Exercise 1: Sets, Functions and Graphs

Answer:
Important notations:

- Curly brackets (i.e. $\{\}$) represent a set.
- The empty set is written as $\varnothing$.
- $x \in A$ means x is an element of the set A, and $y \notin A$ means that y is not an element of A.
- $\wedge$ and $\vee$ mean *and* and *or*, respectively.
- Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$.
  (By some definitions $0 \in \mathbb{N}$, by others $0 \notin \mathbb{N}$)
- Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.
- Rational numbers: $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$.
- Real numbers: $\mathbb{R} = \left\{ x \mid x \in (-\infty, \infty) \right\}$.
  (The formal definition is too complicated for this course[a])
- Complex numbers: $\mathbb{C} = \left\{ x + iy \mid x, y \in \mathbb{R},\ i^2 = -1 \right\}$.

[a] Some formal definitions can be found here: `https://en.wikipedia.org/wiki/Construction_of_the_real_numbers`.

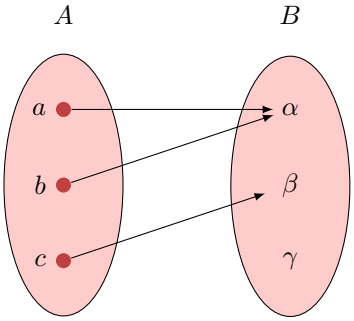
Problem 1: Sets

1. Write the following sets explicitly:
   (a) $A = \left\{ x \in \mathbb{N} \mid 1 < x \leq 7 \right\}$
   (b) $A = \left\{ x \in \mathbb{Z} \mid x < 5 \right\}$
   (c) $A = \left\{ x \in \mathbb{R} \mid x^2 = -1 \right\}$
   (d) $A = \{ x \in \mathbb{N} \wedge x \in \mathbb{Q} \}$
   (e) $A = \left\{ x \in \mathbb{R} \mid x^2 - 3x - 4 = 0 \right\}$
   (f) $A = \left\{ x \in \mathbb{R} \mid x < 5 \wedge x \geq 2 \right\}$
2. Determine the relation between the sets:
   (a) $\mathbb{R},\ \mathbb{Q},\ \mathbb{Z},\ \mathbb{N}$
   (b) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$
   (c) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$
   (d) $A = \mathbb{Z},\ B = \left\{ \pm x \mid x \in \mathbb{N} \cup \left\{ 0 \right\} \right\}$
   (e) $A = \left\{ \pi, e, \sqrt{(2)} \right\},\ B = \mathbb{Q}$
3. Using Venn diagrams, show the following relations:
   (a) $A \cup B = B \cup A$
   (b) $A \cap B = B \cap A$
   (c) $(A \cup B) \cup C = A \cup (B \cup C)$
   (d) $(A \cap B) \cap C = A \cap (B \cap C)$
   (e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (g) $(A \cup B)^C = A^C \cap B^C$
   (h) $(A \cap B)^C = A^C \cup B^C$

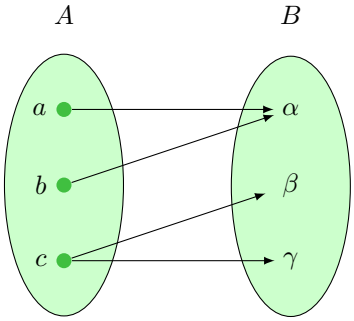
4. Cartesian products:
   (a) What is the Cartesian product of $A = \{x,\ y,\ z\}$ and $B = \{a,\ b,\ c\}$?
   (b) Is the following true: $(x,\ a) = (a,\ x)$?
   (c) What is B^2? What is A^3?
5. Extra: Prove that $\sqrt{2}$ is irrational. (In formal writing: $\nexists p, q \in \mathbb{Z} \to \sqrt{2} = \frac{p}{q}$. Simply written: $\sqrt{2} \notin \mathbb{Q}$)

Problem 2: Functions

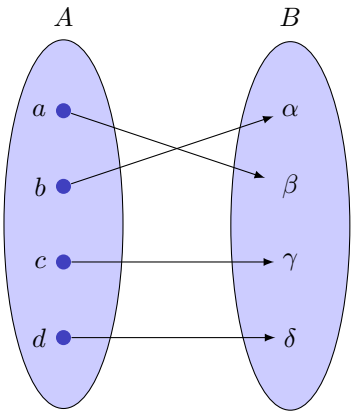
1. Which of the following figures represent functions? Which of these functions are injective, surjective and/or bijective?



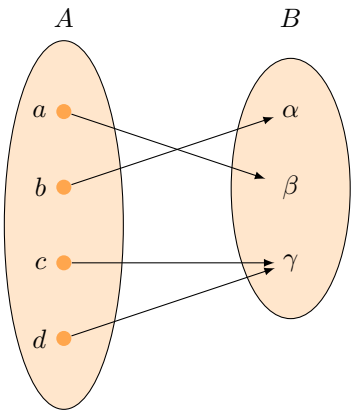
(1)



(2)



(3)



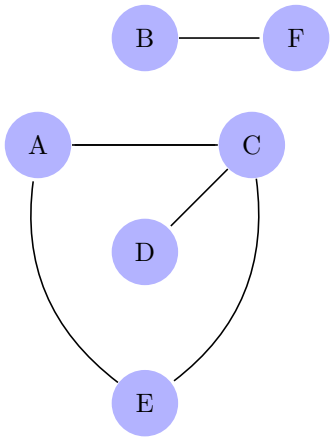
(4)

2. Which of the following functions are injective over $\mathbb{R}$?
   (a) $f(x) = x^2$
   (b) $f(x) = x^3$
   (c) $f(x) = \sin(x)$
   (d) $f(x) = \log(x)$
   (in the context of this course, $\log(x)$ is the natural logarithm, also known as $\ln(x)$)
   (e) $f(x) = |x|$
   (f) $f(x) = \sqrt{x}$
3. Find the images of the functions from 2.
4. For each of the injective functions from 2, find its inverse.
5. For each of the non injective functions from 2, find a domain over which it is injective.

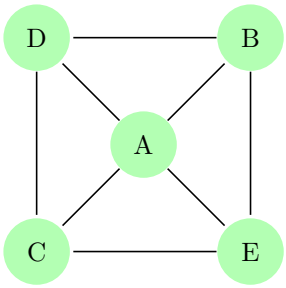
Find the inverse of the following functions (over $\mathbb{R}$):

1. $f(x) = x + 5$
2. $f(x) = x^7$
3. $f(x) = \frac{1}{3x-1}$
4. $f(x) = x^2 + 1$
5. $f(x) = e^{-x}$

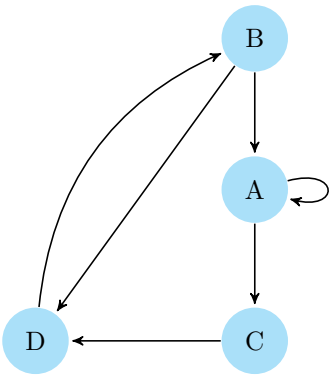
Problem 3: Graphs



(1)



(2)



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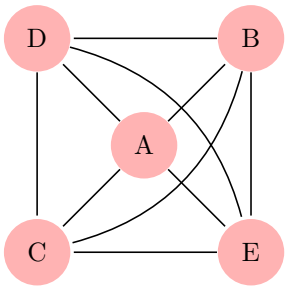

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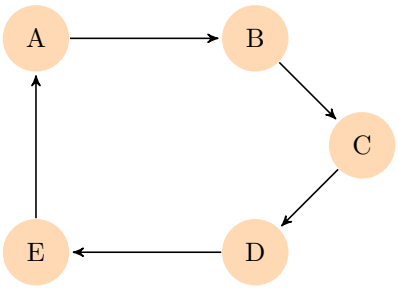
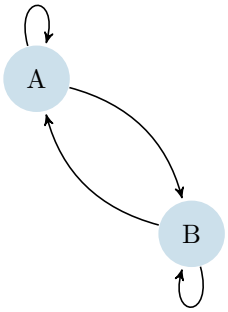


(5)



(6)

1. Which of the above graphs are:
   (1) Connected?
   (2) Complete?
   (3) Directed?
2. What are dist (A, B) and dist (A, E) for these graphs?