Spatial Translations Using Matrices

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May 18, 2020

A translation is a transformation which adds a *displacement vector* $\vec{\Delta}$ to every vector in the space. For example, the following is a translation $T : \mathbb{R}^3 \to \mathbb{R}^3$:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \Delta_x \\ y + \Delta_y \\ z + \Delta_z \end{pmatrix}, \tag{1}$$

where $\vec{\Delta} = \begin{pmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{pmatrix}$ is the displacement vector.

Translations are not linear transformations. The easiest way to see this is by observing the change to the zero vector:

$$T\left(\vec{0}\right) = \vec{0} + \vec{\Delta} = \vec{\Delta}. \tag{2}$$

Since the transformation does not preserve the zero vector, it is not linear.

There is, however, a way to express translations in as matrices, despite their nonlinearity which involves representing the vectors in a higher dimension. For example, consider the 3-dimensional vector $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. By extending the vector with an additional component equaling 1, we get

$$\vec{u}^4 = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}. \tag{3}$$

We can now use the following matrix to perform a translation by the vector $\vec{\Delta} = \begin{pmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \\ 0 \end{pmatrix}$:

$$D = \begin{pmatrix} 1 & 0 & 0 & \Delta_x \\ 0 & 1 & 0 & \Delta_y \\ 0 & 0 & 1 & \Delta_z \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Applying this matrix to the vector $\vec{v}$ yields

$$D\vec{u}^4 = \begin{pmatrix} 1 & 0 & 0 & \Delta_x \\ 0 & 1 & 0 & \Delta_y \\ 0 & 0 & 1 & \Delta_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + \Delta x \\ y + \Delta_y \\ z + \Delta_z \\ 1 \end{pmatrix} = \vec{u}^4 + \vec{\Delta}. \tag{5}$$