Degree bounds for sum of squares representations

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Fix a collection $h_1, \dots, h_r \in \mathbb{R}[x_1, \dots, x_n]$ of polynomials with real coefficients. Let M be the set of all $f \in \mathbb{R}[x_1, \dots, x_n]$ that admit a representation

$$f \;=\; \sum_{i,j} p_{ij}^2 h_j \qquad (*)$$

where $p_{ij} \in \mathbb{R}[x_1, \dots, x_n]$ for all i, j. We ask whether the degree of such a representation can be bounded uniformly in terms of the degree of f alone. More precisely, let $\delta(f)$ be the smallest number $m \geq 0$ such that there exists an identity $(*)$ in which $\deg(p_{ij}^2 h_j) \leq m$ for all i, j. For given degree $d \geq 1$ we say that M is stable in degrees $\leq d$ if the supremum

$$\sup\{\delta(f)\colon f \in M,\ \deg(f) \leq d\}$$

is finite. Deciding whether or not this property holds is often not easy. Even when the supremum is known to be finite, one does not necessarily know any concrete upper bound. On the other hand, this notion has applications to moment problems and to semidefinite optimization. We plan to give an example-oriented introduction to the concept of stability and discuss some open questions along the way.