LOCALLY DEFINABLE SUBGROUPS OF SEMIALGEBRAIC GROUPS

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Abstract. We prove the following instance of a conjecture stated in [7]. Let G be an abelian semialgebraic group and X a semialgebraic subset of G. Then the group $\mathcal{U}$ generated by X contains a generic semialgebraic set. In particular, $\mathcal{U}$ is divisible.

1. Introduction

Let $\mathcal{M}$ be an arbitrary κ-saturated o-minimal structure (for κ sufficiently large). A *locally definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{k\in\mathbb{N}} X_k$ of definable subsets of M^n for some fixed n, and for every $i, j \in \mathbb{N}$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). The dimension of $\mathcal{U}$ is by definition $\dim(\mathcal{U}) = \max\{\dim(X_k) : k \in \mathbb{N}\}$.

A map $\phi : \mathcal{U} \to \mathcal{H}$ between locally definable groups is called *locally definable* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $graph(\phi) \cap (X \times Y)$ is a definable set. Equivalently, the restriction of ϕ to any definable set is a definable map.

For a locally definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a bounded union of definable sets). Note that if $\phi : \mathcal{U} \to \mathcal{V}$ is a locally definable homomorphism between locally definable groups, then $\ker(\phi)$ is a compatible locally definable normal subgroup of $\mathcal{U}$. Compatible subgroups are used in order to obtain locally definable quotients:

Fact 1.1. [5, Theorem 4.2] *If $\mathcal{U}$ is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup then $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \to \mathcal{V}$ whose kernel is $\mathcal{H}$.*

We are mostly interested here in *definably generated* groups, namely locally definable groups $\mathcal{U}$ which are generated as a group by a definable subset

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X. An important example of such groups is the universal cover of a definable group, see [9]. Another relevant class of such groups are the connected locally definable groups with a definable *(left) generic* set X, that is, a definable set such that $AX = \mathcal{U}$ for some countable subset $A \subseteq \mathcal{U}$ (see [6, Fact 3.2(2)]). In [7], the converse of the latter was conjectured in the abelian case:

Conjecture 1.2. *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then $\mathcal{U}$ contains a definable generic set.*

We will use the following notation:

Notation 1.3. For $X \subseteq \mathcal{U}$ we write $\Sigma_n X$ for the set $X \pm \overset{n}{\cdots} \pm X$. The set $X(m)$ denotes the addition of $X - X$ to itself m times.

It has been shown in recent papers that the above conjecture can be restated in several ways. For example, given $\mathcal{U}$ abelian, connected, definably generated group we say that a locally definable normal subgroup $\Gamma < \mathcal{U}$ is a *lattice* if $\dim(\Gamma) = 0$ and $\mathcal{U}/\Gamma$ is definable, that is, there exist a definable group G and a locally definable surjective homomorphism from $\mathcal{U}$ onto G, whose kernel is Γ (note that Γ is necessarily compatible).

Fact 1.4 ([7, Proposition 3.5] and [6, Theorem 2.1])**.** *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then there is k so that the following are equivalent:*

(1) *$\mathcal{U}$ contains a definable generic set.*
(2) *$\mathcal{U}$ admits a lattice.*
(3) *$\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$.*

Moreover, the above conclusions imply that $\mathcal{U}$ is divisible.

Our aim in the present note is to study some cases of the above conjecture. In Section 2 we study the conjecture under the presence of an exact sequence, and we prove the following result:

Theorem. *Assume that we are given an exact sequence of abelian locally definable groups and maps.*

$$0 \longrightarrow \mathcal{H} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \mathcal{V} \longrightarrow 0$$

Assume also that definably generated connected subgroups of $\mathcal{V}$ and of $\mathcal{H}$ contain a definable generic set. Then the same is true for $\mathcal{G}$.

This is a useful criterion that can be applied inductively in certain situations. For example, we will use it in Sections 3 and 4 to study semialgebraically generated subgroups of semialgebraic groups. The following (Theorem 4.4) is one of the main results of the paper, which generalizes the results in [8].

Theorem. *Let G be an abelian semialgebraic group and let X be a semialgebraic subset of G. Then the group generated by X contains a generic semialgebraic set.*

2. Group extensions

In this section we study the existence of generic sets when dealing with abelian group extensions. As a corollary, we will prove that definable generated subgroups of abelian torsion-free definable groups contain generic sets.

Proposition 2.1. *Assume that we are given an exact sequence of abelian locally definable groups and maps,*

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{V} \longrightarrow 0\ ,$$

where $\mathcal{V}$ is connected and admits a lattice. Let $Y \subseteq \mathcal{V}$ be a definable generic set and let $s : Y \to \mathcal{G}$ be a definable section. Then the intersection $\langle s(Y)\rangle \cap \mathcal{H}$ is definably generated.

Proof. By [6, Fact 3.2(2)]), $\mathcal{V} = \langle Y\rangle$. In particular, π sends the group $\langle s(Y)\rangle$ onto $\mathcal{V}$.

Henceforth we will use that given a definable set $Z \subseteq \mathcal{V}$, we can assume that $Z \subseteq Y$. Indeed, by saturation there is n such that $Z \subseteq \Sigma_n Y$ and by definable choice there is a section $r : Z \to \Sigma_n s(Y) \subseteq \langle s(Y)\rangle$. Thus we can extend the section $s : Y \to \mathcal{G}$ to a section $\widetilde{s} : Y \cup (Z \setminus Y) \to \mathcal{G}$ via r in such a way that $\langle s(Y)\rangle = \langle \widetilde{s}(Y \cup Z)\rangle$. Therefore we can work with the generic set $Y \cup Z$ instead of Y, as required.

By Fact 1.4 and since Y is generic, the locally definable group $\mathcal{V}$ admits a lattice $\Gamma \simeq \mathbb{Z}^k$. Since $\mathcal{V}/\Gamma$ is definable and Y generic in $\mathcal{V}$, there is a finite set $A \subseteq \mathcal{V}$ such that $Y + A + \Gamma = \mathcal{V}$. Without loss we can assume that A contains a fixed set of generators $\gamma_1, \dots, \gamma_k$ of Γ. Therefore we can assume that $Y + \Gamma = \mathcal{V}$ and $\gamma_1, \dots, \gamma_k \in Y$ (extending the section s to $Y + A$). Moreover, we can assume that Y is symmetric (again extend the section s to set $-Y$).

Let $\Delta = \langle s(\gamma_1), \dots, s(\gamma_k)\rangle$ and note that $\pi|_\Delta : \Delta \to \Gamma$ is an isomorphism. Let Δ_0 be the symmetric finite set of all $\delta \in \Delta$ such that $\pi(\delta) \in \Sigma_3 Y$ (notice that $\Sigma_3 Y \cap \Gamma$ is finite). Consider the definable set

$$D := (\Delta_0 + \Sigma_3 s(Y)) \cap \mathcal{H} \ \subseteq\ \langle \Delta + s(Y)\rangle \cap \mathcal{H} \ =\ \langle s(Y)\rangle \cap \mathcal{H}.$$

To prove that $D \ni 0$ generates $\langle s(Y)\rangle \cap \mathcal{H}$ is enough to prove the following:

Claim. For all n and for every $\delta_1, \dots, \delta_{2^n} \in \Delta$ and $y_1, \dots, y_{2^n} \in Y$, if $\Sigma_{i=1}^{2^n} \pi(\delta_i) + y_i = 0$ then $\Sigma_{i=1}^{2^n} \delta_i + s(y_i) \in \langle D\rangle$.

Indeed, granted the claim suppose that $\Sigma_{i=1}^m s(y_i) \in \langle s(Y)\rangle \cap \mathcal{H}$. Recall that the generator γ_1 of Γ belongs to Y and $s(\gamma_1) \in \Delta$, so we can write

$$\Sigma_{i=1}^m s(y_i) = \Sigma_{i=1}^m \big(0 + s(y_i)\big) + \Sigma_{i=m+1}^{2^m} \big(s(\gamma_1) - s(\gamma_1)\big).$$

Since $\Sigma_{i=1}^m s(y_i) \in \mathcal{H}$ we have that $\Sigma_{i=1}^m y_i = 0$ and therefore we are in the hypothesis of the claim, so we deduce that $\Sigma_{i=1}^m s(y_i) \in \langle D\rangle$, as required.

Proof. By induction on n. The case $n = 0$ gives $\pi(\delta_1) + y_1 = 0$, hence $\pi(\delta_1) \in Y \subseteq \Sigma_3 Y$, so $\delta_1 \in \Delta_0$. Therefore $\delta_1 + s(y_1) \in D$.

Assume now that $\Sigma_{i=1}^{2^n}\pi(\delta_i)+y_i=0$ in $\mathcal{V}$. We want to show that $\Sigma_{i=1}^{2^n}\delta_i+s(y_i)$ is in $\langle D\rangle$. We write the sum in pairs:

$$\Sigma_{i=1}^{2^n}(\delta_i+s(y_i))=\Sigma_{k=1}^{2^{n-1}}(s(y_{2k-1})+s(y_{2k})+\delta_{2k-1}+\delta_{2k}).$$

Now, because $Y+\Gamma=\mathcal{V}$, for each $k=1,\dots,2^{n-1}$ there is $w_k\in Y$ and $\beta_k\in\Gamma$ such that $y_{2k-1}+y_{2k}=\beta_k+w_k$. Let $\alpha_k\in\Delta$ be such that $\pi(\alpha_k)=\beta_k$. Note that $\beta_k\in\Sigma_3 Y$, so that $\alpha_k\in\Delta_0$. Hence,

$$(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k))\in D.$$

Thus the above sum also equals:

$$\begin{aligned}\Sigma_{i=1}^{2^n}(\delta_i+s(y_i)) =&\Sigma_{k=1}^{2^{n-1}}\big(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k)\big)+\\ &\Sigma_{k=1}^{2^{n-1}}\big(\delta_{2k-1}+\delta_{2k}+\alpha_k+s(w_k)\big).\end{aligned}$$

We already showed that $\Sigma_{k=1}^{2^{n-1}}\big(s(y_{2k-1})+s(y_{2k})-\alpha_k-s(w_k)\big)\in\langle D\rangle$, so if we denote $\widetilde{\delta}_k=\delta_{2k-1}+\delta_{2k}+\alpha_k\in\Delta$ then it remains to prove that

$$\Sigma_{k=1}^{2^{n-1}}\big(\widetilde{\delta}_k+s(w_k)\big)\in\langle D\rangle.$$

Indeed, we have that

$$\Sigma_{k=1}^{2^{n-1}}\big(\pi(\widetilde{\delta}_k)+w_k\big)=-\Sigma_{i=1}^{2^n}(\pi(\delta_i)+y_i)-\Sigma_{k=1}^{2^{n-1}}\big(y_{2k-1}+y_{2k}-\beta_k-w_k\big)=0$$

and therefore by induction we deduce that $\Sigma_{k=1}^{2^{n-1}}\big(\widetilde{\delta}_k+s(w_k)\big)\in\langle D\rangle$, as required. □

Proposition 2.2. *With $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{V}$ as above, assume that $X\subseteq\mathcal{G}$ is a definable set with $\langle\pi(X)\rangle=\mathcal{V}$. Then $\langle X\rangle\cap\mathcal{H}$ is definably generated.*

Proof. Since $\mathcal{V}$ admits a lattice it contains a generic set Y. Without loss we can assume that $\Sigma_2\pi(X)\subseteq Y$. By saturation $Y\subseteq\Sigma_n\pi(X)$ for some n and therefore by definable choice we can pick a section $s:Y\to\langle X\rangle$. Moreover, we can assume that $s(\Sigma_2\pi(X))\subseteq\Sigma_2 X$. By Proposition 2.1 we have that $\mathcal{H}_0=\langle s(Y)\rangle\cap\mathcal{H}$ is definably generated. Let $E=\Sigma_4 X\cap\mathcal{H}$. Therefore to prove that $\langle X\rangle\cap\mathcal{H}$ is definably generated it suffices to show that $\langle X\rangle\cap\mathcal{H}=\langle E\rangle+\mathcal{H}_0$.

We shall prove that for every $x_1\dots,x_{2^n}\in X$, if $\Sigma_{i=1}^{2^n}x_i\in\mathcal{H}$ then $\Sigma_{i=1}^{2^n}x_i\in\langle E\rangle+\mathcal{H}_0$. When $n=1$ then $x_1+x_2\in\Sigma_2X\cap\mathcal{H}\subseteq\Sigma_4X\cap\mathcal{H}=E$ so we are done. For $n>1$, write

$$\Sigma_{i=1}^{2^n}x_i=\Sigma_{k=1}^{2^{n-1}}(x_{2k-1}+x_{2k}-s\big(\pi(x_{2k-1}+x_{2k})\big))+\Sigma_{k=1}^{2^{n-1}}s\big((\pi(x_{2k-1}+x_{2k})\big).$$

Each

$$(x_{2k-1}+x_{2k}-s\big(\pi(x_{2k-1}+x_{2k})\big))\in\big(\Sigma_2X+s(\pi(\Sigma_2X))\big)\cap\mathcal{H}\subseteq E$$

so the sum on the left is in $\langle E\rangle$. Finally, for the sum on the right we have that

$$\begin{aligned}\Sigma_{k=1}^{2^{n-1}}s\big((\pi(x_{2k-1}+x_{2k})\big) =&\Sigma_{i=1}^{2^n}x_i\\ &-\Sigma_{k=1}^{2^{n-1}}(x_{2k-1}+x_{2k}-s\big(\pi(x_{2k-1}+x_{2k})\big))\in\mathcal{H}+\mathcal{H}=\mathcal{H}\end{aligned}$$

so that $\Sigma_{k=1}^{2n-1} s\big((\pi(x_{2k-1} + x_{2k})\big) \in \langle s(Y)\rangle \cap \mathcal{H} = \mathcal{H}_0$, which finishes the proof. □

Before the main corollary we need also:

Lemma 2.3. *Let $\mathcal{H}$ be an abelian locally definable group. If $\mathcal{H}$ is definably generated then its connected component is definably generated by a connected definable set. In particular, if every* connected *definably generated subgroup of $\mathcal{H}$ contains a generic set then every definably generated subgroup of $\mathcal{H}$ contains a generic set.*

Proof. Let $X \subseteq \mathcal{H}$ be a definable set which generates $\mathcal{H}$. Let $X_1, \dots, X_k$ be its connected components. Fix an element a_i in each X_i, and let $\Gamma = \langle a_1, \dots, a_k\rangle$. Consider the connected set $\widetilde{X} = \bigcup X_i - a_i$, and notice that $\langle X\rangle = \langle \widetilde{X}\rangle + \Gamma$. Since $\langle \widetilde{X}\rangle$ is a locally definable subgroup of $\mathcal{H}$ of bounded index, it must be its connected component.

For the second part of the statement, let $\mathcal{G}$ be a definably generated subgroup of $\mathcal{H}$. Then its connected component $\mathcal{G}^0$ is definably generated by the first part and therefore by hypothesis it contains a generic set Y, that is, there is a bounded $A \subseteq \mathcal{G}^0$ such that $A + X = \mathcal{G}^0$. Since $\mathcal{G}^0$ has bounded index in $\mathcal{G}$, there is a bounded $B \subseteq \mathcal{G}$ such that $B + \mathcal{G}^0 = \mathcal{G}$. In particular $A + B + X = \mathcal{G}$, as required. □

Theorem 2.4. *Assume that we are given an exact sequence of abelian locally definable groups and maps.*

$$0 \longrightarrow \mathcal{H} \xrightarrow{\ i\ } \mathcal{G} \xrightarrow{\ \pi\ } \mathcal{V} \longrightarrow 0$$

Assume also that definably generated connected subgroups of $\mathcal{V}$ and of $\mathcal{H}$ contain a generic set. Then the same is true for $\mathcal{G}$.

Proof. Let $X \subseteq \mathcal{G}$ be a definable set which generates a connected subgroup of $\mathcal{G}$. Since $\pi(\langle X\rangle) = \langle \pi(X)\rangle$ is a definably generated connected group, we have the exact sequence of locally definable groups

$$0 \to \langle X\rangle \cap \mathcal{H} \to \langle X\rangle \to \langle \pi(X)\rangle \to 0$$

By hypothesis the connected group $\langle \pi(X)\rangle$ contains a generic set, that is, there exists a definable set $Z_1 \subseteq \langle X\rangle$ such that $\pi(Z_1)$ is generic in $\langle \pi(X)\rangle$. In particular the group $\langle \pi(X)\rangle$ admits a lattice (see Fact 1.4) and therefore by Proposition 2.2 the group $\langle X\rangle \cap \mathcal{H}$ is definably generated. Again by hypothesis and by Lemma 2.3 we have that $\langle X\rangle \cap \mathcal{H}$ contains a generic set Z_2. Finally, it is not hard to see that $Z_1 + Z_2$ is generic in $\langle X\rangle$. □

Next we apply the above result in different contexts. First, we can study definably generated subgroups of abelian torsion-free definable groups (see basic facts on torsion-free groups definable in o-minimal structures in Section 2.1 in [14]).

Corollary 2.5. *Let G be any definable, abelian torsion-free definable group. Then for any definable $X \subseteq G$, the group $\langle X\rangle$ contains a generic set.*

Proof. We prove it by induction on $\dim(G)$. Asume first that $\dim(G) = 1$. Then there is a definable order relation $<$ on G such that G with $<$ is a definable ordered group. By Lemma 2.3 it suffices to study a subgroup generated by a set of the form $(-b, b) := \{x \in G : -b < x < b\}$, that is,

$$\langle(-b, b)\rangle = \bigcup_{n\in\mathbb{N}} (-nb, nb)$$

But this follows straightforward since the zero-dimensional locally definable subgroup $\Gamma = \mathbb{Z}b$ of $\langle(-b, b)\rangle$ is a lattice because $\langle(-b, b)\rangle/\Gamma$ is isomorphic to the definable group $\big([0, b), \bmod b\big)$.

Now, assume that $\dim(G) > 1$. Then, by [16], there exists a subgroup H of G of dimension 1. In particular, we have the exact sequence

$$0 \to H \to G \to G/H \to 0$$

Since both H and G/H are abelian torsion-free definable groups, by induction we have that definably generated connected subgroups of G/H and of H contain a generic set. Thus by Theorem 2.4 and Lemma 2.3 we deduce that definably generated subgroups of G contain a generic set, as required. □

Next, we study algebraic groups defined without parameters (but note that this result is only a particular case of Theorem 4.4).

Corollary 2.6. *Let $\mathcal{R}$ be an expansion of a model of $T_{an,exp}$. Let $H \subseteq R^k$ be an abelian connected algebraic group defined without parameters. Let $X \subseteq H(R)^o$ be a definable connected and definably compact definable set in $\mathcal{R}$, possibly with parameters. The subgroup of H generated by X has a generic definable subset.*

Proof. Denote $G := H(\mathbb{R})^o$. By [10, Lemma 4.10] we have that

$$G(\mathbb{R}) = H_1(\mathbb{R}) \times H_2(\mathbb{R}) \times K$$

where $H_1(\mathbb{R})$ and $H_2(\mathbb{R})$ are semialgebraic subgroups of $G(\mathbb{R})$ semialgebraically isomorphic to $(\mathbb{R}, +)^s$ and $(\mathbb{R}^*, \cdot)^\ell$ respectively, and K is a closed and compact real analytic group. Since K is compact it is definable in $T_{\mathrm{an,exp}}$ and therefore we can transfer the equality above to $\mathcal{R}$. Moreover, the subgroup K is analytically isomorphic to a torus $\mathbb{T}(\mathbb{R}) = (\mathbb{R}/\mathbb{Z})^p$, and $H_1(\mathbb{R})$ and $H_2(\mathbb{R})$ can be identified vie the exponentation. All in all, we have definable in $\mathcal{R}$ an isomorphism

$$G = R^m \times \mathbb{T}.$$

Now, consider the definable universal covering $\pi : \widetilde{G} \to G$, that is,

$$\widetilde{G} = R^m \times \mathrm{Fin}(R)^p,$$

where $\mathrm{Fin}(R) := \{x \in R : |x| < n \text{ for some } n \in \mathbb{N}\}$.

By definably choice there is $S \subseteq \widetilde{G}$ such that $\pi(S) = X$. Moreover, the group generated by S is in particular a definably generated subgroups of R^{m+p} and therefore by Corollary 2.5 we have that $\langle S\rangle$ contains a generic set. In particular, the group generated by X contains a generic set, as required. $\square$

3. Abelian Varieties

For a positive $g \in \mathbb{N}$, by a *complex g-torus* we mean the quotient group $\mathbb{C}^g/\Lambda$ where Λ is a lattice. It is a compact complex Lie group of dimension g. A torus $\mathbb{C}^g/\Lambda$ is called an *abelian variety* if it is biholomorphic with a projective variety in $\mathbb{P}^k(\mathbb{C})$ for some k.

Let us denote by $\mathbb{H}_g$ the set of $g \times g$ symmetric matrices with a positive definite imaginary part. Then the well-known *Riemann criterion* establishes that a complex g-torus E is an abelian variety if and only if it is bi-holomorphic with a torus $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$ where $\tau \in \mathbb{H}_g$ and D is a diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_g)$ with positive integers $d_1|d_2|\cdots|d_g$. We call D a *polarization type* of E.

Note that if E has polarization D, then it also has polarization kD for any positive integer k. Indeed, the map $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g) \to \mathbb{C}^g/(k\tau\mathbb{Z}^g + kD\mathbb{Z}^g) : z \mapsto kz$ is a biholomorphism and therefore being biholomorphic with $\mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$ is equivalent to being biholomorphic with $\mathbb{C}^g/(k\tau\mathbb{Z}^g + kD\mathbb{Z}^g)$. In particular, by taking $k = 6$ above, any abelian variety has a polarization type $D = \mathrm{diag}(d_1, \dots, d_g)$ satisfying $d_1 \geq 4$, $2|d_1$ and $3|d_1$. If a polarization type satisfies the latter we it a *Baily-polarization.*

Let us show that the family of abelian varieties with a fixed Baily-polaritazion is constructible.

First, let us recall the definition of theta functions. In some occasions, we will identify the symmetric $g \times g$ matrices with $\mathbb{C}^n$ for $n := \frac{g(g+1)}{2}$, and therefore we will view $\mathbb{H}_g$ as a subset of $\mathbb{C}^n$. The well-defined function

$$\begin{array}{rcl} \vartheta : \mathbb{C}^g \times \mathbb{H}_g & \to & \mathbb{C} \\ (z, \tau) & \mapsto & \displaystyle\sum_{n \in \mathbb{Z}^g} \exp\bigl(\pi i({}^t n \tau n) + 2{}^t n z)\bigr) \end{array}$$

is holomorhic and $\mathbb{Z}^g$-periodic in z and $(2\mathbb{Z})^n$-periodic in τ. For any $a \in \mathbb{R}^g$ the associated *Riemann Theta function* is

$$\begin{array}{rcl} \vartheta_a : \mathbb{C}^g \times \mathbb{H}_g & \to & \mathbb{C} \\ (z, \tau) & \mapsto & \vartheta_a(z, \tau) = \exp\bigl(\pi i({}^t a \tau a) + 2{}^t a z)\bigr)\vartheta(z + \tau a, \tau). \end{array}$$

A consequence of the classical Lefschetz Theorem is the following. Fix a polarization $D = \mathrm{diag}(d_1, \dots, d_g)$ with $d_1 \geq 3$ and fix a set of representatives $\{c_0, \dots, c_N\}$ of the cosets of $\mathbb{Z}^g$ in the group $D^{-1}\mathbb{Z}^g$. Then

$$\varphi^D : \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C})$$

$$\varphi^D(z, \tau) = \bigl(\vartheta_{c_0}(z, \tau) : \cdots : \vartheta_{c_N}(z, \tau)\bigr)$$

is a well-defined holomorphic map, and given $\tau \in \mathbb{H}_g$ we have that

$$\begin{array}{rcl} \varphi^D_\tau : \mathbb{C}^g & \to & \mathbb{P}^N(\mathbb{C}) \\ z & \mapsto & \varphi^D_\tau(z) := \varphi^D(z, \tau) \end{array}$$

is an $(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$-periodic immersion and induces an analytic embedding of the abelian variety $\mathcal{E}_\tau^D = \mathbb{C}^g/(\tau\mathbb{Z}^g + D\mathbb{Z}^g)$ into $\mathbb{P}^N(\mathbb{C})$.

Moreover, if we denote

$$\Psi^D : \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C}) : \tau \mapsto \varphi^D(0, \tau),$$

and

$$\Phi^D : \mathbb{C}^g \times \mathbb{H}_g \to \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C}) : (z, \tau) \mapsto (\varphi^D(z, \tau), \Psi^D(\tau))$$

then we have the following properties for Baily-polarizations, see [3, Theorem 8.10.1. and Remark 8.10.4].

Fact 3.1. *Let D be a Baily-polarization.*

1) *The map Ψ^D is an immersion, the set $\Psi^D(\mathbb{H}_g)$ is a Zariski open subset of an algebraic subvariety of $\mathbb{P}^N(\mathbb{C})$. Moreover, if $\Psi^D(\tau) = \Psi^D(\tau')$ then*
$$\varphi_\tau^D(\mathcal{E}_\tau^D) = \varphi_{\tau'}^D(\mathcal{E}_{\tau'}^D).$$
2) *The image of Φ^D is a Zariski open subset of an algebraic subvariety of $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$.*

Remark 3.2. In other words, if we denote by $P^D := \Psi^D(\mathbb{H}_g)$ and for each $p := \Psi^D(\tau) \in P^D$ we denote $A_p^D := \varphi_\tau^D(\mathcal{E}_\tau^D)$, then the above result ensures that the family

$$\mathcal{A}^D := \{A_p^D : p \in P^D\}$$

is constructible.

Let $\mathcal{M}$ be a first order structure. Let A, B and T be definable sets in a first order structure $\mathcal{M}$. We say that the family of functions $\{f_t : A \to B : t \in T\}$ is definable if the family $\{(a, b, t) : t \in T, b = f_t(a)\}$ is a definable set. If $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ and X is a definable subset of M^n then we denote by $X(N)$ the realization of X in $\mathcal{N}$.

Let K be an algebraic closed field. Henceforth, when we write that $\mathcal{A} = \{A_t : t \in T\}$ is a constructible family of g-dimensional abelian varieties of $\mathbb{P}^N(K)$, we mean that each A_t is an irreducible projective subvariety of $\mathbb{P}^N(K)$ of dimension g and that there is a constructible family $\{F_t : t \in T\}$ of regular maps $F_t : A_t \times A_t \to A_t$ such that each F_t endows A_t with a group structure.

Proposition 3.3. *Let $\mathcal{A} = \{A_t : t \in T\}$ be a constructible family without parameters of g-dimensional abelian varieties of $\mathbb{P}^M(\mathbb{C})$. Then there exist finitely many polarizations $D_1, \ldots, D_k$ and $d \in \mathbb{N}$ such that for any $t \in T$ there exists an isomorphism of degree less than d between A_t and an abelian variety in $\mathcal{A}^{D_j}$ for some $j = 1, \ldots, k$.*

Proof. We already note above that for each $t \in T$ the abelian variety A_t has a Bayli-polarization D and therefore it is bi-holomorphic isomorphic to an abelian variety in $\mathcal{A}^D$, so in particular they are bi-regularly isomorphic, by Chow's Theorem. We claim that only finitely many Bayli-polarizations $D_1, \ldots, D_k$ are needed to cover all the abelian varieties in $\mathcal{F}$.

Indeed, assume that it is not true. For any fixed Bayli-polarization D and for any $d \in \mathbb{N}$ consider the formula $F_d^D(t)$ in the language of rings that says that $t \in T$ and for every $p \in P^D$ there is no bi-regular isomorphism from A_t to A_p^D of degree d. Now, consider

$$q(t) = \{F_d^D(t) : D \text{ a Bayli-polarization } \& \, d \in \mathbb{N}\}.$$

By our assumption, we get that q is a partial type over a countable set of parameters. On the other hand, the complex field $\mathbb{C}$ in the language of rings is $\aleph_1$-saturated. For, any ω-stable complete theory in a countable language has a saturated model of cardinality $\aleph_1$ (see [11, Thm. 6.5.4]). In particular, since ACF_0 is $\aleph_1$-categorical, it follows that $\mathbb{C}$ is $\aleph_1$-saturated. Thus, there is $t_0 \in T$ that realises the type q. In other words, the abelian variety A_{t_0} is not isomorphic to $\mathcal{A}_p^D$ for any Bayli-polarization D and $p \in P^D$, which is a contradiction. It follows that there exist $D_1, \ldots, D_k$ Bayli-polarizations such that for any $t \in T$ we have that A_t is bi-regularly isomorphic to an abelian variety in $\mathcal{A}^{D_j}$ for some $j \in \{1, \ldots, k\}$. Moreover, we have that the degree of the isomorphims is bounded. Otherwise, we would get a contradiction by considering the partial type

$$q(t) := \{F_d^{D_i}(t) : j = 1, \ldots, k \ \& \ d \in \mathbb{N}\},$$

as required. $\square$

Remark 3.4. Note that with the notation used in Proposition 3.3, for each $j \in \{1, \ldots, k\}$ the set T_j of $t \in T$ such that there exists a bi-regular isomorphism of degree less than d between A_t and an abelian variety in $\mathcal{A}^{D_j}$ is constructible without parameters. Indeed, fix $j \in \{1, \ldots, k\}$. Given an abelian variety $A_p^{D_j} \subseteq \mathbb{P}^{N_j}(\mathbb{C})$ in $\mathcal{A}^{D_j}$ and $t \in T$, the set $I_{t,p}$ of bi-regular isomorphisms of degree less than d from A_t to $A_p^{D_j}$ is clearly a constructible set. Moreover, the family $\{I_{t,p} : t \in T, p \in P^{D_j}\}$ is also constructible without parameters. In particular, the set $T_j = \{t \in T : \exists p \in P^{D_j}, I_{t,p} \neq \emptyset\}$ is constructible without parameters.

In the rest of the paper, we fix a real closed field R, and let $K = R(i)$ be its algebraic closure.

We will use the obvious identification of K^n with R^{2n}. We say that a subset of K^n is semialgebraic over $C \subseteq R$ if it is semialgebraic over C as a subset of R^{2n}. Note that if X is a constructible subset of K^n over $A \subseteq K$ then it is clearly semialgebraic over the real and imaginary parts of the elements in A. Finally, note that R^n can be identified with the real part of K^n, and that by elimination of quantifiers of the theory of real closed fields, a subset of R^n is semialgebraic in the usual sense if and only if it is semialgebraic as a subset of K^n.

Henceforth, given a semialgebraic subset T of K^n, when we write that $\mathcal{A} = \{A_t : t \in T\}$ is a semialgebraic family of g-dimensional abelian varieties of $\mathbb{P}^N(K)$, we mean that each A_t is an irreducible projective subvariety of $\mathbb{P}^N(K)$ of dimension g, that the family $\mathcal{A}$ is semialgebraic in the obvious (complex) sense, that is, the set

$$\{(x, t) : t \in T, x \in A_t\}$$

is a semialgebraic subset of $\mathbb{P}^M(\mathbb{C}) \times T$, and that there is also a semialgebraic family $\{F_t : t \in T\}$ of regular maps $F_t : A_t \times A_t \to A_t$ such that each F_t endow A_t with a group structure. Note that it only makes sense to say that a semialgebraic family is defined over a real tuple.

Lemma 3.5. *Let $g, d \in \mathbb{N}$, let D be a polarization and $\mathcal{F} = \{A_t : t \in T\}$ be a semialgebraic family of subsets of $\mathbb{P}^M(\mathbb{C})$ such that each A_t is a g-dimensional abelian variety for which there exists a bi-regular isomorphism of degree less than d between A_t and an abelian variety in $\mathcal{A}^D$. Then there exists a semialgebraic family $\{g_t : t \in T\}$ of bi-regular isomorphisms g_t from A_t to an abelian variety in A^D.*

Proof. As in Remark 3.4, given $t \in T$ and $p \in P^D$, denote by $I_{t,p}$ the non-empty constructible set of bi-regular isomorphisms of degree less than d from A_t to $\mathcal{A}_p^D$, and note that $\{I_{t,p} : t \in T, p \in P^D\}$ is a semialgebraic family. Thus, by definable Skolem functions of the theory of real closed fields, there is a semialgebraic map

$$\mathtt{p} : T \to P^D$$

and for each $t \in T$ a bi-regular isomorphism

$$g_t : A_t \to A^D_{\mathtt{p}(t)}$$

such that the family $\{g_t : t \in T\}$ is semialgebraic, as required. □

Given a polarization D and $\tau \in \mathbb{H}_g$, we denote by $E_\tau^D \subseteq \mathbb{C}^g$ the fundamental parallelogram of $\mathcal{E}_\tau^D$. The following fact follows from [15, Thm 8.10] (see also the comments above it), and recall that P^D denotes $\Psi^D(\mathbb{H}_g) \subseteq \mathbb{P}^N(\mathbb{C})$.

Fact 3.6. *Let D be a Baily-polarization. Then there is a set $S \subseteq \mathbb{H}_g$ such that*

$$\Psi^D|_S : S \to P^D$$

is a surjective map definable in $\mathbb{R}_{an,exp}$ and such that there is a family $\{h_\tau^D : \tau \in S\}$ definable in $\mathbb{R}_{an,exp}$ with

$$h_\tau^D : E_\tau^D \to \mathbb{P}^N(\mathbb{C})$$

an embedding of the abelian variety $\mathcal{E}_\tau^D$ into the projective space $\mathbb{P}^N(\mathbb{C})$.

Theorem 3.7. *Let $\mathcal{A} = \{A_t : t \in T\}$ be a semialgebraic family without parameters of g-dimensional abelian varieties of $\mathbb{P}^M(\mathbb{C})$. Let $d \in \mathbb{N}$ and D be a polarization such that for each $t \in T$ there exists a bi-regular isomorphism* *of degree less than d between A_t and an abelian variety in $\mathcal{A}^D$. Then there is definable in $\mathbb{R}_{an,exp}$ a family of analytic maps*

$$\{h_t : t \in T\}$$

where $h_t : E_t \to A_t$ is such that E_t is a fundamental parallelogram in $\mathbb{C}^g$ of an abelian variety $\mathcal{E}_t$ and h_t induces an isomorphism between $\mathcal{E}_t$ and A_t.

Proof. By Lemma 3.5 there is a semialgebraic map

$$\mathtt{p} : T \to P^D$$

and for each $t \in T$ a bi-regular isomorphism

$$g_t : A_t \to A^D_{\mathtt{p}(t)}$$

such that the family $\{g_t : t \in T\}$ is semialgebraic. On the other hand, pick S a definable set in $\mathbb{R}_{an,exp}$ as in Fact 3.6 and consider a definable section

$$\mathtt{s} : P^D \to S$$

of $\Psi^D|_S : S \to P^D$. Finally, define

$$h_t := h^D_{\mathtt{s}(\mathtt{p}(t))} \circ g_t^{-1}$$

where $h^D_\tau : E^D_\tau \to \mathbb{P}^N(\mathbb{C})$, $\tau \in S$, are the embeddings given by Fact 3.6. □

We now prove that semialgebraic connected subsets of abelian varieties generate locally definable groups with a generic subset. We will need the following generalization of a result in [8]. Given a subset Y of $\mathbb{R}^n$ we say that it is symmetric if $Y = -Y$. Recall that we write $Y(m)$ for the addition of $Y - Y$ to itself m times. If $0 \in Y$ then $Y \subseteq Y(m)$. Given a point $a \in \mathbb{R}^n$, we denote by $(-a, a)$ the open segment from $-a$ to a.

Definition 3.8. Let $\mathcal{R}$ be an o-minimal expansion of a real closed field R. Let $\mathcal{G} = \{G_t : t \in T\}$ be a locally definable family of locally definable groups such that $G_t \subseteq R^\ell$ for some $\ell \in \mathbb{N}$. We say that $\mathcal{G}$ *has the uniform generic property (UGP)* if for every definable family $\{Y_t : t \in T\}$ of closed, bounded, connected definable sets $Y_t \subseteq G_t$, there are $m, N, s \in \mathbb{N}$ such that for every $t \in T$ there exists a finite set $0 \in A_t \subseteq Y_t(s)$ of cardinality at most N such that $Y_t(m) + Y_t(m) \subseteq A_t + Y_t(m)$.

We say that a locally definable group G has the *UGP* if for any definable set T, the definable family $\{G : t \in T\}$ has the UGP.

Lemma 3.9. *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field R. Let $\{\pi_t : \widetilde{G}_t \to G_t : t \in T\}$ be a locally definable family of maps where $G_t \subseteq R^g$ is a definable group, $\widetilde{G}_t \subseteq R^\ell$ is its universal covering and π_t is its covering map. Let $\{s_t : G_t \to \widetilde{G}_t : t \in T\}$ be a locally definable family of sections of π_t which map bounded subsets to bounded subsets. If the family $\{\widetilde{G}_t : t \in T\}$ has the UGP, then $\{G_t : t \in T\}$ also has the UGP.*

Proof. Let $\{X_t : t \in T\}$ be a definable family of closed bounded connected subsets $X_t \subseteq G_t$. Denote $Z_t := s_t(X_t)$, so that $\{Z_t : t \in T\}$ is a definable family of bounded definable subsets of $\widetilde{G}_t$ such that $\pi_t(Z_t) = X_t$.

We can assume that the number of connected components of Z_t is k for all $t \in T$, say $Z_t^1, \dots, Z_t^k$. By definable choice, there are definable mas

$$a_i : T \to R^\ell$$

such that $a_i(t) \in Z_t^i$ for $i = 1, \dots, k$. Let Y_t be the topological closure of $\bigcup_{i=1}^k [Z_t^i - a_i(t)]$. Note that Y_t is connected, bounded, closed and definable in $\mathcal{R}$. Moreover, if we denote $B_t = \{\pm\pi_t(a_1(t)), \dots, \pm\pi_t(a_k(t))\}$ then clearly

$$\pi_t(Y_t) \subseteq X_t + B_t.$$

Indeed, $\pi(Y_t)$ is contained in $\overline{\pi(\bigcup_{i=1}^k [Z_t^i - a_i(t)])}$, which in turn is contained in the closed subset $X_t + B_t$. On the other hand, note also that

$$X_t \subseteq \pi(Y_t) + B_t.$$

For, pick $x \in X_t$. Then, there is y in Z_t such that $\pi_t(y) = x$. In particular, $y \in Z_t^i$ for some i, and so $y - a_i(t) \in Y_t$. Then

$$x = \pi_t(y) \in \pi_t(Y_t) + \pi_t(a_i(t)) \subseteq \pi(Y_t) + B_t,$$

as required.

By hypothesis there are $m, N, s \in \mathbb{N}$ and a finite set $0 \in A_t \subseteq Y_t(s)$ of cardinality at most N such that

$$Y_t(m) + Y_t(m) \subseteq A_t + Y_t(m).$$

Thus,

$$\begin{aligned} X_t(m) + X_t(m) &\subseteq [\pi_t(Y_t)(m) + \pi_t(Y_t)(m)] + [B_t(m) + B_t(m)] \\ &\subseteq \pi_t(A_t) + \pi_t(Y_t(m)) + [B_t(m) + B_t(m)] \\ &\subseteq \pi_t(A_t) + [X_t(m) + B_t(m)] + [B_t(m) + B_t(m)] \\ &\subseteq [\pi_t(A_t) + B_t(3m)] + X_t(m). \end{aligned}$$

The cardinal of the finite set $\pi_t(A_t)+B_t(3m)$ is bounded by a certain number N_0 for all $t \in T$. Moreover, since $\pi_t(A_t) \subseteq X_t(s)$ and $B_t \subseteq X_t$, we have that $\pi_t(A_t)+B_t(3m) \subseteq X_t(s+3m)$. Therefore, if we define $C_t := \pi_t(A_t)+B_t(3m)$ and we set $s_0 := s + 3m$, we get that for the finite subset C_t of $X_t(s_0)$ of cardinality at most $\leq N_0$ it holds that

$$X_t(m) + X_t(m) \subseteq C_t + X_t(m),$$

as required. □

Lemma 3.10. *Let $\mathcal{R}$ be an o-minimal structure whose universe is the real field. Then R^n has the UGP.*

Proof. Let $\{Y_t : t \in T\}$ be a family definable without parameters of bounded closed connected definable subsets of $\mathbb{R}^n$ containing 0. We will prove that there are $m \in \mathbb{N}$ and, for every $t \in T$, a finite subset $A_t \ni 0$ of $Y_t(m)$ of cardinality less than $2n+1$ such that

$$Y_t(m) + Y_t(m) \subseteq \Sigma_{4m} A_t + Y_t(m).$$

Assume first that Y_t is also symmetric and contains 0. Let Z_t denote the convex hull of Y_t, so by [8, Lemma 3.3 and its proof] the family $\{Z_t : t \in T\}$ is definable. Moreover, there is $\ell \in \mathbb{N}$ such that $Y_t \subseteq Z_t \subseteq Y_t(\ell)$ for all $t \in T$. Otherwise, there is an elementary extension $\mathcal{R}'$ of $\mathcal{R}$ and $t_0 \in T(R')$ such that $Z_{t_0}(R')$ is not contained in $Y_{t_0}(R')(\ell)$ for all $\ell \in \mathbb{N}$, a contradiction with [8, Lem. 3.4].

Claim. There is $m \in \mathbb{N}$ such for every $t \in T$ we have that, up to an isometry of $\mathbb{R}^n$, there are at most $k \leq n$ points $a_1, \dots, a_k \in Z_t(m)$ such that if we denote $I_i = (-a_i, a_i)$ then:

(1) the segments $I_1, \dots, I_k$ are linearly independent,
(2) $Z_t \subseteq I_1 + \cdots + I_k \subseteq Z_t(m)$.

Proof of Claim. We prove it by induction. If $n = 1$ then fix $t_0 \in T$. Since Z_{t_0} is an interval, we can use a translation so that $Z_{t_0} = [-a, a]$, for some $a \in R$. In particular, $Z_{t_0} \subseteq (-2a, 2a) \subseteq Z_{t_0}(2)$, as required.

We prove for $n+1$ assuming that it is true for n. Fix $t_0 \in T$. Consider all line segments contained in Z_{t_0} and let J_0 be such segment of maximal length (it exists by o-minimality and because Z_{t_0} is closed). Since we work in a field we may assume that J_0 is parallel to the x_{n+1}-coordinate and furthermore that $0 \in J_0$ divides it exactly into two equal parts. We can therefore write $J_0 = (a_{k+1}, a_{k+1})$ with $a_{k+1} \in Z_{t_0}$. Let π denote the projection onto the first n coordinates, and note that

$$\{\pi(Z_t) : t \in T\}$$

is a family of connected bounded symmetric subsets of $\mathbb{R}^n$. By induction, there is $N \in \mathbb{N}$ and there are points $a_1, \dots, a_k \in \pi(Z_{t_0})(N)$, $k \leq n$, such that (1) and (2) are true for the intervals $I_i = (-a_i, a_i)$. So if we denote $I = I_1 + \cdots + I_k$ then $I \subseteq \pi(Z_{t_0})(N)$. Then by Claim after the Question in [8], we have that

$$Z_{t_0} \subseteq I + J_0 \subseteq Z_{t_0}(2N).$$

Finally, if we set $m := 2\ell N$, then it follows that $Y_{t_0} \subseteq I + J_0 \subseteq Y_{t_0}(m)$, as required. □

Now, note that for each $t \in T$ we have that $\dim(Z_t(m))$ is the number of intervals of the Claim, and therefore we can assume that it is a constant k. Now, by definable Skolem functions and since conditions (1) and (2) are first-orde definable, there are definable functions

$$a_1, \dots, a_k : T \to Z_t$$

such that for $I_{i,t} := (-a_i(t), a_i(t))$ we have that (1) and (2) hold true. Thus, if we define $I_t := I_{1,t} + \cdots + I_{k,t}$ then the family $\{I_t : t \in T\}$ is definable and $Z_t \subseteq I_t \subseteq Z_t(m)$. It is easy to prove by induction on k that

$$(*) \qquad I_t + I_t = A_t + I_t$$

where A_t are the vertices of the box I_t. Note that the cardinality of $A_t \subseteq Z_t(m)$ is $2k \leq 2n$. Without loss, we can add 0 to A_t. and hence the cardinality is $\leq 2n+1$. Finally, since $Y_t \subseteq Z_t \subseteq I_t$ it is easy to prove by induction on j that:

$$Y_t(j) \subseteq \Sigma_{2j-1} A_t + I_t.$$

On the other hand, since $I_t \subseteq Z_t(m) \subseteq Y_t(m\ell)$ and $0 \in A_t$, it follows that

$$Y_t(m\ell) + Y_t(m\ell) = Y_t(2m\ell) \subseteq \Sigma_{4m\ell-1} A_t + Y_t(m\ell) \subseteq \Sigma_{4m\ell} A_t + Y_t(m\ell),$$

and we are done.

In the general case in which Y_t is not necessarily symmetric, consider $\widetilde{Y}_t := Y_t(1) = Y_t - Y_t$, which is definable, closed, connected symmetric and contains 0. Note that $\widetilde{Y}_t(j) = Y_t(2j)$ for all $j \in \mathbb{N}$. By the previous case there is $m \in N$ such that

$$\widetilde{Y}_t(m) + \widetilde{Y}_t(m) \subseteq \Sigma_{4m} A_t + \widetilde{Y}_t(m)$$

and therefore $Y_t(2m) + Y_t(2m) \subseteq \Sigma_{4m} A_t + Y_t(2m)$. Since $0 \in A_t$ then $Y_t(2m) + Y_t(2m) \subseteq \Sigma_{8m} A_t + Y_t(2m)$ and we are done. □

Proposition 3.11. *Let R be a real closed field and $K = R(i)$ its algebraic closure. Let $A \subseteq \mathbb{P}^N(K)$ be an abelian variety defined over R and let X be a semialgebraic subset of A. Then the group generated by X contains a generic semialgebraic subset.*

Proof. First, note that we can assume that X is connected by Lemma 2.3. Moreover, since the group generated by X is closed, we can replace X by its closure, so we can assume that X is closed.

Let $c \in K^\ell$ be a tuple of coefficients defining algebraically the irreducible projective subvariety A of $\mathbb{P}^N(K)$ and the regular group operation $A \times A \to A$. We can replace the parameter c by a tuple u of free variables and therefore we obtain (without parameters) a constructible family $\mathcal{A} = \{A_u : u \in U\}$ of irreducible projective subvarieties of $\mathbb{P}^N(K)$ of dimension $g := \dim(A)$, and a constructible family $\mathcal{F} = \{F_u : u \in U\}$ of regular maps such that F_u endow A_u with a group structure. Note that $c \in U$, for which we get $A_c = A$ and $F_c = F$. Consider the realization $\mathcal{A}(\mathbb{C})$ of $\mathcal{A}$ in $\mathbb{C}$. By Proposition 3.3 we can assume that there exists $d \in \mathbb{N}$ and a polarization D such that for each $u \in U(\mathbb{C})$ there exists a bi-regular isomorphism of degree less than d between $A_u(\mathbb{C})$ and an abelian variety in $\mathcal{A}^D$.

Next, consider the semialgebraic subset X of A which is semialgebraically defined over the tuple $r \in R^\ell$. Again replace r by a tuple v of free variables, define $A_{(u,v)} := A_u$ and consider the semialgebraic set $X_{(u,v)}$ of $A_{(u,v)}$ obtained when v plays the role of the real parameter r. Let T be the set of

$t := (u, v) \in U \times R^\ell$ for which X_t is connected. Note that T is a semialgebraic subset of $U \times R^\ell \subseteq U \times K^\ell$ defined without parameters and that $(c, r) \in T$.

Consider the semialgebraic family $\mathcal{A}_0 = \{A_t : t \in T\}$ of g-dimensional abelian varieties of $\mathbb{P}^N(K)$. Again, take the realization $\mathcal{A}_0(\mathbb{C})$ of $\mathcal{A}_0$ in the complex numbers, and note that by Theorem 3.7 there is a definable family in $\mathbb{R}_{an,exp}$ of analytic maps

$$\{h_t : t \in T(\mathbb{C})\}$$

where $h_t : E_t \to A_t$ is such that E_t is a fundamental parallelogram in $\mathbb{C}^g$ of an abelian variety $\mathcal{E}_t$, and h_t induces an isomorphism f_t between $\mathcal{E}_t$ and A_t.

Henceforth, we will work over the complex numbers and we omit the script $\mathbb{C}$, for example, we write T instead of $T(\mathbb{C})$. For each $t \in T$, denote by π_t the projection of $\mathbb{C}^g$ over $\mathcal{E}_t$, so that $\pi_t|_{E_t} \circ f_t = h_t$. Note that $(\pi|_{E_t})^{-1} : \mathcal{E}_t \to E_t$ is a definable section of π_t. Therefore by Lemmas 3.9 and 3.10, the definable family $\{\mathcal{E}_t : t \in T\}$ has the UGP. In particular, since $\{f_t : t \in T\}$ is a definably familiy of isomorphisms between $\mathcal{E}_t$ and A_t, it follows that $\{A_t : t \in T\}$ has also the UGP. Thus, there are $m, N, s \in \mathbb{N}$ such that for every $t \in T$ there exists a finite set $0 \in A_t \subseteq X_t(s)$ of cardinality at most N such that $X_t(m) + X_t(m) \subseteq A_t + X_t(m)$. In particular, since $X_{(c,r)} = X$ and $A_{(c,r)} = A$, we obtain that $X(m)$ is a generic subset of the group generated by X, as desired. □

4. Semialgebraically generated subgroups of semialgebraic groups

The main purpose of this section is to show that groups generated by a semialgebraic subset of a semialgebraic group contain a generic semialgebraic subset.

Lemma 4.1. *Let H be an irreducible abelian linear algebraic group defined over R. Then the subgroup generated by a connected semialgebraic subset of H contains a semialgebraic generic set.*

Proof. By [12, Corollary 17.19], the group H is isomorphic to $K^m \times (K^*)^n$ for some $m, n \in \mathbb{N}$ (but note that this isomorphism is defined over K). Thus, by Theorem 2.4 and Corollary 2.5 we can assume that $H = (K^*)^n$. By an inductive argument and Theorem 2.4 again, we may assume $H = K^*$. Next, consider the connected semialgebraic subgroup $R_{>0}$ of K^*, and note that the quotient $K^*/R_{>0}$ is semialgebraically isomorphic to

$$\mathbb{S}^1 = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in R, x^2 + y^2 = 1 \right\}.$$

Thus, we have the exact sequence

$$1 \to R_{>0} \to K^* \to \mathbb{S}^1 \to 1.$$

By Lemma 2.3, Theorem 2.4 and Corollary 2.5, it suffices to prove that every subgroup generated by a connected semialgebraic subset of $\mathbb{S}^1$ contains a semialgebraic generic set.

Denote by X a connected semialgebraic subset of $\mathbb{S}^1$, say definable over $d \in R^\ell$, and note that X is bounded. If we replace d by some free-variables t then we obtain a semialgebraic family $\mathcal{X} := \{X_t : t \in T\}$ defined without parameters of connected semialgebraic bounded subsets of $\mathbb{S}_1$. Consider the realization $\mathcal{X}(\mathbb{R})$ of this family in the real field. Note that the universal covering map of $\mathbb{S}^1(\mathbb{R})$,

$$\mathbb{R} \to \mathbb{S}^1 : t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

is locally definable in $\mathbb{R}_{an,exp}$. By Lemma 3.10, the group $\mathbb{R}^n$ has the UGP, and therefore $\mathbb{S}^1(\mathbb{R})$ also has property the UGP by Lemma 3.9. In particular, there are $m, N, s \in \mathbb{N}$ such that for every $t \in T$ there is a finite subset $0 \in A_t \subseteq X_t(s)$ of cardinality at most N such that $X_t(m)+X_t(m) \subseteq A_t+X_t(m)$. This is a first-order property and therefore for our parameter d we obtain that $X(m)$ is a generic subset of $\langle X \rangle$, as required. $\square$

Theorem 4.2. *Let H be an irreducible algebraic group definable over R. Then the subgroup generated by a connected semialgebraic subset of H contains a semialgebraic generic set.*

Proof. By Chevalley theorem's, there is a linear group L and abelian variety A, both defined over R, such that

$$0 \to L \to H \to A \to 0.$$

Thus, by Corollary 2.4, Proposition 3.11 and Lemma 4.1, we get the result. $\square$

Note that if in the above result, if X is a subset of $H(R)^o$, then the group generated by X is contained in $H(R)^o$ and the result gives a semialgebraic generic subset (in the usual "real" sense) of $H(R)^o$.

Proposition 4.3. *Let G be an abelian definable group and let $p : \widetilde{G} \to G$ be its universal covering. Assume that definably generated connected subgroups of $\widetilde{G}$ contain a definable generic set. Then the same is true for G.*

Proof. Let $s : G \to \widetilde{G}$ be a definable section of p. Consider the definable subset $Z = s(X)$ of $\widetilde{G}$ and let $Z_1, \dots, Z_k$ be its connected components. For each $i = 1, \dots, k$ take $a_i \in Z_i$ and consider the connected definable set $Y = \bigcup_{i=1}^k (Z_i - a_i)$ of $\widetilde{G}$. As in the proof of Lemma 2.3, note that $\langle Y \rangle$ is the connected component of $\langle Z \rangle$. Since $p(\langle Y \rangle)$ is a connected compatible subgroup of $p(\langle Z \rangle)$, and $p(\langle Z \rangle) = \langle X \rangle$ is connected, we deduce that $p(\langle Y \rangle) = \langle X \rangle$. On the other hand, by hypothesis there is a definable subset E of $\langle Y \rangle$ which is generic. Thus, $p(E)$ is a generic definable subset of $\langle X \rangle$, as desired. $\square$

Theorem 4.4. *Let G be an abelian semialgebraic group and let X be a semialgebraic subset of G. Then the group generated by X contains a generic semialgebraic subset.*

Proof. By Lemma 2.3, we can assume that X is connected. By [4, Thm 1.2] the quotient of G by its maximal normal semialgebraic torsion-free subgroup $\mathcal{N}(G)$ is semialgebraically compact. Then by Corollary 2.4 and Corollary 2.5 we can assume that G is semialgebraically compact. By [3, Thm 7.2] there is an irreducible algebraic group H defined over R, an open connected locally semialgebraic subgroup $\mathcal{W}$ of the o-minimal universal covering group $\widetilde{H(R)^o}$ of $H(R)^o$ and a locally semialgebraic homomorphism $\theta : \mathcal{W} \to G$ that is that o-minimal universal covering homomorphism of G.

Denote by $p : \widetilde{H(R)^o} \to H(R)^o$ the universal covering map. We have the exact sequence

$$0 \to \ker(p) \to \widetilde{H(R)^o} \to H(R)^o \to 0.$$

Note that $\ker(p)$ is discrete and therefore its only semialgebraically generated connected subgroup is the trivial one. Thus, by Corollary 2.4 and Theorem 4.2, we deduce that semialgebraically generated connected subgroups of $\widetilde{H^0(R)}$ contain a semialgebraic generic set. In particular, the same is true for $\mathcal{W}$, and by Proposition 4.3, for G, as required. $\square$

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