Cell decomposition in P-minimal structures

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Joint work with Pablo Cubides Kovacsics & Eva Leenknegt

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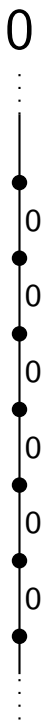
Tree pictures of the p-adic numbers



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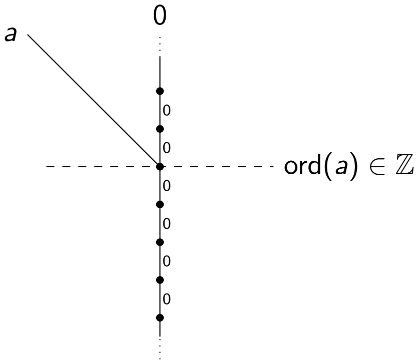

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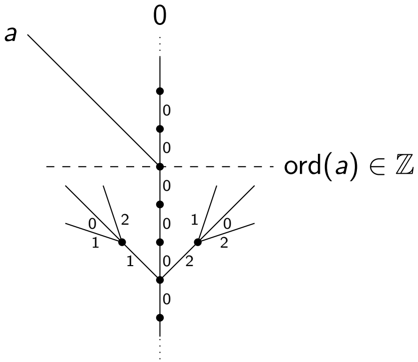

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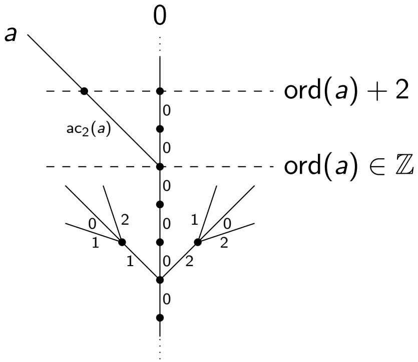

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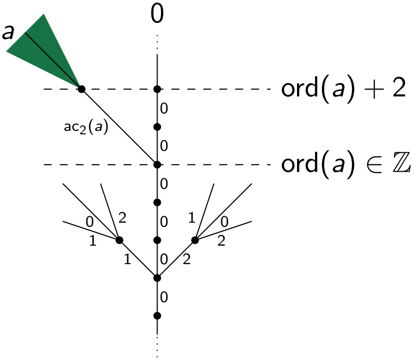

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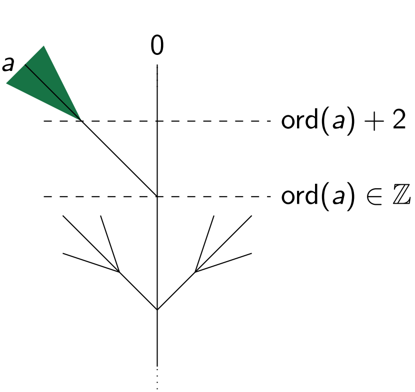

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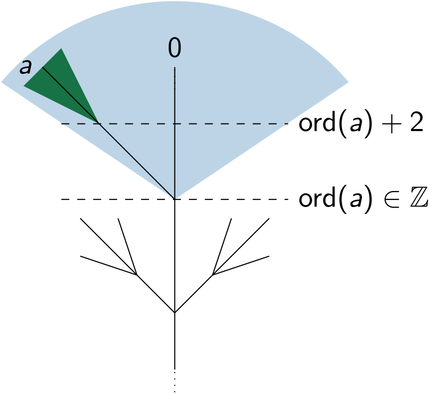

Definitions

Let p be a prime number. $\mathbb{Q}_p$ is the field of the *p-adic numbers*.
$\mathbb{Q}_p^\times = \{\sum_{i=k}^{\infty} a_i p^i \mid k \in \mathbb{Z}, a_i \in \{0, 1, \ldots, p-1\}, a_k \neq 0\}$.

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For each $m \in \mathbb{N}_0$,

$$\mathrm{ac}_m : \mathbb{Q}_p \to \mathbb{Z}_p/p^m\mathbb{Z}_p : a \mapsto \begin{cases} \sum_{i=0}^{m-1} a_{i+k} p^i & \text{if } a = \sum_{i=k}^{\infty} a_i p^i; \\ 0 & \text{if } a = 0. \end{cases}$$

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Clopen balls $B_\gamma(a) := \{x \in \mathbb{Q}_p \mid \mathrm{ord}(x - a) \geq \gamma\}$, $\gamma \in \mathbb{Z}$.

Semialgebraic sets

Let K be a field and $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$.

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o-minimality

- K is a *real closed field* if it is $\mathcal{L}_{\text{ring}}$-elementary equivalent to $\mathbb{R}$.
- $x < y \Leftrightarrow \exists z(x + z^2 = y)$.
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P-minimality

- K is a *p-adically closed field* if it is $\mathcal{L}_{\text{ring}}$-elementary equivalent to a finite field extension of $\mathbb{Q}_p$.
- $x \in P_k \Leftrightarrow \exists y \neq 0(y^k = x)$.
- $\mathcal{L}_{\text{Mac}} := \mathcal{L}_{\text{ring}} \cup \{P_k\}_{k>0}$.
- $(K, \mathcal{L}_{\text{Mac}})$ has QE. (Prestel & Roquette, '84)

Examples of semialgebraic sets

$X \subseteq \mathbb{R}$ semialgebraic $\Rightarrow X$ is a Boolean combination of sets of the form

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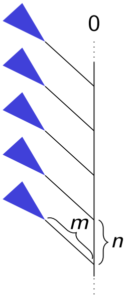
Instead of $(P_k)_{k>0}$ we prefer $(Q_{m,n})_{m,n>0}$, where

$Q_{m,n} := \{x \in \mathbb{Q}_p^\times \mid \mathrm{ord}(x) \equiv 0 \mod n,\ \mathrm{ac}_m(x) = 1\}$.



o-minimality and P-minimality

Definition

Let $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ and let K be a real closed field. The structure $(K, \mathcal{L})$ is *o-minimal* if every $\mathcal{L}$-definable subset $X \subseteq K$ is also $\mathcal{L}_{\text{ring}}$-definable.

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Definition

Let $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$ and let K be a p-adically closed field. The structure $(K, \mathcal{L})$ is *P-minimal* if for every $(K', \mathcal{L}) \equiv (K, \mathcal{L})$, every $\mathcal{L}$-definable subset $X \subseteq K'$ is also $\mathcal{L}_{\text{ring}}$-definable.

Classical cells

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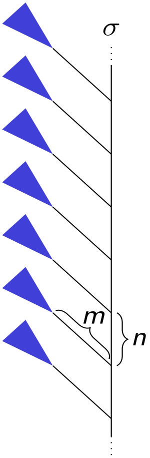

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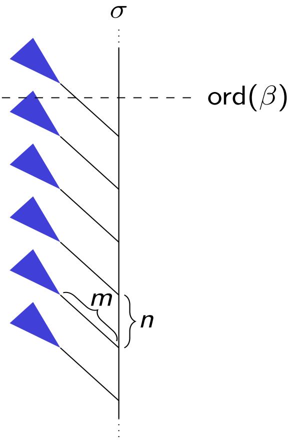

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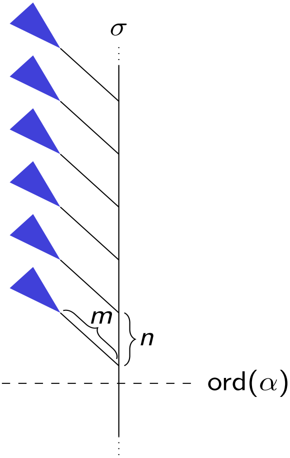

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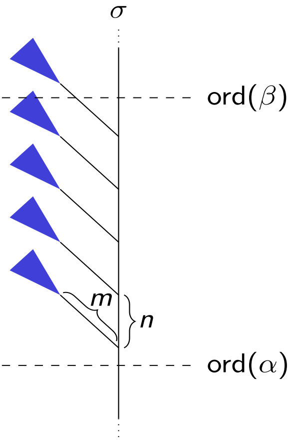

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Classical cells

$$C = \left\{ (s,x) \in S \times K \,\middle|\, \begin{array}{l} \mathrm{ord}(\alpha(s)) \square_1 \mathrm{ord}(x-\sigma(s)) \square_2 \mathrm{ord}(\beta(s)), \\ x-\sigma(s) \in \lambda Q_{m,n} \end{array} \right\}$$

S is a definable set, $\alpha, \beta : S \to K^\times$ and $\sigma : S \to K$ are definable functions, $\square_i$ can be either $<$ or 'no condition', $n, m \in \mathbb{N}_0$ and $\lambda \in K$.
We call σ the *center* of the cell.

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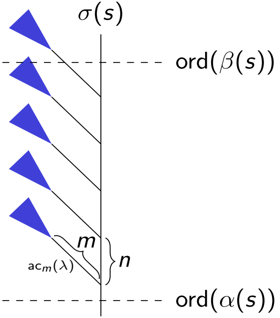
$\lambda = 0.$



$\lambda \neq 0$ and $\square_1 = \square_2 = <$.

Definable Skolem functions

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A structure $(K, \mathcal{L})$ has *definable Skolem functions* if for every definable $X \subseteq K^{r+1}$ there exists a definable section $\sigma : \pi(X) \to K$. This means that $(x, \sigma(x)) \in X$ for all $x \in \pi(X)$. By $\pi : K^{r+1} \to K^r$ we mean the projection onto the first r coordinates.

Theorem (Mourgues, '09)

A P-minimal structure $(K, \mathcal{L})$ has cell decomposition (with classical cells) if and only if it has definable Skolem functions.

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Remarks:

- All o-minimal structures $(K, \mathcal{L})$ have definable Skolem functions (and cell decomposition).
- Cubides and Nguyen have found a P-minimal structure that does not have definable Skolem functions!

Centers of cells

Cell condition: $C(s,y,x) = \begin{array}{l} \mathrm{ord}(\alpha(s)) \square_1 \mathrm{ord}(x-y) \square_2 \mathrm{ord}(\beta(s)) \\ \wedge (x-y) \in \lambda Q_{m,n} \wedge s \in S. \end{array}$

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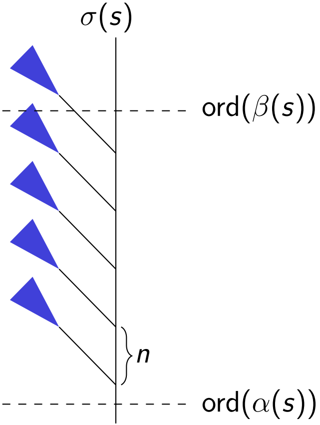

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$$C^\sigma := \{(s,x) \in S \times K \mid C(s,\sigma(s),x)\}.$$

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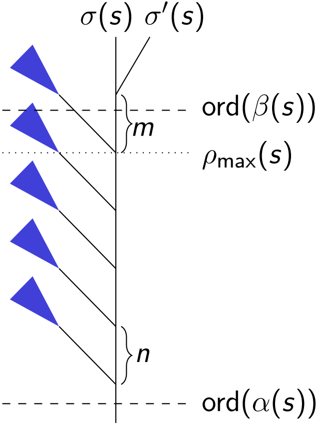

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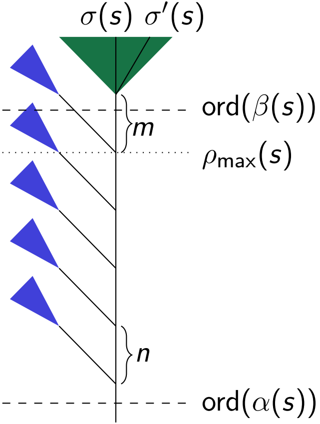

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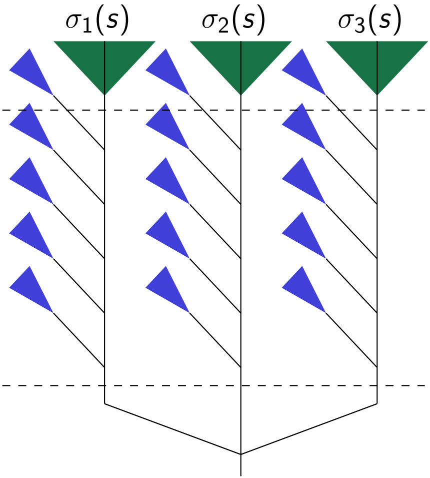
The possible centers are the sections of

$$\Sigma := \bigcup_{s \in S} \{s\} \times B_{\rho_{\max}(s)+m}(\sigma(s)) \subseteq S \times K.$$

Then $C^\sigma = \{(s,x) \mid \exists \sigma' \in \Sigma_s \; C(s,\sigma',x)\}$.

Inseperably clustered cells



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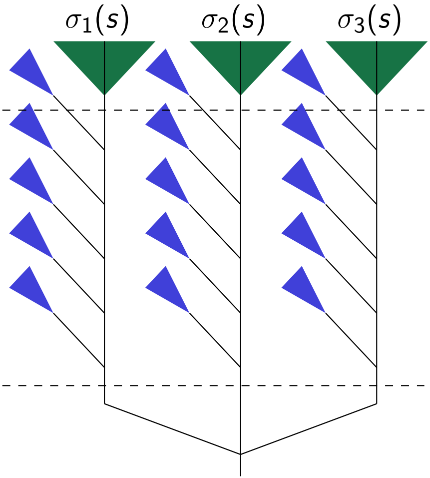

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There exists a definable $\Sigma \subseteq S \times K$ for which each Σ_s is a union of k balls of the same size and such that

$$X = \{(s, x) \mid \exists \sigma' \in \Sigma_s \; C(s, \sigma', x)\}.$$