Computer Science and Mathematics
Extra Material for the course

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Contents

1 Number Bases

1.1 Introduction

When counting numbers, we normally start with 0, then go up to 1, then to 2, then 3... when we reach the number 9 we find ourselves in a small problem: there are no more digits left to count up! We only have ten digits: $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9. What digit are we going to use after 9? We agree that instead of making up a new digit, we cycle the digits back to 0, and add a 1 to left of it to account for that recycling: we end up with 10.

We repeat the process again when reaching 19: we cycle back to 0 and mark the extra cycle with an addition to the extra digit 1, making it a 2 and ending with 20. This process repeats for 30, 40 and so forth, until we reach 99. At this point we cycle both 9s back to 0 and add a 1 to the left of those digits to mark a grand-cycle: 100.

This action of cycling all digits back to 0 and adding a 1 to the left of our digits repeats each time when all of the digits are 9s. What we are essentially doing is to mark tens, hundreds, thousands, etc. by adding new digits to the left. For example, the number 37 has 3 tens plus a 7, while the number 91 has nine tens plus a 1. In the same fashion, the number 271 has 2 hunderds plus 7 tens plus a 1 and the number 1337 has a 1 thousand plus 3 hundreds plus 3 tens plus a 7 (See Figure 1.1).

In essence, each digit counts how many increasing powers of 10 our number is made of (since $1,000 = 10^3$, $100 = 10^2$, $10 = 10^1$ and $1 = 10^0$). We can continue this system indefinetly: the next digit to the left will represent $10,000 = 10^4$ then $100,000 = 10^5$, etc.

1.1.1 Bases Smaller or Equal to 10

Since we are using 10 digits, we call this numbering system *base-10* counting (another common name for a base is the Latin *radix*). Of couere, we can use other bases then base-10: all it means is that the number of digits we use will differ. For example, in base-2 we only have two digits: 0 and 1. This means that as in the case of the digit 9 in base-10, each time we count pass the digit 1 it will cycle back to 0 and the next digit will increase: we start with 0, then 1 - and then we must cycle back to 0 and add a new digit, getting 10. We continue: 11, then cycling both back to 0 and adding an new digit, getting 100. Then come 101, 110, 111, 1000 and so forth.

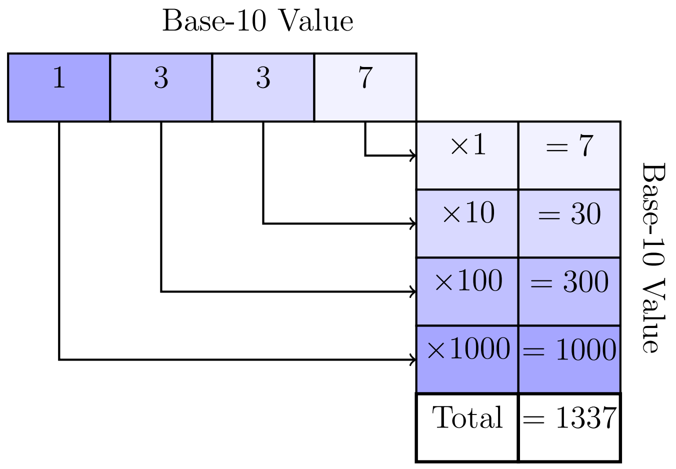

Figure 1.1: The components of the number 1,337: the digits are separated, each representing an increasing power of 10 (1, 10, 100, $1,000$). These values are multiplied by the digit's value (under the arrows). Summing up all the values yields back the number 1,337.

In the case of base-3, for example, the digits cycle after they reach 2: 0, 1, 2, 10, 11, 12, 100, 101, 102, etc. Notice that for each base-N (where $N = 2, 3, 4, 5, 6\ldots$) the number 10 always equals to N. In order to be clear as in which base are counting, we sorround the number with round paranthesis and write the base in lower case to the right of the number (although sometimes the parenthesis are not used). For example $(143)_5$ is the number represented by 143 in base 5, while $(1001)_2$ is the base-2 representation of the number 1001. Most often, when not dealing with numbers in any bases other than base-10, we simply omit the base symbol (as in most day to day life). The same applies for when it is clear which base system are we using.

It is important to understand that for any base-N the digits represent increasing powers of N. As for base-10, where each digit (starting from the right) represented $10^0 = 1$, $10^1 = 10$, $10^2 = 100$ and so on, the digits for base-N represent the numbers N^0(=1), N^1(=N), N^2, N^3, $\ldots$ and so forth. Thus for base-2 the digits will represent the numbers $2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8, 2^4 = 16\ \ldots$ etc. In the case of base-3 the digits will stand for $3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ \ldots$ etc. (See Figure 1.2 for a graphical representation)

In any counting base, the left-most digit is the *Most Significant Digit*, since it represents the biggest part of the number: for example, in the number 961, the digit 9 represents 900, which is much bigger than both 60 and 1 (the numbers represented by the second and third digits). In the same manner, the right most digit is the *Least Significant Digit* (in the example here it is 1). See Figure 1.3. Base-2 is also named *Binary* while base-10 is named *Decimal*.

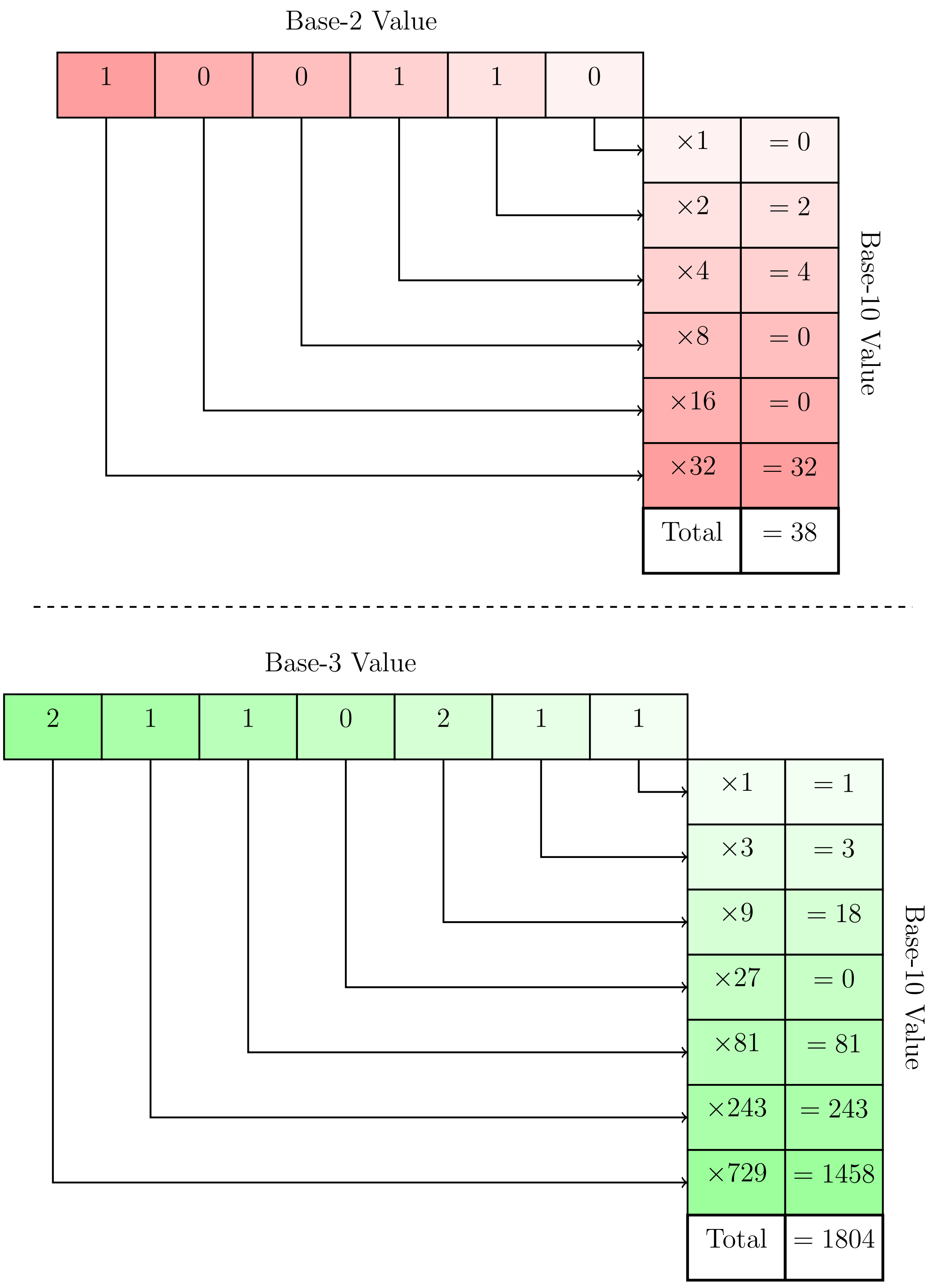

Figure 1.2: Top: the representation of $(100110)_2$. Bottom: the representation of the number $(2110211)_3$.

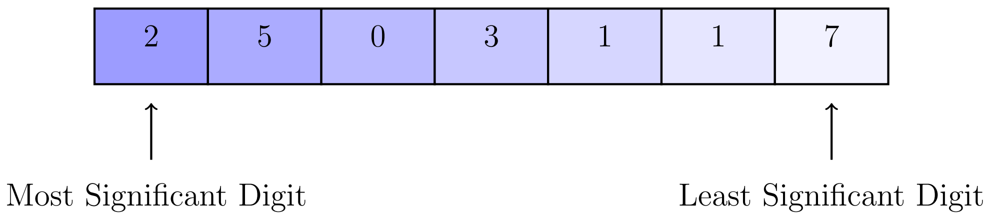

Figure 1.3: The number 2503117_{10} has 2 as the Most Significant Digit (representing $2,000,000$) and 7 as the Least Significant Digit (representing simply 7).

1.1.2 Bases Bigger than 10

When our counting base has more than 10 digits (i.e. base-11, base-12, base-16, base-64, etc.), we use other symbols to represent the new digits past 9. Usually those symbols are the Latin alphabet: A, B, C, etc. For example, when counting in base-16 (also known as *Hexdecimal*), the following digits are used: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$ and F, where the digits $0-9$ have the same meaning as in base-10, and the letters represent the following values: $A = (10)_{10}$, $B = (11)_{10}$, $C = (12)_{10}$, $D = (13)_{10}$, $E = (14)_{10}$ and $F = (15)_{10}$. See Figure 1.4 for some examples.

In programming languages such as C/C++, Python, Java and countless more, the symbol *0x* before a number symbolizes that the number is in hexdecimal form (i.e. *0xFFA01* is the number $FFA01_{16}$ which equals to 1047041_{10}).

1.2 Converting Between Different Bases

While converting between any base and base-10 is pretty straight-forward (as seen in the figures above), converting in the opposite direction is a bit more tricky since we normally count in base-10. In essence, the procedure of converting a number x_{10} to base-N goes as following:

1. Divide the number by the base (N) to get the remainder R. This remainder is the least significant digit of the number in base-N.
2. Repeat the process by dividing the quotient of step 1 by N. This time, the remainder is the second least significant digit.
3. Repeat this process until your quotient becomes 0. This quotient is the most significant digit.

For example, let us look at the number 9_{10} and convert it to binary according to the given process: first, we divide 9 by 2: $\frac{9}{2} = 4.5$. This translates to a quotient 4 with

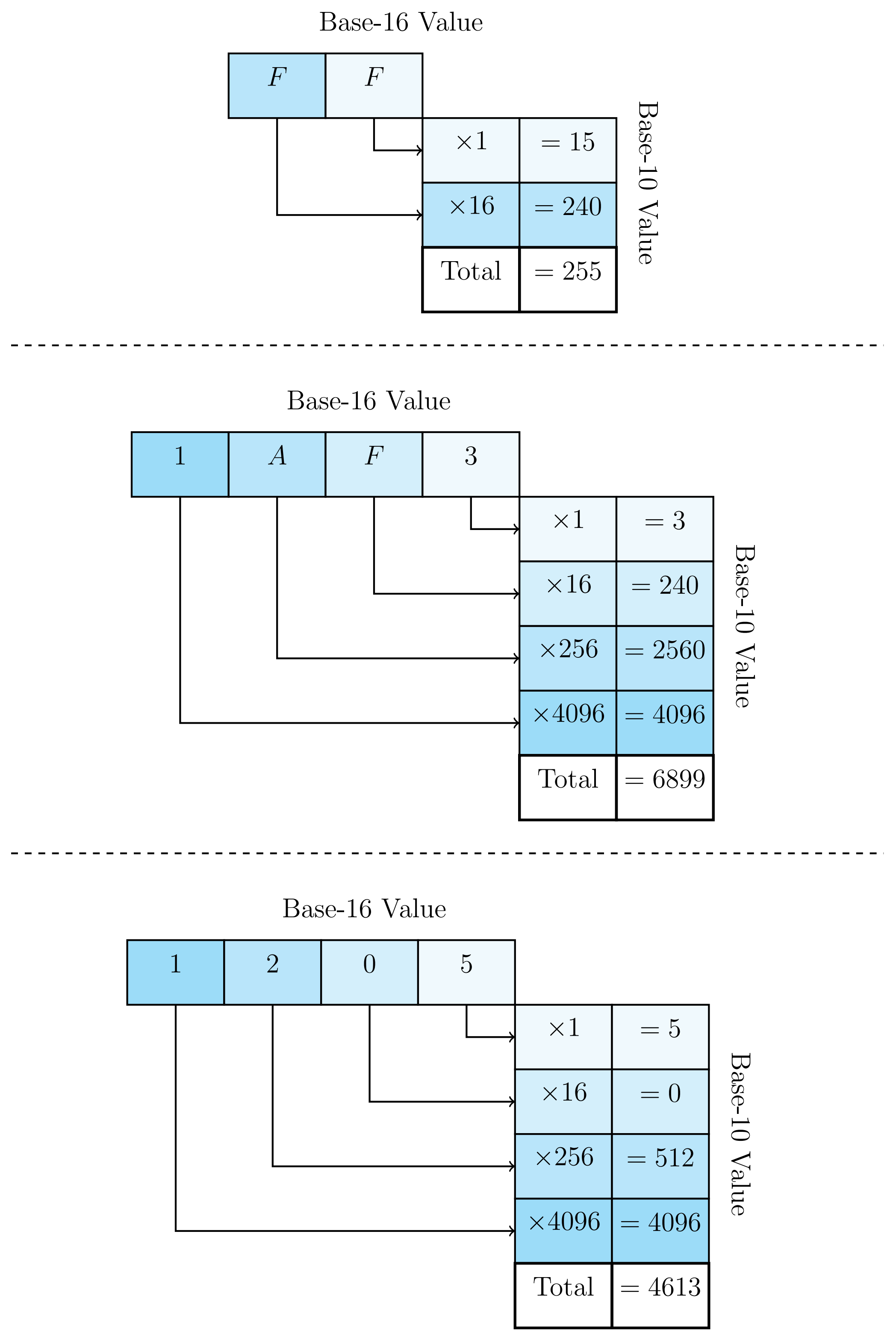

Figure 1.4: Three Hexdecimeal numbers converted to decimal. Remeber that $A = 10_{10}$, $B = 11_{10}$, $\ldots$, $F = 15_{10}$.

Table 1.1: The process for converting 9_{10} to binary.

Step	Quotient	Devision	New quotient	Remainder
(1)	9	$\frac{9}{2} = 4.5$	4	1
(2)	4	$\frac{4}{2} = 2$	2	0
(3)	2	$\frac{2}{2} = 1$	1	0
(4)	1	$\frac{1}{2} = 0.4$	0	1

remainder 1. Next, we divide the quotient 4 by 2: $\frac{4}{2} = 2$, or quotient 2 with a remainder 0. Next, we divide the quotient 2 by 2: $\frac{2}{2} = 1$, which is a quotient of 1 with remainder 0. Goint one more step further, we divide the quotient 1 by 2, yieding quotient 0 with a remainder of 1. At this stop we stop, and count the remainders backwards: 1001. Hence, we get $9_{10} = 1001_2$. See Table 1.1 for a more graphical representation.

1.3 Converting Between Hexdecimal and Binary

One big problem with binary (base-2) representations is that they are very long. Already we see that in base-2, the number 23_{10} is represented by 5 digits (10111), while the number 121_{10} by 7 digits (1111001). As binary code is used a lot in computer science and electronics, a base with shorter-length numbers is required, in order to make numbers more human-readable. The usual choice for such a purpose is base-16 (Hexdecimal). The reason is that $16 = 2^4$ meaning that each 4 binary digits can be represented by just one hexdecimal digit. Using Table 1.2 for conversion, we see that converting the binary number 1011 to hexdecimal is straight-forward: $1011_2 = B_{16}$.

Usually, we wish to keep our convertions easy, and thus we write binary representations with a number of *leading zeros* that complete the represntation to a number of digits of our choice. For example, in order to show that binary and hexdecimal digits are easily interchangeable, we write the binary numbers with 4 digits, as in Table 1.2, third column. This way, we can easily convert hexdecimal numbers with more than one digit to binary: for example, the hexdecimal number $F3_{16}$ has the digits F and 3. If we convert both digits to their binary 4-digit representation, we get $(F)_{16} = (1111)_2$ and $(3)_{16} = (0011)_2$, and thus $(F3)_{16} = (1111\ 0011)_2$ (the space is inserted for readability). See Figure 1.5 for an example.

Table 1.2: Hexdecimal to binary convertion

Hexdecimal	Binary	4-digit Binary
0	0	0000
1	1	0001
2	10	0010
3	11	0011
4	100	0100
5	101	0101
6	110	0110
7	111	0111
8	1000	1000
9	1001	1001
A	1010	1010
B	1011	1011
C	1100	1100
D	1101	1101
E	1110	1110
F	1111	1111

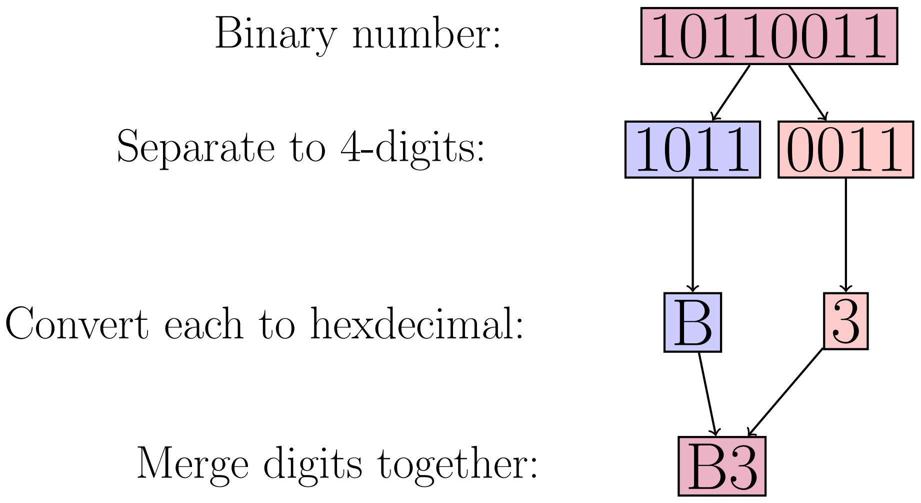

Figure 1.5: Converting the long binary number 10110011 to the hexdecimal number B3.

Appendices

MUCH EMPTY