STRONGLY MINIMAL GROUPS IN O-MINIMAL STRUCTURES

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Abstract. We prove Zilber's Trichotomy Conjecture for strongly minimal expansions of 2-dimensional groups, definable in o-minimal structures:

Theorem. Let $\mathcal{M}$ be an o-minimal expansion of a real closed field, $\langle G; +\rangle$ a 2-dimensional group definable in $\mathcal{M}$, and $\mathcal{D} = \langle G; + \ldots\rangle$ a strongly minimal structure, all of whose atomic relations are definable in $\mathcal{M}$. If $\mathcal{D}$ is not locally modular, then an algebraically closed field K is interpretable in $\mathcal{D}$, and the group G, with all its induced $\mathcal{D}$-structure, is definably isomorphic in $\mathcal{D}$ to an algebraic K-group with all its induced K-structure.

1. Introduction

1.1. Zilber's Conjecture (ZC).

In [33], Boris Zilber formulated the following conjecture.

Zilber's Trichotomy Conjecture. *The geometry of every strongly minimal structure $\mathcal{D}$ is either (i) trivial, (ii) non-trivial and locally modular, or (iii) isomorphic to the geometry of an algebraically closed field K definable in $\mathcal{D}$. Moreover, in (iii) the structure induced on K from $\mathcal{D}$ is already definable in K (that is, the field K is "pure" in $\mathcal{D}$).*

The conjecture reduces by [7] to: *If a strongly minimal structure $\mathcal{D}$ is not locally modular, then it interprets a field K, and the field K is pure in $\mathcal{D}$.*

In the early 1990s, Hrushovski refuted both parts of the conjecture. Using his amalgamation method he showed the existence of a strongly minimal structure which is not locally modular and yet does not interpret any group (so certainly not a field), see [9]. In addition he showed the existence of *a proper* strongly minimal expansion of a field, see [8], thus disproving also the purity of the field. Nevertheless, Zilber's Conjecture stayed alive since it turned out to be true in various restricted settings, and moreover its verification in those settings gave rise to important applications (such as Hrushovski's proof of the function field Mordell-Lang conjecture in all characteristics [10]).

A common feature to many cases where the conjecture is true is the presence of an underlying geometry which puts strong restrictions on the definable sets in the strongly minimal structure $\mathcal{D}$. This is for example the case, when $\mathcal{D}$ is definable in an algebraically closed field ([4], [15] and [30]), differentially closed field (see, for example, [16]), separably

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closed field ([10]), or algebraically closed valued field ([14]). This is also the case when $\mathcal{D}$ is endowed with a Zariski geometry ([11]).

Thus, it is interesting to examine the conjecture in various geometric settings. One such setting is that of o-minimal structures.

1.2. **The connection to o-minimality.** The complex field is an example of a strongly minimal that is definable in the o-minimal structure $\langle \mathbb{R}; +, \cdot, < \rangle$, and indeed, the underlying Euclidean geometry is an important component in understanding complex algebraic varieties. This leads to examining in greater generality those strongly minimal structures definable in o-minimal ones, and to the following restricted variant of Zilber's Conjecture, formulated by the third author in a model theory conference at East Anglia in 2005.

The o-minimal ZC. *Let $\mathcal{M}$ be an o-minimal structure and $\mathcal{D}$ a strongly minimal structure whose underlying set and atomic relations are definable in $\mathcal{M}$. If $\mathcal{D}$ is not locally modular then an algerbaically closed field K is interpretable in $\mathcal{D}$, and moreover, K is a pure field in $\mathcal{D}$.*

Remark 1.1.
(1) Because every algebraically closed field of characteristic zero (ACF_0) is definable in an o-minimal real closed field, Zilber's Conjecture for reducts of algebraically closed fields of characteristic zero is a special case of the o-minimal ZC. This variant of the conjecture is still open for reducts whose universe is not an algebraic curve.
(2) The purity of the field in the o-minimal setting was already proven in [23], thus the o-minimal ZC reduces to proving the interpretability of a field in $\mathcal{D}$.
(3) Since every definable algebraically closed field in an o-minimal structure has dimension 2 (see [25]), it is not hard to see that the above conjecture implies that the underlying universe of $\mathcal{D}$ must be 2-dimensional in $\mathcal{M}$. Therefore, it is natural to consider the o-minimal ZC under the 2-dimensional assumption on $\mathcal{D}$, which is the case of our Theorem 1.3 below.
(4) By [6], if $\mathcal{D}$ is strongly minimal, interpretable in an o-minimal structure and $\dim_{\mathcal{M}} D = 1$, then $\mathcal{D}$ must be locally modular, thus trivially implying the o-minimal ZC in the case when $\dim_{\mathcal{M}} D = 1$.
(5) The theory of compact complex manifolds, denoted by CCM, (see [34]) is the multi-sorted theory of the structure whose sorts are all compact complex manifolds, endowed with all analytic subsets and analytic maps. It is known ([34, Theorems 3.4.3 and 3.2.8]) that each sort in this structure has finite Morley Rank, and also that the structure is interpretable in the o-minimal $\mathbb{R}_{an}$. Hence, every sufficiently saturated structure elementarily equivalent to a CCM is interpretable in an o-minimal structure.

    By [18], every set of Morley rank one in any model of CCM is definably isomorphic to an algebraic curve. Thus, Zilber's conjecture for reducts of CCM whose universe is analytically 1-dimensional reduces to the work in [4]. The higher dimensional cases may also reduce to the conjecture for ACF_0 but this is still open.

In [5] the following case of the o-minimal ZC was proven.

Theorem 1.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a definable function in an o-minimal expansion of the real field. If $\mathcal{D} = \langle \mathbb{C}; +, f\rangle$ is strongly minimal and non-locally modular (equivalently, f is not an affine map) then up to conjugation by an invertible 2×2 real matrix and finitely many corrections, f is a complex rational function. In addition, a function $\odot : \mathbb{C}^2 \to \mathbb{C}$ is definable in $\mathcal{D}$, making $\langle \mathbb{C}; +, \odot\rangle$ an algebraically closed field.*

In our current result below we replace the additive group of $\mathbb{C}$ above by an arbitrary 2-dimensional group G definable in an o-minimal structure. Moreover, we let $\mathcal{D}$ be an arbitrary expansion of G and not only by a map $f : G \to G$. Since strongly minimal groups are abelian ()[29, Corollary 3.1]), we are allowed to write the group below additively. Here is the main theorem of our article.

Theorem 1.3. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R, and let $\langle G; \oplus\rangle$ be a 2-dimensional group definable in $\mathcal{M}$. Let $\mathcal{D} = \langle G; \oplus, \dots\rangle$ be a strongly minimal structure expanding G, all of whose atomic relations are definable in $\mathcal{M}$.*

Then there are in $\mathcal{D}$ an interpretable algebraically closed field K, a K-algebraic group H with $\dim_K H = 1$, and a definable isomorphism $\varphi : G \to H$, such that the definable sets in $\mathcal{D}$ are precisely those of the form $\varphi^{-1}(X)$ for X a K-constructible subset of H^n.

In fact, the structure $\mathcal{D}$ and the field K are bi-interpretable.

Note that the theorem implies in particular that G is definably isomorphic in $\mathcal{D}$ to either $\langle K; +\rangle$, $\langle K^\times; \cdot\rangle$ or to an elliptic curve over K.

1.3. **The general strategy: from real geometry and strong minimality to complex algebraic geometry.** Let $\mathcal{M}$, G and $\mathcal{D}$ be as in Theorem 1.3. Since G is a group definable in an o-minimal expansion of a real closed field R, it admits a differentiable structure which makes it into a Lie group with respect to R (see [27]). We let $\mathfrak{F}_0$ be the collection of all differentiable (with respect to that Lie structure) partial functions $f : G \to G$, with $f(0_G) = 0_G$, such that for some $\mathcal{D}$-definable strongly minimal $S_f \subseteq G^2$, we have $graph(f) \subseteq S_f$. We let $J_0 f$ denote the Jacobian matrix of f at 0. The following is easy to verify, using the chain rule for differentiable functions:

$$J_0(f \oplus g) = J_0 f + J_0 g \ ; \ J_0(f \circ g) = J_0 f \cdot J_0 g,$$

where on the left we use the group operation and functional composition, and on the right the usual matrix operations in $M_2(R)$. Let also

$$\mathfrak{R} = \{J_0 f \in M_2(R) : f \in \mathfrak{F}_0\}.$$

The key observation, going back to Zilber, is that via the above equations we can recover a ring structure on $\mathfrak{R}$ by performing addition and composition of curves in $\mathcal{D}$. Most importantly, for the ring structure to be $\mathcal{D}$-definable, one needs to recognize tangency of curves at a point $\mathcal{D}$-definably. The geometric idea for that goes back to Rabinovich's work [30], and requires us to develop a sufficient amount of intersection theory for $\mathcal{D}$-definable sets, so as to recognize "combinatorially" when two curves are tangent.

This paper establishes in several distinct steps the necessary ingredients for the proof. In each of these steps we prove an additional property of $\mathcal{D}$-definable sets which shows their resemblance to complex algebraic sets. We briefly describe these steps.

We call $S \subseteq G^2$ *a plane curve* if it is $\mathcal{D}$-definable and $\mathrm{RM}(S) = 1$ (we recall the definition of Morley Rank in Section 2.1). In Section 4 we investigate the frontier of plane curves, where the frontier of a set S is $\mathrm{cl}(S) \setminus S$. We prove that every plane curve has finite frontier.

In Section 5 we consider the poles of plane curves, where a pole of $S \subseteq G^2$ is a point $a \in G$, such that for every neighborhood $U \ni a$, the set $(U \times G) \cap S$ is "unbounded". We prove that every plane curve has at most finitely many poles.

As a corollary of the above two results we establish in Section 6 another geometric property which is typically true for complex analytic curves. Namely, we show that every plane curve S whose projection on both coordinates is finite-to-one, is locally, outside finitely many points, the graph of a homeomorphism.

Next, we discuss the differential properties of plane curves, and consider in Section 8 the collection of all Jacobian matrices at 0 of local smooth maps from G to G whose graph is contained in a plane curve. Using our previous results we prove that this collection forms an algebraically closed subfield K of $M_2(R)$, and thus up to conjugation by a fixed invertible matrix, every such Jacobian matrix at 0 satisfies the Cauchy-Riemann conditions.

In Section 9 we establish elements of complex intersection theory, showing that if two plane curves E and X are tangent at some point then by varying E within a sufficiently well-behaved family, we gain additional intersection points with X. This allows us to identify tangency of curves in $\mathcal{D}$ by counting intersection points.

Finally, in Section 10 we use the above results in order to interpret an algebraically closed field in $\mathcal{D}$ and prove our main theorem.

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2. Preliminaries

We review briefly the basic model theoretic notions appearing in the text. We refer to any standard textbook in model theory (such as [17, §6, §7]) for more details. Standard facts on o-minimality can be found in [3] whose Sections 1.1 and 1.2 provide most of the basic background needed on structures and definability.

2.1. **Strong minimality and related notions.** Throughout the text, given a structure $\mathcal{N}$, by $\mathcal{N}$*-definable* we mean definable in $\mathcal{N}$ with parameters, unless stated otherwise. We drop the index '$\mathcal{N}$-' if it is clear from the context. In the next subsection, we will adopt a global convention about this index to be enforced in Sections 4 - 10.

Let $\mathcal{N} = \langle N, \ldots \rangle$ be an ω-saturated structure. A definable set S is *strongly minimal* if every definable subset of S is finite or co-finite. We call $\mathcal{N}$ *strongly minimal* if N is a strongly minimal set.

Let $\mathcal{N} = \langle N, <, \ldots \rangle$ be an expansion of a dense linear order without endpoints. We call $\mathcal{N}$ *o-minimal* if every definable subset of N is a finite union of points from N and open intervals whose endpoints lie in $N \cup \{\pm\infty\}$. The standard topology in $\mathcal{N}$ is the order topology on N and the product topology on N^n.

Now let $\mathcal{N}$ be a strongly minimal or an o-minimal structure. The algebraic closure operator acl in both cases is known to give rise to a pregeometry. We refer to [17, §6.2] and [27, §1] for all details, and recall here only some. Given $A \subseteq N$ and $a \in N^n$, we let $\dim(a/A)$ be the size of a maximal acl-independent subtuple of a over A. Given a definable set $C \subseteq N^n$, we define

$$\dim(C) = \max\{\dim(a/A) : a \in C\},$$

and we call an element $a \in C$ *generic in C over A in $\mathcal{N}$* if $\dim(a/A) = \dim(C)$.

If $\mathcal{N}$ is a strongly minimal structure, then $\dim(C)$ coincides with the Morley rank of C, and we denote $\dim(a/A)$ and $\dim(C)$ by $\mathrm{RM}(a/A)$ and $\mathrm{RM}(C)$, respectively. In the o-minimal case, $\dim C$ coincides with topological dimension of C, and we keep the notation $\dim(a/A)$ and $\dim(C)$.

Let $\mathcal{N}$ be any structure. Given a definable set X, *a canonical parameter for X* is an element in $\mathcal{N}^{eq}$ which is inter-definable with the set X, namely $\bar{a}$ is a canonical parameter for X if $\varphi(\bar{x}, \bar{a})$ defines X and $\varphi(\bar{x}, \bar{a}') \neq X$ for all $\bar{a}' \neq \bar{a}$. Any two canonical parameters are inter-definable over $\emptyset$, and so we use $[X]$ to denote any such parameter. Note that if $X = X_{t_0}$ for some definable family of sets over $\emptyset$, $\{X_t : t \in T\}$, then $[X] \in \mathrm{dcl}(t_0)$, but t_0 need not be a canonical parameter for X.

A structure $\mathcal{N} = \langle N, \ldots \rangle$ is *interpretable* in $\mathcal{M}$ if there is an isomorphism of structures $\alpha : \mathcal{N} \to \mathcal{N}'$, where the universe of $\mathcal{N}'$ and all $\mathcal{N}'$-atomic relations are interpretable $\mathcal{M}$.

If $\mathcal{N}$ is interpretable in $\mathcal{M}$ via α and $\mathcal{M}$ is interpretable in $\mathcal{N}$ via β, and if in addition $\beta \circ \alpha$ is definable in $\mathcal{N}$ and $\alpha \circ \beta$ is definable in $\mathcal{M}$, then we say that *$\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.*

Note that if $\mathcal{M}$ is an o-minimal expansion of an ordered group, then by Definable Choice, every interpretable structure in $\mathcal{M}$ is also definable in $\mathcal{M}$.

2.2. **The setting.** Throughout Sections 4 - 10, we fix a sufficiently saturated o-minimal expansion of a real closed field $\mathcal{M} = \langle R; +, \cdot, <, \ldots \rangle$, and a 2-dimensional definable group G. By [27], the group G admits a C^1-manifold structure with respect to the field R, such that the group operation and inverse function are C^1 maps with respect to it. The topology and differentiable structure which we refer to below is always that of this smooth group structure on G. As noted in [1, Lemma 10.4], the group G is definably isomorphic, as a topological group, to a definable group whose domain is a closed subset of some M^r, endowed with the M^r-topology. Thus, we assume that G is a closed subset of M^r and its topology is the subspace topology.

Throughout Sections 4 - 10, we also fix a strongly minimal non-locally modular structure $\mathcal{D} = \langle G; \ldots \rangle$. We treat $\mathcal{M}$ as the default structure and thus use "definable" to mean "definable in $\mathcal{M}$", and use "$\mathcal{D}$-definable" to mean "definable in $\mathcal{D}$". Similarly, we use acl, dim and 'generic' to denote the corresponding notions in $\mathcal{M}$, and let $\mathrm{acl}_{\mathcal{D}}$, RM, '$\mathcal{D}$-generic' and '$\mathcal{D}$-canonical parameter' denote the corresponding notions in $\mathcal{D}$.

Since the underlying universe of the strongly minimal $\mathcal{D}$ is the 2-dimensional set G, it follows that for every $\mathcal{D}$-definable set $X \subseteq G^n$, we have

$$\dim X = 2\,\mathrm{RM}(X).$$

Also, for $a \in G^n$ and $A \subseteq G$, we have

$$\dim(a/A) \leq 2\,\mathrm{RM}(a/A),$$

and in particular, if $X \subseteq G^n$ is definable in $\mathcal{D}$ and $a \in X$ is generic in X over A then it is also $\mathcal{D}$-generic in X over A. The converse fails: Indeed, let $\mathcal{M}$ be the real field and $\mathcal{D}$ the complex field, interpretable in the real field $\mathcal{M}$. The element $\pi \in \mathbb{C}$ is $\mathcal{D}$-generic in $\mathbb{C}$ over $\emptyset$ but it is not generic in $\mathbb{C}$ over $\emptyset$ because it is contained in the definable, 1-dimensional set $\mathbb{R}$.

2.3. **Notation.** If S is a set in a topological space, its closure, interior, boundary and frontier are denoted by $\mathrm{cl}(S), \mathrm{int}(S), \mathrm{bd}(S) := \mathrm{cl}(S) \setminus \mathrm{int}(S)$ and $\mathrm{fr}(S) := \mathrm{cl}(S) \setminus S$, respectively. Given a group $\langle G, + \rangle$ and sets $A, B \subseteq G$, we denote by $A - B$ the Minkowski difference of the two sets, $A - B = \{x - y : x \in A,\, y \in B\}$. Given a set X and $S \subseteq X^2$, we denote $S^{op} = \{(y, x) \in X^2 : (x, y) \in S\}$. The graph of a function f is denoted by Γ_f. If $\gamma(t)$ is map with image in the domain of a function f, we often write $f(\gamma)$ instead of $f(Im(\gamma))$.

3. Plane curves

In this section, we work in a strongly minimal structure $\mathcal{D}$ and prove some lemmas about the central objects of our study, plane curves. When $\mathcal{D}$ expands a group G and is non-locally modular, we construct in Subsections 3.3 and 3.4 two special definable families of plane curves which will be used in the subsequent sections.

3.1. **Some basic definitions and notations.** Let $\mathcal{D}$ be a strongly minimal structure.

Definition 3.1. A *$\mathcal{D}$-plane curve* (or just *plane curve*) is a $\mathcal{D}$-definable subset of D^2 of Morley Rank 1.

Definition 3.2. For two plane curves C_1, C_2, we write $C_1 \sim C_2$ if $|C_1 \triangle C_2| < \infty$. Note that this gives a $\mathcal{D}$-definable equivalence relation on any $\mathcal{D}$-definable family of plane curves. A $\mathcal{D}$-definable family of plane curves $\mathcal{F} = \{C_t : t \in T\}$ is *faithful* if for $t_1 \neq t_2$ in T, $C_{t_1} \not\sim C_{t_2}$. It is *almost faithful* if all $\sim$-equivalence classes are finite.

Given a $\mathcal{D}$-definable family of plane curves $\mathcal{F}$, there exists – at the price of possibly naming one element not in $\mathrm{acl}_{\mathcal{D}}(\emptyset)$ – a $\mathcal{D}$-definable almost faithful family of plane curves $\mathcal{F}' = \{C'_t : t \in T'\}$, such that every curve in $\mathcal{F}$ has an equivalent curve in $\mathcal{F}'$ and vice versa (see, for example, [8], p.137)[1]. It is not hard to see that $\mathrm{RM}(T')$ is independent of the choice of $\mathcal{F}'$. Thus we can make the following definition.

Definition 3.3. A $\mathcal{D}$-definable family of plane curves $\mathcal{F}$ as above is said to be *n-dimensional*, written also as $\mathrm{RM}(\mathcal{F}) = n$, if in the corresponding almost faithful family $\mathcal{F}'$ as above, we have $\mathrm{RM}(T') = n$. We call $\mathcal{F}$ *stationary* if Morley degree$(T) = 1$.

We call $\mathcal{D}$ a *non-locally modular structure* if there exists a $\mathcal{D}$-definable family of plane curves $\mathcal{F}$ with $\mathrm{RM}(\mathcal{F}) \geq 2$.

In fact, [28, Proposition 5.3.2], if $\mathcal{D}$ is non-locally modular, then for every n there exists an n-dimensional $\mathcal{D}$-definable family of plane curves. We will sketch a proof of a slightly stronger result in Proposition 3.20 below.

Finally, we recall the notion of very ampleness.

Definition 3.4. Let $\mathcal{F} = \{C_t : t \in T\}$ be a $\mathcal{D}$-definable family of plane curves. For every $p \in G^2$, denote

$$T(p) = \{t \in T : p \in C_t\} \ ; \ \mathcal{F}(p) = \{C_t : p \in C_t\}.$$

We say that $\mathcal{F}$ is *(generically) very ample* if for every $p \neq q \in G^2$ (each $\mathcal{D}$-generic over the parameters defining $\mathcal{F}$),

$$\mathrm{RM}(T(p) \cap T(q)) < \mathrm{RM}(T(p)).$$

In the rest of this section, $\mathcal{D} = \langle G; +, \dots \rangle$ denotes a strongly minimal expansion of a group G.

3.2. **Local modularity.** Here we recall some basic facts about local modularity.

Definition 3.5. A $\mathcal{D}$-definable set is *G-affine* if it is a finite boolean combination of cosets of $\mathcal{D}$-definable subgroups of G.

We use the following simple observation without further reference (see, for example, [17, Lemma 7.2.5, Corollary 7.1.6 and Corollary 7.2.4] for details).

Remark 3.6. If $S \subseteq G^2$ is $\mathcal{D}$-definable and strongly minimal, then S is G-affine if and only if $G \sim H + a$ for some $\mathcal{D}$-definable strongly minimal subgroup $H \subseteq G^2$.

Definition 3.7. Given a strongly minimal plane curve C, the *stabilizer of* C is the set

$$Stab^*(C) = \{g \in G^2 : C \sim C + g\}.$$

The stabilizer of C is easily seen to be a $\mathcal{D}$-definable subgroup of G^2. The next properties are easy to verify.

Lemma 3.8. *For C a strongly minimal $\mathcal{D}$-plane curve, and $p, q \in G^2$,*

(1) *$C + p \sim C + q$ if and only if $p - q \in Stab^*(C)$.*
(2) *$Stab^*(C)$ is trivial if and only if $\{C + p : p \in G^2\}$ is a faithful family.*

[1] Allowing imaginary elements, we can always obtain faithful families of plane curves. The point here is to work in the real sort only.

(3) *$Stab^*(C)$ is finite if and only if $\{C+p : p \in G^2\}$ is almost faithful.*
(4) *$Stab^*(C)$ is infinite if and only if C is G-affine.*

We only use here the following characterization of non-local modularity in expansions of groups, which follows from [12].

Fact 3.9. *If $\mathcal{D}$ is a strongly minimal expansion of a group G, then $\mathcal{D}$ is non-locally modular if and only if there exists a $\mathcal{D}$-plane curve which is not G-affine.*

Note that if $\mathcal{F} = \{C+p : p \in T\}$ is a $\mathcal{D}$-definable family of plane curves, with C strongly minimal and $T = G^2$, then for every $p \in G^2$,

$$T(p) = \{q : p \in C+q)\} = p - C.$$

In particular $T(p)$ is strongly minimal so that, in fact, if $\mathrm{RM}(T(p) \cap T(q)) = \mathrm{RM}(T(p))$ then $T(p) \sim T(q)$. We thus have the following lemma.

Lemma 3.10. *If C is a strongly minimal plane curve and $\mathcal{F} = \{C+p : p \in G^2\}$, then the following are equivalent:*

(1) *$\mathcal{F}$ is very ample*
(2) *$\mathcal{F}$ is faithful*
(3) *$Stab^*(C)$ is trivial.*

Finally, given a $\mathcal{D}$-definable family of plane curves $\mathcal{F} = \{C_t : t \in T\}$, we call C_t *a $\mathcal{D}$-generic curve in $\mathcal{F}$* over A, if t is $\mathcal{D}$-generic in T over A. We say that $\mathcal{F}$ is *generically strongly minimal* if every $\mathcal{D}$-generic curve in $\mathcal{F}$ is strongly minimal.

3.3. **Dividing by a finite subgroup of G.** The main goal of this subsection is to prove Lemma 3.11 below, which will be used in the proof of Theorem 1.3 in Section 10. It will also allow us, without loss of generality, to construct a faithful, very ample family of strongly minimal plane curves of Morley rank two (Proposition 3.13 below).

Given a strongly minimal plane curve C which is not G-affine, we plan to work with the family $\mathcal{F} = \{C+p : p \in G^2\}$. We know that $Stab^*(C)$ cannot be infinite but it can be a finite, non-trivial, group in which case $\mathcal{F}$ is neither faithful nor very ample. We next want to prove that dividing the structure $\mathcal{D}$ by a finite group is harmless.

Given a finite group $F \subseteq G$, $\mathcal{D}$-definable over $\emptyset$, we consider the map $\pi_F : G \to G/F$, and still use $\pi_F : G^n \to (G/F)^n$ to denote the map $\pi_F(g_1, \ldots, g_n) = (\pi_F(g_1), \ldots, \pi_F(g_n))$.

We let $\mathcal{D}_F$ be the structure whose universe is G/F and whose atomic relations are all sets of the form $\pi_F(S)$ for $S \subseteq G^n$ a $\emptyset$-definable set in $\mathcal{D}$. The structure $\mathcal{D}_F$ is again an expansion of a group.

The following result implies that for the purpose of our main theorem we may work with $\mathcal{D}_F$ instead of $\mathcal{D}$.

Lemma 3.11. *Assume that the group G has unbounded exponent. Then the structures $\mathcal{D}$ and $\mathcal{D}_F$ are bi-interpretable, without parameters. In particular, $\mathcal{D}$ is bi-interpretable with an algebraically closed field if and only if $\mathcal{D}_F$ is.*

Proof. Because F and π_F are $\emptyset$-definable in $\mathcal{D}$, the structure $\mathcal{D}_F$ is interpretable, with no additional parameters, in $\mathcal{D}$, via the identity interpretation $\alpha(g+F) = g+F$.

Next, let us see how we interpret $\mathcal{D}$ in $\mathcal{D}_F$. Let $n = |F|$, and let $\pi_F^* : G/F \to G$ be the map defined as follows: Given $y \in G/F$, and $x \in G$ for which $\pi_F(x) = y$, let

$$\pi_F^*(y) = nx.$$

Because G is commutative, if $\pi(x) = \pi(x') = y$ then $nx = nx' + ng$ for some $g \in F$. Since $ng = 0$ this proves that π_F^* is a well-defined group homomorphism with kernel $\pi_F(G[n])$, where $G[n] = \{x \in G : nx = 0\}$.

Since G is strongly minimal and has unbounded exponent, the group $G[n]$ is finite and hence $\ker(\pi_F^*)$ is finite, so $\dim \mathrm{Im}(\pi_F^*) = \dim G/F = \dim G$. Because G is definably connected, π_F^* is surjective. Thus the homomorphism π_F^* induces an isomorphism of $(G/F)/\pi_F(G[n])$ with G. Its inverse $\beta : G \to (G/F)/\pi_F(G[n])$ is given by

$$\beta(g) = (\frac{g}{n} + F) + \pi_F(G[n]),$$

where g/n is any element $h \in G$ such that $nh = g$.

By our assumptions, $\pi_F(G[n])$ is $\emptyset$-definable in $\mathcal{D}_F$, and therefore the quotient $(G/F)/\pi_F(G[n])$ is $\emptyset$-definable in $\mathcal{D}_F$. Now, given any $\emptyset$-definable $X \subseteq G^k$ in $\mathcal{D}$, the set $\{(g_i/n)_{i=1}^k \in G^k : g \in X\}$ is also $\emptyset$-definable in $\mathcal{D}$, and hence its image in $(G/F)^k/\pi_F(G[n])^k$ is $\emptyset$-definable in $\mathcal{D}_F$. We therefore showed that $\mathcal{D}$ is interpretable, without parameters, in $\mathcal{D}_F$ via β.

To see that this is indeed bi-interpretation, we first note that the isomorphism between $\mathcal{D}$ and its interpretation in $\mathcal{D}_F$ is $\alpha \circ \beta$, which equals β. It is clearly definable in $\mathcal{D}$.

Let us examine the map induced on G/F by $\beta \circ \alpha$ and prove that it is definable in $\mathcal{D}_F$. We denote by F/n the preimage of F in G under the map $g \mapsto ng$. It is not hard to see that the image of F inside $\beta(G)$ is the group $\pi_F(F/n) + \pi_F(G[n]) = \pi_F(F/n)$, and hence the isomorphism which $\beta \circ \alpha$ induces on G/F is

$$g + F \mapsto g/n + \pi_F(F/n).$$

This map is definable in the group G/F by sending $g+F$ to the unique coset $h + F/n$ such that $nh + F = g + F$.

This completes the proof that $\mathcal{D}$ and $\mathcal{D}_F$ are bi-interpretable over $\emptyset$. ☐

Note that in our case, when the group G is abelian and definable in an o-minimal structure then by [31], the group G has unbounded exponent, so the above result holds.

For the rest of this subsection, assume that $\mathcal{D}$ is non-locally modular, and fix (after possibly absorbing into the language a finite set of parameters) a strongly minimal plane curve $C \subseteq G^2$ which is $\mathcal{D}$-definable over $\emptyset$ and not G-affine. Assume that $F' = Stab^*(C)$ is a finite group and let $F \subseteq G$ be a $\mathcal{D}$-$\emptyset$-definable subgroup such that $F' \subseteq F \times F$. Consider the structure $\mathcal{D}_F$ expanding $\langle G/F, + \rangle$ as above.

Claim 3.12. *$\pi_F(C)$ is strongly minimal in $\mathcal{D}_F$ and $Stab^*(\pi_F(C))$ in $(G/F)^2$ is trivial.*

Proof. The strong minimality of $\pi_F(C)$ is immediate from the strong minimality of C in $\mathcal{D}$.

Assume that $q \in Stab^*(\pi_F(C)) \subseteq (G/F)^2$, namely $q + \pi_F(C) \sim \pi_F(C)$. Let $\tilde{F} = F \times F \subseteq G^2$ and fix $p \in G^2$ such that $\pi_F(p) = q$. Then $p + C + \tilde{F} \cap C + \tilde{F}$ is infinite and since $\tilde{F}$ is finite, there exist $g, h \in \tilde{F}$ such that $C + p + g \cap C + h$ is infinite. But then $p + g - h \in Stab^*(C) \subseteq \tilde{F}$, implying that $p \in \tilde{F}$, and hence $0 = \pi_F(p) = q$.

We thus showed that $Stab^*(\pi_F(C))$ is trivial. □

By Lemma 3.10, Lemma 3.11 and Claim 3.12, we can conclude.

there is a strongly minimal plane curve C with $Stab^*(C) = \{0\}$. By Lemma 3.10, the family $\{C + p : p \in G^2\}$ is faithful and very ample. We can thus conclude the following proposition.

Proposition 3.13. *Assume $\mathcal{D}$ is non-locally modular, expanding a group G of unbounded exponent. Then there exists a finite group $F \subseteq G$, and in the structure $\mathcal{D}_F$ defined above there exists a definable family $\mathcal{L}_0 = \{l_t : t \in Q_0\}$, of strongly minimal plane curves, which is faithful, very ample, and has $\mathrm{RM}(Q_0) = 2$.*

The structures $\mathcal{D}$ and $\mathcal{D}_F$ are bi-interpretable, over the parameters defining F.

Assumption: For the rest of the article, we replace the structure $\mathcal{D}$ with the structure $\mathcal{D}_F$, and thus assume that the family $\mathcal{L}_0$ exists in $\mathcal{D}$.

3.4. **Very ample families of high dimension.** The goal of this subsection is to construct a larger family $\mathcal{L}'$ of plane curves which still has the geometric properties of the family $\mathcal{L}_0$ from Proposition 3.13. The main method is to use composition of binary relations and families of plane curves. Recall the notion of a composition of binary relations, extending composition of functions: Given $S_1, S_2 \subseteq G^2$, we let

$$S_1 \circ S_2 = \{(x, z) \in G^2 : \exists y(x, y) \in S_2 \text{ and } (y, z) \in S_1\}.$$

Clearly, if S_1, S_2 are $\mathcal{D}$-definable, then so is $S_1 \circ S_2$. We will be mostly interested in the composition of plane curves, and even more so, in the composition of families of plane curves: if $\mathcal{L}_1, \mathcal{L}_2$ are $\mathcal{D}$-definable families of plane curves, we let $\mathcal{L}_1 \circ \mathcal{L}_2 := \{C_1 \circ C_2 : C_1 \in \mathcal{L}_1, C_2 \in \mathcal{L}_2\}$.

As a rule, geometric properties are not preserved under compositions of (families of) curves. The composition of two strongly minimal curves has, indeed, Morley rank 1, but it need not be strongly minimal. More generally, a $\mathcal{D}$-generic curve of $\mathcal{L}_1 \circ \mathcal{L}_2$ need not be strongly minimal, and even if it were, $\mathcal{L}_1 \circ \mathcal{L}_2$ need not be faithful. In fact, although the dimension of $\mathcal{L}_1 \circ \mathcal{L}_2$ cannot decrease, it need not be greater than that of $\mathcal{L}_1$ or $\mathcal{L}_2$. For example, if both families are the family of affine lines in $\mathbb{A}^2$ then $\mathcal{L}_1 \circ \mathcal{L}_2 = \mathcal{L}_1$.

We will need a series of lemmas to address these issues. We start with the following easy observation:

Lemma 3.14. *Assume that $\mathcal{L}_1 = \{C_t : t \in T\}$ and $\mathcal{L}_2 = \{D_r : r \in R\}$ are two $\mathcal{D}$-definable faithful, generically very ample families of plane curves. Then $\mathcal{L}_1 \circ \mathcal{L}_2$ is also generically very ample.*

Proof. Let $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2$ and $(a, b), (c, d) \in G^2$ be distinct $\mathcal{D}$-generic points. Let $C = C_1 \circ C_2$ for $C_i \in \mathcal{L}_i$. Then $(a, b) \in C$ if and only if there is $e \in G$ such that $(a, e) \in C_2$ and

$(e,b) \in C_1$. For a $\mathcal{D}$-generic e independent over all the data,

$$\mathrm{RM}(\{(t,r) \in T \times R : (a,e) \in C_t, (e,b) \in C_r\}) = \mathrm{RM}(T) - 1 + \mathrm{RM}(R) - 1.$$

This is true since $\mathcal{D}$-genericity of (a,b) implies that a,b are $\mathcal{D}$-generic in G and by choice of e the points (a,e) and (e,b) are, therefore, $\mathcal{D}$-generic in G^2. So $\mathrm{RM}(T(a,e)) = \mathrm{RM}(T) - 1$, and $\mathrm{RM}(R(e,b)) = \mathrm{RM}(R) - 1$, with the desired conclusion.

Note that if e is non-$\mathcal{D}$-generic, and since (a,e) and (e,b) are not $\mathcal{D}$-algebraic over $\emptyset$, we get – from rank considerations – that $\mathrm{RM}(T(a,e)) < \mathrm{RM}(T) - 1$ for all $e \in G$, and similarly $\mathrm{RM}(R(e,b)) < \mathrm{RM}(R)$. So by the above calculations, varying $e \in G$ we get by faithfulness of the families,

$$\mathrm{RM}((T \times R)(a,b)) = \mathrm{RM}\{(t,r) \in T \times R : C_r \circ C_t \in \mathcal{L}(a,b)\} = \mathrm{RM}(T) + \mathrm{RM}(R) - 1.$$

Similarly, $(c,d) \in C$ if and only if there exists e' such that $(c,e') \in C_2$ and $(e',d) \in C_1$. By assumption, fixing $e, e' \in G$, the Morley rank of $R(a,e) \cap T(c,e')$ is smaller than that of $R(a,e)$, $T(c,e')$ and $R(e',b) \cap T(d,e')$. So

$$\begin{aligned}\mathrm{RM}\{(t,r) \in T \times R : C_r \circ C_t \in \mathcal{L}(a,b) \cap \mathcal{L}(c,d)\} &\leq \mathrm{RM}(T) - 2 + \mathrm{RM}(R) - 2 + 2 = \\ &= \mathrm{RM}(T) + \mathrm{RM}(R) - 2\end{aligned}$$

implying that $\mathcal{L}$ is generically very ample. $\square$

Definition 3.15. Given two $\mathcal{D}$-definable family of plane curves, $\mathcal{L}$ and $\mathcal{L}'$, we say that $\mathcal{L}$ *extends* $\mathcal{L}'$ if for every $C' \in \mathcal{L}'$ there exists $C \in \mathcal{L}$ such that $C' \subseteq C$.

In the next couple of lemmas we show that although the composition of two families of curves need not preserve the properties of the original families (as already discussed), it extends a family of curves which does.

Lemma 3.16. *Let $\mathcal{L}$ be a k-dimensional faithful $\mathcal{D}$-definable family of plane curves. Let E be any non-G-affine plane curve. Then $E \circ \mathcal{L}$ extends a k-dimensional almost faithful $\mathcal{D}$-definable family of plane curves. In fact, if $C \in \mathcal{L}$ is $\mathcal{D}$-generic over $[E]$, then for any strongly minimal $C_E \subseteq E \circ C$ we have* $\mathrm{RM}([C_E]/[E]) = k$.

Proof. Fix some $C \in \mathcal{L}$ which is $\mathcal{D}$-generic over $[E]$ and $C_E \subseteq E \circ C$ strongly minimal. Note that $(E^{-1} \circ C_E) \cap C$ is infinite, and since C is strongly minimal $E^{-1} \circ C_E$ is a set of Morley rank 1, containing the set C, up to a finite set. It follows that $[C] \in \mathrm{acl}_{\mathcal{D}}([C_E][E])$. Since $\mathrm{RM}([C]/[E]) = \mathrm{RM}([C])$ we get, by exchange, that $\mathrm{RM}([C]) = \mathrm{RM}([C]/[E]) = \mathrm{RM}([C_E]/[E])$.

Absorbing $[E]$ into the language, we can find $\bar{c} \in \mathrm{acl}_{\mathcal{D}}([C])$ and a formula $\varphi(x, \bar{c})$ defining C_E. By compactness, there is a formula $\theta \in \mathrm{tp}(\bar{c})$ such that whenever $\bar{c}' \models \theta$ there is some $C' \in \mathcal{L}$ such that $\varphi(x, \bar{c}') \subseteq E \circ C'$, and for all $\mathcal{D}$-generic $\bar{c}' \models \theta$ the formula $\varphi(x, \bar{c}')$ is strongly minimal. We may further require – by compactness, again – that if $\varphi(x, \bar{c}') \wedge \varphi(x, \bar{c}'')$ is infinite then the symmetric difference $\varphi(x, \bar{c}') \triangle \varphi(x, \bar{c}'')$ is finite for all $\bar{c}', \bar{c}'' \models \theta$. By rank considerations the family $\varphi(x, y) \wedge \theta(y)$ is almost faithful, and therefore satisfies the required properties. $\square$

As an immediate application we get the following statement.

Corollary 3.17. *Let $\mathcal{L}_1, \mathcal{L}_2$ be faithful k-dimensional $\mathcal{D}$-definable families of plane curves. Then $\mathcal{L}_1 \circ \mathcal{L}_2$ extends an almost faithful, stationary, generically strongly minimal family of plane curves of dimension at least k.*

We can now formulate a definition.

Definition 3.18. Let $\mathcal{L}_1, \mathcal{L}_2$ be faithful k-dimensional $\mathcal{D}$-definable families of plane curves. We say that $\mathcal{D}$-definable family $\mathcal{L}$ is *a composition subfamily* of $\mathcal{L}_1 \circ \mathcal{L}_2$ if (i) $\mathcal{L}$ is an almost faithful, stationary, generically strongly minimal family, extended by $\mathcal{L}_1 \circ \mathcal{L}_2$, and (ii) $\mathcal{L}$ is such family of maximal dimension.

The last technical lemma we need is the following.

Lemma 3.19. *Let $\mathcal{L}_1, \mathcal{L}_2$ be faithful k-dimensional $\mathcal{D}$-definable families of plane curves for $k \geq 2$.*

(1) *If some composition subfamily of $\mathcal{L}_1 \circ \mathcal{L}_2$ is k-dimensional then $\mathcal{D}$ interprets an infinite field.*
(2) *If $\mathcal{L}_1$, $\mathcal{L}_2$ are generically very ample then the same is true of any composition subfamily of $\mathcal{L}_1 \circ \mathcal{L}_2$.*

Proof. (1) Assume that there exists a k-dimensional composition subfamily $\mathcal{L}$ of $\mathcal{L}_1 \circ \mathcal{L}_2$. We will construct a field configuration (see Definition 10.2). Let S_i be the parameter set of $\mathcal{L}_i$ and S the parameter set of $\mathcal{L}$. By strong minimality of $\mathcal{D}$ and the fact that $\mathcal{L}_1$ and $\mathcal{L}_2$ are both k-dimensional there is a finite-to-finite correspondence α between S_1 and S_2. We may assume that α is $\emptyset$-definable in $\mathcal{D}$ and since field configurations are invariant under inter-algebraicity, we may also assume that $S_1 = S_2$. By the same token there is also a finite-to-finite correspondence between S and S_1, so we may also assume that $S = S_1$. By assumption there is a $\mathcal{D}$-definable correspondence $\beta : S \times S \to S$ mapping (s,t) to a finite subset of S corresponding to $C_s \circ C_t$.

Now let $t, s \in S$ be independent $\mathcal{D}$-generics, $C_s \in \mathcal{L}_1$, $C_t \in \mathcal{L}_2$. Let $u = \beta(s,t)$, $x \in G$ $\mathcal{D}$-generic over all the data, $y \in C_t(x)$ and $z \in C_s^{op}(x)$. Note that $y \in C_t \circ C_s(z)$ so y is inter-algerbraic with z over u. This implies that $\{s,t,u,x,y,z\}$ form a group configuration. The $\mathcal{D}$-canonical base of $\mathrm{tp}_{\mathcal{D}}(x,y/t)$ is inter-algebraic with t, by almost faithfulness of $\mathcal{L}_1$, and similarly for the $\mathcal{D}$-canonical bases of $\mathrm{tp}_{\mathcal{D}}(z,x/s)$ and $\mathrm{tp}_{\mathcal{D}}(z,x/u)$. This implies that $\{s,t,u,x,y,z\}$ is, in fact, a field configuration, and the result follows from Fact 10.3. This ends the proof of (1).

(2) Composition subfamilies are of maximal rank $\mathrm{RM}(\mathcal{L}(q)) = \mathrm{RM}((\mathcal{L}_1 \circ \mathcal{L}_2)(q))$ for any q $\mathcal{D}$-generic in G^2 over $\emptyset$. Thus, if $\mathrm{RM}(\mathcal{L}(q) \cap \mathcal{L}(p)) = \mathrm{RM}(\mathcal{L}(q))$ we would, necessarily get that $\mathrm{RM}[(\mathcal{L}_1 \circ \mathcal{L}_2)(p) \cap (\mathcal{L}_1 \circ \mathcal{L}_2)(q)] = \mathrm{RM}[\mathcal{L}_1 \circ \mathcal{L}_2)(q)]$, which is impossible by Lemma 3.14. □

We can finally conclude the last result of this section.

Proposition 3.20. *There exists a $\mathcal{D}$-definable almost faithful family of generically strongly minimal plane curves, $\mathcal{F} = \{C_t : t \in T\}$, which is generically very ample, and* $\mathrm{RM}(T) \geq 3$.

Proof. Let $\mathcal{L}_0$ be as in Proposition 3.13. By Corollary 3.17, there is a composition subfamily $\mathcal{F}$ of $\mathcal{L}_0 \circ \mathcal{L}_0$. If $\mathrm{RM}(\mathcal{F}) = 2$, then by Lemma 3.19(1) a field K is interpretable in $\mathcal{D}$, and then the conclusion is obvious (take the family of polynomials of degree $d > 1$ over K). If $\mathrm{RM}(\mathcal{F}) \geq 3$, then by Lemma 3.19(2), $\mathcal{F}$ is generically very ample. $\square$

From now on, until the end of the paper, we fix a sufficiently saturated o-minimal expansion of a real closed field $\mathcal{M} = \langle R; +, \cdot, <, \ldots \rangle$, and a 2-dimensional group $G = \langle G; \oplus \rangle$ definable in $\mathcal{M}$. We also fix a strongly minimal non-locally modular structure $\mathcal{D} = \langle G; \oplus \ldots \rangle$. The conventions of Section 2.2 are now in force. In Sections 4 - 7, we denote $\oplus$ by $+$, for simplification.

4. Frontiers of plane curves

4.1. **Strategy.** Our goal is to show (Theorem 4.9) that if $S \subseteq G^2$ is a $\mathcal{D}$-definable set with $\mathrm{RM}(S) = 1$, then its frontier $\mathrm{fr}(S)$ is finite and in fact contained in $\mathrm{acl}_{\mathcal{D}}([S])$. The geometric idea originates in [23] and it is implemented in Lemma 4.7 below, as follows. We consider the family $\mathcal{L}_0$ from the assumption following Proposition 3.13. We also fix $b \in \mathrm{fr}(S)$ and consider a curve $l_q \in \mathcal{L}_0$ going through b with q generic over $[S]$. If all points of $l_q \cap S$ are transversal and b is sufficiently generic in G^2, then by moving l_q to an appropriate $l_{q'}$ close to l_q, the curve $l_{q'}$ will intersect S near all points of $l_q \cap S$, and in addition at a new point near b. Since b itself was not in S it follows that a generic l_q through b intersects S at fewer points than a generic curve in $\mathcal{L}_0$. Thus b is $\mathcal{D}$-algebraic over $[S]$ and in particular $\mathrm{fr}(S)$ is finite.

While this strategy works well when the curves in $\mathcal{L}_0$ are complex lines in $\mathbb{C}^2$, the problem becomes more difficult when they are arbitrary plane curves and b is not necessarily generic in G^2. To get around this problem, the idea in [5] was to replace S by its image under composition with a "generic enough" curve from a new "large" family $\mathcal{L}'$ (Proposition 3.20). We carry out this replacement in Lemma 4.8 below. An additional complication of this strategy in the current setting is that instead of the functional language in [5] we need to work with arbitrary curves, and control their composition

4.2. **Two technical lemmas about 2-dimensional sets in G^2.** The following lemmas will be used in the sequel.

Claim 4.1. *Assume that $\{Y_e : e \in E\}$ is a definable family of 2-dimensional subsets of G^2, with $\dim E = k \geq 2$. Assume that for all $e \in E$ there are at most finitely many $e' \in E$, such that $|Y_e \cap Y_{e'}| = \infty$. Then $\dim(\bigcup_{t \in E} Y_t) = 4$.*

Proof. The set
$$\{(e, s) : e \in E\,,\ s \in Y_e\}$$
has dimension $k + 2$. Therefore, if the union of the Y_e had dimension smaller than 4, then for a generic s in this union, the dimension of $E(s) = \{e \in E : s \in Y_e\}$ is at least $k - 1 \geq 1$, and in particular, is infinite. Hence, there are $e_1, e_2 \in E(s)$, independent and generic over

s. Therefore, $\dim(e_1, e_2/s) = 2k - 2$ and hence $\dim(e_1, e_2, s) = 2k - 2 + 3 = 2k + 1$. But this is impossible since $\dim(e_1, e_2/\emptyset) \le 2k$ and, by our assumption on the family, the set $Y_{e_1} \cap Y_{e_2}$ is finite, so $s \in acl(e_1, e_2)$. $\square$

Definition 4.2. We say that two 2-dimensional sets C_1 and C_2 *intersect transversely at* $p \in C_1 \cap C_2$ if C_1 and C_2 are both smooth at p, and their tangent spaces at p generate the full tangent space of G^2 at p, namely $T_pC_1 + T_pC_2 = T_pG^2$.

Lemma 4.3. *Let $\mathcal{L} = \{l_q : q \in Q\}$ be a definable family of 2-dimensional subsets of G^2, and $S \subseteq G^2$ a definable 2-dimensional set, all $\emptyset$-definable. Let q be generic in Q over $\emptyset$ and assume that l_q and S intersects transversely at s. Then for every neighborhood $U \subseteq G^2$ of s, there exists a neighborhood $V \subseteq Q$ of q, such that for every $q' \in V$, we have $l_{q'} \cap S \cap U \ne \emptyset$.*

Proof. Without loss of generality, U is definable over $\emptyset$ and $l_q \cap U$ is smooth (we can shrink it so that q is generic in Q over the parameters defining it). We may therefore write $l_q \cap U$ as the zero set of a definable C^1-map $F_q : U \to R^2$, and similarly write S as the zero set of a C^1-map $G : U \to R^2$. The transversal intersection of l_q and S implies that the joint map $(F_q, G) : U \to R^4$ is a diffeomorphism at s, so in particular there is $U_0 \subseteq U$ such that (F_q, G) is a diffeomorphism on U_0 and $\bar{0} \in R^4$ is in its open image. We may choose U_0 so q is still generic over the parameters defining U_0. It follows that there is neighborhood $V \subseteq Q$ of q such that for every $q' \in V$, $l_{q'} \cap U = F_{q'}^{-1}(0)$, for some definable $F_{q'} : U_0 \to R^2$, and the map $(F_{q'}, G)$ is still a diffeomorphism on $U_0 \subseteq U$, with $\bar{0}$ in its image. But now, if $(F_{q'}, G)(s') = \bar{0}$ then $s' \in U_0 \cap l_{q'} \cap S$. $\square$

4.3. **Bad points.** Recall that $\mathcal{L}_0 = \{l_q : q \in Q\}$ is a faithful and very ample $\mathcal{D}$-definable family of strongly minimal plane curves, with $\mathrm{RM}(Q) = 2$. Notice that for $b \in G^2$ generic, the set $Q(b) = \{q \in Q : b \in l_q\}$ has Morley rank 1. We denote $\mathcal{L}_0(b) = \{l_q : q \in Q(b)\}$.

Definition 4.4. Let $U \subseteq G^2$ be an open set and $b \in G^2$. We say that $\mathcal{L}_0(b)$ *fibers* U if for every $s \in U$ there exists a unique $q \in Q(b)$ such that $s \in l_q$, the set $Q(b)$ is smooth at q and furthermore the function $s \mapsto q : U \to Q(b)$ is a submersion at s (that is, the differential map between the tangent spaces is surjective).

Definition 4.5. For $b \in G^2$, we say that a point $s = (s_1, s_2) \in G^2$ is *b-good* if

(1) There exists an open neighborhood $U \subseteq G^2$ of s such that the family $\mathcal{L}_0(b)$ fibers U.
(2) For all $q \in Q(b)$ such that $s \in l_q$, the curve l_q is smooth at s.

Otherwise, we that s is a *b-bad*. We denote by $\mathrm{Bad}(b)$ the set of all b-bad points.

Clearly, the set of b-bad points in G^2 in definable over b.

Lemma 4.6. *For every $b \in G^2$, the set of b-bad points has dimension at most 3.*

Proof. Note that since $\mathcal{L}_0(b)$ is faithful, it follows that $\mathrm{RM}(G^2 \setminus \bigcup_{q \in Q(b)} l_q) \le 1$.

By cell decomposition, for a fixed generic $q \in Q$, the set of points $s \in l_q$ failing (2) is at most 1-dimensional. So the set of all points s failing (2) is at most 3-dimensional.

We now fix a generic $s \in G^2$ over b, and show that it satisfies (1). The set of singular points q on $Q(b)$ has dimension one, and for every such q, l_q has dimension 2. Thus, the union of all such l_q has dimension at most 3, and does not contain s. So if $s \in l_q$ for some $q \in Q(b)$ then q is a smooth point on $Q(b)$.

Since s is generic in G^2, there are at most finitely many curves in $\mathcal{L}_0(b)$ containing s. Hence, there is an open neighborhood $W \subseteq Q(b)$ such that $W \cap Q(b) \cap Q(s) = \{q\}$. We may choose W to be definable over generic parameters. Hence the first-order property over b: "$\varphi(s') := |W \cap Q(s') \cap Q(b)| = 1$" must hold for all s' in a neighborhood $U \subseteq G^2$ of s. Let $g : U \to Q(b)$ be the map sending s' to the unique $q' \in W \cap Q(b)$ with $s' \in l_{q'}$. Note that for every $q' \in g(U)$, $g^{-1}(q') = l_{q'} \cap U$. Since the family $\mathcal{L}_0(b)$ is faithful, we have $\dim g(U) = 2 = \dim Q(b)$, and by the genericity of s in $\mathrm{dom}(g)$, the function g is a submersion at s, thus s is a b-good point. $\square$

4.4. **Finiteness of the frontier.** The heart of the geometric argument is contained in the following lemma which shows that under certain assumptions indeed the frontier of S is contained in $\mathrm{acl}_{\mathcal{D}}([S])$.

Lemma 4.7. *Let $\mathcal{F} = \{S_t : t \in T\}$ be a $\mathcal{D}$-definable almost faithful family of plane curves, with $\mathrm{RM}(T) \geq 3$. Assume that $b \in G^2$ with $\dim(b/\emptyset) = 4$, and $t_0 \in T_0$ generic over $\emptyset$. If $b \in \mathrm{fr}(S_{t_0})$ then $b \in \mathrm{acl}_{\mathcal{D}}(t_0)$.*

Proof. We may assume first that S_{t_0} is strongly minimal. Indeed, S_{t_0} is a finite union of strongly minimal sets, each definable over $\mathrm{acl}_{\mathcal{D}}(t_0)$. Clearly, b is in the frontier of one of these so we may replace S_{t_0} by this strongly minimal set, and modify the family $\mathcal{F}$ accordingly.

Denote $S = S_{t_0}$ and $B = \mathrm{Bad}(b)$.

Claim 1. $\dim(S \cap B) \leq 1$.

Proof of Claim 1. Since $\dim(t_0/\emptyset) \geq 6$ and $\dim(b/\emptyset) \leq 4$, we obtain $\dim(t_0/b) \geq 2$. Assume towards a contradiction that $\dim(S \cap B) = 2$. Let

$$I = \{t \in T : \dim(S_t \cap B) = 2\}.$$

Notice that I is defined over b and $t_0 \in I$, so $\dim I \geq 2$. Because $\mathcal{F}$ is almost faithful, $\{S_t \cap B : t \in I\}$ is a definable family of 2-dimensional subsets of G^2 satisfying the assumptions of Claim 4.1. It follows that $\dim \bigcup_{t \in T}(S_t \cap B) = 4$. But $\bigcup(S_t \cap B) \subseteq B$, contradicting Lemma 4.6. $\square$

Claim 2. *For every $q' \in Q$, $S \cap l_{q'}$ is finite.*

Proof of Claim 2. If not then by strong minimality of S, we would have $S_{t_0} \sim l_{q'}$ for some $q' \in Q$, implying that $t_0 \in acl(q')$. However, we assumed that $\dim(t_0/\emptyset) \geq 6$, while $\dim(q'/\emptyset) \leq 4$, a contradiction. $\square$

We fix an element $q \in Q(b)$ generic over t_0 and b. Since $\dim(b/\emptyset) = 4$, q is generic in Q over $\emptyset$, hence we have $\dim(q/\emptyset) = 4$.

Since $\mathcal{L}_0$ is very ample, no two points in G^2 belong to infinitely many curves in $\mathcal{L}_0$, and hence each $s \in S \cap l_q$ is inter-algebraic with q over t_0 and b. Thus such an s is generic in S over t_0 and b. So in particular S is smooth at s. It is not hard to see now (using the fact that $\mathcal{F}$ is almost faithful) that $\dim(s/b) = 4$.

For the rest of this proof, we fix an element $s \in S \cap l_q$.

Claim 3. *The curve l_q is smooth at s, and the intersection of S and l_q is transversal at s.*

Proof of Claim 3. Because $\dim(s/b) = 4$, it follows from Claim 1 that s is b-good, so in particular l_q is smooth at s and there exist neighborhoods $U \subseteq G^2$ of s and $W \subseteq Q(b)$ of q, and a $\mathcal{D}$-definable a parameter choice function $g_b : U \to W$, such that $g(s')$ is the unique $q' \in W$ with $s' \in l_q \cap U$. Restricting U, W if needed we may assume that $l_q \cap U$ (which equals $g_b^{-1}(q)$) is a C^1-sub-manifold of G^2. Thus, the tangent space to l_q at s, $T_s(l_q)$, equals $\ker(d_s(g_b))$. If the intersection is not transversal, then $\dim(T_s(l_q) \cap T_s(S)) \geq 1$. It follows that $\dim(d_s(g_b)(T_s(S))) \leq 1$, and by genericity of s in S over t_0, b, the same is true of any $s' \in S$ in some open neighborhood $U' \ni s$. Thus, the image of $g_b(S \cap U)$ is a 1-dimensional manifold (or finite), and it follows that for some q' in this image, $l_{q'} \cap S$ is infinite. This contradicts Claim 2. □

Claim 4. *For every neighborhood $V \subseteq Q$ of q, there exists a neighborhood $U \subseteq G^2$ of b such that for every $b' \in U$ there are infinitely many $q' \in V$ with $b' \in l_{q'}$.*

Proof of Claim 4. By assumptions, $b \in l_q$ is generic in G^2 over $\emptyset$. Thus, by shrinking V if needed, we may assume b is still generic in G^2 over the parameters defining V. Since $Q(b) \cap V$ is infinite, the first order statement:

$$\varphi(b') := \exists^\infty q' \in V \ b' \in l_{q'}$$

holds for b and therefore there is a neighborhood $U \ni b$ for which it holds. □

Let N be the number of intersection points with S of a generic curve from $\mathcal{L}_0$ over t_0 (recall that $\mathcal{L}_0$ has Morley degree 1, so it has a unique generic type).

Claim 5. The curve l_q intersects S in less than N points.

Proof of Claim 5. We write $l_q \cap S = \{s_1, \ldots, s_n\}$ (note that b is not among them). We first fix some open disjoint neighbourhoods $U_1, \ldots, U_n \subseteq G^2$, of $s_1, \ldots, s_n$, respectively. By Claim 3 and Lemma 4.3, applied to each of the s_i, there is a neighbourhood $V \subseteq Q$ of q such that for every $q' \in V$, the curve $l_{q'}$ intersects S at least n times – at least once in each of the U_i, $i = 1, \ldots, n$. Next, we apply Claim 4 to V and find $U_0 \ni b$, which we may assume is disjoint from all the U_i, as in Claim 4.

Because b is in $\mathrm{cl}(S) \setminus S$, we can find in $S \cap U_0$ a $\mathcal{D}$-generic element s' of S over t_0, and by Claim 4, we can find in V a generic $q' \in Q(s')$ over s' and t_0. But now $l_{q'}$ intersects S at least $n+1$ times: at s' and in each of $U_1, \ldots, U_n$. Since $S \cap l_{q'}$ is finite, the curve $l_{q'}$ is generic in $\mathcal{L}_0$ over t_0. So we have $N \geq n+1 > n = |l_q \cap S|$. □

Finally, let us see that $b \in \mathrm{acl}_{\mathcal{D}}(t_0)$. By Claim 5 no generic curve in $\mathcal{L}_0(b)$ intersects S_{t_0} in a generic number of points. So b is contained in the set Y of all those $b' \in G^2$ such that for all but finitely many $q_1 \in \mathcal{L}_0(b')$, we have $|l_{q_1} \cap S| < N$. The set Y is $\mathcal{D}$-definable over $\emptyset$ and has Morley rank at most 1. Since t_0 is generic in T over $\emptyset$ and $\mathrm{RM}(T) \geq 3$ we get that $\mathrm{RM}(t_0/\emptyset) \geq 3$, and hence $Y \cap S_{t_0}$ is finite. Since $b \in Y \cap S_{t_0}$ it follows that $b \in \mathrm{acl}_{\mathcal{D}}(t_0)$. □

In our next step we show that the assumptions of Lemma 4.7 can be met for a $\mathcal{D}$-definable set S of $\mathrm{RM}(S) = 1$, after replacing S by its composition with a generic enough curve in a family $\mathcal{L}'$ as in Proposition 3.20.

Lemma 4.8. *Let $S \subseteq G^2$ be a $\mathcal{D}$-definable set with* $\mathrm{RM}(S) = 1$, *and assume that c is generic in G^2 over $\emptyset$ and belongs to* $\mathrm{fr}(S)$. *Then there are*

(1) *An almost faithful family of plane curves $\mathcal{S}' = \{S'_t : t \in T\}$, $\mathcal{D}$-definable over $[S]$, with* $\mathrm{RM}(T) \geq 3$.
(2) *t_0 generic in T over* $c \cup \mathrm{acl}_{\mathcal{D}}([S])$
(3) *b is $\mathcal{D}$-interalgebraic with c over $t_0 \cup [S]$.*
(4) $b \in \mathrm{fr}(S'_{t_0})$ *and* $\dim(b/\emptyset) = 4$.

Proof. Let $\mathcal{L}' = \{C_t : t \in T\}$ be a $\mathcal{D}$-definable family of curves as in Proposition 3.20. Recall that, for every $(a, b) \in G^2$,

$$T(a, b) := \{t \in T : (a, b) \in C_t\}.$$

If we write $c = (c_1, c_2)$ then by assumption, c_2 is generic in G over $\emptyset$. Fix an element $b_2 \in G$, which is generic over $c_2 \cup \mathrm{acl}_{\mathcal{D}}([S])$ (abusing notation, in the present proof we will write $[S]$ for $\mathrm{acl}_{\mathcal{D}}([S])$), and let t_0 be generic in $T(c_2, b_2)$ over c_1, c_2, b_2 and $[S]$. Note that $(c_2, b_2) \in G^2$ is generic and C_{t_0} is generic through it. So $\dim(t_0 c_2 b_2) = \dim(T) + 2$, whereas $\dim(t_0/c_2 b_2) = \dim(T) - 2$. Since $b_0 \in \mathrm{acl}_{\mathcal{D}}(t_0 c_2)$ we get that $\dim(t_0/c_2) = \dim(T)$. Because t_0 was chosen generic over $c_1, [S]$ too, we get $\dim(t_0/c_1 c_2 [S]) = \dim(T)$.

We set $b := (c_1, b_2)$. Since $(c_2, b_2) \in C_{t_0}$ and $\mathrm{RM}(C_{t_0}) = 1$, b_2 and c_2 are inter-algebraic in $\mathcal{D}$ over t_0 and $[S]$, and hence so are (c_1, b_2) and (c_1, c_2).

Claim. $b \in \mathrm{fr}(C_{t_0} \circ S)$.

Proof of Claim. Since c_2 is generic in G over $\emptyset$, (c_2, b_2) is generic in G^2 over $\emptyset$ and therefore, by our choice of t_0, the point (c_2, b_2) is also generic in C_{t_0} over t_0. Hence, the curve C_{t_0} is a homeomorphism at (c_2, b_2). Denote this local map by f_0. It follows that the map $(x, y) \mapsto (x, f_0(y))$ is a local homeomorphism on a neighborhood W of (c_1, c_2), sending (c_1, c_2) to (c_1, b_2). It is easy to verify that it sends every point in $S \cap W$ to a point in $C_{t_0} \circ S$, and therefore sends every point in $\mathrm{cl}(S) \cap W$ to a point in $\mathrm{cl}(C_{t_0} \circ S)$. We conclude that $(c_1, b_2) \in \mathrm{cl}(S'_{t_0})$.

It remains to see that $(c_1, b_2) \notin C_{t_0} \circ S$. Let

$$S_{c_1} = \{y \in G : (c_1, y) \in S\} = \{d_1, \ldots, d_k\}.$$

Note that, since $(c_1, c_2) \notin S$, we have $c_2 \notin S_{c_1}$. Also, $(c_1, b_2) \in C_{t_0} \circ S$ if and only if there is some $i = 1, \ldots, k$ for which $(d_i, b_2) \in C_{t_0}$.

Since $\mathcal{L}'$ is very ample, for every $i = 1, \ldots, k$, $\dim(T(c_2, b_2) \cap T(d_i, b_2)) < \dim T$. But t_0 is generic in $T(c_2, b_2)$ over $\{c_1, c_2, d_1, \ldots, d_k, b_2, [S]\}$ and therefore, $t_0 \notin T(c_2, b_2) \cap T(d_i, b_2)$. That is, none of the points (d_i, b_2) is in C_{t_0}. It follows that $(c_1, b_2) \notin C_{t_0} \circ S$, so we may conclude that $b = (c_1, b_2) \in \mathrm{fr}(S'_{t_0})$.

Since $\mathrm{RM}(C_{t_0} \circ S) = 1$ there is a strongly minimal $C \subseteq C_{t_0} \circ S$ such that $b \in \mathrm{fr}(C)$. By Lemma 3.16 $\mathrm{RM}[C] = \mathrm{RM}[C_{t_0}] = \mathrm{RM}(T)$ and is contained, therefore, in an almost faithful family $\mathcal{S}'$ of the same rank. This gives condition (1) of the lemma, (2) is by the choice of t_0, (3) is the line before the above claim, and (4) is what we just showed. So the lemma is proved.

□

We can now conclude the main result of this section.

Theorem 4.9. *Let $S \subseteq G^2$ be a $\mathcal{D}$-definable set with* $\mathrm{RM}(S) = 1$. *Then* $\mathrm{fr}(S) \subseteq \mathrm{acl}_{\mathcal{D}}([S])$ *and hence* $\mathrm{fr}(S)$ *is finite. In particular, S is locally closed, namely every $p \in S$ has a neighborhood $U \ni p$ in G^2 such that $S \cap U$ is closed in U.*

Proof. Fix $c \in \mathrm{fr}(S)$. Replacing S by $S + p$ for p generic in G^2 over c and $[S]$, we may assume that $\dim(c/\emptyset) = 4$. We can now apply Lemma 4.8 and obtain t_0, S'_{t_0} and $b \in \mathrm{fr}(S'_{t_0})$ as in the lemma. Working first in a richer language where $[S]$ is $\emptyset$-definable, we may apply Lemma 4.7, and then conclude that $b \in \mathrm{acl}_{\mathcal{D}}(t_0, [S])$.

By Lemma 4.8, c is interalgebraic with b over t_0 and $[S]$ hence, $c \in \mathrm{acl}_{\mathcal{D}}(t_0, [S])$. Since $\dim(t_0/c, [S]) = \dim(t_0/c)$, we obtain that $c \in \mathrm{acl}_{\mathcal{D}}([S])$.

For $p \in S$, let $U \ni p$ be any neighborhood such that $U \cap \mathrm{fr}(S) = \emptyset$, and then $S \cap U$ is closed in U. □

4.5. **A structural corollary on plane curves.**

Definition 4.10. Let $S \subseteq G^2$ be strongly minimal. Say that $x \in \pi(S)$ is a *an injective point of S* if for all $y \in S_x := \{z : (x, z) \in S\}$, there is an open $U \ni (x, y)$ such that $S \cap U$ is the graph of an injective function in the x variable (we do not specify its domain).

We call $x \in \pi(S)$ *a non-injective point of S* otherwise.

Note that at this point, if x is injective then it might be the case that for some y, the point (x, y) is an isolated point of S, or that S is locally at (x, y) 1-dimensional.

We now prove:

Corollary 4.11. *Let $S \subseteq G^2$ be strongly minimal and assume S is not $\sim$-equivalent to any fiber $G \times \{a\}$ or $\{a\} \times G$. Then the set of non-injective points of S is finite and contained in* $\mathrm{acl}_{\mathcal{D}}([S])$.

Proof. By Theorem 4.9, we may assume that S is closed. Let

$$S_1 := \{(y_1, y_2) \in G^2 : y_1 \neq y_2 \ \& \ \exists x((x, y_1) \in S \wedge (x, y_2) \in S)\} = (S \circ S^{op}) \setminus \Delta.$$

It is definable in $\mathcal{D}$ over the same parameters as S and $\mathrm{RM}(S_1) \leq 1$. Note that $(a, a) \notin \mathrm{fr}(S_1)$ if and only if there exists an open $U \ni a$ such that for all $x \in G$ there exists at most one $y \in U$ such that $(x, y) \in S$. This is equivalent to saying that $S \cap (G \times U)$ is the graph of a function.

Similarly, let

$$S_2 := \{(x_1, x_2) \in G^2 : x_1 \neq x_2 \ \& \ \exists y((x_1, y) \in S \wedge (x_2, y) \in S)\} = (S^{op} \circ S) \setminus \Delta.$$

Then $(b, b) \notin \mathrm{fr}(S_2)$ if and only if there exists an open $V \ni b$ such that for all $y \in G$ there exists at most one $x \in V$ such that $(x, y) \in S$. Equivalently, $S^{op} \cap (G \times V)$ is the graph of a function.

Thus, for $(a, b) \in S$, if $(a, a) \notin \mathrm{fr}(S_1)$ and $(b, b) \notin \mathrm{fr}(S_2)$ then there is an open $U \times V \ni (a, b)$ such that $S \cap U \times V$ is the graph of an injective function (notice that we cannot determine yet the domain of the function). It follows that if $(a, b) \in S$ is a non-injective point then either $(a, a) \in \mathrm{fr}(S_1)$ or $(b, b) \in \mathrm{fr}(S_2)$. By Theorem 4.9, these frontiers are finite and contained $\mathrm{acl}_{\mathcal{D}}([S])$.

Finally, by our assumptions on S, for every $a, b \in G$ the sets S_a and S^b are finite and therefore the set of (a, b) which are non-injective is finite and contained in $\mathrm{acl}_{\mathcal{D}}([S])$. □

5. Poles of plane curves

Recall that we assume that G is a definable closed subset of some R^n, equipped with the subspace topology, making it a topological group.

The goal of this section is to prove that just like affine algebraic curves in $\mathbb{C}^2$, every plane curve has at most finitely many poles. We may assume that $0_G = 0 \in R^n$, and for $x \in G$, we write $B(x; \epsilon)$ for all $g \in G$ whose Euclidean distance from x is smaller than ϵ. We write B_ϵ for $B(0; \epsilon)$. For $A \subseteq G$, and $\epsilon > 0$, we let

$$B(A; \epsilon) = \{y \in G : \exists x \in A \ , \ y \in B(x; \epsilon)\}.$$

Definition 5.1. Let $S \subseteq G^2$ be a definable set. We call $a \in G$ a *pole of S* if for every open $U \subseteq R^n$ containing a, the set $(U \times G) \cap S$ is an unbounded subset of R^n. We denote the set of poles of S by S_{pol}.

Given $S \subseteq G^2$ and $U \subseteq R^n$, we define

$$S(U) := \{y \in G : \exists x \in U \ (x, y) \in S\} \subseteq R^n.$$

Note that then $a \in S_{\mathrm{pol}}$ if and only if for every open $U \subseteq R^n$ containing a, $S(U)$ is unbounded. Another remark is that if S is G-affine then $S_{\mathrm{pol}} = \emptyset$. Indeed, if S is a subgroup of G^2 or its coset then its projection onto the first coordinate is a finite-to-one topological covering map, and hence S has no poles.

The main result of this section is the following.

Theorem 5.2. *If $S \subseteq G^2$ is a $\mathcal{D}$-definable set and* $\mathrm{RM}(S) = 1$*, then S_{pol} is finite.*

Notice that if G is a definably compact group (for example, a complex elliptic curve) then G is a closed and bounded subset of R^n, and hence $S_{\mathrm{pol}} = \emptyset$. So the theorem is about those G which are not definably compact.

Let us first introduce the key notion of "approximated points" and then discuss the strategy of our proof. Recall that for $S \subseteq G^2$ and $x \in G$, we let $S_x = \{y \in G : (x, y) \in S\}$.

Definition 5.3. Let $S \subseteq G^2$, $b, x_1, x_2 \in G$, and $I \subseteq G$. We say that

(1)
- *b is S-attained at (x_1, x_2)* if $b \in S_{x_1} - S_{x_2}$.
- *b is S-attained at I* if there are $x_1, x_2 \in I$ such that $b \in S_{x_1} - S_{x_2}$.

(2)
- *b is S-attained near (x_1, x_2)* if for every $\epsilon > 0$ there are $x_1' \in B(x_1; \epsilon)$, $x_2' \in B(x_2; \epsilon)$ such that $b \in S_{x_1'} - S_{x_2'}$.
- *b is S-attained near I* if for every $\epsilon > 0$, b is attained at $B(I; \epsilon)$.

(3)
- *b is S-approximated near (x_1, x_2)* if for every $\epsilon > 0$ there are $x_1' \in B(x_1; \epsilon)$, $x_2' \in B(x_2; \epsilon)$ such that $B(b; \epsilon) \cap (S_{x_1'} - S_{x_2'}) \neq \emptyset$.
- *b is S-approximated near I* if for every $\epsilon > 0$, there are $x_1, x_2 \in B(I; \epsilon)$ such that $B(b; \epsilon) \cap (S_{x_1'} - S_{x_2'}) \neq \emptyset$. The set of such points b is denoted by $A(S, I)$.

We omit S from the above notation whenever it is clear from the context.

The following claim is immediate from the definitions.

Claim 5.4. *For any $S \subseteq G^2$ and $I \subseteq G$,*

$$b \text{ attained at } I \Rightarrow b \text{ is attained near } I \Rightarrow b \text{ is approximated near } I.$$

If in addition S and I are closed and bounded then they are equivalent and $A(S, I) = S(I) - S(I)$.

Here is a simple example.

Example 5.5. Let $G = \langle \mathbb{C}, + \rangle$ and consider the complex algebraic curve

$$S = \{(z, w) \in \mathbb{C}^2 : zw = 1\}.$$

The following are easy to verify: $S_{\mathrm{pol}} = \{0\}$, every $b \in \mathbb{C}$ is attained near 0, and thus $A(S, \{0\}) = \mathbb{C}$.

The strategy of the proof of Theorem 5.2 is as follows. Assume towards a contradiction that the theorem fails. It is easy to see that we may assume that S is closed, strongly minimal and not G-affine. Now, for any such $\mathcal{D}$-definable set S and infinite definable $I \subseteq G$, we first find an infinite definable set $I_0 \subseteq I$ and an open bounded ball $B \subseteq R^n$, such that the set $A(S, I_0) \setminus B$ is at most 1-dimensional (Proposition 5.6(1)). Then, using further that S_{pol} is infinite, we construct (Proposition 5.10) another $\mathcal{D}$-definable set $\hat{S}$, again closed, strongly minimal and not G-affine, and an infinite definable $\hat{I} \subseteq G$, such that for every infinite definable set $T \subseteq \hat{I}$ and open bounded ball B, the set $A(\hat{S}, T) \setminus B$ is 2-dimensional. A contradiction.

5.1. **Upper bound on dimension of the set of approximated points.** The goal of this subsection is to prove the following proposition.

Proposition 5.6. *Assume that $S \subseteq G^2$ is a $\mathcal{D}$-definable strongly minimal closed set which is not G-affine, and let $I \subseteq G$ be an infinite definable set. Then there is a definable 1-dimensional $I_0 \subseteq I$, such that*

(1) *there exists a bounded $B \subseteq G$, such that the set $A(S, I_0)\setminus B$ is at most 1-dimensional,*
(2) *for every definable open $V \ni 0$ in G there exist $\epsilon > 0$ and a bounded ball $B' \ni 0$ such that for all $x \in G \setminus B'$,*
$$x + V \not\subseteq S(B(I_0, \epsilon)).$$

The rest of this subsection is devoted to the proof of Proposition 5.6. We fix throughout S and I as in its assumptions. By o-minimality $\dim S_{\mathrm{pol}} \leq 1$. Without loss of generality, S is defined over $\emptyset$.

We begin with an observation regarding the notions of Definition 5.3.

Lemma 5.7. *Let $I \subseteq G^2$ be a definable bounded set over $\emptyset$.*

(1) *If $b \in G$ is attained near I then there are $x_1, x_2 \in \mathrm{cl}(I)$ such that b is attained near (x_1, x_2).*
(2) *If b is generic in G over $\emptyset$ and b is approximated near I then b is attained near I.*

Proof. (1) By assumption, for every $\epsilon > 0$ there are $x_1(\epsilon), x_2(\epsilon) \in B(I; \epsilon)$ and $y_1(\epsilon), y_2(\epsilon)$ such that $(x_i(\epsilon), y_i(\epsilon)) \in S$ for $i = 1, 2$ and $b = y_1(\epsilon) - y_2(\epsilon)$. Since I is bounded, the curves $x_i(t)$ have limits $x_1, x_2 \in \mathrm{cl}(I)$, so b is attained near (x_1, x_2).

(2) Fix b generic in G over $\emptyset$, and assume that it is approximated near I. It follows from the definition that for every $\epsilon > 0$, the element b is in the closure of
$$Y_\epsilon = \{y_2 - y_1 : \exists x_1, x_2 \in B(I; \epsilon)\ (x_1, y_1), (x_2, y_2) \in S\}.$$

Notice that the collection of Y_ϵ forms a definable chain of definable sets decreasing with ϵ. We may now take ϵ sufficiently small, so that b is still generic in G over ϵ, and therefore b is generic in $\mathrm{cl}(Y_\epsilon)$ over ϵ. Hence, $b \notin \mathrm{fr}(Y_\epsilon)$, a set of dimension at most 1. It follows that $b \in Y_\epsilon$ for all sufficiently small ϵ, and so b is attained near I. □

The following technical claim about definable and $\mathcal{D}$-definable sets will be used in the subsequent lemma.

Claim 5.8. *Let $R \subseteq G^2 \times G$ be a $\mathcal{D}$-definable set of Morley rank 1, whose projection on the G^2-coordinate is finite-to-one. Then for any definable sets $I, J \subseteq G$ of dimension at most 1,*
$$\dim(R \cap (I \times J \times G)) \leq 1.$$

Proof. Notice that if either I or J are finite then the result follows from our assumption on R. So we may assume that $\dim I = \dim J = 1$. By assumption on R, the projection of R on one of the coordinates of G^2 has infinite image. Let us assume it is the projection on the first coordinate. Hence for every $\mathcal{D}$-generic $a \in G$, the set $\{(w, z) \in G \times G : (a, w, z) \in R\}$ is finite. Since $I \subseteq G$ is infinite every generic of I is also $\mathcal{D}$-generic in G. But then, for such an $a \in I$ the set $\{(w,z) \in J \times G : (a,y,z) \in R\}$ is finite. Because of our assumption on R, it follows that $\dim(R \cap (I \times J \times G)) \leq 1$. $\square$

We proceed with the proof of Proposition 5.6.

Lemma 5.9. *There exists a finite set $F \subseteq G$, $F \subseteq \mathrm{acl}_{\mathcal{D}}([S])$, and a definable set $X \subseteq G$, with $\dim X \leq 1$, such that for every $b \in G \setminus X$ and for every $(x_1, x_2) \in G^2 \setminus F^2$, if b is attained near (x_1, x_2) then b is attained at (x_1, x_2).*

Proof. Consider the $\mathcal{D}$-definable set

$$T = \{(x_1, x_2, b) \in G^3 : b \in S_{x_1} - S_{x_2}\}.$$

Since every generic fiber S_x is finite, $\mathrm{RM}(T) = 2$. Also, it is easy to see that the projection of T on the last coordinate is infinite and hence for every $\mathcal{D}$-generic $b \in G$ the set

$$T^b = \{(x_1, x_2) \in G^2 : (x_1, x_2, b) \in T\}$$

has Morley rank 1. Note that $(x_1, x_2, b) \in T$ if and only if b is attained at (x_1, x_2), and $(x_1, x_2) \in \mathrm{cl}(T^b)$ if and only if b is attained near (x_1, x_2). We also note, although we will not use this, that $(x_1, x_2, b) \in \mathrm{cl}(T)$ if and only if b is approximated near (x_1, x_2).

Claim 1. *For $b \in G$ and $x_1, x_2 \in G$, the following are equivalent:*
1. *b is attained near (x_1, x_2) but not attained at (x_1, x_2).*
2. *$(x_1, x_2) \in \mathrm{fr}(T^b)$ and $x_1, x_2 \in S_{pol}$.*
3. *$(x_1, x_2) \in \mathrm{fr}(T^b)$.*

Proof of Claim 1. (1) $\Rightarrow$ (2): If b is attained near (x_1, x_2) then we can find definable curves $(x_1(t), y_1(t)) \in S$ and $(x_2(t), y_2(t)) \in S$ such that $x_i(t) \to x_i$, for $i = 1, 2$, and $y_1(t) - y_2(t) = b$. It immediately follows that $(x_1, x_2) \in \mathrm{cl}(T^b)$, and by our assumption, $(x_1, x_2) \notin T^b$, so belongs to $\mathrm{fr}(T^b)$.

Notice that $y_1(t)$ is bounded if and only if $y_2(t)$ is bounded, in which case, since S is closed, their limit points y_1, y_2 satisfy $(x_1, y_1), (x_2, y_2) \in S$ and $y_2 - y_1 = b$, so b is attained at (x_1, x_2). Because we assumed that this is not the case, $y_1(t)$ and $y_2(t)$ are unbounded, hence x_1, x_2 are both in S_{pol}.

The other implications are easy, thus ending the proof of Claim 1. $\square$

By Theorem 4.9, for each $b \in G$, every element of $\mathrm{fr}(T^b)$ is contained in $\mathrm{acl}_{\mathcal{D}}(b)$. By compactness we may find then a set $R \subseteq G^2 \times G$, $\mathcal{D}$-definable over $[S]$, such that for every $b \in G$ the set R^b is finite and contains $\mathrm{fr}(T^b)$. It follows that $\mathrm{RM}(R) = 1$. Note however that we do not claim that for every $b \in G$, we have $R^b = \mathrm{fr}(T^b)$. Thus, for example, we allow at this stage the possibility that the set of b for which T^b is not closed is 1-dimensional.

Now, by Claim 1, if b is attained near (x_1, x_2) and not attained at (x_1, x_2) then $(x_1, x_2) \in R^b$.

Assume first that the image of R under the projection onto the G^2-coordinates, call it F_1, is finite, and let $F \subseteq G$ be a finite set, $\mathcal{D}$-definable over $\mathrm{acl}_{\mathcal{D}}([S])$, such that $F_1 \subseteq F^2$. We may take $X = \emptyset$ and complete the proof of the lemma in this case. Assume then that F_1 is infinite.

Let $F_0 \subseteq G^2$ be the set of all $p \in G^2$ such that the fiber $R_p \subseteq G$ is infinite. This is a finite set, $\mathcal{D}$-definable over $\mathrm{acl}_{\mathcal{D}}([S])$, and because we assumed that F_1 is infinite, the set $R^* := (G^2 \setminus F_0^2) \times G$ still Morley rank 1, and the projection map from R^* onto the G^2-coordinate is finite-to-one.

Set

$$Y = \{b \in G : \mathrm{fr}(T^b) \setminus F_0^2 \neq \emptyset\}, \quad \text{a definable set in } \mathcal{M} .$$

Claim 2. $\dim(Y) \leq 1$.

Proof of Claim 2. Assume towards contradiction that $\dim Y = 2$. For every $b \in Y$ there exists $(x_1, x_2) \in \mathrm{fr}(T^b) \setminus F_0^2$. By Claim 1 and our choice of R, $(x_1, x_2) \in (R^*)^b \cap (S_{\mathrm{pol}} \times S_{\mathrm{pol}})$, so since $\dim Y = 2$ it follows that

$$\dim(R^* \cap (S_{\mathrm{pol}} \times S_{\mathrm{pol}} \times Y)) \geq 2.$$

This contradicts Claim 5.8. □

By Claim 1, for every $b \in G$ and for every $(x_1, x_2) \in G^2$, if b is attained near (x_1, x_2) and not at (x_1, x_2) then $(x_1, x_2) \in \mathrm{fr}(T^b) \subseteq R^b$. Now, either $(x_1, x_2) \in F_0^2$, or $b \in Y$. Thus, we may take $X = Y$ and $F = F_0$ and complete the proof of Lemma 5.9. □

Proof of Proposition 5.6 (1) . Fix a finite $F \subseteq G$ as in Lemma 5.9, and a definable 1-dimensional closed set $I_0 \subseteq I$, such that $I_0 \cap F = \emptyset$. Because $S \cap I_0 \times G$ is a 1-dimensional subset of $G \times G$, we may shrink I_0 further and assume that the set $S \cap (I_0 \times G)$ is closed and bounded. Thus, the set

$$B = \{b \in G : b \text{ is attained at } I_0\} = S(I_0) - S(I_0)$$

is a closed and bounded subset of G. By Lemma 5.9 and the choice of I_0, there is a definable $X \subseteq G$ with $\dim X \leq 1$ such that for every $b \in G \setminus X$, if b is attained near $(x_1, x_2) \in I_0^2$ then b is attained at (x_1, x_2). Assume towards contradiction that the set $A(S, I_0) \setminus B$ has dimension 2. By Lemma 5.7 (2), the set L of all $b \in G \setminus B$ which are attained near I_0 has dimension 2, and therefore there is some $b \in L$ which is not in X. By Lemma 5.7 (1), b is attained near some $(x_1, x_2) \in \mathrm{cl}(I_0) = I_0$, and since $b \notin X$ it is attained at (x_1, x_2). Namely, $b \in S(I_0) - S(I_0) = B$, a contradiction. □

The rest of this subsection is devoted to the proof of Proposition 5.6(2). Fix an open $V \subseteq G$ containing 0. We may assume that V is bounded and symmetric, namely $-V = V$. Given $r > 0$, let $P_r = \mathrm{cl}(B_r) \cap G$ and $S_r = \mathrm{fr}(B_r) \cap G$, the intersection of the r-sphere with G. Let B be as in Proposition 5.6(1).

Claim 1. *There are $r_1 > r_0 > 0$ sufficiently large such that $B \subseteq P_{r_0} \subseteq P_{r_1}$ and $S_{r_0} + V \subseteq P_{r_1} \setminus B$.*

Proof of Claim 1. Since $B + V$ is bounded, there exists $r_0 > 0$ such that $B \subseteq P_{r_0}$ and $B + V$ does not intersect S_{r_0}. Since V is symmetric, it follows that $(S_{r_0} + V) \cap B = \emptyset$. Because $S_{r_0} + V$ is bounded there exists $r_1 > r_0$ such that $S_{r_0} + V \subseteq P_{r_1}$. It follows that $S_{r_0} + V \subseteq P_{r_1} \setminus B$. □

For $\epsilon > 0$ again let

$$Y_\epsilon = \{y_2 - y_1 : \exists x_1, x_2 \in B(I_0;\epsilon)\ (x_1, y_1)\,,\,(x_2, y_2) \in S\}.$$

Claim 2. *There exists $\epsilon_0 > 0$ such that no translate of V is contained in $(P_{r_1} \setminus B) \cap Y_{\epsilon_0}$.*

Proof of Claim 2. The family of Y_ϵ decreases with ϵ, and we have already seen above that

$$A(S, I_0) = \bigcap_\epsilon \mathrm{cl}(Y_\epsilon).$$

We restrict our attention to the definably compact set $P_{r_1} \setminus \mathrm{int}(B)$ and let

$$\bar{Y}_\epsilon^{r_1} = \mathrm{cl}(Y_\epsilon) \cap (P_{r_1} \setminus \mathrm{int}(B)) \ \text{ and } \ A_{r_1}(S, I_0) = A(S, I_0) \cap (P_{r_1} \setminus \mathrm{int}(B)).$$

Thus, we have $A_{r_1}(S, I_0) = \bigcap_{\epsilon>0} \bar{Y}_\epsilon^{r_1}$. Each $\bar{Y}_\epsilon^{r_1}$ is definably compact, and hence $A_{r_1}(S, I_0)$ is also definably compact.

By Proposition 5.6(1), $\dim(A(S, I_0) \setminus B) \leq 1$ and hence, since the boundary of B is at most 1-dimensional, also $\dim(A(S, I_0) \setminus \mathrm{int}(B)) \leq 1$. It follows that $A_{r_1}(S, I_0)$ is a definably compact set which is at most 1-dimensional. Using that, it is not hard to see that for sufficiently small open $W \ni 0$ the set $A_{r_1}(S, I_0) + W$ does not contain any translate of our open set V. Fix such a set W.

Because $A_{r_1}(S, I_0) = \bigcap_\epsilon \bar{Y}_\epsilon^{r_1}$ it is not hard to see that there exists $\epsilon_0 > 0$, $\bar{Y}_{\epsilon_0}^{r_1} \subseteq A_{r_1}(S, I_0) + W$. It follows that the set $\bar{Y}_{\epsilon_0}^{r_1}$ does not contain any translate of V, thus proving Claim 2. $\square$

It is left to show that setting $\epsilon := \epsilon_0$ for ϵ_0 as provided by Claim 2, the requirements of Proposition 5.6 (2) are satisfied.

Claim 3. *There exists $r > 0$ such that for all $x \in G \setminus P_r$, $x + V \nsubseteq S(B(I_0, \epsilon_0))$.*

Proof of Claim 3. Assume towards a contradiction that no such r exists. Then we can find an unbounded, definably connected curve $\Gamma \subseteq G$ such that $\Gamma + V \subseteq S(B(I_0, \epsilon_0))$. It follows from the definition that $(\Gamma + V) - (\Gamma + V) \subseteq Y_{\epsilon_0}$.

Fix any $\gamma_0 \in \Gamma$ and let $\Gamma_0 = \Gamma - \gamma_0$. The curve Γ_0 is unbounded, definably connected, with $0 \in \Gamma_0$ and in addition $\Gamma_0 + V \subseteq (\Gamma + V) - (\Gamma + V) \subseteq Y_{\epsilon_0}$. Note that $\Gamma_0 \cap S_{r_0} \neq \emptyset$. Indeed, although $S_{r_0} = \mathrm{fr}(B_{r_0}) \cap G$ need not be definably connected, by considering the curve Γ_0 in R^n which is definably connected and unbounded, containing 0 we observe that Γ_0 must intersect the sphere $\mathrm{fr}(B_{r_0}) \subseteq R^n$. This intersection point necessarily lies in S_{r_0}. Fix $x_0 \in \Gamma_0 \cap S_{r_0}$.

By our choice of Γ_0, $x_0 + V \subseteq \Gamma_0 + V \subseteq Y_{\epsilon_0}$ and by our choice of r_0 in Claim 1, $x_0 + V \subseteq P_{r_1} \setminus B$. However, by Claim 2, no translate of V is contained in $Y_{\epsilon_0} \cap (P_{r_1} \setminus B)$, contradiction. $\square$

Choose r as in Claim 3. Setting $B' = P_r$ and $\epsilon = \epsilon_0$ finishes the proof of Proposition 5.6 (2).

5.2. **Lower bound on dimension of the set of approximated points.** In this subsection, we modify the set S from Lemma 5.11, using an idea from [5, Section 4]. The proof of Theorem 5.2 in the next subsection is by contradiction, and towards that we prove here the following lemma.

Lemma 5.10. *Let $S \subseteq G^2$ be a $\mathcal{D}$-definable strongly minimal, closed set which is not G-affine, and assume that S_{pol} is infinite. Then there is a strongly minimal closed set $\hat{S} \subseteq G^2$ which is not G-affine, definable in $\mathcal{D}$ (over additional parameters), and there exists an infinite definable $\hat{I} \subseteq G$, such that for every infinite definable set $T \subseteq \hat{I}$ and any bounded ball B, the set $A(\hat{S}, T) \setminus B$ is 2-dimensional.*

The rest of this subsection is devoted to the proof of Lemma 5.10. We apply Proposition 5.6 to the fixed set S and the infinite set S_{pol}. We thus obtain and fix a definable one dimensional $I_0 \subseteq S_{\mathrm{pol}}$ satisfying Clause (1) and (2) of the proposition.

Lemma 5.11. *There is a definable smooth 1-dimensional $I_1 \subseteq I_0$ and*

(1) *a definably connected bounded open $U \subseteq G$,*
(2) *a definable continuous function $f : U \to G$ with $\Gamma_f \subseteq S$, and*
(3) *a definable family $\{\gamma_x : x \in I_1\}$ of curves $\gamma_x : (0,1) \to U$ with $\lim_{t\to 0} \gamma_x(t) = x$, $\lim_{t\to 0} f(\gamma_x(t)) = \infty$, and for every $x_1, x_2 \in I_1$,*

$$\lim_{t\to 0} f(\gamma_{x_1}(t)) - f(\gamma_{x_2}(t)) = 0.$$

Proof. Using o-minimality and the fact that the projection of S onto G is finite-to-one, we may partition S and I_0 into finitely many cells and reach the following situation. There is a definable, definably connected bounded open $U \subseteq G$ and a definable 1-dimensional smooth $I_1 \subseteq I_0$, with I_1 on the boundary of U and $U \cup I_1$ a manifold with a boundary. We may assume that $\mathrm{cl}(U) \cap S_{\mathrm{pol}} = \mathrm{cl}(I_1)$. Furthermore, there is a definable, injective, continuous function $f : U \to G$ whose graph is contained in S, such that for every $x_0 \in I_1$ and every curve $\gamma : (0,1) \to U$ tending to x_0 at 0, the image of γ under f is unbounded.

After applying a definable local diffeomorphism, we may assume that $I_1 = (a,b) \times \{0\} \subseteq R^2$ and $U = (a,b) \times (0,1) \subseteq R^2$. By shrinking I_1 if needed we may assume that f is defined on the box $[a,b] \times (0,1]$. For $\epsilon \leq 1$, let

$$U_\epsilon = (a,b) \times (0,\epsilon) \subseteq U$$

and

$$C_\epsilon = f([a,b] \times \{\epsilon\}),\ \Gamma_{\epsilon,1} = f(\{a\} \times (0,\epsilon)),\ \Gamma_{\epsilon,2} = f(\{b\} \times (0,\epsilon)).$$

When $\epsilon = 1$, we denote $C_1, \Gamma_{1,1}$ and $\Gamma_{1,2}$ by C, Γ_1 and Γ_2, respectively. For every $\epsilon \leq 1$, the set C_ϵ is a bounded set and $\Gamma_{\epsilon,i}$, $i = 1, 2$, are unbounded curves. Recall that $\partial f(U_\epsilon)$ denotes the boundary of $f(U_\epsilon)$ (which is contained in G). Because $f : U \to G$ is continuous and injective it is in fact a homeomorphism, by [13], hence

$$\partial f(U_\epsilon) = \Gamma_{\epsilon,1} \cup \Gamma_{\epsilon,2} \cup C_\epsilon$$

(we use here the fact that the limit of $|f(x)|$ as x tends to any point in I_1 is ∞).

The next claim roughly says that for an infinitesimal ϵ, the set $f(U_\epsilon)$ is contained in two infinitesimal tubes around the Γ_1 and Γ_2.

Claim 1. *For every $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that*

$$f(U_{\epsilon_2}) \subseteq \bigcup_{i=1}^{2} \Gamma_i + B_{\epsilon_1}.$$

Proof of Claim 1. We fix $\epsilon_1 > 0$. Using Proposition 5.6(2), we can find $\epsilon > 0$ and a bounded neighborhood of 0, $B' \subseteq G$, such that for every $y \in G \setminus B'$, $y + B_{\epsilon_1} \not\subseteq f(U_\epsilon)$. Next, choose $0 < \epsilon_2 < \min\{\epsilon, \epsilon_1\}$, such that $f(U_{\epsilon_2})$ does not intersect the bounded sets B' and $C_\epsilon + B_{\epsilon_1}$. This can be done since the limit of $|f(x)|$ is ∞ as x tends in U to any point in I_1. We claim that this ϵ_2 satisfies our requirements.

Indeed, given $x \in U_{\epsilon_2}$, we have $f(x) \notin B'$ and hence $f(x) + B_{\epsilon_1} \not\subseteq f(U_\epsilon)$. However, clearly $f(x) \in f(U_\epsilon)$ (since $\epsilon_2 < \epsilon$) and so, because $f(x) + B_{\epsilon_1}$ is definably connected, we must have $(f(x)+B_{\epsilon_1}) \cap \partial f(U_\epsilon) \neq \emptyset$. Since $f(x) \notin C_\epsilon + B_{\epsilon_1}$, we have $(f(x)+B_{\epsilon_1}) \cap C_\epsilon = \emptyset$, and therefore $f(x) + B_{\epsilon_1}$ must intersect $\Gamma_{\epsilon,1} \cup \Gamma_{\epsilon,2}$, and hence also $\Gamma_1 \cup \Gamma_2$. It now follows that for some $i = 1, 2$, $f(x) \in \Gamma_i + B_{\epsilon_1}$. □

Claim 2. *There is a definable 1-dimensional subset $I_2 \subseteq I_1$, and a definable family $\{\gamma_x : x \in I_2\}$ of curves $\gamma_x : (0,1) \to U$ with $\lim_{t\to 0} \gamma_x(t) = x$, such that for every $x_1, x_2 \in I_2$,*

$$\lim_{t\to 0} f(\gamma_{x_1}(t)) - f(\gamma_{x_2}(t)) = 0.$$

Proof of Claim 2. Consider the unbounded curves $\Gamma_1, \Gamma_2 \subseteq \partial f(U)$, and for each $i = 1, 2$ fix a definable parametrization $\gamma_i(t) : (0,1) \to G$ for Γ_i, such that $\lim_{t\to 0} |\gamma_i(t)| = \infty$.

Now fix a definable family $\{\gamma_x : x \in I_1\}$ of curves $\gamma_x : (0,1) \to U$ with $\lim_{t\to 0} \gamma_x(t) = x$. By Claim 1, for each $x \in I_1$, the curve $f(\gamma_x(t))$ approaches one of the Γ_i as t tends to 0, and therefore, after possibly re-parameterizing γ_x, we can find γ_i, $i = 1, 2$, such that $\lim_{t\to 0} f(\gamma_x(t)) - \gamma_i(t) = 0$. The re-parametrization can be done uniformly in x. We can now find an infinite subinterval $I_2 \subseteq I_1$ and $i \in \{1, 2\}$ such that if $x \in I_2$ then $\lim_{t\to 0} f(\gamma_x(t)) - \gamma_i(t) = 0$. □

Replacing I_1 by I_2 finishes the proof of Lemma 5.11. □

The rest of this subsection is devoted to the proof of Lemma 5.10. We fix $I_1 \subseteq I_0, U, f, \{\gamma_x : x \in I_0\}$ as in Lemma 5.11. Without loss of generality, S is defined over $\emptyset$.

Because I_0 is smooth on the boundary of U, we can find infinite sub-cell $\hat{I} \subseteq I_0$ and $c \in G$ generic over $\emptyset$ such that $\mathrm{cl}(\hat{I} + c)$ is contained in U. We fix such c. A key initial observation is the following.

Claim 5.12. *For any infinite definable set $T \subseteq \hat{I}$, the set $V_c = f(T + c) - f(T + c)$ is a 2-dimensional bounded set.*

Proof. Since f is continuous and $\mathrm{cl}(\hat{I}+c) \subseteq U$, it follows that V_c is bounded. Assume now towards contradiction that $\dim V_c = 1$. By [19, Lemma 2.7], the one dimensional set $A_c = f(T+c)$ is a translate of local subgroup H_c of G (by that we mean that for all $x, y, z \in A_c$ sufficiently close to each other, $x - y + z \in A_c$).

By shrinking T if needed we may assume that c is still generic in G over the parameters defining T. It follows that for all c' in some open neighborhood $W \ni c$, the set $f(T + c')$ is a translate of a 1-dimensional local subgroup $A_{c'}$ of G.

We say that A_{c_1} and A_{c_2} have the same germ at 0 if there exists a neighborhood $W_0 \ni 0$ such that $W_0 \cap A_{c_1} = W_0 \cap A_{c_2}$. Note that A_{c_1} and A_{c_2} have the same germ if and only if their intersection is infinite. This is an equivalence relation on (definable families of) definable sets. The collection of germs of the $A_{c'}$'s at 0 is the collection of equivalence classes of this equivalence relation. Because G is a two dimensional group, the collection of germs at 0 of the $A_{c'}$'s can be at most 1-dimensional. Indeed, we may choose V a sufficiently small neighborhood of 0 such that every element of V has infinite order in G. Thus, if A_1 and A_2 are definable and inequivalent local subgroups of G then $A_1 \cap A_2 \cap V = \{0\}$. Thus, the family of germs of the $A_{c'}$'s is at most one dimensional.

Because c' varies in a two dimensional set, there are infinitely many c' for which the germ of $A_{c'}$ is the same. If we now fix generic and independent $x, y, z \in T$ sufficiently close to each other, then there is $w \in T$ and there are infinitely many c' such that

$$f(x + c') - f(y + c') + f(z + c') = f(w + c').$$

It easily follows (see Lemma 3.10) that S is G-affine, contradicting our assumptions. □

Consider now the $\mathcal{D}$-definable set

$$S' = \{(x, y_1 - y_2) : (x + c, y_1), (x, y_2) \in S\}.$$

and the continuous function $\hat{f} : U \to G$

$$\hat{f}(x) = f(x + c) - f(x).$$

Clearly, $\mathrm{RM}(S') = 1$ and $\Gamma(\hat{f}) \subseteq S'$. We now want to replace S' with a $\mathcal{D}$-definable strongly minimal subset containing the graph of $\hat{f}$. This is not hard to do and the details will given in Lemma 7.2. Thus we may conclude $\Gamma(\hat{f})$ is contained in a strongly minimal set $\hat{S} \subseteq G^2$. Clearly, $\Gamma(\hat{f})_{\mathrm{pol}} \subseteq \hat{S}_{\mathrm{pol}}$. Since $\mathrm{fr}(\hat{S})$ is finite, we may assume that $\hat{S}$ is closed.

Claim 5.13. $\hat{I} \subseteq \hat{S}_{pol}$.

Proof. It suffices to prove $\hat{I} \subseteq \Gamma(\hat{f})_{\mathrm{pol}}$. Let $\gamma(t) \subseteq U$ tend to $x \in \hat{I}$ as $t \to 0$, then $\hat{f}(\gamma(t)) = f(\gamma(t)+c) - f(\gamma(t))$. Since $\lim_{t\to 0} \gamma(t)+c = x+c$, it follows that $\lim_{t\to 0} f(\gamma(t)+c) = f(x+c)$, and because $\lim_{t\to 0} |f(\gamma(t))| = \infty$, and $x \notin S_{\mathrm{pol}}$ also $\lim_{t\to 0} |\hat{f}(\gamma(t))| = \infty$, so x is a pole of $\hat{S}$. □

Since $\hat{S}_{\mathrm{pol}} \neq \emptyset$ it follows that $\hat{S}$ not G-affine.

We can now proceed with the proof of Lemma 5.10. Let T be any infinite definable subset of $\hat{I}$, and B any open bounded ball. We want to prove that $A(\hat{S}, T) \setminus B$ has dimension 2.

Claim 1. *There is a definable unbounded* 1*-dimensional group* $H \subseteq G$*, such that for every* $x \in T$ *and* $h \in H$*, there is a definable* $\pi : (0,1) \to (0,1)$*, with* $\pi(0^+) = 0^+$ *and*

$$\lim_{t\to 0} f(\gamma_x(\pi(t))) - f(\gamma_x(t)) = h.$$

Proof of Claim 1. We first recall a theorem from [25]: Given a definable curve $\sigma : (0,1) \to G$ with $\lim_{t\to 0} |\sigma(t)| = \infty$, the set of all limit points of $\sigma(t) - \sigma(s)$, as s and t tend to 0, forms an 1-dimensional torsion-free unbounded subgroup $H_\sigma \subseteq G$. In particular, for each $h \in H_\sigma$ there is a definable function $\pi_h : (0,1) \to (0,1)$ with $\pi_h(0^+) = 0^+$ such that $\lim_{t\to 0} \sigma(\pi_h(t)) - \sigma(t) = h$. It follows from the definition of H_σ that for every other definable curve $\sigma' : (0,1) \to G$, if $\lim_{t\to 0} \sigma'(t) - \sigma(t) = 0$ then $H_\sigma = H_{\sigma'}$. We now apply this result to the unbounded curves $f(\gamma_x(t))$, $x \in T$, and obtain the desired H. $\square$

Claim 2. *For every* $b \in V_1 := f(T + c) - f(T + c)$ *and* $h \in H$*, we have* $b + h \in A(\hat{S}, T)$*.*

Proof of Claim 2. Let $b = f(x_1 + c) - f(x_2 + c) \in V_1$, where $x_1, x_2 \in T$, and let π be as in Claim 1, for $x = x_2$ and h. Hence $h = \lim_{t\to 0} f(\gamma_{x_2}(\pi(t))) - f(\gamma_{x_2}(t))$. We have:

$$\begin{aligned}
&\hat{f}(\gamma_{x_1}(t)) - \hat{f}(\gamma_{x_2}(\pi(t))) = \hat{f}(\gamma_{x_1}(t)) - \hat{f}(\gamma_{x_2}(\pi(t))) + f(\gamma_{x_2}(t)) - f(\gamma_{x_2}(t))\\
&= [f(\gamma_{x_1}(t) + c) - f(\gamma_{x_1}(t)))] - [f(\gamma_{x_2}(\pi(t)) + c) - f(\gamma_{x_2}(\pi(t)))] + f(\gamma_{x_2}(t)) - f(\gamma_{x_2}(t))\\
&= [f(\gamma_{x_1}(t) + c) - f(\gamma_{x_2}(\pi(t)) + c)] + [f(\gamma_{x_2}(t)) - f(\gamma_{x_1}(t))] + [f(\gamma_{x_2}(\pi(t))) - f(\gamma_{x_2}(t))].
\end{aligned}$$

As t tends to 0, for $i = 1, 2$, the curve $\gamma_{x_i}(\pi(t)) + c)$ still tends to $x_i + c$ so its image under f tends to $f(x_i + c)$. By Lemma 5.11(3), $\lim_{t\to 0} f(\gamma_{x_2}(t)) - f(\gamma_{x_1}(t)) = 0$. Thus, the above expression tends to $f(x_1 + c) - f(x_2 + c) + h = b + h$, proving that $b + h$ can be approximated near T. $\square$

We can now conclude the proof of Lemma 5.10, as follows. Because V_1 and B are bounded, we can find $r_0 > 0$ such that for every $h \in G \setminus B_{r_0}$, the set $V_1 + h \subseteq G \setminus B$. In particular, for every $b \in V_1$, $b + (H \setminus B_{r_0}) \subseteq G \setminus B$. Moreover, since H is unbounded, $H \setminus B_{r_0}$ has dimension 1. Hence, by Claim 2, the 2-dimensional set $V_1 + (H \setminus B_{r_0})$ is contained in $A(\hat{S}, T) \setminus B$, as needed.

5.3. **Proof of Theorem 5.2.** Assume towards a contradiction that S_{pol} is infinite. Since for any $S_1, S_2 \subseteq G^2$, $(S_1 \cup S_2)_{\text{pol}} = S_{1\text{pol}} \cup S_{2\text{pol}}$, and $S_{\text{pol}} = \text{cl}(S)_{\text{pol}}$, we may assume that S is strongly minimal and closed. Since $S_{\text{pol}} \neq \emptyset$, we have that S is not G-affine. By Lemma 5.10, there is a $\mathcal{D}$-definable set $\hat{S}$ which is closed, strongly minimal and not G-affine, and an infinite definable $\hat{I} \subseteq G$, such that for every infinite set $T \subseteq \hat{I}$ and open bounded ball B, $A(\hat{S}, T) \setminus B$ is 2-dimensional. This contradicts Proposition 5.6(1) for $\hat{S}$ and $\hat{I}$.

Example 5.14. One of the difficulties in the above proof was the need to replace the initial set S with a set $\hat{S}$, in order to reach a situation where $\dim(A(\hat{S}, T) \setminus B) = 2$, for every infinite $T \subseteq \hat{I} \subseteq \hat{S}_{\text{pol}}$. The following example shows that the initial S can indeed have infinitely many poles and yet $\dim A(S, I_0) = 1$ for some (in fact, any bounded) infinite $I_0 \subseteq S_{\text{pol}}$. Consider the graph of function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = \begin{cases} (x,0) & \text{if } y = 0 \\ (xy, 1/y) & \text{if } y \neq 0 \end{cases},$$

with $G = \langle \mathbb{C}, + \rangle$. The function f is a bijection of $\mathbb{C}$ which is its own inverse. Its set of poles is the x-axis. For every $x \in \mathbb{R}$, as $(x,y) \to (x,0)$, $f(x,y)$ approaches the y-axis, with $|f(x,y)| \to \infty$. Thus, for any bounded $I_0 \subseteq \mathbb{R} \times \{0\}$, $A(S, I_0) = y$-axis. After moving to $\hat{S}$ as in the proof of Lemma 5.10, we can see that $\dim(A(\hat{S}, T) \setminus B) = 2$, for any infinite $T \subseteq \hat{S}_{\text{pol}}$ and bounded ball B.

6. Topological corollaries

We establish here several topological properties of plane curves, typically true for complex algebraic plane curves. These properties are used later on in our proof of the main theorem.

Definition 6.1. For $S \subseteq G^2$ and $a = (a_1, a_2) \in S$, we say that S *is an open relation at* a if for every open box $B = B_1 \times B_2 \ni a$, $a_1 \in \text{int}(\pi_1(B \cap S))$ and $a_2 \in \text{int}(\pi_2(B \cap S))$ (here π_1 and π_2 are the projections onto the first and second coordinates).

We say that S *is open over* $a_1 \in \pi_1(S)$ if for every a_2 such that $(a_1, a_2) \in S$ and every open box $B = B_1 \times B_2 \ni (a_1, a_2)$, $a_1 \in \text{int}(\pi_1(B \cap S))$.

Note that if there exists an open box $B \ni a = (a_1, a_2)$ such that $a_1 \notin \text{int}(\pi_1(B \cap S))$ then the same remains true for all smaller open boxes.

Lemma 6.2. *Assume that $S \subseteq G^2$ is a plane curve. Then there are at most finitely many $a_1 \in \pi_1(S)$ such that S is not open over a_1. In particular, S does not contain any 1-dimensional components.*

If S is strongly minimal and its projection on both coordinates is finite-to-one then there are at most finitely points $a \in S$ such that S is not open at a.

Proof. First note that if $S = S_1 \cup S_2$ and S is not open over $a_1 \in G$ then either S_1 is not open over a_1 or S_2 not open over a_1. Thus we may assume that S is strongly minimal. Without loss of generality, S is defined in $\mathcal{D}$ over $\emptyset$.

Assume towards contradiction that the set N of all a in $\pi_1(S)$ over which S is not open is infinite. Pick a_1 generic in N over $\emptyset$. Because $\text{RM}(\pi_1(S)) = 1$, the point a_1 is $\mathcal{D}$-generic in $\pi_1(S)$ over $\emptyset$.

Fix $a = (a_1, a_2) \in S$ and $B = B_1 \times B_2 \ni a$ such that $a_1 \notin \text{int}(\pi_1(S \cap B))$. Let $B_S = S \cap B$ and write $\bar{B}_S := \text{cl}(B_S)$. Note that a is $\mathcal{D}$-generic in S over $\emptyset$.

By Theorem 5.2, S has finitely many poles and since $\dim(a_1/\emptyset) \geq 1$, the point a_1 is not a pole of S. By Corollary 4.11, there are at most finitely many points in $\pi_1(S)$ at which S is is non-injective and each one of these is in $\text{acl}_{\mathcal{D}}([S])$. Thus a_1 is an injective point of S. By Theorem 4.9, $\text{fr}(S) \subseteq \text{acl}_{\mathcal{D}}([S])$ and hence we have $(\{a_1\} \times G) \cap \text{fr}(S) = \emptyset$.

Since $a_1 \notin \text{int}(\pi_1(B_S))$ there exists a definable curve $\gamma : (0,1) \to B_1 \setminus \pi_1(B_S)$ such that $\lim_{t \to 0} \gamma(t) = a_1$. Notice that for t small enough $\gamma(t)$ must be $\mathcal{D}$-generic in G, and therefore, because $\pi_1(S)$ is co-finite in G, $\gamma(t)$ is $\mathcal{D}$-generic in $\pi(S)$ over $\emptyset$. So, we may assume that the fiber $S_{\gamma(t)}$ has constant size $n \geq 1$. For each t, let $y_1(t), \ldots, y_n(t) \in G$ be distinct such that $(\gamma(t), y_i(t)) \in S$. Because $\gamma(t) \notin \pi_1(B_S)$, none of the $y_i(t)$ is in B_2.

Since $a_1 \notin S_{\mathrm{pol}}$, each of the curves $\gamma_i(t)$ is bounded, and hence has a limit $y_i \in G \setminus B_2$. Since $(\{a_1\} \times G) \cap \mathrm{fr}(S) = \emptyset$ each of the limit points (a_1, y_i) is in S and in addition $(a_1, a_2) \in S$, with $a_2 \neq y_i$ for all i. However, since a_1 is $\mathcal{D}$-generic we must have $|S_{a_1}| = n$. This implies that for some $i \neq j$, we have $y_i = y_j$, so S is non-injective at (a_1, y_i), contradiction.

Assume now that S is strongly minimal and its projection on both coordinates is infinite. We apply the above to both S and S^{op}, and then by removing from $\pi_1(S)$ and $\pi_1(S^{op})$ finitely many points, we remain, by our assumption on S, with a co-finite subset of S at which it is an open relation. $\square$

We are ready to deduce two useful topological corollaries:

Corollary 6.3. *Assume that $S \subseteq G^2$ is a plane curve. If $a = (a_1, a_2)$ is not an isolated point of S and both $(\{a_1\} \times G) \cap S$ and $(G \times \{a_2\}) \cap S$ are finite then S is an open relation at a.*

If, in addition, a is an injective point of S^{op} then there exists an open box $B = B_1 \times B_2 \ni a$ such that $S \cap B$ is the graph of an open continuous function $f : B_1 \to B_2$.

Proof. Assume towards contradiction that S is not an open relation at a, say not open over a_1 (since our assumptions on a are symmetric, the argument for a_2 is identical). In order to reach a contradiction it is sufficient to conclude that there are infinitely many points in $\pi_1(S)$ over which S is not open.

Note that since a is not isolated and $\{a\} \times G$ is finite, the set $\pi_1(B \cap S)$ is infinite for every open $B \ni a$. By Lemma 6.2, we may find $B \ni a$ sufficiently small such that S is open over all but finitely many points in $\pi_1(B \cap S)$. It follows that $\dim(B \cap S) = \dim(\pi_1(B \cap S)) = 2$ for all such B.

By Theorem 4.9, we may find an open box $B = B_1 \times B_2$ containing a such that $S \cap \mathrm{cl}(B)$ is closed, and $a_1 \notin \mathrm{int}(\pi_1(B \cap S))$. Let $B_S = B \cap S$ and denote $\bar{B}_S = \mathrm{cl}(B_S)$. It is not hard to see that we also have $a_1 \notin \mathrm{int}(\pi_1(\bar{B}_S)) = \mathrm{int}(\pi_1(S \cap \mathrm{cl}(B)))$. Because $S \cap (\{a_1\} \times G)$ is finite, we may also assume that $S \cap (\{a_1\} \times \mathrm{cl}(B_2)) = \{a\}$.

The set $\pi_1(\bar{B}_S)$ is closed in G and we already saw that it is 2-dimensional. The point a_1 belongs to the boundary of $\pi_1(\bar{B}_S)$, so by o-minimality, there exists a definable curve $\gamma_1 : (0,1) \to \partial(\pi_1(\bar{B}_S))$, with $a_1 = \lim_{t \to 0} \gamma_1(t)$. Since $\bar{B}_S = S \cap \mathrm{cl}(B)$, there exists a definable curve $\gamma_2 : (0,1) \to \mathrm{cl}(B_2)$ such that for every t, $(\gamma_1(t), \gamma_2(t)) \in S \cap \mathrm{cl}(B)$. Let $b = \lim_{t \to 0} \gamma_2(t) \in \mathrm{cl}(B_2)$.

Since $S \cap \mathrm{cl}(B)$ is closed it follows that $(a_1, b) \in S$, and therefore by our assumptions, $b = a_2$. But then the curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ tends to a, so for small enough t, it must belong to the open set B, and its projection is not in $\mathrm{int}(\pi_1(B_S))$. Therefore S is not open over every $\gamma_1(t)$ for t small enough. This contradicts Lemma 6.2 and ends the proof of the first clause.

Assume now that in addition a is an injective point of S^{op}. Then there exists a box $B = B_1 \times B_2$ such that no point in B_S is isolated, and the projection of B_S on B_1 is at most one-to-one. By what we just saw, we may shrink B so that S is open over every point in $\pi_1(B_S)$ and therefore B_S is the graph of an open map $f : B_1 \to B_2$. Since S is closed in B for a sufficiently small B, the function f is continuous on B_1. $\square$

Notice that we still do not know at this point that the set of isolated points of a curve S is finite. However, by Corollary 4.11, the non-injective points of S and of S^{op} are in $\mathrm{acl}_{\mathcal{D}}([S])$ so we may conclude from the above:

Corollary 6.4. *Let $S \subseteq G^2$ be a plane curve whose projection on both coordinates is finite-to-one. If $a \in S$ is a non-isolated point which is $\mathcal{D}$-generic in S over $[S]$, then there exists an open box $B = B_1 \times B_2 \ni a$ such that $B \cap S$ is the graph of a homeomorphism $f : B_1 \to B_2$.*

7. On $\mathcal{D}$-functions

Every plane curve $S \subseteq G^2$ gives rise to a definable partial function from G into G, around almost every point in S (except when S is contained in finitely many fibers $\{a\} \times G$). The goal of this section is to establish the basic theory of such functions.

7.1. Basic theory.

Definition 7.1. Let $U \subseteq G$ be a definable open set and let $f :\to G$ be a definable continuous function.

(1) We say that $f : U \to G$ is *a $\mathcal{D}$-function* if there exists a plane curve $S \subseteq G^2$ such that $\Gamma_f \subseteq S$. We say in this case that S *represents* f.
(2) We say that f is $\mathcal{D}$-represented over A is there exists S representing f which is $\mathcal{D}$-definable over A.
(3) We say that a plane curve S *represents the germ of f at $x_0 \in U$* if there exists an open neighborhood $W \ni x_0$, $W \subseteq \mathrm{dom}(f)$, such that $\Gamma_{f|W} \subseteq S$.

Note that our definition does not require that S is, locally at $(x_0, f(x_0))$, the graph of a function, but only that it *contains* the graph of f. Indeed, at least for some of the $\mathcal{D}$-functions which we need to consider we do not know if this stronger notion is true as well.

Lemma 7.2. *Let $U \subseteq G$ be a definably connected set and $f : U \to G$ a continuous $\mathcal{D}$-function, $\mathcal{D}$-represented over A. Then f can be $\mathcal{D}$-represented over $\mathrm{acl}_{\mathcal{D}}(A)$ by a strongly minimal set.*

Proof. Assume that $f : U \to G$ is $\mathcal{D}$-represented over A by S. We let $S = S_1 \cup \cdots \cup S_r$ be a decomposition of S into strongly minimal sets, definable in $\mathcal{D}$ over $\mathrm{acl}(A)$. By Theorem 4.9, we may assume, by adding finitely many points in $\mathrm{acl}_{\mathcal{D}}(A)$, that each S_i is closed in G^2, but now the intersection $S_i \cap S_j$ for $i \neq j$ may be non-empty and finite. We claim that one of the S_i must contain Γ_f. Indeed, for each $i = 1, \ldots r$, let $C_i = \pi(S_i \cap \Gamma_f) \subseteq U$, where $\pi : G^2 \to G$ is the projection on the first coordinate. By the continuity of f, these are definable, relatively closed subsets of U, whose pair-wise intersection is at most finite.

Let $U' := U \setminus \bigcup_{i \neq j} C_i \cap C_j$. Because U is open and definably connected so is U'. For $i = 1, \ldots, r$ let $C_i' = C_i \cap U'$. Now the C_i's are pairwise disjoint and still relatively closed in U'. But now each of the C_i' is clopen in U' so for some j, $C_j' = U'$. Because C_j is closed in U it follows that $C_j = U$. □

Proposition 7.3. *Let $\{S_t : t \in T\}$ be family of plane curves $\mathcal{D}$-definable over A. Then there exists a family of functions in $\mathfrak{F}_0$, $\mathcal{F} = \{f_s : s \in T_0\}$, definable in $\mathcal{M}$ over A, such that:*

1. *For every $t \in T$, if S_t represents the germ at 0 of a $\mathcal{D}$-function $f \in \mathfrak{F}_0$ then there exists $s \in T_0$ and an open $W \ni 0$ such that $f|W = f_s|W$.*
2. *For every $s \in T_0$ there exists $t \in T$ such that S_t represents the germ at 0 of f_s.*

Proof. Let us see first that we may assume that each S_t is a closed subset of G^2. By Theorem 4.9, there exists a $\mathcal{D}$-definable family of finite sets $\{R_t : t \in T\}$, such that for every $t \in T$, $\mathrm{cl}(S_t) \subseteq S_t \cup R_t$. Since R_t is finite it follows that $S_t \cup R_t$ is closed, so we can replace the original family with $\{S_t \cup R_t : t \in T\}$ (note that we do not claim that $R_t = \mathrm{fr}(S_t)$).

By fixing a coordinate system near 0 we can identify some neighbourhood $W \ni 0$ in G with an open subset of R^2. For each $r > 0$, we consider the disc B_r centered at 0, and let $S_t^r = S_t \cap (B_r \times W)$. By o-minimality, there exists a uniform cell decomposition of the sets $\{S_t^r : t \in T, r > 0\}$. In particular, there is a bound $k \in \mathbb{N}$ such that every such decomposition contains at most k cells. By allowing cells to be empty we obtain a definable collection of cells $\{C_{t,i}^r : t \in T\,,\ r > 0, i = 1, \ldots, k\}$, such that for every $t \in T$, $r > 0$,

$$S_t^r = \bigcup_{i=1}^{k} C_{t,i}^r.$$

Recall that the notion of a decomposition implies that for $C_{t,i}^r, C_{t,j}^r$, if $\pi : G^2 \to G$ is the projection onto the first coordinate then either $\pi(C_{t,i}^r) = \pi(C_{j,j}^r)$ or $\pi(C_{t,i}^r) \cap \pi(C_{j,j}^r) = \emptyset$.

Claim. *For every $t \in T$, and a $\mathcal{D}$-function $f \in \mathfrak{F}_0$, the following are equivalent:*

1. *S_t represents the germ of f at 0.*
2. *There exist $r > 0$, and $A \subseteq \{1, \ldots, k\}$, such that*
$$\Gamma_{f|B_r} = \bigcup_{i \in A} C_{t,i}^r.$$

Proof of Claim. (1) ⇒ (2). We assume that $S_t \cap (B_r \times G)$ contains the graph of $f|B_r$, for $r > 0$. To simplify notation we omit r and consider the cell decomposition $S_t = C_{t,1} \cup \cdots \cup C_{t,k}$.

We let $A \subseteq \{1, \ldots, k\}$ be all i such that $C_i \cap \Gamma_f \neq \emptyset$. We fix a cell $C = C_i$ with $i \in A$ and claim that $C \subseteq \Gamma_f$. Without loss of generality $\dim C > 0$, and since $C \subseteq \Gamma_f$, the projection $\pi : C \to G$ is injective. Since C is definably connected it is sufficient to prove that $C \cap \Gamma_f$ is clopen inside C. Because C is locally closed and f is continuous, it follows that $C \cap \Gamma_f$ is closed in C, so we need to prove that it is also open in C.

Let $U \subseteq B_r$ be an open neighborhood of x_0, and consider $U \cap \pi(C)$. Since $\Gamma_f \subseteq S_t$, there exists a cell C' in the decomposition of S_t which contains $\Gamma_f \cap [(U \cap \pi(C)) \times G]$ for some open set $U \ni x_0$. But then $\pi(C) \cap \pi(C') \neq \emptyset$ and therefore $\pi(C) = \pi(C')$. By the continuity of f it follows that $(x_0, f(x_0)) \in C'$, forcing $C' = C$. It follows that $C \cap \Gamma_f$ is clopen in C, and therefore $C \subseteq \Gamma_f$.

We showed that for each $i \in A$, $C_i \subseteq \Gamma_f$ and hence $\Gamma_f = \bigcup_{i \in A} C_i$.

(2) $\Rightarrow$ (1). This is immediate, since $\Gamma_{f|B_r} \subseteq S_t$. □

We now return to the proof of Proposition 7.3 and consider the uniform decomposition

$$S_t^r = \bigcup_{i=1}^{k} C_{t,i}^r.$$

For each $A \subseteq \{1, \ldots, k\}$, we consider

$$G_{t,A}^r = \bigcup_{i \in A} C_{t,i}^r.$$

The family

$$\mathcal{F} = \{G_{t,A}^r : G_{t,A}^r \text{ is the graph of a continuous function on } B_r\}$$

is definable in $\mathcal{M}$, as t varies in T, A varies among subsets of $\{1, \ldots, k\}$ and $r > 0$. By the above claim, this family satisfies our requirements. □

Remark. (1) Note that in the above family $\mathcal{F}$ of $\mathcal{D}$-functions, each germ of a function appears infinitely often since we allow arbitrarily small r. One can divide the family, definably in $\mathcal{M}$, by the equivalence of germs at 0 and then, using Definable Choice in o-minimal structures, obtain a unique $\mathcal{D}$-function in the family representing each germ. Thus, if $f \in \mathfrak{F}_0$ is represented by plane curve S_t then there exists $g \in \mathfrak{F}_0$ which has the same germ as f at 0 and is definable in $\mathcal{M}$ over t.

(2) It follows from the above that if S_t represents $f \in \mathfrak{F}_0$ then $J_0(f)$ is in $\mathrm{dcl}(t)$.

Notation. For a $\mathcal{D}$-function f, we reserve the notation S_f for a strongly minimal set representing f.

Definition 7.4. We say that a $\mathcal{D}$-function $f : U \to G$ is *G-affine* if there exist non-empty open sets $V \subseteq U$ and $W \ni 0$ such that for every $x_1, x_2 \in V$ and $x \in W$,

$$f(x + x_1) - f(x_1) = f(x + x_2) - f(x_2).$$

As we already noted for G-affine subsets of G^2, we have:

Claim 7.5. *A $\mathcal{D}$-function f is G-affine if and only if S_f is G-affine if and only if $Stab^*(S_f)$ is infinite.*

Remark 7.6. If f is G-affine and $f(0) = 0$ then f is a partial group homomorphism, in a neighborhood of 0. Hence, if f is G-affine and $J_0 f = 0$ then f vanishes at 0.

As we already saw in Fact 3.9, since $\mathcal{D}$ is not locally modular there exists at least one $\mathcal{D}$-function which is not G-affine.

7.2. **The open mapping theorem for $\mathcal{D}$-functions.**

Theorem 7.7. *Let $U \subseteq G$ be an open definably connected set and $f : U \to G$ a continuous non-constant D-function. Then f is an open map on U.*

Proof. By Lemma 7.2, there exists a strongly minimal $S_f \subseteq G^2$ representing f. Because f is not constant, the projection of S_f onto both coordinates is finite-to-one.

By o-minimality, if f is continuous and not open then there are infinitely many points where f is not open.

By Corollary 6.4, this definable, infinite set of points is contained in $\mathrm{acl}_{\mathcal{D}}([S_f])$, which is impossible. Hence, f is an open map at a. ☐

8. The ring of Jacobian matrices

8.1. **The ring $\mathfrak{R}$.**

Our next goal is to show that if f is a $\mathcal{D}$-function then its Jacobian matrix vanishes at 0 if and only if f is not locally invertible at 0. This will be done in this and the next section. In the present section we prove that similarly to a complex analytic function, the Jacobian matrix is non-zero if and only if it is an invertible matrix.

Throughout this section we fix a definable local coordinate system for G near 0_G, identifying 0_G with $0 \in R^2$. From now on we identify G locally with an open subset of R^2. For a differentiable $\mathcal{D}$-function f in a neighborhood of 0, with $f(0) = 0$, the Jacobian matrix at x, denoted by $J_x f$, is computed with respect to this fixed coordinate system, and we denote by $|J_x f|$ its determinant. We use $d_x f$ to denote the differential of f, viewed as a map from the tangent space of G at x, denoted by $T_x(G)$, to $T_{f(x)}(G)$. As we soon observe, the collection of all matrices $J_0 f$ forms a ring, and the main goal of this section is to show that it is in fact a field (thus every nonzero matrix is invertible).

We first observe the following:

Lemma 8.1. *Let $f : U \to G$ be a non-constant $\mathcal{D}$-function. Then*
1. *The set of $a \in U$ at which $|J_a f| = 0$ is at most 1-dimensional.*
2. *The set of $a \in U$ at which $J_a f = 0$ is finite.*

Proof. (1) By strong minimality, for every open $V \subseteq U$, we have $\dim f(V) = 2$, for otherwise the pre-image of some point is infinite and co-infinite. By the o-minimal Sard's Theorem, it follows that the set of singular points of f is at most 1-dimensional. For (2), note that if $J_a f = 0$ on a definably connected path then it must be constant there, which by strong minimality implies that f is constant on U. ☐

Definition 8.2. Recall that $\mathfrak{F}_0$ the collection of all $\mathcal{D}$-functions f which are C^1 in a neighborhood of 0, with $f(0) = 0$. We let

$$\mathfrak{R} = \{J_0 f \in M_2(R) : f \in \mathfrak{F}_0\}.$$

It is important here to distinguish between the group operation in G and the usual ring operations in $M_2(R)$. Thus we reserve the additive notation $\pm$ for matrix addition, and let $\oplus, \ominus$ denote the group operations in G.

Lemma 8.3. *The set $\mathfrak{R}$ is a subring of $M_2(R)$ and for every $A \in \mathfrak{R}$ which is invertible, $A^{-1} \in \mathfrak{R}$.*

Proof. We first note that the collection of germs of functions in $\mathfrak{F}_0$ is closed under $\oplus$ and functional composition. Indeed, if S_f and S_g represent $\mathcal{D}$-functions f and g in $\mathfrak{F}_0$ then the plane curve $S_f \circ S_g$ represents $f \circ g$ and the plane curve

$$S_f \boxplus S_g = \{(x, y_1 \oplus y_2) : (x, y_1) \in S_f \ , \ \ (x, y_2) \in S_g\}$$

represents $f \oplus g$.

Using the chain rule it is easy to verify that for $f, g \in \mathfrak{F}_0$, $J_0(f \oplus g) = J_0 f + J_0 g$, and $J_0(f \circ g) = J_0 f \cdot J_0 g$. Since the germs in $\mathfrak{F}_0$ are closed under $\oplus$ and functional composition, it follows that $\mathfrak{R}$ is a ring. If $J_0 f$ is invertible then f is a locally invertible function in which case it is clear that f^{-1} is also in $\mathfrak{F}_0$, and therefore $(J_0 f)^{-1} \in \mathfrak{R}$. □

Note. Even if S_f and S_g are locally at $(0,0)$ the graphs of functions, the sets $S_f \circ S_g$ and $S_f \boxplus S_g$ need not be such, but they still represent $f \circ g$ and $f \oplus g$, respectively. It is in fact possible that no strongly minimal set representing $f \circ g$ will be locally at $(0,0)$ the graph of a function.

8.2. **Definability and dimension of $\mathfrak{R}$.** Our aim is to show that $\mathfrak{R}$ is a definable field isomorphic to $R(\sqrt{-1})$. This is achieved in several steps. We first show (Theorem 8.11) that $\mathfrak{R}$ is a definable ring of one of two kinds, and then – by eliminating one of these possibilities – we deduce the desired result.

We are going to use extensively the following operation:

Definition 8.4. For a $\mathcal{D}$-function f which is C^1 in a neighborhood of some $a \in G$, we let

$$\tilde{J}_a f = J_0(f(x \oplus a) \ominus f(a)) \ ; \ \tilde{d}_a f = d_0(f(x \oplus a) \ominus f(a)).$$

Note that $f(x \oplus a) \ominus f(a)$ is in $\mathfrak{F}_0$ and thus $\tilde{J}_0 f \in \mathfrak{R}$. If we let $\ell_a(x) = x \oplus a$ then we have:

Lemma 8.5. (1) $\tilde{d}_a f = (d_0 \ell_{f(a)})^{-1} \circ d_a f \circ d_0 \ell_a$.

(2) *For every $a \in dom(f)$, $J_a f$ is invertible if and only if $\tilde{J}_a f$ is invertible, and $J_a f = 0 \Leftrightarrow \tilde{J}_a f = 0$.*

(3) *For any two differentiable $\mathcal{D}$-functions $f, g : U \to G$ and $x_0 \in U$, $\tilde{J}_{x_0}(f \ominus g) = \tilde{J}_{x_0} f - \tilde{J}_{x_0} g$.*

Proof. (1) is easy to verify and (2) follows, so we prove (3): Note that

$$\tilde{J}_{x_0}(f \ominus g) = J_0[(f \ominus g)(x_0 \oplus x) \ominus (f \ominus g)(x_0)],$$

which equals

$$J_0[((f(x_0 \oplus x) \ominus f(x_0)) \ominus (g(x_0 \oplus x) \ominus g(x_0))].$$

As we noted in the proof of Lemma 8.3, $J_0(h_1 \ominus h_2) = J_0 h_1 - J_0 h_2$, therefore the above equals

$$J_0(f(x_0 \oplus x) \ominus f(x_0)) - J_0(g(x_0 \oplus x) \ominus g(x_0)) = \tilde{J}_{x_0}(f) - \tilde{J}_{x_0}(g).$$

□

We are going to need the following:

Lemma 8.6. *There are invertible matrices in $\mathfrak{R}$ arbitrarily close to the* 0 *matrix.*

Proof. We go via the following claim which is also used later in the text.

Claim 8.7. *There exists $g \in \mathfrak{F}_0$ which is not G-affine, with $J_0 g = 0$.*

Proof. For $y \in G$ and $n \in \mathbb{N}$ we write $ny := \overbrace{y \oplus \cdots \oplus y}^{n\text{-times}}$. Fix $f : U \to G$ in $\mathfrak{F}_0$ which is not G-affine, and for $n \in \mathbb{N}$, let $g_n(x) = f(nx) - nf(x)$. It is easy to see that $g_n \in \mathfrak{F}_0$ and $J_0 g_n = nJ_0 f - nJ_0 f = 0$. We want to show that for some $n \in \mathbb{N}$, the function g_n not G-affine, so gives the desired g.

Notice that if g_n is G-affine, then since $J_0 g_n = 0$, the function g_n must vanish on its domain. Assume towards contradiction that for every $n \in \mathbb{N}$, the function g_n vanishes on its domain, namely $f(nx) = nf(x)$ whenever $nx \in U$. Pick a $\mathcal{D}$-generic $x_0 \in U$ sufficiently close to 0 so that for all n, $nx \in U$ and $nf(x) \in U$ (we can do it by saturation). For all n we have

$$f(x_0 + nx_0) = f((n+1)x_0) = (1+n)f(x_0) = f(x_0) + nf(x_0) = f(x_0) + f(nx_0).$$

Thus, there are infinitely many $y \in G$ such that $f(x_0 + y) = f(x_0) + f(y)$. Because f is a $\mathcal{D}$-function it follows that for almost all y with $x_0 + y \in U$, $f(x_0 + y) = f(x_0) + f(y)$. Since x_0 is $\mathcal{D}$-generic, the function f must be G-affine, contradiction. □

We return to Lemma 8.6, take the function $g : V \to G$ from the above claim. By Lemma 8.1, for every $x \in V$ generic, $J_x g$ and hence $\tilde{J}_x g$ is invertible. Because g is smooth and $J_0 g = 0$ there are invertible matrices of the form $\tilde{J}_x g \in \mathfrak{R}$ arbitrarily close to the 0 matrix. □

Definition 8.8. Given a set $W \subseteq \mathfrak{R}$ and a family $\mathcal{F} = \{f_t : t \in T_0\}$ of $\mathcal{D}$-functions, we say that W *is realized by* $\mathcal{F}$ if

$$W = \{J_0 f_t : t \in T_0\}.$$

Proposition 8.9. *The ring $\mathfrak{R}$ is a definable subset of $M_2(R)$.*

Proof. Let us see first that $\mathfrak{R}$ can be viewed as a locally definable subring of $M_2(R)$, namely a bounded union of definable subsets of $M_2(R)$. By Proposition 7.3 and the remarks which follow, any $\mathcal{D}$-definable, over $\emptyset$, family $\{S_t : t \in T\}$ of plane curves all containing $(0,0)$ gives rise to a $\emptyset$-definable family of $\mathcal{D}$-functions in $\mathfrak{F}_0$, $\mathcal{F}_0 = \{f_t : t \in T_0\}$ containing precisely all functions in $\mathfrak{F}_0$ whose germ at 0 is represented by the S_t, $t \in T$. The family $\mathcal{F}_0$ realizes in the above sense the definable set $J = \{J_0 f_t : t \in T_0\} \subseteq \mathfrak{R}$.

As pointed out in the proof of Lemma 8.3, the families

$$\{f_t \oplus f_s : t, s \in T_0\}\ ,\ \{f_s \circ f_t : s, t \in T_0\}\ ,\ \{f_t^{-1} : t \in T_0 \text{ and } f_t \text{ invertible near } 0\}.$$

are also a families of $\mathcal{D}$-functions. Hence, the sets of matrices $J + J$, $J \cdot J$, and $J^{-1} = \{A^{-1} : A \in J \text{ invertible}\}$ can be realized by definable families of $\mathcal{D}$-functions. Thus $\mathfrak{R}$ can be first identified as a bounded $\bigvee$-definable subring of $M_2(R)$, where the set of disjuncts is bounded in size by the cardinality of the language of $\mathcal{D}$. It follows that there is a definable open neighborhood $U \subseteq M_2(R)$ of the zero matrix, such that $U \cap \mathfrak{R}$ is definable (for more on locally definable groups and rings see [22]). More precisely, there exists a $\emptyset$-definable family of $\mathcal{D}$-functions which realizes U.

We now proceed to show that $\mathfrak{R}$ is actually a *definable* subset of $M_2(R)$. Let $U \ni 0$ be a neighborhood of 0 in $M_2(R)$ such that $U \cap \mathfrak{R}$ is definable as above. We claim that

$$\mathfrak{R} = \{AB^{-1} : A, B \in U \cap \mathfrak{R} \text{ , } B \text{ is invertible}\}.$$

Indeed, for every $C \in \mathfrak{R}$ we can find, by Lemma 8.6, an invertible matrix $B \in U \cap \mathfrak{R}$, sufficiently close to 0, such that $CB \in U \cap \mathfrak{R}$. It follows that $\mathfrak{R}$ is definable. □

Proposition 8.10. *Let $U \subseteq G$ be an open neighborhood of* 0*, and assume that $f : U \to G$ is a continuously differentiable $\mathcal{D}$-function which is not G-affine. Then the set $\tilde{J}(U) = \{\tilde{J}_a f \in M_2(R) : a \in U\}$ has dimension* 2*. In particular,* $\dim \mathfrak{R} \geq 2$.

Proof. Since $\dim U = 2$ we have $\dim \tilde{J}(U) \leq 2$. Assume towards contradiction, that $\dim \tilde{J}(U) \leq 1$.

Claim. *There exists $g_0 \in G$, $g_0 \notin \operatorname{dcl}(\emptyset)$, and infinitely many $a \in G$ such that $\tilde{J}_a f = \tilde{J}_a(f(x \oplus g_0))$.*

Proof of Claim. For every matrix $A \in \tilde{J}(U)$ let $C_A := \{x \in U : \tilde{J}_x f = A\}$. By our assumptions, there exists $A \in \tilde{J}(U)$ such that $\dim C_A \geq 1$, and by possibly shrinking it, we assume that C_A is definably connected. Consider $B_A = C_A \ominus C_A \subseteq G$. There are two cases to consider:

Case 1 There exists $A \in \tilde{J}(U)$ such that $\dim B_A = 1$.

We may apply [19, Lemma 2.7] and conclude that the set C_A consists of a subset of a coset of a locally definable one dimensional subgroup $\mathcal{H}$ of G. It follows that for $g_0 \in \mathcal{H}$ sufficiently small there are infinitely many $a \in C_A$ such that $a \oplus g_0 \in C_A$, and thus $\tilde{J}_a f = \tilde{J}_a f(x \oplus g_0) = \tilde{J}_{a+g_0} f$.

Case 2 For all $A \in \tilde{J}(U)$, $\dim B_A = 2$, so B_A contains an open subset of G.

Given A generic in $\tilde{J}(U)$ we may find an open set $W \subseteq G$ in B_A such that A is still generic in $\tilde{J}(U)$ over the parameters defining W. Thus there are infinitely many $A \in \tilde{J}(U)$ for which $W \subseteq B_A$. Pick g_0 generic in W and then for each A such that $g_0 \in B_A$ there are $a, b \in C_A$ such that $a \ominus b = g_0$, so $a = b \oplus g_0$. By definition of C_A we know that for every such pair (a, b) we have $\tilde{J}_a f = \tilde{J}_b f$, so $\tilde{J}_{b \oplus g_0} f = \tilde{J}_b f$. We get:

$$\tilde{J}_b(f(x \oplus g_0) = J_0(f(x \oplus b \oplus g_0) \ominus f(b \oplus g_0)) = \tilde{J}_{b \oplus g_0} f$$

So

$$\tilde{d}_b f(x \oplus g_0) = \tilde{d}_{b \oplus g_0} f = \tilde{d}_b f$$

whenever b and $b \oplus g_0$ are in C_A. Since there are infinitely many such pairs $b, b \oplus g_0$, as A varies, we are done. $\square$

To conclude the proof, fix g_0 as provided by the Claim and infinitely many a such that $\tilde{J}_a f = \tilde{J}_a f(x \oplus g_0)$. By Lemma 8.5 (3), for each such a, $\tilde{d}_a(f(x \oplus g_0) \ominus f(x)) = 0$. But then, by Lemma 8.5 (2), the $\mathcal{D}$-function $k(x) = f(x \oplus g_0) \ominus f(x)$, has infinitely many a where $J_a k = 0$, so $k(x)$ is constant on that set, say of value d. By strong minimality of $\mathcal{D}$, (g_0, d) is in $Stab^*(S_f)$. Since g_0 is not in $\operatorname{dcl}(\emptyset)$, it is not a torsion-element so $Stab^*(S_f)$ is infinite and therefore f is G-affine, contradiction. $\square$

8.3. **The structure of $\mathfrak{R}$.** The main result of this section is:

Theorem 8.11. *There exists a fixed invertible matrix $M \in M_2(R)$ such that one of the following two holds:*

(1)
$$\mathfrak{R} = \{M^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M : a, b \in R\}.$$

In particular, $\mathfrak{R}$ is a field isomorphic to $R(\sqrt{-1})$. Or,

(2)
$$\mathfrak{R} = \{M^{-1} \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} M : a, b \in R\}.$$

We need some preliminaries.

Lemma 8.12. *Let $U \subseteq G$ be a definably connected open neighbourhood of 0. Let $f : U \to G$ be a non-constant $\mathcal{D}$-function. Then $|J_x f|$ has constant sign at all $x \in U$ where f is differentiable and $J_x f$ is invertible.*

Proof. By Corollary 4.11, we may assume – possibly removing finitely many points from U – that f is locally injective. The result now follows from [24, Theorem 3.2]. $\square$

Now, for $f \in \mathfrak{F}_0$ non-constant we denote by $\sigma(f)$ the sign of $|J_x(f)|$ for all x sufficiently close to 0 at which $J_x f$ is invertible.

Proposition 8.13. *Every invertible $A \in \mathfrak{R}$ has positive determinant.*

Proof. Fix $A_0 \in \mathfrak{R}$ generic over $\emptyset$, and $W \subseteq \mathfrak{R}$ an open neighborhood of A_0. Fix also a definable family of $\mathcal{D}$-functions, $\{f_t : t \in T\}$ realizing W. Let a_0 be generic in T, such that $J_0 f_{a_0} = A_0$. We may assume that T is a cell in some M^k, and by definable choice in o-minimal structures further assume that the map $t \mapsto J_0 f_t$ is a homeomorphism of T and W. By Proposition 8.10, $\dim T = \dim W = \dim \mathfrak{R} \geq 2$.

For every $t \in T$, let $U_t \subseteq G$ be the domain of f_t (containing 0). We may find now a definably connected neighborhood $U_0 \ni 0$ and a definably connected neighborhood $T_0 \ni a_0$ in T, such that for every $t \in T_0$, $U_0 \subseteq U_t$. The sets U_0 and T_0 might use additional parameters but we may choose these so that a_0 is still generic in T_0 over these parameters. Let $W_0 = \{J_0 f_t : t \in T_0\}$ be the corresponding neighborhood of A_0 in $\mathfrak{R}$.

Consider now the set of matrices $\hat{W}_0 = W_0 - A_0 \subseteq \mathfrak{R}$. It is an open neighborhood of 0 in $\mathfrak{R}$, which is realized by the family $\{f_t \ominus f_{a_0} : t \in T_0\}$.

Our goal is to show that every invertible matrix in $\hat{W}_0$ has positive determinant. Let us first see that they all have the same determinant sign. Note that for all $t \in T_0 \setminus \{a_0\}$, the function $f_t \ominus f_{a_0}$ is non-constant on U_0, thus by Theorem 7.7 it is an open map. We now show, using our above notation, that $\sigma(f_t \ominus f_{a_0})$ is constant as t varies in a punctured neighborhood of a_0.

Fix $x_0 \in U_0$ which is generic over a_0. Since (a_0, x_0) is generic in $T_0 \times U_0$ there exist an open $T_0' \ni a_0$ inside T_0 and an open $\hat{U}_0 \ni x_0$ inside U_0 such that the map $F(t,x) = f_t(x) \ominus f_{a_0}(x)$ is continuous on $T_0' \times U_0$. Because $\dim T_0 = \dim W \geq 2$, the set $\hat{T}_0 = T_0' \setminus \{a_0\}$ is still definably connected, and for each $t \in \hat{T}_0$, the function $f_t \ominus f_{a_0}$ is open on $\hat{U}_0$. Given, $t_1 \neq t_2 \in \hat{T}_0$, there exists a definable path $p : [0,1] \to \hat{T}_0$ connecting t_1 and t_2, and by possibly shrinking $\hat{U}_0$, the induced map $(s,x) \mapsto F(p(s),x)$ is a proper homomotopy of $f_{t_1} \ominus f_{a_0}$ and $f_{t_2} \ominus f_{a_0}$, hence by [24, Theorem 3.19], for every x generic in $\hat{U}_0$, $|J_x(f_{t_1} \ominus f_{a_0})|$ and $|J_x(f_{t_2} \ominus f_{a_0})|$ have the same sign. It follows that

$$\sigma(f_{t_1} \ominus f_{a_0}) = \sigma(f_{t_2} \ominus f_{a_0}).$$

Thus, every invertible matrix in $\hat{W}_0$ has the same determinant sign.

Next, note that for every invertible $A \in \hat{W}_0$ sufficiently close to 0, the matrix A^2 is also in $\hat{W}_0$ and clearly has positive determinant. Thus all invertible matrices in $\hat{W}_0$ have positive determinant.

Finally, as we saw in Proposition 8.9, $\mathfrak{R} = \{AB^{-1} : A, B \in \hat{W}_0, B \text{ invertible}\}$, and hence all invertible matrices in $\mathfrak{R}$ have positive determinant. ☐

Proof of Theorem 8.11. Assume first that every non-zero $A \in \mathfrak{R}$ is invertible, namely that $\mathfrak{R}$ is a definable division ring. It follows from [26, Theorem 4.1] that $\mathfrak{R}$ is definably isomorphic to either R or $R(\sqrt{-1})$ or the ring of quaternions over R. Because $\dim \mathfrak{R} \geq 2$, we are left with the last two possibilities. The ring of quaternions is not isomorphic to a definable subring of $M_2(R)$, so $\mathfrak{R}$ is necessarily isomorphic to $R(\sqrt{-1})$. It is now not difficult to see that there exists an invertible $M \in M_2(R)$ such that $\mathfrak{R}$ is of the form (1).

We thus assume that there exists at least one matrix A which is not invertible, of rank 1. We want to show that there exists an invertible $M \in M_2(R)$ such that $\mathfrak{R}$ has form as in (2).

We conjugate $\mathfrak{R}$ by some fixed matrix so that A, written in columns, has the form $(w, 0)$ for some $w \in R^2$. We now show that every matrix in $\mathfrak{R}$ is of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ for some $a, b \in R$. Consider the set

$$H = \{A' \in \mathfrak{R} : A' = (u, 0), u \in R^2\}.$$

It is a definable subring of $\mathfrak{R}$ of positive dimension (since H is a non-trivial subring of $\mathfrak{R}$) with $\dim H \leq 2$.

Claim 1. *The ring H is 1-dimensional.*

Proof of Claim 1. Write the matrices in H in the form $B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, and note that for $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$,

$$|B + C| = |C| + (af - bd). \tag{1}$$

Assume towards a contradiction that $\dim H = 2$, and then H consists of *all* matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. We may now take $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$ invertible, sufficiently close to 0, and since d, f cannot be both 0, it easy to see that by choosing a, b appropriately, we may obtain a matrix $B + C \in \mathfrak{R}$ whose determinant is negative, a contradiction. □

Thus, H is a 1-dimensional ring.

Claim 2. *The matrices in H are not of the form $B = \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix}$, for some fixed $\alpha \in \mathbb{R}$.*

Proof of Claim 2. For every invertible $C = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$, and $B = \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix} \in H$, we have $|B + C| = |C| + (af - \alpha ad)$. By choosing a appropriately, we obtain $|B + C| < 0$ unless $f = \alpha d$. We now consider

$$\begin{pmatrix} c & d \\ e & \alpha d \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ \alpha a & 0 \end{pmatrix} = \begin{pmatrix} a(c + \alpha d) & 0 \\ a(e + \alpha^2 d) & 0 \end{pmatrix}.$$

Since this matrix must be in H, we must have $\alpha(c + \alpha d) = e + \alpha^2 d$, which implies that $e = \alpha c$. However, this would make C non-invertible, a contradiction. □

We are thus left with the case that matrices in H are of the form $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$. If we now take an arbitrary $\begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathfrak{R}$ and multiply it on the right by a non-zero element of H, we obtain another element of H, forcing d to be 0. Thus all matrices in $\mathfrak{R}$ are lower triangular. Because H contains all matrices of the form $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, every matrix in $\mathfrak{R}$ can be written as the sum of a diagonal matrix in $\mathfrak{R}$ and a matrix in H.

Finally, we note that the diagonal matrices in $\mathfrak{R}$ form a sub-ring, and since the determinant of each such non-zero matrix must be positive, it follows that they are of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Thus, the matrices in $\mathfrak{R}$ are of the form $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$, as required. This ends the proof of Theorem 8.11. □

8.4. **From ring to field.**

Definition 8.14. We say that $\mathfrak{R}$ is of *analytic form* if it satisfies (1) of Theorem 8.11.

Our goal in this section is to prove that Case (2) of Theorem 8.11 contradicts the strong minimality of $\mathcal{D}$. Thus, our negation assumption is that there exists a matrix $M \in GL(2, R)$ such that all matrices in $M^{-1}\mathfrak{R}M$ are of the form

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \tag{2}$$

for $a, b \in R$.

Let us first note that we may assume that all matrices in $\mathfrak{R}$ itself are in form (2). Indeed, we fix a small definable open chart $U \ni 0$ in G and identify it with an open subset of R^2. We may also assume that $MU = U$. Now consider the definable bijection $h : G \to G$ which is the identity outside U and $h(x) = Mx$ on U. The push-forward of the structure $\mathcal{D}$ under h is an isomorphic, definable, strongly minimal structure $\mathcal{D}'$, and it is easy to verify that for any $\mathcal{D}$-function $f \in \mathfrak{F}_0$, its image in $\mathcal{D}'$ is a function whose differential at 0 is $M^{-1}J_0fM$. Thus the ring of Jacobians at 0 of all smooth $\mathcal{D}'$-functions f with $f(0) = 0$ consists of matrices as in (2). We now replace $\mathcal{D}$ with $\mathcal{D}'$.

In the case where $G = \langle R^2, +\rangle$ then [5, Corollary 2.18] would immediately yield the desired result. The goal of this sub-section is to prove an analogue of that result in the context of an arbitrary group G.

We first need the following version of the uniqueness of definable solutions to definable ODE's. It can be easily deduced from [20, Theorem 2.3]:

Proposition 8.15. *Let $Gr(k, n)$ be the space of all k-dimensional linear subspaces of R^n. Let $U \subseteq R^n$ be an open set and assume that $L : U \to Gr(k, n)$ is a definable $\mathcal{C}^3$-function assigning to each $p \in U$ a k-linear space L_p. Assume that $C_1, C_2 \subseteq U$ are definable k-dimensional smooth manifolds such that for every $p \in C_1 \cap C_2$, the tangent space of C_i at p equals L_p. Then for every $p \in C_1 \cap C_2$ there exists a neighbourhood $V \ni p$ such that $C_1 \cap V = C_2 \cap V$.*

Definition 8.16. *A definable vector field* on an open $U \subseteq G$, is given by a definable partial function $F : U \to T(U)$ from U to its tangent bundle $T(U)$, such for every $g \in G$, $F(g) \in T_g(G)$.

Every definable non-vanishing vector field F on U gives rise to a *a definable line field*, still denoted by F, where to each $g \in U$ we assign the 1-dimensional subspace of $T_g(U)$ spanned $F(g)$.

We say that a line field F is (left) G-invariant if if for every $g, h \in U$,

$$F(h) = d_g(\ell_{hg^{-1}}) \cdot F(g).$$

Given a line field F, we say that a definable smooth 1-dimensional set $C \subseteq U$ is *a trajectory of F* if for every $g \in U$, the tangent space to C at g is $F(g)$.

Fact 8.17. *Let $T_0(G)$ be the Lie algebra of G and F a definable non-vanishing G-invariant line field. Assume that $C \subseteq G$ is a definably connected smooth 1-dimensional trajectory of F. Then C is a coset of a definable local subgroup of G.*

Proof. Recall that we identify an open neighborhood U of 0 with an open subset of R^2, and $T(U)$ is identified with $U \times R^2$. The line field can be viewed as a map $F : U \to Gr(1, 2)$.

Without loss of generality $0 \in C$. Since F is left-invariant, for any $g \in G$, $g \oplus C$ is also a solution to F. By Proposition 8.15, C and $g \oplus C$ coincide on some neighborhood of g. It follows that every $x \in C$ and $g \in C$ sufficiently small, we also have $x \oplus g \in C$. Thus C is a local subgroup of G. $\square$

We can now return to our main goal: proving that $\mathfrak{R}$ is of analytic form. Recall that we assume that for a $\mathcal{D}$-function f and $b \in \mathrm{dom}(f)$ we can write $\tilde{J}_b(f) = \begin{pmatrix} \alpha_f(b) & 0 \\ \beta_f(b) & \alpha_f(b) \end{pmatrix}$. When f is clear from the context we omit the subscript f.

Let $v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and consider the non-vanishing G-invariant vector field F given by

$$\{d_0\ell_b(v_0) : b \in G\}.$$

For $b \in G$, let $v_b = d_0\ell_b(v_0) \in T_b(G)$.

Lemma 8.18. *For every $\mathcal{D}$-function $f : U \to G$ and $b \in \mathrm{dom}(f)$, we have*

$$d_b f \cdot v_b = \alpha(b) v_{f(b)} \in R v_{f(b)},$$

namely the line-field induced by F is invariant under df.

If in addition $\alpha(b) = 0$ then

$$d_b f \cdot T_b(G) \subseteq R v_{f(b)}.$$

Proof. By assumption on the form of matrices in $\mathfrak{R}$, we have $\tilde{d}_b f \cdot v_0 = \alpha(b) \cdot v_0$. Writing $\tilde{d}_b f$ explicitly (and composing on the left with $d_0\ell_{f(b)}$), we obtain

$$(d_b f)(d_0\ell_b) \cdot v_0 = \alpha(b)(d_0\ell_{f(b)} \cdot v_0),$$

which implies the first clause.

For the second clause, notice the special form of $\tilde{d}_b f$ implies that when $\alpha(b) = 0$, then for every $v \in T_0(G)$, we have $\tilde{d}_b f \cdot v \in R v_0$. The result easily follows. $\square$

Lemma 8.19. *Assume that $f : U \to G$ is a $\mathcal{D}$-function, and that $C \subseteq G$ is a definable smooth curve which is a trajectory of F. Then so is $f(C)$.*

Proof. By the first clause of Lemma 8.18, the image of C under f is also a solution to F, hence a trajectory. $\square$

Lemma 8.20. *Assume that f is a $\mathcal{D}$-function, and $C \subseteq G$ is a definable smooth curve, such that at every $b \in C$ we have $\alpha(b) = 0$ (in the above notation). Then for every generic $b \in C$, the tangent space of $f(C)$ at $f(b)$ is the R-span of $v_{g(b)}$. Namely, $f(C)$ is a trajectory of F, in a neighborhood of $f(b)$.*

Proof. Consider the restriction of f to C, and pick a generic b in C. Since b is generic, the map $f|C : C \to f(C)$ is a submersion, namely $T_{f(b)}(C) = d_b f \cdot T_b(C)$. By the second clause of Lemma 8.18, we conclude that $T_{f(b)}(f(C))$ equals $R v_{f(b)}$. $\square$

Lemma 8.21. *There exists a $\mathcal{D}$-function h and an definable curve $C \subseteq G$ such that $h(C)$ is a trajectory of F.*

Proof. This is similar to the proof of Claim in Proposition 8.10. Fix any $\mathcal{D}$-function f, with α, β as above. We claim that there is $a_0 \in G$ such that for infinitely many $b \in G$, we have $\alpha_f(b) = \alpha_f(b \oplus a_0)$.

Indeed, for $r \in R$, let $C_r = \{b \in G : \alpha_f(b) = r\}$. Pick r generic in the image of α_f so that C_r is 1-dimensional, and consider $D_r = C_r \ominus C_r$. If D_r is still 1-dimensional then as we already saw several times, C_r is contained in a coset of a locally definable subgroup H_r and then picking $a_0 \in H_r$ small enough will work with any $b \in C_r$.

Otherwise, D_r is 2-dimensional. We may now pick $a_0 \in D_r$ generic over r. Since r is still generic over a_0 there are infinitely many r' such that $a_0 \in D_{r'}$. For each such r', there exists $b \in C_{r'}$ with $b \oplus a_0 \in C_{r'}$.

Fix a_0 as above, and consider the $\mathcal{D}$-function $h(x) = f(x \oplus a_0) \ominus f(x)$. It is easily verified that for each $b \in G$ we have

$$\tilde{J}_b h = \tilde{J}_b(f(x \oplus a_0) \ominus f(x)) = \tilde{J}_b f(x \oplus a_0) - \tilde{J}_b f(x).$$

It follows that $\alpha_h(b) = 0$ for every $b \in G$ such that $\alpha_g(b \oplus a_0) = \alpha_g(b)$. Let C be the collection of all those elements b. By Lemma 8.20, the curve $h(C)$ is a trajectory of F near b. $\square$

We can now conclude:

Theorem 8.22. *The ring $\mathfrak{R}$ is of analytic form.*

Proof. We still work under the negation assumption that we are in Case 2 of Theorem 8.11. Using Lemma 8.21 and Lemma 8.17 we obtain a definable local subgroup H which is a solution to the vector field F, and thus all of its cosets are also trajectories of F. Let U be a neighbourhood of 0 which can be covered by cosets of H, all solutions to F.

Fix any $\mathcal{D}$-function $f \in \mathfrak{F}_0$ which is not G-affine. By Lemma 8.19, for every $a \in U$ such that $f(a) \in U$, the image $f(H \oplus a)$ is also a coset of H. Fix $a_0 \in H$ close enough to 0 and consider the $\mathcal{D}$-function $k(x) = f(x) \ominus f(x \oplus a_0)$. Since f is not G-affine, the function k is not constant.

Notice that for every x sufficiently close to 0, the elements x and $x \oplus a_0$ belong to the coset $x \oplus H$, and therefore as we just noted, $f(x)$ and $f(x \oplus a_0)$ belong to the same coset of H. It follows that $k(x) \in H$ and therefore k sends an open subset of G into H, contradicting strong minimality (the pre-image of some point will be infinite). $\square$

Note that the above argument does not really use the definability of the trajectory C but merely its existence. Thus, if we worked over the reals then we could have used the usual existence theorem for solutions to differential equations in order to derive a contradiction.

9. Some intersection theory for $\mathcal{D}$-curves

Our ultimate goal is to show, under suitable assumptions, that if two plane curves $C, D \subseteq G^2$ are tangent at some point p, and C belongs to a $\mathcal{D}$-definable family $\mathcal{F}$ of plane curves then by varying C within $\mathcal{F}$ one gains additional intersection points with D, near the point p (see Proposition 9.12 (2)). This will allow us to detect tangency $\mathcal{D}$-definably.

The main tool towards this end is the following theorem, whose proof will be carried out in this section via a sequence of lemmas.

Theorem 9.1. *Assume that f is in $\mathfrak{F}_0$. If $J_0(f) = 0$ then there is no neighborhood of 0 on which f is injective.*

We now digress to report on an unsuccessful strategy, which nevertheless may be of some interest.

9.1. **Digression: on almost complex structures.** Let $K = R(\sqrt{-1})$. In analogy to the notion of an almost complex structure on a real manifold, we call *a definable almost K-structure* on a definable R-manifold M, a definable smooth linear $J : TM \to TM$ sending each $T_x(M)$ to $T_x(M)$, such that $J^2 = -1$.

Note that every definable K-manifold admits a natural almost K-structure, induced by multiplication of each $T_x(M)$ by $i = \sqrt{-1}$. It is known that when $K = \mathbb{C}$ any 2-dimensional almost complex structure is isomorphic, as an almost complex structure, to a complex manifold. The proof of this result seems to be using integration and thus we do not expect it to hold for almost K-structures in arbitrary o-minimal expansions of real close fields.

Returning now to our 2-dimensional group G, we can endow G with a definable almost K-structure in the following way: Just as we did at the beginning of Section 8.4, we may first assume that every matrix in $\mathfrak{R}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Next, we identify naturally $T_0(G)$ with $R^2 \sim K$ and let $J : T_0(G) \to T_0(G)$ be defined by $J(x, y) = (x, -y)$. Next, use the differential of ℓ_a to obtain $J : TG \to TG$ as required. Note that since TG is a trivial tangent bundle, this step can be carried out for any definable group of even dimension. However in the case of G, our choice of J and the fact that for each $\mathcal{D}$-function f, $\tilde{J}_a f$ has analytic form, implies that f is so-called J-holomorphic, namely that each each $a \in \operatorname{dom} f$ we have

$$J \circ d_a f = d_{f(a)} \circ J.$$

Now, if our underlying real closed field R were the field of real numbers then G would be isomorphic as an almost complex structure to a complex manifold $\hat{G}$, and this isomorphism would send every J-holomorphic function from G to G to a holomorphic function from $\hat{G}$ to $\hat{G}$. In particular, by our above observation every $\mathcal{D}$-function would be sent to a holomorphic function. This would give an immediate proof of Theorem 9.1, due to the fact that the result is true for holomorphic maps.

Unfortunately, we do not know how to prove for arbitrary K that every 2-dimensional almost K-manifold is (definably) isomorphic to a K-manifold, and hence we cannot use the theory of K-holomorphic maps in order to deduce Theorem 9.1. We thus use a different strategy.

9.2. **A motivating example.** If f were holomorphic then the above theorem would follow from the argument principle and the open mapping theorem. However, our functions are not necessarily holomorphic and hence we need a different strategy. In order to describe it, let us first give an alternative proof for holomorphic f: We let $h(z) = f(z)/z$ (complex division) for $z \neq 0$ and $h(0) = 0$. The assumption that $J_0 f = 0$ implies that h is continuous at 0 and hence holomorphic. Thus h is either locally constant or an open map in a neighborhood of 0. Now, if h were locally constant then $f \equiv 0$ near 0 and thus clearly non-injective, so assume that h is an open map.

We now consider the complex function $M(z, w) = z \cdot w$, and for $a, b \in \mathbb{C}$ near 0, let $M_{a,b}(z) = M(z - a, h(z) - b)$. Notice that $M_{0,0}(z) = f(z)$. Let $\deg_0(f)$ be the local degree of f at 0 (see details below). Since the local degree is preserved under definable homotopy (see Fact 9.2 below), it follows from the general theory that $\deg_0(M_{a,b}) = \deg_0(f)$ for sufficiently small a, b. Because each $M_{a,b}$ is holomorphic, the sign of $|J_z M_{a,b}|$ is positive at a generic z in $\mathrm{int}(C)$, and therefore

$$\deg_0(M_{a,b}) \geq |M_{a,b}^{-1}(w)|,$$

for all w close to 0.

If we take $w = 0$, then we get

$$|M_{a,b}^{-1}(0)| \geq 2$$

(the points a and $h^{-1}(b)$ being two such pre-images), implying $\deg_0(f) = \deg_0(M_{a,b}) \geq 2$. This implies that f is not locally injective near 0.

Our objective is to imitate the above proof, using $\mathcal{D}$-functions instead of holomorphic ones. The main obstacle is the fact that we do not have multiplication or division in $\mathcal{D}$, so we want to produce a $\mathcal{D}$-function which sufficiently resembles the multiplication function M.

9.3. **Topological preliminaries.** Throughout this section we will be using implicitly the o-minimal version of Jordan's plane curve theorem (see [32]) . We recall some definitions and results (see [23, Section 2.2-2.3]): Given a circle $C \subseteq R^2$, a definable continuous $f : R^2 \to R^2$ and $w \notin f(C)$, we let $W_C(f, w)$ denote the winding number of f along C around w. If $f^{-1}(w)$ is finite, $p \in R^2$ and $f(p) = w$ then $\deg_p(f)$ is defined to be $W_C(f, f(p))$ for all sufficiently small C around p. We need the following results:

Fact 9.2.

1. *Let $C \subseteq R^2$ be a circle oriented counter clockwise. If $\{f_t : t \in T\}$ is a definable continuous family of functions with $w \notin f_t(C)$ for any $t \in T$ and T definably connected, then $W_C(f_{t_1}, w) = W_C(f_{t_2}, w)$ for all $t_1, t_2 \in T$.*
2. *Assume that C is a circle around p, $f : C \to R^2$ definable and continuous, and w_1, w_2 are in the same component of $R^2 \setminus f(C)$. Then $W_C(f, w_1) = W_C(f, w_2)$.*
3. *If f is definable and R-differentiable at p and $J_p(f)$ is invertible, then $deg_p(f)$ is either 1 or -1, depending on whether $|J_p(f)|$ is positive or negative.*
4. *Assume that f is a definable R-smooth, open map, finite-to-one in a neighbourhood U of p and that $f(z) \neq f(p)$ for all $z \neq p$ in U. Assume also that $J_z(f)$ is invertible of positive determinant for all generic $z \in U$.*

Let $C \subseteq U$ be a circle around p. Then for all $w \in f(\mathrm{int}(C))$, if w and $f(p)$ are in the same component of $R^2 \setminus f(C)$ then $W_C(f, f(p)) \geq |f^{-1}(w) \cap \mathrm{int}(C)|$, and if w is also generic then $W_C(f, f(p)) = |f^{-1}(w) \cap \mathrm{int}(C)|$.

Proof. (1) follows from [23, Lemma 2.13(4)]. (2) is just [23, Lemma 2.15]. The proof of (3) is the same as the classical one, so we omit it.

(4) It follows from (2) that $W_C(f, f(p)) = W_C(f, w)$. We let $\{z_1, \ldots, z_k\} = f^{-1}(w) \cap \mathrm{int}(C)$. By [23, Lemma 2.25], $W_C(f, w) = \Sigma_{i=1}^k deg_{z_i}(f)$, so it is sufficient to see that $deg_{z_i}(f) \geq 1$, for each i. We fix a small circle, C_i, around z_i such that $W_{C_i}(f, w) = deg_{z_i}(f)$, and then fix a generic $w_0 \in f(\mathrm{int}(C_i))$ sufficiently close to w, so in particular, the Jacobian of f at each pre-image of w_0 is invertible of positive determinant. By [23, Lemma 2.25], $deg_{z_i}(f) = \Sigma_j deg_{p_j}(f)$, where the p_j are the pre-images of w_0 in $\mathrm{int}(C)$. By (3), for each p_j, we have $deg_{p_j}(f) = 1$, thus $deg_{z_i}(f) = |f^{-1}(w_0)| \geq 1$.

The same argument shows that for generic w_0 near p, we have $W_C(f, f(p)) = |f^{-1}(w_0)|$. □

9.4. **Back to $\mathcal{D}$-functions.** We still identify an open neighborhood of G with an open subset of R^2 and identify 0_G with $0 = (0,0)$. For an open set $U \subseteq G$ and a function $f : U \to G$, sending x_0 to y_0, we say that f *is generically k-to-1 at x_0* if for every open $V \ni x_0$ and $W \ni y_0$ there exists an open $y_0 \in W_0 \subseteq W$ such that for any generic $y \in W_0$, $|f^{-1}(y) \cap V| = k$.

Below we use the notion of a $\mathcal{D}$-function M from an open $U \subseteq G^2$ into G. By that we mean that there exists a $\mathcal{D}$-definable set $S \subseteq G^2 \times G$ of Morley rank 2, which contains the graph of M.

Lemma 9.3. *Let $U \subseteq G^2$ be a definable open neighborhood of $(0,0)$. Assume that $M : U \to G$ is a continuous $\mathcal{D}$-function such that $M(0, y) = M(x, 0) = 0$ for all x, y close enough to 0. Assume that $f, h \in \mathfrak{F}_0$ and we have:*

1. *For every a, b in some neighborhood of 0, the function $g_{a,b}(x) = M(f(x) \ominus a, h(x) \ominus b)$ is not locally constant near 0.*
2. *f and h are, respectively, generically k-to-one and m-to-one near 0,*

Then $g(x) = M(f(x), h(x))$ is, generically, at least $k + m$-to-one near 0.

Proof. By Corollary 4.11 each $g_{a,b}$ is open, thus, since it is a $\mathcal{D}$-function, it is finite-to-one near 0. Also, it follows from Corollary 8.13 and Proposition 8.22 that $g_{a,b}$ has positive determinant of the Jacobian at every point where the Jacobian matrix does not vanish, which by Lemma 8.1 is a co-finite set.

We now fix C around 0 such that $0 \notin g(C) = g_{0,0}(C)$ and $deg_0(g) = W_C(g, 0)$. By continuity of M and g we can find an open $U_1 \ni 0$, and an open disc $U_2 \ni 0$, such that for all $a, b \in U_1$, $g_{a,b}(0) \in U_2$ and $g_{a,b}(C) \cap U_2 = \emptyset$. It follows that $g_{a,b}(0)$ and 0 are in the same component of $R^2 \setminus g_{a,b}(C)$.

By Fact 9.2,

$$\deg_0(g) = W_C(g, 0) = W_C(g_{a,b}, 0) = W_C(g_{a,b}, g_{a,b}(0)) \geq |g_{a,b}^{-1}(0)|.$$

If we take a, b generic and independent near 0, then $f^{-1}(a) \cap h^{-1}(b) = \emptyset$. Also, by our assumptions on M and the definition of $g_{a,b}$, we have

$$f^{-1}(a) \cup h^{-1}(b) \subseteq g_{a,b}^{-1}(0).$$

Hence, $|g_{a,b}^{-1}(0)| \geq m + k$. It follows from Fact 9.2 (4) that $deg_0(g) \geq m + k$ and that g is generically $k + m$-to-one near 0. $\square$

9.5. **Producing the function M.** We now proceed to construct the desired $\mathcal{D}$-function M as in Lemma 9.3. We start with a $\mathcal{D}$-function $k(x)$ which is not G-affine and fix an generic $a_0 \in \operatorname{dom} k$. Define

$$M(x, y) = (k(a_0 \oplus x \oplus y) \ominus k(a_0 \oplus x)) \ominus (k(a_0 \oplus y) \ominus k(a_0)).$$

We write $M_a(y) = M(a, y)$.

By definition, we have

(A): For x, y near 0, $M(0, y) = M(x, 0) = 0$.

Our next goal is to show that M can be used, similarly to multiplication, to "divide (an appropriate) function f by x". Namely, that we can implicitly solve $M(x, y) = f(x)$ in some neighborhood of $x = 0$. This is the purpose of the next few results.

By Theorem 8.22 and the discussion in Section 8.4, we may assume that for a smooth $f \in \mathfrak{F}_0$, the matrix $J_0(f)$ has the form

$$\begin{pmatrix} c & -e \\ e & c \end{pmatrix},$$

with $c, e \in R$.

We consider the partial definable map d on G, given by $d(a) = J_0(M_a)$. It is convenient to identify every $d(a)$ with the first row of the above matrix, so we view d as a map from G into R^2. By Claim 2 in the proof of Proposition 8.10 (and using the fact that $\tilde{J}_0(f) = J_0(f)$), we get that $d(a)$ is equal to:

$$(\dagger) \quad J_0 M_a = \tilde{J}_0([k(a_0 \oplus a \oplus y) \ominus k(a_0 \oplus a)]) - \tilde{J}_0([k(a_0 \oplus y) \ominus k(a_0)]) = \tilde{J}_{a_0 \oplus a}(k) - \tilde{J}_{a_0}(k).$$

By Proposition 8.10, applied to $k(x)$, the image of every open $U \ni 0$ under $x \mapsto \tilde{J}_x(k)$ is a 2-dimensional subset of $\mathfrak{R}$, hence by o-minimality this map is locally injective near the generic a_0. Equivalently, the map $x \mapsto \tilde{J}_{a_0 \oplus x}(k)$ is locally injective near 0. Since $\tilde{J}_{a_0}(k)$ is constant, it follows that $d(x)$ is locally injective at 0. In particular, we have

(B): $d(0) = 0$, and there is a neighborhood of 0 where $d(a) \neq 0$ for all $a \neq 0$.

.

We are going to use several different norms in the next argument, so we set

$$||(x, y)|| = \sqrt{x^2 + y^2},$$

and for a linear map T we denote the operator norm by

$$||T||_{op} = max\{||T(x)||/||x|| : x \in dom(T)\}.$$

It is well-known (and easy to see) that if we identify every linear map with a 2×2 matrix then $||T||_{op}$ and $||T||$ are equivalent norms.

We need an additional property of M. Given two functions $\alpha, \beta : U^* \to R^{\geq 0}$ on a punctured neighborhood $U^* \subseteq R^2$ of 0, we write $\alpha \sim \beta$ if $\lim_{t \to 0} \alpha(t)/\beta(t)$ is a positive element of R. We will show:

(C): There are definable $R^{>0}$-valued functions $e(a)$ and $\delta(a)$, in some punctured neighborhood U^* of 0, with $e(a) \sim ||d(a)||$ and $\delta(a) \sim ||d(a)||^2$, such that for every $a \in U^*$, the function $M_a = M(a, -)$ is invertible on the disc $B_{e(a)}$ and its image contains the disc $B_{\delta(a)}$ (recall that for $a = 0$ we have $M_a(x) \equiv 0$ near 0).

In order to prove (C), we use an effective version of the inverse function theorem, as appearing in [3, §7.2]. We give the details, with references to [3].

Proposition 9.4. *There exist definable functions $e(a)$ and $\delta(a)$ from a punctured neighborhood of 0 into $R^{>0}$, such that for $a \neq 0$ in a small neighborhood of 0, the function $M_a(y)$ is injective on $B(0; e(a))$ and the its image contains a ball of radius $\delta(a)$ around 0. Furthermore, there is a constant $C > 0$ such that $e(a) = ||d(a)||/4C$ and $\delta(a) = e(a)^2/2$.*

Proof. We start with some observations. If

$$A = J_0(M_a) = \begin{pmatrix} c & -e \\ -e & c \end{pmatrix}$$

then $||A||_{op} = \sqrt{c^2 + e^2} = ||d(a)||$. And if A is invertible then $||A^{-1}||_{op} = 1/||d(a)||$.

Consider the partial map $D : G \times G \to R^4$, defined by $D(a, y) = J_y(M_a) \in M_2(R)$. For each a, y, we view $D(a, y)$ both as a linear operator and a vector in R^4. Since M is a $\mathcal{C}^2$-function, $||J_{(a,y)}D||_{op}$ is bounded by some constant C, as (a, y) varies in a neighborhood $B_1 \times B_2$ of $(0, 0)$, and by normalizing M we may assume that $C > 1$. By [3, Lemma 7.2.8] applied to D, for every $(a_1, y_1), (a_2, y_2) \in B_1 \times B_2$ we have

$$(*) \qquad ||J_{y_1}(M_{a_1}) - J_{y_2}(M_{a_2})|| < C||(a_1, y_1) - (a_2, y_2)||.$$

Note also that $D(0, 0) = J_0 M_0 = 0$, so restricting further B_1, B_2 we may also assume that $||D(a, y)|| < 1$ for all $(a, y) \in B_1 \times B_2$.

We now need a version of [3, Lemma 7.2.10]:

Lemma 9.5. *For every $a \in B_1$ such that $J_0 M_a$ is invertible, and for all $y_1, y_2 \in B_2$, there exists $e(a) \in R^{>0}$, such that if $||y_1||, ||y_2|| \leq e(a)$ then*

(1) *the matrices $J_{y_1} M_a, J_{y_2} M_a$ are invertible.*
(2) $||M_a(y_1) - M_a(y_2)|| \geq e(a)||y_1 - y_2||$. *In particular, M_a is injective on the disc $B_{e(a)}$.*

Moreover, we can choose $e(a) = ||d(a)||/C$ for some constant $C > 0$.

Proof. We fix a with $J_0(M_a)$ invertible and we write $J_0 M_a = \begin{pmatrix} c & e \\ -e & c \end{pmatrix}$. By (*), for every $y \in B_2$ and for every $E > 0$, if $||y|| < E/2C$ then

$$||J_y M_a - J_0 M_a|| \leq C||y|| \leq C||(y, a) - (0, a)|| < \frac{E}{2}.$$

In particular, since $J_0M_a \neq 0$, we may take $E < ||d(a)|| = ||J_0M_a||_{op}$ and then J_yM_a must be non-zero. Because M_a is a $\mathcal{D}$-function it follows that J_yM_a is invertible.

Let $c' = 1/||J_0(M_a)^{-1}||_{op}$. As we pointed out earlier, in our case

$$||J_0(M_a^{-1})||_{op} = ||J_0(M_a)^{-1}||_{op} = 1/||d(a)||,$$

hence $c' = ||d(a)||$. Now, for all non-zero vectors w, we have $||J_0(M_a)^{-1}(w)|| \leq \frac{1}{c'}||w||$, so by substituting w with $J_0(M_a)^{-1}(z)$, we get $c'||z|| = ||d(a)|| \cdot ||z|| \leq ||J_0M_a(z)||$.

Hence, for any two $y_1, y_2 \in R^2$:

$$(**) \qquad ||J_0M_a \cdot (y_1 - y_2)|| \geq ||d(a)|| \cdot ||y_1 - y_2||.$$

By [3, Lemma 7.2.9], applied to the function M_a, we also have for all $y_1, y_2 \in B_2$,

$$||M_a(y_1) - M_a(y_2) - J_0M_a(y_1 - y_2)|| \leq ||y_1 - y_2|| \max_{t\in[y_1,y_2]} ||J_tM_a - J_0M_a||_{op},$$

where $[y_1, y_2]$ is the line segment in R^2 connecting y_1 and y_2. Hence, by the triangle inequality,

$$||M_a(y_1) - M_a(y_2)|| \geq ||J_0M_a(y_1 - y_2)|| - ||y_1 - y_2|| \max_{t\in[y_1,y_2]} ||J_tM_a - J_0M_a||_{op}.$$

Putting this together with (*) and (**), we have: If $y_1, y_2 \in B_2$ and $||y_i|| < E/2C$, for $i = 1, 2$, then

$$||M_a(y_1) - M_a(y_2)|| \geq (||d(a)|| - C||y_1 - y_2||)||y_1 - y_2||.$$

If in addition $||y_1 - y_2|| < \frac{||d(a)||}{2C}$ then

$$(***) \qquad ||M_a(y_1) - M_a(y_2)|| \geq (||d(a)|| - \frac{||d(a)||}{2})||y_1 - y_2|| = \frac{||d(a)||}{2}||y_1 - y_2||.$$

We summarize what we have shown so far: for any $E < ||d(a)||$, if $||y_1||, ||y_2|| < E/2C$ and $||y_1 - y_2|| < ||d(a)||/2C$, then $J_y(M_a)$ is invertible and (***) holds.

We now fix the parameters as follows: Set $E = ||d(a)||/2$, $e(a) = ||d(a)||/4C = E/2C$. So, if $||y_1||, ||y_2|| < E/2C$ then $||y_1 - y_2|| < E/C = ||d(a)||/2C$, so we may apply (***) and conclude that J_{y_i} are invertible for $i = 1, 2$ and

$$||M_a(y_1) - M_a(y_2)|| \geq \frac{||d(a)||}{2}||y_1 - y_2|| \geq e(a)||y_1 - y_2||.$$

$\square$

By the proof of [3, Theorem 2.11],

$$\{y : ||y - M_a(0)|| < \frac{e^2(a)}{2}\} \subseteq \{M_a(z) : ||z|| < e(a)\}$$

(apply the claim on the second line of p.113 with ϵ, c there both substituted with $e(a)$ here, and our M_a substituting f there). Thus, the image of the disc $B_{e(a)}$ under M_a contains a disc of radius $\frac{e(a)^2}{2}$ around $M_a(0) = 0$. We do not repeat the proof here.

Setting $\delta(a) = \frac{e(a)^2}{2}$ completes the proof of Proposition 9.4. $\square$

9.6. **Proving Theorem 9.1.** We now fix a $\mathcal{D}$-function $M : G^2 \to G$ satisfying conditions (A), (B) and (C) as above, with $d(x) = J_0 M_x$. We first need a simple observation:

Fact 9.6. *Assume that $f : U \subseteq R^2 \to R^2$ is a definable C^2-function sending 0 to 0. If $J_0 f = 0$ then $\lim_{x\to 0} ||f \circ f(x)||/||x||^2 = 0$.*

Proof. As already mentioned above, the operator norm and the Euclidean norm on R^2 are equivalent – so we may work with either.

Using the fact that $f(0) = 0$, we may apply [3, Lemma 7.2.8] to f and conclude that $||f(x)|| \leq C||x||$, for all x in an open disc U centered at 0, where C is a bound on $||J_a(f)||_{op}$ in U. By the equivalence of norms we may assume that C is also a bound on $||J_a(f)||$ as a varies in U.

Consider now the map $a \mapsto J_a(f)$, as a map from U into R^4, and let C' be a bound on the norm of the differential of this map in U. We apply again [3, Lemma 7.2.8] to this map, and using the fact that $J_0(f) = 0$, we conclude that $||J_a(f)|| < C'||a||$, for x in some $U' \ni 0$, and $a \in B(0; ||x||)$

Thus, for all $x \in U$,

$$||f(x)|| \leq C||x|| \leq C'||a|| \cdot ||x|| \leq C'||x||^2.$$

It follows that $||f \circ f(x)|| \leq C'||x||^4$ and hence $\lim_{x\to 0} ||f \circ f(x)||/||x||^2 = 0$. □

We also need:

Lemma 9.7. *If $x(t) : (a, \epsilon) \to R^2$ is a definable curve tending to 0 as $t \to 0$, then $\lim_{t\to 0} \frac{||d(x(t))||}{||x(t)||} \neq 0$.*

Proof. Recall that d is a map from U into R^2, defined by $d(a) = (c, e)$, where $J_0(M_a) = \begin{pmatrix} c & -e \\ e & c \end{pmatrix}$. Notice that since $\mathfrak{R}$ is 2-dimensional we may also view d as a map from G into $\mathfrak{R}$.

We claim that the Jacobian at 0 of d is invertible. Indeed, we have seen in (†) above (Subsection 9.5), $J_0(M_a) = \tilde{J}_{a_0 \oplus a} k - \tilde{J}_{a_0} k$. By Proposition 8.10, the function $a \mapsto \tilde{J}_a f$ is a diffeomorphism in a small neighbourhood of the generic point a_0 onto an open subset of $\mathfrak{R}$. Since $x \mapsto a_0 \oplus x$ is a diffeomorphism (between open subsets of G) in a neighbourhood of 0 we get that $a \mapsto \tilde{J}_{a_0 \oplus a} k$ is a diffeomorphism near 0 between an open subset of G and $\mathfrak{R}$. Since $\tilde{J}_{a_0} k$ is a constant matrix it follows that $a \mapsto J_0(M_a)$ is a diffeomorphism near 0. So $J_0 d$ is invertible.

It follows from the definition of the differential that

$$\lim_{t\to 0} \left(\frac{d(x(t))}{||x(t)||} - \frac{J_0(d) \cdot x(t)}{||x(t)||} \right) = 0.$$

Since $J_0(d)$ is invertible the limit of $\frac{J_0(d) \cdot x(t)}{||x(t)||}$ is a non-zero vector, and hence $\lim_{t\to 0} \frac{||d(x(t))||}{||x(t)||} \neq 0$. □

Corollary 9.8. *Let $e(a)$ and $\delta(a)$ be as provided by Proposition 9.4. Assume that $f : G \to G$ is such that $f(0) = 0$ and $J_0 f = 0$. Let $g = f \circ f$. Then there is an open neighborhood $U \ni 0$, such that for all non-zero $a \in U$, we have*
(i) $||g(a)|| < \delta(a)$.
(ii) There exists a unique $y \in B(0; e(a))$ such that $M(a, y) = g(a)$.

Proof. We first prove (i) for some neighborhood U, so we assume that (i) fails. Then there exists an definable function $x(t)$ tending to 0 in G, such that for all t,

$$||g(x(t))|| \geq \delta(x(t)) = ||d(x(t))||^2/32C^2.$$

Because $J_0 f = 0$, Fact 9.6 implies that $\lim_{t\to 0} ||g(x(t))||/||x(t)||^2 = 0$. Combined with the above inequality we get $\lim_{x(t)\to 0} ||d(x(t))||^2/||x(t)||^2 = 0$, hence $\lim_{t\to 0} ||d(x(t))||/||x(t)|| = 0$, contradicting Lemma 9.7. Thus there exists $U \ni 0$ such that for all $a \in U$ we have $||g(a)|| < \delta(a)$. It now follows from our choice of $\delta(a)$ that there exists a unique $y \in B(0; e(a))$ such that $M(a, y) = g(a)$. □

Corollary 9.9. *Let f and g be as above, and $e(a)$, $\delta(a)$ as in* **(C)** *above. Let $U = \{x : |g(x)| < \delta(a)\}$ and $U^* = U \setminus \{0\}$. For every $x \in U^*$, let $h(x)$ be the unique y in $B_{e(a)}$ such that $M(x, y) = g(x)$. Then*
(i) U contains an open disc around 0.
(ii) h is differentiable on U^* and $\lim_{x\to 0} h(x) = 0$ (so it extends continuously to 0). In particular, if g is not constant then neither is h.
(iii) The continuous extension of h to U is a $\mathcal{D}$-function.

Proof. Clause (i) is just Corollary 9.8. To see that h is differentiable everywhere we apply the Implicit Function Theorem to $M(x, y) - g(x)$. By Lemma 9.5, $J_y M_x$ is invertible for every $x \in U^*$ and $|y| < e(a)$, so indeed $h(x)$, the solution to $M(x, y) - g(x) = 0$, is differentiable at x.

To see that the limit of h at 0 is 0, we compute the limit along an arbitrary curve $x(t)$ tending to 0. By definition, $|h(x(t))| < ||e(t)|| \sim ||d(t)||$, so since $d(0) = 0$, also $h(x(t))$ must tend to 0. The second clause of (ii) follows since if h were constant with $h(0) = 0$ necessarily h would vanish on its domain, implying that g was identically 0.

For (iii), note that the graph of h is contained in the plane curve $B = \{(x, y) : (x, y, g(x)) \in \hat{M}(x, y)\}$ where $\hat{M}$ is the $\mathcal{D}$-definable set of Morley rank 2 containing the graph of M. □

We note that locally near the point $(0, 0)$ itself, the $\mathcal{D}$-definable set B need not be the graph of a function, but this does not come up in the argument.

Proof of Theorem 9.1: Assume that $f \in \mathfrak{F}_0$ and $J_0(f) = 0$. We will show that f is not injective near 0.

Proof. Consider $g(x) = f(f(x))$, and assume towards contradiction that f and thus also g is injective. near 0. By Corollary 9.9, there exists a $\mathcal{D}$-function h in a neighborhood U of

0, with $h(0) = 0$ such that for all $x \in U$,

$$M(x, h(x)) = g(x).$$

We now wish to apply Lemma 9.3 to the functions x and $h(x)$. For that we just need to note that for a and b near 0 the function $g_{a,b}(x) = M(x \ominus a, h(x) \ominus b)$ is non-constant near 0. Indeed, we can find a fixed definably connected open $W \ni 0$, such that for a, b close to 0, $W \subseteq dom(g_{a,b})$. Since each $g_{a,b}$ is a $\mathcal{D}$-function, its graph is contained in a strongly minimal set, and hence if it were constant near 0 then it would have to be constant on the whole of W. But then, by the continuity of M, the function $g = g_{0,0}$ must also be constant on W, contradiction.

By applying Lemma 9.3 we conclude that $g(x)$ is at least $1+k$-to-one near 0, where $k \geq 1$. This contradicts the assumption that f and thus g were locally injective. Contradiction. $\square$

The following example shows that the proof of Theorem 9.1 uses more than just the basic geometric properties of the function f.

Example 9.10. A crucial point in our above argument was that $f(z)/z$, or in the language of our proof, the implicitly defined function h, is an open map. This followed from the fact that it was a $\mathcal{D}$-function.

Consider the function $f(z) = |z|^2 z$ from $\mathbb{C}$ to $\mathbb{C}$. The function is smooth everywhere, $J_0 f = 0$, and yet it is injective everywhere. However, the function $f(z)/|z|$ is clearly not an open map.

9.7. **Intersection theory in families.** Based on the topological properties we established thus far we can develop some intersection theory resembling that of complex analytic curves.

Definition 9.11. Given two plane curves X, Y, and $p = (p_1, p_2) \in X \cap Y$. We say that *$X$ and Y are tangent at p* if there are $\mathcal{D}$-functions f, g which are C^1 in a neighborhood of p_1, with $\Gamma_f \subseteq X$ and $\Gamma_g \subseteq Y$, such that

$$f(p_1) = p_2 = g(p_1) \quad \text{and} \quad J_{p_1} f = J_{p_1} g.$$

Proposition 9.12. *Let $\mathcal{F} = \{E_a : a \in T\}$ be a $\mathcal{D}$-definable almost faithful family of plane curves, $\mathcal{D}$-definable over $\emptyset$, and let X be a strongly minimal plane curve whose projections on both coordinates are finite-to-one.*

Assume that a is generic in T over $\emptyset$, E_a strongly minimal, $X \cap E_a$ is finite and $p = (x_0, y_0) \in E_a \cap X$.

(1) *If p is generic in E_a over a, not topologically isolated in X and also $\mathcal{D}$-generic in X over $[X]$ then for every neighborhood $U \ni p$, there is a neighborhood $V \ni a$ in T, such that for every $a' \in V$, $E_{a'}$ intersects X in U.*

(2) *(Here we do not make any genericity assumptions on p). Assume that for some open $V \ni a$ whenever $a' \in V$ the set $E_{a'}$ represents a $\mathcal{D}$-function $f_{a'}$ in a neighborhood of x_0 and that the map $(a', x) \mapsto f_{a'}(x)$ is continuous at (a, x_0). Assume also that X represents a function g at p and that $J_{x_0}(f_a) = J_{x_0}(g)$. Then for every neighborhood $U \ni p$ there is a neighborhood $V \ni a$ in T, such that for every $a' \in V$, either $E_{a'}$ and X are tangent at some point in U or $|E_{a'} \cap X \cap U| > 1$.*

Proof. (1) Fix an open $U = U_1 \times U_2 \ni p$ definably connected. Since p is generic in E_a over a, we may choose U so that E_a is locally the graph of a continuous function $f_a : U_1 \to U_2$ or $f_a : U_2 \to U_1$. Since our assumptions are symmetric with respect to both coordinates we may assume the former.

Since, in addition, a is generic in T over $\emptyset$ we may shrink U and find an open definably connected $V_0 \ni a$ in T such that for every $a' \in V$, the set $E_{a'} \cap U_1 \times U_2$ is the graph of a $\mathcal{D}$-function $f_{a'} : U_1 \to U_2$ and furthermore, the map $(a', x) \mapsto f_{a'}(x)$ is continuous on $V_0 \times U_1$.

Since p is not isolated in X, $\mathcal{D}$-generic in X over $[X]$, and the projections of X on both coordinates are finite-to-one, it follows from Corollary 6.4 that, after possibly shrinking U further, the set $X \cap U$ is the graph of an open continuous map $g : U_1 \to U_2$.

Notice that for every $a' \in V_0$, and $(x, y) \in U$,

$$(x, y) \in E_{a'} \cap X \Leftrightarrow f_{a'}(x) \ominus g(x) = 0.$$

Because $X \cap E_a$ is finite, the function $f_a \ominus g$ is not constant on its domain, so by Theorem 7.7, $f_a \ominus g$ is open on U_1.

Claim There exists $V \ni a$ such that for every $a' \in V \setminus \{a\}$, the function $f_{a'} \ominus g$ is an open map on U_1.

Indeed, assume towards a contradiction that for $a' \in V_0$ arbitrarily close to a the map $f_{a'} \ominus g$ is not an open map. Thus, by Theorem 7.7, it is constant on U_1. It follows from continuity that $f_a \ominus g$ is constant on U_1, contradicting out assumption.

Thus, we showed that there exists $V \ni a$ such that for all $a' \in V$, the function $f_{a'} \ominus g$ is open and finite-to-one on U_1. In addition, the the map $(a', x) \mapsto (f_{a'} \ominus g)(x)$ is continuous in a neighborhood (a, x_0). Because $0 \in (f_a \ominus g)(U)$ it follows from Fact 9.2(1), (4) that for some open $V_0 \ni a$ small enough and for all $a' \in V_0$, the set $(f_{a'} \ominus g)(U)$ contains 0, namely $X \cap E_{a'} \cap U \neq \emptyset$. This ends the proof of (1).

(2) Let g be a $\mathcal{D}$-function with $g(x_0) = y_0$ such that $\Gamma_g \subseteq X$, and $J_{x_0} f_a = J_{x_0} g$. Note that $(f_a \ominus g)(x_0) = 0$ and $f_a \neq g$. So, for $C \subseteq G$ a sufficiently small circle around x_0 the only zero of $f_a \ominus g$ in the closed ball B determined by C is x_0. By continuity of $(x, a') \mapsto f_{a'}(x)$, we may find some neighborhood $V \subseteq V_0$ of a such that for every $a' \in V$, $0 \notin (f_{a'} \ominus g)(C)$. It follows from Fact 9.2(1) that

$$W_C(f_{a'} \ominus g, 0) = W_C(f_a \ominus g, 0)$$

for every $a' \in V$.

By our assumptions, $J_{x_0}(f_a \ominus g) = 0$ and therefore by Theorem 9.1, $f_a \ominus g$ is not injective in any neighborhood of x_0, that is, for every generic y near 0, $|(f_a \ominus g)^{-1}(y)| > 1$. It follows from Fact 9.2(4) that $W_C(f_a \ominus g, 0) > 1$. Thus, for every $a' \in V$, $W_C(f_{a'} \ominus g, 0) > 1$.

We can now conclude that for every a', either 0 is a regular value of the function $f_{a'} \ominus g$ on $\operatorname{int}(C)$, in which case it has more than one pre-image and then $E_{a'}$ and X intersect more than once in $\mathrm{int}(C)$, or 0 is a singular value, in which case the curves $E_{a'}$ and X are tangent at some point in $\mathrm{int}(C)$. □

10. The main theorem

We are now ready to prove our main result. Our proof follows closely that of [5, Theorem 7.3]. As some of the technicalities of that proof were dealt with in earlier stages of the present paper, the proof will be somewhat simpler. We begin with a series of useful technical facts. Throughout this section we let $K := \mathfrak{R}$.

Lemma 10.1. *There exist $\mathcal{D}$-definable families $\mathcal{C}_0 = \{E^0_a : a \in T_0\}$, $\mathcal{C}_1 = \{E^1_b : b \in T_1\}$, of plane curves all passing through $(0,0)$ such that:*

1. *For $i = 0{,}1$, T_i is strongly minimal and $\mathcal{C}_i$ is almost faithful.*
2. *Every generic curve in $\mathcal{C}_i$, $i = 0{,}1$, is closed, strongly minimal and has no isolated points.*
3. *There are definable open neighborhoods $U \subseteq G$ of 0, and there are definable open sets $T'_0 \subseteq T_0$, $T'_1 \subseteq T_1$ such that for every $i = 0{,}1$, and $a \in T'_i$, the curve E^i_a represents a function $f^i_t : U \to G$ in $\mathfrak{F}_0$,*
4. *For $i = 0{,}1$, the sets*
$$W_i := \{J_0\, f^i_a : a \in T'_i\}$$
are open subsets of K, with $0 \in \mathrm{cl}(W_0)$ and $1 \in \mathrm{cl}(W_1)$.
5. *For each $i = 0{,}1$, the map $(a, x) \mapsto f^i_a(x)$ is continuous on $T'_i \times U$.*

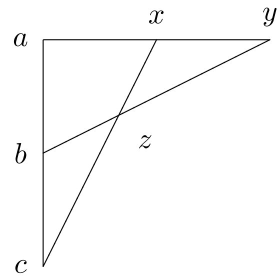
Proof. By Claim 8.7, there exists a $\mathcal{D}$-function $f : U \to G$ which is not G-affine, such that $J_0 f = 0$. Let $S \subseteq G^2$ be a strongly minimal set representing f. By Theorem 4.9, we may assume that S is closed, and by allowing parameters we may assume that S has no isolated points. Let
$$\mathcal{C}_0 = \{S \ominus p : p \in S\}.$$
Let $T_0 := S$ and for $a \in S$ let $E^0_a := S \ominus a$.

For every $a = (x_0, f(x_0)) \in S$, the curve S_a represents the $\mathcal{D}$-function $f(x \oplus x_0) \ominus f(x_0)$. By Proposition 8.10, the set of elements of K
$$W = \{J_0(f(x \oplus x_0) \ominus f(x_0)) : x_0 \in U\}$$
has dimension 2, and by applying the same proposition to a smaller U, we see that $J_0 f = 0$ is in the closure of a 2-dimensional component of W. By o-minimality, we may find an open $U' \subseteq U$ such that the set $W_0 = \{J_0(f(x \oplus x_0) \ominus f(x_0)) : x_0 \in U'\}$ is an open subset of K with the 0 matrix in its closure. We let $T'_0 := \{(x_0, f(x_0)) : x_0 \in U'\}$. By its definition, the sets U', T'_0 and W_0 satisfy all clauses of the lemma.

In order to obtain $\mathcal{C}_1$, we replace f with the function $h(x) = f(x) \oplus x$. It is a $\mathcal{D}$-function which is not G-affine, with $J_0 h = 1 \in K$. We repeat the above process and obtain the rest of the lemma. □

Recall the following definition.

Definition 10.2. Let $\mathcal{N}$ be strongly minimal. A set $\{a, b, c, x, y, z\}$ of tuples is called a *field configuration in $\mathcal{N}$* if



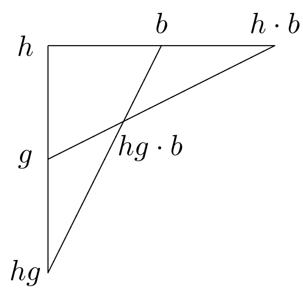
(1) all elements of the diagram are pairwise independent and $\mathrm{RM}(a, b, c, x, y, z) = 5$;
(2) $\mathrm{RM}(a) = \mathrm{RM}(b) = \mathrm{RM}(c) = 2$, $\mathrm{RM}(x) = \mathrm{RM}(y) = \mathrm{RM}(z) = 1$;
(3) all triples of tuples lying on the same line are dependent, and moreover, $\mathrm{RM}(a, b, c) = 4$, $\mathrm{RM}(a, x, y) = \mathrm{RM}(b, z, y) = \mathrm{RM}(c, x, z) = 3$;
(4) there are no $a' \in \mathrm{acl}(a)$, $b' \in \mathrm{acl}(b)$ and $c' \in \mathrm{acl}(c)$ with $\mathrm{RM}(a') = \mathrm{RM}(b') = \mathrm{RM}(c') = 1$ such that the same dependencies hold with a', b', c' replacing a, b, c.

For a proof of the following theorem, see [2, Main Theorem, Proposition 2] and the discussion following Proposition 2 there.

Fact 10.3. *(Hrushovski) If a strongly minimal structure $\mathcal{N}$ admits a field configuration, then $\mathcal{N}$ interprets an algebraically closed field.*

Let $\mathbb{G}_m$ and $\mathbb{G}_a$ denote the multiplicative and additive groups of the algebraically closed field K. The action of $\mathbb{G}_m \ltimes \mathbb{G}_a$ on $\mathbb{G}_a$ (defined by $(a, c) \cdot b = ab + c$) gives rise, naturally, to a field configuration on the structure $(K, +, \cdot)$ as follows: take $g, h \in \mathbb{G}_m \ltimes \mathbb{G}_a$ independent generics (in $\mathcal{M}$), and $b \in \mathbb{G}_a$ generic over g, h (in $\mathcal{M}$). Then

$$\mathcal{F} := \{g, h, gh, b, h \cdot b, hg \cdot b\}$$



where $\cdot$ denotes the action of $\mathbb{G}_m \ltimes \mathbb{G}_a$ on $\mathbb{G}_a$ is readily verified to be a field configuration in the field K.

Our aim is to pull this field configuration back into the structure $\mathcal{D}$. First note that although W_0 and W_1 from Lemma 10.1, are not neighborhoods of 0 and 1, respectively, it is still the case that for every $B \in W_0$, if $A \in W_1$ and $C \in W_0$ are sufficiently close to 1 and 0, respectively, then $AB + C$ is still in W_0 (since W_0 is open). Similarly, for every $A \in W_1$, if $C \in W_1$ is sufficiently close to 1 then $AC \in W_1$.

Let $e = (1, 0)$ be the identity of $\mathbb{G}_m \ltimes \mathbb{G}_a$, and choose $b \in W_0$ and h, g in $W_1 \times W_0 \subseteq \mathbb{G}_m \ltimes \mathbb{G}_a$ sufficiently close to e so that $gh \in W_1 \times W_0$, and $h \cdot b$ and $hg \cdot b$ are in W_0. Note that we may choose g, h, b to be independent generics in the sense of $\mathcal{M}$ (and then also independent in the sense of K).

Abusing notation, we sometimes write $f \in \mathcal{C}_i$ for a $\mathcal{D}$-function f which is represented by a curve in $\mathcal{C}_i$. In particular, let us denote, for $i = 1, 2$,

$$\mathcal{C}_i' = \{f_t^i : t \in T_i'\}.$$

We are going to reconstruct $\mathcal{F}$ as a set of jacobian matrices of $\mathcal{D}$-functions in $\mathcal{C}_0'$ and $\mathcal{C}_1'$, and show that it is, in fact, a field configuration in $\mathcal{D}$.

As a corollary of Lemma 10.1 we get:

Corollary 10.4. *There are $a_1, a_2 \in W_1$ and $b, b_1, b_2 \in W_0$, such that $g = (a_1, b_1), h = (a_2, b_2) \in W_1 \times W_0$ and the following hold:*

(1) *There exist $g_1, g_2 \in \mathcal{C}_1'$ and $f_1, f_2, k_1 \in \mathcal{C}_0'$ with $J_0 g_i = a_i$ (for $i = 1, 2$) and $J_0 f_i = b_i$ (for $i = 1, 2$) and $J_0 k_1 = b$.*
(2) *$hg \in W_1 \times W_0$, and there are $f_3 \in \mathcal{C}_0'$ and $g_3 \in \mathcal{C}_1'$ with $(J_0 g_3, J_0 f_3) = hg$.*
(3) *There are $k_2, k_3 \in \mathcal{C}_0'$ such that $J_0 k_2 = h \cdot b$ and $J_0 k_3 = hg \cdot b$.*

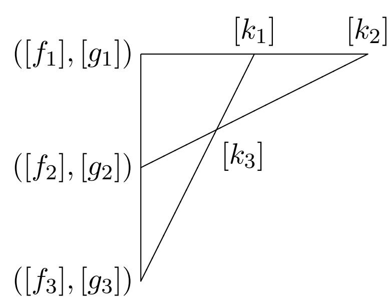
For a $\mathcal{D}$-function Ψ, we denote by $[\Psi]$ the $\mathcal{D}$-canonical parameter of some fixed strongly minimal set representing it. Our goal is to prove the following proposition.

Proposition 10.5. *Keeping the above notation:*

$$\mathcal{Y} := \{([f_1], [g_1]), ([f_2], [g_2]), ([f_3], [g_3]), [k_1], [k_2], [k_3]\} \tag{*}$$

is a field configuration in $\mathcal{D}$.

Proof. We have to verify the following diagram (in the sense of $\mathcal{D}$):



The families $\mathcal{C}_0$ and $\mathcal{C}_1$ are almost faithful, so for a function $f_a \in \mathcal{C}_0'$ we have $\mathrm{acl}_{\mathcal{D}}(a) = \mathrm{acl}_{\mathcal{D}}([f_a])$. Since field configurations are stable with respect to $\mathcal{D}$-inter-algebraicity we may assume that $a = [f_a]$. The same is true for $\mathcal{C}_1'$.

Note that for every $z \in \mathcal{C}_i'$, $J_0 z$ is interdefinable (in $\mathcal{M}$) with $[z]$. Therefore, if $(J_0 z_1, J_0 z_2)$ is an independent tuple (in $\mathcal{M}$), then the tuple $([z_1], [z_2])$ is also independent (in $\mathcal{M}$), and hence also $\mathcal{D}$-independent. For example, since the above tuple (a_1, b_1) is independent, we obtain that $\mathrm{RM}([f_1], [g_1]) = 2$. Similarly, we can see that clauses (1), (2) and (4) of Definition 10.2 are true. So in order to verify the diagram, we have to show that the dependencies implied by (3) of that definition remain valid in $\mathcal{D}$. That is, we have to show that $\{[k_3],[k_2],([f_2],[g_2])\}$ is $\mathcal{D}$-dependent, and similarly $\{[k_1],[k_3],([f_3],[g_3])\}$. Since the arguments are similar, we only prove in detail the latter case.

It will suffice to show:

Lemma 10.6. $k_3 \in \mathrm{acl}_{\mathcal{D}}([f_3],[k_1],[g_3])$.

Proof. The geometric idea behind it goes back to Eugenia Rabinovich's work [30]. Write $f_a = k_3$, with $a \in T_0$. By our assumptions, $J_0 f_a$ is generic in K over $\emptyset$. To simplify the notation, we denote the curves in $\mathcal{C}_0$ by $E_{a'}$, $a' \in T_0$, and the curves in $\mathcal{C}_1$ by C_g, $g \in T_1$.

Let X be a strongly minimal subset of $S := (E_{f_3} \circ E_{k_1}) \boxplus C_{g_3}$, representing the function $(f_3 \circ k_1) \oplus g_3$ (see Lemma 8.3 for the notation). We want to show that $a \in \mathrm{acl}_{\mathcal{D}}([S])$. Assume towards a contradiction that this is not the case.

Claim 1 *The projections of X on both coordinates is infinite and all isolated points of X are in* $\mathrm{acl}_{\mathcal{D}}([X])$.

Proof of Claim. By our choice of $\mathcal{C}_0$, the curves E_{f_3} and E_{k_1} are strongly minimal without isolated points. It follows that each of these curves has a finite intersection with every fiber in G^2, and thus as we already observed, $E_{f_3} \circ E_{k_1}$ has no isolated points. Also, the curve C_{g_3} has no isolated points. It is not hard to see that in this case their $\boxplus$-sum S has no isolated points either. It follows that all isolated points of X belong in $\mathrm{fr}(S \setminus X)$. Since $[X] \in \mathrm{acl}_{\mathcal{D}}(S)$ we from Theorem 4.9 that $\mathrm{fr}(S \setminus X) \subseteq \mathrm{acl}_{\mathcal{D}}([S])$.

Since X is a strongly minimal set representing a function $(f_3 \circ k_1) \oplus g_3$ its projection on the first coordinate is finite-to-one. Since the function has a non-zero jacobian at 0, it is non-constant and hence has an infinite projection in the second coordinate as well. ☐

It follows from Claim 1 that the set of isolated points of X is contained in a $\mathcal{D}$-algerbaic set $\mathcal{D}$-definable over $[X]$. Thus, by removing this $\mathcal{D}$-definable set there is no loss of generality in assuming that X contains no isolated points.

Note that the assumption that $a \notin \mathrm{acl}_{\mathcal{D}}([S])$ implies that $a \notin \mathrm{acl}_{\mathcal{D}}([X])$. We will ultimately show that this leads to a contradiction. Because E_a is strongly minimal, it follows that $E_a \cap X$ is finite.

Since T_0 is strongly minimal there exists some natural number n such that $|X \cap E_b| = n$ for all $b \in T_0$ which is $\mathcal{D}$-generic over $[X]$. Thus, the set

$$F = \{b \in T_0 : |X \cap E_b| < n\}$$

is a finite set, defined in $\mathcal{D}$ over $[X]$. We will show that $a \in F$, thus reaching a contradiction.

By our choice of a, $\dim(J_0 f_a/\emptyset) = 2 = \dim G$ and since $J_0 f_a \in \mathrm{dcl}(a)$ we also have $\dim(a/\emptyset) = 2$. Thus we also have $a \in \mathrm{dcl}(J_0 f_a) = \mathrm{dcl}(J_0 X)$.

Claim 2 *Let $\{x_1, \dots, x_k\} := X \cap E_a$. Then for every $i = 1, \dots, k$, either* $\dim(x_i/a) = 2$, *or* $x_i \in \mathrm{acl}_{\mathcal{D}}(\emptyset)$.

Proof. We consider the family

$$\mathcal{F}' = \{(E_{a_1} \circ E_{a_2}) \boxplus C_b : a_1, a_1 \in T_0\ ,\ b \in T_1\},$$

and for simplicity write the members of $\mathcal{F}'$ as $\{X_t : t \in T\}$. By our choice of X, there is t_0 generic in T such that X is a strongly minimal subset of X_{t_0}, so definable over $\mathrm{acl}_{\mathcal{D}}(t_0)$. We may now replace $\mathcal{F}'$ by another family of the same dimension, defined over $\emptyset$, such that the generic member of $\mathcal{F}'$ is strongly minimal and X belongs to the family. Without loss of generality, we still call this new family $\mathcal{F}$.

Thus $X = X_{t_0}$, with $\mathcal{F} = \{X_t : t \in T\}$ a $\mathcal{D}$-definable almost faithful family of plane curves, and t_0 generic in T over $\emptyset$. So, if we let $d = \mathrm{RM}(T)$ then $\dim(t_0/\emptyset) = 2d$. Because $a \in \mathrm{dcl}(J_0 X)$, we have $a \in \mathrm{dcl}(t_0)$. Also, by our underlying negation assumption, $\mathrm{RM}(a\, t_0/\emptyset) = d + 1$, hence $\mathrm{RM}(t_0/a) = \mathrm{RM}(t_0/\emptyset) = d$. Since $x_i \in E_a$, we have $\dim(x_i/a) \le 2$, so we assume that $\dim(x_i/a) \le 1$ and show that $a \in \mathrm{acl}_{\mathcal{D}}(\emptyset)$.

We first prove: $\mathrm{RM}(t_0/a, x_i) = \mathrm{RM}(t_0/\emptyset)$.

Clearly, $x_i \in \mathrm{dcl}(t_0, a)$, so since $a \in \mathrm{dcl}(t_0)$ we have $2d = \dim(t_0/\emptyset) = \dim(t_0, a, x_i/\emptyset)$, and therefore

$$2d - 2 = \dim(t_0/\emptyset) - \dim(a/\emptyset) = \dim(t_0/a) = \dim(t_0, x_i/a) =$$
$$= \dim(t_0/a, x_i) + \dim(x_i/a) \le \dim(t_0/a, x_i) + 1.$$

Hence,

$$\dim(t_0/a, x_i) \ge 2d - 1,$$

and therefore

$$\mathrm{RM}(t_0/a_i, x_i) = d = \mathrm{RM}(t_0/\emptyset).$$

We now consider the $\mathcal{D}$-type of t_0 over x_i. Since $\mathrm{RM}(t_0/x_i) = \mathrm{RM}(t_0/\emptyset)$ it follows that almost all curves X_t contain x_i. Because the family $\{X_t : t \in T\}$ is almost faithful, it follows that there are only finitely many points $x \in G^2$ which belong to almost all curves X_t, and therefore $x_i \in \mathrm{acl}_{\mathcal{D}}(\emptyset)$. This ends the proof of Claim 10. $\square$

We now return to the proof of Lemma 10.6. By Claim 2 we may assume that for $i = 1, \ldots, r$, we have $\dim(x_i/a) = 2$ and for $i = r+1, \ldots, k$, we have $x_i \in \mathrm{acl}_{\mathcal{D}}(\emptyset)$. Without loss of generality, $x_k = 0$.

In order to show that $a \in F$, we have to show that $k < n$. Towards that end, we will show that there are infinitely many $a' \in T_0$ such that $n = |X \cap E_{a'}| \ge k + 1$.

Let $U_1 \ldots U_r, U_k$ be pairwise disjoint open neighborhoods of $x_1, \ldots, x_r, x_k$, respectively. Since $x_{r+1}, \ldots, x_k$ are in $\mathrm{acl}_{\mathcal{D}}(\emptyset)$ then each of these points belongs to all but finitely many $E_{a'}$.

Because X has no isolated points we may apply proposition 9.12. We first apply Proposition 9.12 (2) to $0 = x_k$, and obtain $V \ni a$ such that for every $a' \in V$, $|E_{a'} \cap X \cap U_k| \ge 2$, counted with multiplicity. Because $J_0(E_a)$ is generic in K, it is attained at most finitely many times and hence by choosing V sufficiently small and $a' \in V$, $a' \neq a$, the curves $E_{a'}$ and X are not tangent at 0, so there exists $p \in E_{a'} \cap X \cap U_k$ which is different than 0. It follows that for all but finitely many $a' \in V$, $|E_{a'} \cap X \cap U_k| \ge 2$.

We now apply Proposition 9.12(1) to $x_1, \ldots, x_r$, and obtain a sub-neighborhood V' of a such that for every $a' \in V'$, and $i = 1, \ldots, r$, $E_{a'} \cap X \cap U_i \neq \emptyset$.

Summarizing, we see that for every $a' \neq a$ close to a, we have $|E_{a'} \cap X| \geq k+1$, and therefore a is in the finite set F which we defined above. This ends the proof of Lemma 10.6, and with it the proof of Proposition 10.5. □

We can now prove our main result:

Theorem 10.7. *Let $\mathcal{D} = \langle G; \oplus, \cdots \rangle$ be a strongly minimal expansion of a group G, interpretable in an o-minimal expansion $\mathcal{M}$ of a field R, with $\dim_{\mathcal{M}}(G) = 2$. If $\mathcal{D}$ is not locally modular then there exists in $\mathcal{D}$ an interpretable algebraically closed field $K \simeq R((\sqrt{-1})$, and there exists a K-algebraic group H, such that G and H are definably isomorphic in $\mathcal{D}$ and every $\mathcal{D}$-definable subset of H^n is K-constructible.*

Moreover, the structure $\mathcal{D}$ and the field K are bi-interpretable.

Proof. By Proposition 10.5, the configuration $\mathcal{Y}$ of $(*)$ is a field configuration in $\mathcal{D}$.

By Fact 10.3, an algebraically closed field K is interpretable in $\mathcal{D}$. By strong minimality there exists a $\mathcal{D}$-definable function $f : G \to K$ with finite fibres (this is standard using the symmetric functions on K). By [21, Lemma 4.6] (and using strong minimality of G) there exists a finite subgroup $F \leq G$ such that G/F is internal to K in the structure $\mathcal{D}$ (it is in fact, the proof of the lemma which provides us with the finite subgroup F). By [23, Theorem 3.1], every $\mathcal{D}$-definable subset of K^n is K-constructible, and therefore G/F is $\mathcal{D}$-definably isomorphic to a K-constructible group. By Weil-Hrushovski, it is therefore definably isomorphic to a K-algebraic group H (of algebraic dimension 1). It is known s that H, with all its induced K-algebraic structure, is bi-interpretable with K, and therefore the same holds for G/F, with all its induced $\mathcal{D}$-structure.

Finally, by Lemma 3.11, the structure $\mathcal{D}$ is also bi-interpretable with K. □

□

10.1. **Concluding remarks.** Note that as a result of the main theorem, the almost K-structure on G which we introduced in Section 9.1 turns out to be definably isomorphic to the K-structure of the algebraic group H. Thus, in this very special setting, we are able to mimic the classical result about the integrability of 2-dimensional almost complex curves.

Also, note that the general o-minimal version of Zilber's conjecture remains open for general strongly minimal structures whose universe has dimension 2. As we noted earlier, the more general conjecture, allowing underlying sets of arbitrary dimension is open even for reducts of the complex field.

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