Simple Ball Collision Calculations

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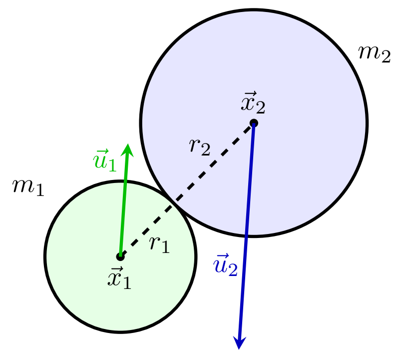
October 18, 2023

1 Ball-Ball Collision

1.1 Problem

Two perfectly symmetrical balls with radii r_i and masses m_i ($i \in 1, 2$) collide head-on. At the moment of collision, their positions are $\vec{x}_i$ and velocities $\vec{u}_i$. What would be their velocities $\vec{v}_i$ following the collision?



1.2 Solution

Note: the solution given here is based on the text in this webpage.

Conservation of momentum gives

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2, \tag{1}$$

i.e.

$$m_1\left(\vec{u}_1 - \vec{v}_1\right) = -m_2\left(\vec{u}_2 - \vec{v}_2\right). \tag{2}$$

Conservation of (kinetic) energy gives

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2, \tag{3}$$

which can be further simplified using the dot product (since for any vector $\vec{w}$, $w^2 = \vec{w}\cdot\vec{w}$) and eliminating the $\frac{1}{2}$ coefficient, leaving us with

$$m_1\left(\vec{u}_1\cdot\vec{u}_1 - \vec{v}_1\cdot\vec{v}_1\right) = -m_2\left(\vec{u}_2\cdot\vec{u}_2 - \vec{v}_2\cdot\vec{v}_2\right), \tag{4}$$

i.e. (utilizing $a^2 - b^2 = (a-b)(a+b)$)

$$m_1\left(\vec{u}_1 - \vec{v}_1\right)\cdot\left(\vec{u}_1 + \vec{v}_1\right) = m_2\left(\vec{u}_2 - \vec{v}_2\right)\cdot\left(\vec{u}_2 + \vec{v}_2\right). \tag{5}$$

The change in momentum must happen along the line connecting the centers of the balls. We can easily derive the vector connecting $\vec{x}_2$ to $\vec{x}_1$:

$$\vec{X} = \vec{x}_1 - \vec{x}_2, \tag{6}$$

and normalize it to

$$\hat{X} = \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|} = \frac{\vec{X}}{r_1 + r_2}. \tag{7}$$

The change in momentum is this given by

$$m_1 (\vec{u}_1 - \vec{v}_1) = -m_2 (\vec{u}_2 - \vec{v}_2) = \alpha \hat{X}, \tag{8}$$

where $\alpha \in (0, \infty)$.
We can use Equation 8 to express Equation 5 as

$$\hat{X} \cdot (\vec{u}_1 + \vec{v}_1) = \hat{X} \cdot (\vec{u}_2 + \vec{v}_2), \tag{9}$$

and using Equation 8 we get the following expressions for the velocities $\vec{v}_i$:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 - \frac{\alpha}{m_1}\hat{X}, \\ \vec{v}_2 &= \vec{u}_2 + \frac{\alpha}{m_2}\hat{X}. \end{aligned} \tag{10}$$

The only thing left to do is to find α. Applying Equation 10 to Equation 5 we find that

$$\hat{X} \cdot \left(2\vec{u}_1 - \frac{\alpha}{m_1}\hat{X}\right) = \hat{X} \left(2\vec{u}_2 + \frac{\alpha}{m_2}\hat{X}\right). \tag{11}$$

and since $\hat{X}$ is a unit vector, i.e. $\hat{X} \cdot \hat{X} = 1$, the above reduces to

$$2\hat{X} \cdot (\vec{u}_1 - \vec{u}_2) = \alpha \left(\frac{1}{m_1} + \frac{1}{m_2}\right), \tag{12}$$

and thus

$$\alpha = \frac{2\hat{X} (\vec{u}_1 - \vec{u}_2)}{\frac{1}{m_1} + \frac{1}{m_2}}. \tag{13}$$

To simplify things further, we can define two more quantities:

$$\Delta\vec{u} = \vec{u}_1 - \vec{u}_2, \tag{14}$$

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}, \tag{15}$$

and then Equation 13 simplifies to

$$\alpha = 2\mu\hat{X}\Delta\vec{u}. \tag{16}$$

2 Ball-Wall Collision