PRODUCT CONES IN DENSE PAIRS

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Abstract. Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be an o-minimal expansion of an ordered group, and $P \subseteq M$ a dense set such that certain tameness conditions hold. We introduce the notion of a *product cone* in $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$, and prove: if $\mathcal{M}$ expands a real closed field, then $\widetilde{\mathcal{M}}$ admits a product cone decomposition. If $\mathcal{M}$ is linear, then it does not. In particular, we settle a question from [10].

1. Introduction

Tame expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ of an o-minimal structure $\mathcal{M}$ by a set $P \subseteq M$ have received lots of attention in recent literature ([1, 2, 3, 4, 6, 7, 12, 14]). One important category is when every definable open set is already definable in $\mathcal{M}$. Dense pairs and expansions of $\mathcal{M}$ by a dense independent set or by a dense multiplicative group with the Mann Property are of this sort. In [10], all these examples were put under a common perspective and a cone decomposition theorem was proved for their definable sets and functions. This theorem provided an analogue of the cell decomposition theorem for o-minimal structures in this context, and was inspired by the cone decomposition theorem established for semi-bounded o-minimal structures ([8, 9, 15]). The central notion is that of a *cone*, and, as its definition in [10] appeared to be quite technical, in [10, Question 5.14], we asked whether it can be simplified in two specific ways. In this paper we refute both ways in general, showing that the definition in [10] is optimal, but prove that if $\mathcal{M}$ expands a real closed field, then a *product cone* decomposition theorem does hold.

In Section 2, we provide all necessary background and definitions. For now, let us only point out the difference between product cones and cones, and state our main theorem. Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be an o-minimal expansion of an ordered group in the language $\mathcal{L}$, and $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ an expansion of $\mathcal{M}$ by a set $P \subseteq M$ such that certain tameness conditions hold (those are listed in Section 2). For example, $\widetilde{\mathcal{M}}$ can be a dense pair ([6]), or P can be a dense independent set ([5]) or a multiplicative group with the Mann Property ([7]). By 'definable' we mean 'definable in $\widetilde{\mathcal{M}}$, and by $\mathcal{L}$-definable we mean 'definable in $\mathcal{M}$'. The notion of a *small* set is given in Definition 2.1 below, and it is equivalent to the classical notion of being P-internal from geometric stability theory ([10, Lemma 3.11 and Corollary 3.12]). A *supercone* generalizes the notion of being co-small in an interval (Definition 2.2). Now, and roughly speaking, a cone is then defined as a set of the

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form

$$h\left(\bigcup_{g\in S}\{g\}\times J_g\right),$$

where h is an $\mathcal{L}$-definable continuous map with each $h(g,-)$ injective, $S\subseteq M^m$ is a small set, and $\{J_g\}_{g\in S}$ is a definable family of supercones. In Definition 2.4 below, we call a cone a *product cone* if we can replace the above family $\{J_g\}_{g\in S}$ by a product $S\times J$. That is, C has the form

$$h(S\times J),$$

with h and S as above and J a supercone. Let us say that $\widetilde{\mathcal{M}}$ *admits a product cone decomposition* if every definable set is a finite union of product cones. Our main theorem below asserts whether $\widetilde{\mathcal{M}}$ admits a product cone decomposition or not based solely on assumptions on $\mathcal{M}$.

Theorem 1.1.

(1) *If $\mathcal{M}$ is linear, the $\widetilde{\mathcal{M}}$ does not admit a product cone decomposition.*
(2) *If $\mathcal{M}$ expands a real closed field, then $\widetilde{\mathcal{M}}$ admits a product cone decomposition theorem.*

Theorem 1.1(1), in particular, refutes [10, Question 5.14(2)]. [10, Question 5.14(1)] further asked whether one can define a supercone as a product of co-small sets in intervals, and still obtain a cone decomposition theorem. In Proposition 4.2 we also refute that question in general, by constructing a counterexample whenever $\mathcal{M}$ expands a real closed field.

Remark 1.2. Theorem 1.1 deals with the two main categories of o-minimal structures; namely, $\mathcal{M}$ is linear or it expands a real closed field. In the 'intermediate', semi-bounded case ([9]), where $\mathcal{M}$ defines a field on a bounded interval but not on the whole of M, the answer to [10, Question 5.14] is rather unclear. Indeed, in the presence of two different notions of cones in this setting, the semi-bounded cones (from [9]) and the current ones, the methods in Sections 3.1 and 3.2 do not seem to apply and a new approach is needed.

Notation. The topological closure of a set $X\subseteq M^n$ is denoted by $cl(X)$. Given any subset $X\subseteq M^m\times M^n$ and $a\in M^n$, we write X_a for

$$\{b\in M^m\ :\ (b,a)\in X\}.$$

If $m\le n$, then $\pi_m : M^n\to M^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1}, unless stated otherwise. A family $\mathcal{J}=\{J_g\}_{g\in S}$ of sets is called definable if $\bigcup_{g\in S}\{g\}\times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g\in S}\{g\}\times J_g$.

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2. Preliminaries

In this section we lay out all necessary background and terminology. Most of it is extracted from [10, Section 2], where the reader is referred to for an extensive account. We fix an o-minimal theory T expanding the theory of ordered abelian groups with a distinguished positive element 1. We denote by $\mathcal{L}$ the language of T

and by $\mathcal{L}(P)$ the language $\mathcal{L}$ augmented by a unary predicate symbol P. Let $\widetilde{T}$ be an $\mathcal{L}(P)$-theory extending T. If $\mathcal{M} = \langle M, <, +, \dots \rangle \models T$, then $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of $\mathcal{M}$ that models $\widetilde{T}$. By 'A-definable' we mean 'definable in $\widetilde{\mathcal{M}}$ with parameters from A'. By '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{M}$ with parameters from A'. We omit the index A if we do not want to specify the parameters. For a subset $A \subseteq M$, we write $\mathrm{dcl}(A)$ for the definable closure of A in $\mathcal{M}$, and for an $\mathcal{L}$-definable set $X \subseteq M^n$, we write $\dim(X)$ for the corresponding pregeometric dimension. The following definition is taken essentially from [7].

Definition 2.1. Let $X \subseteq M^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : M^{nm} \to M$ such that $f(X^m)$ contains an open interval in M. We call X *small* if it is not large.

Consider the following properties.

Tameness Conditions ([10]):

(I) P is small.

(II) Every A-definable set $X \subseteq M^n$ is a boolean combination of sets of the form

$$\{x \in M^n : \exists z \in P^m \varphi(x, z)\},$$

where $\varphi(x, z)$ is an $\mathcal{L}_A$-formula.

(III) (Open definable sets are $\mathcal{L}$-definable) For every parameter set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^s$, its topological closure $cl(V) \subseteq M^s$ is $\mathcal{L}_A$-definable.

From now on, we assume that every model $\widetilde{\mathcal{M}} \models \widetilde{T}$ satisfies Conditions (I)-(III) above. We fix a sufficiently saturated model $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle \models \widetilde{T}$.

We next turn to define the central notions of this paper. As mentioned in the introduction, the notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both notions, supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way.

Definition 2.2 ([10])**.** A *supercone* $J \subseteq M^k$, $k \geq 0$, is defined recursively as follows:

- $M^0 = \{0\}$ is a supercone.
- A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : M^n \to M \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it.

Abusing terminology, we call a supercone *A-definable* if it is an A-definable set and its closure is $\mathcal{L}_A$-definable.

Note that, for $k > 0$, the interior U of $cl(J)$ is an open cell.

Recall that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g \in S}$ with $\bigcup_{g \in S}\{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 2.3 (Uniform families of supercones [10])**.** Let $\mathcal{J} = \bigcup_{g \in S}\{g\} \times J_g \subseteq M^{m+k}$ be a definable family of supercones. We call $\mathcal{J}$ *uniform* if there is a cell

$V \subseteq M^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for $\mathcal{J}$. Abusing terminology, we call $\mathcal{J}$ *A-definable*, if it is an A-definable family of sets and has an $\mathcal{L}_A$-definable shell.

In particular, if $\mathcal{J}$ is uniform, then so is each projection $\pi_{m+j}(\mathcal{J})$. Moreover, if V is a shell for $\mathcal{J}$, then $\pi_{m+j}(V)$ is a shell for $\pi_{m+j}(\mathcal{J})$. Observe also that if V is a shell for $\mathcal{J}$, then for every $x \in \pi_{m+k-1}(\mathcal{J})$, $\mathcal{J}_x$ is co-small in V_x.

A shell for $\mathcal{J}$ need not be unique. Whenever we say that $\mathcal{J}$ is a uniform family of supercones with shell V, we just mean that V is one of the shells for $\mathcal{J}$.

Definition 2.4 (Cones [10] and product cones)**.** A set $C \subseteq M^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq M^m$, a uniform family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in M^k, and an $\mathcal{L}$-definable continuous function $h : V \subseteq M^{m+k} \to M^n$, where V is a shell for $\mathcal{J}$, such that

(1) $C = h(\mathcal{J})$, and
(2) for every $g \in S$, $h(g, -) : V_g \subseteq M^k \to M^n$ is injective.

We call C a *k-product cone* if, moreover, $\mathcal{J} = S \times J$, for some supercone $J \subseteq M^k$. A *(product) cone* is a k-(product) cone for some k. Abusing terminology, we call a cone $h(\mathcal{J})$ *A-definable* if h is $\mathcal{L}_A$-definable and $\mathcal{J}$ is A-definable.

The cone decomposition theorem below (Fact 2.6) is a statement about definable sets and functions. The notion of a 'well-behaved' function in this setting is given next.

Definition 2.5 (Fiber $\mathcal{L}$-definable maps [10])**.** Let $C = h(\mathcal{J}) \subseteq M^n$ be a k-cone with $\mathcal{J} \subseteq M^{m+k}$, and $f : D \to M$ a definable function with $C \subseteq D$. We say that f is *fiber $\mathcal{L}$-definable with respect to C* if there is an $\mathcal{L}$-definable continuous function $F : V \subseteq M^{m+k} \to M$, where V is a shell containing $\mathcal{J}$, such that

- $(f \circ h)(x) = F(x)$, for all $x \in \mathcal{J}$.

We call f *fiber $\mathcal{L}_A$-definable with respect to C* if F is $\mathcal{L}_A$-definable.

As remarked in [10, Remark 4.5(4)], the terminology is justified by the fact that, if f is fiber $\mathcal{L}_A$-definable with respect to $C = h(\mathcal{J})$, then for every $g \in \pi(\mathcal{J})$, f agrees on $h(g, J_g)$ with an $\mathcal{L}_{Ag}$-definable map; namely $F \circ h(g, -)^{-1}$. Moreover, the notion of being fiber $\mathcal{L}$-definable with respect to a cone $C = h(\mathcal{J})$, depends on h and $\mathcal{J}$ ([10, Example 4.6]). However, it is immediate from the definition that if f is fiber $\mathcal{L}_A$-definable with respect to a cone $C = h(\mathcal{J})$, and $h(\mathcal{J}') \subseteq h(\mathcal{J})$ is another cone (but with the same h), then f is also fiber $\mathcal{L}_A$-definable with respect to it.

We are now ready to state the cone decomposition theorem from [10].

Fact 2.6 (Cone decomposition theorem [10, Theorem 5.1])**.**

(1) Let $X \subseteq M^n$ be an A-definable set. Then X is a finite union of A-definable cones.
(2) Let $f : X \to M$ be an A-definable function. Then there is a finite collection $\mathcal{C}$ of A-definable cones, whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each cone in $\mathcal{C}$.

Another important notion from [10] is that of 'large dimension', which we recall next. The proof of Theorem 1.1(2) runs by induction on large dimension.

Definition 2.7 (Large dimension [10])**.** Let $X \subseteq M^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of X by $\operatorname{ldim}(X)$.

Remark 2.8. The tameness conditions that we assume in this paper guarantee that the notion of large dimension is well-defined; namely, the above maximum k always exists ([10, Section 4.3]). In fact, everything that follows only uses the following two assumptions (instead of the tameness conditions): (a) Fact 2.6 and (b) the notion of large dimension is well-defined.

3. Product cone decompositions

In this section we prove Theorem 1.1.

3.1. **The linear case.** The following definition is taken from [13].

Definition 3.1 ([13])**.** Let $\mathcal{N} = \langle N, +, <, 0, \dots \rangle$ be an o-minimal expansion of an ordered group. A function $f : A \subseteq N^n \to N$ is called *affine*, if for every $x, y, x + t, y + t \in B$,

$$f(x + t) - f(x) = f(y + t) - f(y). \tag{1}$$

We call $\mathcal{N}$ *linear* if for every definable $f : A \subseteq N^n \to N$, there is a partition of A into finitely many definable sets B, such that each $f_{\restriction B}$ is affine.

The typical example of a linear o-minimal structure is that of an ordered vector space $\mathcal{V} = \langle V, <, +, 0, \{d\}_{d\in D}\rangle$ over an ordered division ring D. In general, if $\mathcal{N}$ is linear, then there exists a reduct $\mathcal{S}$ of such $\mathcal{V}$, such that $\mathcal{S} \equiv \mathcal{N}$ ([13]). Using this description, it is not hard to see that every affine function has a continuous extension to the closure of its domain.

Assume now that our fixed structure $\mathcal{M}$ is linear.

Lemma 3.2. *Let $h : [a, b] \times [c, d] \to M$ be an $\mathcal{L}$-definable continuous function, such that for every $t \in (a, b)$, $h(t, -) : [c, d] \to M$ is strictly increasing. Then*

$$h(b, d) - h(b, c) > 0.$$

Proof. Let $\mathcal{W}$ be a cell decomposition of $[a, b] \times [c, d]$ such that for every $W \in \mathcal{W}$, $h_{\restriction W}$ is affine. Since $d - c > 0$, there must be some $W = (f, g)_I \in \mathcal{W}$, where I is an interval with $\sup I = b$, and $r \in I$, such that the map $\delta(t) := g(t) - f(t)$ is increasing on $[r, b)$. We claim that for every $t \in (r, b)$,

$$h(t, g(t)) - h(t, f(t)) \geq h(r, g(r)) - h(r, f(r)).$$

Indeed, there is $k \geq 0$, such that

$$h(t, f(t) + \delta(t)) - h(t, f(t)) = h(t, f(t) + \delta(r) + k) - h(t, f(t)) =$$

$$h(t, f(t) + \delta(r) + k) + h(t, f(t) + \delta(r)) - h(t, f(t) + \delta(r)) + h(t, f(t)) \geq$$

$$\geq h(t, f(t) + \delta(r)) - h(t, f(t)) = h(r, f(r) + \delta(r)) - h(r, f(r)),$$

where the inequality is because $h(t,-)$ is increasing, and the last equality because h is affine on W. We conclude that

$$\begin{aligned} h(b,d) - h(b,c) &= \lim_{t\to b} \big(h(t,d) - h(t,c)\big) \\ &\geq \lim_{t\to b} \big(h(t,g(t)) - h(t,f(t))\big) \\ &\geq h(r,g(r)) - h(r,f(r)) \\ &> 0, \end{aligned}$$

where the first and last inequalities are because $h(t,-)$ and $h(r,-)$ are strictly increasing. $\square$

Counterexample to product cone decomposition. Let $S \subseteq M$ be a small set such that 0 is in the interior of its closure (by translating P to the origin, such an S exists). Let

$$X = \bigcup_{a\in S^{>0}} \{a\} \times (0,a).$$

Claim 3.3. *X is not a finite union of product cones.*

Proof. First of all, X cannot contain any k-cones for $k > 1$, since $\mathrm{ldim}(X) = 1$, by [10, Lemma 4.24 and 4.27]. Now let $H(T \times J)$ be an 1-product cone contained in X, with $H = (H_1, H_2) : Z \subseteq M^{l+1} \to M^2$, such that the origin is in its closure. Since H is $\mathcal{L}$-definable and continuous, and for each $g \in T$, $H_2(g,-)$ is injective, we may assume that the latter is always strictly increasing. By [10, Lemma 5.10] applied to J, $f(-) = \pi_1 H(g,-)$ and S, we have

- for every $g \in T$, there is $a \in S$, such that $H(g, J) \subseteq \{a\} \times (0,a)$.

By continuity of H, it follows that

- for every $g \in cl(T) \cap \pi(Z)$, there is $a \in M$, such that

$$H(g, cl(J)) \subseteq \{a\} \times [0,a].$$

Let $f : \pi(Z) \to M$ be the $\mathcal{L}$-definable map given by

$$f(g) = \pi_1(H(g, cl(J))).$$

Since the origin is in the closure of $H(T \times J)$, there must be an affine $\gamma : (a,b) \to cl(T) \cap \pi(Z)$ with $\lim_{t\to b} f(\gamma(t)) = 0$. Now the map

$$H_2(\gamma(-),-) : (a,b) \times (c,d) \to M$$

is affine and hence has a continuous extension h to $[a,b] \times [c,d]$. By definition of X,

$$h(b,c) = h(b,d) = 0.$$

But, by Lemma 3.2,

$$h(b,d) - h(b,c) > 0,$$

a contradiction. $\square$

3.2. **The field case.** We now assume that $\mathcal{M}$ expands an ordered field. The main idea behind the proof in this case is as follows. By Fact 2.6, it suffices to write every cone as a finite union of product cones. We illustrate the case of an 1-cone $C = h(\mathcal{J})$, for some $\mathcal{J} = \{J_g\}_{g\in S}$.

Step I (Lemma 3.4). Replace $\mathcal{J}$ by a cone $\mathcal{J}' = \{J'_g\}_{g\in S}$, such that for some fixed interval I, each J'_g is contained in I and it is co-small in it. Here we use the field structure of $\mathcal{M}$, so this step would fail in the linear case.

Step II (Lemma 3.5). By [10, Lemma 4.25], the intersection $J = \bigcap_{g\in S} J'_g$ is co-small in I. Moreover, if we let $L = S \times J$, then, by [10, Lemma 4.29], we obtain that the large dimension of $\mathcal{J} \setminus L$ is 0.

Step III (Theorem 3.6). Use Steps I and II and induction on large dimension. Here, the inductive hypothesis is only applied to sets of large dimension 0. In general, $\operatorname{ldim}(\mathcal{J} \setminus L) < \operatorname{ldim}(\mathcal{J})$.

To achieve Step I, we first need to make an observation and fix some notation. Using the field multiplication, one can define an $\mathcal{L}_\emptyset$-definable continuous $f : M^3 \to M$, such that for every $b, c \in M$,

$$f(b, c, -) : (b, c) \to (0, 1)$$

is a bijection. Similarly, there are $\mathcal{L}_\emptyset$-definable continuous maps $f_1, f_2 : M^2 \to M$, such that for every $b, c \in M$, the maps

$$f_1(b, -) : (b, +\infty) \to (0, 1)$$

and

$$f_2(c, -) : (-\infty, c) \to (0, 1)$$

are bijections. To give all these maps a uniform notation, we write $f(b, +\infty, x)$ for $f_1(b, x)$, and $f(-\infty, c, x)$ for $f_2(c, x)$ and . We fix this f for the next proof. Observe that if $J \subseteq (b, c)$ is co-small in (b, c), for $b, c \in M \cup \{\pm\infty\}$, then $f(b, c, J)$ is co-small in $(0, 1)$.

Lemma 3.4. *Let $\mathcal{J} \subseteq M^{m+k}$ be an A-definable uniform family of supercones, with shell $Z \subseteq M^{m+k}$. Then there are*

- *an A-definable uniform family $\mathcal{J}' = \{J'_g\}_{g\in S}$ of supercones $J'_g \subseteq M^k$, with a shell $\pi(Z) \times (0, 1)^k$,*
- *and an $\mathcal{L}_A$-definable continuous and injective map $F : Z \to M^{m+k}$, such that*
$$F(\mathcal{J}) = \mathcal{J}'.$$

Proof. For every $g \in \pi(\mathcal{J})$, since J_g is a supercone, it follows that Z_g is an open cell. Hence, for every $0 < j \le k$, there are $\mathcal{L}_A$-definable continuous maps $h^j_1, h^j_2 : \pi_{l+j-1}(Z) \to M$ such that

$$\pi_{m+j}(Z) = (h^j_1, h^j_2)_{\pi_{m+j-1}(Z)}.$$

We define

$$F = (F_1, \dots, F_{m+k}) : Z \to M^{m+k},$$

as follows. Let $I = (0, 1)$ and f be the map fixed above. Let $(g, t) \in Z \subseteq M^{m+k}$

If $1 \leq i \leq m$,
$$F_i(g,t) = g_i.$$
If $i = m+j$, with $0 < j \leq k$,
$$F_{m+j}(g,t) = f(h_1^j(g,t_1,\dots,t_{j-1}), h_2^j(g,t_1,\dots,t_{j-1}), t_j).$$
Clearly, F is injective, $\mathcal{L}_A$-definable and continuous. Let
$$\mathcal{J}' = F(\mathcal{J}).$$
That is, $\mathcal{J}' = \{J'_g\}_{g\in S}$, where for every $g \in S$, $J'_g = F(g, J_g)$. It is not hard to check, by induction on m, that for every $0 < m \leq k$, $\pi_{m+j}(\mathcal{J}')$ is an A-definable uniform family of supercones with shell $F(Z) = \pi(Z) \times I^m$. □

Lemma 3.5. *Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq M^{m+k}$ be an A-definable uniform family of supercones in M^k with shell Z. Suppose that $Z = \pi(Z) \times I^k$, where $I = (0,1)$. Then $\mathcal{J}$ is a disjoint union*
$$(S \times J) \cup Y,$$
where $S \times J$ is an A-definable uniform family of supercones with shell Z, and Y is an A-definable set of large dimension $< k$.

Proof. By induction on k. For $k = 0$, the statement is trivial. We assume the statement holds for k, and prove it for $k+1$. Let $\pi : M^{m+k+1} \to M^{m+k}$ be the projection onto the first $m+k$ coordinates. Since $\pi(\mathcal{J})$ is also an A-definable uniform family of supercones with shell $\pi(Z)$, by inductive hypothesis we can write $\pi(\mathcal{J})$ as a disjoint union
$$\pi(\mathcal{J}) = (S \times T) \cup Y,$$
where $T \subseteq M^k$ is an A-definable supercone with $cl(T) = cl(I^k)$, and Y is an A-definable set of large dimension $< k$. By [10, Corollary 5.5], the set $\bigcup_{t\in Y}\{t\} \times \mathcal{J}_t \subseteq \mathcal{J}$ has large dimension $< k+1$, and hence we only need to bring its complement X in $\mathcal{J}$ into the desired form. We have
$$X = \bigcup_{t\in S\times T} \{t\} \times \mathcal{J}_t.$$
Define, for every $a \in T$,
$$K_a = \bigcap_{g\in S} \mathcal{J}_{g,a}.$$
Since each $\mathcal{J}_{g,a}$ is co-small in I, by [10, Lemma 4.25] K_a is co-small in I. Hence, the set
$$L = \bigcup_{a\in T}\{a\} \times K_a$$
is a supercone in M^{k+1}. Since $cl(T) = cl(I^k)$ and for each $a \in T$, $cl(K_a) = cl(I)$, it follows that $cl(L) = cl(I^{k+1})$. In particular, Z is a shell for $S \times L$. Since $S \times L \subseteq X$, it remains to prove that $\operatorname{ldim}(X \setminus (S \times L)) < k+1$. We have
$$X \setminus (S \times L) = \bigcup_{(g,a)\in S\times T} \{(g,a)\} \times (\mathcal{J}_{g,a} \setminus K_a).$$
But $\mathcal{J}_{g,a} \setminus K_a$ is small, and hence, by [10, Lemma 4.29], the above set has large dimension $= \operatorname{ldim}(S \times T) = k$. □

We can now conclude the main theorem of the paper.

Theorem 3.6 (Product cone decomposition in the field case)**.** *Let $X \subseteq M^n$ be an A-definable set. Then*

1. *X is a finite union of A-definable product cones.*
2. *If $f : X \to M$ is an A-definable function, then there is a finite collection $\mathcal{C}$ of A-definable product cones, whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each cone in $\mathcal{C}$.*

Proof. (1). We do induction on the large dimension of X. By Fact 2.6, we may assume that X is a k-cone. Every 0-cone is clearly a product cone. Now let $k > 0$. By induction, it suffices to write X as a union of an A-definable product cone and an A-definable set of large dimension $< k$. Let $X = h(\mathcal{J})$ be as in Definition 2.4, and $Z \subseteq M^{m+k}$ a shell for $\mathcal{J}$.

Claim. *We can write X as a k-cone $h'(\mathcal{J}')$, such that for every $g \in \pi(\mathcal{J}')$, $cl(\mathcal{J}')_g = (0,1)^k$.*

Proof of Claim. Let $\mathcal{J}'$ and $F : Z \to M^{m+k}$ be as in Lemma 3.4, and define $h' = h \circ F^{-1} : F(Z) \to M^n$. Then

$$h(\mathcal{J}) = hF^{-1}(F(\mathcal{J})) = h'(\mathcal{J}')$$

is as required. □

By the claim, we may assume that for every $g \in S$, $cl(\mathcal{J})_g = (0,1)^k$. By Lemma 3.5, we have $\mathcal{J} = (S \times J) \cup Y$, where $J \subseteq M^k$ is an A-definable supercone, and $\text{ldim}Y < k$. Thus $h(\mathcal{J}) = h(S \times J) \cup h(Y)$ has been written in the desired form.

(2). By Fact 2.6, we may assume that X is a k-cone and that f is fiber $\mathcal{L}_A$-definable with respect to it. So let again $X = h(\mathcal{J})$ with shell $Z \subseteq M^{m+k}$, and in addition, $\tau : Z \subseteq M^{m+k} \to M$ with $\mathcal{J} \subseteq Z$, be $\mathcal{L}_A$-definable so that for every $x \in \mathcal{J}$,

$$(f \circ h)(x) = \tau(x).$$

By induction on large dimension, it suffices to show that X is the union of a product cone C and a set of large dimension $< k$, such that f is fiber $\mathcal{L}_A$-definable with respect to C. Let $X = h'(\mathcal{J}')$ be as in Claim of Item (1) and $F : Z \to M^{m+k}$ as in its proof. So $h' = h \circ F^{-1} : F(Z) \to M^n$. Define $\tau' : F(Z) \to M^n$ as $\tau' = \tau \circ F^{-1}$. We then have, for every $x' \in \mathcal{J}'$,

$$fh'(x') = fh'F(x) = fh(x) = \tau(x) = \tau F^{-1}(x) = \tau'(x),$$

witnessing that f is fiber $\mathcal{L}_A$-definable with respect to $h'(\mathcal{J}')$.

Therefore, we may replace h by h and $\mathcal{J}$ by $\mathcal{J}'$. Now, as in the proof of Item (1), we can write $h(\mathcal{J})$ as the union of a product cone $h(S \times J)$ and a set of large dimension $< k$. By the remarks following Definition 2.5, f is also fiber $\mathcal{L}$-definable with respect to $h(S \times J)$. □

Remark 3.7. The above proof yields that in cases where we have disjoint unions in Fact 2.6 (such as in [10, Theorem 5.12]), so do we in Theorem 3.6.

4. Refined supercones

In this section we refute [10, Question 5.14 (1)]. The question asked whether the Structure Theorem holds if we strengthen the notion of a supercone as follows.

Definition 4.1. A supercone $\mathcal{J}$ in M^k is called *refined* if it is of the form

$$\mathcal{J} = J_1 \times \cdots \times J_k,$$

where each J_i is a supercone in M. Let us call a (k-)cone $C = h(\mathcal{J})$ a *(k-)refined cone* if $\mathcal{J}$ is refined.

Our result is the following.

Proposition 4.2. *Assume $\mathcal{M}$ expands a real closed field. Then there is a supercone in M^2 which contains no 2-refined cone. In particular, it is not a finite union of refined cones.*

Proof. [1] The 'in particular' clause follows from [10, Corollaries 4.26 and 4.27]. Now, for every $a \in M$, let

$$J_a = M \setminus (P + aP)$$

and define $\mathcal{J} = \bigcup_{a\in M}\{a\} \times J_a$. Towards a contradiction, assume that $\mathcal{J}$ contains a 2-cone. That is, there are supercones $J_1, J_2 \subseteq M$, an open cell $U \subseteq M^2$ with $cl(J_1 \times J_2) = cl(U)$, and an $\mathcal{L}$-definable continuous and injective map $f : U \to M^2$, such that $C = f(J_1 \times J_2) \subseteq \mathcal{J}$. We write $X = f(U)$, and for each $a \in M$, $X_a \subseteq M$ for the fiber of X above a. Suppose C is A-definable.

By saturation, there is $a \in M$ which is dcl-independent over $A \cup P$, and further $g, h \in P$ which are dcl-independent over a. So

$$\dim(g, h, a) = 3.$$

By assumption, there are $(p, q) \in U \setminus (J_1 \times J_2)$, such that

$$f(p, q) = (a, g + ha).$$

Observe that $a \in \mathrm{dcl}(p, q)$. Also, one of p, q must be in $\mathrm{dcl}(AP)$. Indeed, we have $p \notin J_1$ or $q \notin J_2$. If, say, the former holds, then $p \in \pi(U) \setminus J_1$. Since the last set is A-definable and small, we obtain by [10, Lemma 3.11], that $p \in \mathrm{dcl}(AP)$.

We may now assume that $p \in \mathrm{dcl}(AP)$. If we write $f = (f_1, f_2)$, we obtain

$$f_2(p, q) = g + hf_1(p, q). \tag{2}$$

Since a is dcl-independent over $A \cup P$, there must be an open interval $I \subseteq M$ of p, such that for every $x \in I$,

$$f_2(x, q) = g + hf_1(x, q).$$

Viewing both sides of the equation as functions in the variable $f_1(x, q)$, and taking their derivatives with respect to it, we obtain:

$$\frac{\partial f_2(x, q)}{\partial f_1(x, q)} = f_1(x, q) + h.$$

Evaluated at p, the last equality gives $h \in \mathrm{dcl}(p, q)$. By (2), also $g \in \mathrm{dcl}(p, q)$. All together, we have proved that $g, h, a \in \mathrm{dcl}(p, q)$. It follows that

$$\dim(g, h, a) \leq \dim(p, q) \leq 2,$$

a contradiction. □

[1] The proof is based on an idea suggested by P. Hieronymi.

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