

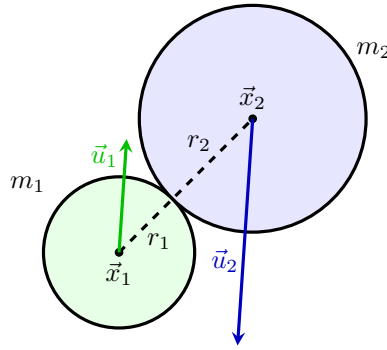
Simple Ball Collision Calculations

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1 Ball-Ball Collision

Two perfectly symmetrical balls with radii r_i and masses m_i ($i \in 1, 2$) collide head-on. At the moment of collision, their positions are \vec{x}_i and velocities \vec{u}_i . What would be their velocities \vec{v}_i following the collision?



Conservation of momentum gives

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{v}_1 + m_2 \vec{v}_2, \quad (1)$$

i.e.

$$m_1 (\vec{u}_1 - \vec{v}_1) = -m_2 (\vec{u}_2 - \vec{v}_2). \quad (2)$$

Conservation of (kinetic) energy gives

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2, \quad (3)$$

which can be further simplified using the dot product (since for any vector \vec{w} , $w^2 = \vec{w} \cdot \vec{w}$) and eliminating the $\frac{1}{2}$ coefficient, leaving us with

$$m_1 (\vec{u}_1 \cdot \vec{u}_1 - \vec{v}_1 \cdot \vec{v}_1) = -m_2 (\vec{u}_2 \cdot \vec{u}_2 - \vec{v}_2 \cdot \vec{v}_2), \quad (4)$$

i.e. (utilizing $a^2 - b^2 = (a - b)(a + b)$)

$$m_1 (\vec{u}_1 - \vec{v}_1) \cdot (\vec{u}_1 + \vec{v}_1) = m_2 (\vec{u}_2 - \vec{v}_2) \cdot (\vec{u}_2 + \vec{v}_2). \quad (5)$$

The change in momentum must happen along the line connecting the centers of the balls. We can easily derive the vector connecting \vec{x}_2 to \vec{x}_1 :

$$\vec{X} = \vec{x}_1 - \vec{x}_2, \quad (6)$$

and normalize it to

$$\hat{X} = \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|} = \frac{\vec{X}}{r_1 + r_2}. \quad (7)$$

The change in momentum is this given by

$$m_1 (\vec{u}_1 - \vec{v}_1) = -m_2 (\vec{u}_2 - \vec{v}_2) = \alpha \hat{X}, \quad (8)$$

where $\alpha \in (0, \infty)$.

We can use Equation 8 to express Equation 5 as

$$\hat{X} \cdot (\vec{u}_1 + \vec{v}_1) = \hat{X} \cdot (\vec{u}_2 + \vec{v}_2), \quad (9)$$

and using Equation 8 we get the following expressions for the velocities \vec{v}_i :

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 - \frac{\alpha}{m_1} \hat{X}, \\ \vec{v}_2 &= \vec{u}_2 + \frac{\alpha}{m_2} \hat{X}. \end{aligned} \quad (10)$$

The only thing left to do is to find α . Applying Equation 10 to Equation 5 we find that

$$\hat{X} \cdot \left(2\vec{u}_1 - \frac{\alpha}{m_1} \hat{X} \right) = \hat{X} \cdot \left(2\vec{u}_2 + \frac{\alpha}{m_2} \hat{X} \right). \quad (11)$$

and since \hat{X} is a unit vector, i.e. $\hat{X} \cdot \hat{X} = 1$, the above reduces to

$$2\hat{X} \cdot (\vec{u}_1 - \vec{u}_2) = \alpha \left(\frac{1}{m_1} + \frac{1}{m_2} \right), \quad (12)$$

and thus

$$\alpha = \frac{2\hat{X} \cdot (\vec{u}_1 - \vec{u}_2)}{\frac{1}{m_1} + \frac{1}{m_2}}. \quad (13)$$

To simplify things further, we can define two more quantities:

$$\Delta\vec{u} = \vec{u}_1 - \vec{u}_2, \quad (14)$$

$$M = \frac{1}{m_1} + \frac{1}{m_2}. \quad (15)$$

2 Ball-Wall Collision