Research Statement

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Model theory is a branch of mathematical logic where we study mathematical structures by making essential use of the language that underlies them. A *structure* is a set M equipped with a collection of distinguished subsets of various cartesian powers of M, called *basic relations*. We always assume that equality is a basic (binary) relation. For example, we could consider the real numbers $\mathbb{R}$ and declare as basic relations the order relation and the graphs of addition and multiplication. A set is *definable* in a given structure if it is obtained from the basic relations by taking finite unions, complements, finite cartesian products, and images and fibers of co-ordinate projections. In the example of $\langle \mathbb{R}, <, +, \cdot \rangle$, the definable sets are (by the Tarski-Seidenberg theorem) the *semi-algebraic* sets; namely, boolean combinations of sets defined by polynomial equations and inequalities. Model theory is concerned with the study of definable sets in a given structure. O-minimal structures provide a rigid framework to study real analytic and algebraic geometry. While broad enough to include structures that expand $\langle \mathbb{R}, <, +, \cdot \rangle$ by the graph of the exponential function (Wilkie [23]) or even by all globally sub-analytic sets (Dries-Macintyre-Marker [3]), o-minimality imposes a tameness condition on the definable sets thus favoring them with many of the good topological and geometric properties of the semi-algebraic sets. O-minimality is often regarded as the correct formalization of Grothendieck's 'topologie modérée' and, together with geometric stability theory, provide the two most successful trends of interaction between model theory and other areas of mathematics. The powerful model-theoretic techniques applied to known mathematical structures have yielded a range of exciting applications outside logic, such as Hrushovski's proof of the function field Modell-Lang conjecture in all characteristics [14] and Pila's recent solution of certain cases of the André-Oort Conjecture [20]. General survey articles on the subject can be found in the Notices of the AMS, such as Pillay [22].

In my research I have analyzed *groups* definable in various o-minimal structures and established their connection to real Lie groups. Parallel to transferring tools from algebraic topology in the o-minimal setting, I have introduced new techniques and developed a program that I plan to employ further in the analysis of sets and groups definable in more general structures, such as expansions of the real field with *o-minimal open core* and structures with *dependent theories*. This program has several aspects in common with the analysis of types from geometric stability theory and I envision it will develop into a powerful tool of model theory in the upcoming years.

1. Background on definable groups

A group is *definable* in a structure $\mathcal{M}$ if both its domain and the graph of its multiplication are definable in $\mathcal{M}$. Classical examples include algebraic groups, which are definable in algebraically closed fields, and compact real Lie groups, which are definable in o-minimal expansions of the real field. Definable groups have always been at the core of model theory, largely because of their prominent role in important applications of the subject, such as the aforementioned theorem by Hrushovski.

Groups definable in o-minimal structures strikingly resemble real Lie groups. The starting point for their study was Pillay's theorem [21] that every such group admits a definable manifold topology that makes it into a topological group. Since then, an increasing number of theorems reinforced this resemblance culminating in the solution of Pillay's Conjecture

(PC) and Compact Domination Conjecture (CDC) in recent years. (PC) can be viewed as a non-standard analogue of Hilbert's fifth problem. In its simplified form, it asserts that every definably compact group G admits a surjective homomorphism π onto a real Lie group, whose dimension (as a Lie group) is equal to the o-minimal dimension of G. (CDC) carries this connection further and implies that π induces a unique left-invariant *Keisler measure* on the collection of all definable subsets of G.

When the ambient o-minimal structure expands a real closed field, a rich machinery from *o-minimal algebraic topology* is at our disposal. For example, the triangulation theorem is known to hold in this case, giving rise to (co)homology theory. Using this heavy machinery, (PC) and (CDC) in this setting were solved in [16], [17] and [18]. The analysis brought into light new, far reaching developments for structures with dependent theories.

On the other hand, when the o-minimal structure does not expand a real closed field, the triangulation theorem fails. I illustrated a serious consequence of this failure in [5] with an example of a semi-linear group which cannot be definably, homeomorphically embedded in the affine space. In the absence of machinery from algebraic topology, I initiated a program of directly analyzing definable sets and groups in this case. The outcome has been:

- the solution of (PC) and (CDC) for *semi-linear groups* in [12] and [6], respectively.
- the solution of (CDC) for *semi-bounded groups* in [10]; (PC) was already solved in [19] by Peterzil.

In fact, strong structure theorems for these definable groups were obtained, with one aspect in common: each time a *lattice* was recovered capturing some significant invariant of the group at hand. We proceed to define the notion of a lattice. Let us fix a structure $\mathcal{M}$. A *locally definable group* is a group whose domain is a countable union of definable sets, and its multiplication, when restricted to a definable set, is a definable map. Accordingly, a *locally definable map* between locally definable groups is a map which, when restricted to a definable set, is definable. The following notion of a lattice grew out of my program of analyzing definable groups.

Definition 1.1 ([8])**.** Let $\mathcal{U}$ be a locally definable group and m an integer. A normal subgroup $L \subseteq \mathcal{U}$ is called a *lattice in $\mathcal{U}$ of rank m* if L is generated by m $\mathbb{Z}$-independent elements, and there are a definable group K and a surjective locally definable homomorphism $\phi : \mathcal{U} \to K$ whose kernel is L. We write $K = \mathcal{U}/L$.

An early theorem I proved with Edmundo is that for every abelian group G definable in an o-minimal structure $\mathcal{M}$, there is a divisible torsion-free locally definable group $\mathcal{U}$ and a lattice L in $\mathcal{U}$ such that $G = \mathcal{U}/L$. The group $\mathcal{U}$ is the *o-minimal universal cover* of G and the lattice L is isomorphic to the fundamental group of G. As general and useful this theorem proved to be, it does not provide any information about how $\mathcal{U}$ or L relate to $\mathcal{M}$. The original motivation for my structure theorems for definable groups was to recover an explicit connection between G and $\mathcal{M}$.

2. Description of main program, results and projects

The program of proving structure theorems for definable groups has been put into practice in my work in the semi-linear and semi-bounded settings. Its potential to apply to other settings became evident in the work described in Project 2 below. My proposal is to build a solid machinery for applying my program to questions of rather general nature, concerning definable groups and more.

When analyzing definable groups in an o-minimal structure, the foremost task in my analysis is to describe the local behavior of the group operation around *suitable* generic elements. The desired notion of genericity should appeal to the particular aspect of the structure we study. I have devised a machinery to define pregeometries that capture exactly that aspect. What is most significant in my definition is that it is given uniformly in a prefixed ideal of sets attached to the structure in question. This uniformity makes it possible to state a list of distinct goals that one might want to achieve in different contexts.

To start with, let $\mathcal{R} = \langle R, <, \dots \rangle$ be an o-minimal structure and fix an ideal $\mathcal{I}$ of distinguished subsets of R. Let $\mathcal{M} = \langle \mathcal{R}, \mathcal{I} \rangle$ be the expansion of $\mathcal{R}$ by all sets in $\mathcal{I}$. By 'definable' we mean 'definable in $\mathcal{M}$', whereas '$\mathcal{R}$-definable' means 'definable in $\mathcal{R}$'. We define the *$\mathcal{I}$-closure operator*

$$cl_{\mathcal{I}}(A) = \{a \in R : \text{there is a definable } X \in \mathcal{I} \text{ over } A \text{ that contains } a\}.$$

The desired goals are expressed via the following principles:

(P1) *$cl_{\mathcal{I}}$ defines a pregeometry*, yielding the corresponding notions of $cl_{\mathcal{I}}$-generic elements and $cl_{\mathcal{I}}$-dimension.

(P2) *Structural characterization of $cl_{\mathcal{I}}$-dimension*: the $cl_{\mathcal{I}}$-dimension of a definable set $X \subseteq M^n$ is the maximum k such that there are an $\mathcal{R}$-definable injective map $f : M^k \to M^n$ and definable sets $J_1, \dots, J_k \subseteq M$, not in $\mathcal{I}$, with $f(J_1 \times \cdots \times J_k) \subseteq X$.

Now let G be a definable group of $cl_{\mathcal{I}}$-dimension k.

(P3) *Local analysis*: every $cl_{\mathcal{I}}$-generic element of G is contained in a definable set $V \subseteq G$ of $cl_{\mathcal{I}}$-dimension k on which the group operation is the restriction of an $\mathcal{R}$-definable map.

(P4) *Structure Theorem*: $G = \mathcal{U}/L$, where
- $\mathcal{U}$ is a group extension of a definable group of $cl_{\mathcal{I}}$-dimension 0 by a locally definable group of $cl_{\mathcal{I}}$-dimension k whose operation is the restriction of an $\mathcal{R}$-definable map.
- L is a lattice in $\mathcal{U}$ of rank k.

Comment. There is an analogy of the proposed analysis with the analysis of types in geometric stability theory that I recently realized. Sets of $cl_{\mathcal{I}}$-dimension 0 can be thought of as *$\mathcal{I}$-internal* ones and Principle (P4) suggests a splitting theorem of the cover $\mathcal{U}$ of G into its $\mathcal{R}$-definable and '$\mathcal{I}$-internal components'. I am currently investigating the precise meaning of this connection.

Moreover, after defining my first $\mathcal{I}$-closure in the semi-bounded setting [7], I became aware of a similar notion in [1] intended for certain expansions of the real field. In Project 1 below I propose to apply my analysis to definable groups in that setting.

In the next two paragraphs I summarize how this program was successfully implemented in the semi-linear and semi-bounded settings. Then I list specific projects and objectives. The first three pertain to this program, whereas the rest concern problems in relevant areas.

Let $\mathcal{R} = \langle R, <, +, 0, \{d\}_{d \in D} \rangle$ be an ordered vector space over an ordered division ring D. Sets definable in $\mathcal{R}$ are called *semi-linear*. We let $\mathcal{I}$ be the ideal of all finite subsets of R. Then $\mathcal{M} = \mathcal{R}$ and $cl_{\mathcal{I}}$ is the usual *dcl* operator, for which (P1) and (P2) follow from the cell decomposition theorem. In my Ph.D. thesis I made substantial use of these properties and proved strengthened forms of (P3) and (P4): $\mathcal{U}$ is the universal cover of G and, assuming G is definably connected, $\mathcal{U}$ is a subgroup of $\langle R^k, + \rangle$. Two important consequences were the solutions of the conjectures (PC) and (CDC) mentioned in Section 1.

Let $\mathcal{R} = \langle R, <, +, 0, \{d\}_{d \in D} \rangle$ be as above. By the Trichotomy Theorem, a proper expansion $\mathcal{M}$ of $\mathcal{R}$ that does not expand a field must contain a definable field whose domain F

is a bounded interval in R. Let $\mathcal{I}$ be the collection of all subsets of R that are in definable bijection with F. Peterzil observed that $\mathcal{I}$ is an ideal. Clearly, $\langle \mathcal{R}, \mathcal{I}\rangle$ has the same definable structure with $\mathcal{M}$. Sets definable in $\mathcal{M}$ are called *semi-bounded.* In [7], I introduced $cl_{\mathcal{I}}$ and established (P1). I also proved a structure theorem for definable sets, answering a conjecture made by Peterzil, and obtained (P2) and (P3). In collaboration with Peterzil, I proved (P4) in [10], establishing a splitting theorem for covers of definable groups into their semi-linear and 'field-internal parts'. This splitting is optimal since we know it cannot hold for definable groups themselves. As an application, we solved (CDC) in this setting by reducing it to the semi-linear and field cases.

1. Prove (P1)-(P4) in expansions of real closed fields with o-minimal open core. These structures include dense pairs, as well as expansions of the real field $\overline{\mathbb{R}}$ by a subgroup of the multiplicative group satisfying the Mann Property, such as $\langle \overline{\mathbb{R}}, 2^{\mathbb{Q}}\rangle$. Let us describe the project for $\mathcal{M} = \langle \overline{\mathbb{R}}, 2^{\mathbb{Q}}\rangle$. This structure has a dependent theory. Consider the ideal $\mathcal{I}$ of all deifnable $X \subseteq \mathbb{R}$ for which there are no m and definable $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(X^m)$ contains an open interval in $\mathbb{R}$. In [1], (P1) was established for $cl_{\mathcal{I}}$. Quantifier elimination results known for $\mathcal{M}$ imply that $\mathcal{I}$ consists exactly of the $2^{\mathbb{Q}}$-*bound* subsets of $\mathbb{R}$, a notion from [2] whose definition is almost identical to that of '$2^{\mathbb{Q}}$-internal'. However, it is far from clear how one could use the existing literature to obtain (P2). Together with Günaydin, I plan to prove a structure theorem for definable sets as an intermediate step towards (P2), saying that every definable set is a finite *union* of sets of the form

$$\exists y\, P(y) \wedge \varphi_1(x_1, y) \wedge \cdots \wedge \varphi_n(x_n, y),$$

where each $\varphi_i(x_i, b)$ defines a definable subset of $\mathbb{R}$ not in $\mathcal{I}$. Similarly, no groups definable in this setting have been previously studied. I believe (P3) will follow easily from (P2). Currently, I am trying to extract a general procedure for proving (P4) given (P2) and (P3), guided by the successful techniques already developed in the semi-bounded setting. (P4) would provide a splitting theorem for covers of definable groups into their semi-algebraic and '$2^{\mathbb{Q}}$-bound parts'. For general structures with o-minimal open core, even less is known, and proving that a suitable $cl_{\mathcal{I}}$ defines a pregeometry would already be a result of innovative character. It is worth mentioning here that a similar analysis for definable groups in pairs of algebraically closed fields is also pending and perhaps even more direct to obtain from the known analysis of types in geometric stability theory.

2. Prove or disprove elimination of imaginaries (EI) for o-minimal structures. Surprisingly, my ideas recently found application in a somewhat different context. Let $\mathcal{R}$ be any o-minimal structure. If $\mathcal{R}$ expands an ordered group, then it has EI. In general, it is not known if $\mathcal{R}$ has EI after naming finitely many parameters. In joint work with Peterzil and Ramakrishnan [11], we let $\mathcal{I}$ be the ideal of all subsets of R that are finite unions of *group-intervals* and proved (P1), and (P2) for $k = n$. The outcome was tri-fold: every interpretable group is definable, every one-dimensional quotient can be eliminated, and every definable group lives in a product of group-intervals. I expect that proving the full (P2) will have an impact to deciding whether $\mathcal{R}$ has EI or not.

3. Axiomatize $\mathcal{I}$ and extend the program. It is natural to seek axioms for $\mathcal{I}$ that guarantee the validity of each of the principles (P1)-(P4). For example, such axioms should include $\mathcal{I}$ being closed under definable images, and that a definable family of sets in $\mathcal{I}$ parametrized by a set in $\mathcal{I}$ has its union in $\mathcal{I}$. Currently, I am testing the sufficiency of several lists of axioms and I believe that upon completion of my Project 1, I will be well-qualified to solicit the right one.

On the long term, I aspire to modify suitably my program and make it applicable to a larger range of problems. For example, one could ask whether we can analyze groups definable in $\mathbb{R}_{exp}$ in terms of their semi-algebraic and '*exp*-internal parts'. In order to meaningfully approach this question, I would first investigate, in general, what expansions $\langle \mathcal{R}, P\rangle$ of a structure $\mathcal{R}$ by a predicate P yield a robust notion of P-internality. For such $\langle \mathcal{R}, P\rangle$, I would let $\mathcal{I} = \bigcup_n \mathcal{I}_n$, where $\mathcal{I}_n$ is the ideal of all P-internal subsets of R^n. Finally, I would define $a \in cl_{\mathcal{I}}(A)$ if there are a number n, elements $a_1, \dots, a_{n-1} \in A$ and a definable set X in $\mathcal{I}_n$ with parameters in A, such that $(a, a_1, \dots, a_{n-1}) \in X$. I am eager to test these definitions on $\mathbb{R}_{exp}$ and check whether $cl_{\mathcal{I}}$ defines indeed a pregeometry.

4. Describe locally definable groups as covers of definable groups. By my early theorem with Edmundo in the o-minimal setting, every abelian definable group admits a universal cover $\mathcal{U}$. The conjecture that every abelian locally definable group $\mathcal{U}$, which is definably generated, is a cover of a definable group is of fundamental importance. Indeed, together with Peterzil [9], I proved that it is equivalent to the existence of a definable generic set in $\mathcal{U}$, as well as to the existence of $\mathcal{U}^{00}$. Moreover, it implies that $\mathcal{U}$ is divisible. I am currently working towards proving the conjecture. Combining my work with recent results by Berarducci-Mamino-Edmundo, I have reduced it to the question whether a single element in $\mathcal{U}$ exists such that the subgroup it generates is *compatible*; namely it intersects every definable subset of $\mathcal{U}$ at finitely many points.

5. Semi-linear geometry: prove embedding theorems and define homology. There are intriguing open geometric questions for a manifold X definable in $\mathcal{R} = \langle R, <, +, 0, \{d\}_{d\in D}\rangle$. The model theory of this structure becomes important when studying definable groups in algebraically closed valued fields [15]. Hrushovski and Loeser have asked for conditions on X that would ensure its definable homeomorphic embedding into the affine space. For a semi-linear group $\mathcal{U}/L$ of dimension two, I have found simple such conditions on the generators of L which I believe can generalize to arbitrary dimensions. Another question is whether one can develop homology in this setting. Together with Edmundo and Prelli [4], I proved stratification results for semi-linear sets which I plan to exploit as a substitute to the triangulation theorem.

6. Complex analytic geometry: prove Zilber dichotomy in the o-minimal setting. This project concerns understanding expansions $\mathbb{C}_S = \langle \mathbb{C}, +, S\rangle$ of the complex group by a set S definable in some o-minimal expansion $\mathcal{R}$ of the real field. A central question is whether Zilber Dichotomy holds in this setting: if $\mathbb{C}_S$ is strongly minimal and non-locally modular, then it is bi-interpretable to the complex field. By standard model-theoretic methods, it is enough to consider $S \subseteq \mathbb{C}^2$. Hasson-Kowalski [13] proved the dichotomy in case S is the graph of a map $f : \mathbb{C} \to \mathbb{C}$. An easy argument reduces it to the case where S is a finite union of definable (in $\mathcal{R}$) partial functions from $\mathbb{C}$ to $\mathbb{C}$. One could try to reduce this to the single function case, by constructing a $\mathbb{C}_S$-definable function as the 'average' of these functions 'with respect to a rational slope'. However, I have found examples where despite the non-local modularity of $\mathbb{C}_S$, all such averages are linear. I am trying to handle these examples separately. Another strategy would involve showing that one of the above functions is holomorphic on an open set. Together with Moosa, we view this approach as a special case of the more general understanding of various complex analytic notions in pure o-minimal terms, such as compact complex manifolds and Kähler manifolds. An independent short term objective is to characterize the semi-linear sets S such that $\mathbb{C}_S$ is strongly minimal; contrary to the content of Zilber dichotomy in the non-locally modular case, it is plausible that there are examples of such S that are not definable in the complex field.

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