Exercise 4: Determinants, Systems of Linear Equations

Problem 1: Determinants

1. Find the determinants of the following matrices:

   (a) $\begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & 7 \\ 4 & 5 & 5 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 4 & 7 \end{pmatrix}$

   (e) $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 7 \end{pmatrix}$

2. What does it mean when the determinant of a 2×2 matrix is negative?

Problem 2: Row Operations and Rank

A matrix is said to be in its *Row Echelon Form* if:

1. All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (all zero rows, if any, belong at the bottom of the matrix)
2. The leading coefficient (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

If the leading coefficients are all 1, then the form is called a *Reduced Row Echelon Form.*
For example, the following matrices are all presented in reduced row echelon form:

$$\begin{pmatrix} 1 & -2 & 0 & 5 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1. Use row operations to bring the following two matrices to their reduced row echelon form:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 6 & 2 & 1 \\ 3 & 6 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -3 & 0 & -3 \end{pmatrix}.$$

2. What are the ranks of A and B?

Problem 3: Systems of Linear Equations

The following system of linear equations is given:

$$\begin{cases} -x + 3z = 20 \\ 3x + y + 3z = 15 \\ 9x + 3y = -18 \end{cases}.$$

Solve the system using the Gaussian elimination method.