Definability-theoretic dividing lines

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Throughout this talk, we will consider expansions of

- $(\mathbb{R}, <, +)$,
- $(\mathbb{R}, <, +, x \mapsto \lambda x$ for each $\lambda \in \mathbb{R})$.
- $\overline{\mathbb{R}} := (\mathbb{R}, <, +, \cdot)$.

Motivating goal. Classify such expansions!

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A disclaimer. Two perspectives on first-order expansions of $(\mathbb{R}, <, +)$:

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More realistic goal. Classify such expansions according to the geometric/topological complexity/tameness of its definable set?

This is an instance of a larger program:

'A lot of model theory is concerned with discovering and charting the "tame" regions of mathematics, where wild phenomena like space filling curves and Gödel incompleteness are absent, or at least under control. As Hrushovski put it recently: Model Theory = Geography of Tame Mathematics.' - Lou van den Dries

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So what is tameness precisely?

'The notion of tameness carries with it some judgement by the user as to what constitutes acceptable behavior of the definable sets, as well as what constitutes acceptable knowledge of the behavior.' - Chris Miller

There are large number of expansions of $(\mathbb{R}, <, +)$ known to be tame:

- $(\mathbb{R}, <, +)$, $(\mathbb{R}, <, +, \cdot)$, $\mathbb{R}_{an}$, (o-minimal)
- $(\mathbb{R}, <, +, \mathbb{Z})$, (locally o-minimal)
- $(\mathbb{R}, <, +, \mathbb{Q})$, (o-minimal open core)
- $(\mathbb{R}, <, +, \cdot, 2^{\mathbb{Z}})$, (d-minimal)
- $(\mathbb{R}, <, +, \cdot, 2^{\mathbb{Z}}3^{\mathbb{Z}}, 2^{\mathbb{Z}})$, (d-minimal open core)
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Compare to this, the expansion of $(\mathbb{R}, <, +)$ by predicates for every compact subset of $\mathbb{R}^n$ - call it $\mathbb{R}_{cph}$. It defines every projective subset of $[0, 1]^k$ for every $k \in \mathbb{N}$.

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For future reference: $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$ defines all compact sets.

Observation: There is a large class of structures whose definable sets can be understood by model-theoretic (tame topology) methods. But there are also structures whose definable sets can only be studied in terms of descriptive set theory.

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In the rest of the talk, I will discuss Condition 3 and 4.

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In our setting: Classify (non-elementary) classes of expansions of the real field by objects of the same type according to common tameness conditions they satisfy.

Tameness conditions that appear in such classifications, should be consider significant.

Classification of expansions of $(\mathbb{R}, <, +)$ by cyclic multiplicative subgroups of $\mathbb{C}^*$.

Let $S := (ae^{i\varphi})^{\mathbb{Z}} \subseteq \mathbb{R}^2$ be infinite, where $a \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{R}$. Then exactly one of the following holds:

1. the open core of $(\overline{\mathbb{R}}, S)$ is o-minimal,
2. $(\overline{\mathbb{R}}, S)$ is d-minimal,
3. $(\overline{\mathbb{R}}, S)$ defines $\mathbb{Z}$.

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Caulfield (2016): fails for finitely generated subgroups. There is a subgroup S generated by two elements such that $(\overline{\mathbb{R}}, S)$ has d-minimal open core, but is not d-minimal.

Classification of expansions of $(\mathbb{R}, <, +)$ by discrete multiplicative subgroups of $\mathrm{Gl}_n(\mathbb{C})$. (H.-Walsberg 2018)

Let Γ be an infinite discrete subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then either

- $(\overline{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ or
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If Γ is not virtually abelian, then $(\overline{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$.

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Conjecture. Let G be a finitely generated subgroup of $\mathrm{Gl}_n(\mathbb{C})$. Then one of the following holds:

1. $(\overline{\mathbb{R}}, G)$ defines $\mathbb{Z}$,
2. $(\overline{\mathbb{R}}, G)$ is d-minimal,
3. $(\overline{\mathbb{R}}, G)$ has o-minimal open core,
4. $(\overline{\mathbb{R}}, G)$ has d-minimal open core.

Conjecture (Miller). Let K be a subfield of $\mathbb{R}$. Then one of the following holds:

- $(\overline{\mathbb{R}}, K)$ defines $\mathbb{Z}$,
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Wide open. Alexi Block Gorman will report on some further evidence that the conjecture could actually be true.

Tameness? For $r \in \mathbb{N}_{\geq 2}$, consider a ternary predicate $V_r(x, u, k)$ that holds if and only if u is an integer power of r, $k \in \{0, \dots, r-1\}$, and the digit of some base r representation of x in the position corresponding to u is k.

Set $\mathcal{T}_r := (\mathbb{R}, <, +, V_r)$.

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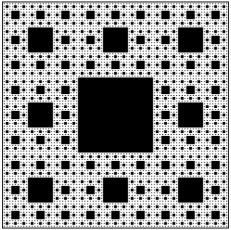
Observation 3: The middle-thirds Cantor set is $\emptyset$-definable in $\mathcal{T}_3$ (it is the set of those numbers in $[0, 1]$ admitting a base 3 expansion that omits the digit 1.) and so is...

... the Sierpinski carpet



It is the set of pairs of real numbers $(x, y) \in [0, 1]^2$ such that for every positive integer n the n-th digit of two ternary expansions of x and of y are not both equal to 1.

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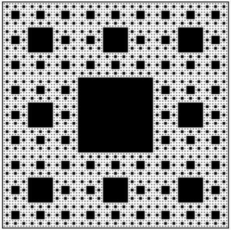

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Same for the Menger sponge and other famous fractals. See also Adamczewski, Bell, *'An analogue of Cobham's Theorem for Fractals'*, TAMS 2011

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Define $\tau_r \colon D_r \to r^{-\mathbb{N}_{>0}}$ so that $\tau_r(d)$ is the least $u \in r^{-\mathbb{N}_{>0}}$ appearing with nonzero coefficient in the finite base r expansion of d.

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For $d, e \in D_r$, let

$$d \prec_r e \iff \tau_r(d) > \tau_r(e) \text{ or } (\tau_r(d) = \tau_r(e) \text{ and } d < e).$$

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Observation: $(D_r, \prec)$ is definable in $\mathcal{T}_r$ and has order type ω.

Definition. An infinite definable subset of $\mathbb{R}^n$ is ω-**orderable** if it admits a definable ordering with order type ω. We say an expansion $\mathcal{R}$ of $(\mathbb{R}, <, +)$ **defines a dense** ω-**orderable set** if it defines an ω-orderable subset of $\mathbb{R}$ that is dense in some open interval.

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Example. $(\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \lambda x)$ defines $n \in \mathbb{Z} \mapsto \lambda n - \lfloor \lambda n \rfloor$, and thus a dense ω-orderable set.

Trivial Trichotomy. An expansion $\mathcal{R}$ of $(\mathbb{R}, <, +)$ satisfies exactly one of the following three conditions:

(A) $\mathcal{R}$ does not define a dense ω-orderable set.

(B) $\mathcal{R}$ defines a dense ω-orderable, but avoids a compact set.

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'What about decidability of the theory? Just as biological taxonomy does not tell us whether a species is tasty, the classificaton here does not deal with decidability.' - Saharon Shelah

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- Every nowhere dense definable subset of $\mathbb{R}^n$ is Lebesgue null.
- There is no continuous definable bijection from $\mathbb{R}^m$ to $\mathbb{R}^n$ when $m < n$.

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The previous results all depend on the following

Strong Baire Category Theorem.
Let $\{X_{r,s} : r, s > 0\}$ be a definable family of closed subsets of $\mathbb{R}^n$ such that $X_{r,s} \subseteq X_{r',s'}$ when $r \leq r'$ and $s \geq s'$. If $\bigcup_{r,s} X_{r,s}$ is somewhere dense, then there are $r', s' > 0$ such that $X_{r',s'}$ has interior.

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Special case. Let $\{X_r : r > 0\}$ be a definable family of *finite* subsets of $\mathbb{R}^n$ such that $X_r \subseteq X_{r'}$ when $r \leq r'$. Then $X_\infty := \bigcup_r X_r$ is nowhere dense.

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Proof. For $x \in X_\infty$, let $\delta(x) := \inf\{r > 0 \ : \ x \in X_r\}$.

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Proof. For $x \in X_\infty$, let $\delta(x) := \inf\{r > 0 \ : \ x \in X_r\}$. Then for $x, y \in X_\infty$ set

$$x \prec y \text{ iff } (\delta(x), x) <_{\text{lex}} (\delta(y), y).$$

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The order $\prec$ has order type ω. Thus X_∞ is nowhere dense.

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Elgot-Rabin '66. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f^{-1}(n)$ is infinite for every $n \in \mathbb{N}$. Then the theory of $(\mathbb{N}, \mathcal{P}(\mathbb{N}), +1, \in, f)$ is undecidable.

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H.-Tychonievich 2014. If $\mathcal{R}$ defines an order $(D, \prec)$ and a function $g : \mathbb{R}^3 \times D \to D$ such that

1. $(D, \prec)$ has order type ω and D is dense in $[0, 1]$,
2. for every $a, b \in [0, 1]$ and $e, d \in D$ with $a < b$ and $e \preceq d$,

$$\{c \in \mathbb{R} : g(c, a, b, d) = e\} \cap (a, b) \text{ has nonempty interior,}$$

then $\mathcal{R}$ is type C. Example

The previous theorem puts strong restriction on what kind of function are definable in type B expansions.

Corollary. Suppose $\mathcal{R}$ defines a dense ω-orderable set and a function $f : [0, 1] \to \mathbb{R}$ that is C^2, but not affine. Then $\mathcal{R}$ is type C.

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Corollary. Expansions of $\mathcal{M}$ are either type A or type C.

Future direction.

Connections to neostability. NTP2, NIP $\Rightarrow$ Type A. Does NIP (NTP2) imply stronger geometric tameness than Type A? (cp. Dolich and Goodrick's work on strong theories)

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Tame geometry. Let $\mathcal{R}$ be a type A expansions of $\mathcal{M}$ by closed sets. Does every definable subset of $\mathbb{R}$ has interior or is nowhere dense?

Thank you!