Simple Ball Collision Calculations

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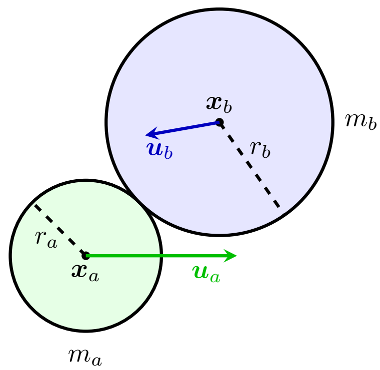
October 19, 2023

1 Ball-Ball Collision

1.1 Problem

Two perfectly spherical n-balls with radii r_a, r_b and masses m_a, m_b are moving at constant velocities $\boldsymbol{u}_a, \boldsymbol{u}_b$. At time $t=0$ their positions are $\boldsymbol{x}_a, \boldsymbol{x}_b$. At time $t=\tau$ they collide in an elastic manner.

2-dimentional illustration of the collision:



1. What is the time $t=\tau$ of the collision?
2. What are the positions $\boldsymbol{x}'_a, \boldsymbol{x}'_b$ of the centers of the balls at time $t=\tau$?
3. What are the velocities $\boldsymbol{v}_a, \boldsymbol{v}_b$ immediately following the collision?

1.2 Solution

1. At the time of collision $t=\tau$ the distance between the balls, R is the sum of their radii:

$$R = r_a + r_b. \tag{1}$$

On the other hand, the vector connecting the two balls (i.e. from $\boldsymbol{x}_a$ to $\boldsymbol{x}_b$) is

$$\Delta\boldsymbol{x} = \boldsymbol{x}_a - \boldsymbol{x}_b = \begin{bmatrix} x_a^1 - x_b^1 \\ x_a^2 - x_b^2 \\ \vdots \\ x_a^n - x_b^n \end{bmatrix}, \tag{2}$$

and thus its square length is

$$R^2 = \langle \Delta\boldsymbol{x}, \Delta\boldsymbol{x} \rangle = \sum_{k=1}^{n} \left(\Delta x^k\right)^2 = \sum_{k=1}^{n} \left(x_a^k - x_b^k\right)^2. \tag{3}$$

At each time along their trajectory, the positions of the (center of the) balls are

$$\begin{aligned} \boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a t, \\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b t, \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{x}}_\alpha$ means the position of the α ball at time $t = 0$.

Therefore, at the time of the collision the positions are

$$\begin{aligned}\boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a\tau,\\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b\tau,\end{aligned} \tag{5}$$

which can be written in components form as

$$x_\alpha^k = \tilde{x}_\alpha^k + u_\alpha^k\tau. \tag{6}$$

Subtituting Equation 6 into Equation 3 gives the following expression for R^2:

$$R^2 = \sum_{k=1}^{n}\left(\tilde{x}_a^k + u_a^k\tau - \tilde{x}_b^k - u_b^k\tau\right)^2 = \tag{7}$$

Combining all similar terms and expanding the parentheses give us an explicit expression for the distance squared in terms of powers of τ:

$$\begin{aligned}R^2 &= \sum_{k=1}^{n}\left(\tilde{x}_a^k - \tilde{x}_b^k + \tau\left[u_a^k - u_b^k\right]\right)^2\\ &= \sum_{k=1}^{n}\left(\Delta\tilde{x}^k + \tau\Delta u^k\right)^2\\ &= \sum_{k=1}^{n}\left(\left(\Delta\tilde{x}^k\right)^2 + 2\Delta\tilde{x}^k\Delta u^k\tau + \tau^2\left(\Delta u^k\right)^2\right).\end{aligned} \tag{8}$$

Since in Equation 8 we have a sum of three expressions in each term, we can rewrite it as

$$R^2 = \sum_{k=1}^{n}\left(\Delta\tilde{x}^k\right)^2 + 2\tau\sum_{k=1}^{n}\Delta\tilde{x}^k\Delta u^k + \tau^2\sum_{k=1}^{n}\left(\Delta u^k\right)^2. \tag{9}$$

Now each of these sums can be viewed in terms of an inner product:

$$R^2 = \langle\Delta\tilde{\boldsymbol{x}},\Delta\tilde{\boldsymbol{x}}\rangle + 2\tau\langle\Delta\tilde{\boldsymbol{x}},\Delta\boldsymbol{u}\rangle + \tau^2\langle\Delta\boldsymbol{u},\Delta\boldsymbol{u}\rangle. \tag{10}$$

This is a simple quadratic equation in τ, with

$$\begin{aligned}a &= \langle\Delta\boldsymbol{u},\Delta\boldsymbol{u}\rangle,\\ b &= 2\langle\Delta\tilde{\boldsymbol{x}},\Delta\boldsymbol{u}\rangle,\\ c &= \langle\Delta\tilde{\boldsymbol{x}},\Delta\tilde{\boldsymbol{x}}\rangle - R^2.\end{aligned} \tag{11}$$

For Equation 10 to have a real solution, the first condition is that $b^2 - 4ac \geq 0$, or

$$\langle\Delta\tilde{\boldsymbol{x}},\Delta\boldsymbol{u}\rangle^2 \geq \langle\Delta\boldsymbol{u},\Delta\boldsymbol{u}\rangle\langle\Delta\tilde{\boldsymbol{x}},\Delta\tilde{\boldsymbol{x}}\rangle - R^2. \tag{12}$$

2. The positions of the balls at the time of impact can be found by subtituting $t = \tau$ into Equation 5:

$$\begin{aligned}\boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a\tau,\\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b\tau,\end{aligned} \tag{13}$$

3. Note: the solution given here is based on the text in this webpage.

   Conservation of momentum gives

$$m_a\boldsymbol{u}_a + m_b\boldsymbol{u}_b = m_a\boldsymbol{v}_a + m_b\boldsymbol{v}_b, \tag{14}$$

i.e.

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right). \tag{15}$$

Conservation of (kinetic) energy gives

$$\frac{1}{2}m_au_a^2+\frac{1}{2}m_bu_b^2=\frac{1}{2}m_av_a^2+\frac{1}{2}m_bv_b^2, \tag{16}$$

which can be further simplified using the dot product (since for any vector $\boldsymbol{w}$, $w^2=\boldsymbol{w}\cdot\boldsymbol{w}$) and eliminating the $\frac{1}{2}$ coefficient, leaving us with

$$m_a\left(\boldsymbol{u}_a\cdot\boldsymbol{u}_a-\boldsymbol{v}_a\cdot\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b\cdot\boldsymbol{u}_b-\boldsymbol{v}_b\cdot\boldsymbol{v}_b\right), \tag{17}$$

i.e. (utilizing $a^2-b^2=(a-b)(a+b)$)

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)\cdot\left(\boldsymbol{u}_a+\boldsymbol{v}_a\right)=m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right)\cdot\left(\boldsymbol{u}_b+\boldsymbol{v}_b\right). \tag{18}$$

The change in momentum must happen along the line connecting the centers of the balls. We can easily derive the vector connecting $\boldsymbol{x}_b$ to $\boldsymbol{x}_a$:

$$\boldsymbol{x}=\boldsymbol{x}_a-\boldsymbol{x}_b, \tag{19}$$

and normalize it to

$$\hat{X}=\frac{\boldsymbol{x}_a-\boldsymbol{x}_b}{\|\boldsymbol{x}_a-\boldsymbol{x}_b\|}=\frac{\boldsymbol{x}}{r_a+r_b}. \tag{20}$$

The change in momentum is this given by

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right)=\alpha\hat{X}, \tag{21}$$

where $\alpha\in(0,\infty)$.

We can use Equation 21 to express Equation 18 as

$$\hat{X}\cdot\left(\boldsymbol{u}_a+\boldsymbol{v}_a\right)=\hat{X}\cdot\left(\boldsymbol{u}_b+\boldsymbol{v}_b\right), \tag{22}$$

and using Equation 21 we get the following expressions for the velocities $\boldsymbol{v}_i$:

$$\begin{aligned}\boldsymbol{v}_a&=\boldsymbol{u}_a-\frac{\alpha}{m_a}\hat{X},\\ \boldsymbol{v}_b&=\boldsymbol{u}_b+\frac{\alpha}{m_b}\hat{X}.\end{aligned} \tag{23}$$

The only thing left to do is to find α. Applying Equation 23 to Equation 18 we find that

$$\hat{X}\cdot\left(2\boldsymbol{u}_a-\frac{\alpha}{m_a}\hat{X}\right)=\hat{X}\left(2\boldsymbol{u}_b+\frac{\alpha}{m_b}\hat{X}\right). \tag{24}$$

and since $\hat{X}$ is a unit vector, i.e. $\hat{X}\cdot\hat{X}=1$, the above reduces to

$$2\hat{X}\cdot\left(\boldsymbol{u}_a-\boldsymbol{u}_b\right)=\alpha\left(\frac{1}{m_a}+\frac{1}{m_b}\right), \tag{25}$$

and thus

$$\alpha=\frac{2\hat{X}\left(\boldsymbol{u}_a-\boldsymbol{u}_b\right)}{\frac{1}{m_a}+\frac{1}{m_b}}. \tag{26}$$

To simplify things further, we can define two more quantities:

$$\Delta\boldsymbol{u}=\boldsymbol{u}_a-\boldsymbol{u}_b, \tag{27}$$

$$\mu=\frac{1}{\frac{1}{m_a}+\frac{1}{m_b}}, \tag{28}$$

and then Equation 26 simplifies to

$$\alpha=2\mu\hat{X}\Delta\boldsymbol{u}. \tag{29}$$