Quasianalytic Ilyashenko Algebras

Zeinab GALAL
Supervisor: Prof. P. Speissegger
McMaster University

Abstract

A recent result was the construction of a quasianalytic class containing all transition maps at hyperbolic singularities with logarithmic monomials in their series expansions. The end goal being obtaining o-minimality of this structure, we need an extension to several variables stable under certain operations (such as blow-up substitutions). As a first step towards the several variables extension, we construct a quasianalytic Hardy field extending the previous class where the monomials are now allowed to be any definable function in $\mathbb{R}_{\text{an,exp}}$.

1. Motivation

Hilbert's 16th problem (2nd part)

"*This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form $\frac{dy}{dx} = \frac{Y}{X}$ where X and Y are rational integral functions of the n-th degree in x and y.*" (Hilbert, 1900)

Question 1 Is it true that a polynomial vector field on the real plane has a finite number of limit cycles?

Question 2 Is it true that the number of limit cycles of a polynomial vector field of degree d on the real plane is bounded by a constant depending on d only?

Question 2 bis: Is there a uniform bound on the number of limit cycles in analytic families of real analytic planar vector fields?

Dulac's Problem

- **Dulac**:
  - If an analytic vector field $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ has infinitely many limit cycles, then they must converge to a **polycycle** Γ.
  - The **Poincaré first return map** can be written as a finite composition of correspondence maps and real analytic maps.
  - Each correspondence map σ_i (and hence the Poincaré map) has an asymptotic expansion of the form

$$\widehat{\sigma_i} = \sum_{j=0}^{+\infty} P_j(\log(x))x^{\nu_j}$$

where $P_j \in \mathbb{R}[x]$ with $P_0 \in \mathbb{R}^{>0}$ and $0 < \nu_0 < \nu_1 \nearrow +\infty$.

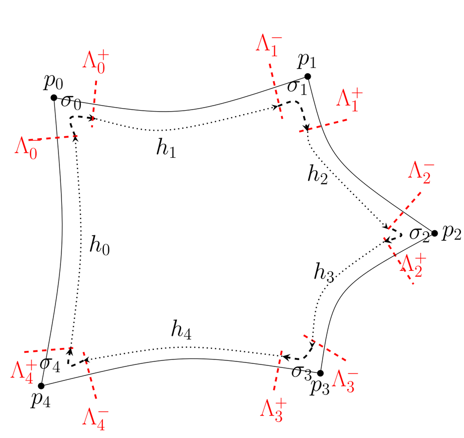


Figure: Example of a polycycle for ξ

Let ξ be a real analytic vector field in $\mathbb{R}^2$ and Γ a polycycle of ξ with singular points $p_0, p_1, \ldots, p_k$ and trajectories $\gamma_0, \ldots, \gamma_k$ connecting the p_i's in the order following the flow. For each i, it is possible to choose transverse segments Λ_i^-, Λ_i^+ sufficiently close to p_i such that:

1. there exists **real analytic maps** $h_i : \Lambda_i^+ \to \Lambda_{i+1}^-$ representing the flow of ξ from Λ_i^+ to Λ_{i+1}^-,
2. each trajectory starting on Λ_i^- sufficiently close to p_i crosses Λ_i^+ near p_i. We can therefore define **correspondence maps** $\sigma_i : \Lambda_i^- \to \Lambda_i^+$.

The Poincaré map $\boldsymbol{\sigma}$ is then represented by the finite composition
$\boldsymbol{\sigma = h_k \circ \sigma_k \circ \cdots \circ \sigma_1 \circ h_0 \circ \sigma_0}$.

- **Ilyashenko:**
  If ξ is analytic and all the singularities of Γ are hyperbolic then each $\sigma_i \circ e^{-x}$ is an ***almost regular map***.

2. Background

Almost Regular Maps and the Phragmén-Lindelöf Principle

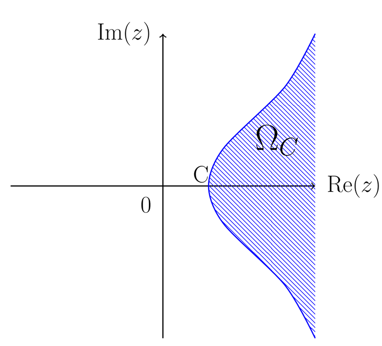
Definition. A map f is said to be ***almost regular*** if it has a holomorphic extension $\mathfrak{f}$ to some standard quadratic domain Ω and can be expanded in an asymptotic *Dulac exponential series* in this domain, i.e., if:

$$\forall N \in \mathbb{N}, \left|\mathfrak{f}(z) - \sum_{j=0}^{N} P_j(z)e^{-\nu_j z}\right| = o(e^{-\nu_N z}) \text{ as } |z| \to +\infty \text{ in } \Omega$$

where
- $P_j \in \mathbb{R}[x]$ with $P_0 \in \mathbb{R}^{>0}$
- $0 < \nu_0 < \nu_1 \nearrow +\infty$.

The class of all almost regular maps is denoted by $\mathcal{I}$.

➢ A subset of $\mathbb{C}$ is called a ***standard quadratic domain*** if it is of the form $\Omega_C := \{z + C\sqrt{1+z} \mid Re(z) > 0\}$ for some $C > 0$.



➢ A set $\Omega \subset \mathbb{C}^{>0}$ is said to be a ***standard domain*** if there exists $a > 0$ and a continuous function $f :]a, +\infty[\to]0, +\infty[$ such that:
$\Omega = \{z \in \mathbb{C} | \operatorname{Re}(z) > a \text{ and } |\operatorname{Im}(z)| < f(\operatorname{Re}(z))\}$.

Phragmén-Lindelöf Principle:
Let $\Omega \subset \mathbb{C}$ be a standard quadratic domain and $f : \overline{\Omega} \to \mathbb{C}$ be a holomorphic function. If f is bounded and for all $n \in \mathbb{N}$ and $z \in \Omega$, $f(z) = o(e^{-nz})$ as $|z| \to +\infty$ then $f \equiv 0$.

Quasianalytic Asymptotic Algebras

Definition. A tuple $(K, \mathcal{M}, T)$ is called a ***quasianalytic asymptotic (qaa) algebra*** if:
1. K is an $\mathbb{R}$-algebra of $\mathcal{C}^\infty$ germs at $+\infty$.
2. $\mathcal{M}$ is a multiplicative subgroup of some Hardy field $\mathcal{H}$ of $\mathcal{C}^\infty$ germs at $+\infty$ and the valuation map is injective on $\mathcal{M}$.
3. T is an injective $\mathbb{R}$-algebra homomorphism from K to $\mathbb{R}((\mathcal{M}))$ such that:
   - $T(K)$ is truncation closed, i.e., for every $f \in K$ and $m \in \mathcal{M}$, there exists $h \in K$ such that $T(h) = T(f)_m$,
   - $\forall f \in K, \forall m \in \mathcal{M}, |f(x) - T^{-1}((T(f)_m))(x)| = o(m(x))$ as $x \to +\infty$.

Objective

Objective: Given an analytic family ξ of real analytic planar vector fields with hyperbolic singularities, we want to construct a multivariable quasianalytic algebra (stable under the operations needed for o-minimality) containing $\mathcal{I}$ and prove that the obtained stucture is o-minimal.

T. Kaiser, J.-P. Rolin and P. Speissegger (2009): If all the singularities are in addition non-resonant, the P_j's are actually real numbers and the Poincaré map is definable in an o-minimal structure which leads to uniform bounds on the number of limit cycles (see [2]).

Steps of the proof:

Step 1. extend $\mathcal{I}$ into a quasianalytic asymptotic algebra

Step 2. obtain a multivariate version of $\mathcal{I}$ (stable under the operations needed for o-minimality)

Step 3. prove o-minimality

3. Approach

Step 1: Ilyashenko Algebras with log Monomials

P. Speissegger (2016): Step 1 using logarithmic generalized power series (see [3]).
Set of monomials: $\mathcal{L} := \{(\log_{-1}^{r_{-1}}) \ldots (\log_k^{r_k}) \mid r_{-1}, \ldots, r_k \in \mathbb{R}\}$

Theorem.
- There exists a qaa field $(\mathcal{F}, \mathcal{L}, T)$ that contains the class $\mathcal{I}$.
- The field $\mathcal{F}$ is closed under differentiation and log-composition.

➢ Problem: multivariable extension, for example the blow-up substitution $x \mapsto xy$ will map $\log_2(x)$ to $\log_2(xy) = \log(\log(x) + \log(y))$.

➢ Question: what is a good multivariable version of $\mathbb{R}((\mathcal{L}))$?

Candidate for the monomials: $\mathcal{H}_{an,exp}$

Approach: allow any element of $\mathcal{H}_{an,exp}$ the Hardy field of all germs at $+\infty$ of one-variable functions definable in $\mathbb{R}_{an,\text{exp}}$ as a monomial.

Problem. Not all elements of $\mathcal{H}_{an,\text{exp}}$ have the extensions to standard quadratic domains that we want!

Geometrically Pure Functions

(Work in progress by T. Kaiser and P. Speissegger)
Let f be an element of $\mathcal{H}_{an,\text{exp}}$.
- f has ***level*** $s \in \mathbb{Z}$ if there exists $k \in \mathbb{N}$ such that $\log_k(f) \sim \log_{k-s}$. Then, s is unique and we write $\operatorname{level}(f) = s$.
- We can associate to every element f in $\mathcal{H}_{an,\text{exp}}$ a complexity corresponding to the number of times *exp* is used in the construction of the term in $\mathcal{L}_{an,\text{exp,log}}$. We call this complexity the ***exponential height*** of f and denote it by $\operatorname{eh}(f)$.

Examples.
- For all $k \in \mathbb{N}$, $\operatorname{eh}(\log_k) = \operatorname{level}(\log_k) = -k$ and $\operatorname{eh}(exp_k(x)) = \operatorname{level}(exp_k(x)) = k$, in particular $\operatorname{eh}(x) = \operatorname{level}(x) = 0$.
- $\operatorname{eh}(x + exp(-x)) = 1$ but $\operatorname{level}(x + exp(-x)) = 0$.
- For all $f \in \mathcal{H}_{an,\text{exp}}$, $\operatorname{level}(f) \leq \operatorname{eh}(f)$.

Definition. We say that $f \in \mathcal{H}_{an,\text{exp}}$ is ***geometrically pure*** if $\operatorname{level}(f) = \operatorname{eh}(f)$.

Facts
- For all $f \in \mathcal{H}_{an,\text{exp}}$, there exist geometrically pure $g_1, \ldots, g_k$ such that $f = g_1 + \cdots + g_k$.
- For all $x > f$ with $\operatorname{eh}(f) \leq 0$, f has a holomorphic extension $\mathfrak{f}$ on the right half-plane that maps standard quadratic domains into standard domains.
- For all $x > f > g$ geometrically pure, $\operatorname{eh}(f \circ g^{-1}) \leq 0$.

4. Construction

4.1 Construction (Base Case)

Setup. $x = f_0 > f_1 > \cdots > f_k \in \mathcal{H}_{an,\text{exp}}$ where
- **P1.** The f_i's are infinite and in distinct archimedean classes
- **P2.** For all $0 \leq i < j \leq k$, $\operatorname{eh}(f_j \circ f_i^{-1}) \leq 0$.

Define $M(f_0) := \{e^{-rx} | r \in \mathbb{R}\}$.

Let $\mathcal{A}_{f^{<0>}}$ be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:
1. f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain.
2. there exists a series $F = \sum_{m \in M(f_0)} a_n n$ with natural support included in $M(f_0)$ and $a_n \in \mathbb{R}$ such that: $\forall m \in M(f_0), \left|\mathfrak{f} - \sum_{n \geq m} a_n \mathfrak{n}\right| = o(|\mathfrak{m}|)$ as $|z| \to +\infty$ in Ω.

A subset $S \subset \mathcal{H}_{an,\text{exp}}$ is ***natural*** if for all $m \in \mathcal{H}_{an,\text{exp}}$, $S \cap [m, +\infty[$ is finite (natural is stronger than anti well-ordered).

➢ Let $T_{f^{<0>}} : \mathcal{A}_{f^{<0>}} \to \mathbb{R}((M(f_0)))$ be the map defined by $T_{f^{<0>}}(f) := \Sigma\, a_n n$.

➢ Let $\mathcal{F}_{f^{<0>}}$ be the fraction field of $\mathcal{A}_{f^{<0>}}$.

$(\mathcal{F}_{f^{<0>}}, M(f_0), T_{f^{<0>}})$ is a qaa field.

Remark. The set $\mathcal{A}_{f^{<0>}} \circ (-\log)$ contains all analytic functions near 0^+ and all correspondence maps near non-resonant hyperbolic singularities of planar real analytic vector fields.

4.2 Construction (Inductive Case)

$$\mathcal{A}_{f^{<k>}} : x = f_0 > f_1 > \cdots > f_k$$
$$\circ(f_1 \circ f_0^{-1})^{-1} \Big\downarrow \equiv \circ f_1^{-1}$$
$$\mathcal{A}_{f^{<k-1>}} : x = f_1 \circ f_1^{-1} > \cdots > f_k \circ f_1^{-1}$$
$$\circ(f_2 \circ f_1^{-1})^{-1} \Big\downarrow \equiv \circ f_1 \circ f_2^{-1}$$
$$\mathcal{A}_{f^{<k-2>}} : x = f_2 \circ f_2^{-1} > f_3 \circ f_2^{-1} > \cdots > f_k \circ f_2^{-1}$$
$$\vdots$$
$$\mathcal{A}_{f^{<1>}} : x = f_{k-1} \circ f_{k-1}^{-1} > f_k \circ f_{k-1}^{-1}$$
$$\circ(f_k \circ f_{k-1}^{-1})^{-1} \Big\downarrow \equiv \circ f_{k-1} \circ f_k^{-1}$$
$$\mathcal{A}_{f^{<0>}} : x = f_k \circ f_k^{-1}$$

Figure: Shifts for the induction

Let $\mathcal{A}_{f^{<k>}}$ be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:
1. f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain
2. there exists a series $F = \sum_{m \in M(f_0)} (a_n \circ f_1)n$ with natural support included in $M(f_0)^{\leq 1}$ and $a_n \in \mathcal{F}_{f^{<k-1>}}$ such that:

$$\forall m \in M(f_0), \left|\mathfrak{f} - \sum_{n \geq m} (\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n}\right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

➢ Define $T_{f^{<k>}}(f) := \Sigma\left(\left(T_{f^{<k-1>}}(a_n)\right) \circ f_1\right) n$.

➢ Let $\mathcal{F}_{f^{<k>}}$ be the fraction field of $\mathcal{A}_{f^{<k>}}$ and $M(f_0, \ldots, f_k)$ be the group $\{e^{-\alpha_0 f_0} \ldots e^{-\alpha_k f_k} \mid \alpha_i \in \mathbb{R}\}$.

$(\mathcal{F}_{f^{<k>}}, M(f_0, \ldots, f_k), T_{f^{<k>}})$ is a qaa field.

➢ The set of all fields $\mathcal{F}_{f^{<k>}}$ is a directed set (with respect to inclusion on the set of all finite tuples $(f_0, \ldots, f_k)$).

Taking the direct limit, we obtain a field $\mathcal{F}$ and a common extension T such that $(\mathcal{F}, \mathcal{M}, T)$ is a qaa field where $\mathcal{M} := \{\prod_{i=0}^{k} e^{-\alpha_i f_i} \mid f_0 > \cdots > f_k \text{ verify } P1, P2 \text{ and } \alpha_i \in \mathbb{R}\}$.

Remark. In particular, $(\mathcal{F}, \mathcal{M}, T)$ is an extension of the qaa field of logarithmic generalized power series where the construction is done with $x = f_0 > \log > \log_2 > \ldots$.

4.3 General Construction

In general, the f_i's are not in distinct archimedean classes so we generalize the notions of truncation, qaa algebra and asymptotic expansion.

New setup. Let $f_0, f_1, \ldots, f_k$ be elements of $\mathcal{H}_{an,\text{exp}}$ verifying:
- **P1.** $x = f_0 > f_1 > \cdots > f_k$ infinite
- **P2.** the f_i's are in $l \leq k+1$ distinct archimedean classes
- **P3.** for all $0 \leq i < j \leq k$, $\operatorname{eh}(f_j \circ f_i^{-1}) \leq 0$

Result

The direct limit of the integral domains obtained with the generalized construction, $(\mathcal{F}, \mathcal{M}, \mathcal{T})$, is a qaa Hardy field containing $\mathcal{I}$, Ilyashenko's class of almost regular maps.

References

[1] Yu. S. Ilyashenko. *Finiteness theorems for limit cycles*, volume 94 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by H. H. McFaden.

[2] T. Kaiser, J.-P. Rolin, and P. Speissegger. Transition maps at non-resonant hyperbolic singularities are o-minimal. *J. Reine Angew. Math.*, 636:1-45, 2009.

[3] P. Speissegger. Quasianalytic Ilyashenko algebras. *ArXiv e-prints*, June 2016.