# Definable Extension Theorems in O-minimal Structures

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## Overview

1. Nonconstructive nature of extension theorems
2. Definable extension theorems
3. Open questions

# Nonconstructive nature of extension theorems

## Question

Let $f\colon A \to \mathbb{R}^n$ ($A \subseteq \mathbb{R}^m$) be $L$-LIPSCHITZ, i.e.,

$$||f(x) - f(y)|| \leqslant L \cdot ||x - y|| \quad \text{for all } x, y \in A.$$

Can one extend $f$ to an $L$-LIPSCHITZ map $\mathbb{R}^m \to \mathbb{R}^n$?

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$$f_a(x) := f(a) + L \cdot ||x - a|| \qquad (a \in A)$$

is $L$-LIPSCHITZ on $\mathbb{R}^m$, hence so is $F(x) := \inf\{f_a(x) : a \in A\}$, and clearly $F \restriction A = f$.

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Applying this to the coordinate functions of $f$ yields an $L\sqrt{n}$-LIPSCHITZ extension. But one can do better!

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Let $f\colon A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^m$, be $1$-LIPSCHITZ, and let $x \in \mathbb{R}^m \setminus A$.

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$$\bigcap_{a \in A} \underbrace{\{y \in \mathbb{R}^n : ||y - f(a)|| \leqslant ||x - a||\}}_{\overline{B}_{||x-a||}(f(a))} \qquad \text{is non-empty.}$$

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By compactness one first reduces to the case of a *finite* set $A$.

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$$\overline{B}_{r_1}(x_1) \cap \cdots \cap \overline{B}_{r_k}(x_k) \neq \emptyset \quad \Rightarrow \quad \overline{B}_{r_1}(y_1) \cap \cdots \cap \overline{B}_{r_k}(y_k) \neq \emptyset.$$

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Idea of the proof: the function

$$\mathbb{R}^m \to \mathbb{R} \colon y \mapsto G(y) := \max \left\{ ||y - y_i||/r_i : i = 1, \dots, k \right\}$$

is LIPSCHITZ with $\lim\limits_{||y|| \to \infty} G(y) \to \infty$.

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is LIPSCHITZ with $\lim\limits_{||y|| \to \infty} G(y) \to \infty$. Thus $G$ achieves a minimum at some $w \in \mathbb{R}^m$; show $G(w) \leqslant 1$.

# Nonconstructive nature of extension theorems

By an **extension problem** we will mean a situation of the following kind:

Let $\mathcal{C}$ be a class of maps $\mathbb{R}^m \to \mathbb{R}^n$. Find a necessary and sufficient condition for some given map $X \to \mathbb{R}^n$ ($X \subseteq \mathbb{R}^m$), possibly equipped with additional data, to have an extension to a map $\mathbb{R}^m \to \mathbb{R}^n$ from $\mathcal{C}$.

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We'll look at some other examples, associated with WHITNEY.

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From now on, $X \subseteq \mathbb{R}^n$ is closed, and

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## Definition

A **jet of order** $m$ **on** $X$ is a family $F = (F^\alpha)_{|\alpha| \leqslant m}$ of continuous functions $F^\alpha \colon X \to \mathbb{R}$.

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## Question

Let $F$ be a jet of order $m$ on $X$. *What is a necessary and sufficient condition to guarantee the existence of a $C^m$-function $f \colon \mathbb{R}^n \to \mathbb{R}$ such that $J_X^m(f) = F$?*

# Nonconstructive nature of extension theorems

Let $F = (F^\alpha)_{|\alpha| \leqslant m}$ be a jet of order $m$ on $X$ and $a \in X$. Put

$$T_a^m F(x) := \sum_{|\alpha| \leqslant m} \frac{F^\alpha(a)}{\alpha!}(x-a)^\alpha, \quad R_a^m F := F - J_X^m(T_a^m F).$$

## Definition

A jet $F$ of order $m$ is a $C^m$-WHITNEY **field** ($F \in \mathscr{E}^m(X)$) if for $x_0 \in X$ and $|\alpha| \leqslant m$,

$$(R_x^m F)^\alpha(y) = o(|x-y|^{m-|\alpha|}) \quad \text{as } X \ni x, y \to x_0.$$

By TAYLOR's Formula, for each $f \in C^m(\mathbb{R}^n)$,

$$J_X^m(f) := (D^\alpha f \restriction X)_{|\alpha| \leqslant m} \in \mathscr{E}^m(X).$$

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## WHITNEY Extension Theorem (H. WHITNEY, 1934)

For every $F \in \mathscr{E}^m(X)$, there is an $f \in C^m(\mathbb{R}^n)$ with $J_X^m(f) = F$.

## Proof outline

- Decompose $\mathbb{R}^n \setminus X$ into countably many cubes with disjoint interior satisfying some inequality regarding their diameter and distance from $X$. ("WHITNEY decomposition")
- Use this to get a "special" partition of unity $(\phi_i)$ on $\mathbb{R}^n \setminus X$.
- Pick $x_i \in X$ such that $d(x_i, \operatorname{supp}(\phi_i)) = d(X, \operatorname{supp}(\phi_i))$.
- $f(x) = \begin{cases} F^0(x), & \text{if } x \in X; \\ \sum_i \phi_i(x) T_{x_i}^m F(x), & \text{if } x \notin X. \end{cases}$

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Again, this is very non-explicit!

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WHITNEY actually asked a somewhat different question (and answered it for $n = 1$):

**WHITNEY's Extension Problem**

How can we determine whether a function $X \to \mathbb{R}$ is the restriction of a $C^m$-function $\mathbb{R}^n \to \mathbb{R}$?

A complete answer was only given by C. FEFFERMAN in the early 2000s.

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More about that later.

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Of course, one can also ask this question for maps definable in an o-minimal expansion of a real closed ordered field.

## Definable extension theorems

Assuming o-minimality turned out to be unnecessarily strong:

**Theorem (A.-FISCHER, 2011)**

*Let $\boldsymbol{R}$ be any definably complete expansion of a real closed ordered field. Every definable $L$-LIPSCHITZ map $A \to R^n$ ($A \subseteq R^m$) has a definable $L$-LIPSCHITZ extension $R^m \to R^n$.*

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The heavy lifting for the proof of this theorem was already done in convex analysis, based on a relationship between Lipschitz maps and (certain) set-valued maps (Minty; more recently, Bauschke & Wang).

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What about a definable Whitney Extension Theorem?

Let $X \subseteq \mathbb{R}^n$ be closed.

**Theorem (Kurdyka & Pawłucki, 1997)**

Let $F \in \mathscr{E}^m(X)$ be subanalytic. Then there is a subanalytic $C^m$-function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $J^m_X(f) = F$.

Their proof used tools very specific to the subanalytic context (e.g., reduction to the case $X$ compact; quasiconvexity).

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**Theorem (Pawłucki, 2008)**

Suppose $X$ is definable in an o-minimal expansion $\boldsymbol{R}$ of the ordered field of reals. There is a linear extension operator

$$\mathscr{E}^m_{\mathrm{def}}(X) \to C^m(\mathbb{R}^n)$$

which is a finite composition of operators each of which preserves definability, or is integration w.r.t. a parameter.

# Definable extension theorems

Let now $\boldsymbol{R}$ be an o-minimal expansion of a real closed field.

**Theorem** (THAMRONGTHANYALAK, 2012; KURDYKA-PAWŁUCKI, 2014)

*Let $F \in \mathscr{E}^m(X)$ be definable. Then there is a definable $C^m$-function $f\colon R^n \to R$ such that $J^m_X(f) = F$.*

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THAMRONGTHANYALAK's proof follows the outline of the construction of PAWŁUCKI, combining it with results on $\Lambda^m$-stratifications by FISCHER, which help to gain control on the growth of derivatives.

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**Definition**

Let $f\colon X \to R^n$ $(n \geqslant 1)$ be definable and $C^m$. One says that $f$ is $\Lambda^m$**-regular** if there exists $L > 0$ such that

$$\|D^\alpha f(x)\| \leqslant \frac{L}{d(x, \partial X)^{|\alpha|-1}} \quad \text{for all } x \in X,\ \alpha \in \mathbb{N}^d,\ 1 \leqslant |\alpha| \leqslant m.$$

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Also declare each map $R^0 \to R^n$ and the constant functions $\pm\infty$ to be $\Lambda^m$-regular.

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Also declare each map $R^0 \to R^n$ and the constant functions $\pm\infty$ to be $\Lambda^m$-regular.

For example, $f(x) = \frac{1}{x}$ on $X = (0, +\infty)$ is *not* $\Lambda^1$-regular.

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A **standard $\Lambda^m$-regular cell in $R^n$** is either

1. a standard open $\Lambda^m$-regular cell in $R^n$; or
2. the graph of a definable $\Lambda^m$-regular map $D \to R^{n-d}$, where $D$ is a standard open $\Lambda^m$-regular cell in $R^d$, and $0 \leqslant d < n$.

**Standard open $\Lambda^m$-regular cells** in $R^n$ are defined inductively just like open cells, requiring the continuous definable functions arising in the definition to be $\Lambda^m$-regular.

A **standard $\Lambda^m$-regular cell in $R^n$** is either

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The image of a standard $\Lambda^m$-regular cell in $R^n$ under an $R$-linear orthogonal isomorphism of $R^n$ is called a **$\Lambda^m$-regular cell in $R^n$**.

**Theorem (Fischer, 2007)**

*Let $S_1, \ldots, S_k$ be definable subsets of $R^n$. Then there exists a finite partition $\mathscr{D}$ of $R^n$ into $\Lambda^m$-regular cells such that*

- *each $\partial D$ $(D \in \mathscr{D})$ and*
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At the root of this (and also of the PILA-WILKIE parametrization theorem) is a calculus lemma due to GROMOV.

**Lemma**

*Let $h\colon I \to R$ be a definable $C^2$-function on an interval $I$ in $R$ such that $h$, $h''$ are semidefinite. Let $t \in I$ and $r > 0$ with $[t-r, t+r] \subseteq I$. Then*

$$\left|h'(t)\right| \leqslant \frac{1}{r} \sup \left\{ |h(\xi)| : \xi \in [t-r, t+r] \right\}.$$

Now on to the definable WHITNEY extension problem for $m = 1$.

# Definable extension theorems

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**Theorem (A. & Thamrongthanyalak)**

*Let $(f_a)_{a\in R^N}$ be a definable family of functions $f_a\colon X_a \to R$, where $X_a \subseteq R^n$ is closed. Then*

$$A_* := \{a \in R^N : f_a \text{ extends to a definable } C^1\text{-function } R^n \to R\}$$

*is definable. Moreover, there is a definable family $(\widetilde{f}_a)_{a\in A_*}$ of $C^1$-functions $R^n \to R$ such that $\widetilde{f}_a \restriction X_a = f_a$ for each $a \in A_*$.*

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**Corollary ($C^1$ case of a question of VAN DEN DRIES, 1997)**

*Let $f \colon X \to R$ be a definable function where $X \subseteq R^n$ is closed. Suppose that $f$* locally *extends to a definable $C^1$-function. Then $f$ extends to a definable $C^1$-function on $R^n$.*

The proof of this theorem (following KLARTAG-ZOBIN) uses the method of *affine bundles*:

### Definition

A definable set-valued map $H\colon X \rightrightarrows R^m$ is an **affine bundle** if for every $x \in X$, $H(x)$ is an affine subspace of $R^m$ or $H(x) = \emptyset$.

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This method is actually easier to explain when applied to solving a different problem:

Let $f, g_1, \ldots, g_m\colon X \to R$ be definable. Are there *definable* continuous functions $y_1, \ldots, y_m\colon X \to R$ such that

$$f = g_1 y_1 + \cdots + g_m y_m? \qquad (*)$$

Now $(*)$ has a solution iff the affine bundle $H^{(0)}$ on $R^m$ given by

$$H^{(0)}(x) := \{(y_1, \ldots, y_m) \in R^m : f(x) = g_1(x)y_1 + \cdots + g_m(x)y_m\}.$$

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Inductively define $H^{(l+1)} := (H^{(l)})'$.

Here, $H'$ is the GLAESER **refinement** of $H$:

$$H'(x_0) := \{y_0 \in H(x_0) : d\big(y_0, H(x)\big) \to 0 \text{ as } x \to x_0 \text{ in } X\}.$$

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Easy to check:

- $H' \subseteq H$, so every selection of $H'$ is a selection of $H$; conversely, each *continuous* selection of $H$ is also a selection of $H'$;
- $H$ is an affine bundle $\Rightarrow$ $H'$ is an affine bundle.

## Lemma

*Let* $x_0 \in X$*. Then* $\dim H'(x_0) \leqslant \liminf_{X \ni x \to x_0} \dim H(x)$.

Indeed, let $p_0, \ldots, p_d \in H'(x_0)$ be such that $p_1 - p_0, \ldots, p_d - p_0$ are linearly independent, $d = \dim H'(x_0)$, and $\varepsilon > 0$.
By definition of $H'$, there is a $\delta > 0$ such that for all $x \in B_\delta(x_0)$ we obtain $q_0, \ldots, q_d \in H(x)$ with $\|p_i - q_i\| < \varepsilon$. For small $\varepsilon$, $q_1 - q_0, \ldots, q_d - q_0$ are linearly independent: $\dim H(x) \geqslant d$. $\square$

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Another little extra argument based on this lemma yields

$$H^{(*)} := H^{(2m+1)} = H^{(2m+2)} = \cdots$$

So we have:

$$H \text{ has a continuous selection}$$

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Can we reverse this implication?

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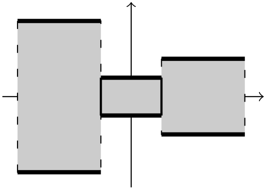
Can we reverse this implication? To answer this we note that

$H^{(*)}$ is *lower semi-continuous*
in the sense of the following definition.

Let $T\colon X \rightrightarrows R^m$ be a set-valued map.

### Definition

One says that $T$ is **lower semi-continuous (l.s.c.)** if for every $x \in X$, $y \in T(x)$, and neighborhood $V$ of $y$, there is a neighborhood $U$ of $x$ such that $T(x') \cap V \neq \emptyset$ for all $x' \in U \cap X$.



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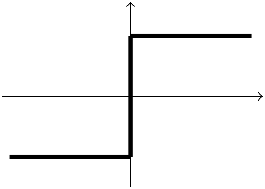

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**Theorem (Definable MICHAEL's Selection Theorem)**

*Suppose $X$ is closed and $T : X \rightrightarrows R^m$ is definable and l.s.c. such that for every $x \in X$, $T(x)$ is nonempty, closed, and convex. Then $T$ has a continuous definable selection.*

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Classically, this theorem is shown by a nonconstructive iterative procedure. Our proof relies on Cell Decomposition.

It does also hold for *bounded* $X$ in the category of semilinear sets and maps (using a different proof).
[A. & THAMRONGTHANYALAK; simplified by CZAPLA & PAWŁUCKI]

## Corollary

*Let $f, g_1, \ldots, g_m$ be definable maps $X \to R$. If there are continuous functions $y_1, \ldots, y_m \colon X \to R$ such that*

$$f = g_1 y_1 + \cdots + g_m y_m, \tag{$*$}$$

*then there are also* definable *continuous function $y_i$ solving ($*$).*

For polynomials $f, g_1, \ldots, g_m$, this was shown by FEFFERMAN and KOLLÁR by other means. (KOLLÁR-NOWAK: in this case one cannot always take the $y_i$ to be rational.)

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## Question (GABRIELOV)

If a definable set-valued map $X \rightrightarrows \mathbb{R}^m$ has a continuous selection, does it always have one that is definable?

Back to the $C^1$-WHITNEY Extension Problem.

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## Definition

Let $f\colon X \to R$. A **holding space for** $f$ is an affine bundle

$$H\colon X \rightrightarrows (R \times R^n)$$

such that whenever $F \in C^1(R^n)$ is definable, $F \restriction X = f$, then

$$\left(F(x), \frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)\right) \in H(x) \quad \text{for all } x \in X.$$

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Identify $R \times R^n$ with the space $\mathscr{P}_n$ of linear polynomials in $n$ indeterminates. Think of a holding space for $f$ as a collection of potential TAYLOR polynomials of definable $C^1$-extensions of $f$.

**Definition (the $C^1$-GLAESER refinement $\widetilde{H}$ of $H$)**

$p_0 \in \widetilde{H}(x_0) :\Longleftrightarrow p_0 \in H(x_0)$ and

$$(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x_1, x_2 \in X \cap B_\delta(x_0))\,(\exists p_1 \in H(x_1), p_2 \in H(x_2))$$

$$|D^\alpha(p_i - p_j)(x_i)| \leqslant \varepsilon \, \|x_i - x_j\|^{1-|\alpha|} \text{ for } i, j = 0, 1, 2 \text{ and } |\alpha| \leqslant 1.$$

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As before, the sequence $(H_l)$ where

$$H_0 = \text{trivial holding space for } f, \quad H_{l+1} := \widetilde{H_l}$$

eventually stabilizes. With $H_*$ its eventual value, by definable WHITNEY extension,

$f$ extends to a definable $C^1$-function on $R^n$ $\Longleftrightarrow$ $H_*$ has a *continuous* definable selection.

Let $\boldsymbol{R}$ be an o-minimal structure on $\mathbb{R}$.

1. Suppose $f\colon A \to \mathbb{R}^n$ ($A \subseteq \mathbb{R}^m$) is $L$-LIPSCHITZ and *locally definable* in $\boldsymbol{R}$. Does $f$ extend to a locally definable $L$-LIPSCHITZ function $\mathbb{R}^m \to \mathbb{R}^n$? (*Yes* if $f$ is bounded, or if in the conclusion $L$ is replaced by $L + \varepsilon$.)

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2. What about the $C^m$ case of WHITNEY extension for definable functions in $\boldsymbol{R}$? [BIERSTONE & MILMAN, 2009]

# Open questions

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2. What about the $C^m$ case of WHITNEY extension for definable functions in $\boldsymbol{R}$? [BIERSTONE & MILMAN, 2009]
3. The $p$-adic case; e.g.: Let $f, g_1, \dots, g_m\colon X \to \mathbb{Q}_p$ ($X \subseteq \mathbb{Q}_p^m$) be semialgebraic such that there are continuous $y_i$ with

$$f = g_1 y_1 + \cdots + g_m y_m.$$

   Are there also semialgebraic continuous $y_i$ with this property?
   (FEFFERMAN-KOLLÁR: *yes* if $f$, $g_i$ are polynomials.)