Research Proposal

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My main research area is model theory and its connections to algebra and geometry. I work in o-minimality, which provides a rigid framework to study real algebraic geometry. I have proved structure theorems for semi-linear and semi-bounded groups. I am planning to use my techniques in the analysis of groups definable in other settings, such as in expansions of real closed fields with o-minimal open core.

1. Description

1.1. **Context.** The study of groups definable in first-order structures has been a core subject in model theory spanning over at least a period of thirty years. Classical examples are algebraic groups, which are definable in algebraically closed fields, and compact real Lie groups, which are definable in o-minimal expansions of the real field. The importance of definable groups in model theory is two-fold. On the one hand, definable groups are present whenever non-degenerate phenomena occur inside a structure and their study has played a prominent role in Shelah's classification theory. On the other hand, the model-theoretic treatment of groups definable in known mathematical structures has found striking applications outside model theory, such as Hrushovski's proof of the function field Modell-Lang conjecture in all characteristics ([11]).

The main line of my research concerns the study of groups definable in o-minimal structures, henceforth called 'o-minimal groups'. The standard example of an o-minimal structure is a real closed field. O-minimal structures provide a rich, yet tame, model-theoretic setting where definable sets enjoy many of the nice topological properties that hold for semi-algebraic sets. For example, a topological notion of *dimension* can be defined for every definable set. It is often said that o-minimality is the correct formalization of Grothendieck's 'topologie modérée' ([2]).

O-minimal groups vividly resemble real Lie groups. The starting point for their study was Pillay's theorem in [18] that every such group admits a definable manifold topology that makes it into a topological group. Since then, an increasing number of theorems reinforced the resemblance of o-minimal groups with real Lie groups, culminating in the solution of Pillay's Conjecture (PC) and Compact Domination Conjecture (CDC) in recent years. (PC) can be viewed as a non-standard analogue of Hilbert's fifth problem. Consider, for example, an abelian variety in characteristic zero. If it is defined over the field of complex numbers, then it is proved by analytic methods that it is a Lie group. If it is defined over parameters in a bigger algebraic closed field, then (PC) establishes its connection to a Lie group in a purely model theoretic language. In its simplified form, (PC) says that every definably connected, definably compact o-minimal group G admits a surjective homomorphism π onto a real Lie group, whose dimension (as a Lie group) is equal to the o-minimal dimension of G. (CDC) carries this connection even further, formalizing the intuition that the homomorphism π from (PC) is a kind of an intrinsic 'standard part map'.

1.2. **My research.** Depending on what kind of ambient o-minimal structure we are dealing with, different methods for analyzing definable groups have been developed. The main dichotomy has been whether the structure expands a real closed field or not. Let us recall some definitions. An *o-minimal* structure $\mathcal{M} = \langle M, <, \dots \rangle$ is a structure with a dense linear order $<$ such that every definable subset of M is a finite union of open intervals and points. The standard setting where definable groups have been studied is that of an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group (although, newly in [9], the assumption of the ambient group structure is being removed). It follows from the Trichotomy Theorem that we can have exactly the following three cases:

(1) (Field case) $\mathcal{M} = \langle M, <, +, \cdot, 0, 1, \dots \rangle$ expands a real closed field.
(2) (Semi-bounded case) $\mathcal{M}$ does not expand a real closed field, but it contains a definable real closed field whose domain is a bounded interval $I \subseteq M$.
(3) (Linear case) No real closed field is definable.

The typical example of a linear structure is an ordered vector space $\mathcal{M} = \langle M, <, +, 0, \{d\}_{d\in D}\rangle$ over an ordered division ring D. The model theory of this structure becomes important when studying definable groups in algebraically closed valued fields (see Hrushovski and Loeser ([12]).

An important example of a semi-bounded structure is the expansion $\mathcal{B}$ of the real ordered vector space $\mathbb{R}_{vect} = \langle \mathbb{R}, <, +, 0, \{d\}_{d\in\mathbb{R}}\rangle$ by all bounded semi-algebraic sets. Every bounded interval in $\mathcal{B}$ admits the structure of a definable real closed field. For example, the field structure on $(-1, 1)$ induced from $\mathbb{R}$ via the semi-algebraic bijection $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is definable in $\mathcal{B}$. It is known that $\mathcal{B}$ is the unique structure that lies strictly between $\mathbb{R}_{vect}$ and the real field.

When $\mathcal{M}$ expands a real closed field, a rich machinery from *o-minimal algebraic topology* is at our disposal. For example, the triangulation theorem is known to hold in this case, giving rise to (co)homology theory. Using this heavy machinery, (PC) and (CDC) were solved in [14], [13] and [15]. On the contrary, if $\mathcal{M}$ does not expand a field, the triangulation theorem fails. I illustrated a serious consequence of this failure in [5] by an example of a semi-linear group which cannot be definably, homeomorphically embedded in the affine space.

Under the lack of the machinery from algebraic topology, I have initiated a *straightforward* analysis of definable groups in the linear and semi-bounded cases. The analysis consists of two steps:

Step I. Study the group behavior locally, around suitable generic elements.
Step II. Extend the local analysis to a global one.

The structure theorems for definable groups that are obtained from this approach have one aspect in common: each time a *lattice* is recovered which captures some significant invariant of the group at hand. These structure theorems implied, in particular:

- (PC) and (CDC) in the linear case ([10] and [6], respectively)
- (CDC) in the semi-bounded case ([8]) − (PC) was already solved in [16].

In the next section, I analyze the two new elements of this program: lattices and the local analysis mentioned in Step I. I propose to apply this program to attack problems in general o-minimal theories, and even beyond. In the last section, I focus on two specific projects in this direction.

2. The two main elements

2.1. **Lattices.** A successful way to describe uniformly the results of this program has been via the use of the notion of a lattice. Classically, a lattice L in $\mathbb{R}^n$ of rank m $(\leq n)$ is a subgroup of $\langle \mathbb{R}^n, +\rangle$ which satisfies any of the following equivalent statements:

(1) L is discrete and spans an m-dimensional subspace of $\mathbb{R}^n$ (over $\mathbb{R}$).
(2) L is generated by m elements of $\mathbb{R}^n$ and the quotient group $\mathbb{R}^n/L$ is a connected Lie group (equipped with the quotient topology).

As discreteness is too weak a condition in the non-standard setting, our focus is on generalizing (2) to an arbitrary o-minimal structure. In the next definitions, we assume that $\mathcal{M}$ is a sufficiently saturated first-order structure, not necessarily o-minimal. We refer the reader to [17] for the notion of a $\bigvee$-definable group or map.

Definition 2.1. Given a $\bigvee$-definable group $\mathcal{U}$ and a normal subgroup $L \subseteq \mathcal{U}$, the quotient group $\mathcal{U}/L$ is called *definable* if there is a definable group K and a surjective $\bigvee$-definable homomorphism $\phi : \mathcal{U} \to K$ whose kernel is L. We write $K = \mathcal{U}/L$.

Definition 2.2. Let $\mathcal{U}$ be a $\bigvee$-definable group. A subgroup $L \subseteq \mathcal{U}$ is called a *lattice in* $\mathcal{U}$ *of rank* m if it is generated by m $\mathbb{Z}$-independent elements and the quotient $\mathcal{U}/L$ is definable.

For the rest of this research proposal, unless stated otherwise, we work in an o-minimal expansion of an ordered group $\mathcal{M} = \langle M, <, +, \ldots \rangle$, and denote by G an abelian, definably connected definable group of dimension n.

Definition 2.2 suggests that a lattice presupposes a definable group. I have shown, in collaboration with Edmundo, that the converse is also true: given a definable group, we can always recover a lattice.

Theorem 2.3. [3] *There is a divisible torsion-free $\bigvee$-definable group $\mathcal{U}$ and a lattice L in $\mathcal{U}$ such that $G = \mathcal{U}/L$.*

This theorem is very general in nature and was proved imitating the classical proof of the existence of universal covers. Indeed, $\mathcal{U}$ is the 'o-minimal universal cover' of G and the lattice L is isomorphic to the fundamental group of G. Observe, however, that no information is given about how $\mathcal{U}$ or L are related to the ambient o-minimal structure. One motivation for going through the local analysis, described in the next section, is to recover more information about $\mathcal{U}$ and L in the linear and semi-bounded cases.

As a side remark, my collaborators and I have recently proved the existence of universal covers for definable o-minimal *manifolds*, in general ([4]).

2.2. **Local analysis.** The central technique in the local analysis of o-minimal groups is to obtain a sharp description of the group operation around suitable 'generic' elements. Loosely speaking, the generic elements are used to 'pick out' the nice behavior of the definable group at hand.

The fundamental notion here is that of a *pregeometry.* Associated to a pregeometry is the notion of generic elements and that of dimension. The classical pregeometry in model theory is the one based on the algebraic closure operator *acl*.

Linear case. A result from my Ph.D. thesis is that every *acl*-generic element of a definable group G is contained in an open definable set on which the group operation is locally isomorphic (in fact, coincides!) with the addition $+$ in M^n. The effect of this result to the global picture is two-fold:

- The universal cover $\mathcal{U}$ of G from Theorem 2.3 is a subgroup of $\langle M^n, + \rangle$
- The rank of L equals the *compact dimension of* G; namely, $\operatorname{rank}(L) = \dim(G/H)$, where H is a maximal torsion-free definable subgroup of G.

That is, the lattice L 'captures' the compact dimension of G.

Semi-bounded case. The local behavior of the definable group here cannot be as linked to the ambient group structure as in the linear case, due to the presence of the bounded field. To pick out the correct nice behavior of the definable group, I introduced in [7] a new pregeometry based on the *short* closure operator, defined below. Let us call an interval $J \subseteq M$ *short* if there is a definable real closed field whose domain is J. The *short closure operator* is defined as:

$$shcl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that}$$

$$\phi(\mathcal{M}, \bar{b}) \text{ is a } \textbf{short} \text{ interval and } \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

I then showed in [7] that every *shcl*-generic element of a definable group G is contained in a definable set V of *schl*-dimension equal to the *schl*-dimension of G, such that on V the group operation is locally isomorphic with the addition $+$ in M^k, where k is the *schl*-dimension of G. The effect of this result to the global picture is again two-fold ([8]):

- The universal cover $\mathcal{U}$ of G is an extension of a $\bigvee$-definable group $\mathcal{K}$ that lives in some o-minimal expansion of a real closed field, by a $\bigvee$-definable open subgroup $\mathcal{H}$ of $\langle M^k, + \rangle$.

(1) $$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{U} \longrightarrow \mathcal{K} \longrightarrow 0$$

- We can replace the previous $\mathcal{K}$ by a **definable** group $\overline{K}$, so that the resulting extension $\overline{\mathcal{U}}$ contains a lattice $\overline{L}$ with $G = \overline{\mathcal{U}}/\overline{L}$, and $\operatorname{rank}(\overline{L}) = k$.

$$0 \longrightarrow \mathcal{H} \longrightarrow \overline{\mathcal{U}} \longrightarrow \overline{K} \longrightarrow 0 \tag{2}$$

That is, the lattice $\overline{L}$ captures the *shcl*-dimension of G. The first item solves a conjecture from [16]. The second item was used in the solution of (CDC) in [8].

Varying the pregeometry by considering different closure operators is potentially a very strong tool for addressing problems in different settings. I have put this into practice in my work on semi-bounded groups, but a similar approach also appears in other settings (see for example [9] and [1]). In my first proposed project below, I intend to exemplify this potential in a specific research problem.

3. Projects

3.1. **Beyond the o-minimal setting.** My ultimate goal here is to analyze groups definable in expansions of a real closed field that have o-minimal open core. These structures include dense pairs, as well as expansions of the real field $\overline{\mathbb{R}}$ by a multiplicative subgroup of $\overline{\mathbb{R}}^{\times}$ satisfying the Mann Property.

I present my project in the following setting. Let $\mathcal{M}$ be a real closed field and $P \subseteq M$. We call P *large* if there is some m and a definable function $f : M^m \to M$ such that $f(P^m)$ contains an open interval in M. We call P *small* if it is not large. In [1] the 'small closure operator'

$$\begin{gathered} smcl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and a formula } \phi(x, \bar{y}), \text{ such that} \\ \phi(\mathcal{M}, \bar{b}) \text{ is a } \textbf{small} \text{ set and } \mathcal{M} \vDash \phi(a, \bar{b})\}. \end{gathered}$$

was shown to define a pregeometry, under certain conditions. Together with Günaydin, I propose:

Conjecture 1a. If an expansion $\langle \mathcal{M}, P \rangle$ of a real closed field by a small predicate has o-minimal open core, then *smcl* is a pregeometry.

At the local level, we propose:

Conjecture 1b. Given a definable group G in this structure, every *smcl*-generic element is contained in a definable set V of *smcl*-dimension equal to the *smcl*-dimension of G, such that on V the group operation is the restriction of some semi-algebraic function.

We anticipate this analysis will have the following global effect:

Conjecture 1c. Under some condition of 'connectedness' for G, we can write it as a quotient by lattice

$$G = \overline{\mathcal{U}}/\overline{L},$$

for some divisible $\bigvee$-definable group $\mathcal{U}$, which is an extension of a $\bigvee$-definable group $\mathcal{K}$ of *smcl*-dimension 0, by a group $\mathcal{H}$, $\bigvee$-definable in $\mathcal{M}$, whose dimension equals *smcl*-dimension of G,

$$0 \longrightarrow \mathcal{H} \longrightarrow \overline{\mathcal{U}} \longrightarrow \overline{K} \longrightarrow 0$$

and a lattice L of rank equal to the *smcl*-dimension of G.

3.2. **Lattices and the divisibility conjecture.** The theory of lattices in arbitrary o-minimal structures has not yet been studied. Motivated by the above structure theorems, I believe that it has great potential in attacking problems for definable or $\bigvee$-definable groups in general model theory. It is therefore worth studying lattices in their own right.

In the analysis of semi-bounded groups, the passage from the exact sequence (1) to (2) was done by finding a lattice Λ in $\mathcal{K}$, resulting to the definable $\overline{K} = \mathcal{K}/\Lambda$. This suggests the following conjecture.

Conjecture 2a. *Let $\mathcal{U}$ be $\bigvee$-definable connected abelian group, generated by a definable set. Then $\mathcal{U}$ contains a lattice.*

This statement is very strong. By [8], it is equivalent to a number of other statements, such as the existence of $\mathcal{U}^{00}$, and the existence of a definable subset of $\mathcal{U}$ whose countably many translates cover $\mathcal{U}$. Surprisingly, it also implies a pure algebraic condition on $\mathcal{U}$, namely, that $\mathcal{U}$ is divisible.

Conjecture 2b. Let $\mathcal{U}$ be $\bigvee$-definable connected abelian group. Then $\mathcal{U}$ is divisible.

In order to attack Conjecture 2a, I have introduced a new notion that counts how 'non-definable' the group $\mathcal{U}$ is. A $\bigvee$-definable subgroup A of $\mathcal{U}$ is called *compatible in* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap A$ is a definable set. It is easy to see that a lattice L in $\mathcal{U}$ is compatible. On the other hand, a compatible subgroup of $\mathcal{U}$, isomorphic to some $\mathbb{Z}^m$, can always be extended to a lattice.

Definition 3.1. The $\bigvee$*-dimension* of $\mathcal{U}$, denoted by $\mathrm{vdim}(\mathcal{U})$, is the maximum k such that $\mathcal{U}$ contains a compatible subgroup isomorphic to $\mathbb{Z}^k$, if such k exists, and ∞, otherwise.

I have reduced Conjecture 2a to the following equivalent one: For every $\mathcal{U}$, as in Conjecture 2a, the following hold:

(1) If $\mathcal{U}$ is not definable, then $\mathrm{vdim}(\mathcal{U}) > 0$.
(2) $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$. (In particular, $\mathrm{vdim}(\mathcal{U})$ is finite.)

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