Classifying expansions of the real field by complex subgroups

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Previous work

- $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$: van den Dries, *The field of reals with a predicate for the powers of two* (1985)
- $(\overline{\mathbb{R}}, \Gamma)$, Γ an infinite finite rank subgroup of $\mathbb{S}^1$:

Theorem (Belegradek/Zilber, 2008)

Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma)$ is a Boolean combination of subsets of $\mathbb{R}^m$ defined in $(\overline{\mathbb{R}}, \Gamma)$ by formulas of the form

$$\exists x_1 \exists y_1 \dots \exists x_n \exists y_n \left(\bigwedge_{i=1}^{n} (x_i, y_i) \in \Gamma \wedge \phi(x_1, y_1, \dots, x_n, y_n, v_1, \dots, v_m) \right)$$

where $\phi(x, y, v)$ is a quantifier free $\mathcal{L}_{or}(\mathbb{R})$-formula.

- $(\overline{\mathbb{R}}, 2^{\mathbb{Z}}, 2^{\mathbb{Z}}3^{\mathbb{Z}})$: Günaydın, *Model theory of fields with multiplicative groups* (2008)

Motivation

Theorem (Hieronymi, 2010)

Let S be an infinite cyclic subgroup of $(\mathbb{C}^\times, \cdot)$. Then exactly one of the following holds:

1. $(\overline{\mathbb{R}}, S)$ *defines* $\mathbb{Z}$
2. $(\overline{\mathbb{R}}, S)$ *is d-minimal*
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- Let $S = (ae^{i\varphi})^{\mathbb{Z}}$. If $a = 1$, we are in case (3). If $\varphi \in \pi\mathbb{Q}$, we are in case (2). If $a \neq 1$ and $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$, we are in case (1).

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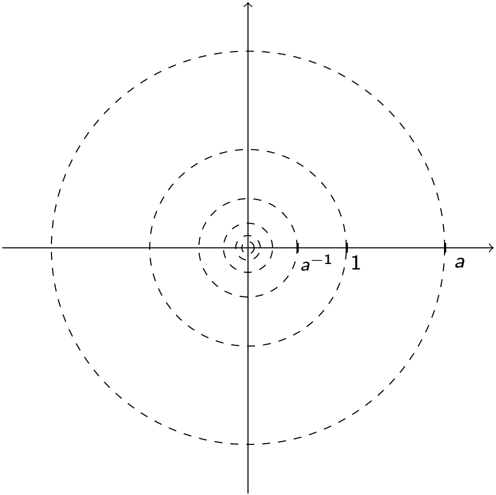
- If S is an arbitrary infinite finite rank subgroup of $\mathbb{C}$, must $(\overline{\mathbb{R}}, S)$ satisfy one of (1)-(3)?

Let $a > 1$ and $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$. The structure $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ does not satisfy any of (1)-(3).



Definable sets

Let $\Delta = \varepsilon^{\mathbb{Z}}$ $(\varepsilon > 1)$ and let Γ be an infinite finite rank subgroup of $\mathbb{S}^1$.

Theorem A (C.)

Every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, \Gamma\Delta)$ is a Boolean combination of sets of the form

$$\{x \in \mathbb{R}^m : \exists y \in (\Gamma\Delta)^n \textit{ s.t. } (x, y) \in W\}$$

for some semialgebraic set $W \subseteq \mathbb{R}^{m+2n}$.

Let $H = a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$. From Theorem A, it follows that $(\overline{\mathbb{R}}, H)$ does not satisfy any of (1)-(3).

Definable sets

- $(\overline{\mathbb{R}}, H)$ does not define $\mathbb{Z}$
  - By Theorem A, if $X \subseteq \mathbb{R}$ is definable in $(\overline{\mathbb{R}}, H)$, then X is a Boolean combinations of sets $X_1, \ldots, X_k$, where for $1 \leq i \leq k$

    $$X_i = \bigcup_{y \in H^{n_i}} \{x \in \mathbb{R} : (x, y) \in W_i\}$$

    for some $n_i \geq 1$ and semialgebraic set W_i.

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- $(\overline{\mathbb{R}}, H)$ is not d-minimal
  - $\mathrm{proj}_{\mathbb{R}}(H)$ is dense and codense in $\mathbb{R}$
- Not every open set definable in $(\overline{\mathbb{R}}, H)$ is semialgebraic
  - $\mathbb{R}^{>0} \setminus a^{\mathbb{Z}}$ is open and definable in $(\overline{\mathbb{R}}, H)$

Definable sets

A natural next step is to consider $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}})$, where $a, b > 1$ and $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$. We assume for now that $(ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$ is dense in $\mathbb{C}$.

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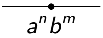
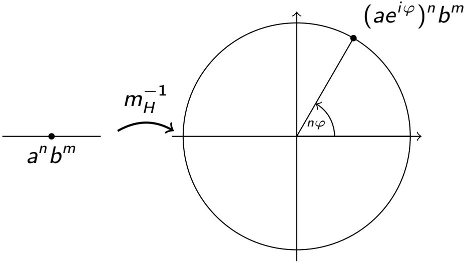
In order to study $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}})$, we instead study

$$(\overline{\mathbb{R}}, (e^{i\varphi})^{\mathbb{Z}}, a^{\mathbb{Z}} b^{\mathbb{Z}}, b^{\mathbb{Z}}, \rho)$$

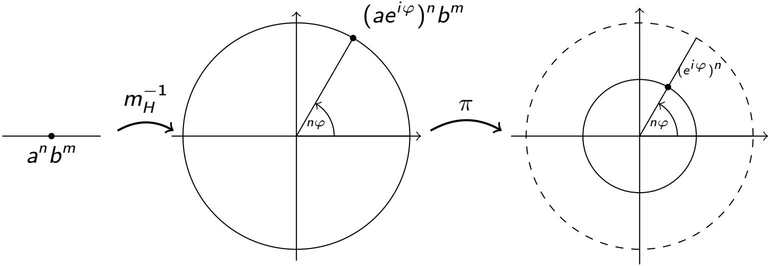
where $\rho : a^{\mathbb{Z}} b^{\mathbb{Z}} \to (e^{i\varphi})^{\mathbb{Z}}$ is defined by $\rho(a^n b^m) = e^{i\varphi n}$.

Definable sets




$a^n b^m$
m_H^{-1}
$(ae^{i\varphi})^n b^m$
$n\varphi$

Definable sets



Definable sets

Theorem B (C.)

Let $a, b > 1$ and let $\varphi \in \mathbb{R}$. Let $H := (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$ and suppose that H is dense in $\mathbb{C}$. Then every subset of $\mathbb{R}^m$ definable in $(\overline{\mathbb{R}}, H)$ is a Boolean combination of sets of the form

$$\{x \in \mathbb{R}^m : \exists z \in H^n \text{ such that } (x, z, \rho(|z|)) \in W\}$$

for some set $W \subseteq \mathbb{R}^{m+4n}$ definable in $\overline{\mathbb{R}}$.

Definable sets

We also study expansions of the form $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}})$ when $(ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$ is not dense in $\mathbb{C}$.

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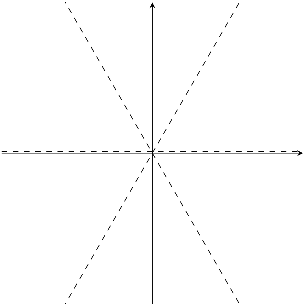

$\varphi = \frac{\pi}{3}$, $a^{\mathbb{Z}} b^{\mathbb{Z}}$ dense in $\mathbb{R}^{>0}$

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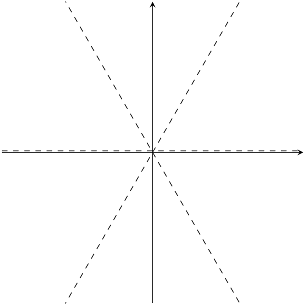

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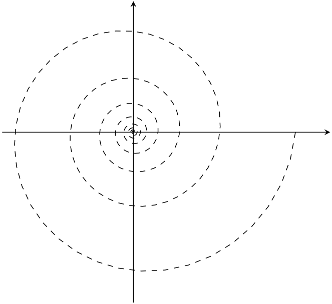

$a = e^{\varphi}$,
$b = e^{(1+i)2\pi}$

Definable sets

Definition (Logarithmic spiral)

Let $\omega \in \mathbb{R}^{\times}$. The logarithmic spiral S_ω is defined as

$$S_\omega := e^{(i+\omega)\mathbb{R}}.$$

The spiral S_ω is parameterized by the equations $(x(t), y(t)) = (e^{\omega t}\cos(t), e^{\omega t}\sin(t))$.

Definable sets

Theorem C (C.)

Let $H = (ae^{i\varphi})^{\mathbb{Z}} b^{\mathbb{Z}}$, where $a, b \neq 1$. Suppose that for all $\omega \in \mathbb{R}^{\times}$, $(\overline{\mathbb{R}}, H)$ does not define S_{ω}. Then exactly one of the following holds:

1. $(\overline{\mathbb{R}}, H)^{o} =_{df} \overline{\mathbb{R}}$.
2. $(\overline{\mathbb{R}}, H)^{o} =_{df} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$.

Open cores

Theorem D (C.)

The open core of $(\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$ *is interdefinable with* $(\overline{\mathbb{R}}, a^{\mathbb{Z}})$.

Open cores

Theorem F (C.)

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Theorem G (C.)

Suppose $(ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}}$ *is dense in* $\mathbb{C}$. *The open core of* $(\overline{\mathbb{R}}, (ae^{i\varphi})^{\mathbb{Z}}b^{\mathbb{Z}})$ *is interdefinable with* $(\overline{\mathbb{R}}, b^{\mathbb{Z}})$.

Expansions of $\overline{\mathbb{R}}$ by complex subgroups with two generators

The following table summarizes what we know about definable sets in expansions of $\overline{\mathbb{R}}$ by two complex generators. Let $\Gamma := (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ and let $\mathcal{R} := (\overline{\mathbb{R}}, \Gamma)$.

	$\varphi, \psi \in \pi\mathbb{Q}$	$\varphi \in \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \in \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$
$a = 1, b = 1$	$\mathcal{R} =_{\mathrm{df}} \overline{\mathbb{R}}$	Γ is dense in $\mathbb{S}^1$; $(\overline{\mathbb{R}}, \Gamma)$ has PNMC	$(\overline{\mathbb{R}}, \Gamma)$ has PNMC	$(\overline{\mathbb{R}}, \Gamma)$ has PNMC
$a = 1, b \neq 1$	$\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	$\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, (be^{i\psi})^{\mathbb{Z}})$; $\mathcal{R}$ defines $\mathbb{Z}$	$\psi = 0$: $\mathcal{R}$ has PNMC	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \in \mathbb{Q}$	$\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	$\varphi = 0$: $\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}}(e^{i\psi})^{\mathbb{Z}})$	$\psi = 0$: $\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}})$	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$	$\mathcal{R} =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}}b^{\mathbb{Z}})$	$\varphi = 0$: ▪ Γ dense in $\mathbb{C}$: $\mathcal{R}$ has ρ-PNMC ▪ Γ not dense in $\mathbb{C}$: $\mathcal{R}$ defines S_ω	$\psi = 0$: ▪ Γ dense in $\mathbb{C}$: $\mathcal{R}$ has ρ-PNMC ▪ Γ not dense in $\mathbb{C}$: $\mathcal{R}$ defines S_ω	Unknown

The following table summarizes what we know about the open cores of expansions of $\overline{\mathbb{R}}$ by two complex generators. Again, let $\Gamma := (ae^{i\varphi})^{\mathbb{Z}}(be^{i\psi})^{\mathbb{Z}}$ and let $\mathcal{R} := (\overline{\mathbb{R}}, \Gamma)$.

	$\varphi, \psi \in \pi\mathbb{Q}$	$\varphi \in \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \in \pi\mathbb{Q}$	$\varphi \notin \pi\mathbb{Q}, \psi \notin \pi\mathbb{Q}$
$a = 1, b = 1$	$\mathcal{R}^o =_{\mathrm{df}} \overline{\mathbb{R}}$	$\mathcal{R}^o =_{\mathrm{df}} \overline{\mathbb{R}}$	$\mathcal{R}^o =_{\mathrm{df}} \overline{\mathbb{R}}$	$\mathcal{R}^o =_{\mathrm{df}} \overline{\mathbb{R}}$
$a = 1, b \neq 1$	$\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	all open sets of all arities are definable in $\mathcal{R}$	$\psi = 0$: $\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \in \mathbb{Q}$	$\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	$\varphi = 0$: $\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	$\psi = 0$: $\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	Unknown
$a \neq 1, b \neq 1$ and $\frac{\ln(a)}{\ln(b)} \notin \mathbb{Q}$	$\mathcal{R}^o =_{\mathrm{df}} \overline{\mathbb{R}}$	$\varphi = 0$, Γ dense in $\mathbb{C}$: $\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$	$\psi = 0$, Γ dense in $\mathbb{C}$: $\mathcal{R}^o =_{\mathrm{df}} (\overline{\mathbb{R}}, b^{\mathbb{Z}})$	Unknown

A new classification

Conjecture

If H is an infinite finitely generated subgroup of $\mathbb{C}^\times$, then exactly one of the following holds:

1. $(\overline{\mathbb{R}}, H)$ defines $\mathbb{Z}$ (ex. $H = (ae^{i\varphi})^{\mathbb{Z}}$, $a \neq 1$, $\varphi \in \mathbb{R} \setminus \pi\mathbb{Q}$);

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4. $(\overline{\mathbb{R}}, H)$ is not d-minimal and there is $a > 1$ such that $(\overline{\mathbb{R}}, H)^o =_{\mathrm{df}} (\overline{\mathbb{R}}, a^{\mathbb{Z}})$ (ex. $H = a^{\mathbb{Z}}(e^{i\varphi})^{\mathbb{Z}}$, $a > 1$, $\varphi \notin \pi\mathbb{Q}$);

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5. the open core of $(\overline{\mathbb{R}}, H)$ has the form $(\overline{\mathbb{R}}, S_\omega)$ ($\omega \in \mathbb{R}^\times$).