TAME EXPANSIONS - PROBLEMS

Problem 1. Consider the structure $(\mathbb{R}, <, +, -, 0, 1, (f_q)_{q\in\mathbb{Q}}, \lambda)$, where $f_q : \mathbb{R} \to \mathbb{R}$ maps x to qx and $\lambda : \mathbb{R} \to \mathbb{R}$ denotes the greatest integer function on $\mathbb{R}$, that is $\lambda(x) = \max(\infty, x] \cap \mathbb{Z}$. Show that the theory of this structure has quantifier elimination.

Problem 2. Show that every Borel sets is definable in $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.

Problem 3. Show that the theory of $(\mathbb{R}, <, +, -, 0, 1, \mathbb{Q})$ admits quantifier elimination. What happens if you replace $\mathbb{Q}$ by a Hamel basis (ie a basis for the $\mathbb{Q}$-vector space $\mathbb{R}$)?

Problem 4. (Open) Let $f_r : \mathbb{R} \to \mathbb{R}$ maps x to rx. Let $r \in \mathbb{R} \setminus \mathbb{Q}$. What can you say about the definable sets in $(\mathbb{R}, <, +, -, 0, 1, f_r, \mathbb{Q})$?

Problem 5. (Open) Let $\mathcal{R}$ be an expansion of a ordered field such that every bounded definable subset of $\mathcal{R}$ has a supremum. Let $X \subseteq \mathcal{R}$ be a definable set that is closed, bounded and discrete, and let $f : X \to X$ be definable and injective. Is f surjective?

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