Computer Science and Mathematics
Extra Material for the course

Peleg Bar Sapir

Contents

1 Number Bases

1.1 Introduction

When counting numbers, we normally start with 0, then go up to 1, then to 2, then 3... when we reach the number 9 we find ourselves in a small problem: there are no more digits left to count up! We only have ten digits: $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9. What digit are we going to use after 9? We agree that instead of making up a new digit, we recycle the digits back to 0, and add a 1 to left of it to account for that recycling: we end up with 10.

We repeat the process again when reaching 19: we cycle back to 0 and mark the extra cycle with an addition to the extra digit 1, making it a 2 and ending with 20. This process repeats for 30, 40 and so forth, until we reach 99. At this point we cycle both 9s back to 0 and add a 1 to the left of those digits to mark a grand-cycle: 100.

This action of cycling all digits back to 0 and adding a 1 to the left of our digits repeats each time all our counting reaches a point where all of the digits are 9s. What we are essentially doing is to mark tens, hundreds, thousands, etc. by adding new digits to the left. For example, the number 37 has 3 tens plus a 7, while the number 91 has nine tens plus a 1. In the same fashion, the number 271 has 2 hunderds plus 7 tens plus a 1 and the number 1337 has a 1 thousand plus 3 hundreds plus 3 tens plus a 7 (See Figure 1.1).

In essence, each digit counts how many increasing powers of 10 our number is made of (since $1,000 = 10^3$, $100 = 10^2$, $10 = 10^1$ and $1 = 10^0$). We can continue this system indefinetly: the next digit to the left will represent $10,000 = 10^4$ then $100,000 = 10^5$, etc.

1.1.1 Bases Smaller or Equal to 10

Since we are using 10 digits, we call this numbering system *base-10* counting (another common name for a base is the Latin *radix*). Of couere, we can use other bases then base-10: all it means is that the number of digits we use will differ. For example, in base-2 we only have two digits: 0 and 1. This means that as in the case of the digit 9 in base-10, each time we count pass the digit 1 it will cycle back to 0 and the next digit will increase: we start with 0, then 1 - and then we must cycle back to 0 and add a new digit, getting 10. We continue: 11, then cycling both back to 0 and adding an new digit, getting 100. Then come 101, 110, 111, 1000 and so forth.

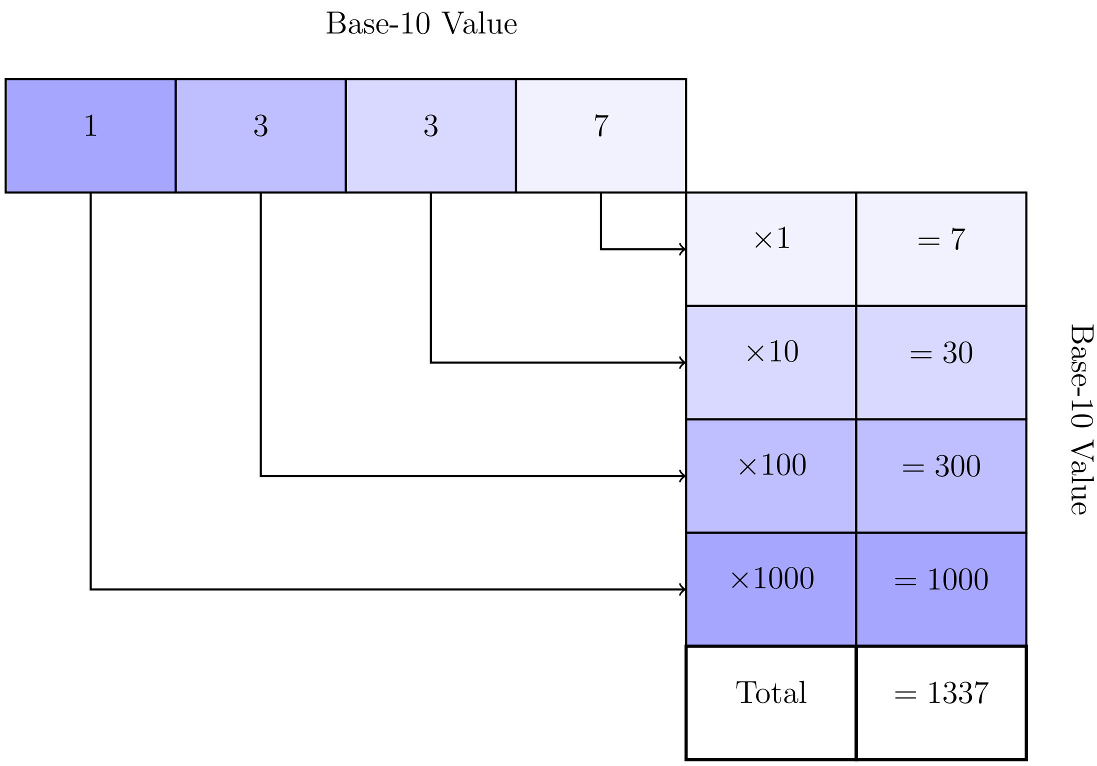

Figure 1.1: The components of the number 1,337: the digits are separated, each representing an increasing power of 10 (1, 10, 100, 1,000). These values are multiplied by the digit's value (under the arrows). Summing up all the values yields back the number 1,337.

In the case of base-3, for example, the digits cycle after they reach 2: 0, 1, 2, 10, 11, 12, 100, 101, 102, etc. Notice that for each base-N (where $N = 2, 3, 4, 5, 6 \ldots$) the number 10 always equals to N. In order to be clear as in which base are counting, we sorround the number with round paranthesis and write the base in lower case to the right of the number (although sometimes the parenthesis are not used). For example $(143)_5$ is the number represented by 143 in base 5, while $(1001)_2$ is the base-2 representation of the number 1001. Most often, when not dealing with numbers in any bases other than base-10, we simply omit the base symbol (as in most day to day life). The same applies for when it is clear which base system are we using.

It is important to understand that for any base-N the digits represent increasing powers of N. As for base-10, where each digit (starting from the right) represented $10^0 = 1$, $10^1 = 10$, $10^2 = 100$ and so on, the digits for base-N represent the numbers N^0(=1), N^1(=N), N^2, N^3, $\ldots$ and so forth. Thus for base-2 the digits will represent the numbers $2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8, 2^4 = 16\ \ldots$ etc. In the case of base-3 the digits will stand for $3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ \ldots$ etc. (See Figure 1.1.1 for a graphical representation)

1.1.2 Bases Bigger than 10

When our counting base has more than 10 digits (i.e. base-11, base-12, base-16, base-64, etc.), we use other symbols to represent the new digits past 9. Usually those symbols are the Latin alphabet: A, B, C, etc. For example, when counting in base-16 (also known as *Hexdecimal*), the following digits are used: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$ and F, where the digits $0 - 9$ have the same meaning as in base-10, and the letters represent the following values: $A = (10)_{10}$, $B = (11)_{10}$, $C = (12)_{10}$, $D = (13)_{10}$, $E = (14)_{10}$ and $F = (15)_{10}$. See Figure 1.1.2 for some examples.

In programming languages such as C/C++, Python, Java and countless more, the symbol *0x* before a number symbolizes that the number is in hexdecimal form (i.e. *0xFFA01* is the number $FFA01_{16}$ which equals to 1047041_{10}).

Table shows the Hexdecimal and binary values for the numbers 0-15 (in base-10).

1.2 Converting Between Different Bases

While converting between any base and base-10 is pretty straight-forward (as seen in the figures above). Converting in the opposite direction is a bit more tricky.

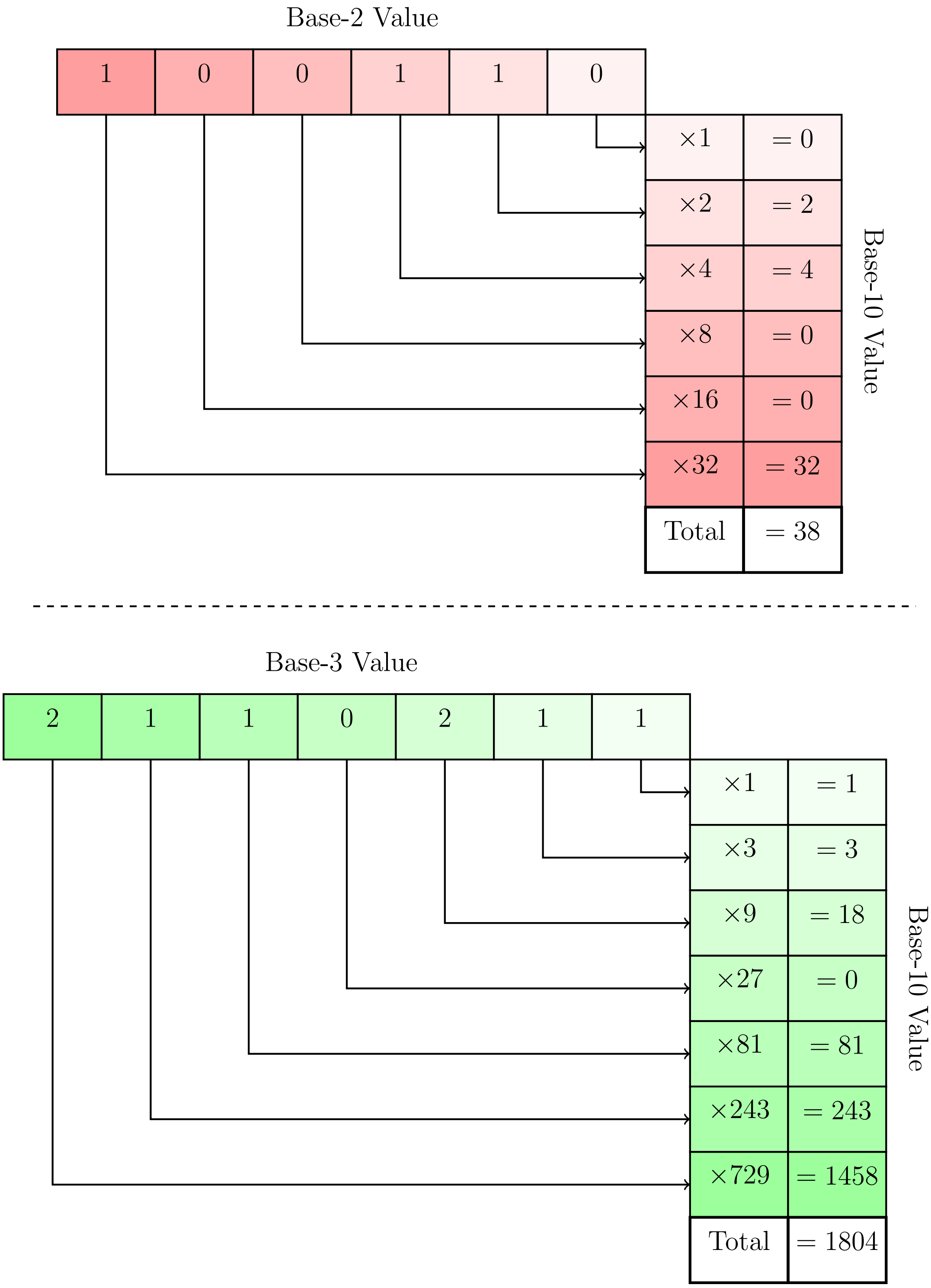

Figure 1.2: Top: the representation of $(100110)_2$. Bottom: the representation of the number $(2110211)_3$.

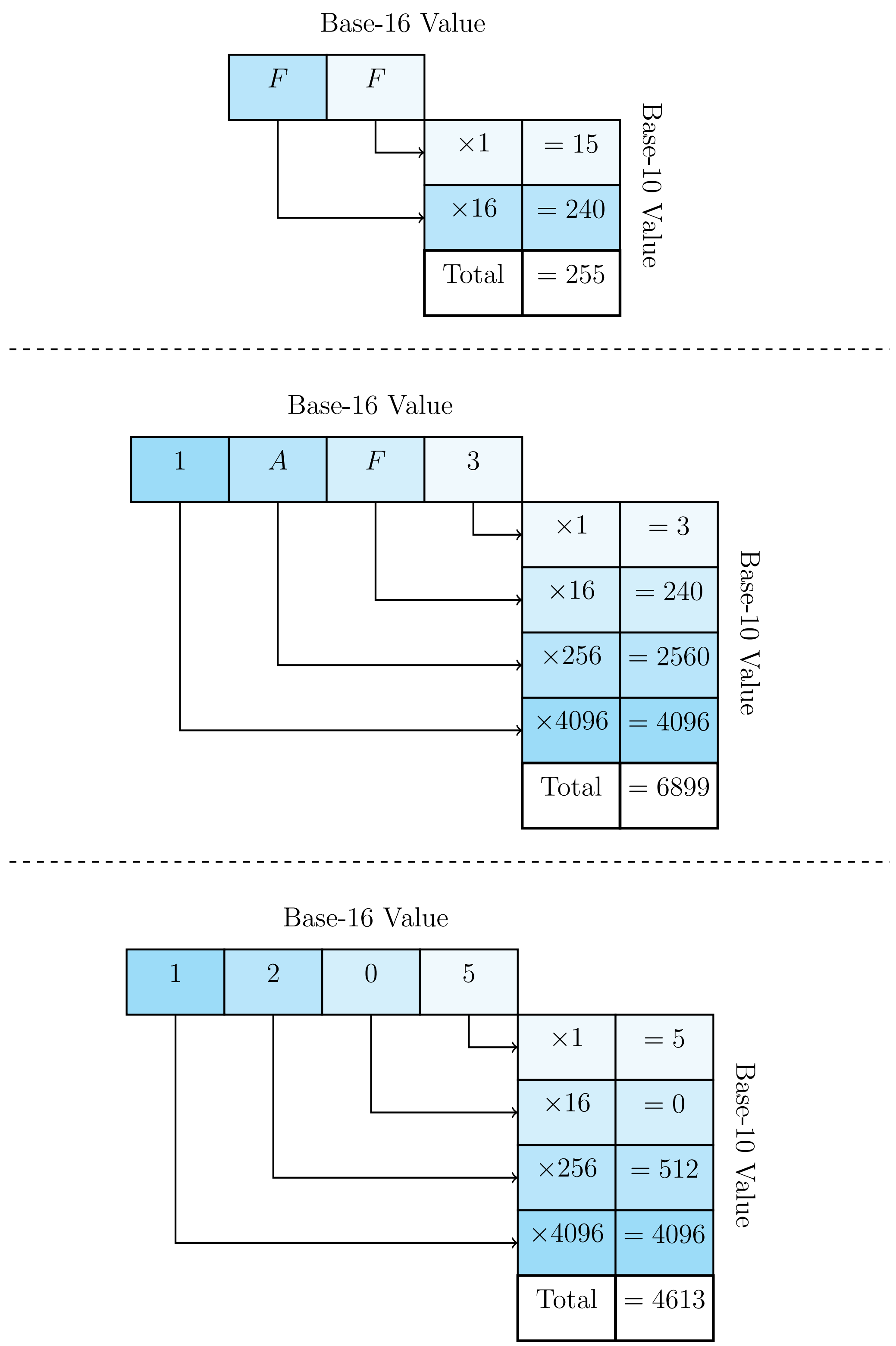

Figure 1.3: Three Hexdecimeal numbers converted to decimal. Remeber that $A = 10_{10},\ B = 11_{10},\ \ldots,\ F = 15_{10}$.

Table 1.1: Hexdecimal and binary values of the numbers 0-15

Decimal	Hexdecimal	Hexdecimal
0	0	0
1	1	1
2	10	2
3	11	3
4	100	4
5	101	5
6	110	6
7	111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F

Appendices

MUCH EMPTY