Distality in Pairs

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The concept of **distality** was introduced by Simon to attempt to classify *purely unstable* behavior in an NIP theory. This definition is a strong contrast to being generically stable.

Setup

Throughout all theories are complete and dependent (NIP), $\mathcal{U}$ will serve as a monster model with underlying set U. All sets and indiscernible sequences are small relative to the saturation and homogeneity of $\mathcal{U}$. While we will not concern ourselves too directly with multiple sorted structures, however all concepts can be straightforwardly adapted. We will sometimes abuse notation and consider realizations of global types.

Distality, ctd.

Definition

Let $\pi(x)$ be a partial type over A. We say that $\pi(x)$ is *distal* (over A) if for every indiscernible sequence $(a_i)_{i\in\mathbb{Q}+(c)+\mathbb{Q}}$ from $\pi(U)$ and $b \in U$:

$$(a_i)_{i\in I_1+I_2} \text{ is } b-\text{indiscernible} \Leftrightarrow (a_i)_{i\in I_1+(c)+I_2} \text{ is } b-\text{indiscernible}$$

Fact

In the fact $\mathbb{Q}$ can be any infinite linear order without endpoints.

Definition

A theory T is *distal* if the partial type $x = x$ is distal.

Generically Stable Types

A distal type is in essence the strongest contrast to a generically stable type.

Definition

Let $x = (x_1, \ldots, x_n)$ be a tuple of variables. A type $p(x) \in S(U)$ is *generically stable* if there is a small $A \subset U$ such that the following occur:

1. (p is definable over A) For each $\varphi(x; b) \in p$, the set $\{b' : \varphi(x, b') \in p\}$ is A-definable.
2. (p is finitely satisfiable in A) For each $\varphi(x; b) \in p$, there is $a \in A$ such that $\mathcal{U} \models \varphi(a, b)$.

Fact

If p is generically stable, then any Morley sequence of p is totally indiscernible.

Our Examples Today

Let $\mathcal{A}$ be an o-minimal $\mathcal{L}$-structure expanding an ordered group. We will be considering (theories of) structures of the form $(\mathcal{A}, B)$ where B is a unary predicate. When relevant, T will refer to the original o-minimal theory of $\mathcal{A}$ and T_P for the pair. We are concerned with the following examples where:

1. $\mathcal{A}$ is the real field and B is a cyclic multiplicative subgroup of $\mathbb{R}_{>0}$ (discrete subgroup),
2. $\mathcal{A}$ expands a real closed field, B is an elementary substructure such that there is a unique way to define a standard part map from A to B (tame pair),
3. B is the universe of a proper elementary substructure of $\mathcal{A}$ (dense pair),
4. $\mathcal{A}$ is the real field and B is a dense multiplicative subgroup of $\mathbb{R}_{>0}$ with the Mann property (dense subgroup),
5. B is a dense $\mathrm{dcl}_{\mathcal{L}}$-independent set (independent pair).

Distal and non-Distal Pairs

Theorem (Hieronymi, N.)

The theories of discrete subgroups and tame pairs are distal, while the theories of dense pairs, dense subgroups, and independent pairs are non-distal.

The proof for discrete subgroups uses a criterion for expansions by a function symbol later used by Gehret and Kaplan to prove distality for the asymptotic couple of $T_{\log}$. For tame pairs it follows from dp-minimality by a result of Simon. For the dense examples, the picture is more interesting.

Definition

Let $\mathcal{M} \models T$. A *Keisler Measure* on $\mathcal{M}$ is a finitely additive probability measure on the Boolean algebra of M-definable subsets of M.

Definition

Let μ be a Keisler measure on $\mathcal{M}$. We say μ is *smooth* (over $\mathcal{M}$) if for each $\mathcal{N} \succeq \mathcal{M}$ there is a unique extension of μ to a Keisler Measure on $\mathcal{N}$.

Fact

Considering a type as a $\{0,1\}$-valued probability measure, a smooth type is a realized type.

Keisler Measures, Continued

Definition

Let μ be a Keisler measure on $\mathcal{U}$. We say that μ is *generically stable* if there is a small $\mathcal{M} \prec \mathcal{U}$ such that:

1. (μ is definable over M) For each $\epsilon > 0$ and definable $R \subseteq U \times U_y$, there is an M-definable partition $S_1, \ldots, S_n$ of U_y such that for each $i \in \{1, \ldots, n\}$ and $b, b' \in S_i$, $|\mu(R(b)) - \mu(R(b'))| < \epsilon$.
2. (μ is finitely satisfiable in M) For each definable $S \subseteq \mathcal{U}$, if $\mu(S) > 0$ then $S \cap M \neq \emptyset$.

Fact

This agrees with the definition for generic stability of a type if one replaces $p(x)$ with the corresponding $\{0, 1\}$-valued Keisler measure.

Distality and Measures

Definition

A theory T is *distal* if for each $\mathcal{M} \models T$ all generically stable Keisler Measures on $\mathcal{M}$ are smooth over $\mathcal{M}$.

Simon showed this is equivalent to the indiscernible sequence definition. Furthermore, he also showed that if T is distal, then T^{eq} is distal. Thus to show a theory is not distal, it suffices to find an unrealized generically stable type in T_P^{eq}.

Large Dimension and Small Closure

In recent work, Eleftheriou, Günaydin, and Hieronymi introduce a notion of *large dimension* that makes sense in a class of structures including our dense examples. They also prove equivalence with the rank coming from a natural pre-geometry.

Definition

For $S \subseteq U$, we define the *small closure* of S, $\mathrm{scl}(S)$, by $\mathrm{dcl}_{\mathcal{L}}(S \cup P(U))$.

We then will consider the large dimension of an A-definable set S to be its scl-rank over A. For unary sets, dimension 0 will be called *small*, while dimension 1 sets are *large*.

A Generically Stable Type

In the examples of dense pairs and dense subgroups, the group operation allows one to find an interval $(a, b) \subseteq U \cup \{\pm\infty\}$ and a definable equivalence relation E with small, dense clases. If $S \subseteq U$ is a union of E classes, we call it E-invariant.

Theorem (N.)

The collection of large, E-invariant definable sets form an ultrafilter on the Boolean Algebra of E-invariant definable sets. The corresponding type $q(y)$ on U/E is generically stable.

This shows that the examples of dense pairs and dense subgroups are non-distal. Furthermore, this allows us to answer a question of Simon in these examples.

A Question of Simon

Following the determination that the dense examples were not distal, Simon asked whether there is a family of generically stable types (in $\mathcal{U}^{\mathrm{eq}}$ such that if $p(x)$ is orthogonal to these, then it is distal. We answer this positively.

Theorem (N.)

A type $p(x) \in S(\mathcal{U})$ is distal if and only if it is weakly orthogonal to $q(y)$. That is, $p(x)$ is distal if and only if $p(x) \cup q(y)$ implies a complete (x, y)-type in $\mathcal{U}^{eq}$.

Theorem (N.)

A type $p(x) \in S(\mathcal{U})$ is weakly orthogonal to $q(y)$ if and only if there is $\varphi(x, b) \in p(x)$ such that $\varphi(U, b)$ is small.

The equivalence of smallness and distality also holds for independent pairs.

Distal Expansions

The reduct of a distal theory need not be distal, as the theory of equality is stable. However, distality has some combinatorial consequences that are preserved under reducts.

Definition

Let X, Y be sets and $R \subset X \times Y$. A pair $A \subseteq X$, $B \subseteq Y$ is said to be R-homogeneous if either $A \times B \subseteq R$ or $(A \times B) \cap R = \emptyset$.

Theorem (Chernikov, Starchenko)

Let $\mathcal{M}$ be a distal structure, and let $R \subseteq M^n \times M^m$ a definable relation. Then there is a constant $\delta = \delta(R)$ such that for any generically stable measures μ_1 and μ_2 on M^n and M^n respectively, there are definable sets $A \subseteq M^n$ and $B \subseteq M^m$ with $\mu_1(A) > \delta$ and $\mu_2(B) > \delta$, such that the pair A, B is R-homogeneous.

Theorem (N.)

The structure $(\mathbb{R}; +, <, \mathbb{Q})$ *admits a distal expansion.*

This follows from expanding by a relation $\prec$ on $\mathbb{R}/\mathbb{Q}$ that makes $\mathbb{R}/\mathbb{Q}$ into an ordered $\mathbb{Q}$-vector space.

Theorem (Gehret, N.)

The theory of independent pairs of ordered abelian groups admits a distal expansion.

Proof Sketch of Gehret, N.

- Each $r \in \mathbb{R}$ can be uniquely written as $q_1 h_1 + \ldots + q_m h_m$ where $h_1 < \ldots < h_m$ and each $q_i \in \mathbb{Q}$.
- Thus we can also view $\mathbb{R}$ as $\bigoplus_{h \in H} h\mathbb{Q}$.
- This gives a natural valuation by $v(r) = h_1$, where h_1 is the same as earlier.
- Furthermore, we define an ordering $<_1$ by $r >_1 0$ if $q_1 > 0$.
- We then show the resulting structure $(\mathbb{R}; +, <, H, v, <_1)$ has a distal theory.