{-1}(U)$ then $h(g_1) + h(g_2) = h(g_1 + g_2)$. Indeed,

$$\begin{aligned} h(g_1 + g_2) - h(g_1) + h(g_2) &= (\pi_{\widehat{\mathcal{H}}}(sf(g_1) + sf(g_2) - s(f(g_1) + f(g_2)), 0), \\ &= (\pi_{\widehat{\mathcal{H}}}(\sigma(f(g_1), f(g_2))), 0) \end{aligned}$$

and since $\sigma(f(W) \times f(W)) \subseteq \widehat{\mathcal{H}}^{00}$, we have $\pi_{\widehat{\mathcal{H}}}(\sigma(f(g_1), f(g_2)) = 0$.

Claim 8.1. *There is a group homomorphism $\widehat{h} : \widehat{G} \to \mathbb{R}^k \times \mathbb{R}^n$ extending h, such that* $\ker(\widehat{h}) = \widehat{G}^{00}$ *and the induced map* $h^* : \widehat{G}/\widehat{G}^{00} \to \mathbb{R}^k \times \mathbb{R}^n$ *is a topological group isomorphism.*

Proof. We first show:
Claim For every $g \in \widehat{G}$ there is a real number $r > 0$ such that for every rational $q \leq r$ we have $q\,g \in f^{-1}(U)$ (this is well-defined since $\mathcal{U}$ is torsion-free and divisible).

Indeed, given $g \in \widehat{G}$ consider $\pi_{\mathcal{U}}(f(g))$. Because U contains $\mathcal{U}^{00}$ there is, by logical compactness, an open neighborhood $U' \subseteq \mathbb{R}^n$ of 0 such that $\pi_{\mathcal{U}}^{-1}(U') \subseteq U$. Because U' is open in $\mathbb{R}^n$, there is $r \in \mathbb{R}$ such that for all $r' \leq r$ we have $r'\pi_{\mathcal{U}}(f(g)) \in U'$. But then, for all rational $q \in \mathbb{Q}$, if $q \leq r$, we must have $q\,f(g) \in U$ which implies that $q\,g \in f^{-1}(U)$.

We extend h as follows: given $g \in \widehat{G}$, it follows from the claim that there is $n \in \mathbb{N}$ such that for all rational $q \leq 1/n$, we have $q\,g \in f^{-1}(U)$. We let $\widehat{h}(g) = nh(g/n)$. We need to see that $\widehat{h}$ is well-defined. Namely, we need to see that if g/n and g/m are in $f^{-1}(U)$, as above, then $nh(g/n) = mh(g/m)$. If we take $k \geq n, m$ then by our assumption, $g/kn, g/km \in f^{-1}(U)$ and moreover, for all $i = 1, \ldots, k$, we have $ig/nk\,,\, ig/km \in f^{-1}(U)$. In particular, if $m \geq n$, then for all $i = 1, \ldots, m$ we have $ig/nm \in f^{-1}(U)$. Since h is a partial homomorphism on the set $f^{-1}(U)$, we have for all $i = 1, \ldots, m$, $h(ig/nm) = ih(g/nm)$. In particular,

$$h(g/m) = h(ng/nm) = nh(g/nm) \text{ and } h(g/n) = h(mg/nm) = mh(g/nm).$$

It follows that $mh(g/m) = nh(g/n)$, as we wanted.

Let us see that the map $\widehat{h} : \widehat{G} \to \mathbb{R}^k \times \mathbb{R}^n$ is a group homomorphism. For $x, y \in \widehat{G}$, we take n large enough so that for all $q \leq 1/n$, we have $qx, qy, q(x+y) \in f^{-1}(U)$, and by our definition we have $\widehat{h}(x+y) = n\,h((x+y)/n)$, $\widehat{h}(x) = nh(x/n)$ and $\widehat{h}(y) = nh(x/n)$. Since h is a partial homomorphism we have $\widehat{h}(x+y) = \widehat{h}(x) + \widehat{h}(y)$, therefore $\widehat{h}$ is a group homomorphism.

Our next goal is to show:

$$\text{the pre-image of every closed subset of } \mathbb{R}^k \times \mathbb{R}^n \text{ under } \widehat{h} \text{ is a type-definable subset of } \widehat{G}. \tag{26}$$

Let us first see it for closed subsets of $\widehat{h}(f^{-1}(U))$. Because there is a $\bigvee$-definable bijection between $f^{-1}(U)$ and $\widehat{\mathcal{H}} \times U$ it is sufficient to check it for the induced map, call it $h' : \widehat{\mathcal{H}} \times U \to \mathbb{R}^k \times \mathbb{R}^n$. But this is just the map $(x, u) \mapsto (\pi_{\widehat{\mathcal{H}}}(h), \pi_{\mathcal{U}}(u))$, so clearly the pre-image of every closed subset is type-definable. Finally, note that since $\widehat{\mathcal{H}} \times U$ contains $\widehat{\mathcal{H}}^{00} \times \mathcal{U}^{00}$, it follows from compactness that there is an open neighborhood $V \subseteq \mathbb{R}^k \times \mathbb{R}^n$ of $(0,0)$ such that $h'^{-1}(V) \subseteq \widehat{\mathcal{H}} \times U$. In particular, $\widehat{h}^{-1}(V) \subseteq f^{-1}(U)$. It is now easy to see that the pre-image of every open subset of $\mathbb{R}^k \times \mathbb{R}^n$ under $\widehat{h}$ is

$\bigvee$-definable (since this is true on V) and therefore the pre-image of every closed set is type-definable.

Next, we claim that $\ker(\widehat{h}) = \widehat{G}^{00}$. Indeed, since $\widehat{h}$ is a map into $\mathbb{R}^k \times \mathbb{R}^n$, its kernel has bounded index. But then, since $\widehat{G}$ is torsion-free we can conclude from Proposition 7.5, that $\ker(\widehat{h}) = \widehat{G}^{00}$.

We now have two surjective group homomorphisms $\pi_{\widehat{G}} : \widehat{G} \to \widehat{G}/\widehat{G}^{00}$ and $\widehat{h} : \widehat{G} \to \mathbb{R}^k \times \mathbb{R}^n$, both with kernel $\widehat{G}^{00}$. We therefore obtain an induced group isomorphism $h^* : \widehat{G}/\widehat{G}^{00} \to \mathbb{R}^k \times \mathbb{R}^n$.

$$\begin{tikzcd}
\widehat{G} \arrow[r, "\widehat{h}"] \arrow[d, "\pi_{\widehat{G}}"'] & \mathbb{R}^k \times \mathbb{R}^n \\
\widehat{G}/\widehat{G}^{00} \arrow[ur, dotted, "h^*"'] &
\end{tikzcd}$$

Since $\widehat{G}/\widehat{G}^{00}$ is equipped with the logic topology, [19, Remark 2.3(i)] and (26) imply that h^* is also a topological homeomorphism. This ends the prof of Claim 8.1. □

Let us return to our problem and recall that we wanted to see that $m(\pi_{\widehat{G}}(X)) = 0$. Because of the uniqueness of the Haar measure, the map h^* sends sets of positive Haar measure to sets of positive Lebesgue measure, so it is sufficient to see that $m(\widehat{h}(X)) = 0$.

By Lemma 10.1, we can decompose $f(X)$ into two definable sets $Y_1 \cup Y_2$ such that for every $u \in Y_1$, we have $\operatorname{lgdim}(f^{-1}(u) \cap X) < k$ and for every $u \in Y_2$, $\operatorname{lgdim}(f^{-1}(u) \cap X) = \dim(f^{-1}(u) \cap X) = k$. Because $\dim X < \dim \widehat{G}$ and $\dim f(X) = \dim \mathcal{U}$, the dimension of Y_2 must be smaller than $\dim(\mathcal{U})$. By Case (1), we can ignore Y_2 and assume now that for every $u \in f(X)$, $\operatorname{lgdim}(f^{-1}(u) \cap X) < k$.

Using the identification of $f^{-1}(U)$ and $\widehat{\mathcal{H}} \times U$, we may assume that $X \subseteq \widehat{\mathcal{H}} \times U$ (recall that $X \subseteq f^{-1}(U)$). If we denote by $p_1(X), p_2(X)$ the projections of X into $\widehat{\mathcal{H}}$ and $\mathcal{U}$, respectively, then, because $\widehat{h}$ induces on $\widehat{H} \times U$ the map $(h, u) \mapsto (\pi_{\widehat{\mathcal{H}}}(h), \pi_{\mathcal{U}}(u))$, the image of X under this map is contained in $\pi_{\widehat{H}}(p_1(X)) \times \pi_{\mathcal{U}}(p_2(X))$. By Fubbini's Theorem, it is sufficient to prove that the Lebesgue measure of $\pi_{\widehat{H}}(p_1(X))$ is zero.

Because U is a short set and for every $y \in U$ we have $\operatorname{lgdim}(f^{-1}(u) \cap X) < k$, we must have $\operatorname{lgdim}(p_1(X)) < k$ ([13, Corollary 3.4]). It is thus sufficient to prove the following final claim:

Claim 8.2. *If $Y \subseteq \widehat{\mathcal{H}}$ is definable and* $\operatorname{lgdim}(Y) < k$ *then* $m(\pi_{\widehat{\mathcal{H}}}(Y)) = 0$ *(where we now take the measure in $\mathbb{R}^k$).*

Proof. Recall that $\widehat{\mathcal{H}}$ is a subgroup of $\langle M^k, + \rangle$ and that the set of all short elements of M^k is contained in $\widehat{\mathcal{H}}^{00}$. Hence, if B is any definably connected short set, then $\pi_{\widehat{\mathcal{H}}}(B) = \{b\}$ is a singleton.

The set Y is a finite union of m-long cones, with $m < k$, hence we may assume that Y is such a cone $C = B + \langle C\rangle^+$, where $\langle C\rangle^+ = \left\{\sum_{i=1}^{k} \lambda_i(t_i) : t_i \in I_i\right\}$, for some I_i and linear $\lambda_i : I_i \to M^k$. We have

$$\pi_{\widehat{\mathcal{H}}}(C) = b + \sum_{i=1}^{m} \pi_1(\lambda_i(t_i)).$$

Because $\pi_{\widehat{\mathcal{H}}}$ is a homomorphism from $\langle \widehat{\mathcal{H}}, +\rangle$ onto $\langle \mathbb{R}^k, +\rangle$, it follows that for each $i = 1, \dots, m$, $t_i \mapsto \pi_1(\lambda_i(t_i))$ is a partial homomorphism from I_i into $\langle \mathbb{R}^k, +\rangle$. Hence, the image of the $\widehat{G}$-linear set $\{\lambda_i(t) : t \in I_i\}$ is a closed affine subset of $\mathbb{R}^k$. In particular, the set $\pi_{\widehat{\mathcal{H}}}(\langle C\rangle^+)$ is an affine subset of $\mathbb{R}^k$ of dimension m. Since $m < k$ we have $m(\pi_{\widehat{\mathcal{H}}}(Y)) = m(\pi_{\widehat{\mathcal{H}}}(C)) = 0$. $\square$

This ends the proof of Compact Domination for abelian G.

II. The general case (G not necessarily abelian).

Assume now that G is an arbitrary definably compact group. By [20], G is the almost direct product of a definably connected abelian group G_0 and a definable semi-simple group S. It is enough to prove the result for a finite cover of G hence we may assume that $G = G_0 \times S$. By [20, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short, and therefore $\mathrm{lgdim} G = \mathrm{lgdim} G_0 = k$. To simplify the diagram, we use $\overline{G_0} = G_0/G_0^{00}$, $\overline{S} = S/S^{00}$, so we have $G/G^{00} = \overline{G_0} \times \overline{S}$.

We have

$$\begin{tikzcd}
0 \arrow[r] & G_0 \arrow[r, "i"] \arrow[d, "\pi_{G_0}"'] & G_0 \times S \arrow[r, "f"] \arrow[d, "\pi_G"'] & S \arrow[r] \arrow[d, "\pi_S"] & 1 \\
0 \arrow[r] & \overline{G_0} \arrow[r, "\widehat{i}"] & \overline{G_0} \times \overline{S} \arrow[r, "\widehat{f}"] & \overline{S} \arrow[r] & 0
\end{tikzcd}$$

Assume now that $X \subseteq G$ is a definable set and $\dim(X) < \dim(G)$. If $\dim(f(X)) < \dim(S)$ then by Dompact Domination in expansions of fields, the Haar measure of $\pi_S(f(X))$ in $\overline{S}$ is 0 and therefore $m(\pi_G(X))$ in G/G^{00} is 0.

If $\dim(X) = \dim(S)$ then, as in the abelian case, we may assume, after partition, that for every $s \in S$, $\mathrm{lgdim}(f^{-1}(s) \cap X) < k$. Because S is short, it follows that $\mathrm{lgdim}(X) < k$ and therefore the projection of X into G_0, call it X', has long dimension smaller than k. But now, similarly to Claim 8.2, we can prove that the Haar measure in $\overline{G_0}$ of $\pi_{G_0}(X')$ equals to 0. This implies that the Haar measure of $\pi_G(X)$ must be zero.

This ends the proof of Compact Domination for definably compact groups in o-minimal expansions of ordered groups. $\square$

9. Appendix A - pullback and pushout

9.1. **Pushout.**

Proof of Proposition 2.3. We start with

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \\ C & \end{tikzcd}$$

and prove the existence of the pushout D. We first review the standard construction of D (without verifying the algebraic facts). We consider the direct product $B \times C$ and take $D = (B \times C)/H$ where H is the subgroup $H = \{(\alpha(a), -\beta(a)) : a \in A\}$. If we denote by $[b, c]$ the coset of (b, c) mod H then the maps γ, δ are defined by $\gamma(b) = [b, 0]$ and $\delta(c) = [0, c]$. Assume now that we also have

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d, "\gamma'"] \\ C \arrow[r, "\delta'"] & D' \end{tikzcd}$$

We define $\phi : D \to D'$ by $\phi([b, c]) = \gamma'(b) + \delta'(c)$. Clearly, if all data is definable then so are $B \times C$ and H, and therefore, using definable choice, D and the associated maps are definable.

If α is injective then δ is also injective, and if β is surjective then so is γ (see observation (b) on p. 53 in [16])

Suppose that A, B, C and α, β are $\bigvee$-definable and that $\alpha(A)$ is compatible subgroup of B. Clearly $B \times C$ is $\bigvee$-definable and it is easy to see that H is a $\bigvee$-definable subgroup. We want to show that H is a compatible subgroup of $B \times C$. For that we write $A = \bigcup A_i$, $B = \bigcup B_j$ and $C = \bigcup C_k$. It follows that $B \times C = \bigcup_{j,k} B_j \times C_k$. To show compatibility of H it is enough to show that for every j, k, the intersection $(B_j \times C_k) \cap H$ is definable. Because $\alpha(A)$ is compatible in B, the set $B_j \cap \alpha(A)$ is definable. Hence, there is some i_0 such that $\alpha(A_{i_0}) \supseteq B_j \cap \alpha(A)$. Moreover, because α is injective $\alpha^{-1}(B_j) \subset A_{i_0}$. It follows that the intersection $H \cap (B_j \times C_k)$ equals

$$\{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A\} = \{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A_{i_0}\}.$$

The set on the right is clearly definable, hence H is a compatible subgroup of $B \times C$, so $D = (B \times C)/H$ is $\bigvee$-definable (see Fact 2.1). It is now easy to check that $\gamma : B \to D$ and $\delta : C \to D$ are $\bigvee$-definable.

If $E = B/\alpha(A)$ then, by the compatibility of $\alpha(A)$, we see that E is $\bigvee$-definable. If $\pi : B \to E$ is the projection then we define $\pi' : D \to E$ by $\pi'([b, c]) = \pi(b)$. It is routine to verify that π' is a well-defined surjective homomorphism whose kernel is $\delta(C)$. It follows, using Fact 2.1, that $\delta(C)$

is a compatible subgroup of D. Finally, it is routine to verify commutation of all maps. □

Proof of Lemma 2.4. We have

$$\begin{tikzcd} B \arrow[r, "\gamma"] & D \arrow[r, "\mu"] & F \\ A \arrow[u, "\alpha"] \arrow[r, "\beta"] & C \arrow[u, "\delta"'] \arrow[r, "\eta"] & E \arrow[u, "\xi"'] \end{tikzcd} \tag{27}$$

with D the pushout of B and C over A and F the pushout of B and E over A and we want to see that F is also the pushout of D and E over C.

It is sufficient to show that for every given commutative diagram

$$\begin{tikzcd} D \arrow[r, "\mu'"] & F' \\ C \arrow[u, "\delta"'] \arrow[r, "\eta"] & E \arrow[u, "\xi'"'] \end{tikzcd} \tag{28}$$

there is a map $\phi' : F \to F'$ such that $\phi'\mu = \mu'$ and $\phi'\xi = \xi'$ (according to the definition we also need to prove uniqueness but this follows).

By commutativity we have $\mu'\delta = \xi'\eta$ and hence $\mu'\delta\beta = \xi'\eta\beta$. Since $\delta\beta = \gamma\alpha$ we also have $(\mu'\gamma)\alpha = (\xi')\eta\beta$. We now use the fact that F is the pushout of B and E over A and conclude that there is $\phi' : F \to F'$ such that

$$(i)\phi'\xi = \xi' \quad \text{and} \quad (ii)\phi'\mu\gamma = \mu'\gamma \tag{29}$$

(i) gives half of what we need to show so it is left to see that $\phi'\mu = \mu'$.

Consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "\mu'\gamma"] & F' \\ A \arrow[u, "\alpha"] \arrow[r, "\beta"] & C \arrow[u, "\xi'\eta"'] \end{tikzcd} \tag{30}$$

Because D is the pushout of B and C over A, there is a unique map $\psi : D \to F'$ with the property

$$(i)\psi\delta = \xi'\eta \quad \text{and} \quad (ii)\psi\gamma = \mu'\gamma.$$

If we can show that both maps μ' and $\phi'\mu$ from D into F' satisfy these properties of ψ then by uniqueness we will get their equality. For $\psi = \mu'$, (i) is part of the assumptions, and (ii) is obvious. For $\psi = \phi'\mu$, we obtain (ii) directly from (29)(ii). To see (i), start from (29)(i), $\phi'\xi = \xi'$, and conclude $\phi'\xi\eta = \xi'\eta$. By commutation, $\xi\eta = \mu\delta$ so we obtain $\phi'\mu\delta = \xi'\eta$, as needed. We therefore conclude that $\mu' = \phi'\mu$ and hence F is the pushout of E and D over C. □

9.2. **Pullback.**

Proof of Proposition 2.6. Consider the diagram

$$\begin{tikzcd} & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & A \end{tikzcd}$$

We again review the algebraic construction of a pullback (which is simpler because we take no quotients). We let

$$D = \{(b, c) \in B \times C : \alpha(b) = \beta(c)\},$$

and the maps are just $\gamma(b, c) = b$ and $\delta(b, c) = c$. Given

$$\begin{tikzcd} D' \arrow[r, "\gamma'"] \arrow[d, "\delta'"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & A \end{tikzcd}$$

we define $\phi(d') = (\gamma'(d'), \delta'(d')) \in D$.

Clearly, if all data is definable then so is D and the associated maps. Similarly, if all data is $\bigvee$-definable then so are D and the associated maps.

If $G = \ker(\gamma)$ then

$$G = \{(b, c) \in D : b = 0\} = \{(0, c) \in B \times C : \beta(c) = 0\},$$

and then clearly $j(0, c) = c$ is an isomorphism of G and $H = \ker(\beta)$. If all given data is $\bigvee$-definable then so are G, H and the associated maps. Furthermore, since G and H are kernels of $\bigvee$-definable maps they are clearly compatible in D, C, respectively.

If β is surjective then so is γ and the sequences in the diagram are exact (and the diagram is commutative). $\square$

10. Appendix B - Short and long set

We assume that $\mathcal{M}$ is an o-minimal semi-bounded expansion of an ordered group

Lemma 10.1. *Let $S \subseteq M^r$ be a definable short set and let $A \subseteq S \times M^n$ be a definable set. For $s \in S$, we let $A_s = \{x \in M^n : (s, x) \in A\}$. Then, for every $\ell \geq 0$, the set $\ell(A) = \{s \in S : \operatorname{lgdim}(A_s) = \ell\}$ is definable.*

Proof. By [13], the set A can be written as a union of long cones $\bigcup C_i$. Since $\operatorname{lgdim}(X_1 \cup \cdots \cup X_m) = max_i(\operatorname{lgdim}(X_i))$, we may assume that A itself is a long cone $A = B + \sum_{i=1}^k \lambda_i(t_i)$, where $B \subseteq M^{r+n}$ is a short cell, $\lambda_1, \ldots, \lambda_k$ are M-independent partial linear maps $\lambda_i : I_i \to M^{r+n}$ and $I_i = (0, a_i)$ are long intervals. We write $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{r+n})$, for $i = 1, \ldots, k$, so each λ_i^j is a partial endomorphism from I_i into M.

We claim that for every $s \in S$, $\operatorname{lgdim}(A_s) = k$. This clearly implies what we need.

For $b = (b_1, \dots, b_{r+n}) \in B$, $i = 1, \dots, k$ and $t_i \in I_i$, we have $b_i + \lambda_i(t_i) : I_i \to A$. Therefore, we have $(b_1, \dots, b_r) + (\lambda_i^1(t_i), \dots, \lambda_i^r(t_i)) \in S$. Each λ_i^j is either injective or constantly 0 and hence, because S is short and each I_i is long, for each $j = 1, \dots, r$ and $i = 1, \dots, k$, we have $\lambda_i^j \equiv 0$. It follows that for every $b \in B$, we have $(b_1, \dots, b_r) \in S$.

For $i = 1 \dots, k$, we let

$$\hat{\lambda}_i = (\lambda_i^{r+1}, \dots, \lambda_i^{r+n}) : I_i \to M^n.$$

Because $\lambda_1, \dots, \lambda_k$ were M-independent, it is still true that $\hat{\lambda}_1, \dots, \hat{\lambda}_k$ are M-independent. We now have, for every $s \in S$,

$$A_s = \left\{ b + \sum_{i=1}^{k} \hat{\lambda}_i(t_i) : b \in B_s, t \in I_i \right\}$$

and therefore the set A_s is k-long cone, so $\operatorname{lgdim}(A_s) = k$. □

References

[1] Elías Baro and Mário J. Edmundo, *Corrigendum to: "Locally definable groups in o-minimal structures" by Edmundo*, J. Algebra **320** (2008), no. 7, 3079–3080.

[2] Elías Baro and Margarita Otero, *Locally definable homotopy*, Annals of Pure and Applied Logic **161** (2010), no. 4, 488–503.

[3] Alessandro Berarducci, Margarita Otero, Yaa'cov Peterzil, and Anand Pillay, *A descending chain condition for groups definable in o-minimal structures*, Ann. Pure Appl. Logic **134** (2005), no. 2-3, 303–313.

[4] Lou van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998.

[5] Mario J. Edmundo, *Structure theorems for o-minimal expansions of groups*, Ann. Pure Appl. Logic **102** (2000), no. 1-2, 159–181.

[6] Mário J. Edmundo, *Locally definable groups in o-minimal structures*, J. Algebra **301** (2006), no. 1, 194–223.

[7] ———, *Covers of groups definable in o-minimal structures*, Illinois J. Math. **49** (2005), no. 1, 99–120 (electronic).

[8] Mário J. Edmundo and Pantelis E. Eleftheriou, *Definable group extensions in semi-bounded o-minimal structures*, Math. Log. Quart. **55** (2009), no. 6, 598–604.

[9] Mário J. Edmundo, *Solvable groups definable in o-minimal structures*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 103–145.

[10] Mário J. Edmundo and Pantelis E. Eleftheriou, *The universal covering homomorphism in o-minimal expansions of groups*, Math. Log. Quart. **53** (2007), 571–582.

[11] Edmundo M. and M. Otero, *Definably compact abelian groups*, Journal of Math. Logic **4** (2004), 163–180.

[12] Pantelis E. Eleftheriou, *Compact domination for groups definable in linear o-minimal structures*, Archive for Mathematical Logic **48** (2009), no. 7, 607–623.

[13] ______, *Local analysis for semi-bounded groups*, preprint (2010).

[14] ______, *Affine embeddings for semi-linear tori*, in preparation.

[15] Pantelis E. Eleftheriou and Sergei Starchenko, *Groups definable in ordered vector spaces over ordered division rings*, J. Symbolic Logic **72** (2007), no. 4, 1108–1140.

[16] László Fuchs, *Infinite abelian groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York, 1970.

[17] Karl H. Hofmann and Sidney A. Morris, *The structure of compact groups*, de Gruyter Studies in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 1998. A primer for the student—a handbook for the expert.

[18] Ehud Hrushovski and Anand Pillay, *On NIP and invariant measures*, preprint.

[19] Ehud Hrushovski, Ya'acov Peterzil, and Anand Pillay, *Groups, measures, and the NIP*, J. Amer. Math. Soc. **21** (2008), no. 2, 563–596.

[20] ______, *On central extensions and definably compact groups in o-minimal structures*, Journal of Algebra **327** (2011), 71–106.

[21] Péter Komjáth and Vilmos Totik, *Problems and theorems in classical set theory*, Problem Books in Mathematics, Springer, New York, 2006.

[22] Margarita Otero and Ya'acov Peterzil, *G-linear sets and torsion points in definably compact groups*, Arch. Math. Logic **48** (2009), 387–402.

[23] Ya'acov Peterzil, *Returning to semi-bounded sets*, J. Symbolic Logic **74** (2009), no. 2, 597–617.

[24] Ya'acov Peterzil and Anand Pillay, *Generic sets in definably compact groups*, Fund. Math. **193** (2007), no. 2, 153–170.

[25] Ya'acov Peterzil and Charles Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, Journal of London Math. Soc. **69** (1999), no. 2, 769–786.

[26] Ya'acov Peterzil and Sergei Starchenko, *A trichotomy theorem for o-minimal structures*, Proceedings of London Math. Soc. **77** (1998), no. 3, 481–523.

[27] ______, *Definable homomorphisms of abelian groups in o-minimal structures*, Ann. Pure Appl. Logic **101** (2000), no. 1, 1–27.

[28] A. Pillay, *Type-definability, compact Lie groups, and o-minimality*, J. Math. Logic **4** (2004), 147–162.

[29] A. Strzebonski, *Euler charateristic in semialgebraic and other o-minimal groups*, J. Pure Appl. Algebra **96** (1994), 173–201.

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