Function field Mordell-Lang in positive characteristic
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- The topic concerns one of main historical applications of logic to other parts of mathematics.
- Around 20 years ago, Hrushovski gave a proof of the theorem in the title of this talk, using essentially all the machinery of geometric model theory that been developed up to that point.
- The proof was correct, but among the ingredients was a "type-definable" version of the "Zariski geometries" theorem of Hrushovski-Zilber, for which the existing proofs/expositions were not optimal, and in fact rather impenetrable for non model theorists as well as many model theorists.

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- Recently we managed to succeed in the positive characteristic too (which was really Hrushovski's new contribution), carrying out a certain strategy of reducing Mordell-Lang to Manin-Mumford which I had sketched around 2010.

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- Recently we managed to succeed in the positive characteristic too (which was really Hrushovski's new contribution), carrying out a certain strategy of reducing Mordell-Lang to Manin-Mumford which I had sketched around 2010.
- This is what I want to talk about. Among the interesting things for me is that the new approach is rather like a classical "nonstandard analysis" proof: going up to a nonstandard model, performing a model-theoretic analysis, then pulling the information down to the standard model.

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- Subsets of $\mathbb{C}$, $\mathbb{C} \times \mathbb{C}$,... defined by systems of polynomial equations are called complex algebraic varieties, and come with a "dimension". Algebraic curves are by definition one-dimensional algebraic varieties, for example subsets of $\mathbb{C} \times \mathbb{C}$ defined by a single equation $P(x, y) = 0$.

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- So one of the precise modern statements of the **Mordell conjecture** is that if X is an irreducible, smooth, projective curve of genus ≥ 2, defined over $\mathbb{Q}$, then $X(\mathbb{Q})$ is finite.

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- If C is defined over $\mathbb{Q}$, then so is $J(C)$, and a standard theorem, Mordell-Weil, says that $J(C)(\mathbb{Q})$ is a finitely generated group.
- Hence a generalization of the Mordell conjecture is the so-called **Mordell-Lang conjecture**, also proved later by Faltings:
- Let A be an abelian variety, X an algebraic subvariety of A, and Γ a finitely generated subgroup of A. Assume $X \cap \Gamma$ is "large" , more precisely Zariski dense, in X. Then X is an algebraic subgroup of A, up to translation.
- EXERCISE: show that this statement implies the Mordell conjecture.

Variants: positive characteristic and function fields I

- It is natural to ask what happens to the Mordell-Lang conjecture in positive characteristic, where the complex field $\mathbb{C}$ is replaced by an algebraically closed field of characteristic $p > 0$ such as $\mathbb{F}_p^{alg}$ or $\mathbb{F}_p(t)^{alg}$.

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- It is *false* when the data is defined over a finite field, because of the *Frobenius* map Fr. Namely, if C is a curve of genus ≥ 2 defined over $\mathbb{F}_p$ and p is a transcendental point on C, then the subgroup Γ of $J(C)$ generated by the infinite set $\{Fr^n(p) : n = 1, 2., ..\}$ is finitely generated, its intersection with C contains this infinite set, so is Zariski-dense in C, but C is not a translate of a subgroup of $J(C)$.

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- So a function field version of ML, in positive characteristic, will at least have to include the hypothesis that the data is *not* defined over a finite field.

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- Let $k = \mathbb{F}_p^{alg}$ and $K = k(t)^{sep}$. Let A be an abelian variety defined over K, and assume A has no abelian subvariety isomorphic to an abelian variety defined over k. Let X be an algebraic subvariety of A defined over K, and Γ a finitely generated subgroup of $A(K)$. Assume $\Gamma \cap X$ is (Zariski) dense in X. Then X is a subgroup of A, up to translation.

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- There is another related statement, **Function field Manin-Mumford in characteristic** $p > 0$: The statement is exactly as in ML above, except the hypothesis on Γ is *replaced* by: Γ is the set of torsion points, i.e. points of finite order, in $A(K)$.

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- This function field Manin-Mumford statement *does* have a transparent but nontrivial algebraic -geometric proof, given by Pink and Roessler around 2002.

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- This reduction or implication, has several additional ingredients:
  – a certain positive characteristic "Theorem of the kernel", proved on request by Damian Roessler,
  – a theorem on commutative groups of finite Morley rank without proper infinite definable subgroups (Frank Wagner), and,
  – a "quantifier elimination" theorem for a certain associated type-definable, in a suitable structure, group $A^\sharp$.

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  – a "quantifier elimination" theorem for a certain associated type-definable, in a suitable structure, group $A^\sharp$.
- The rest of the talk will be a sketch of the proof of the MM implies ML theorem, explaining first these ingredients.

Model theory I

- The relevant first order theory is $Th(\mathbb{F}_p(t)^{sep}, +, \cdot)$, also known as the theory $SCF_{p,1}$ of separably closed fields of characteristic p and Ersov invariant 1. $K = \mathbb{F}_p(t)^{sep}$ is the "standard" model, and we let $\mathcal{U}$ be a saturated elementary extension of K.

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- If A is the relevant abelian variety over K, then $A(K)$ is definable in K and $A(\mathcal{U})$ definable in $\mathcal{U}$.
- $SCF_{p,1}$ is stable, but not superstable. Nonsuperstability is witnessed by the the $p^n th$ powers of K, or $\mathcal{U}$, forming an infinite descending chain of definable subfields, the intersection of which is $\mathbb{F}_p^{alg}$, which was called k earlier.

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- Likewise, with additive notation, the $p^n th$ multiples of $A(K)$, or $A(\mathcal{U})$, form an infinite descending chain of definable subgroups.

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- To "see" all of the points of a type-definable set we in general need to be situated in a ω_1-saturated model. So working in $\mathcal{U}$, we let $A^\sharp$ denote $\cap_n(p^n A(\mathcal{U}))$, a type-definable subgroup of $A(\mathcal{U})$. $A^\sharp$ is the key model-theoretic object to be understood.

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- $A^\sharp$ is the maximal divisible subgroup of $A(\mathcal{U})$ and also $A^\sharp(K)$ identifies with $\cap_n(p^n(A(K))$, the maximal divisible subgroup of $A(K)$.
- Can $A^\sharp$ with its "induced structure" from $\mathcal{U}$ be meaningfully viewed as a first order structure in its own right?
- The answer turns out to be yes and this is the "quantifier elimination" theorem mentioned earlier.

Model theory III

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- We call the relevant language L_1 and for simplicity call the relevant structure $A^\sharp$. We summarise (to be discussed) the afore-mentioned ingredients by:

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Lemma 0.1

(i) $Th(A^\sharp)$ has quantifier elimination in the language L_1, and in fact has finite Morley rank,
(ii) The L_1-substructure $A^\sharp(K)$ of $A^\sharp$ is an elementary substructure (application of Wagner's theorem),
(iii) $A^\sharp(K)$ consists of torsion points of A. (Theorem of the kernel.)

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- Hence we find a strictly descending chain $\Gamma = C_0 \geq C_1 \geq C_2 .....$, where each C_i is a coset of $p^i\Gamma$ in Γ and with $X \cap C_i$ dense in X.

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- Let D_i be the unique coset of $p^iA(K)$ in $A(K)$ which contains C_i. Then we have $A(K) = D_0 \geq D_1 \geq D_2...$ and $X \cap D_i$ is dense in X for each i.

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- Now $\cap D_i$ may be empty but passing to or working in the "nonstandard model" $\mathcal{U}$, $\cap D_i = D$ is nonempty, is a translate of $A^\sharp$ and moreover $X \cap D$ is dense in X.

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- As $X \cap D$ is dense in X, also $X \cap D_0$ is Zariski-dense in X. So after replacing X by a translate $X \cap A^\sharp(K)$ is dense in X.
- By Lemma 0.1 (iii), $A^\sharp(K)$ is contained in the torsion points of K, so $X \cap Tor(A(K))$ is dense in X, and we conclude by Manin-Mumford that X is (up to translation) a subgroup of A.

Additional work

- Function field ML for semiabelian varieties.
- Reduction to the abelian variety case using model theory of finite rank groups.
- The model-theoretic socle and algebraic-geometric analogues.