Simple Ball Collision Calculations

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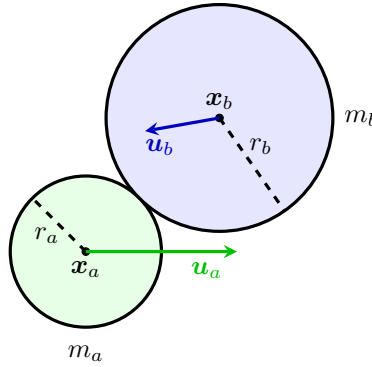
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1 Ball-Ball Collision

1.1 Problem

Two perfectly spherical n-balls with radii r_a, r_b and masses m_a, m_b are moving at constant velocities $\boldsymbol{u}_a, \boldsymbol{u}_b$. At time $t = 0$ their positions are $\boldsymbol{x}_a, \boldsymbol{x}_b$. At time $t = \tau$ they collide in an elastic manner.

2-dimentional illustration of the collision:

1. What is the time $t = \tau$ of the collision?

2. What are the positions $\boldsymbol{x}_a', \boldsymbol{x}_b'$ of the centers of the balls at time $t = \tau$?

3. What are the velocities $\boldsymbol{v}_a, \boldsymbol{v}_b$ immediately following the collision?

1.2 Solution

1. At the time of collision $t = \tau$ the distance between the balls, R is the sum of their radii:

$$R = r_a + r_b. \tag{1}$$

On the other hand, the vector connecting the two balls (i.e. from $\boldsymbol{x}_a$ to $\boldsymbol{x}_b$) is

$$\Delta\boldsymbol{x} = \boldsymbol{x}_a - \boldsymbol{x}_b = \begin{bmatrix} x_a^1 - x_b^1 \\ x_a^2 - x_b^2 \\ \vdots \\ x_a^n - x_b^n \end{bmatrix}, \tag{2}$$

and thus its square length is

$$R^2 = \langle \Delta\boldsymbol{x}, \Delta\boldsymbol{x} \rangle = \sum_{k=1}^{n} \left(\Delta x^k \right)^2 = \sum_{k=1}^{n} \left(x_a^k - x_b^k \right)^2. \tag{3}$$

At each time along their trajectory, the positions of the (center of the) balls are

$$\boldsymbol{x}_a = \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a t,$$
$$\boldsymbol{x}_b = \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b t, \tag{4}$$

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where $\tilde{\boldsymbol{x}}_\alpha$ means the position of the α ball at time $t = 0$.

Therefore, at the time of the collision the positions are

$$\begin{aligned}
\boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a \tau, \\
\boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b \tau,
\end{aligned} \tag{5}$$

which can be written in components form as

$$x_\alpha^k = \tilde{x}_\alpha^k + u_\alpha^k \tau. \tag{6}$$

Substituting **??** into Equation 3 gives the following expression for R^2:

$$R^2 = \sum_{k=1}^{n} \left(\tilde{x}_a^k + u_a^k \tau - \tilde{x}_b^k - u_b^k \tau \right)^2 = \tag{7}$$

Combining all similar terms and expanding the parentheses give us an explicit expression for the distance squared in terms of powers of τ:

$$\begin{aligned}
R^2 &= \sum_{k=1}^{n} \left(\tilde{x}_a^k - \tilde{x}_b^k + \tau \left[u_a^k - u_b^k \right] \right)^2 \\
&= \sum_{k=1}^{n} \left(\Delta \tilde{x}^k + \tau \Delta u^k \right)^2 \\
&= \sum_{k=1}^{n} \left(\left(\Delta \tilde{x}^k \right)^2 + 2 \Delta \tilde{x}^k \Delta u^k \tau + \tau^2 \left(\Delta u^k \right)^2 \right).
\end{aligned} \tag{8}$$

Since in **??** we have a sum of three expressions in each term, we can rewrite it as

$$R^2 = \sum_{k=1}^{n} \left(\Delta \tilde{x}^k \right)^2 + 2\tau \sum_{k=1}^{n} \Delta \tilde{x}^k \Delta u^k + \tau^2 \sum_{k=1}^{n} \left(\Delta u^k \right)^2. \tag{9}$$

Now each of these sums can be viewed in terms of an inner product:

$$R^2 = \langle \Delta \tilde{\boldsymbol{x}}, \Delta \tilde{\boldsymbol{x}} \rangle + 2\tau \langle \Delta \tilde{\boldsymbol{x}}, \Delta \boldsymbol{u} \rangle + \tau^2 \langle \Delta \boldsymbol{u}, \Delta \boldsymbol{u} \rangle. \tag{10}$$

This is a simple quadratic equation in τ, with

$$\begin{aligned}
a &= \langle \Delta \boldsymbol{u}, \Delta \boldsymbol{u} \rangle, \\
b &= 2 \langle \Delta \tilde{\boldsymbol{x}}, \Delta \boldsymbol{u} \rangle, \\
c &= \langle \Delta \tilde{\boldsymbol{x}}, \Delta \tilde{\boldsymbol{x}} \rangle - R^2.
\end{aligned} \tag{11}$$

For **??** to have a real solution, the first condition is that $b^2 - 4ac \geq 0$, or

$$\langle \Delta \tilde{\boldsymbol{x}}, \Delta \boldsymbol{u} \rangle^2 \geq \langle \Delta \boldsymbol{u}, \Delta \boldsymbol{u} \rangle \langle \Delta \tilde{\boldsymbol{x}}, \Delta \tilde{\boldsymbol{x}} \rangle - R^2. \tag{12}$$

2. The positions of the balls at the time of impact can be found by substituting $t = \tau$ into **??**:

$$\begin{aligned}
\boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a \tau, \\
\boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b \tau,
\end{aligned} \tag{13}$$

3. <u>Note</u>: the solution given here is based on the text in this webpage.

   Conservation of momentum gives
$$m_a \boldsymbol{u}_a + m_b \boldsymbol{u}_b = m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b, \tag{14}$$

i.e.
$$m_a \left(\boldsymbol{u}_a - \boldsymbol{v}_a \right) = -m_b \left(\boldsymbol{u}_b - \boldsymbol{v}_b \right).\tag{15}$$

Conservation of (kinetic) energy gives
$$\frac{1}{2}m_a u_a^2 + \frac{1}{2}m_b u_b^2 = \frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2,\tag{16}$$

which can be further simplified using the dot product (since for any vector $\boldsymbol{w}$, $w^2 = \boldsymbol{w} \cdot \boldsymbol{w}$) and eliminating the $\frac{1}{2}$ coefficient, leaving us with
$$m_a \left(\boldsymbol{u}_a \cdot \boldsymbol{u}_a - \boldsymbol{v}_a \cdot \boldsymbol{v}_a \right) = -m_b \left(\boldsymbol{u}_b \cdot \boldsymbol{u}_b - \boldsymbol{v}_b \cdot \boldsymbol{v}_b \right),\tag{17}$$

i.e. (utilizing $a^2 - b^2 = (a - b)(a + b)$)
$$m_a \left(\boldsymbol{u}_a - \boldsymbol{v}_a \right) \cdot \left(\boldsymbol{u}_a + \boldsymbol{v}_a \right) = m_b \left(\boldsymbol{u}_b - \boldsymbol{v}_b \right) \cdot \left(\boldsymbol{u}_b + \boldsymbol{v}_b \right).\tag{18}$$

The change in momentum must happen along the line connecting the centers of the balls. We can easily derive the vector connecting $\boldsymbol{x}_b$ to $\boldsymbol{x}_a$:
$$\boldsymbol{x} = \boldsymbol{x}_a - \boldsymbol{x}_b,\tag{19}$$

and normalize it to
$$\hat{X} = \frac{\boldsymbol{x}_a - \boldsymbol{x}_b}{\|\boldsymbol{x}_a - \boldsymbol{x}_b\|} = \frac{\boldsymbol{x}}{r_a + r_b}.\tag{20}$$

The change in momentum is this given by
$$m_a \left(\boldsymbol{u}_a - \boldsymbol{v}_a \right) = -m_b \left(\boldsymbol{u}_b - \boldsymbol{v}_b \right) = \alpha \hat{X},\tag{21}$$

where $\alpha \in (0, \infty)$.

We can use **??** to express **??** as
$$\hat{X} \cdot \left(\boldsymbol{u}_a + \boldsymbol{v}_a \right) = \hat{X} \cdot \left(\boldsymbol{u}_b + \boldsymbol{v}_b \right),\tag{22}$$

and using **??** we get the following expressions for the velocities $\boldsymbol{v}_i$:
$$\boldsymbol{v}_a = \boldsymbol{u}_a - \frac{\alpha}{m_a}\hat{X},$$
$$\boldsymbol{v}_b = \boldsymbol{u}_b + \frac{\alpha}{m_b}\hat{X}.\tag{23}$$

The only thing left to do is to find α. Applying **??** to **??** we find that
$$\hat{X} \cdot \left(2\boldsymbol{u}_a - \frac{\alpha}{m_a}\hat{X} \right) = \hat{X} \left(2\boldsymbol{u}_b + \frac{\alpha}{m_b}\hat{X} \right).\tag{24}$$

and since $\hat{X}$ is a unit vector, i.e. $\hat{X} \cdot \hat{X} = 1$, the above reduces to
$$2\hat{X} \cdot \left(\boldsymbol{u}_a - \boldsymbol{u}_b \right) = \alpha \left(\frac{1}{m_a} + \frac{1}{m_b} \right),\tag{25}$$

and thus
$$\alpha = \frac{2\hat{X} \left(\boldsymbol{u}_a - \boldsymbol{u}_b \right)}{\frac{1}{m_a} + \frac{1}{m_b}}.\tag{26}$$

To simplify things further, we can define two more quantities:
$$\Delta \boldsymbol{u} = \boldsymbol{u}_a - \boldsymbol{u}_b,\tag{27}$$
$$\mu = \frac{1}{\frac{1}{m_a} + \frac{1}{m_b}},\tag{28}$$

and then **??** simplifies to
$$\alpha = 2\mu \hat{X} \Delta \boldsymbol{u}.\tag{29}$$