Simple Ball Collision Calculations

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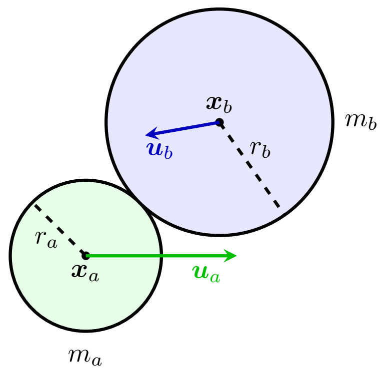
1 Ball-Ball Collision

1.1 Problem

Two perfectly spherical n-balls ($n \in 2, 3$) with radii r_a, r_b and masses m_a, m_b are moving at constant velocities $\boldsymbol{u}_a, \boldsymbol{u}_b$. At time $t = 0$ their positions are $\boldsymbol{x}_a, \boldsymbol{x}_b$. At time $t = \tau$ they collide in an elastic manner.

2-dimentional illustration of the collision:



1. What is the time $t = \tau$ of the collision?
2. What are the positions $\boldsymbol{x}'_a, \boldsymbol{x}'_b$ of the centers of the balls at time $t = \tau$?
3. What are the velocities $\boldsymbol{v}_a, \boldsymbol{v}_b$ immediately following the collision?

1.2 Solution

1. At the time of collision $t = \tau$ the distance between the balls, R is the sum of their radii:

$$R = r_a + r_b. \tag{1}$$

On the other hand, the vector connecting the two balls (i.e. from $\boldsymbol{x}_a$ to $\boldsymbol{x}_b$) is

$$\Delta\boldsymbol{x} = \boldsymbol{x}_a - \boldsymbol{x}_b = \begin{bmatrix} x_a^1 - x_b^1 \\ x_a^2 - x_b^2 \\ \vdots \\ x_a^n - x_b^n \end{bmatrix}, \tag{2}$$

and thus its square length is

$$R^2 = \langle \Delta\boldsymbol{x}, \Delta\boldsymbol{x} \rangle = \sum_{k=1}^{n} \left(\Delta x^k\right)^2 = \sum_{k=1}^{n} \left(x_a^k - x_b^k\right)^2. \tag{3}$$

At each time along their trajectory, the positions of the (center of the) balls are

$$\begin{aligned} \boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a t, \\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b t, \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{x}}_\alpha$ means the position of the α ball at time $t = 0$.

Therefore, at the time of the collision the positions are

$$\begin{aligned} \boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a \tau, \\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b \tau, \end{aligned} \tag{5}$$

which can be written in components form as

$$x_\alpha^k = \tilde{x}_\alpha^k + u_\alpha^k \tau. \tag{6}$$

Subtituting Equation 6 into Equation 3 gives the following expression for R^2:

$$R^2 = \sum_{k=1}^{n} \left(\tilde{x}_a^k + u_a^k \tau - \tilde{x}_b^k - u_b^k \tau\right)^2 = \tag{7}$$

Combining all similar terms and expanding the parentheses give us an explicit expression for the distance squared in terms of powers of τ:

$$\begin{aligned} R^2 &= \sum_{k=1}^{n} \left(\tilde{x}_a^k - \tilde{x}_b^k + \tau \left[u_a^k - u_b^k\right]\right)^2 \\ &= \sum_{k=1}^{n} \left(\Delta\tilde{x}^k + \tau \Delta u^k\right)^2 \\ &= \sum_{k=1}^{n} \left(\left(\Delta\tilde{x}^k\right)^2 + 2\Delta\tilde{x}^k \Delta u^k \tau + \tau^2 \left(\Delta u^k\right)^2\right). \end{aligned} \tag{8}$$

Since in Equation 8 we have a sum of three expressions in each term, we can rewrite it as

$$R^2 = \sum_{k=1}^{n} \left(\Delta\tilde{x}^k\right)^2 + 2\tau \sum_{k=1}^{n} \Delta\tilde{x}^k \Delta u^k + \tau^2 \sum_{k=1}^{n} \left(\Delta u^k\right)^2. \tag{9}$$

Now each of these sums can be viewed in terms of an inner product:

$$R^2 = \langle \Delta\tilde{\boldsymbol{x}}, \Delta\tilde{\boldsymbol{x}} \rangle + 2\tau \langle \Delta\tilde{\boldsymbol{x}}, \Delta\boldsymbol{u} \rangle + \tau^2 \langle \Delta\boldsymbol{u}, \Delta\boldsymbol{u} \rangle. \tag{10}$$

This is a simple quadratic equation in τ, with

$$\begin{aligned} a &= \langle \Delta\boldsymbol{u}, \Delta\boldsymbol{u} \rangle, \\ b &= 2\langle \Delta\tilde{\boldsymbol{x}}, \Delta\boldsymbol{u} \rangle, \\ c &= \langle \Delta\tilde{\boldsymbol{x}}, \Delta\tilde{\boldsymbol{x}} \rangle - R^2. \end{aligned} \tag{11}$$

For Equation 10 to have a real solution, the first condition is that $b^2 - 4ac \geq 0$, or

$$\langle \Delta\tilde{\boldsymbol{x}}, \Delta\boldsymbol{u} \rangle^2 \geq \langle \Delta\boldsymbol{u}, \Delta\boldsymbol{u} \rangle \langle \Delta\tilde{\boldsymbol{x}}, \Delta\tilde{\boldsymbol{x}} \rangle - R^2. \tag{12}$$

2. The positions of the balls at the time of impact can be found by subtituting $t = \tau$ into Equation 5:

$$\begin{aligned} \boldsymbol{x}_a &= \tilde{\boldsymbol{x}}_a + \boldsymbol{u}_a \tau, \\ \boldsymbol{x}_b &= \tilde{\boldsymbol{x}}_b + \boldsymbol{u}_b \tau, \end{aligned} \tag{13}$$

3. Note: the solution given here is based on the text in this webpage.

   Conservation of momentum gives

$$m_a \boldsymbol{u}_a + m_b \boldsymbol{u}_b = m_a \boldsymbol{v}_a + m_b \boldsymbol{v}_b, \tag{14}$$

i.e.

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right). \tag{15}$$

Conservation of (kinetic) energy gives

$$\frac{1}{2}m_a u_a^2+\frac{1}{2}m_b u_b^2=\frac{1}{2}m_a v_a^2+\frac{1}{2}m_b v_b^2, \tag{16}$$

which can be further simplified using the dot product (since for any vector $\boldsymbol{w}$, $w^2=\boldsymbol{w}\cdot\boldsymbol{w}$) and eliminating the $\frac{1}{2}$ coefficient, leaving us with

$$m_a\left(\boldsymbol{u}_a\cdot\boldsymbol{u}_a-\boldsymbol{v}_a\cdot\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b\cdot\boldsymbol{u}_b-\boldsymbol{v}_b\cdot\boldsymbol{v}_b\right), \tag{17}$$

i.e. (utilizing $a^2-b^2=(a-b)(a+b)$)

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)\cdot\left(\boldsymbol{u}_a+\boldsymbol{v}_a\right)=m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right)\cdot\left(\boldsymbol{u}_b+\boldsymbol{v}_b\right). \tag{18}$$

The change in momentum must happen along the line connecting the centers of the balls. We can easily derive the vector connecting $\boldsymbol{x}_b$ to $\boldsymbol{x}_a$:

$$\boldsymbol{x}=\boldsymbol{x}_a-\boldsymbol{x}_b, \tag{19}$$

and normalize it to

$$\hat{\boldsymbol{X}}=\frac{\boldsymbol{x}_a-\boldsymbol{x}_b}{\|\boldsymbol{x}_a-\boldsymbol{x}_b\|}=\frac{\boldsymbol{x}}{r_a+r_b}. \tag{20}$$

The change in momentum is this given by

$$m_a\left(\boldsymbol{u}_a-\boldsymbol{v}_a\right)=-m_b\left(\boldsymbol{u}_b-\boldsymbol{v}_b\right)=\alpha\hat{\boldsymbol{X}}, \tag{21}$$

where $\alpha\in(0,\infty)$.

We can use Equation 21 to express Equation 18 as

$$\hat{\boldsymbol{X}}\cdot\left(\boldsymbol{u}_a+\boldsymbol{v}_a\right)=\hat{\boldsymbol{X}}\cdot\left(\boldsymbol{u}_b+\boldsymbol{v}_b\right), \tag{22}$$

and using Equation 21 we get the following expressions for the velocities $\boldsymbol{v}_i$:

$$\begin{aligned}\boldsymbol{v}_a&=\boldsymbol{u}_a-\frac{\alpha}{m_a}\hat{\boldsymbol{X}},\\ \boldsymbol{v}_b&=\boldsymbol{u}_b+\frac{\alpha}{m_b}\hat{\boldsymbol{X}}.\end{aligned} \tag{23}$$

The only thing left to do is to find α. Applying Equation 23 to Equation 18 we find that

$$\hat{\boldsymbol{X}}\cdot\left(2\boldsymbol{u}_a-\frac{\alpha}{m_a}\hat{\boldsymbol{X}}\right)=\hat{\boldsymbol{X}}\left(2\boldsymbol{u}_b+\frac{\alpha}{m_b}\hat{\boldsymbol{X}}\right). \tag{24}$$

and since $\hat{\boldsymbol{X}}$ is a unit vector, i.e. $\hat{\boldsymbol{X}}\cdot\hat{\boldsymbol{X}}=1$, the above reduces to

$$2\hat{\boldsymbol{X}}\cdot\left(\boldsymbol{u}_a-\boldsymbol{u}_b\right)=\alpha\left(\frac{1}{m_a}+\frac{1}{m_b}\right), \tag{25}$$

and thus

$$\alpha=\frac{2\hat{\boldsymbol{X}}\left(\boldsymbol{u}_a-\boldsymbol{u}_b\right)}{\frac{1}{m_a}+\frac{1}{m_b}}. \tag{26}$$

To simplify things further, we can define two more quantities:

$$\Delta\boldsymbol{u}=\boldsymbol{u}_a-\boldsymbol{u}_b, \tag{27}$$

$$\mu=\frac{1}{\frac{1}{m_a}+\frac{1}{m_b}}, \tag{28}$$

and then Equation 26 simplifies to

$$\alpha=2\mu\hat{\boldsymbol{X}}\Delta\boldsymbol{u}. \tag{29}$$

2 Ball-Wall Collision

2.1 Problem

Given a single ball moving with a constant velocity $\boldsymbol{u}$ and a wall defined as a plane with normal $\hat{\boldsymbol{n}}$ and containing the point $\boldsymbol{p}_0$ -

1. will the ball collide with the wall at any time $t \geq 0$? If so - what is the position of the ball at the moment of collision?

2. what is the velocity $\boldsymbol{v}$ of the ball immediately following the collision?

2.2 Solution

1. We can answer the first question by calculating the distance between the ball and the wall at any given time t. The first question then reduces to whether the distance is at any time equal to the radius of the ball, r, and what is the position of the ball at that time. To find the distance...

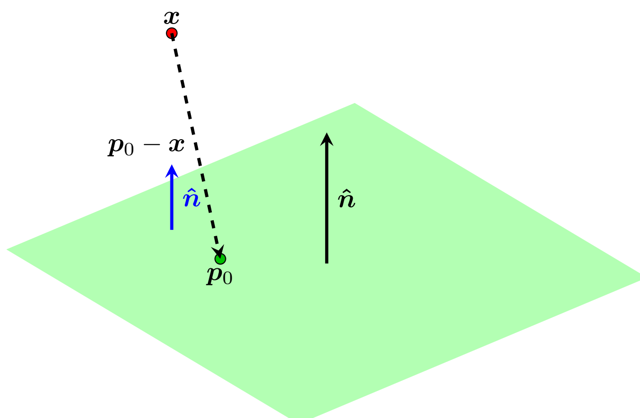


Figure 1: