COUNTING ALGEBRAIC POINTS IN EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

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Abstract. The Pila-Wilkie theorem states that if a set $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structures $\mathcal{R}$ and contains 'many' rational points, then it contains an infinite semialgebraic set. In this paper, we extend the theorem to two important model theoretic settings. Let $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ be an expansion of $\mathcal{R}$ by a dense set P, which is either an elementary substructure of $\mathcal{R}$, or it is independent. We prove that if a set X is definable in $\widetilde{\mathcal{R}}$ and contains many rational points, then it is dense in an infinite semialgebraic set. Along the way, we introduce the notion of the *algebraic trace part* of any set X.

1. Introduction

Point counting theorems have recently occupied an important part of model theory, mainly due to their pivotal role in applications of o-minimality to number theory and Diophantine geometry. Arguably, the biggest breakthrough was the Pila-Wilkie theorem [20], which roughly states that if a definable set in an o-minimal structure contains "many" rational points, then it contains an infinite semialgebraic set. Pila employed this result together with the so-called Pila-Zannier strategy to give an unconditional proof of certain cases of the André-Oort Conjecture [19]. An excellent survey on the subject is [21]. Although several strengthenings of these theorems have since been established within the o-minimal setting, the topic remains largely unexplored in more general tame settings. In this paper, we establish point counting theorems in tame expansions of o-minimal structures by a dense set.

Recall that, for a set $X \subseteq \mathbb{R}^n$, the *algebraic part* X^{alg} of X is defined as the union of all infinite connected semialgebraic subsets of X. Pila in [19], generalizing [20], proved that if a set X is definable in an o-minimal structure, then $X \setminus X^{alg}$ contains "few" algebraic points of fixed degree (see Fact 2.3 below for a precise statement). This statement immediately fails if one leaves the o-minimal setting. For example, the set $\mathcal{A}$ of algebraic points itself contains many algebraic points, but $\mathcal{A}^{alg} = \emptyset$. However, adding $\mathcal{A}$ as a unary predicate to the language of an o-minimal structure results in a well-behaved model theoretic structure, and it is desirable to retain point counting theorems in that setting. We achieve this goal by means of the following definition.

Definition 1.1. Let $X \subseteq \mathbb{R}^n$. The *algebraic trace part* of X, denoted by X^{alg}_t, is the union of all traces of infinite connected semialgebraic sets in which X is dense.

Date: August 13, 2017.

2010 *Mathematics Subject Classification.* Primary 03C64, 11G99, Secondary 06F20.

Key words and phrases. o-minimal structure, algebraic point, dense pair, independent set.

Research supported by a Research Grant from the German Research Foundation (DFG) and a Zukunftskolleg Research Fellowship.

That is,

$$X_t^{alg} = \bigcup \{X \cap T : T \subseteq \mathbb{R}^n \text{ infinite connected semialgebraic, and } T \subseteq cl(X \cap T)\}.$$

The density requirement $T \subseteq cl(X \cap T)$ is essential: without it, we would always have $X_t^{alg} = X$, as witnessed by $T = \mathbb{R}^n$.

We first show in Section 2 that the above notion is a natural generalization of the usual notion of the algebraic part of a set, in the following sense.

Proposition 1.2. *Suppose $X \subseteq \mathbb{R}^n$ is definable in an o-minimal expansion of the real field. Then $X^{alg} = X_t^{alg}$.*

Then, in Sections 3 and 4, we establish point counting theorems in two main categories of structures that go beyond the o-minimal setting: dense pairs and expansions of o-minimal structures by a dense independent set. Namely, we prove that if X is a definable set in these settings, then $X \setminus X_t^{alg}$ contains "few" algebraic points of fixed degree (Theorem 1.3 below). We postpone a discussion about the general tame setting until the end of this introduction, as we now proceed to fix our notation and state the main theorem. Some familiarity with the basic notions of model theory, such as definability and elementart substructures, is assumed. The reader can consult [11, 16, 18].

For the rest of this paper, and unless stated otherwise, we let $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ be an o-minimal expansion of the real field, whose language is $\mathcal{L}$, and $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ an expansion of $\mathcal{R}$ by a set $P \subseteq \mathbb{R}$, in the expanded language $\mathcal{L}(P) = \mathcal{L} \cup \{P\}$. By '$A$-definable' we mean 'definable in $\widetilde{\mathcal{R}}$ with parameters from A', and by '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{R}$ with parameters from A'. We omit the index A if we do not want to specify the parameters. For a subset $A \subseteq \mathbb{R}$, we write $\text{dcl}(A)$ for the definable closure of A in $\mathcal{R}$, and $\text{dcl}_{\mathcal{L}(P)}(A)$ for the definable closure in $\widetilde{\mathcal{R}}$. We call a set $X \subseteq \mathbb{R}$ dcl-*independent over* A, if for every $x \in X$, $x \notin \text{dcl}((X \setminus \{x\}) \cup A)$, and simply dcl-*independent* if it is dcl-independent over $\emptyset$. An example of a dcl-independent set in the real field is a transcendence basis over $\mathbb{Q}$. Following [18], we define the *(multiplicative) height* $H(\alpha)$ of an algebraic point α as $H(\alpha) = \exp h(\alpha)$, where $h(\alpha)$ is the *absolute logarithmic height* from [6, page 16]. For a set $X \subseteq \mathbb{R}^n$, $k \in \mathbb{Z}^{>0}$ and $T \in \mathbb{R}^{>1}$, we define

$$X(k,T) = \{(\alpha_1, \dots, \alpha_n) \in X : \max_i [\mathbb{Q}(\alpha_i) : \mathbb{Q}] \leq k, \max_i H(\alpha_i) \leq T\}$$

and

$$N_k(X,T) = \#X(k,T).$$

Our main result is the following.

Theorem 1.3. *Suppose $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ is an o-minimal expansion of the real field, and $P \subseteq R$ is a dense set such that one of the following two conditions holds:*

(A) *$P \preccurlyeq \mathcal{R}$ is an elementary substructure.*
(B) *P is a* dcl-*independent set.*

Let $X \subseteq \mathbb{R}^n$ be definable in $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$, $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is a definable set $W \subseteq X_t^{alg}$ such that

$$N_k(X \setminus W, T) = O_{X,k,\epsilon}(T^\epsilon).$$

A few words about the general tame setting are in order. As o-minimality can only be used to model phenomena that are in general locally finite, many authors have early on sought expansions of o-minimal structures which escape from the o-minimal context, yet preserve the tame geometric behavior on the class of all definable sets. These expansions have recently seen significant growth ([1, 2, 5, 8, 10, 12, 15, 17]) and are by now divided into two important classes of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. Cases (A) and (B) from Theorem 1.3 belong to the first category, whereas further examples of this sort can be found in [8] and [13]. Certain point counting theorems in the second category have recently appeared in [7]. In both categories, sharp *cone decomposition theorems* are by now at our disposal ([13] and [22]), in analogy with the cell decomposition theorem known for o-minimal structures.

Expansions of $\mathcal{R}$ of type (A) are called *dense pairs* and were first studied in van den Dries [10], whereas expansions of type (B) were recently introduced in Dolich-Miller-Steinhorn [9]. These two examples are representative of the first category, and they are often thought of as "orthogonal" to each other, mainly because in the former case $\mathrm{dcl}(\emptyset) \subseteq P$, whereas in the latter, $\mathrm{dcl}(\emptyset) \cap P = \emptyset$. This orthogonality is vividly reflected in our proof of Theorem 1.3. Indeed, since the set $\mathcal{A}$ of algebraic points is contained in $\mathrm{dcl}(\emptyset)$, we have $\mathcal{A} \subseteq P$ and $\mathcal{A} \cap P = \emptyset$ in the two cases, respectively. Based on this observation, the proof for (A) becomes almost immediate, assuming facts from [10], whereas the one for (B) makes an essential use of the aforementioned cone decomposition theorems from [13]. More details about these proofs are postponed until the corresponding sections. Let us finally point out that far reaching generalizations of the two settings have already been developed, such as lovely pairs [3] and H-structures [4], respectively. Those settings can accommodate also structures coming from geometric stability theory, such as pairs of algebraically closed fields, or SU-rank 1 structures. Point counting theorems in those, and other interesting model-theoretic settings, are wildly unknown. and pending to be explored.

Notation. The topological closure of a set $X \subseteq \mathbb{R}^n$ is denoted by $cl(X)$. If $X, Z \subseteq \mathbb{R}^n$, we call X dense in Z, if $Z \subseteq cl(X \cap Z)$. Given any subset $X \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and $a \in \mathbb{R}^m$, we write X_a for

$$\{b \in \mathbb{R}^n \ : \ (a, b) \in X\}.$$

If $m \leq n$, then $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1}, unless stated otherwise. A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g \in S} \{g\} \times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g \in S} \{g\} \times J_g$. If $X, Y \subseteq \mathbb{R}$, we sometimes write XY for $X \cup Y$. By $\mathcal{A}$ we denote the set of algebraic points. If $M \subseteq \mathbb{R}$, by $M \preccurlyeq \mathcal{R}$ we mean that M is an elementary substructure of $\mathcal{R}$ in the language of $\mathcal{R}$.

Acknowledgments. The author wishes to thank Gal Binyamini, Chris Miller, Ya'acov Peterzil, Jonathan Pila, Patrick Speissegger, Pierre Villemot and Alex Wilkie for several discussions on the topic, and the Fields Institute for its generous support and hospitality during the Thematic Program on Unlikely Intersections, Heights, and Efficient Congruencing, 2017.

2. The algebraic trace part of a definable set

In this section, we introduce the notion of the *algebraic trace part* of a set, and prove that it generalizes the notion of the algebraic part of a set definable in an o-minimal structure. We also state a version of Pila's theorem [18], Fact 2.3 below, suitable for our purposes. Finally, in Subsection 2.1, we make a notational convention and state some basic facts.

The proof of Theorem 1.3, in both cases (A) and (B), is by reducing it to Pila's theorem, Fact 2.3 below. It is important to point out that the formulation of that fact involves a refined version of the algebraic part of a set, which prompts the following definitions.

Definition 2.1. Let $A \subseteq \mathbb{R}$ be a set. An *A-set* is an infinite connected semialgebraic set definable over A. If it is, in addition, a cell, we call it an *A-cell*.

We are mainly interested in $\mathbb{Q}$-sets. One important observation is that the set $\mathcal{A}$ of algebraic points is dense in every $\mathbb{Q}$-set. This fact will be crucial in the proofs of Lemma 3.1 and Theorem 4.15.

Definition 2.2. Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$. The *algebraic part of X over A*, denoted by X^{alg_A}, is the union of all A-subsets of X. That is,

$$X^{alg_A} = \bigcup\{T \subseteq X : T \text{ is an } A\text{-set}\}.$$

It is an effect of the proof in [18] that the following statement holds.

Fact 2.3. *Let $X \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable, $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is a definable set $W \subseteq X^{alg_\mathbb{Q}}$ such that*

$$N_k(X \setminus W, T) = O_{X,k,\epsilon}(T^\epsilon).$$

It is this version of Pila's theorem in [18] that we will reduce our Theorem 1.3 to. Let us now also refine Definition 1.1 from the introduction, as follows.

Definition 2.4. Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}$. The *algebraic trace part of X over A*, denoted by $X_t^{alg_A}$ is the union of all traces of A-sets in which X is dense. That is,

$$X_t^{alg_A} = \bigcup\{X \cap T : T \text{ an } A\text{-set, } X \text{ dense in } T\}$$

The goal of this section is to prove the following proposition. We note that this result is independent of the rest of the paper, but it provides canonicity for our definitions.

Proposition 2.5. *Let $X \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set, and $A \subseteq \mathbb{R}$. Then*

$$X^{alg_A} = X_t^{alg_A}.$$

The proof of the proposition requires several lemmas. The first one asserts that, under certain assumptions, the property of being dense in a set passes to its subsets.

Lemma 2.6. *Let $X, Z \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable sets, with $Z \subseteq cl(Z \cap X)$. Suppose that $Z_0 \subseteq Z$ is a cell with $\dim Z_0 = \dim Z$. Then $Z_0 \subseteq cl(Z_0 \cap X)$.*

Proof. Let $x \in Z_0$, and suppose towards a contradiction that $x \notin cl(Z_0 \cap X)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x such that $B \cap Z_0 \cap X = \emptyset$. It follows immediately that for every $x' \in B \cap Z_0$, $x' \notin cl(Z_0 \cap X)$. Since $Z \subseteq cl(Z \cap X)$,

$$B \cap Z_0 \subseteq cl((Z \setminus Z_0) \cap X) \subseteq cl(Z \setminus Z_0)$$

and, hence,

$$B \cap Z_0 \subseteq cl(Z \setminus Z_0) \setminus (Z \setminus Z_0).$$

Moreover, since Z_0 is a cell, $\dim(Z_0) = \dim(B \cap Z_0)$. It follows that

$$\dim(Z_0) < \dim(Z \setminus Z_0) \leq \dim Z,$$

a contradiction. □

We will need a local version of Lemma 2.6. First, a definition.

Definition 2.7. Let $Z \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set and $x \in Z$. The *local dimension of Z at x* is defined to be

$$\dim_x(Z) = \min\{\dim(B \cap Z) : \; B \subseteq \mathbb{R}^n \text{ an open box containing } x\}.$$

Lemma 2.8. *Let $X, Z \subseteq \mathbb{R}^n$ be infinite $\mathcal{L}$-definable sets with $Z \subseteq cl(Z \cap X)$, and $x \in Z$. Suppose $Z_0 \subseteq Z$ is an A-cell with $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Moreover, $B \cap Z_0$ is an A-cell.*

Proof. Let $Z \setminus Z_0 = Z_1 \cup \cdots \cup Z_m$ be a decomposition into cells. It is not hard to see from the definition of $\dim_x(Z)$, that there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that for every $1 \leq i \leq m$, if $B \cap Z_i \neq \emptyset$, then $\dim_x(Z) \geq \dim Z_i$. We may shrink B if needed so that $B \cap Z_0$ becomes an A-cell. Let I be the set of indices $1 \leq i \leq m$ such that $B \cap Z_i \neq \emptyset$. Set

$$Z' := B \cap Z.$$

Since $Z \subseteq cl(Z \cap X)$, we easily obtain that $Z' \subseteq cl(Z' \cap X)$. Moreover, since $x \in cl(Z)$, we have

$$Z' = (B \cap Z_0) \cup \bigcup_{i \in I} (B \cap Z_i),$$

and hence $\dim Z' = \dim(Z_0 \cap B)$. Therefore, by Lemma 2.6 (for Z' and $B \cap Z_0 \subseteq Z'$),

$$B \cap Z_0 \subseteq cl(B \cap Z_0 \cap X) \subseteq cl(Z_0 \cap X),$$

as needed. □

We also need the following lemma.

Lemma 2.9. *Let $Z \subseteq \mathbb{R}^n$ be an A-cell, $T \subseteq Z$ a definable set, and $x \in cl(Z)$. Suppose that $\dim T < \dim Z$. Then there is an A-set $W \subseteq Z \setminus T$ with $x \in cl(W)$.*

Proof. We work by induction on $n > 0$. For $n = 0$, it is trivial. Let $n > 0$. We split into two cases:

Case I: $\dim Z = n$. Since $\dim T < \dim Z$, it follows easily, by cell decomposition, that there is a line segment $W \subseteq Z$ with initial point x, staying entirely outside T.

Case II: $\dim Z = k < n$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a suitable coordinate projection such that $\pi_{\restriction Z}$ is injective. Then $\pi(Z)$ is an A-cell, $\pi(T) \subseteq \pi(Z)$, $\dim \pi(T) < \dim \pi(Z)$ and $\pi(x) \in cl(\pi(Z))$. By induction hypothesis, there is an A-set $W_1 \subseteq \pi(Z) \setminus \pi(T)$, such that $\pi(x) \in cl(W_1)$. Let

$$W = \pi^{-1}(W_1) \cap Z.$$

Then W is clearly an A-set with $W \subseteq Z \setminus T$, and it is also easy to check that $x \in cl(W)$. □

We are now ready to prove Proposition 2.5.

Proof of Proposition 2.5. It is immediate that $X^{alg_A} \subseteq X_t^{alg_A}$. For the opposite inclusion, let Z be an A-set with $Z \subseteq cl(Z \cap X)$. We need to prove that every point $x \in Z \cap X$ is contained in an A-set W contained in X. By cell decomposition in the real field, and since Z is connected, there is a semialgebraic cell $Z_0 \subseteq Z$ over A, such that $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. By Lemma 2.8, there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0$ is an A-cell and $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Let

$$T = (B \cap Z_0) \setminus (Z_0 \cap X) \subseteq cl(Z_0 \cap X) \setminus (Z_0 \cap X).$$

Then

$$\dim T < \dim(Z_0 \cap X) \leq \dim Z_0 = \dim(B \cap Z_0).$$

Also, $x \in Z \setminus T$. Therefore, by Lemma 2.9 (for $Z = B \cap Z_0$), there is an A-set $W_0 \subseteq (B \cap Z_0) \setminus T$ with $x \in cl(W_0)$. But

$$(B \cap Z_0) \setminus T = B \cap Z_0 \cap X,$$

so $W_0 \subseteq X$. Since $x \in cl(W_0)$, the set $W = W_0 \cup \{x\}$ is connected, and hence the desired A-set. $\square$

2.1. **Convention - terminology.** As the rest of this paper exclusively deals with algebraic trace parts of sets over $\mathbb{Q}$, we adopt the following convention.

For a set $X \subseteq \mathbb{R}^n$, X^{alg} denotes $X_t^{alg_{\mathbb{Q}}}$, the algebraic trace part of X over $\mathbb{Q}$.

Note that then Theorems 3.2 and 4.15 below provide stronger versions of Theorem 1.3. Moreover, whenever $X \subseteq \mathbb{R}^n$ is $\mathcal{L}$-definable, there is no ambiguity in talking of X^{alg}, in view of Proposition 2.5.

In preparation of the point counting theorems, we introduce some further terminology.

Definition 2.10. Let $X \subseteq \mathbb{R}^n$. We say that X *has controlled algebraic points* if for every $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$, there is a definable set $W \subseteq X^{alg}$ such that

$$N_k(X \setminus W, T) = O_{X,k,\epsilon}(T^\epsilon).$$

We say that W *(k,ϵ)-controls* the algebraic points of X.

We finish with a basic lemma.

Lemma 2.11. *Let $X, Y \subseteq \mathbb{R}^n$ be two definable sets.*

(1) *If $X \subseteq Y$, then $X^{alg} \subseteq Y^{alg}$.*
(2) *If X and Y have controlled algebraic points, then so does $X \cup Y$.*

Proof. (1) Let Z be a $\mathbb{Q}$-set contained in $cl(X \cap Z)$. It is then contained in $cl(Y \cap Z)$.

(2) Let $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$, and let W and Z (k,ϵ)-control the algebraic points of X and Y, respectively. Of course, $X_0 \cup Y_0 \subseteq X^{alg} \cup Y^{alg} \subseteq (X \cup Y)^{alg}$. Since

$$X \cup Y \setminus X_0 \cup Y_0 = (X \setminus X_0 \cup Y_0) \cup (Y \setminus X_0 \cup Y_0) \subseteq (X \setminus X_0) \cup (Y \setminus Y_0),$$

we obtain that that $W \cup Z$ (k,ϵ)-controls the algebraic points of $X \cup Y$. $\square$

3. Dense pairs

In this section, we let $\widetilde{\mathcal{R}} = \langle \mathbb{R}, P \rangle$ be a dense pair. As mentioned in the introduction, since $P \preccurlyeq \mathcal{R}$, we have $\mathcal{A} \subseteq \mathrm{dcl}(\emptyset) \subseteq P$. Recall that the convention from Section 2.1 is in force.

Lemma 3.1. *Let $X = Y \cap P^n$, for some $\mathcal{L}$-definable set $Y \subseteq \mathbb{R}^n$. Then*

$$X \cap Y^{alg} \subseteq X^{alg}.$$

Proof. Let $x \in X \cap Y^{alg}$. So x is contained in a $\mathbb{Q}$-set $Z \subseteq Y$. We prove that X is dense in Z. Observe that $Z \cap X = Z \cap P^n$. Since the set of algebraic points $\mathcal{A}$ is dense in Z, and $\mathcal{A} \subseteq P^n$, we have

$$Z \subseteq cl(Z \cap \mathcal{A}^n) \subseteq cl(Z \cap P^n) = cl(Z \cap X),$$

and hence X is dense in Z. □

Theorem 3.2. *Every definable set has controlled algebraic points.*

Proof. Let $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. We first observe that if the statement holds for $X \cap P^n$, then it holds for X. Indeed, let $W \subseteq (X \cap P^n)^{alg} \subseteq X^{alg}$ be an $\mathcal{L}$-definable set that (k, ϵ)-controls the algebraic points of $X \cap P^n$. Since $\mathcal{A}^n \subseteq P^n$, the set X has the same algebraic points as $X \cap P^n$, and hence W also (k, ϵ)-controls those points.

We may thus assume that $X \subseteq P^n$. By [10, Theorem 2], there is an $\mathcal{L}$-definable $Y \subseteq \mathbb{R}^n$, such that $X = Y \cap P^n$. By Fact 2.3, there is $\mathcal{L}$-definable $Z \subseteq Y^{alg}$ that (k, ϵ)-controls the algebraic points of Y. Observe that, by Lemma 3.1,

$$X \cap Z \subseteq X \cap Y^{alg} \subseteq X^{alg}.$$

Since

$$X \setminus (X \cap Z) \subseteq Y \setminus Z,$$

we obtain that $X \cap Z$ (k, ϵ)-controls the algebraic points of X. □

4. Dense independent sets

In this section (except for Subsection 4.3), $P \subseteq \mathbb{R}$ is a dense dcl-independent set. The proof of Theorem 4.15 runs by induction on the *large dimension* of a definable set X (Definition 4.8), by making essential use of the *cone decomposition theorem* from [13] (Fact 4.1). As mentioned in the introduction, since P contains no elements of $\mathrm{dcl}(\emptyset)$, we obtain $P \cap \mathcal{A} = \emptyset$. The base step of the aforementioned induction is to show a generalization of this fact; namely, that for a *small* set X (Definition 4.1), $X \cap \mathcal{A}$ is finite (Corollary 4.12). We remind the reader that the convention from Section 2.1 is still in force here.

4.1. Cone decomposition theorem.

In this subsection we recall the necessary background from [13]. The following definition is taken essentially from [12].

Definition 4.1. Let $X \subseteq \mathbb{R}^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : \mathbb{R}^{nm} \to \mathbb{R}$ such that $f(X^m)$ contains an open interval in $\mathbb{R}$. We call X *small* if it is not large.

The notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both notions, supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way. Although this uniformity plays no particular role in this paper, we include it here to match the definitions from [13].

Definition 4.2 ([13])**.** A *supercone* $J \subseteq \mathbb{R}^k$, $k \geq 0$, is defined recursively as follows:

- $\mathbb{R}^0 = \{0\}$ is a supercone.
- A definable set $J \subseteq \mathbb{R}^{n+1}$ is a supercone if $\pi(J) \subseteq \mathbb{R}^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it.

Note that, for $k > 0$, the interior U of $cl(J)$ is an open cell. We call U *the shell of* J. It is the unique open cell in $\mathbb{R}^k$ with $cl(U) = cl(J)$. For $k = 0$, the shell of J is defined to be $J = \mathbb{R}^0$.

Recall that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g \in S}$ with $\bigcup_{g \in S}\{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 4.3 (Uniform families of supercones [13])**.** Let $\mathcal{J} = \bigcup_{g \in S}\{g\} \times J_g \subseteq \mathbb{R}^{m+k}$ be a definable family of supercones. We call $\mathcal{J}$ *uniform* if there is a cell $V \subseteq \mathbb{R}^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for $\mathcal{J}$.

Remark 4.4. A shell for a uniform family of supercones $\mathcal{J}$ need not be unique. One can easily see a supercone $J \subseteq \mathbb{R}^k$ as a uniform family of supercones $\mathcal{J} \subseteq M^{m+k}$ with $\pi_m(\mathcal{J})$ a singleton; in that case, a shell for $\mathcal{J}$ is unique and equals that of J.

Definition 4.5 (Cones [13] and H-cones)**.** A set $C \subseteq \mathbb{R}^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq \mathbb{R}^m$, a uniform family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in $\mathbb{R}^k$, and an $\mathcal{L}$-definable continuous function $h : V \subseteq \mathbb{R}^{m+k} \to \mathbb{R}^n$, where V is a shell for $\mathcal{J}$, such that

(1) $C = h(\mathcal{J})$, and
(2) for every $g \in S$, $h(g, -) : V_g \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is injective.

We call C a *k-H-cone* if, in addition, $S \subseteq P^m$ and $h : \mathcal{J} \to \mathbb{R}^n$ is injective. An *(H-)cone* is a k-(H-)cone for some k.

The cone decomposition theorem [13, Theorem 5.1] is a statement about definable sets and functions. Here we are only interested in a decomposition of sets into H-cones. Before stating the H-cone decomposition theorem, we need the following fact.

Fact 4.6. *Let $S \subseteq \mathbb{R}^n$ be an A-definable small set. Then S is a finite union of sets of the form $f(X)$, where*

- *$f : Z \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is an $\mathcal{L}_A$-definable continuous map,*
- *$X \subseteq P^m \cap Z$ is A-definable, and*
- *$f : X \to \mathbb{R}^l$ is injective.*

Proof. By [13, Lemma 3.11], there is an $\mathcal{L}_A$-definable map $h : \mathbb{R}^m \to \mathbb{R}^n$ such that $X \subseteq h(P^m)$. The result follows from [14, Theorem 2.2]. $\square$

Fact 4.7 (H-cone decomposition theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be an A-definable set. Then X is a finite union of A-definable H-cones.*

Proof. By [13, Theorem 5.12] and [14, Theorem 2.2], X is a finite union of A-definable cones $h(\mathcal{J})$ with $h : \mathcal{J} \to \mathbb{R}^n$ injective (such $h(\mathcal{J})$ is called *strong cone* in the above references). By Fact 4.6, it is not hard to see that $h(\mathcal{J})$ is a finite union of A-definable H-cones. $\square$

We next recall the notion of 'large dimension' from [13].

Definition 4.8 (Large dimension [13])**.** Let $X \subseteq \mathbb{R}^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of X by $\mathrm{ldim}(X)$.

Some basic properties of the large dimension that will be used in the sequel are the following (see [13, Lemma 6.11]): for every two definable sets $X, Y \subseteq \mathbb{R}^n$,

- if $X \subseteq Y$, then $\mathrm{ldim}X \leq \mathrm{ldim}Y$.
- if X is $\mathcal{L}$-definable, then $\mathrm{ldim}X = \dim X$.
- X is small if and only if $\mathrm{ldim}X = 0$.

4.2. **Point counting.** We now proceed to the proof of Theorem 1.3 (2). We need several preparatory lemmas. First, a very useful fact.

Fact 4.9. *Suppose* (OP) *holds. Then for every $A \subseteq \mathbb{R}$ with $A \setminus P$* dcl*-independent over P, we have $\mathrm{dcl}_{\mathcal{L}(P)}(A) = \mathrm{dcl}(A)$.*

Proof. Take $x \in \mathrm{dcl}_{\mathcal{L}(P)}(A)$. That is, the set $\{x\}$ is A-definable in $\langle \mathcal{R}, P\rangle$. By (OP), since $A \setminus P$ is dcl-independent over P, we have that $cl(\{x\})$ is $\mathcal{L}_A$-definable. But $cl(\{x\}) = \{x\}$. $\square$

The following lemma is crucial and uses the fact that P is dcl-independent.

Lemma 4.10. *Let $h : Z \subseteq P^m \times \mathbb{R}^k \to \mathbb{R}$ be a definable injective map. Let $B \subseteq \mathbb{R}$ be a finite set. Then there is a finite set $S_0 \subseteq P^m$ such that*

$$h\left(\bigcup_{g \in P^m \setminus S_0} \{g\} \times Z_g\right) \cap \mathrm{dcl}(B) = \emptyset.$$

Proof. Suppose $h(Z)$ is A-definable, with A finite. Let $A_0 \subseteq A \cup B$ and $P_0 \subseteq P$ finite be so that $A \cup B \subseteq \mathrm{dcl}(A_0 P_0)$ and A_0 is dcl-independent over P. Suppose $q = h(g, t)$, where $g \in P^m$, $t \in Z_g$ and $q \in \mathrm{dcl}(B)$. By injectivity of h, all coordinates of g are in

$$\mathrm{dcl}_{\mathcal{L}(P)}(Aq) \subseteq \mathrm{dcl}_{\mathcal{L}(P)}(AB) \subseteq \mathrm{dcl}_{L(P)}(A_0 P_0) = \mathrm{dcl}(A_0 P_0).$$

Since P is dcl-independent, there can only be $|A_0 P_0|$ many such g's, and so can q's. $\square$

Two particular cases of the above lemma are the following (recall, $\mathcal{A} \subseteq \mathrm{dcl}(\emptyset)$).

Corollary 4.11. *Let $C = h(\bigcup_{g \in S} \{g\} \times J_g)$ be an H-cone. Then there is a finite set $S_0 \subseteq S$ such that $h\left(\bigcup_{g \in S \setminus S_0} \{g\} \times J_g\right)$ contains no algebraic points.*

Corollary 4.12. *Every small set contains only finitely many algebraic points.*

Proof. By Lemma 4.10, for $k = 0$, and Fact 4.6. □

The key lemma in the inductive step of the proof of Theorem 4.15 is the following.

Lemma 4.13. *Let $J \subseteq \mathbb{R}^k$ be a supercone with shell Z, and $B \subseteq \mathbb{R}$ finite. Then there is an $\mathcal{L}$-definable set $F \subseteq Z$ with $\dim(F) < k$, such that*

$$Z \cap \operatorname{dcl}(B) \subseteq J \cup F.$$

Proof. By induction on k. For $k = 0$, the statement is trivial. For $k > 0$, assume $J = \bigcup_{g\in\Gamma}\{g\} \times J_g$, where $\Gamma \subseteq \mathbb{R}^{k-1}$ is a supercone. By inductive hypothesis, there is $F_1 \subseteq \pi(Z)$, such that

$$\pi(Z) \cap \operatorname{dcl}(B) \subseteq \Gamma \cup F.$$

Since $\dim(F_1 \times \mathbb{R}) < k$, it suffices to write $\left(\bigcup_{g\in\Gamma}\{g\} \times Z_g\right) \cap \operatorname{dcl}(B)$ as a subset of $J \cup F_2$, for some $F_2 \subseteq Z$ with $\dim(F_2) < k$. Let

$$X = \bigcup_{g\in\Gamma}\{g\} \times (Z_g \setminus J_g).$$

So we need to prove that $X \cap \operatorname{dcl}(B)$ is contained in an $\mathcal{L}$-definable set $F_2 \subseteq Z$ with $\dim(F_2) < k$. By [14, Theorem 2.2] and [13, Corollary 5.11], X is a finite union of sets $X_1, \dots, X_l$, each of the form

$$X_i = f\left(\bigcup_{g\in S}\{g\} \times U_g\right),$$

where

- $f : V \subseteq \mathbb{R}^{m+k-1} \to \mathbb{R}^n$ is an $\mathcal{L}$-definable continuous map,
- $U \subseteq (S \times \Gamma) \cap V$ is a definable set, and
- $f_{\restriction U}$ is injective.

Using Fact 4.6, we may further assume that $S \subseteq P^m$. By Lemma 4.10, for $h = f$, there is a finite set $S_0 \subseteq P^m$ such that

$$f\left(\bigcup_{g\in S\setminus S_0}\{g\} \times U_g\right) \cap \operatorname{dcl}(B) = \emptyset.$$

For each $i = 1, \dots, l$, and X_i as above, set Set

$$D_i = f\left(\bigcup_{g\in S_0}\{g\} \times U_g,\right).$$

Then $F_2 = \bigcup_{i=1}^{l} D_i$ satisfies the required properties. □

Corollary 4.14. *Let $C = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone with shell Z, and $h : Z \to \mathbb{R}^n$ an $\mathcal{L}$-definable and injective map. Then there is a definable set $F \subseteq Z$ with $\dim(F) < k$, such that all algebraic points of $h(Z)$ are contained in $h(J \cup F)$.*

Proof. Suppose h is $\mathcal{L}_B$-definable, and take F be as in Lemma 4.13. Let $x = h(y) \in h(Z)$ be an algebraic point. In particular, $x \in \operatorname{dcl}(\emptyset)$. Since h is $\mathcal{L}$-definable and injective, $y \in \operatorname{dcl}(B) \subseteq J \cup F$. □

Theorem 4.15. *Every definable set has controlled algebraic points.*

Proof. Let $X \subseteq \mathbb{R}^n$ be a definable set. We work by induction on large dimension of X. If $\mathrm{ldim}(X) = 0$, then X is small and the statement follows from Corollary 4.12. Assume $\mathrm{ldim}(X) = k > 0$. By Facts 4.7 and 2.11(2), we may assume that X is a k-H-cone, say $h(\mathcal{J})$ with $\mathcal{J} \subseteq \mathbb{R}^{m+k}$. By Corollary 4.11, we may further assume that $\pi_m(\mathcal{J})$ is a singleton, and hence, that $X = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone. Let Z be the shell of J, and $F \subseteq Z \setminus J$ as in Corollary 4.14. We have that $X \subseteq h(Z \setminus F) \cup h(F)$. By Fact 2.11(2), it suffices to show the statement for each of $X \cap h(Z \setminus F)$ and $X \cap h(F)$.

$X \cap h(F)$. We have

$$\mathrm{ldim}(X \cap h(F)) \leq \mathrm{ldim}\, h(F) = \dim h(F) < k,$$

and hence we conclude by inductive hypothesis.

$X \cap h(Z \setminus F)$. Let $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. By Fact 2.3, there is an $\mathcal{L}$-definable set $W \subseteq h(Z \setminus F)^{alg}$ that (k, ϵ)-controls the algebraic points of $h(Z \setminus F)$. Observe that

$$X \cap W \subseteq X \cap h(Z \setminus F)^{alg} \subseteq (X \cap h(Z \setminus F))^{alg}.$$

Indeed, for the latter inclusion, let $Y \subseteq h(Z \setminus F)$ be a $\mathbb{Q}$-set. We need to show that $Y \subseteq cl(X \cap h(Z \setminus F))$. By the conclusion of Corollary 4.14, $Y \cap \mathcal{A}^n \subseteq Y \cap X$. Since the set of algebraic points $\mathcal{A}$ is dense in Y, we obtain that

$$Y \subseteq cl(Y \cap \mathcal{A}^n) \subseteq cl(Y \cap X) \subseteq cl(h(Z \setminus F) \cap X),$$

as required. Since

$$(X \cap (Z \setminus F)) \setminus (X \cap W) \subseteq (Z \setminus F) \setminus W,$$

$X \cap W$ (k, ϵ)-controls the algebraic points of $X \cap (Z \setminus F)$. □

4.3. **Concluding remarks.** We finish with two remarks regarding Theorem 1.3.

(1) For $\mathcal{R} = \mathbb{R}$, Theorem 1.3 is trivial. Indeed, it is known in cases (A) and (B) that if X is a definable set, then $cl(X)$ is $\mathcal{L}$-definable (see, for example, [13, Section 2]). So, if $\mathcal{R} = \mathbb{R}$, $cl(X)$ is semialgebraic, and hence $X^{alg} = X$.

(2) A parametric version of Theorem 1.3 is also possible, see Theorem 4.16 below. Indeed, a parametric version of Fact 2.3 was established in [18, Theorem 5.3] (and was necessary in its proof). Here, Theorem 4.16 is proved by reducing it to [18, Theorem 5.3] via the exact same steps we took for Theorem 1.3. A parametric version of Lemma 4.13 is also needed and possible to establish. To avoid burdening the paper with further technicality, we include the parametric version without a proof.

Theorem 4.16. *Suppose $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ is an o-minimal expansion of the real field, and $P \subseteq R$ is a dense set such that one of the following two conditions holds:*

1. *$P \preccurlyeq \mathcal{R}$ is an elementary substructure.*
2. *P is a* dcl*-independent set.*

Let $X \subseteq \mathbb{R}^{m+n}$ be definable in $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$, $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. Then there is a definable set $W \subseteq \mathbb{R}^{m+n}$, such that for every $a \in \mathbb{R}^m$, $W_a \subseteq X_a^{alg}$, and

$$N_k(X_a \setminus W_a, T) = O_{X,k,\epsilon}(T^\epsilon).$$

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