On the Antichain Tree Property

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Abstract

We are investigating a new model theoretical tree property, which is called the antichain tree property (ATP). In this poster session we list what we have found about ATP, including some combinatorial observations, and algebraic examples.

Notations 1

Let κ and λ be cardinals.

- By κ^λ, we mean the set of all functions from λ to κ.
- By ${}^{\lambda>}\kappa$, we mean $\bigcup_{\alpha<\lambda}\kappa^\alpha$ and call it a *tree*. If $\kappa = 2$, we call it a *binary tree*. If $\kappa \geq \omega$, then we call it an *infinite tree*.
- By $\emptyset$ or $\langle\rangle$, we mean the empty string in ${}^{\lambda>}\kappa$, which means the empty set.

Let $\eta, \nu \in {}^{\lambda>}\kappa$.

- By $\eta \trianglelefteq \nu$, we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say η and ν are *comparable*.
- By $\eta \perp \nu$, we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We say η and ν are *incomparable* if $\eta \perp \nu$.
- By $\eta \wedge \nu$, we mean the maximal $\xi \in {}^{\lambda>}\kappa$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
- By $l(\eta)$, we mean the domain of η.
- By $\eta <_{lex} \nu$, we mean that either $\eta \trianglelefteq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.
- By $\eta^\frown\nu$, we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $X \subseteq {}^{\lambda>}\kappa$.

- We say X is an *antichain* if the elements of X are pairwise incomparable.
- We say an antichain X is *maximal* if there is no antichain Y in ${}^{\lambda>}\kappa$ such that $X \subsetneq Y$.

Definition 2

Let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

- We say $\varphi(x, y)$ has the *tree property* (TP) if there exists a tree-indexed set $(a_\eta)_{\eta \in {}^{\omega>}\omega}$ of parameters and $k \in \omega$ such that
  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in {}^\omega\omega$ (path consistency), and
  $\{\varphi(x, a_{\eta^\frown i})\}_{i\in\omega}$ is k-inconsistent for all $\eta \in {}^{\omega>}\omega$, *i.e.*, any subset of $\{\varphi(x, a_{\eta^\frown i})\}_{i\in\omega}$ of size k is inconsistent.
- We say $\varphi(x, y)$ has the *tree property of the first kind* (TP_1) if there is a tree-indexed set $(a_\eta)_{\eta\in{}^{\omega>}\omega}$ of parameters such that
  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in {}^\omega\omega$, and
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in {}^{\omega>}\omega$.
- We say $\varphi(x, y)$ has the *tree property of the second kind* (TP_2) if there is an array-indexed set $(a_{i,j})_{i,j\in\omega}$ of parameters such that
  $\{\varphi(x, a_{n,\eta(n)})\}_{n\in\omega}$ is consistent for all $\eta \in {}^\omega\omega$, and
  $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k}))\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.
- We say $\varphi(x, y)$ has the *1-strong order property* (SOP_1) if there is a binary-tree-indexed set $(a_\eta)_{\eta\in{}^{\omega>}2}$ of parameters such that
  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in 2^\omega$,
  $\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in {}^{\omega>}2$.
- We say $\varphi(x, y)$ has the *2-strong order property* (SOP_2) if there is a binary-tree-indexed set $(a_\eta)_{\eta\in{}^{\omega>}2}$ of parameters such that
  $\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in 2^\omega$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in {}^{\omega>}2$.
- We say $\varphi(x, y)$ has the *antichain tree property* (ATP) if there is a binary-tree-indexed set $(a_\eta)_{\eta\in{}^{\omega>}2}$ of parameters such that
  $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent for all antichain $X \subseteq 2^\omega$,
  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \trianglelefteq \nu \in {}^{\omega>}2$.
- We say a theory has TP if there is a formula having TP with respect to its monster model of the theory. Sometimes we say that the theory is TP, and we call the theory an TP theory. We define TP_1 theory, TP_2 theory, SOP_1 theory, SOP_2 theory, and ATP theory in the same manner.
- We say a theory is NTP if the theory is not TP, and we call the theory NTP theory. We define NTP_1 theory, NTP_2 theory, NSOP_1 theory, NSOP_2 theory, and NATP theory in the same manner.

The following facts are well known (*cf.* [3], [4], [6], and [7]).

Fact 3

- A theory has TP_1 if and only if it has SOP_2.
- A theory has TP if and only if it has TP_1 or TP_2.
- A theory has TP if and only if it has SOP_1 or TP_2.
- If a theory has SOP_2, then it has SOP_1.

And by definition of ATP, it is easy to check that:

Proposition 4 [1, Proposition 4.4, 4.6]

If a theory has ATP, then it has TP_2 and SOP_1.

So we have the following diagram:

$$\begin{array}{ccccccccc} & & \mathrm{NIP} & \Rightarrow & \mathrm{NTP}_2 & \Rightarrow & \mathrm{NATP} & & \\ & & \Uparrow & & \Uparrow & & \Uparrow & & \\ \text{stable} & \Rightarrow & \text{simple} & \Rightarrow & \mathrm{NSOP}_1 & \Rightarrow & \mathrm{NSOP}_2 & \Rightarrow & \cdots . \end{array}$$

Roughly speaking, NATP can be regarded as a common extension of NTP_2 and NSOP_1 (NATP=NTP_2+NSOP_1). On the other hand, NATP is a dual concept of NSOP_2 (NATP$\perp$$\mathrm{NSOP}_2$) in the following sense:

Remark 5

- If $\varphi(x, y)$ witnesses ATP with $(a_\eta)_{\eta\in 2^{<\omega}}$, then for each $X \subseteq 2^{<\omega}$, the set $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is pairwise **incomparable**.
- If $\varphi(x, y)$ witnesses SOP_2 with $(a_\eta)_{\eta\in 2^{<\omega}}$, then for each $X \subseteq 2^{<\omega}$, the set $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent if and only if X is pairwise **comparable**.

Example 6 (examples of ATP) [2, Example 4.31, 4.33]

Skolem arithmetic $(\mathbb{N}, \cdot)$ has ATP. Let $\varphi(x, y)$ be a formula saying "x divides y and $x \neq 1$." For an arbitrary positive integer n, let S_n be the set of all maximal antichain in $2^{<n}$ and $\alpha_n := |S_n|$. Let $\{X_1, ..., X_{\alpha_n}\}$ be an enumeration of S_n and p_i be the i-th prime numbers for each $0 < i \leq |S_n|$. For each $\eta \in 2^{<n}$, let $a_\eta := \prod_{\eta\in X_k\in S_n} p_k$. Then $\varphi(x, y)$ and $(a_\eta)_{\eta\in 2^{<n}}$ satisfy the conditions of ATP. By compactness, $(\mathbb{N}, \cdot)$ has ATP. By the same argument, one can construct antichain trees in any given Atomless Boolean algebras (ABA). So ABA also has ATP.

Combinatorial Observations

Theorem 7 (One variable theorem for ATP) [2, Theorem 3.17]

If a theory has ATP, then there exists a formula $\varphi(x, y)$ and a set of parameters $(a_\eta)_{\eta\in 2^{<\omega}}$ with $|a_\eta| = |y|$ such that φ witnesses ATP with $(a_\eta)_{\eta\in 2^{<\omega}}$, and $|x| = 1$.

Lemma 8 (NATP is preserved under taking disjunction of formulas) [2, Lemma 3.18]

For formulas $\varphi(x, y)$ and $\psi(x, z)$, if $\varphi \vee \psi$ witnesses ATP, then φ witnesses ATP or ψ witnesses ATP.

Definition 9

For a formula $\varphi(x; y)$ and a positive integer $k \geq 2$, we say that $\varphi(x; y)$ has the *k-antichain tree property (k-ATP)* if there is a tree of parameters $(a_\eta)_{\eta\in 2^{<\omega}}$ such that

- For any antichain $X \subset 2^{<\omega}$, $\{\varphi(x; a_\eta) : \eta \in X\}$ is consistent.
- For any pairwise comparable $\eta_0, \ldots, \eta_{k-1} \in 2^{<\omega}$, $\{\varphi(x; a_{\eta_i}) : i < k\}$ is inconsistent.

We say that T has k-ATP if there is a formula having k-ATP.

Lemma 10 (k-ATP $\Leftrightarrow$ ATP) [2, Lemma 3.20]

A complete theory T has k-ATP for some $k \geq 2$ if and only if T has ATP.

Theorem 11 (A criterion for NATP) [2, Theorem 3.25]

Assume κ and κ' are cardinals such that $2^{|T|} < \kappa < \kappa'$ and $\mathrm{cf}(\kappa) = \kappa$. The following are equivalent.

- T is NATP.
- For any strongly indiscernible tree $(a_\eta)_{\eta\in 2^{<\kappa'}}$ and finite tuple b, there exist $\rho \in 2^\kappa$ and b' such that
  (i) $(a_{\rho^\frown 0^i})_{i<\kappa'}$ is indiscernible over b',
  (ii) $b \equiv_{a_\rho} b'$.

Algebraic Examples of ATP/NATP

Definition 12

A graph is called *nice* if

- for any vertex a and b, there exists vertex $c \neq a, b$ such that c is adjacent to a but not to b,
- the graph has no cycle of order 3 nor 4.

Fact 13 [5, Theorem 5.5.1]

Any structure in a countable language is bi-interpretable with a nice graph.

Definition 14

Fix an odd prime number p. Given a nice graph A, the Mekler group $G(A)$ of A is generated freely in the variety of 2-nilpotent groups of exponent p by the vertices of A by imposing that two generators commute if and only if they are connected by an edge in A.

Theorem 15 (Mekler's Construction preserves NATP) [2, Theorem 4.16]

For any infinite nice graph A, $\mathrm{Th}(G(A))$ is NATP if and only if $\mathrm{Th}(A)$ is NATP.

Definition 16

Let $\mathcal{L}_G$ be a ω-sorted language consisted with the following relations:

- a binary relation $\leq_{n,m}$ for $n \geq m$;
- a binary relation $C_{n,m}$ for $n \geq m$;
- a ternary relation P_n for $n \in \omega$.

For a profinite group G, the *complete system* of G is given by an $\mathcal{L}_G$-structure, as follows:

- For each $n \in \omega$, the sort n is the disjoint union of G/N for each open normal subgroup N of G with $[G : N] \leq n + 1$ so that the sort 0 consists of the single element G.
- The relations $\leq_{n,m}$, C, and P_n are interpreted as follows:
  (i) $gN \leq_{n,m} hM \Leftrightarrow N \subset M$,
  (ii) $C(gN, hM) \Leftrightarrow gN \subset hM$,
  (iii) $P_n(g_1N_1, g_2N_2, g_3N_3) \Leftrightarrow N_1 = N_2 = N_3(=: N),\ g_1g_2N = g_3N$,
  for all $gN \in n$, $hM \in m$, and $g_1N_1, g_2N_2, g_3N_3 \in n$.

For a field K, we write $SG(K)$ for the complete system of the absolute Galois group $G(K^s/K)$, where K^s is the separable closure of K.

Theorem 17 (Chatzidakis' Criterion for PAC fields) [2, Theorem 4.20]

For a PAC field F, $\mathrm{Th}(F)$ is NATP if $\mathrm{Th}(SG(F))$ is NATP.

Definition 18

Let $\mathcal{K} = (K, \Gamma, k, v : K \to \Gamma, ac : K \to k)$ be a henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language $\mathcal{L}_{Pas} = \mathcal{L}_K \cup \mathcal{L}_{\Gamma,\infty} \cup \mathcal{L}_k \cup \{v, ac\}$, where $\mathcal{L}_K$, $\mathcal{L}_k$ are languages of rings, $\mathcal{L}_{\Gamma,\infty}$ is the language of ordered abelian group expanded by a constant symbol ∞, v and ac satisfy the following:

- $v(a) = \infty \Leftrightarrow ac(a) = 0 \Leftrightarrow a = 0$.
- If $v(a) = v(b) \neq \infty$, then $\big[ac(a) + ac(b) = ac(a + b) \wedge ac(a + b) \neq 0\big] \Leftrightarrow \big[v(a) = v(b) = v(a + b)\big]$.
- If $v(a) < v(b)$, then $ac(a) = ac(a + b)$.

Theorem 19 (AKE-style principle for valued fields preserves NATP) [2, Theorem 4.29]

$\mathrm{Th}(\mathcal{K})$ has NATP if and only if $\mathrm{Th}(k)$ has NATP.

A Forthcoming Work: Preservation of NATP

In a joint work with JinHoo Ahn, Hyoyoon Lee, and Junguk Lee, we examine how well model-theoretic constructions and expansions preserve NATP. This work includes studies about Fraisse class with SAP (strong amalgamation property), Dense/Co-dense expansions, generic predicate expansions, and ACFO, the theory of algebraic closed fields with circular orders, which appears in [8].

References

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