Exercise 5: Eigenvectors and Eigenvalues (Solution)

Problem 1: Geometric Interpretation

What are the eigenvectors and corresponding eigenvalues for the following transformations (answer without direct calculations)?

1. $T(\vec{v}) = -3\vec{v}$.

Answer:
The transformation scales all vectors by $\lambda = -3$. Since all vectors keep their directions, this is the only eigenvalue, and all vectors are eigenvectors of the transformation.

2. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Answer:
The transformation flips the y-component of any vector in $\mathbb{R}^2$. The only vectors that remain on the same direction before and after application of the transformation are vectors on the x-axis, that do not change and thus have corresponding eigenvalues $\lambda_1 = 1$, and vectors on the y-axis which are flipped, and thus have a corresponding eigenvalue $\lambda_2 = -1$.

3. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.

Answer:
The transformation exchanges the x- and y-components of any vector in $\mathbb{R}^2$. The only vectors which do not change their directions before and after application of the transformation are the vectors with angles $\theta = \pm 45°$ or $\theta = \pm 135°$ (where $-135° = 225°$ and $-45° = 315°$) in respect to the x-axis. For example:

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Vectors at $45°, 225°$ relative to the x-axis are unchanged (e.g. $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, while vectors with angle $135°, 315°$ relative to the x-axis are scaled bt $\lambda = -1$.
Thus, the two families of eigenvectors are represented by

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with corresponding eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = -1.$$

Problem 2: Calculating Eigenvectors and Eigenvalues

Calculate all eigenvectors and corresponding eigenvalues for the following transformations:

1. $\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

Answer:
As always, we start with solving

$$|A - \lambda I| = 0,$$

which in this case is

$$\begin{vmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = (-3-\lambda)^2 = 0,$$

for which the solution is $\lambda = -3$.
Subtituting $\lambda = -3$ into the equation $A\vec{v} = \lambda\vec{v}$ we get

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix},$$

which translates to

$$-3x = -3x,$$

and

$$-3y = -3y,$$

which translate both to any vector in $\mathbb{R}^2$, as any x and y solve these equations.
This is exactly what we expect from an isometric scaling matrix.

2. $\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$

Answer:
Again, strating with

$$0 = \begin{vmatrix} -3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (-3-\lambda)(3-\lambda) = -9 + 3\lambda - 3\lambda + \lambda^2 = \lambda^2 - 9,$$

which has the solutions

$$\lambda_{1,2} = \pm 3.$$

For $\lambda_1 = -3$,

$$\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix},$$

which translates to

$$-3x = -3x, \quad 3y = -3y,$$

i.e. any non zero $x \in \mathbb{R}$, and $y = 0$. This corresponds to the vectors of the family $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e. vectors on the x-axis. When plugged back into the equation,

$$\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

as expected.
Similarily, for $\lambda_2 = 3$, we get the family of vectors $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, i.e. vectors lying on the y-axis.

3. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Answer:
Starting with

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

we get

$$(1-\lambda)(-1-\lambda) = -1 - \lambda + \lambda + \lambda^2 = \lambda^2 - 1,$$

for which the solution is

$$\lambda_{1,2} = \pm 1.$$

For $\lambda_1 = 1$,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

means

$$x = x \text{ and } -y = y,$$

i.e. any (non-zero) x-value and $y = 0$. A representative of this family is $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Verifying:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 \\ 0 \cdot 1 - 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_1 = -1$,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix}$$

means

$$x = -x \text{ and } y = y,$$

i.e. $x = 0$ and any (non-zero) y-value. A representative of this family is $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
Verifying:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 0 \cdot 1 \\ 0 \cdot 0 - 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Answer:
As always, we start with

$$0 = \begin{vmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1,$$

for which the solution is $\lambda_{1,2} = \pm 1$.
For $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e. $x = y$. A representive vector for this family is $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Verifying:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_1 = -1$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e. $y = -x$. A representive for this family is $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Verifying:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + 1 \cdot (-1) \\ 1 \cdot 1 + 0 \cdot (-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

5. $\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}$

Answer:
We start with

$$0 = \begin{vmatrix} 5 - \lambda & 4 \\ 2 & -2 - \lambda \end{vmatrix} = (5 - \lambda)(-2 - \lambda) - 8 = -10 - 5\lambda + 2\lambda + \lambda^2 - 8 = \lambda^2 - 3\lambda - 18,$$

Solving the equation using the quadratic formula yields

$$\lambda_{1,2} = \frac{3 \pm \sqrt{3^2 + 4 \cdot 18}}{2} = \frac{3 \pm \sqrt{81}}{2} = \frac{3 \pm 9}{2} = -3, 6.$$

For $\lambda_1 = -3$,

$$\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix},$$

which translates to

$$5x + 4y = -3x,$$

i.e. $8x + 4y = 0$ or an $x : y$ ratio of $1 : -2$. We can use $\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as a representive of this vector family. Verifying:

$$\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 1 - 4 \cdot 2 \\ 2 \cdot 1 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Now for $\lambda_2 = 6$:

$$\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix},$$

i.e. $5x + 4y = 6x$, or $x = 4y$, which can be represented by $\vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Verifying:

$$\begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \cdot 4 + 4 \cdot 1 \\ 2 \cdot 4 - 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Summary:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} (\lambda_1 = -3), \quad \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} (\lambda_2 = 6).$$

Problem 3: Challange

What do you expect would are the eigenvectors and eigenvalues of the 3-dimensional rotation matrices by φ, ψ around the y- and z-axes, respectively? Explain and then calculate them directly. The two matrices are:

$$R^y_\varphi = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}, \quad R^z_\psi = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Answer:
For R^y_φ,

$$\begin{aligned} 0 &= \begin{vmatrix} \cos(\varphi) - \lambda & 0 & \sin(\varphi) \\ 0 & 1-\lambda & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) - \lambda \end{vmatrix} \\ &= \big(\cos(\varphi) - \lambda\big)(1-\lambda)\big(\cos(\varphi) - \lambda\big) + \sin(\varphi)(1-\lambda)\sin(\varphi) \\ &= \Big(\cos^2(\varphi) - 2\lambda\cos(\varphi) + \lambda^2\Big)(1-\lambda) + (1-\lambda)\sin^2(\varphi) \\ &= (1-\lambda)\Big[\cos^2(\varphi) - 2\lambda\cos(\varphi) + \lambda^2 + \sin^2(\varphi)\Big] \\ &= (1-\lambda)\Big[\lambda^2 - 2\lambda\cos(\varphi) + 1\Big]. \end{aligned}$$

The solution for this polynomial is either $\lambda_1 = 1$ (from the left parantheses), and

$$\lambda_{2,3} = \frac{2\cos(\varphi) \pm \sqrt{4\cos^2(\varphi) - 4}}{2} = \frac{2\cos(\varphi) \pm 2\sqrt{\cos^2(\varphi) - 1}}{2} = \cos(\varphi) \pm \sqrt{\cos^2(\varphi) - 1}.$$

Since the image of $\cos(\varphi)$ is $[-1, 1]$, the image of $\cos^2(\varphi)$ is $[0, 1]$, and thus $\lambda_{2,3}$ exist only for $\cos(\varphi) = \pm 1$, i.e. for $\varphi = 0°$ or $\varphi = 180°$. These angles correspond to either the identity matrix (no rotation), and to a flip in the xz-plane(180°), respectively. Both actions have all vectors as eigenvectors.

The general case, therefore, is for the eigenvalue $\lambda_1 = 1$. Let's calculate its corresponding eigenvectors:

$$\begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which corresponds to

$$\begin{cases} \cos(\varphi)x + \sin(\varphi)z = x, \\ y = y, \\ -\sin(\varphi)x + \cos(\varphi)z = z. \end{cases}$$

The first and third equations force one of two cases:

1. If $\varphi \neq 0°$, then both $\cos(\varphi)$ and $\sin(\varphi)$ are different than 0, and the only possible solution is $x = z = 0$, which gives vectors of the form $\vec{v} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$, and can be represented by the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
2. If $\varphi = 0°$ then $\cos(\varphi) = 1,\ \sin(\varphi) = 0$, and we get $x = x,\ y = y,\ z = z$, which means that any vector is an eigenvector.