A category-theoretic reformulation of Shelah's dividing lines in model theory.

P&S'60

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1. Summary

We interpret category-theoretically several of the Shelah dividing lines in model theory, using a simplicial category of generalised topological spaces. This makes formal the intuition that these properties of models and formulas are defined in terms of avoiding certain "bad" infinite combinatorial structures: the same diagram chasing "trick", the lifting property, applied to (a morphism associated with) a combinatorial structure defines the associated no-tree- or no-order- property of (objects associated with) models. The list of properties includes NOP, NTP, NATP, NTP$_i$, NSOP$_i$ $(i \geqslant 1)$ and NIP.

A logician should find amusing a metamathematical aspect—how trivial is all we do here: take the text of the usual definition of NTP in Tent-Ziegler [5] and "transcribe" it line by line into the simplicial language in a straightforward manner. [9] does the same with a few other textbook definitions including that of a topological space and a limit of a filter in (Bourbaki, General Topology).

2. A category of generalised topological spaces

We define our generalised topological spaces to be simplicial objects of the category of filters on sets, or, equivalently, the category of finitely additive measures taking values 0 and 1 only. Thus a generalised topological space is a simplicial set equipped, for each $n \geq 0$, with a filter on the set of n-simplices such that under any face or degeneration map the preimage of a large set is large.

These spaces generalise uniform and topological spaces, filters, and simplicial sets, and the concept is designed to be flexible enough to formulate categorically a number of standard basic elementary definitions in various fields, e.g. in analysis, limit, (uniform) continuity and convergence, equicontinuity of sequences of functions; in algebraic topology, being locally trivial and geometric realisation; in geometry, quasi-isomorphism; in model theory, stability, simplicity and several Shelah's dividing lines [3,9].

3. The definition of generalised topological spaces

Definition (Continuous maps of filters). *Let X and Y be sets equipped with filters (resp. measures). Call a map $f : X \to Y$* continuous *iff the preimage of a big (resp. full measure) set is necessary big (resp. has full measure).*

Definition (Generalised topological spaces). *Let $\mathcal{F}$ denote the category formed by sets equipped with filters, and their continuous maps. Its category of simplicial objects*

$$s\mathcal{F} := \textit{Functors}(\textit{Non-emptyFiniteLinearOrders}^{op}, \mathcal{F})$$

is our category of generalised topological spaces.

In $s\mathcal{F}$ a topological, resp. uniform, space X is the simplicial set represented by the set of points of X, where $X \times X$ is equipped with the filter of non-uniform neighbourhoods of the diagonal of form $\sqcup_{U_x \text{ a neighbourhood of } x \in X}\{x\} \times U_x$, resp. the uniformity filter, and each X^n is equipped with the coarsest filter such that all the simplicial maps $X^n \to X \times X$ are continuous. Geometric realisation is a space of maps in $s\mathcal{F}$ from $[0,1]^{\leqslant}$ to a simplicial set [3,8,9].

4. A precise meaning for "n-tuple being sufficiently small" for $n > 2$

A topological structure on a set enables one to give an exact meaning to the phrase "whenever x is *sufficiently near* a, x has the property $P(x)$", whereas a generalised topological space enables one to give an exact meaning to the phase "every n-tuple of *sufficiently similar* points $x_1, x_2, ..., x_n$ has property $P(x_1, ..., x_n)$" for $n > 1$. (Uniform spaces were introduced to do this for $n = 2$ and "similar" meaning "at small distance", as explained in (Bourbaki, General Topology; Introduction)). In a topological space, this exact meaning is that the set $\{x \mid P(x)\}$ belongs to the neighbourhood filter of a point a. Similarly, in a generalised topological space, it is that the set $\{(x_1, ..., x_n) \mid P(x_1, ..., x_n)\}$ belongs to the "neighbourhood" filter defined on n-simplices.

In model theoretic examples, similarity may mean either indiscernability or realising sufficiently many instances of a formula: n-simplices are tuples of elements of a model, and the "neighourhood filter" on n-tuples consists of all subsets containing all "sufficiently" indiscernible tuples or realising "sufficiently many" instances of a formula. This exact meaning enables us to bring the standard intuition of topology to model theory.

5. The definition of NTP (no tree property)

For a binary formula $\varphi(x, y)$, call a tuple $(a_i : 0 < i < n)$ *φ-consistent* iff the set $(\varphi(x, a_i) \mid 0 < i < n)$ is consistent. Below ${}^{<\omega}\omega$, resp. ${}^{\omega}\omega$, denotes the tree formed by finite, resp. countably infinite, sequences of natural numbers.

Definition (Tent-Ziegler, 7.2.1). *A formula $\varphi(x, y)$ has* the tree property (with respect to k) *if there is a tree of parameters $(a_s \mid \varnothing \neq s \in {}^{<\omega}\omega)$ such that:*

a) For all $s \in {}^{<\omega}\omega$, each k-tuple of distinct siblings $(a_{si} \mid i < \omega)$ is φ-inconsistent.

b) For all branches $\sigma \in {}^{\omega}\omega$ the tuple $(a_s \mid \varnothing \neq s \subseteq \sigma)$ is φ-consistent.

What do you gain by pretending so ?

Read the definition of NTP line by line
and rephrase it in terms of generalised topological spaces

6. The space associated with a formula

The definition cares about *φ-consistency of tuples in a model M*. Therefore we consider the simplicial set $M_\bullet$: Non-emptyFiniteLinearOrders^{op} $\to$ Sets

$$M_\bullet(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{sets}}(n^{\leqslant}, |M|) = |M|^n, n > 0$$

and equip each $|M|^n$ with the filter generated by a single subset:

the subset of φ-consistent n-tuples.

7. Rewriting the positive requirement of item b)

View a tree as a partial order under prefix relation.
The definition also cares about branches of the tree $T := {}^{<\omega}\omega$, i.e. sets of increasing tuples

$$T^{\leqslant}_\bullet(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{preorders}}(n^{\leqslant}, T^{\leqslant}), n > 0.$$

An easy simplicial argument (as both are representable) shows that

a tree of parameters $(a_s \mid \varnothing \neq s \in T)$ is the same as a map $T^{\leqslant}_\bullet \to M_\bullet$

We want this map to be continuous iff the tree of parameters satisfies the positive requirement on a witness of the tree property, i.e. item b). By definition, this means that the subset of φ-consistent tuples in each $|M|^n$ contains all the linearly ordered (in the tree) tuples of parameters. This becomes precisely what is required by the definition of continuity if we

equip each $T^{\leqslant}_\bullet(n^{\leqslant})$ with the indiscrete filter.

8. Rewriting the negative requirement of item a)

The negative requirement of the definition (i.e. item a)) cares about unordered tuples of parameters. To capture arbitrary tuples, consider the simplicial set

$$|T|_\bullet(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{sets}}(n^{\leqslant}, |T|) = |T|^n, n > 0$$

We want the map $|T|_\bullet \longrightarrow M_\bullet$ to be continuous iff there is a witness of failure of the negative requirement (i.e. a φ-consistent k-tuple of distinct siblings) in each subtree of shape required by the definition (i.e. a copy of ${}^{<\omega}\omega$). We turn this requirement into a definition of filters on $|T|_\bullet$: a subset of $|T|_\bullet(n^{\leqslant})$ is *big* iff

it contains a tuple with k distinct siblings (in the subtree) from each subtree of T isomorphic to ${}^{<\omega}\omega$, and is closed under permutations.

A Ramsey theory argument verifies this indeed defines a filter. (A technicality: we need to slightly modify these filters to ensure $T^{\leqslant}_\bullet \to |T|_\bullet$ is continuous, and insist that big subsets also contain all the linearly ordered tuples of vertices.)

9. Finally, a reformulation via the lifting property

Finally, we see that

a continuous map $T^{\leqslant}_\bullet \to M_\bullet$ extends continuously to $|T|_\bullet \supset T^{\leqslant}_\bullet$
iff
the tree of parameters $(a_s \mid \varnothing \neq s \in T)$ satisfying item b),
has a subtree of shape ${}^{<\omega}\omega$ failing item a).

The latter is equivalent to failure of the tree property when quantified over a_i's; the former is represented by a commutative diagram known as *lifting property:*

$$\begin{tikzcd} T^{\leqslant}_\bullet \arrow[r, "\forall"] \arrow[d] & M_\bullet \arrow[d] \\ |T|_\bullet \arrow[r, "\forall"] \arrow[ur, dashed, "\exists"] & \top \end{tikzcd}$$

This is our category-theoretic interpretation of NTP (no tree property). The same method leads to similar diagrams defining NTP$_i$, NATP, NSOP$_1$ (new item a) leads to new filters), and NOP, NSOP$_i$, $i \geq 3$ (also need to modify $M_\bullet$).

10. Research directions

- The lifting property is used to axiomatise homotopy theory as the notion of a Quillen model category. Is there a homotopy theoretic interpretation of the Shelah dividing lines in model theory ? Say, what is the model theoretic meaning of the number of connected components of the space of maps from $T^{\leqslant}_\bullet$ to $M_\bullet$ where, perhaps, the space of maps is as defined in the $s\mathcal{F}$ reformulation of geometric realisation ?
- Pillay-Starchenko [1,Cor.1.2] uses φ-consistent tuples (take $\varphi := E$ there), i.e. data captured by $M_\bullet$, and so does Simon [2,Def.1.1] to define generically stable measures. Malliaris [6] studies Szemerédi regularity of $M_\bullet$ viewed as a multigraph. Reformulate this in terms of $s\mathcal{F}$. Can definable and Szemerédi regularity be reformulated in terms of our generalised spaces ?
- Reformulating in $s\mathcal{F}$ the Bourbaki definition of a limit of a filter on a topological space defines a notion of limit of an arbitrary morphism in $s\mathcal{F}$ reminiscent of homotopy, see §23. A combinatorial structure on a model, say an NTP tree, is a morphism in $s\mathcal{F}$, hence there is a notion of a limit of a combinatorial structure on a model. Is it useful? Can one define homotopy of models?

11. Appendix: The lifting property

$$\begin{tikzcd} A \arrow[r, "\forall t"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "\forall b"'] \arrow[ur, dashed, "\exists d"] & D \end{tikzcd}$$

Definition. A morphism $A \xrightarrow{f} B$ in a category has *the left lifting property with respect to a morphism* $C \xrightarrow{g} D$ and g *also has the right lifting property with respect to* f, denoted $f \rightthreetimes g$, iff for each map $A \xrightarrow{t} C$ and $B \xrightarrow{b} D$ such that $f \circ b = t \circ g$, there exists $B \xrightarrow{d} C$ such that $t = f \circ d$ and $b = d \circ g$.

For a class C of morphisms in a category, its left orthogonal $C^{\rightthreetimes l}$ with respect to the lifting property, respectively its right orthogonal $C^{\rightthreetimes r}$, is the class of all morphisms which have the left, respectively right, lifting property with respect to each morphism in the class C.

It is clear that $C^{\rightthreetimes lr} \supset C$, $C^{\rightthreetimes rl} \supset C$, $C^{\rightthreetimes l} = C^{\rightthreetimes lrl}$, and $C^{\rightthreetimes r} = C^{\rightthreetimes rlr}$. The class $C^{\rightthreetimes r}$ is always closed under retracts, pullbacks, (small) products (whenever they exist in the category) and composition of morphisms, and contains all isomorphisms. Meanwhile, $C^{\rightthreetimes l}$ is closed under retracts, pushouts, (small) coproducts and transfinite composition (filtered colimits) of morphisms (whenever they exist in the category), and also contains all isomorphisms. A desirable property, sometimes proved by a Quillen small object argument, is that each morphism f decomposes as $f = f_l \circ f_{lr}$ and $f = f_{rl} \circ f_r$ where $f_l \in C^{\rightthreetimes l}$, $f_{lr} \in C^{\rightthreetimes lr}$, $f_r \in C^{\rightthreetimes r}$, and $f_{rl} \in C^{\rightthreetimes rl}$.

Taking the orthogonal ("negation") of a class C is a simple way to define a class of morphisms excluding non-isomorphisms from C, in a way which is useful in a diagram chasing computation. A useful intuition is to think that the property of left-lifting against a class C is a kind of negation of the property of being in C, and that right-lifting is also a kind of negation, and for this reason it is convenient to say that

property $C^{\rightthreetimes l}$, resp. *$C^{\rightthreetimes r}$*, *is left*, resp. *right*, *Quillen negation of property C.*

12. Appendix: Examples of the lifting property

Iterated lifting property (=iterated Quillen negation) is used to define a number of textbook notions starting from an explicitly given list of (counter)examples or a simple property, see [4] for a list. We sketch the simplest examples.

Example. *It is easy to verify that*

- *A map is surjective iff it has the right lifting property with respect to the map from the emptyset to a singleton (the archetypal example of non-surjection $\varnothing \to \{\bullet\}$), and it is injective iff it has the right lifting property with respect to the map gluing together two points (the archetypal non-injection example of $\{\bullet,\bullet\} \to \{\bullet\}$)*

These statements are represented by the following commutative diagrams:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "\therefore(surj)"] \\ \{\bullet\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} \{\bullet,\bullet\} \arrow[r] \arrow[d] & X \arrow[d, "\therefore(inj)"] \\ \{\bullet\} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $\therefore$ marks the property being defined.

13. Rewriting combinatorial characterisations of other dividing lines

Let us now clarify what we said earlier: The same method leads to the same diagram defining NTP$_i$, NATP, NSOP$_1$ (new item a) leads to new filters), and NOP, NSOP$_i, i \geq 3$ (also need to modify $M_\bullet$)'. We also observe that NFCP (no finite cover property) means "of infinite simplicial dimension".

14. Rewriting NTP$_1$ (no tree property)

Let us show the same method leads us to rewrite NTP$_1$ in $s\boldsymbol{\varphi}$. The cases of NATP and NSOP$_1$ are similar.

Definition. *A formula $\varphi(x,y)$ has* TP$_1$ *(tree property 1) iff there is a tree of parameters $(a_s \mid \varnothing \neq s \in {}^{<\omega}\omega)$ such that:*

a) *for each pair $\eta,\mu \in {}^{\leqslant\omega}\omega$ of incomparable elements, the pair $\{a_\mu, a_\eta)$ is φ-inconsistent.*

b) *For all branches $\sigma \in {}^{\omega}\omega$ the tuple $(a_s \mid \varnothing \neq s \subseteq \sigma)$ is φ-consistent.*

Note that the only difference from the definition of NTP is in item a). Therefore, the only the only difference from the reformulation of the definition of NTP is in the definition of filters on $|T|_\bullet$.

As in the case of NTP, we want the map $|T|_\bullet \longrightarrow M_\bullet$ in $s\boldsymbol{\varphi}$ to be continuous iff there is a witness of failure of the negative requirement of item a). In case of NTP$_1$, it means that we define the filters on $|T|_\bullet$ by:

a subset of $|T|_\bullet(n^\leqslant)$ is *big* iff it contains a pair of incomparable vertices from each subtree of T isomorphic to ${}^{\leqslant\omega}\omega$ and all of $T_\bullet^\leqslant(n^\leqslant)$, and is closed under permutations.

15. NSOP$_1$

Definition. *A formula $\varphi(x,y)$ has SOP$_1$ iff there are $(a_\eta : \eta \in {}^{<\omega}2)$ such that*

a) *$\forall\ \eta,\mu \in {}^{<\omega}2$, if $\mu\hat{\ }0 \prec \eta$, then $\{\varphi(x,a_{\mu\hat{\ }1}),\ \varphi(x,a_\eta)\}$ is inconsistent.*

b) *for all branches $\sigma \in {}^{\omega}2$, $\{\varphi(x,a_{\sigma|_n}) : n < \omega\}$ is consistent.*

The reformulation is exactly the same except that we take the tree $T := {}^{<\omega}2$, and equip $|T|_\bullet(n^\leqslant)$ with the filter

a subset U of $|T|_\bullet(n^\leqslant)$ is *big* iff for each subtree of T isomorphic to ${}^{<\omega}2$ there is a pair $\eta,\mu \in {}^{<\omega}2$, such that $\mu_0 \prec \eta$ and U contains all tuples containing both η and μ_1, where μ_0 and μ_1 denote 0- and 1- descendants of μ in the subtree.

16. Rewriting No-Order-Properties

The analysis of No-Order-Properties is similar but different mathematical structures arise. Instead of an infinitely branching tree $T^\leqslant$ No-Order-Properties consider the tree consisting of a single branch, i.e. an increasing sequence.

The order properties care about consistency of instances of a formula with ordered variables, hence we need to modify the filters on $|M|_\bullet$ to reflect that.

17. Rewriting NOP (no order property)

Recall a formula $\varphi(-,-)$ has NOP (no order property) iff there no sequence $(a_i)_{i\in\omega}$ such that $\varphi(a_i,a_j) \Leftrightarrow i \leq j$.

Because NOP cares about *consistency of instances of a formula on tuples of ordered variables*, equip $M_\bullet(n^\leqslant) := M^n$ with the filter of subsets containing

$$\{(a_1,..,a_n) \in M^n : i < j \,\&\, a_i \neq a_j \implies \varphi(a_i,a_j)\}$$

Then rewrite the standard definition of NOP in a form with two items representing positive and negative requirements.

Definition. *A formula $\varphi(x,y)$ has* NOP *iff there is no sequence $(a_i)_{i\in\omega}$ such that*

a) *for $i < j$, $\varphi(a_j,a_i)$ is false.*

b) *for $i < j$, $\varphi(a_i,a_j)$ holds.*

In analogy with NTP, think of the sequence as a tree $T := \omega^\leqslant$ with a single branch, and equip $\omega_\bullet^\leqslant(n^\leqslant) := \mathrm{Hom}_{\text{preorders}}(n^\leqslant,\omega^\leqslant), n > 0.$ with indiscrete filters. Let $|\omega|_\bullet^{\text{NOP}}(n^\leqslant) := \mathrm{Hom}_{\text{Sets}}(n^\leqslant,|\omega|) = |\omega|^n, n > 0.$ where a subset U of $|\omega|_\bullet^{\text{NOP}}(n^\leqslant) = |\omega|^n$ is *large* iff it is closed under permutations and

each infinite subsequence $(a_{i_j})_j$ of ω has a subsequence $(a_{i_{j_1}},..,a_{i_{j_n}}) \in U$ where $j_1 > ... > j_n$

With these definitions, $\omega_\bullet^\leqslant \to |\omega|_\bullet^{\text{NOP}} \rightthreetimes M_\bullet \to \top$ defines NOP.

18. NSOP$_\ell$ for $\ell \geq 3$ (no strict order property)

Definition. *For $\ell \geqslant 3$, a formula $\varphi(x,y)$ has SOP$_\ell$ (ℓ-strong order property) iff*

a) *the set $\{\varphi(x_1,x_2),\ldots,\varphi(x_{n-1},x_n),\varphi(x_\ell,x_1)\}$ is inconsistent.*

b) *there are $(a_i)_{i<\omega}$ such that $\models \varphi(a_i,a_j)$ for all $i < j$*

A theory is NSOP$_\ell$ if no formula has SOP$_\ell$.

Equivalently, NSOP$_\ell$ means that "item a) implies NOP".

Let $|\ell|_\bullet(n^\leqslant) := \mathrm{Hom}_{\text{Sets}}(n^\leqslant,|\ell|) = |\ell|^n$, $n > 0$ denote the simplicial set represented by the set $\{1,2,...\ell\}$ with ℓ elements which we denote by $|\ell|$. Equip $|\ell|_\bullet(2^\leqslant) = |\ell|^2$ with the filter of subsets containing

$$\{(1,2),(2,3),...,(\ell-1,\ell),(\ell,1),(1,1),..,(\ell,\ell)\},$$

and for each $n > 0$ equip $|\ell|_\bullet(n^\leqslant) = |\ell|^n$ with the coarsest filter such that all simplicial maps $|\ell|_\bullet(n^\leqslant) \to |\ell|_\bullet(2^\leqslant)$ are continuous, i.e. a subset is big iff it contains each element of form $(i,i,..,i)$ and $(i,..,i,i+1,..,i+1)$, $1 \leqslant i \leqslant \ell$, and $(\ell,..\ell,1,...,1)$.

Then a map $|\ell|_\bullet^{\ell\text{-cycle}} \longrightarrow M_\bullet, i \mapsto a_i$ is continuous iff either $a_1 = ... = a_\ell$ or $M \models \{\varphi(a_{j_1},a_{j_2}),\ldots,\varphi(a_{j_{\ell'-1}},a_{\ell'}),\varphi(x_{a'_\ell},a_{j_1})\}$ holds for the subsequence of $(a_{j_k})_k$ of distinct elements. Hence, item a) holds iff $|\ell|_\bullet^{\ell\text{-cycle}} \longrightarrow \top \rightthreetimes M_\bullet \longrightarrow \top$. Then NSOP$_{\leqslant\ell}$:= NSOP$_1 \vee .. \vee$ NSOP$_\ell$ can be stated as:

$$|\ell|_\bullet^{\ell\text{-cycle}} \longrightarrow \top \rightthreetimes M_\bullet \longrightarrow \top \text{ implies } \omega_\bullet^\leqslant \to |\omega|_\bullet^{\text{NOP}} \rightthreetimes M_\bullet \to \top$$

19. NFCP (no finite cover property)

We also note that in the category $s\boldsymbol{\varphi}$ of generalised topological spaces NFCP (no finite cover property) means that $M_\bullet$ is of infinite dimension. Recall that "$M_\bullet$ is of finite dimension" (as a simplicial object) means that the whole of $M_\bullet$ is fully determined by a finite piece of $M_\bullet$, namely there is $N > 0$ such that for each $n > N$ $M_\bullet(n^\leqslant)$ is the pullback of the diagram consisting of simplicial maps $M_\bullet(n^\leqslant) \to M_\bullet(m^\leqslant)$ and $M_\bullet(m_1^\leqslant) \to M_\bullet(m_2^\leqslant)$ where $m,m_1,m_2 < N$. In terminology of Malliaris [8,Remark 2.6] this is expressed by saying that "the characteristic sequence $< P_n >$ of φ has finite support".

20. Stability and NIP

Both stability and NIP can be expressed in terms of indiscernible sequences. Hence, we modify the filters on $|M|_\bullet$ to reflect that: the filter on $|M|_\bullet(n^\leqslant) = M^n$ is generated by sets of φ-*indiscernible sequences with repetitions*, i.e. sequences $(a_i)_i$ such that $\varphi(a_{i_1},...,a_{i_r}) \Leftrightarrow \varphi(a_{j_1},...,a_{j_r})$ whenever $i_1 < ... < i_r$, $j_1 < ... < j_r$, all the $a_{i_1},..,a_{i_r}$ are distinct, and all the $a_{j_1},...,a_{j_r}$ are distinct. (If you feel it is more natural, you may require each subsequence with distinct elements to be part of an *infinite* φ-indiscernible sequence.)

This construction of $M_\bullet$ generalises to formulas of arbitrary arity, and, moreover, for arbitrary collections of formulas.

There is a forgetful functor $s\boldsymbol{\varphi} \longrightarrow \mathrm{Top}$ taking a generalised topological space into a topological space (possibly empty). For a unary formula $\varphi(-)$, it takes $M_\bullet$ defined above into the (usual) Stone space of φ-types.

21. Each indiscernible sequence is a set.

Let

$$\omega_\bullet^{\leqslant,\mathrm{cof}}(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{preorders}}(n^{\leqslant}, \omega^{\leqslant}), n > 0,$$
$$|\omega|_\bullet^{\mathrm{cof}}(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{sets}}(n^{\leqslant}, |\omega|) = |\omega|^n, n > 0$$

equip $|\omega| = \omega_\bullet^{\leqslant,\mathrm{cof}}(1^{\leqslant}) = |\omega|_\bullet^{\mathrm{cof}}(1^{\leqslant})$ with the filter of cofinite subsets, and equip each $\omega_\bullet^{\leqslant,\mathrm{cof}}(n^{\leqslant})$, resp. $|\omega|_\bullet^{\mathrm{cof}}(n^{\leqslant})$, with the coarsest filter such that all simplicial maps $\omega_\bullet^{\leqslant,,\mathrm{cof}}(n^{\leqslant}) \to \omega_\bullet^{\leqslant,\mathrm{cof}}(2^{\leqslant})$, resp. $|\omega|_\bullet^{\mathrm{cof}}(n^{\leqslant}) \to |\omega|_\bullet^{\mathrm{cof}}(2^{\leqslant})$, are continuous. Then

each (infinite) φ-indiscernible sequence is a φ-indiscernible set iff

$$\omega_\bullet^{\leqslant,\mathrm{cof}} \longrightarrow |\omega|_\bullet^{\mathrm{cof}} \rightthreetimes M_\bullet \longrightarrow \top$$

22. Each indiscernible sequence is eventually indiscernible over any parameter

Let

$$M[+1]_\bullet(n^{\leqslant}) := \mathrm{Hom}_{\mathrm{sets}}((n+1)^{\leqslant}, |\omega|) = |M| \times |M|^n, n > 0,$$

and equip $|M|^{n+1} = M[+1]_\bullet(n^{\leqslant})$ with the filter generated by subsets

$$\{(a_0, a_1, ..., a_n) : \text{ the seq. } (a_1, ..., a_n) \text{ is } \varphi\text{-indiscernible over } a_0\}, \varphi \text{ a formula}$$

One can check that the map $M[+1]_\bullet \to M_\bullet, (a_0, a_1, ..., a_n) \mapsto (a_1, .., a_n)$ forgetting the first coordinate, is continuous.

Each indiscernible sequence in M is eventually indiscernible over any parameter iff

$$\{0\}_\bullet \to \omega_\bullet^{\leqslant,\mathrm{cof}} \rightthreetimes M[+1]_\bullet \to M_\bullet$$

where $\{0\}_\bullet(n^{\leqslant}) = \{0\}$ is equipped with the filter containing the empty set, and $\{0\}_\bullet \to \omega_\bullet^{\leqslant,\mathrm{cof}}$ is the map $(0, .., 0) \mapsto (0, .., 0)$. Note that the map $\{0\}_\bullet \to M[+1]_\bullet$ picks an arbitrary element of M.

23. Speculations: limits and contractible models ?

In $s\mathcal{F}$ the notions of homotopy and limit are closely related:
the same construction in $s\mathcal{F}$ describes both

- picking a homotopy contracting a topological space (via singular complexes)
- taking a limit of a filter on a topological space,

and

- applies to an arbitrary morphism in $s\mathcal{F}$.

Can one somehow apply this construction to models ? One may perhaps hope that this construction can somehow be used to define a useful notion of a *contractible* (generalised topological space associated with a) *model*, or that the notion of a *limit of a combinatorial structure on a model*, say a tree as in NTP, is useful (i.e. taking a limit of the morphism associated with a combinatorial structure on a model, such as those arising in the lifting properties describing NTP).

Below we sketch this construction; see [9,§3] for more details. We warn the reader that our considerations here are unusually preliminary, and apologise for including them; our excuse is that we are likely unable to pursue them.

Definition ((Limit in a generalised space)). *Let $\mathfrak{F}_\bullet : F_\bullet \longrightarrow X_\bullet$ be a morphism in $s\mathcal{F}$. A morphism $x_\bullet : X_\bullet \longrightarrow X_\bullet \circ [+1]$ is said to be a* limit morphism *(or simply* a limit*) of $\mathfrak{F}_\bullet$ iff the following diagram commutes:*

$$\begin{tikzcd} & X_\bullet \circ [+1] \arrow[d, "pr_{2,3,...}"] \\ F_\bullet \arrow[ur, dashed, "x_\bullet"] \arrow[r, "\mathfrak{F}_\bullet"] & X_\bullet \end{tikzcd}$$

where $[+1] : \Delta \longrightarrow \Delta$ is the shift

$$n \mapsto n+1, \quad f : n \to m \longmapsto f' : n+1 \to m+1, f'(0) := 0; f'(i+1) := f(i) \text{ for } i \geqslant 0,$$

and $X_\bullet \circ [+1] \longrightarrow X_\bullet$ is the expected map "forgetting the first coordinate".

To recover the Bourbaki definition of a limit of a filter $\mathfrak{F}$ on a topological space X, associate with $\mathfrak{F} \in \mathrm{Ob}\,\mathcal{F}$ the simplical set represented by X

$$F_\bullet(n^{\leqslant}) := \mathrm{Hom}(n^{\leqslant}, X)$$

equip $F_\bullet(1^{\leqslant}) = X$ with $\mathfrak{F}$, and equip each $F_\bullet(n^{\leqslant}) := \mathrm{Hom}(n^{\leqslant}, X)$ with the finest filter such that the diagonal map $F_\bullet(1^{\leqslant}) \longrightarrow F_\bullet(n^{\leqslant})$ is continuous. A verification shows that (possibly discontinuous) liftings correspond to points of X (indeed, as simplicial sets, $F_\bullet$ is connected, and $X_\bullet[+1]$ is the disjoint union of connected component which are copies of $X_\bullet$ parametrised by X), and the continuity requirement means precisely that they are limit points.

Example (Limit of the tree in NTP). *Let $T_\bullet^{\leqslant}$, $|T|_\bullet^{NTP}$, and $M_\bullet^{\varphi\text{-}NTP}$ denote the objects corresponding to a tree T and a model M appearing in the lifting property for NTP of formula φ. A verification shows the following.*
A limit of $T_\bullet^{\leqslant} \longrightarrow M_\bullet^{\varphi\text{-}NTP}$ is a point $a \in M$ such that the map $T_\bullet^{\leqslant} \longrightarrow M_\bullet^{\varphi'\text{-}NTP}$ is continuous where $\varphi'(x, y) := \varphi(x, a) \,\&\, \varphi(x, y)$.

To see this, first note that, as simplicial sets (i.e. if we ignore the filters), $M_\bullet^{\varphi\text{-NTP}}[+1]$ is the disjoint union of connected component which are copies of $M_\bullet^{\varphi\text{-NTP}}$ parametrised by elements of M, that both $T_\bullet^{\leqslant}$, $|T|_\bullet$ are connected. Thus to give a map of simplicial sets $T_\bullet^{\leqslant} \longrightarrow M_\bullet^{\varphi\text{-NTP}}$ or $|T|_\bullet \longrightarrow M_\bullet^{\varphi\text{-NTP}}$ is the same as to pick a point $a \in M$. The continuity requirement on $T_\bullet^{\leqslant} \longrightarrow M_\bullet^{\varphi\text{-NTP}}$ means precisely that $\varphi(x, a_1), ..., \varphi(x, a_n)$ is consistent whenever $a_1, ..., a_n$ lie on the same branch of T. The continuity requirement on $T_\bullet^{\leqslant} \longrightarrow M_\bullet^{\varphi\text{-NTP}}[+1]$ means precisely that $\varphi(x, a), \varphi(x, a_1), ..., \varphi(x, a_n)$ is consistent whenever $a_1, ..., a_n$ lie on the same branch of T.
The case of $|T|_\bullet^{\mathrm{NTP}} \longrightarrow M_\bullet^{\varphi\text{-NTP}}$ is similar.
Can localisation and persistence of configurations in $M_\bullet^{\varphi\text{-NTP}}$ of Malliaris [7,8] be rephrased in terms of $s\mathcal{F}$ limits of this kind ? Can NIP be rephrased in terms of limits, e.g. something like compactness or completeness ? Will a homotopy viewpoint on such reformulations be illuminating ?

24. Contracting homotopy as a limit

Now let F and X denote topological spaces.

Remark (Homotopy as limit of singular complexes). *A map $h_0 : F \longrightarrow X$ is contractible, i.e. it factors through the cone of X as $F \xrightarrow{x \mapsto (x,0)} F \times [0,1]/F \times \{1\} \xrightarrow{h} X$, iff in sSets or $s\mathcal{F}$ the map $\mathrm{sing}\, F_\bullet \longrightarrow \mathrm{sing}\, X_\bullet$ of singular complexes has a limit i.e. there is a commutative diagram*

$$\begin{tikzcd} & \mathrm{sing}\, X_\bullet \circ [+1] \arrow[d, "pr_{2,3,...}"] \\ \mathrm{sing}\, F_\bullet \arrow[ur, dashed, "h_\bullet"] \arrow[r] & \mathrm{sing}\, X_\bullet \end{tikzcd}$$

Recall that the singular complex is defined using simplices $\Delta^n = \mathrm{Hom}_{\mathrm{preorders}}([0,1]^{\leqslant}, (n+1)^{\leqslant})$ as "test spaces":

$$\mathrm{sing}\, F_\bullet((n+1)^{\leqslant}) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, F),$$
$$\mathrm{sing}\, X_\bullet((n+1)^{\leqslant}) := \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X),$$
$$\mathrm{sing}\, X_\bullet \circ [+1]((n+1)^{\leqslant}) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n \times [0,1]/\Delta^n \times \{1\}, X)$$

where $n \geqslant 0$ and $\Delta^n \times [0,1]/\Delta^n \times \{1\}$ is the cone of n-simplex Δ^n.
To define a limit(=lifting) $h_\bullet$, take each $\delta : \Delta^n \to F$ in $F_\bullet((n+1)^{\leqslant})$ to $h_*(\delta) : \Delta^n \times [0,1]/\Delta^n \times \{1\} \to X$ in $X_\bullet((n+2)^{\leqslant})$ defined by

$$h_*(\delta)(x, t) := h(\delta(x), t).$$

To see the other direction, note that $h_\bullet : F_\bullet \longrightarrow X_\bullet[+1]$ takes a singular simplex $\delta : \Delta^n \longrightarrow F$ into $h_\bullet(\delta) : \Delta^{n+1} = \Delta^n \times [0,1]/\Delta^n \times \{1\} \longrightarrow X$ such that $\delta \circ h_0 = h_\bullet(\delta)_{|\Delta^n \times \{0\}}$, i.e. each $\delta : \Delta^n \longrightarrow F \longrightarrow X$ factors through the cone of Δ^n. A verification using functoriality shows that the same factorisation holds for $\mathbb{S}^n = \partial\Delta^{n+1}$, which means exactly that h_0 is weakly contractible, and for "nice" topological spaces contractible and weakly contractible are equivalent.

Remark (Homotopy as limit of the fibre). *A map $h : F \times [0,1]/F \times \{1\} \longrightarrow X$ continuous in a neighbourhood of "the top of the cone" point $F \times \{1\}$ is the same as a map in $s\mathcal{F}$*

$$F_\bullet \times ([0,1]_1)_\bullet \longrightarrow X_\bullet[+1]$$
$$(x_1, .., x_n, t_1, .., t_n) \mapsto (h(x_1, 1), h(x_1, t_1), h(x_2, t_2), ..., h(x_n, t_n))$$

where $[0,1]_1$ denotes the interval $[0,1]$ equipped with the filter of neighbourhoods of point 1, and $F_\bullet, X_\bullet$ denote the generalised topological spaces corresponding to F and X.

In other words, $s\mathcal{F}$ can express infinitesimally/"sufficiently" short" homotopies: in an expressive language, we may say that $h_t : F \longrightarrow X$, $t \in [0,1]$, *converges at* 0 iff there is $\varepsilon > 0$ such that $h_{|[0,\varepsilon]}$ is a homotopy contracting F in X.

25. Homotopy theory for a model ?

A standard way to modify the definition of $\mathrm{sing} : \mathrm{Top} \longrightarrow \mathrm{sSets}$ (or $\mathrm{nerve} : \mathrm{Cat} \longrightarrow \mathrm{sSets}$) is to use instead of simplices another cosimplicial "test" space, i.e. take a functor $\mathfrak{d}^\bullet$: Non-emptyFiniteLinearOrders $\longrightarrow s\mathcal{F}$ and set

$$\mathrm{sing}\, X_\bullet(n^{\leqslant}) := \mathrm{Hom}(\mathfrak{d}^\bullet(n^{\leqslant}), X)$$

where Hom may mean a space (rather than merely a set) of maps, e.g. internal hom or as defined in [8]. This suggest that we find a cosimplicial "test" object in $s\mathcal{F}$ such that $\mathrm{Hom}(\mathfrak{d}^\bullet(n^{\leqslant}), M_\bullet)$ is meaningful in model theory, perhaps by finding first a sequence of interesting combinatorial structures.

Alternatively, do something like what we did with NTP: find a definition in model theory that has the pattern of the diagram defining limit in $s\mathcal{F}$. Say, you'd like to see (force?) that NTP_k has pattern of $\pi_n(X) = 0$ e.g. that $T_\bullet^{\leqslant} \longrightarrow |T|_\bullet^{\mathrm{NTP}_k} \rightthreetimes M_\bullet^{\varphi\text{-NTP}} \to \top$ has the pattern of $\partial\Delta^{n+1} \to \Delta^{n+1} \rightthreetimes X \to \top$. defining $\pi_n(X) = 0$ in sSets. Hence, you'd want to see how monotone maps $(n+2)^{\leqslant} \to (n+2)^{\leqslant}$ would act on $|T|_\bullet^{\mathrm{NTP}_k}$ (but not $T_\bullet^{\leqslant}$). You'd also want this action if you want $|T|_\bullet^{\mathrm{NTP}_k}$ to be part of a cosimplicial object used to define sing. (Note that in topology there is no *natural* action of these maps on $\mathbb{D}^{n+1}$ occurring in $\mathbb{S}^n \to \mathbb{D}^{n+1} \rightthreetimes X \to \top$ defining $\pi_n(X) = 0$: you need to pick an arbitrary homeomorphism of ball $\mathbb{D}^{n+1}$ with the standard $(n+1)$-simplex.)

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