Introduction to Complex Numbers and Their Uses

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Important note! These notes are intended for me, the lecturer, and are therefore very brief and don't convey the material in the same way the lecture does. They can be used by students with this in mind.

1 Why Use Complex Numbers?

Topics:

1. From natural numbers to real numbers via extensions (to solve equations).
2. Square roots for negative real numbers as a justification for imaginary numbers.
3. The real line as scaling of the unit $\mathbf{1}$ and the imaginary line as scaling of the unit $\mathbf{i} = \sqrt{-1}$
4. The real and imaginary sets as orthogonal sets except for 0.

2 Algebraic Properties

2.1 The Complex Plane as a Vector Plane

2.2 Addition, multiplication and conjugation of complex numbers

Addition: let $z = a + b\mathbf{i}$ and $w = c + d\mathbf{i}$, then

$$z + w = a + b\mathbf{i} + c + d\mathbf{i} = (a + c) + (b + d)\mathbf{i}. \tag{1}$$

Multiplication: let $z = a + b\mathbf{i}$ and $w = c + d\mathbf{i}$, then

$$\begin{aligned} z \cdot w &= (a + b\mathbf{i})(c + d\mathbf{i}) \\ &= ac + ad\mathbf{i} + b\mathbf{i}c + b\mathbf{i}d\mathbf{i} \\ &= ac + (ad + bc)\mathbf{i} + bd\mathbf{i}^2 \\ &= ac + (ad + bc)\mathbf{i} - bd \\ &= ac - bd + (ad + bc)\mathbf{i}. \end{aligned} \tag{2}$$

Dividing by i: to find what is $\frac{1}{\mathbf{i}}$ we can just multiply the fraction by $\frac{\mathbf{i}}{\mathbf{i}}$ (since $\frac{x}{x} = 1$ for any $x \neq 0$):

$$\frac{1}{\mathbf{i}} = \frac{1}{\mathbf{i}} \cdot \frac{\mathbf{i}}{\mathbf{i}} = \frac{\mathbf{i}}{\mathbf{i} \cdot \mathbf{i}} = \frac{\mathbf{i}}{\mathbf{i}^2} = \frac{\mathbf{i}}{-1} = -\mathbf{i}. \tag{3}$$

Norm: we can define the norm of a complex number in a similar way to that of vectors in $\mathbf{R}^2$, i.e. for $z = a + b\mathbf{i}$

$$|z|^2 = a^2 + b^2. \tag{4}$$

Conjugate: we define a new property of a complex number $z = a + b\mathbf{i}$ called its **conjugate**, z^*:

$$z^* = a - b\mathbf{i}. \tag{5}$$

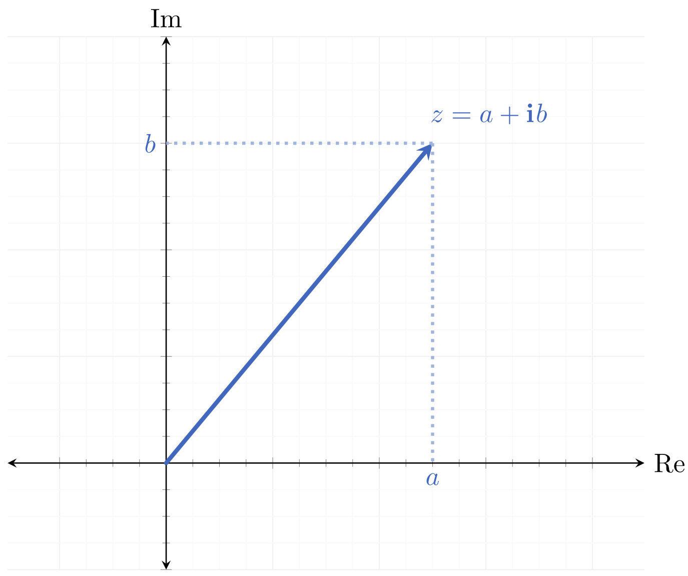

Figure 1: This is a test figure.

(see Figure 2)

We can then define the square norm of a complex number z to be

$$|z|^2 = zz^*, \tag{6}$$

since for $z = a + b\mathbf{i}$ we get

$$zz^* = (a + b\mathbf{i})(a - b\mathbf{i}) = a^2 - \cancel{ab\mathbf{i} + b\mathbf{i}a} - b^2\mathbf{i}^2 = a^2 + b^2 = |z|^2 . \tag{7}$$

3 Polar (Geomteric) Representation

3.1 Norm and angle

Just like with vectors in $\mathbb{R}^2$, we can use the polar coordinates system to describe a complex number $z = a + b\mathbf{i}$: we take the norm of the number $|z|$ as the first coordinate, and its angle with the real axis as the second coordinate (see Figure 1).

The correspondence between the Cartesian coordinates and polar coordinates are as follows:

$$\begin{aligned} \mathrm{Re}\{z\} &= r\cos(\varphi), \\ \mathrm{Im}\{z\} &= r\sin(\varphi). \end{aligned} \tag{8}$$

Thus, instead of writing $z = a + b\mathbf{i}$, we can write

$$z = r\cos(\varphi) + r\sin(\varphi)\mathbf{i} = r\left[\cos(\varphi) + \mathbf{i}\sin(\varphi)\right]. \tag{9}$$

This gives as the following Transformation rules between the coordinate systems:

$$\begin{aligned} r = |z| &= \sqrt{a^2 + b^2}, \\ \varphi &= \arctan\left(\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}\right). \end{aligned} \tag{10}$$

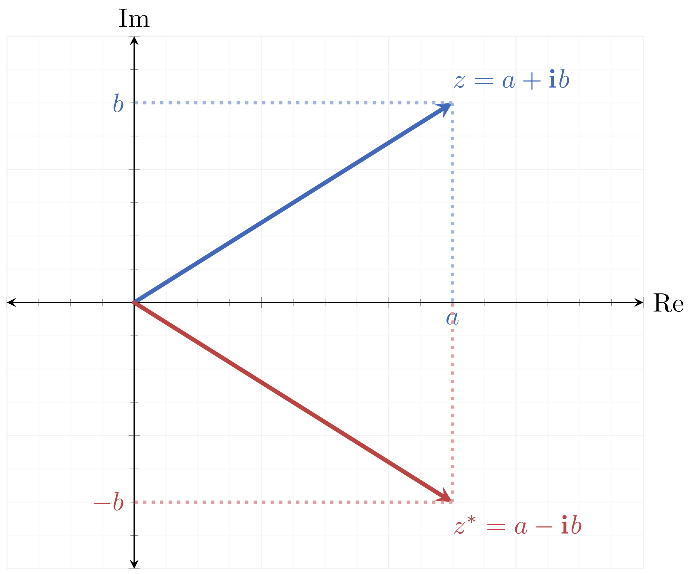

Figure 2: Complex conjugate of $z = a + b\mathbf{i}$, $z^* = a - b\mathbf{i}$.

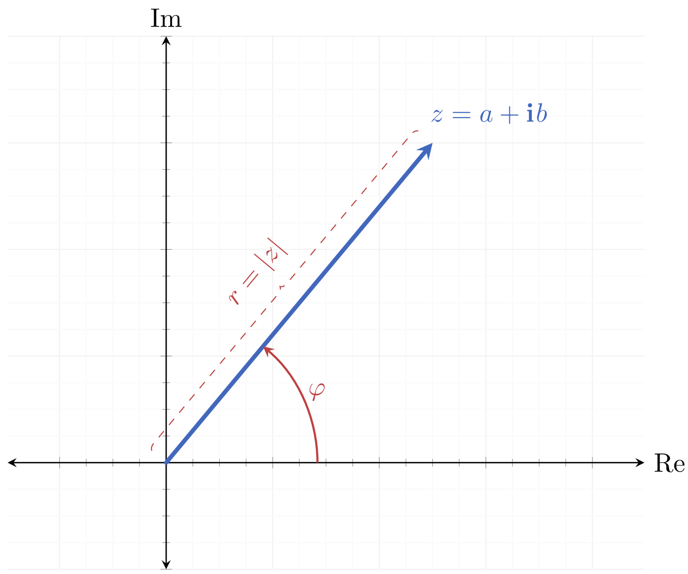

Figure 3: Complex number $z = a + b\mathbf{i}$ in its polar form $r\left[\cos(\varphi) + \mathbf{i}\sin(\varphi)\right]$.

3.2 Product of Complex Numbers

Multiplication of complex numbers becomes very simple in polar form: let $z_1 = r_1 \left[\cos(\varphi_1) + \mathbf{i}\sin(\varphi_1)\right]$ and $z_2 = r_2 \left[\cos(\varphi_2) + \mathbf{i}\sin(\varphi_2)\right]$. We then get

$$\begin{aligned} z_1 \cdot z_2 &= r_1 \left[\cos(\varphi_1) + \mathbf{i}\sin(\varphi_1)\right] \cdot r_2 \left[\cos(\varphi_2) + \mathbf{i}\sin(\varphi_2)\right] \\ &= r_1 r_2 \Big[\cos(\varphi_1)\cos(\varphi_2) + \mathbf{i}\cos(\varphi_1)\sin(\varphi_2) + \mathbf{i}\sin(\varphi_1)\cos(\varphi_2) + \mathbf{i}^2\sin(\varphi_1)\sin(\varphi_2)\Big] \\ &= r_1 r_2 \Big[\cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) + \mathbf{i}\left[\cos(\varphi_1)\sin(\varphi_2) + \sin(\varphi_1)\cos(\varphi_2)\right]\Big]. \end{aligned} \tag{11}$$

Recall the following two trigonometric identites:

$$\begin{aligned} \cos(a)\cos(b) - \sin(a)\sin(b) &= \cos(a+b), \\ \cos(a)\sin(b) + \sin(a)\cos(b) &= \sin(a+b). \end{aligned} \tag{12}$$

Using these two identites, Equation 11 becomes very simple:

$$z_1 z_2 = r_1 r_2 \left[\cos(\varphi_1 + \varphi_2) + \mathbf{i}\sin(\varphi_1 + \varphi_2)\right]. \tag{13}$$

We can interpret the product of two complex numbers using Equation 13 as **scaling** z_1 by the norm of z_2, and **rotating** it by the angle of z_2 (or vice-versa, since the product is commutative).

A general product of $n \in \mathbb{N}$ non-zero complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$ (each with norm r_k and angle φ_k) takes the following polar form:

$$\begin{aligned} \prod_{k=1}^{n} z_k &= r_1 r_2 \ldots r_n \Big[\cos\big(\varphi_1 + \varphi_2 + \cdots + \varphi_n + \mathbf{i}\sin(\varphi_1 + \varphi_2 + \cdots + \varphi_n)\big)\Big] \\ &= \prod_{k=1}^{n} r_k \left[\cos\left(\sum_{k=1}^{n}\varphi_k\right) + \mathbf{i}\sin\left(\sum_{k=1}^{n}\varphi_k\right)\right] \\ &= R\left[\cos(\Phi) + \mathbf{i}\sin(\Phi)\right], \end{aligned} \tag{14}$$

where $R = \prod\limits_{k=1}^{n} r_k$, and $\Phi = \sum\limits_{k=1}^{n} \varphi_k$.

3.3 Roots of Complex Numbers

If $w = r_w \left[\cos(\varphi_w) + \mathbf{i}\sin(\varphi_w)\right]$ is the n-th root of $z = r_z \left[\cos(\varphi_z) + \mathbf{i}\sin(\varphi_z)\right]$ then for $w^n = z$ we get that $r_w^n = r_z$ and that $n\varphi_w = \varphi_z + 2\pi k$, where $k \in \mathbb{Z}$, and therefore

$$z^{\frac{1}{n}} = \sqrt[n]{r_z}\left[\cos\left(\frac{\varphi_z + 2\pi k}{n}\right) + \mathbf{i}\sin\left(\frac{\varphi_z + 2\pi k}{n}\right)\right]. \tag{15}$$

4 Exponential Representation

4.1 Euler's Formula

Exponentiating $\mathbf{i}$ with non-negative integers yields the following pattern:

$$\begin{aligned} \mathbf{i}^0 &= 1 \\ \mathbf{i}^1 &= \mathbf{i} \\ \mathbf{i}^2 &= -1 \\ \mathbf{i}^3 &= -\mathbf{i} \\ \mathbf{i}^4 &= 1 \\ \mathbf{i}^5 &= \mathbf{i} \\ \mathbf{i}^6 &= -1 \\ \mathbf{i}^7 &= -\mathbf{i} \\ &\vdots \end{aligned}$$

i.e. we get the infinitely repeating sequence $1, \mathbf{i}, -1, -\mathbf{i}$.

Now, the Taylor series of sine and cosine around $x_0 = 0$:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \tag{16}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{17}$$

And of the exponential function $f(x) = \mathbf{e}^x$ is

$$\mathbf{e}^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \tag{18}$$

Therefore, a generic complex number has the polar form

$$\begin{aligned}
z &= r\left[\cos(\varphi) + \mathbf{i}\sin(\varphi)\right] \\
&= r\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots + \mathbf{i}\left[\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots\right]\right) \\
&= r\left(1 + \mathbf{i}\varphi - \frac{\varphi^2}{2!} - \frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + \mathbf{i}\frac{\varphi^5}{5!} - \frac{\varphi^6}{6!} - \frac{\varphi^7}{7!} + \dots\right) \\
&= r\left(\mathbf{i}^0 + \mathbf{i}^1\varphi + \mathbf{i}^2\frac{\varphi^2}{2!} + \mathbf{i}^3\frac{\varphi^3}{3!} + \mathbf{i}^4\frac{\varphi^4}{4!} + \mathbf{i}^5\frac{\varphi^5}{5!} + \mathbf{i}^6\frac{\varphi^6}{6!} + \mathbf{i}^7\frac{\varphi^7}{7!} + \dots\right) \\
&= r\left((\mathbf{i}\varphi)^0 + (\mathbf{i}\varphi)^1 + \frac{(\mathbf{i}\varphi)^2}{2!} + \frac{(\mathbf{i}\varphi)^3}{3!} + \frac{(\mathbf{i}\varphi)^4}{4!} + \frac{(\mathbf{i}\varphi)^5}{5!} + \frac{(\mathbf{i}\varphi)^6}{6!} + \frac{(\mathbf{i}\varphi)^7}{7!} + \dots\right) \\
&= r\mathbf{e}^{\mathbf{i}\varphi}.
\end{aligned} \tag{19}$$

The last equality is called **Euler's formula**. Here it is explicitly:

$$\mathbf{e}^{\mathbf{i}\varphi} = \cos(\varphi) + \mathbf{i}\sin(\varphi). \tag{20}$$

For the specific case $\varphi = \pi$[1] we get another famous relation:

$$\mathbf{e}^{\mathbf{i}\pi} = \cos(\pi) + \cancel{\mathbf{i}\sin(\pi)} = -1 + 0 = -1, \tag{21}$$

or in a "prettier" form:

$$\mathbf{e}^{\mathbf{i}\pi} + 1 = 0. \tag{22}$$

4.2 Benefits of the Exponential Representation

4.2.1 The S-plane

Exponenting by a complex number: let $z = a + \mathbf{i}b$, then

$$\mathbf{e}^z = \mathbf{e}^{a+\mathbf{i}b} = \mathbf{e}^a\mathbf{e}^{\mathbf{i}b}. \tag{23}$$

This shows that a, the real part of z, controls scaling - while b, its imaginary part, controls rotation.

Proving trigonometric identities becomes almost trivial using the exponential representation. We start by showing how $\sin(x)$ and $\cos(x)$ can be expressed using exponentials. Let's write

$$\begin{aligned}
\mathbf{e}^{\mathbf{i}x} &= \cos(x) + \mathbf{i}\sin(x), \\
\mathbf{e}^{-\mathbf{i}x} &= \cos(x) - \mathbf{i}\sin(x).
\end{aligned} \tag{24}$$

[1]recall that π radians $= 180°$.

By adding and subtracting the two equations, we get, respectively:

$$\begin{aligned} \cos(x) &= \frac{\mathbf{e}^{\mathbf{i}x} + \mathbf{e}^{-\mathbf{i}x}}{2}, \\ \sin(x) &= \frac{\mathbf{e}^{\mathbf{i}x} - \mathbf{e}^{-\mathbf{i}x}}{2\mathbf{i}}. \end{aligned} \tag{25}$$

Using these definitions the process of proving so-called *trigonometric identities* becomes almost trivial. For example, let's use it to prove the identity $\sin(2x) = 2\sin(x)\cos(x)$: we start by writing $\sin(2x)$ in the exponent form:

$$\sin(2x) = \frac{1}{2\mathbf{i}}\left(\mathbf{e}^{2\mathbf{i}x} - \mathbf{e}^{-2\mathbf{i}x}\right). \tag{26}$$

Similarly for the RHS:

$$\begin{aligned} 2\sin(x)\cos(x) &= \cancel{2}\frac{\mathbf{e}^{\mathbf{i}x} + \mathbf{e}^{-\mathbf{i}x}}{\cancel{2}}\,\frac{\mathbf{e}^{\mathbf{i}x} - \mathbf{e}^{-\mathbf{i}x}}{2\mathbf{i}} \\ &= \frac{1}{2\mathbf{i}}\mathbf{e}^{\mathbf{i}x}\mathbf{e}^{\mathbf{i}x} - \cancel{\mathbf{e}^{\mathbf{i}x}\mathbf{e}^{-\mathbf{i}x} + \mathbf{e}^{-\mathbf{i}x}\mathbf{e}^{\mathbf{i}x}} - \mathbf{e}^{-\mathbf{i}x}\mathbf{e}^{-\mathbf{i}x} \\ &= \frac{1}{2\mathbf{i}}\left(\mathbf{e}^{2\mathbf{i}x} - \mathbf{e}^{-2\mathbf{i}x}\right), \end{aligned} \tag{27}$$

just like the LHS.

Let's do another one: $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$. LHS:

$$\cos(x) + \cos(y) = \frac{1}{2}\left(\mathbf{e}^{\mathbf{i}x} + \mathbf{e}^{-\mathbf{i}x} + \mathbf{e}^{\mathbf{i}y} + \mathbf{e}^{-\mathbf{i}y}\right) \tag{28}$$

RHS:

$$\begin{aligned} 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) &= \frac{1}{2}\left(\mathbf{e}^{\mathbf{i}\frac{x+y}{2}} + \mathbf{e}^{-\mathbf{i}\frac{x+y}{2}}\right)\left(\mathbf{e}^{\mathbf{i}\frac{x-y}{2}} + \mathbf{e}^{-\mathbf{i}\frac{x-y}{2}}\right) \\ &= \frac{1}{2}\left(\mathbf{e}^{\frac{\mathbf{i}}{2}(x+y+x-y)} + \mathbf{e}^{\frac{\mathbf{i}}{2}(x+y-x+y)} + \mathbf{e}^{\frac{\mathbf{i}}{2}(-x-y+x-y)} + \mathbf{e}^{\frac{\mathbf{i}}{2}(x-x-y-y)}\right) \\ &= \frac{1}{\cancel{2}}\left(\mathbf{e}^{\frac{\cancel{2}\mathbf{i}x}{\cancel{2}}} + \mathbf{e}^{\frac{\cancel{2}\mathbf{i}y}{\cancel{2}}} + \mathbf{e}^{-\frac{\cancel{2}\mathbf{i}y}{\cancel{2}}} + \mathbf{e}^{-\frac{\cancel{2}\mathbf{i}x}{\cancel{2}}}\right) \\ &= \frac{1}{2}\left(\mathbf{e}^{\mathbf{i}x} + \mathbf{e}^{\mathbf{i}y} + \mathbf{e}^{-\mathbf{i}y} + \mathbf{e}^{-\mathbf{i}x}\right) \\ &= \frac{1}{2}\left(\mathbf{e}^{\mathbf{i}x} + \mathbf{e}^{-\mathbf{i}x} + \mathbf{e}^{\mathbf{i}y} + \mathbf{e}^{-\mathbf{i}y}\right) \\ &= \cos(x) + \cos(y). \end{aligned} \tag{29}$$