

EXPANSIONS OF REAL CLOSED FIELDS WHICH INTRODUCE NO NEW SMOOTH FUNCTIONS

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ABSTRACT. Let $\tilde{\mathcal{R}}$ be an expansion of the real field $\overline{\mathbb{R}}$, such that every definable set (I) is a uniform countable union of semialgebraic sets, and (II) contains a “semialgebraic chunk”. Then every definable smooth function $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with open semialgebraic domain is semialgebraic.

Conditions (I) and (II) hold for various d-minimal expansions $\tilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P \rangle$ of the real field, such as when $P = 2^{\mathbb{Z}}$, or $P \subseteq \mathbb{R}$ is an iteration sequence. A generalization of the theorem to d-minimal expansions $\tilde{\mathcal{R}}$ of \mathbb{R}_{an} fails. Finally, the conclusion of our theorem also holds for certain expansions with d-minimal open core, such as $\langle \overline{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$.

1. INTRODUCTION

In this paper we provide conditions under which an expansion $\tilde{\mathcal{R}}$ of the real field $\overline{\mathbb{R}}$ by a discrete set introduces no new smooth definable functions. This property can be viewed as a tameness condition for $\tilde{\mathcal{R}}$, much alike the property of introducing no new open definable sets for expansions of $\overline{\mathbb{R}}$ by a dense-codense set ([1]). Note that neither of these properties is captured by the usual notions of tameness, since both $(\overline{\mathbb{R}}, \exp)$ and $(\overline{\mathbb{R}}, \mathbb{Z})$ define new smooth functions and open sets, yet they are at the two extremes of the model-theoretic complexity hierarchy, the first one being o-minimal, the second defining the whole projective hierarchy.

The targeted example of the current work is the expansion $\tilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P \rangle$ of the real field by $P = 2^{\mathbb{Z}}$, first studied by L. van den Dries [4]. It was proven there that every definable set is given by an existential formula, where the quantifier ranges over P and the rest of the formula is semialgebraic. As a consequence, every definable subset of \mathbb{R} is the union of an open set and finitely many discrete sets. Subsequently, C. Miller [12] extended this study to the context of *d-minimal* structures, which are defined exactly as the expansions of $\langle \overline{\mathbb{R}}, < \rangle$ that satisfy the last property, uniformly in parameters. By now, many structures are known to be d-minimal, such as expansions of the real field by an iteration sequence [13], a fast sequence [7] or a spiral [12].

In this paper, we take a different approach and set as our starting point a decomposition property for definable sets, which we introduce next. Although our main interest is in expansions of the real field, we work here in the general setting of expansions $\tilde{\mathcal{R}}$ of a real closed field \mathcal{R} .

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From the rest of this paper, we fix an expansion $\tilde{\mathcal{R}} = \langle \mathcal{R}, \dots \rangle$ of a real closed field $\mathcal{R} = \langle R, <, +, \cdot \rangle$. By ‘definable’ (respectively, ‘semialgebraic’), we mean definable in $\tilde{\mathcal{R}}$ (respectively, in \mathcal{R}), with parameters.

The real closed field \mathcal{R} is an o-minimal structure. We assume familiarity with the basics of o-minimality, in particular with the o-minimal topology and the notions of a cell and cell decomposition. We call a semialgebraic set *connected* if it is not the union of two proper relatively open semialgebraic subsets. Equivalently, any two points in it can be connected via a semialgebraic path.

For any set $X \subseteq R^n$, we define its *dimension* as the maximum k such that some projection of X to k coordinates has non-empty interior.

Definition 1.1. Let $Y \subseteq X \subseteq R^n$ be two sets. We say that Y is a *semialgebraic chunk* of X if it is a semialgebraic cell, $\dim Y = \dim X$, and for every $y \in Y$, there is an open box $B \subseteq R^n$ containing y such that $B \cap X \subseteq Y$.

For example, if $X = 2^{\mathbb{Z}} \cup \{0\}$, then every singleton contained in X is a semialgebraic chunk, except for $\{0\}$.

Definition 1.2. We say that $\tilde{\mathcal{R}}$ has the *decomposition property* (DP) if for every definable set $X \subseteq R^n$,

- (I) there is a semialgebraic family $\{Y_t\}_{t \in R^m}$ of subsets of R^n , and a definable set $S \subseteq R^m$ with $\dim S = 0$, such that $X = \bigcup_{t \in S} Y_t$,
- (II) X contains a semialgebraic chunk.

Definition 1.3. We say that $\tilde{\mathcal{R}}$ has the *dimension property* (DIM) if for every definable family $\{X_t\}_{t \in S}$ of semialgebraic sets, with $\dim S = 0$, we have

$$\dim \bigcup_{t \in S} X_t = \max_{t \in S} \dim X_t.$$

Our main theorem is the following (Theorem 3.12).

Theorem 1.4. Let $\tilde{\mathcal{R}}$ be an expansion of a real closed field \mathcal{R} , satisfying (DP) and (DIM). Let $f : X \subseteq R^n \rightarrow R$ be a definable smooth function, with an open semialgebraic domain X . Then f is semialgebraic.

Remark 1.5.

- (1) If $R = \mathbb{R}$ and $\tilde{\mathcal{R}}$ is d-minimal, then (DIM) becomes redundant. Indeed, in this case, a definable set is 0-dimensional if and only if it is discrete if and only if it is countable. Hence (DIM) holds by the Baire Category Theorem, see Fact 4.1 below.
- (2) If $R = \mathbb{R}$ and (DP)(I) holds with S countable instead of $\dim S = 0$, then every definable set X is countable if and only if it is 0-dimensional. Indeed, if $\dim X = 0$, then by the adapted (DP)(I), it is a countable union of finite sets, and hence countable. If $\dim X > 0$, then by (DP)(II), it must contain an infinite semialgebraic set, and hence it is uncountable. In particular, (DIM) in this case again becomes redundant, by Fact 4.1, and we also obtain the “countable version” of the theorem stated in the abstract (where by a “uniform countable decomposition”, we mean a decomposition as in (DP)(I) with S countable.)

- (3) If we replace “smooth” by “analytic” in Theorem 1.4, then the conclusion becomes clear, since an analytic function which is semialgebraic on an open subset of its domain is semialgebraic. On the other hand, if $\tilde{\mathcal{R}}$ is d-minimal and not o-minimal, then it is possible to define new \mathcal{C}^n functions, for every $n \in \mathbb{N}$, see Remark 4.8(3) below.

In Section 4, we apply our theorem to various examples.

Proposition 1.6. *Let $\tilde{\mathcal{R}}$ be any of the following structures:*

- (1) $\langle \mathbb{R}, \alpha^{\mathbb{Z}} \rangle$, where $1 < \alpha \in \mathbb{R}$,
- (2) $\langle \mathbb{R}, P \rangle$, where P is an iteration sequence,

Then $\tilde{\mathcal{R}}$ satisfies (DP). Hence (by Remark 1.5(1)), the conclusion of Theorem 1.4 holds.

We suspect that (DP) holds in more d-minimal expansions of the real field, such as expansions by fast sequences (Question 4.12). Here we apply our theorem to structures that go even beyond the d-minimal setting, and where (DP)(II) actually fails (Corollary 4.6).

Corollary 1.7. *Let $\tilde{\mathcal{R}} = \langle \mathbb{R}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, where \mathbb{R}_{alg} is the field of real algebraic numbers. Then every definable smooth function with open semialgebraic domain is semialgebraic.*

Finally, by Example 4.7, due to P. Hieronymi, if we replace \mathcal{R} by an o-minimal expansion \mathcal{R}' of a real closed field (such as \mathbb{R}_{an}), and ‘semialgebraic’ by ‘definable in \mathcal{R}' ’, then Theorem 1.4 fails.

Strategy of the proof. In the rest of this introduction, we describe the strategy of our proof of Theorem 1.4, and provide a basic example. The proof uses a key fact from real algebraic geometry ([2, Proposition 8.4.1] or Fact 3.7 below), that the Zariski closure of a connected Nash-submanifold is irreducible. This allows us to establish that under certain conditions, the Zariski closures of certain graphs of two functions coincide (Corollary 3.8).

Example 1.8. Let $\tilde{\mathcal{R}} = \langle \mathbb{R}, 2^{\mathbb{Z}} \rangle$ and $f : \mathbb{R}^{>0} \rightarrow \mathbb{R}$ be a definable smooth function which is semialgebraic on each $(2^n, 2^{n+1})$, $n \in \mathbb{Z}$. Denote $f_n = f|_{(2^n, 2^{n+1})}$. By induction on $|n|$, and Lemma 3.10, it suffices to prove that the graphs $\Gamma(f_n)$ and $\Gamma(f_{n+1})$ are contained in the same irreducible algebraic set of dimension 1 (namely, that their Zariski closures $\overline{\Gamma(f_n)}^{zar}$ and $\overline{\Gamma(f_{n+1})}^{zar}$ coincide). If f_n and f_{n+1} are given by polynomials, then it is not hard to show that the smoothness condition on f forces (the degrees and coefficients of) those polynomials to be the same. In general, the graphs $\Gamma(f_n)$ and $\Gamma(f_{n+1})$ are contained in some 1-dimensional algebraic sets, and the smoothness condition on f forces the “concatenation” of their graphs to be a connected Nash-submanifold (Definition 3.5). By Fact 3.7, its Zariski closure is irreducible. This implies $\overline{\Gamma(f_n)}^{zar} = \overline{\Gamma(f_{n+1})}^{zar}$ (Corollary 3.8), as needed.

For arbitrary definable functions in $\langle \mathbb{R}, 2^{\mathbb{Z}} \rangle$, the above inductive proof cannot work because, even though by (DP)(I) the graph of f is a countable union of graphs of semialgebraic maps $f_t : Z_t \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in S$, it need not be true that every f_t can be “concatenated” with some other $f_{t'}$ and yield a connected Nash-submanifold. For example, suppose that the above f is extended to the whole \mathbb{R} ,

via some semialgebraic map $g : (-\infty, 0) \rightarrow \mathbb{R}$. Then there is no $n \in \mathbb{Z}$, such that the set $[-\infty, 0] \cup [2^n, 2^{n+1}]$ is connected.

We are thus led to prove Theorem 1.4 by contradiction, as follows. Using (DP)(I) and Lemma 3.11, we can find a 0-dimensional definable family of algebraic sets $\{Y_t : t \in S\}$ in \mathbb{R}^n , such that for every $t \in S$, the graph of f_t is contained in Y_t . Using a basic dimension-theoretic argument, it suffices to prove that for any $t, t' \in S$ with $\dim Y_t = \dim Y_{t'} = n$, we have $\overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f_{t'})}^{zar}$. Assume not. We apply Proposition 3.1, which is proved using (DP)(II), to find distinct f_t and $f_{t'}$, which are Nash-concatenable. Thus, again by Corollary 3.8, we obtain $\overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f_{t'})}^{zar}$, a contradiction.

Structure of the paper. In Section 2, we fix notation and collect some basic facts. In Section 3, we prove Theorem 1.4, after some preparatory work and an interlude of real algebraic geometry. In Section 4, we apply our theorem to certain examples and state some open questions.

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2. PRELIMINARIES

We assume familiarity with the basics of o-minimality, as they can be found for example in [3]. By a k -cell, we mean a semialgebraic cell of dimension k . If $S \subseteq R^n$ is a set, its closure, interior and boundary are denoted by \overline{S} , $\text{int}(S)$, $\text{bd}(S) := \overline{S} \setminus \text{int}(S)$, respectively. Sole exception to this notation is that of the real field \mathbb{R} . We call a family $\{X_t\}_{t \in S}$ of sets $X_t \subseteq R^n$ semialgebraic (respectively, definable) if the set $\bigcup_{t \in S} \{t\} \times X_t$ is semialgebraic (respectively, definable). By an open box $B \subseteq R^n$, we mean a set of the form

$$B = (-a_1, b_1) \times \cdots \times (a_n, b_n),$$

for some $a_i, b_i \in R \cup \{\pm\infty\}$. By an open set we always mean a non-empty open set. Unless stated otherwise, $\pi : R^n \rightarrow R^{n-1}$ denotes the coordinate projection onto the first $n-1$ coordinates. By a ‘smooth’ function on an open domain we mean a function of class C^∞ ; namely, it is n -differentiable for every $n \in \mathbb{N}$.

We prove three basic lemmas that will be used in the sequel.

Lemma 2.1. *Let $X \subseteq R^n$ be an open semialgebraic set and $Y \subseteq \overline{X}$ a definable set, such that $\dim(X \setminus Y) < n$. Then $\overline{Y} = \overline{X}$.*

Proof. Let $x \in \overline{X}$. Let $B \subseteq R^n$ be any open box containing x . Since X is open, there is an open box $B' \subseteq B \cap X$. Since $\dim(X \setminus Y) < n$, $B' \cap Y \neq \emptyset$, and hence $B \cap Y \neq \emptyset$. Thus $x \in \overline{Y}$, as needed. \square

Lemma 2.2. *Assume (DP)(II). Let $Y \subseteq R^n$ be an open definable set. Then $\dim \text{bd}(Y) < n$.*

Proof. Let us assume for contradiction that $\dim \text{bd}(Y) = n$. By (DP)(II), $\text{bd}(Y)$ contains an open box B . Since $B \subseteq \text{bd}(Y)$, we have $B \cap Y = \emptyset$. But then $X \cap \overline{Y} = \emptyset$, a contradiction. \square

Lemma 2.3. *Let $A_1, A_2 \subseteq R^n$ be two open, connected and disjoint semialgebraic sets. Suppose there is an open box $B \subseteq \text{int}(\overline{A_1} \cup \overline{A_2})$, such that $B \cap A_i \neq \emptyset$, for $i = 1, 2$. Then $\text{int}(\overline{A_1} \cup \overline{A_2})$ is connected.*

Proof. Denote $X = \text{int}(\overline{A_1} \cup \overline{A_2})$, and take any $x, y \in X$. We need to find a semialgebraic path in X connecting x and y . Take an open box $B_1 \subseteq X$ containing x . Clearly, B_1 must intersect A_i , for some $i = 1, 2$, and since B_1 is connected, we may assume that $x \in A_i$. Similarly, we may assume $y \in A_i$, for some $i = 1, 2$. By assumption, there is an open box $B \subseteq X$ with each $B \cap A_i \neq \emptyset$. Since all of B , A_1 and A_2 are connected and contained in X , the result easily follows. \square

3. PROOF OF THEOREM 1.4

The proof takes place in Subsection 3.3 below. As mentioned in the introduction, a crucial step of our strategy is to show that certain functions $f_i : X_i \rightarrow R$ are “Nash-concatenable”. This notion is introduced in Subsection 3.2. At a first step, one needs to establish that $\text{int}(\overline{X_1} \cup \overline{X_2})$ is connected. Towards that goal, we prove Proposition 3.1 below.

3.1. Towards concatenation. A simple instance of Proposition 3.1 is the following. Suppose that $\{W_t : t \in S\}$ is a definable family of disjoint open intervals in R , such that $R = \bigcup_{t \in S} W_t$. Then there are $t \neq t' \in S$ such that $\text{int}(\overline{W_t} \cup \overline{W_{t'}})$ is connected.

Proposition 3.1. *Assume (DP)(II). Let $\{W_t : t \in S\}$ be a definable family of open sets in R^n , such that:*

- (i) *for every $t \in S$, $W_t = \text{int}(\overline{W_t})$,*
- (ii) *there is an open cell $X \subseteq R^n$, such that $\overline{X} = \overline{\bigcup_{t \in S} W_t}$,*
- (iii) *for every $t, t' \in S$, $W_t \neq W_{t'} \Rightarrow W_t \cap W_{t'} = \emptyset$,*
- (iv) *there are $t, t' \in S$, $W_t \neq W_{t'}$.*

Then there are an open box $B \subseteq X$ and $W_{t_1} \neq W_{t_2}$, such that

- (1) *$B \cap W_{t_i}$ is semialgebraic, open and connected, $i = 1, 2$,*
- (2) *$\text{int}(\overline{B \cap W_{t_1}} \cup \overline{B \cap W_{t_2}})$ is connected.*

Proof. By (iii) and since each W_t is open, we have $\bigcup_{t \in S} \text{bd}(W_t) \subseteq \text{bd}(\bigcup_{t \in S} W_t)$. By Lemma 2.2, $\dim \bigcup_{t \in S} \text{bd}(W_t) \leq n - 1$.

Claim. $\dim \bigcup_{t \in S} \text{bd}(W_t) \cap X = n - 1$.

Proof of Claim. It suffices to find $t \in S$, such that $\dim \text{bd}(W_t) \cap X = n - 1$. By induction on n . For $n = 1$, X is an open interval and the statement follows easily from (iv). Let $n > 0$. We split into two cases.

Case I: For all $t \in S$, $\pi^{-1}(\pi(W_t)) \cap X \subseteq W_t$. It is then not hard to see that for all $t \in S$ and $x \in \text{bd}(\pi(W_t))$,

$$\pi^{-1}(x) \cap X \subseteq \text{bd}(W_t) \cap X,$$

and the family $\{\pi(W_t) : t \in S\}$ satisfies Conditions (i)-(iv). By inductive hypothesis, there is $t \in S$, such that $\text{bd}(\pi(W_t)) \cap \pi(X)$ has dimension $n - 2$. Thus, since X is an open cell, the set

$$\pi^{-1}(\text{bd}(\pi(W_t)) \cap \pi(X)) \cap X \subseteq \text{bd}(W_t) \cap X$$

has dimension $n - 1$.

Case II: Otherwise, there are $t_1, t_2 \in S$ with $W_{t_1} \neq W_{t_2}$ and $x_i \in W_{t_i}$, such that $\pi(x_1) = \pi(x_2)$. Since each W_{t_i} is open, the set $A \subseteq \pi(W_{t_1}) \cap \pi(W_{t_2}) \subseteq R^{n-1}$ is open. It suffices to show that for all $x \in A$, there is $y \in \text{bd}(W_{t_1}) \cap X$, such that $\pi(y) = x$. Since X is an open cell, $\pi^{-1}(x) \cap X$ is connected. The sets W_{t_i} being disjoint by (iii), and each $\pi^{-1}(x) \cap W_{t_i} \neq \emptyset$ open, we obtain the result. \square

By the claim and (DP)(II), there is a semialgebraic set $Y \subseteq \bigcup_{t \in S} \text{bd}(W_t) \cap X$ of dimension $n - 1$, and an open box $B \subseteq R^n$, such that

$$\emptyset \neq B \cap \bigcup_{t \in S} \text{bd}(W_t) \cap X \subseteq Y.$$

Since X is open, we may assume $B \subseteq X$. Moreover, since Y is semialgebraic and has dimension $n - 1$, by taking B small enough, we may assume that $B \setminus Y$ has two semialgebraic connected components, B_1 and B_2 . By (ii) and since each $B_i \subseteq X$, we can find $t_1, t_2 \in S$, such that $B_i \cap W_{t_i} \neq \emptyset$, $i = 1, 2$.

We next show that $W_{t_1} \neq W_{t_2}$. First observe that since $B \cap \bigcup_{t \in S} \text{bd}(W_t) \subseteq Y$ and $B_i \cap Y = \emptyset$, we have $B_i \cap \bigcup_{t \in S} \text{bd}(W_t) = \emptyset$. In particular, $B_i \cap \text{bd}(W_{t_i}) = \emptyset$. Therefore, $B_i \subseteq W_{t_i}$, $i = 1, 2$. Since B is an open box and $\dim(B \setminus (B_1 \cup B_2)) < n$, we have

$$(*) \quad B \subseteq \text{int}(\overline{B_1} \cup \overline{B_2}) \subseteq \text{int}(\overline{W_{t_1}} \cup \overline{W_{t_2}}).$$

If $W_{t_1} = W_{t_2}$, then by (i), we obtain

$$B \subseteq \text{int}(\overline{B_1} \cup \overline{B_2}) = \text{int}(\overline{W_{t_1}} \cup \overline{W_{t_2}}) = \text{int}(\overline{W_{t_1}}) = W_{t_1},$$

contradicting $B \cap \bigcup_{t \in S} \text{bd}(W_t) \neq \emptyset$, (iii) & (iv).

Now we are ready to show (1). It suffices to prove that $B \cap W_{t_i} = B_i$, for $i = 1, 2$. As shown earlier, we have $B_i \subseteq W_{t_i} \cap B$. On the other hand, observe that

$$B \cap W_{t_1} = (B_1 \cap W_{t_1}) \cup (B_2 \cap W_{t_1}) \cup (B \cap Y \cap W_{t_1}).$$

Since $B_2 \subseteq W_{t_2}$ and $W_{t_1} \cap W_{t_2} = \emptyset$, the second part of the above union is empty. Moreover, since $Y \subseteq \bigcup_{t \in S} \text{bd}(W_t)$, we have $Y \cap W_{t_1} = \emptyset$. Therefore, the third part of the above union is also empty. Hence $B \cap W_{t_1} = B_1$. Similarly, one shows $B \cap W_{t_2} = B_2$, as needed.

Finally, (2) follows from (1), (*) and Lemma 2.3, for $A_i = B \cap W_{t_i}$. \square

3.2. Real algebraic geometry. In this section we recall some basic facts from real algebraic geometry. A standard reference is Bochnak-Coste-Roy [2]. The key fact is Fact 3.7 below which states that the Zariski closure of a connected Nash-submanifold is irreducible. This allows us to establish that under certain conditions, the Zariski closures of the graphs of two Nash-concatenable functions coincide (Corollary 3.8).

Definition 3.2. By an *algebraic set* $A \subseteq R^n$, we mean the zero set $P^{-1}(0)$ of a polynomial $P \in R[X]$. The *Zariski closure* of a set $V \subseteq R^n$ is the intersection of every algebraic set containing V , denoted by $\overline{V}^{\text{zar}}$. Note that $\overline{V}^{\text{zar}}$ is algebraic, because $R[X_1, \dots, X_n]$ is Noetherian.

Let V be an algebraic set. We say that V is *irreducible* if, whenever $V = V_1 \cup V_2$, with each V_i algebraic, we have $V = V_i$, for $i = 1$ or 2 . The *Krull-dimension* of V , denoted by $\dim_K(V)$, is the maximum n such that there is a sequence of irreducible algebraic sets $V_0 \subsetneq \dots \subsetneq V_n \subseteq V$.

Fact 3.3. *Let X be a semialgebraic set. Then*

$$\dim(X) = \dim_K(X) = \dim(\overline{X}^{zar}).$$

Proof. For the first equality: by [2, Proposition 2.8.4], if $A \subseteq R^n$ is an open semialgebraic set, then $\dim_K(A) = n$. By [2, Proposition 2.8.7], the graph of a semialgebraic function has Krull-dimension equal to the Krull-dimension of its domain. By [2, Proposition 2.8.5], the Krull-dimension of a finite union of semialgebraic sets equals the sum of their Krull-dimensions. Now, let X be a semialgebraic set. By cell decomposition, X is a finite union of graphs of semialgebraic functions with open domains, and $\dim X$ equals the maximum of their dimensions. Combining the above three propositions, the first equality follows.

The second equality is by [2, Proposition 2.8.2]. \square

Lemma 3.4. *Let $U \subseteq V \subseteq R^n$ be two algebraic sets of the same dimension. If V is irreducible, then $U = V$.*

Proof. By Fact 3.3, we have $\dim_K(U) = \dim_K(V)$, and hence the result follows from the definition of Krull-dimension. \square

Definition 3.5. ([2, Definitions 2.9.3, 2.9.9]) A *Nash function* $f : X \subseteq R^n \rightarrow R^m$ is a semialgebraic smooth function with open domain. A *Nash-diffeomorphism* $f : X \rightarrow Y$ is a Nash function which is a bijection and whose inverse is also Nash.

A semialgebraic set $V \subseteq R^m$ is a *Nash-submanifold of dimension d* if, for every $x \in V$, there is a Nash-diffeomorphism ϕ from an open semialgebraic neighborhood U of the origin in R^m onto an open semialgebraic neighborhood U' of x in R^m , such that $\phi(0) = x$ and $\phi((R^d \times \{0\}) \cap U) = V \cap U'$.

Note that the graph of a Nash function with connected domain is a connected Nash-submanifold.

We introduce the following notion.

Definition 3.6. Let $f : X \subseteq R^n \rightarrow R^m$, $g : Y \subseteq R^n \rightarrow R^m$ be two semialgebraic maps with open connected domains. We say that f and g are *Nash-concatenable*, if the set $A = \text{int}(\overline{X} \cup \overline{Y})$ is connected, and there is a (unique) Nash function $h : A \rightarrow R^m$ that extends both f and g . We denote $h = f \frown g$.

The uniqueness of the above map h follows from its continuity and the fact that $X \cup Y$ is dense in its domain A . Note also that since A is connected, $\Gamma(h)$ is a connected Nash-submanifold.

The key fact from [2] is the following.

Fact 3.7. *Let $V \subseteq R^m$ be a connected Nash-submanifold. Then \overline{V}^{zar} is irreducible.*

Proof. See [2, Proposition 8.4.1]. \square

Corollary 3.8. *Let f and g be two Nash-concatenable functions. Then*

$$\overline{\Gamma(f)}^{zar} = \overline{\Gamma(g)}^{zar}.$$

Proof. Since $\Gamma(f \frown g)$ is a connected Nash-submanifold, by Fact 3.7, $\overline{\Gamma(f \frown g)}^{zar}$ is irreducible. Note that $\dim \Gamma(f) = \dim \Gamma(f \frown g) = \dim \Gamma(g)$, and hence, by Lemma 3.3,

$$\dim \overline{\Gamma(f)}^{zar} = \dim \overline{\Gamma(f \frown g)}^{zar} = \dim \overline{\Gamma(g)}^{zar}.$$

By Lemma 3.4, $\overline{\Gamma(f)}^{zar} = \overline{\Gamma(f \frown g)}^{zar} = \overline{\Gamma(g)}^{zar}$. \square

Corollary 3.9. *Let $f : X \subseteq R^n \rightarrow R$ be a Nash function with connected domain, and let $U \subseteq X$ be open. Then*

$$\overline{\Gamma(f)}^{zar} = \overline{\Gamma(f|_U)}^{zar}.$$

Proof. By Corollary 3.8, for $g = f|_U$. \square

3.3. Proof of Theorem 1.4. We will need two preliminary lemmas. If \mathcal{C} is a collection of subsets in R^{n+1} , by $\pi(\mathcal{C})$ we denote the collection of their projections on R^n .

Lemma 3.10. *Let $f : X \subseteq R^n \rightarrow R$ be a continuous map, such that X is a k -cell and $\Gamma(f) \subseteq Y$, with Y a semialgebraic set of dimension k . Then f is semialgebraic.*

Proof. We may clearly assume that $\pi(Y) = X$. Let \mathcal{D} be a cell decomposition of R^{n+1} partitioning Y , and $\mathcal{C} \subseteq \mathcal{D}$ the collection of those cells that intersect Y . So $\pi(\mathcal{C})$ is a finite partition of X . It suffices to show that for every $S \in \pi(\mathcal{C})$, $f|_S$ is semialgebraic. Fix such S . By definition of cell decomposition, we have

$$Y \cap (S \times R) = Y_1 \cup \dots \cup Y_m,$$

for some disjoint k -cells $Y_i \subseteq R^{n+1}$, with $\pi(Y_i) = S$. Of course, since $\Gamma(f) \subseteq Y$, we have $\Gamma(f|_S) \subseteq Y \cap (S \times R)$. Since the Y_i 's are disjoint and $f|_S$ is continuous, $\Gamma(f|_S)$ must equal one of the Y_i 's. Hence it is semialgebraic. \square

Lemma 3.11. *Let $X \subseteq R^{m+l}$ be a semialgebraic set and A its projection onto the last l coordinates. Then there are finitely many polynomials $P_i \in R[X, T]$ and semialgebraic sets A_i , $i = 1, \dots, s$, such that $A = \bigcup_i A_i$, and for every i and $t \in A_i$,*

$$(*) \quad X_t \subseteq P_i(-, t)^{-1}(0) \quad \text{and} \quad \dim X_t = \dim P_i(-, t)^{-1}(0).$$

Proof. By partitioning A into finitely many sets, we may assume that there is n with each $\dim X_t = n$. We perform induction on $k = \dim A$. For $k = 0$, the result follows from Fact 3.3. Let $k > 0$. By Fact 3.3 again, there is $P_0 \in R[X, T]$ such that $X \subseteq P_0^{-1}(0)$ and both sets have dimension $n + k$. Let

$$A_0 = \{t \in A : \dim P_0(-, t)^{-1}(0) > n\}.$$

Then clearly, for every $t \in A \setminus A_0$, $(*)$ holds, with $i = 0$. Moreover, by o-minimality, $\dim A_0 < k$. Hence, by inductive hypothesis, the statement of the lemma also holds for $X \cap (R^m \times A_0)$, and hence it holds for X . \square

We are now ready to prove our main theorem.

Theorem 3.12. *Assume (DP) and (DIM). Let $f : X \subseteq R^n \rightarrow R$ be a definable smooth function, where X is an open semialgebraic set. Then f is semialgebraic.*

Proof. By cell decomposition, we may assume that X is an open cell. By (DP)(I), there is a semialgebraic family $\{Y_t\}_{t \in R^m}$ of sets $Y_t \subseteq R^{n+1}$, and a definable $S \subseteq R^m$ with $\dim S = 0$, such that $\Gamma(f) = \bigcup_{t \in S} Y_t$. By cell decomposition again, we may assume that each Y_t is a cell. For every $t \in S$, write $Z_t = \pi(Y_t)$ and $f_t = f|_{Z_t}$. So $Y_t = \Gamma(f_t)$. Observe that $\dim Z_t = n$ if and only if Z_t is an open cell. Let

$$A = \{t \in S : \dim Z_t = n\}.$$

Since $X = \bigcup_{t \in S} Z_t$, and for every $t \in S \setminus A$, we have $\dim Z_t < n$, by (DIM), we obtain

$$(1) \quad \dim \left(X \setminus \bigcup_{t \in A} Z_t \right) < n.$$

Since X is an open cell, by Lemma 2.1, we have $X \subseteq \overline{\bigcup_{t \in A} Z_t}$. By continuity of f ,

$$(2) \quad \Gamma(f) \subseteq \overline{\bigcup_{t \in A} \Gamma(f_t)}.$$

By Lemma 3.11, there are finitely many polynomials $P_i \in R[X, T]$ and semialgebraic sets C_i , $i = 1, \dots, s$, such that $R^n = \bigcup_i C_i$, and for every i and $t \in R_i$,

$$Y_t \subseteq P_i(-, t)^{-1}(0) \text{ and } \dim P_i(-, t)^{-1}(0) = \dim Y_t.$$

In particular, for every i and $t \in C_i \cap A$,

$$\Gamma(f_t) = Y_t \subseteq P_i(-, t)^{-1}(0) \text{ and } \dim P_i(-, t)^{-1}(0) \leq n.$$

For every i , denote $A_i = C_i \cap A$. So $A = \bigcup_i A_i$. For every $t \in A$, define

$$B_t = \bigcup_i \{t' \in A_i : \Gamma(f_t) \subseteq P_i(-, t')^{-1}(0)\} \subseteq A.$$

Observe that for every $t \in A$, we have $t \in B_t$, and hence $\bigcup_{t \in A} B_t = A$.

Our goal is to show that for every $t, t' \in A$, $B_t = B_{t'}$. This will imply the conclusion of the theorem. Indeed, in this case, we obtain that for every $t \in A$, $B_t = A$. Now, fix $a \in A$, say $a \in A_i$. Take any $t \in A$. So $a \in B_t$. That is, $\Gamma(f_t) \subseteq P_i(-, a)^{-1}(0)$. Together with (2), this implies that

$$\Gamma(f) \subseteq \overline{\bigcup_i P_i(-, a)^{-1}(0)},$$

and since the set on the right is semialgebraic and has dimension n , the conclusion of the theorem follows from Lemma 3.10.

We achieve our goal in Claim 3 below. First, we need two preliminary claims.

Claim 1. *Let $t, t' \in A$. Then*

$$Z_t \cap Z_{t'} \neq \emptyset \Rightarrow \overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f_{t'})}^{zar} \Rightarrow B_t = B_{t'}.$$

Proof of Claim 1. For the first implication, let $U = Z_t \cap Z_{t'}$. Since U is open, by Corollary 3.9, $\overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f|_U)}^{zar} = \overline{\Gamma(f_{t'})}^{zar}$.

For the second implication, let $x \in B_t$. Hence $\Gamma(f_t) \subseteq P_i(-, x)^{-1}(0)$. By assumption $\Gamma(f_{t'}) \subseteq P_i(-, x)^{-1}(0)$. Hence $x \in B_{t'}$. \square

Let us define the equivalence relation \sim on A , by $t \sim t'$ if and only if $B_t = B_{t'}$. For $t \in A$, let

$$W_t = \text{int} \left(\overline{\bigcup_{t \sim t' \in A} Z_{t'}} \right).$$

Clearly, $\{W_t : t \in A\}$ is a definable family of open sets in R^n .

Claim 2. *We have:*

- (i) *for every $t \in A$, $W_t = \text{int}(\overline{W_t})$,*
- (ii) *$\overline{X} = \bigcup_{t \in T_i} W_t$,*

(iii) for every $t, t' \in A$,

$$W_t \neq W_{t'} \Rightarrow t \not\sim t' \in A \Rightarrow W_t \cap W_{t'} = \emptyset.$$

Proof of Claim 2. (i) Clear, by definition of W_t .

(ii) For every $t \in A$, since X contains $\bigcup_{t \sim t' \in A} Z_{t'}$, we have that $fW_t \subseteq \overline{X}$. Moreover, since Z_t is open, by definition of W_t we have $Z_t \subseteq W_t$. Hence, by (1),

$$\dim \left(X \setminus \bigcup_{t \in A} W_t \right) < n,$$

and by Lemma 2.1, $\overline{X} = \overline{\bigcup_{t \in A} W_t}$.

(iii) The first implication is clear. So we need to prove that if $t \not\sim t' \in A$, then $W_t \cap W_{t'} = \emptyset$. Assume otherwise. Note that $W_t \cap W_{t'}$ is open. Hence the intersection

$$\overline{\bigcup_{t \sim s \in A} Z_s} \cap \overline{\bigcup_{t' \sim s' \in A} Z_{s'}}$$

contains an open set. But then, by (DIM), there must be $s \not\sim s'$ with $Z_s \cap Z_{s'} \neq \emptyset$, contradicting Claim 1. \square

In particular, the family $\{W_t : t \in A\}$ satisfies properties Lemma 3.1(i)-(iii).

Claim 3. For every $t, t' \in A$, $B_t = B_{t'}$.

Proof of Claim 3. Assume not. By Claim 2(iii), the family $\{W_t : t \in A\}$ satisfies all properties of Lemma 3.1. By that lemma, there are an open box $B \subseteq R^n$ and $t_1, t_2 \in A$ with $W_{t_1} \neq W_{t_2}$, such that each $B \cap W_{t_j}$ is semialgebraic, open and connected, and $\text{int}(\overline{B \cap W_{t_1}} \cup \overline{B \cap W_{t_2}})$ is connected. That is, $f|_{B \cap W_{t_1}}$ and $f|_{B \cap W_{t_2}}$ are Nash-concatenable. By Corollary 3.8, $\overline{\Gamma(f|_{B \cap W_{t_1}})}^{zar} = \overline{\Gamma(f|_{B \cap W_{t_2}})}^{zar}$. Since each $B \cap W_{t_j}$ is open, by (DIM), there are $a_1, a_2 \in A$ with $a_j \sim t_j$, such that $B \cap Z_{a_j}$ is open, $j = 1, 2$. Since $B \cap W_{t_j}$ is connected, open and semialgebraic, by Lemma 3.9, we obtain $\overline{\Gamma(f|_{B \cap W_{t_j}})}^{zar} = \overline{\Gamma(f|_{B \cap Z_{a_j}})}^{zar}$, $j = 1, 2$, and hence

$$\overline{\Gamma(f|_{B \cap Z_{a_1}})}^{zar} = \overline{\Gamma(f|_{B \cap Z_{a_2}})}^{zar}.$$

By Claim 1, $t_1 \sim a_1 \sim a_2 \sim t_2$, contradicting Claim 2(iii). \square

This completes the proof of the theorem. \square

4. EXAMPLES

In this section we establish Conditions (DP) and (DIM) in the examples of Proposition 1.6, and state some open questions. In those examples, $R = \mathbb{R}$ and $\tilde{\mathcal{R}}$ is d-minimal. Hence, as stated in Remark 1.5(1), Condition (DIM) becomes redundant, based on the following fact.

Fact 4.1. For every countable family $\{X_t\}_{t \in S}$ of semialgebraic sets $X_t \subseteq R^n$, we have

$$\dim \bigcup_{t \in S} X_t = \max_{t \in S} \dim X_t.$$

Proof. Towards a contradiction, suppose $\dim \bigcup_{t \in S} X_t = m > l = \max_{t \in S} \dim X_t$. After projecting onto some m coordinates, we may assume that $m = n$. Hence, $\bigcup_{t \in S} X_t$ has non-empty interior. By the Baire category theorem, some X_t must be dense in some open box, and, hence, since it is semialgebraic, it must have dimension n . \square

Condition (DP)(I) is known for our targeted example, namely $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$, and we provide precise references below. For the second example of Proposition 1.6, it can easily be extracted from the literature, and we only provide a sketch (due to P. Hieronymi).

Fact 4.2. *Let $\tilde{\mathcal{R}}$ be one of the structures in Proposition 1.6. The (DP)(I) holds.*

Proof. (1). This is [16, Corollary 4.1.7]. See also [12, Section 8.6, Remark].

(2). As the proof of [16, Corollary 4.1.7] shows, (DP)(I) there follows from quantifier elimination and the fact that definable functions are given piecewise by terms. These two ingredients are also available in the case when P is an iteration sequence ([13, Theorem 1]). \square

There are more expansions of $\overline{\mathbb{R}}$ where quantifier elimination, and hence (DP)(I), might also be possible to establish, see Questions 4.12 and 4.13 below.

We next focus on proving (DP)(II) in the examples of Proposition 1.6. This will follow from the next general statement.

Proposition 4.3. *Assume $\tilde{\mathcal{R}}$ is d -minimal. Then*

$$(\text{DP})(\text{I}) \text{ and } (\text{DIM}) \Rightarrow (\text{DP})(\text{II}).$$

Before starting the proof of Proposition 4.3, we need some prerequisites. Let $X \subseteq R^n$ be a definable set. Following Fornasiero [6, Definition 4.1], we say that X is an *embedded manifold of dimension m* if for every $x \in X$, there is an open box $B \subseteq R^n$ containing x , such that, after permuting coordinates, $X \cap B$ is the graph of a continuous function $f : U \subseteq R^m \rightarrow V \subseteq R^{n-m}$, where $B = U \times V$.

Fact 4.4. [6, Fact 2.2] *Every definable set is a finite union of finitely many embedded manifolds.*

Lemma 4.5. *Let $X \subseteq R^n$ be a definable set of dimension k . Let $Y \subseteq X^n$ be a k -cell. Assume that for every $y \in Y$,*

$$\begin{aligned} (*)_{y,X} \quad & \text{there is an open box } B \subseteq R^n \text{ containing } y, \text{ such that } X \cap B \text{ is} \\ & \text{the graph of a continuous function } f : U \subseteq R^k \rightarrow V \subseteq R^{n-k}, \\ & \text{where } B = U \times V. \end{aligned}$$

Then Y is a semialgebraic chunk of X .

Proof. Since Y is a k -cell, there is a projection $p : Y \rightarrow R^k$ onto some k coordinates, which is injective. Let $\pi : R^n \rightarrow R^k$ be the projection onto the first k coordinates. Then the map $\pi \circ p^{-1} : p(Y) \rightarrow R^k$ is a continuous injective semialgebraic map, and hence, by [8], its image $\pi(Y) \subseteq R^k$ is open.

Now take $y \in Y$, and let $B = U \times V$ and f be as in our assumptions. Since $\pi(y) \in U \cap \pi(Y)$, there is an open box $U' \subseteq U \cap \pi(Y)$ containing $\pi(y)$. Let $B' = U' \times V$. So $y \in B'$. Moreover, for every $a \in U'$, there is $b \in R^{n-k}$ such that $(a, b) \in Y$. Since also $(a, b) \in X \cap B' = \Gamma(f|_{B'})$, we have that $(X \cap B')_a$ is a singleton, namely, $\{b\}$. Hence, $X \cap B' \subseteq Y$, as needed. \square

Proof of Proposition 4.3. Let $X \subseteq R^n$ be a definable set of dimension k . By Fact 4.4, X is a finite union of embedded manifolds X_1, \dots, X_l . We split two (non-exclusive) cases:

Case I: $\dim \bigcap_i X_i = k$. In this case, by (DP)(I) and (DIM), there is a k -cell $Y \subseteq \bigcap_i X_i$. Take $y \in Y$. We claim that property $(*)_{y,X}$ from Lemma 4.5 holds. Indeed, for every $i = 1, \dots, l$, since $Y \subseteq X_i$ and X_i is an embedded manifold, $(*)_{y,X_i}$ holds. Let $B_i \subseteq R^n$ be the corresponding open box containing y . Then $B = \bigcap B_i$ witnesses $(*)_{y,X}$.

Case II: For some $i \in \{1, \dots, l\}$,

$$\dim \left(X_i \setminus \bigcup_{j \neq i} \overline{X_j} \right) = k.$$

In this case, by (DP)(I) and (DIM), there is a k -cell $Y \subseteq X_i \setminus \bigcup_{j \neq i} \overline{X_j}$. Take $y \in Y$. We claim that property $(*)_{y,X}$ from Lemma 4.5 holds. Indeed, since $Y \subseteq X_i$ and X_i is an embedded manifold, $(*)_{y,X_i}$ holds. Namely, there is an open box $B \subseteq R^n$ containing y , such that $X_i \cap B$ is the graph of a continuous function $f : U \subseteq R^k \rightarrow V \subseteq R^{n-k}$, where $B = U \times V$. On the other hand, since $y \in Y$ and $Y \cap \bigcup_{j \neq i} \overline{X_j} = \emptyset$, there is an open box $B' = U' \times V' \subseteq B$ containing y , such that $X \cap B' = X_i \cap B'$. Since $X_i \cap B' = \Gamma(f|_{U'})$, we are done. \square

Proof of Proposition 1.6. By Remark 1.5, Fact 4.2 and Proposition 4.3. \square

We next apply our theorem to a structure beyond the d-minimal setting.

Corollary 4.6. *Let $\tilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, where \mathbb{R}_{alg} is the field of real algebraic numbers. Then every definable smooth function with open semialgebraic domain is semialgebraic.*

Proof. Let $Y = \overline{\Gamma(f)}$. It is easy to see that $Y \cap (X \times \mathbb{R}) = \Gamma(f)$. By [9], every open definable set in $\tilde{\mathcal{R}}$ is definable in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. We can thus apply Theorem 1.4 to f . \square

Observe that the above structure $\tilde{\mathcal{R}}$ itself does not satisfy (DP). See Question 4.10 below for further discussion.

4.1. Optimality. The following example, due to P. Hieronymi, shows that if we remove the assumption that \mathcal{R} is a reduct of \overline{R} from Theorem 1.4, then the theorem fails. Note that the structure $\langle \mathbb{R}_{an}, e^{2\pi\mathbb{Z}} \rangle$ is d-minimal, by [12, Theorem 3.4.2], and satisfies (DP)(I), by [16, Corollary 4.1.7]. Thus, by Remark 1.5 and Proposition 4.3, it satisfies all the assumptions of Theorem 1.4.

Example 4.7. The structure $\tilde{\mathcal{R}} = \langle \overline{\mathbb{R}}_{an}, e^{2\pi\mathbb{Z}} \rangle$ defines new smooth functions with domain \mathbb{R} .

Proof. Let $\lambda : \mathbb{R} \rightarrow e^{2\pi\mathbb{Z}}$ be the function $\lambda(x) =$ the biggest element of $e^{2\pi\mathbb{Z}}$ lower than x . Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the map $f(x) = \sin \log x$. Clearly, f is not definable in the o-minimal \mathbb{R}_{an} , since its zero set is an infinite discrete set. We show that f is

definable in $\widetilde{\mathcal{R}}$. For every $x \in \mathbb{R}$, we have:

$$\begin{aligned} f(x) &= \sin \log \left(\lambda(x) \frac{x}{\lambda(x)} \right) \\ &= \sin \log \lambda(x) + \sin \log \left(\frac{x}{\lambda(x)} \right) \\ &= \sin \log \left(\frac{x}{\lambda(x)} \right). \end{aligned}$$

But $\frac{x}{\lambda(x)} \in [1, e^{2\pi}]$ and $\log([1, e^{2\pi}]) = [0, 2\pi]$. Therefore, f is definable in $\langle \mathbb{R}_{an}, e^{2\pi\mathbb{Z}} \rangle$. \square

Remark 4.8. We make three further comments towards the optimality of our results:

(1) The assumption that the domain of f is open in Theorem 1.4 is not essential. For example, one could consider definable smooth functions between semialgebraic manifolds instead. Working via the chart maps, a generalization of our theorem in that setting eventually boils down to the current one.

(2). The reader may wonder whether assuming only $R = \mathbb{R}$ (and not d-minimality) can yield more of our properties. Condition (DIM) fails: consider, for example, $\langle \mathbb{R}, P \rangle$, where P is the set of non-algebraic numbers. Let $f(t, s) = t + s : \mathbb{R}^2 \rightarrow \mathbb{R}$. Then the definable family of singletons $\{(f(t, s))\}_{(t,s) \in P^2}$ violates (DIM).

(3). Assume $\widetilde{\mathcal{R}}$ is d-minimal and not o-minimal. We can define new \mathcal{C}^n -functions for all n . Indeed, it is easy to see that such a structure defines an infinite discrete set $A \subseteq R$. Let $s : A \rightarrow A$ be the successor function in A . For $n \in \mathbb{N}$, let $f_n : (0, \infty) \rightarrow R$ be given via, for every $t \in A$, $x \in [t, s(t)] \mapsto (x - t)^n (x - s(t))^n$. It is easy to see that f_n is \mathcal{C}^{n-1} , but not \mathcal{C}^n . It is also clear that f_n is not semi-algebraic since its zero set is A .

4.2. Open questions. We conclude with a list of open questions that arise naturally from this work.

Question 4.9. Strong decomposition property. Let us call a set $X \subseteq R^n$ a *strongly embedded manifold* if there is a semialgebraic family $\{Y_t\}_{t \in R^m}$ of subsets of R^n , and a definable set $S \subseteq R^m$ with $\dim S = 0$, such that $X = \bigcup_{t \in S} Y_t$, and every Y_t is a semialgebraic chunk of X .

Assume (DP) and (DIM). Is every definable set a finite union of strongly embedded manifolds?

Question 4.10. Necessary conditions. Clearly, (DP) and (DIM) are not necessary conditions for Theorem 1.4. For example, in the dense pair $\langle \overline{\mathbb{R}}, \mathbb{R}_{alg} \rangle$, or in the example of Corollary 4.6, there are again no new smooth definable functions, but (DP)(II) fails.

Is there a reasonable way to relax Conditions (DP) and (DIM) into necessary and sufficient conditions for Theorem 1.4?

Question 4.11. Smooth functions with arbitrary domains. Here we relax the assumption on the domain of f in Theorem 1.4.

Let $f : X \subseteq R^n \rightarrow R$ be a definable smooth function, with open connected domain X . Is it true that f is the restriction to X of a semialgebraic function?

The additional connectedness assumption is necessary. Indeed, let $\tilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$, and $f : \bigcup_{t \in 2^{\mathbb{Z}}} (t, 2t) \rightarrow \mathbb{R}$ be given via: $f(x) = 0$, if $x \in (t, 2t)$ for some $t \in 2^{\mathbb{Z}}$, and $f(x) = 1$ if $x \in (t, 2t)$ for some $t \in 2^{2\mathbb{Z}+1}$. Then f is definable in $\tilde{\mathcal{R}}$, smooth but not semialgebraic.

Question 4.12. Expansions by fast sequences. As noted after Fact 4.2, it might be worth to explore whether quantifier elimination holds in further d-minimal expansions of the real field. We mention here one:

Assume $\tilde{\mathcal{R}}$ is an expansion of the real field by a fast sequence [7]. Does it have quantifier elimination (and hence satisfies (DP)(I))?

Question 4.13. Expansions of semibounded structures. By Example 4.7, a generalization of Theorem 1.4, where \mathcal{R} is replaced by an o-minimal expansion of it, fails. It is naturally to ask whether we can replace it by a proper reduct of it. Walsberg asked the following question:

Let $\tilde{\mathcal{R}} = \langle \mathbb{R}, <, \cdot|_{(0,1)^2}, +, \mathbb{Z} \rangle$. Is every definable smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ affine?

A proper reduct of \mathcal{R} is also known in the literature as a *semibounded* structure (see, for example, [5]). We recall from [14] that a *long* interval $I \subseteq R$ is an interval on which no real closed field can be defined. We ask the following general question:

Let $\mathcal{M} = \langle R, <, +, \dots \rangle$ be a semibounded reduct of \mathcal{R} , and $\tilde{\mathcal{R}}$ an expansion of \mathcal{M} that satisfies (DP) and (DIM) with ‘semialgebraic’ replaced by ‘definable in \mathcal{M} ’. Let $f : I \subseteq R \rightarrow R$ be a definable smooth map, where $I \subseteq R$ is a long interval. Is f affine?

Currently, the assumptions of the last question are not known for the example $\tilde{\mathcal{R}} = \langle \mathbb{R}, <, \cdot|_{(0,1)^2}, +, \mathbb{Z} \rangle$, although we expect them to be true. Indeed, (DP)(I) would follow from a quantifier elimination result, which perhaps can be proved following the strategy from [11, Appendix] for $\langle \mathbb{Q}, <, +, \mathbb{Z} \rangle$. Note that Proposition 4.3 still holds with the adjusted assumptions with an analogous proof.

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