Applications of o-minimality to some problems in Diophantine Geometry

Kobi Peterzil

Department of Mathematics
University of Haifa

Summer school in tame geometry
U. Konstanz July 2016

Some Bibliography

J. Pila and U. Zannier, *Rational points in periodic analytic sets and the Manin-Mumford conjecture,* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (2008), no. 2, 149-162.

J. Pila, O-minimality and the André–Oort conjecture for $\mathbb{C}^n$. Ann. of Math. (2), 172(3), 2011, 1779–1840.

Survey papers

T. Scanlon, *A proof of the André–Oort conjecture via mathematical logic [after Pila, Wilkie and Zannier]*, Sèminaire BOURBAKI Avril 2011 63ème année, 2010–2011, no 1037.

T. Scanlon, *Counting special points: Logic, diophantine geometry, and transcendence theory*, Bull. AMS (N.S.) 49 (2012), no. 1, 51 – 71.

An exercise

Setting

Let $\mathcal{C} =$ the family of all cosets of $\mathbb{C}$-linear subspaces of $\mathbb{C}^n$.

Let $\mathcal{S} \subseteq \mathcal{C}$ all cosets $H + b$, such that H has a basis in $\mathbb{Q}^n$ and $b \in \mathbb{Q}^n$. Call these **"special varieties"**.

Let $\mathcal{S}_0 =$ all 0-dimensional $X \in \mathcal{S}$ (note: $\mathcal{S}_0 = \mathbb{Q}^n$). Call these **"special points"**.

Problem

If $X \in \mathcal{C}$ and the special points are (Zariski) dense in X (i.e. $X \cap \mathcal{S}_0$ dense in X) then X is special (i.e. $X \in \mathcal{S}$).

Solution

An exercise. Also, find a "quantitative" assumption* on $X \cap \mathbb{Q}^n$ which ensures that X is special (e.g. $\#$ of points of height ... is ...).

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A general problem scheme

Setting

$\mathcal{C}$ = an underlying family of sets

$\mathcal{S} \subseteq \mathcal{C}$ a marked collection of so-called "special" $\mathcal{C}$-sets

$\mathcal{S}_0$ = a set of so-called "special" points, often these are the $\mathcal{S}$-sets of dimension zero.

The problem scheme

Start with an ambient $\mathcal{S}$-set V and consider an arbitrary $\mathcal{C}$-set $X \subseteq V$. Assume that X has "many" special points ($X \cap \mathcal{S}_0$ is Zariski dense in V).

Show that X contains a special set of positive dimension. Under additional assumptions, show that X itself is a special set.

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The Pila-Wilkie results (viewed in this scheme)

Fix $\mathcal{M} = \langle \mathbb{R}, <, +, \cdot, ... \rangle$ an o-minimal expansion of the real field.

$\mathcal{C} =$ the family of all definable sets in $\mathcal{M}$.

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$\mathcal{S}_0 =$ points in $(\mathbb{Q}^{alg})^n \cap \mathbb{R}^n$.

The Pila-Wilkie theorem(s)

Assume that $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{M}$. If $X \cap (\mathbb{Q}^{alg})^n$ is *large** then X contains a connected infinite semi-algebraic set defined over $\mathbb{Q}$. More precisely, if one removes **all** infinite connected semi-algebraic subsets of X then a *small** number of $\mathbb{Q}^{alg}$-points remains.

$X \cap (\mathbb{Q}^{alg})^n$ is **large*** if exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that

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From now on-the algebraic general problem scheme

The algebraic presentation

$\mathcal{C} =$ a family of complex algebraic (irreducible) varieties, (quasi) affine or projective.

$\mathcal{S} =$ a specified subfamily of "special" varieties.

$\mathcal{S}_0 =$ 0-dimensional $\mathcal{S}$-sets: special points.

$V =$ an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ an irreducible complex algebraic subvariety (so $X \in \mathcal{C}$)

Assumption

The special points ($X \cap \mathcal{S}_0$) are Zariski dense in X.

Goal

The variety X is special ($X \in \mathcal{S}$).

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A test case-the multiplicative group (algebraic torus)

The algebraic side

Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X$ an irreducible algebraic variety$\}$.

$\mathcal{S} = \{p * A : A$ a conn. algebraic subgrp of $\mathbb{G}_m^n$ & p a torsion point$\}$.

$\mathcal{S}_0 =$ Torsion points in $(\mathbb{G}_m)^n$

Goal-a theorem of Laurent-1984)

If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = p * A$ for some connected $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.

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If $X \subseteq (\mathbb{G}_m)^n$ an irreducible algebraic variety and $X \cap Tor(\mathbb{G}_m)^n$ is Zariski dense in X then $X = p * A$ for some connected $A \leqslant (\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$.

Namely,

If $X \in \mathcal{C}$ and $X \cap \mathcal{S}_0$ is Zariski dense in X then $X \in \mathcal{S}$.

A test case-the multiplicative group (algebraic torus)

The algebraic side

Let $V = (\mathbb{C}^*)^n = (\mathbb{G}_m)^n$ (so here V admits the structure of an algebraic group, which is also a complex Lie group).

$\mathcal{C} = \{X \subseteq (\mathbb{G}_m)^n : X \text{ an irreducible algebraic variety}\}$.

$\mathcal{S} = \{p * A : \; A \text{ a conn. algebraic subgrp of } \mathbb{G}_m^n \;\&\; p \text{ a torsion point}\}$.

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The Pila-Zannier strategy-vague description

First note: The field $\langle \mathbb{C}, +, \cdot \rangle$ is definable in $\mathbb{R}$, (via $\mathbb{C} \sim \mathbb{R}^2$). Hence, every complex algebraic variety is definable in the o-minimal structure $\bar{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot \rangle$.

But the strategy will force us to move to a different o-minimal structure:

An analytic presentation of the algebraic problem

As we'll see, in all cases there is a natural analytic covering map, call it $\Theta : \widetilde{V} \to V$, from an open set $\widetilde{V} \subseteq \mathbb{C}^n$ onto the algebraic variety V.

The idea-vague description

1. Using Θ, translate the algebraic problem from V to a problem about sets in $\widetilde{V}$, definable in some o-minimal expansion $\mathcal{M}$ of $\bar{\mathbb{R}}$.
2. Apply a Pila-Wilkie theorem in $\mathcal{M}$.
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Back to $(\mathbb{C}^*)^n$: the analytic presentation

- We have $V = (\mathbb{C}^*)^n = \mathbb{G}_m^n$.
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$$\Theta(\bar{z}) = exp(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n}).$$

- So $\Theta : (\mathbb{C}^n, +) \to ((\mathbb{G}_m)^n, *)$ is a holomorphic group homomorphism.
- Let $\Gamma := Ker(\Theta) = (2\pi i\mathbb{Z})^n$. So, Θ is Γ-invariant, namely:

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“Special analytic” points and varieties

$\widetilde{\text{special}}$ points

Call $\bar{z} \in \mathbb{C}^n$ a $\widetilde{\text{special}}$ point **if** $\Theta(\bar{z})$ is a torsion element (namely if $\Theta(\bar{z})$ is a special point). Let $\widetilde{\mathcal{S}_0}$ = all $\widetilde{\text{special}}$ points.

exp is a homomorphism, so $\Theta(\bar{z})$ is a torsion point of order k iff $k\bar{z} \in \Gamma$.

So, $\widetilde{\mathcal{S}_0} := \{\bar{z} \in \mathbb{C}^n : \exists k\ k\bar{z} \in (2\pi i\mathbb{Z})^n\} = (2\pi i\mathbb{Q})^n$.

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An irreducible **analytic** $Y \subseteq \mathbb{C}^n$ is $\widetilde{\text{special}}$ if $\Theta(Y) = p * A$, where A is a connected algebraic subgroup of $(\mathbb{G}_m)^n$ and $p \in Tor(\mathbb{G}_m)^n$. Namely, if $\Theta(Y) \in \mathcal{S}$.

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The multiplicative group-“weakly special”

Weakly special varieties

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And $\Theta(Y) = \Theta(H) * \Theta(\bar{z}) \subseteq (\mathbb{C}^*)^n$ is called **a weakly special** subvariety. It is (an arbitrary) coset of a conn. algebraic subgroup of $(\mathbb{C}^*)^n$

Note

If $Y \subseteq \mathbb{C}^n$ is weakly special then both Y and $\Theta(Y)$ are algebraic varieties (although $\Theta = exp$ is a transcendental map).

The multiplicative group-“weakly special”

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A key observation

If $Y \subseteq \mathbb{C}^n$ and $\Theta(Y) \subseteq (\mathbb{C}^*)^n$ are both algebraic varieties then necessarily Y is weakly special. (will prove it later in certain settings)

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Multiplicative group continues

Analytic presentation of Laurent's theorem

Assume that $Y \subseteq \mathbb{C}^n$ is an irreducible **analytic** variety, and $X = \Theta(Y) \subseteq (\mathbb{C}^*)^n$ is an algebraic variety, on which the torsion points are Zariski dense. Then Y is special, namely $Y = \bar{q} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$, and $\bar{q} \in (2\pi i\mathbb{Q})^n$.

The Pila-Zannier method

- Using the fact that $X \subseteq \mathbb{C}^n$ has many torsion points we shall conclude that $Y \subseteq (\mathbb{C}^*)^n$ has large*-number of "$2\pi i\mathbb{Q}^n$-points".
- Using Pila-Wilkie for Y, we shall conclude that Y contains an infinite semialgebraic subset of Y, and then also and algebraic subset A.
- Using ideas such as **the key observation** we conclude that A is special, so $\Theta(A)$ a special subvariety of X.
- With slightly more work, $\Theta(A) = X$.

Multiplicative group continues

Analytic presentation of Laurent's theorem

Assume that $Y \subseteq \mathbb{C}^n$ is an irreducible **analytic** variety, and $X = \Theta(Y) \subseteq (\mathbb{C}^*)^n$ is an algebraic variety, on which the torsion points are Zariski dense. Then Y is special, namely $Y = \bar{q} + H$, where H is a $\mathbb{C}$-linear subspace of $\mathbb{C}^n$ **defined over** $\mathbb{Q}$, and $\bar{q} \in (2\pi i\mathbb{Q})^n$.

The Pila-Zannier method

- Using the fact that $X \subseteq \mathbb{C}^n$ has many torsion points we shall conclude that $Y \subseteq (\mathbb{C}^*)^n$ has large*-number of "$2\pi i\mathbb{Q}^n$-points".
- Using Pila-Wilkie for Y, we shall conclude that Y contains an infinite semialgebraic subset of Y, and then also and algebraic subset A.
- Using ideas such as **the key observation** we conclude that A is special, so $\Theta(A)$ a special subvariety of X.
- With slightly more work, $\Theta(A) = X$.

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The multiplicative case I. the (non)definability of Θ

We have $\Theta : \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $\Theta(z_1, \ldots, z_n) = (e^{z_1}, \ldots, e^{z_n})$.

The difficulty

Because $\Gamma = ker\Theta \subseteq \mathbb{C}^n$ is infinite and discrete, the map Θ, as well as $\Theta^{-1}(X)$ **cannot be definable in any o-minimal structure.**

We thus need to "truncate" Θ:

Fundamental sets

A **fundamental set** for Θ, is a set $\mathfrak{F} \subseteq \mathbb{C}^n$, such that (1) $\mathfrak{F} + \Gamma = \mathbb{C}^n$ $(\Rightarrow \Theta(\mathfrak{F}) = (\mathbb{C}^*)^n.)$

and (2) Only finitely many Γ-translates of $\mathfrak{F}$ intersect $Cl(\mathfrak{F})$ (a technical requirement).

- Here is one:

$$\mathfrak{F} = \{\bar{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : 0 \leqslant |Im(z_i)| \leqslant \pi\}.$$

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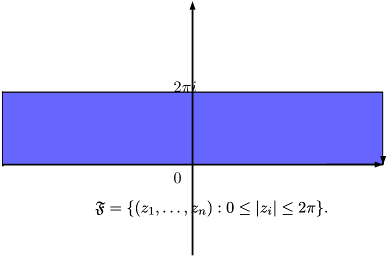
The multiplicative case: the definability of $\Theta \upharpoonright \mathfrak{F}$



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$\Theta \upharpoonright \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$:

We have $e^z = e^{x+iy} = e^x(cosy + i\, siny)$.

The map e^x is definable in $\mathbb{R}_{exp}$; the maps $cos, sin \upharpoonright [0, 2\pi]$ are definable in $\mathbb{R}_{an}$, hence:

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It follows that $\Theta \upharpoonright \mathfrak{F} = exp \upharpoonright \mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.
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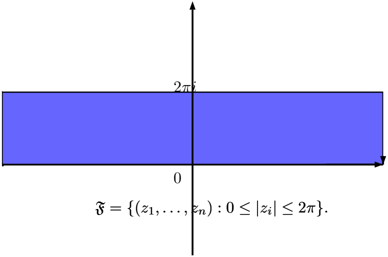

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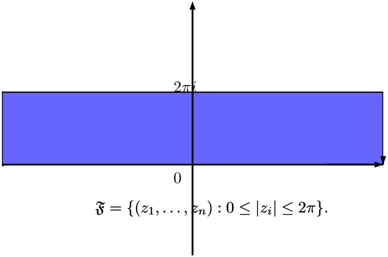

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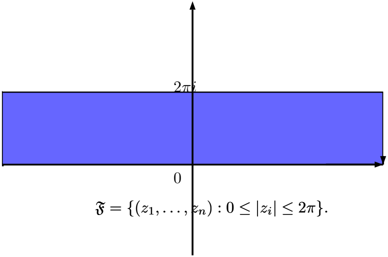

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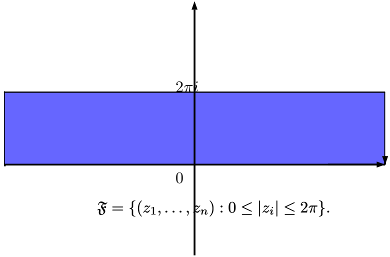

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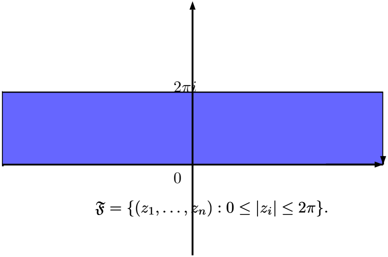

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The multiplicative case II. From infinite to “large”

We assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is infinite.

Claim The set $\Theta^{-1}(X) \cap 2\pi i\mathbb{Q}^n \cap \mathfrak{F}$ is large*:

Proof X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.

• Since $X \cap Tor(\mathbb{C}^*)^n$ is infinite there are natural numbers $m_1 < m_2 < \ldots$ and elements $g_i \in X$, with $ord(g_i) = m_i$.

• Assume $g = (g_1, \ldots, g_n) \in X \subseteq (\mathbb{C}^*)^n$, and $ord(g) = m$. Then each g_j is an d_j-th primitive root of unity, with $d_j | m$ and $l.c.m(d_j) = m$.
• (By some basic Galois theory) $[\mathbb{Q}(g) : \mathbb{Q}] = \phi(m)$, where $\phi(m) = \#\{k \leqslant m : (k, m) = 1\}$ is the Euler totient function.
• So g has $\phi(m)$ conjugates over $\mathbb{Q}$, all lying in X.

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We assume that $X \subseteq (\mathbb{C}^*)^n$ is an irreducible algebraic variety and that $X \cap Tor(\mathbb{C}^*)^n$ is infinite.

Claim The set $\Theta^{-1}(X) \cap 2\pi i\mathbb{Q}^n \cap \mathfrak{F}$ is large*:

Proof X is defined over a number field k. For simplicity, $k = \mathbb{Q}$.

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The multiplicative case III from algebraic to (weakly) special

The Pila-Wilkie input

- The analytic set $\Theta^{-1}(X) \subseteq \mathbb{C}^n$ contains an infinite semi algebraic set S.
- The Zariski closure of S is a complex algebraic subset of $\Theta^{-1}(X)$, of positive dimension.
- Take a maximal such irreducible algebraic set $A \subseteq \Theta^{-1}(X)$.
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Goal: A **is weakly special** $\overset{\frown}{=}$ a coset of a linear s.space of $\mathbb{C}^n$ over $\mathbb{Q}$.

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A proof of III using the classical Ax-Lindemann theorem

A-L Theorem

If $\xi_1, \ldots, \xi_n \in \mathbb{C}(A)$ and $lin.dim_{\mathbb{Q}}(\bar{\xi}/\mathbb{C}) = m$ then $tr.deg(\mathbb{C}(e^{\xi_1}, \ldots, e^{\xi_n})/\mathbb{C}) = m$.

Proof using A-L

- Take $H \subseteq \mathbb{C}^n$ a minimal subspace $/\mathbb{Q}$ with $A \subseteq H + p$ for $p \in \mathbb{C}^n$. Let $m = \dim H$.
- We have $\Theta(A) \subseteq \Theta(H) \star \Theta(p)$, and $\Theta(H) \leqslant (\mathbb{C}^*)^n$ algebraic.
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Summary of proof in the multiplicative case

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- Using number theory we concluded that $\Theta^{-1}(X)$ contains large*-many rational points.
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Recall the general problem scheme:
$\mathcal{C}$ = a family of complex algebraic (irreducible) varieties, (quasi) affine or projective.

$\mathcal{S}$ = a specified subfamily of "special" varieties.

$\mathcal{S}_0$ = 0-dimensional $\mathcal{S}$-sets: special points.

V = an irreducible $\mathcal{S}$-variety.

$X \subseteq V$ an irreducible complex algebraic subvariety (so $X \in \mathcal{C}$)

Assumption

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An analytic covering map

In all our settings we have $\widetilde{V}=$ a (semi-algebraic) open subset of $\mathbb{C}^n$ (with $n=\dim V$). And $\Theta:\widetilde{V}\to V$ a **holomorphic**, **transcendental,** surjection.

General strategy

Instead of V and $X\subseteq V$ consider $\widetilde{V}$ and the complex analytic subvariety $\Theta^{-1}(X)\subseteq\widetilde{V}$.

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In general, Θ and $\Theta^{-1}(X)$ are not definable in any "tame" structure. We will need to "truncate" it.

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An underlying group action

We have $G =$ a real algebraic group acting semi-algebraically and transitively on $\widetilde{V}$. In some cases $\widetilde{V} = G$.

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From special to $\widetilde{\textit{special}}$

An irreducible analytic subvariety $Y \subseteq \widetilde{V}$ **is called a $\widetilde{\text{special}}$ variety** if $\Theta(Y)$ is a special subvariety of V. In particular, $\Theta(Y)$ is algebraic (**!**).

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The ingredients for the Pila-Zannier method

We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\mathcal{S}_0 \subseteq V$ the set of special points.

I. Definability requirements (from algebraic to o-minimal)

One needs to establish the existence of a semialgebraic fundamental set $\mathfrak{F} \subseteq \widetilde{V}$ for Γ and the definability of $\Theta \restriction \mathfrak{F}$ in some o-minimal structure $\mathcal{M}$. In all examples, $\mathcal{M}$ is $\mathbb{R}_{an,exp}$.

For $X \subseteq V$ algebraic, let $\widetilde{X} \subseteq \widetilde{V}$ be an irreducible analytic component of $\Theta^{-1}(X)$. Note that $\widetilde{X} \cap \mathfrak{F} = (\Theta \restriction \mathfrak{F})^{-1}(X)$ is definable in $\mathcal{M}$.

II. Number theory goal

- The set $\widetilde{\mathcal{S}_0} = \Theta^{-1}(\mathcal{S}_0)$ is contained in $\mathbb{Q}_k^{alg}$ for some k.

▶ If $X \cap \mathcal{S}_0$ is Zariski dense in X then $\widetilde{\mathcal{S}_0} \cap (\widetilde{X} \cap \mathfrak{F})$ is large*. This is "the lower bound".

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II. Number theory goal

- The set $\widetilde{\mathcal{S}_0} = \Theta^{-1}(\mathcal{S}_0)$ is contained in $\mathbb{Q}_k^{alg}$ for some k.

▶ If $X \cap \mathcal{S}_0$ is Zariski dense in X then $\widetilde{\mathcal{S}_0} \cap (\widetilde{X} \cap \mathfrak{F})$ is large*. This is "the lower bound".

The ingredients for the Pila-Zannier method

We have $\Theta : \widetilde{V} \to V \sim \Gamma \backslash \widetilde{V}$. $\mathcal{S}_0 \subseteq V$ the set of special points.

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- Assume that we established that $\widetilde{\mathfrak{S}}_0 \cap (\widetilde{X} \cap \mathfrak{F})$ is large*.
- By PW, There exists a connected semi-algebraic nontrivial curve $C \subseteq \widetilde{X} \cap \mathfrak{F}$.
- Let $\overline{C} \subseteq \mathbb{C}^n$ be the Zariski closure of C. It is a complex algebraic curve, and by dimension considerations $(\overline{C} \cap \widetilde{V}) \subseteq \widetilde{X}$.
- So $\widetilde{X}$ **contains a complex algebraic curve** (relative to the open semialgebraic $\widetilde{V}$).

The ingredients of the Pila-Zannier method (cont)

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We have $\Theta : \widetilde{V} \to V \sim \Gamma\backslash\widetilde{V}$. $\widetilde{X} \subseteq \widetilde{V}$ a component of $\Theta^{-1}(X)$.

Ingredient III, the "Ax-Lindemann" goal

Assume that $\widetilde{A}$ is a maximal irreducible algebraic (relative to $\widetilde{V}$) subset of $\widetilde{X}$.

Then $\widetilde{A}$ is a weakly special variety. Namely,

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Application of the Pila-Wilkie Theorem.

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If $\widetilde{A} \subseteq \Theta^{-1}(X)$ is maximal irreducible algebraic then it is weakly special. (So, if in addition $\widetilde{A} \cap \widetilde{\mathcal{S}_0} \neq \varnothing$ then $\widetilde{A}$ is special).

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Another application of the Pila-Zannier method: The Manin-Mumford conjecture

Background: Abelian varieties

- An (complex) **abelian variety** is a smooth projective algebraic variety $V \subseteq \mathbb{P}^n(\mathbb{C})$, together with an algebraic binary operation $*$, which makes $\langle V, * \rangle$ an algebraic group.
- In complex dimension one, these are non-singular elliptic curves: $y^3 = x^2 + ax + b$ (the affine equation).
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Important properties of abelian varieties

- The group $(V, *)$ is (indeed) abelian, written as $(V, +)$ from now on.
- (over $\mathbb{C}$) The group $(V, +)$ admits the structure of a complex Lie group. Since $\mathbb{P}^n(\mathbb{C})$ is compact that group is compact.
- As a real Lie group, $(V, +)$ is isomorphic to a direct product of $(S^1)^{2m}$, where $m =$ complex dimension of V.
- For every $m \in \mathbb{N}$, the m-torsion subgroup of V is $(\mathbb{Z}/m\mathbb{Z})^{2n}$.

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The setting

V = an abelian variety in $\mathbb{P}^n(\mathbb{C})$, written additively $(V, +)$.

$\mathcal{C}$ = all irreducible algebraic subvarieties of V.

$\mathcal{S}$ = all cosets of the form $A + p$, where $p \in Tor(V)$ and A a connected algebraic subgroup (i.e. abelian subvariety) of V.

$\mathcal{S}_0 = Tor(V)$ the torsion elements of the group $(V, +)$.

The Manin-Mumford conjecture (Raynaud's Theorem, 1983)

Assume that V is a complex abelian variety defined over a number field, and $X \subseteq V$ an irreducible algebraic subvariety. If $X \cap Tor(V)$ is Zariski dense in V then $X = A + p$ as above.

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The analytic presentation

- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \to V$.
- $\Gamma := Ker(\Theta)$ is a $2n$-lattice. I.e., $\Gamma = \Sigma_{i=1}^{2n} \mathbb{Z}\omega_i$, where $\omega_1, \ldots, \omega_{2n}$ are linearly independent over $\mathbb{R}$.
(Note: While every $2n$-lattice gives rise to a complex torus, it might not give rise, if $n > 1$, to an **projective** complex torus, i.e. abelian variety.)
- **special points** $= \Theta^{-1}(Tor(V)) = \mathbb{Q}\Gamma = \Sigma_{i=1}^{2n} \mathbb{Q}\omega_i$.
- **special varieties** are cosets of the form $\bar{z} + H$, where H a complex linear subspace defined over $\mathbb{Q}$ and $\bar{z} \in \mathbb{Q}\Gamma$.
- **Weakly special varieties** are arbitrary cosets of such H.

(weakly) special varieties as orbits

The weakly special varieties are exactly those orbits (i.e., cosets) of real subgroups of $(\mathbb{C}^n, +)$ which project onto algebraic subvarieties of V.

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- There exists a holomorphic group homomorphism $\Theta : (\mathbb{C}^n, +) \to V$.
- $\Gamma := Ker(\Theta)$ is a $2n$-lattice. I.e., $\Gamma = \Sigma_{i=1}^{2n} \mathbb{Z}\omega_i$, where $\omega_1, \ldots, \omega_{2n}$ are linearly independent over $\mathbb{R}$.
(Note: While every $2n$-lattice gives rise to a complex torus, it might not give rise, if $n > 1$, to an **projective** complex torus, i.e. abelian variety.)
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The Pila-Zannier method for Manin-Mumford

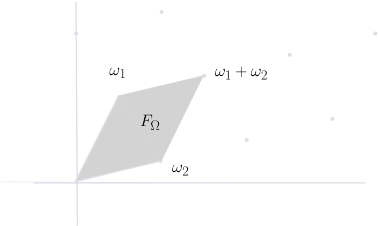
I. The fundamental set and definability of $\Theta \restriction \mathfrak{F}$

Consider the compact semilinear parallelogram $\mathfrak{F} = \{\sum_{i=1}^{2n} t_i\omega_i : 0 \leqslant t_i \leqslant 1\}$. Then:

(i) $\Gamma + \mathfrak{F} = \mathbb{C}^n$.

(ii) The set $\{\gamma \in \Gamma : (\gamma + \mathfrak{F}) \cap \mathfrak{F} \neq \varnothing\}$ is finite.

$\mathfrak{F}$ is a fundamental set for Θ.



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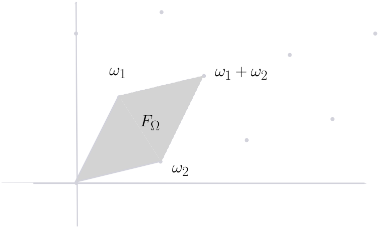

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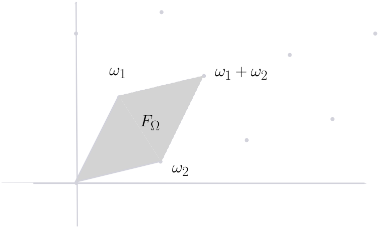

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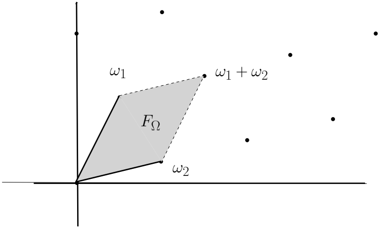

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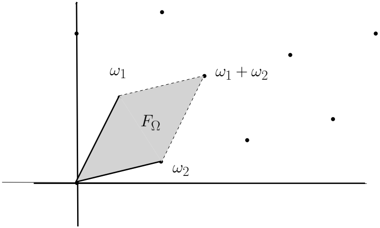

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- V is an abelian variety defined over a number field F.
- $X \subseteq V$ is irreducible algebraic, with $X \cap Tor(V)$ Zariski dense in X.
- So, X is also defined over a number field $k \supseteq F$.

Number theoretic input (Masser)

There exists $\rho = \rho(V) > 0$ and a constant c, such that for every $p \in V$, if $ord(p) = T$ then $[F(p) : \mathbb{Q}] \geqslant cT^{\rho}$.

By conjugating $X \cap Tor(V)$ over k we conclude: if $\epsilon < \rho(V)$ then

$$limsup_T \frac{|\{P \in X : ord(P) \leqslant T\}|}{T^{\epsilon}} = \infty.$$

Conclusion: on the analytic side

The set $\{(q_1, \ldots, q_{2n}) \in \mathbb{Q}^{2n} : \Sigma_{j=1}^{2n} q_j \omega_i \in \Theta^{-1}(X) \cap \mathfrak{F}\}$ is **large***.

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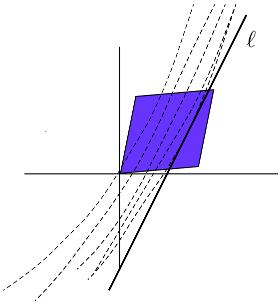
The Pila-Wilkie input

The analytic variety $\Theta^{-1}(X)$ contains an unbounded semialgebraic curve σ.

By the o-minimality of σ, when we translate it into $\mathfrak{F}$ by elements of Γ we get (inside $\widetilde{X}$) curves which are more and more “linear”. Since $\widetilde{X} \cap \mathfrak{F}$ is compact, at the limit we get an affine line $\ell \subseteq \widetilde{X}$.



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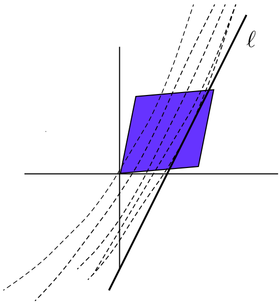

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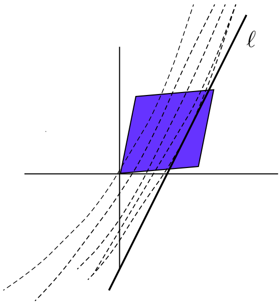

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On the analytic side

We saw that $\Theta^{-1}(X)$ contains a real affine line $\ell \subseteq \mathbb{C}^n$.

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The general analytic setting for Shimura varieties (simplified)

- $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.
- $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This is our $\widetilde{V}$.
- $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$, and consider the quotient $V = \Gamma\backslash\widetilde{V}$.

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There exists a holomorphic embedding $\Theta : \Gamma\backslash\widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

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- $G(\mathbb{R})$ is the $\mathbb{R}$-points of an algebraic semisimple group G over $\mathbb{R}$.
- $K \leqslant G(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$.
- (with additional assumptions) the quotient space $G(\mathbb{R})/K$ admits the structure of an open semi-algebraic subset of $\mathbb{C}^n$. This is our $\widetilde{V}$.
- $G(\mathbb{R})$ acts on $\widetilde{V}$. Actually, for every $g \in G(\mathbb{R})$, $g : \widetilde{V} \to \widetilde{V}$ is a biholomorphism.
- Let $\Gamma = G(\mathbb{Z})$, and consider the quotient $V = \Gamma \backslash \widetilde{V}$.

The Baily-Borel Theorem

There exists a holomorphic embedding $\Theta : \Gamma \backslash \widetilde{V} \to \mathbb{P}^m(\mathbb{C})$ whose image is a quasi-projective variety.

$Im(\Theta) = V$ is ***a Shimura variety*** (a non-specialist viewpoint).

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We start with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$.

The group $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ (transitively) as follows:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\tau \in \mathbb{H}$ then $A \cdot \tau = \frac{a\tau+b}{c\tau+d}$.

Connection to elliptic curves

$\mathbb{H}$ is a parameter space for elliptic curves, namely, every τ represents the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$ where Λ_τ the lattice $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$.

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Special varieties and points

Again, the definition begins on the analytic side.

Definition of special points: The set $\widetilde{\mathbb{S}}_0$

$(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ is **special**, if for every i, the elliptic curve E_{τ_i} has complex multiplication ($End(E_\tau) \neq \mathbb{Z}$).
Equivalently, τ_i belongs to an imaginary quadratic extension of $\mathbb{Q}$.
(abstract definition of special points in Shimura varieties-omitted here).

Definition of special varieties

Recall: An irreducible analytic variety $Y \subseteq \mathbb{H}^n$ is **special** if
(i) Y is an orbit of a real algebraic group $H \leqslant SL(2, \mathbb{R})^n$.
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The Image under Θ of $\widetilde{\text{special}}$ variety is **special** in $\mathbb{C}^n$.

Examples of special varieties

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$\Theta(\widetilde{X}) = Z(\Phi_N) \times \mathbb{C}^{n-2}$ where Φ_N is the zero set of a modular polynomial.

Moonen's work

Every special variety in $\mathbb{C}^n$ is obtained from the above examples by permutation of variables and cartesian products.

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Moonen's work

Every special variety in $\mathbb{C}^n$ is obtained from the above examples by permutation of variables and cartesian products.

Special varieties and points in $V = \mathbb{C}^n$

The image under Θ of a $\widetilde{\text{special}}$ point is **special** in $\mathbb{C}^n$. $\mathcal{S}_0 := \Theta(\widetilde{\mathcal{S}}_0)$.

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The statement of theorem

The André-Oort Conjecture for $\mathbb{C}^n$ (a theorem of Pila)

If $X \subseteq \mathbb{C}^n$ is an irreducible algebraic variety and $X \cap \mathcal{S}_0$ is Zariski dense in X then X is special.

Notice that by the nature of the definitions, we immediately have an analytic presentation of the problem:

- We have $\Theta : \mathbb{H}^n \to \mathbb{C}^n$ given by the J function in each coordinate.
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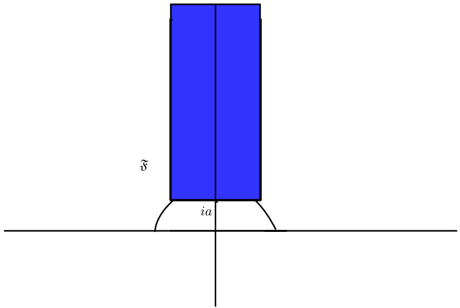
The Pila Zannier method: I. The fundamental set

By the basic theory of elliptic curves, the following is a fundamental set for $SL(2,\mathbb{Z})$ (for every $0 < a < \sqrt{3}/2$):

$$\mathfrak{F} = \{z \in \mathbb{H}; -1/2 \leqslant Re(z) \leqslant 1/2 \,\&\, Im(z) > a\}.$$

So $\mathfrak{F}^n$ is a fundamental set for $SL(2,\mathbb{Z})^n$.



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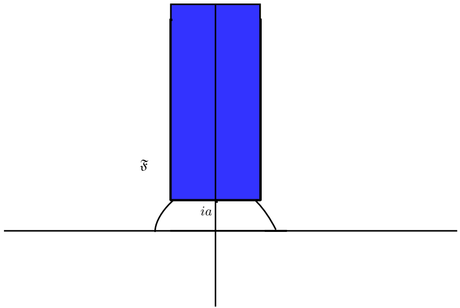
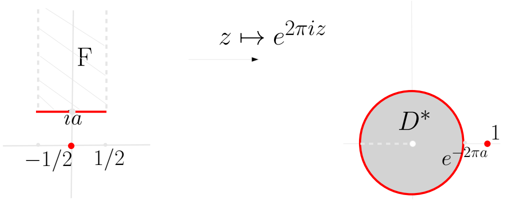


Pila-zanner method I: Definability of $J \restriction \mathfrak{F}$

Theorem

The restriction of J to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

Proof Consider first the map $z \mapsto e^{2\pi iz}$. It sends $\mathfrak{F}$ onto a punctured disc D^*. The “point” $Im(z) = \infty$ is sent to 0.



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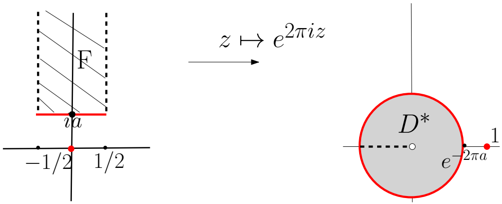

The proof continues

Because J is $\mathbb{Z}$-periodic ($z \mapsto z+1 \ \in SL(2,\mathbb{Z})$) it factors through $e^{2\pi iz}$.

$$\begin{tikzcd}
\mathfrak{F} \arrow[d, "e^{2\pi iz}"'] \arrow[dr, "J"] & \\
D^* \arrow[r, "\widetilde{J}"] & \mathbb{P}(\mathbb{C})
\end{tikzcd}$$

As before, the restriction of

$$e^{2\pi iz} = e^{2\pi i(x+iy)} = e^{-2\pi y}(cosx + isinx)$$

to $\mathfrak{F}$ is definable in $\mathbb{R}_{an,exp}$.

It is known that as $Im(z) \to +\infty$, $J(z) \to +\infty$. Hence, $\widetilde{J}(q)$ tends to ∞ as $q \to 0$ in D^*, so $\widetilde{J}$ is meromorphic on the punctured disc. Hence, $\widetilde{J}$ is definable in $\mathbb{R}_{an}$.

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Let $\widetilde{X} \subseteq \mathbb{H}^n$ be an irreducible **analytic** component of $\Theta^{-1}(X)$.
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Using a theorem of Siegel on imaginary quadratic fields, Pila proves:

Largness of special points

The set $\widetilde{\mathcal{S}}_0 \cap \widetilde{X} \cap \mathfrak{F}$ is large*.

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II. Number Theory

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III. The Ax-Lindemann statement

The Pila-Wilkie input

$\widetilde{X}$ contains an algebraic set of positive dimension (relative to $\mathbb{H}^n$). Let A be maximal irreducible such set.

Goal

A is weakly special. Namely
(i) it is the orbit of a real algebraic subgroup of $SL(2,\mathbb{R})^n$, and
(ii) $\Theta(A)$ is algebraic.

Ax-Lindemann for $\mathbb{H}^n$ (third type of proof)

We have $\widetilde{X} \subseteq \mathbb{H}^n$ an analytic irreducible component of $\Theta^{-1}(X)$ and $A \subseteq \widetilde{X}$ is a maximal, relatively algebraic subset, of positive dimension. Namely, there exists an algebraic $\bar{A} \subseteq \mathbb{C}^n$ such that $A = \bar{A} \cap \mathbb{H}^n$.

Write $G := SL(2, \mathbb{R})^n$, and $\Gamma = SL(2, \mathbb{Z})^n$.

Without loss of generality $\dim(A \cap \mathfrak{F}) = \dim A$ (if not, replace $\widetilde{X}$ and A by $\gamma\widetilde{X}$ and γA, for some $\gamma \in \Gamma$).

Fact A is not contained in finitely many Γ-translates of $\mathfrak{F}$.

WHY?

Otherwise $A \subseteq \bigcup_{i=1}^{k} \gamma_i \mathfrak{F}$. Because the real part of $\mathfrak{F}$ is bounded, it follows that $Re(z)$ is bounded for $z \in \bar{A} \cap \mathbb{H}^n$. This would imply (?) that A must be compact. But a compact complex analytic subset of $\mathbb{H}^n$ is finite. Contradiction.

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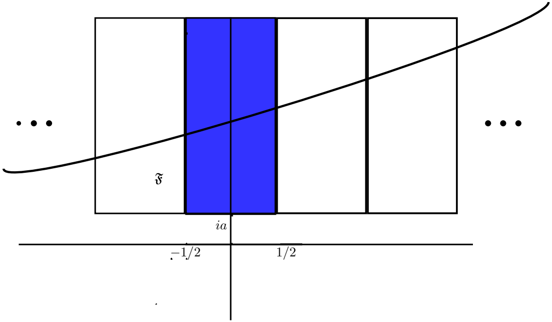
We showed that A is not contained in finitely many Γ-translates of $\mathfrak{F}$



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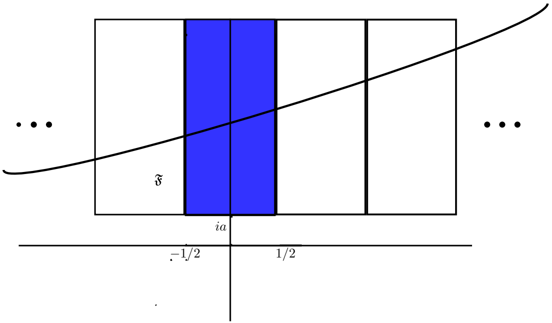

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The $\{\gamma \in SL(2,\mathbb{Z})^n : \gamma \in G(A)$ is large*.

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Consider the real algebraic group $Stab_G(A) \subseteq G$. It is thus infinite and contains infinitely many Γ points (by a finer use of Pila-Wilkie).

Let H be the Zariski closure of $G(A) \cap \Gamma$. It is a real algebraic group defined over $\mathbb{Q}$ which stabilizes A. Using induction and decomposition of Shimura varieties, one can show that A is an orbit of H and that $\Theta(A)$ is algebraic, hence A is weakly special.
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Let H be the Zariski closure of $G(A) \cap \Gamma$. It is a real algebraic group defined over $\mathbb{Q}$ which stabilizes A. Using induction and decomposition of Shimura varieties, one can show that A is an orbit of H and that $\Theta(A)$ is algebraic, hence A is weakly special.
It follows that X contains a special variety. End of Pila's proof.

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I. Definability: P-Starchenko.

II. Number Theory: Uses results of Tsimerman.

III. A-L: using strongly the low dimension of $\mathcal{A}_2$ ($\dim \mathcal{A}_2 = 3$).

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Recent work of Klingler, Yafaev and Ullmo (2013)

I. The restriction of the Baily-Borel embedding of Shimura varieties to Siegel fundamental sets is definable in $\mathbb{R}_{an,exp}$ (!).
III. Ax-Lindemann holds for arbitrary Shimura varieties.

what is missing?

The number Theory part

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