EXISTENTIAL CLOSURE OF SOME TOPOLOGICAL EXPONENTIAL DIFFERENTIAL FIELDS

Nathalie Regnault

Mathematics Department, University of Mons and Mathematics Department, Free University of Brussels, Belgium

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Abstract:

Question: Does $Th(\mathbb{R}, <, exp, D)$, the theory of the real ordered field with exponentiation and derivation, have a model-completion? We answer in a more general setting, by axiomatising the model-completion of some topological exponential differential fields of characteristic 0, as it encompasses the p-adics case too.

SETTING and AIMS

Setting

Let $(K, \mathcal{V})$ be a topological field, where K is a field of characteristic 0 with eventually extra structure and $\mathcal{V}$ is a base of neighborhoods of 0, and let $(K, D, \mathcal{V})$ be a generic differential expansion.
(see $[T]$ for the field language case, and $[GP]$, $[So]$ for the case where relations are added in the language)
We consider the case where K is also endowed with a partial exponential map:

Let R be a commutative unitary characteristic 0 ring. Let $\mathcal{U}(R)$ denote the group of units of $(R^*, \cdots, 1)$.
Let E be a group morphism from $(R, +, 0)$ to $(\mathcal{U}(R), ., 1)$. Then (R, E) is called an **exponential ring**.
Examples:$[W]$ $(\mathbb{R}, exp)$, $[NM]$ $(\mathbb{Z}_p, E_p)$, where $exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $E_p := exp(px), p \neq 2$.

We first define $TFES$ to be topological field K of characteristic 0 with a base of neighborhoods of 0 $\mathcal{V}$ and an exponential subring (R, E) on which E is continuous.

We then define $TDFES$ to be $TFES$ $(K, R, E, \mathcal{V})$ equipped with an exponential derivation D, i.e. a derivation on K s.t. $\forall x \in R, D(E(x)) = E(x).D(x)$

Let $\mathcal{L} \supseteq \{+, -, ., ^{-1}, 0, 1, E, D\}$. We assume that the topology is $\mathcal{L}$-definable.

$[M]$ introduces a notion of **exponential algebraicity**:

Let $R[\bar{X}]^E$ be the ring of E-polynomials on R, that is to say, polynomials composed with iterations of E, e.g.

$$P(X_1, X_2) = E(3rX_2 + E(X_1X_2^3)) + kX_2, \text{ where } r, k \in R$$

If (A, E) is an exponential subring of (B, E), an element $b \in B$ is E-algebraic over A if it is a coordinate of a tuple solution of a Hovanskii system defined over A:
there is an integer n, a uple $(b_1, ..., b_n) \subseteq B$ with $b = b_1$, and $P_1, ..., P_n \in A[\bar{X}]^E$ such that:
$P_i(b_1, ..., b_n) = 0, i = 1, \cdots, n$, and the Jacobian of the system $\{P_i(X_1, ..., X_n) = 0 | i = 1, \cdots, n\}$ is different from 0 at $(b_1, ..., b_n)$.
Generalising results of $[W]$, $[K]$ shows that the E-algebraic closure operator $ecl^B(.)$ induces a good notion of dimension–an exponential transcendence degree–.

Motivation

In $[GP]$, the authors consider theories of fields T (without an exponential) that are model-complete and axiomatise the model-completion T_D^c of their differential expansions T_D–which also proves model completeness of T_D^c–.
$Th(\mathbb{R}, <, exp)$, $Th(\mathbb{C}_p, |.|, exp, |)$ and $Th(\mathbb{Q}_p, |.|, exp, P_n, n \in \mathbb{N}, f, f \in F)$, the theories of the valued fields $\mathbb{C}_p$ and $\mathbb{Q}_p$ with a partially defined exponentiation, (where $P_n(x) \equiv \exists y, y^n = x$ and F is a family of restricted analytic functions) are model-complete, ($[W]$ and $[NM]$), thus candidates for an adaptation of the transfer results of $[GP]$ and their strategy to the exponential case:

Strategy

$$\begin{tikzcd}
T_E \arrow[r, phantom, "\subseteq"] \arrow[d, dashed] & T_{E,D} \arrow[d, "1"] \\
 & \cdot \arrow[d, "2"] \\
 & \cdot \arrow[d, "3"] \\
(T_E)^c & (T_{E,D})^{c?}
\end{tikzcd}$$

Steps

1. Formalise an hypothesis $(I)_E$–kind of existential closedness in the field of Laurent series $K((t))$– and show that a given $TFES$ $(K, R, E, \mathcal{V})$ is embeddable in a $TFES$ containing $K((t))$ and thus satisfies $(I)_E$.
2. Axiomatise geometrically a "Differential Lifting" axiom scheme $(DL)_E$ using E-algebraic varieties; which expresses that every differential E-polynomial has a zero close to a zero of the associated E-algebraic polynomial (if it has one). Show that a structure satisfying $(I)_E$ can be embedded in a structure satisfying $(DL)_E$.
3. Show that $T_{E,D}^c := T_E^c \cup T_{E,D} \cup (DL)_E$ is the model-completion of $T_{E,D}$ the theory of those $TDFES$ for which the restricted $TFES$ has a theory T_E that admits a model-completion T_E^c.

MAIN RESULTS

Ordered field of E-series: known results

- Let R be a ring and let G be an ordered abelian group. Let $R((G))$ be the set of formal power series with monomials in G and reverse well-ordered support: $R((G))$ is the set of elements of the form $s = \sum_{g \in G} c_g g$, where $c_g \in R, g \in G$, and Supp $s := \{g \in G \;:\; c_g \neq 0\}$ is reverse well-ordered in G.
- Beginning with an ordered E-field $(K, exp, <)$, $[DMM]$, $[KKS]$ construct an ordered field of E-series:

Let $(x^{\bar{K}}, .)$ be a multiplicative copy of $(K, +)$. Let $K_0 := K((x^K))$, $A_n := \{s \in K_n \;:\; Supp\, s > 1\}$, $K_{n+1} := K_n((M(A_n)))$,where $M(A_n)$ is a multiplicative copy of the group A_n.
$K((t))^E := \cup K_n$
Let $s = k + a + \epsilon \in K_n = K_{n-1} \oplus A_n \oplus \{s \in K_n \;:\; Supp\, s < 1\}$.
Then $E(s) = exp(k)M(a) \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \in K_{n+1}$.

$(I)_E$

We say that a $TFES$ $(K, R, E, \mathcal{V})$ satisfies hypothesis $(I)_E$ if:
Considering $K((t)) := K((x^{\mathbb{Z}}))$ as a topological field extension of K and given G a Hovanskii system with coefficients in $K[[t]]$ (where E applies only on elements of $R[[t]]$);
if we have $G(\bar{a}) \sim_K \bar{0}$ and $Jac_G(\bar{a}) \nsim_K 0$ for an $\bar{a} \subseteq K[[t]]$, then there is a $TFES$ $(K', R', E, \mathcal{W})$ containing $K((t))$ such that:

1. There is a subset $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ which satisfies $t \sim_{\widetilde{\mathcal{W}}} 0$.
2. There is some $\bar{b} \in K'$ such that $G(\bar{b}) = \bar{0}$, $Jac_G(\bar{b}) \nsim_{\widetilde{\mathcal{W}}} 0$ and $\bar{a} \sim_{\widetilde{\mathcal{W}}} \bar{b}$.

$(DL)_E$

Let $A \subseteq K^n$ be an irreducible variety defined over K. The *torseur* of A is the set
$\tau(A) := \{(\bar{a}, \bar{b}) \;:\; \bar{a} \in A \text{ and } \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(\bar{a}).b_i + P^D(\bar{a}) = 0 \text{ for all } P(\bar{X}) \in I(A(K))\}$
In $[PP]$ (DCF_0), $[MR]$ $(CODF)$, $[GP]$ the authors axiomatise geometrically a differential lifting scheme that adapts to our setting:

Let $(K, R, E, D, \mathcal{V})$ be a $TDFES$. We say that it satisfies the scheme $(DL)_E$ if:
For any $V \in \mathcal{V}$, for any irreducible E-algebraic varieties A, B defined over K such that A is regular,B is finitely generated, with $A \subseteq \tau(B)$ and if A projects generically on B and if there is a tuple $(\bar{a}, \bar{c}) \in K^{2n} \cap A$, then there is $\bar{b} \in K^n$ such that $(\bar{b}, D(\bar{b})) \in A$ and

$$(\bar{a}, \bar{c}) - (\bar{b}, D(\bar{b})) \in V^{2n}$$

Our results

Let F be a topological E-field of characteristic 0.
- We notice that the additive group of F is torsion free hence orderable.
- We thus adapt the construction of $[DMM]$, $[KKS]$ to construct a topological (unordered) E-field $F((t))^E$ (we extend the topology step by step following $[GP]$).

Let $(K, R, E, D, \mathcal{V})$ be a $TDFES$.
- We construct a $TDFES$ $(K((t))^E, R((t))^E, E, D, \mathcal{W})$ extending $(K, R, E, D, \mathcal{V})$ and containing $K((t))$.
- $(K, R, E, \mathcal{V})$ satisfies $(I)_E$.

Using results of $[M]$, we adapt a result of $[S]$ to our setting:
- Let $P \subseteq R[\bar{X}]^E$. Let $\bar{a} \in V(P) \subseteq R^{|\bar{X}|}$. There exists $Q = (Q_1, \cdots, Q_m) \subseteq R[[t]][\bar{X}]^E$ and a clopen definable subset S of $V^{reg}(Q) \subseteq R[[t]]^{|\bar{X}|}$ such that $\bar{a} \in S \subseteq V(P) \subseteq R[[t]]^{|\bar{X}|}$.

Let (K, R, E, D) be a $DFES$ and (K', R', E) be a FES such that $(K, R, E) \subseteq (K', R', E)$. We prove, using results of $[K]$:
- It is possible to extend D on E-algebraic transcendant elements of (K', R', E) (relatively to (K, R, E))
- It is possible to extend D uniquely on E-algebraic elements of (K', R', E) (relatively to (K, R, E))
- $ecl^{K'}$ is closed under D

Let $\mathcal{K} := (K, R, E, D, \mathcal{V})$ be a $TDFES$ satisfying $(I)_E$. We prove, using adapted results from $[PP]$ to the E-algebraic setting:
- $\mathcal{K}$ has a $\mathcal{L}$-extension $\hat{\mathcal{K}} := (\hat{K}, \hat{R}, \hat{E}, \hat{D}, \hat{\mathcal{V}})$ satisfying $(DL)_E$.

Let $T_{E,D}$ the theory of those $TDFES$ for which the restricted $TFES$ has a theory T_E that admits a model-completion T_E^c. We prove, using adapted results from $[S]$:
- $T_{E,D}^c := T_E^c \cup T_{E,D} \cup (DL)_E$ is the model-completion of $T_{E,D}$.

- Corollary 1: If $T_E^c = Th(\mathbb{R}, <, exp)$, then $T_{E,D}^c$ is the model-completion of the theory of ordered exponential differential fields of characteristic 0.
- Corollary 2: If $T_E^c = Th(\mathbb{C}_p, |.|, exp, |)$, then $T_{E,D}^c$ is the model-completion of the theory of algebraically closed valued differential fields of characteristic 0 and residual characteristic p with an exponential differential valuation ring.
- Corollary 3: If $T_E^c = Th(\mathbb{Q}_p, |.|, exp, ...)$, then $T_{E,D}^c$ is the model-completion of the theory of p-valued differential fields of characteristic 0 with an exponential differential valuation ring.

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