Mathematics and Computer Science (B.MES.108)
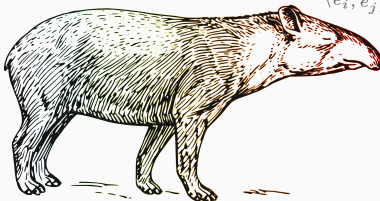# Summer Semester, 2020
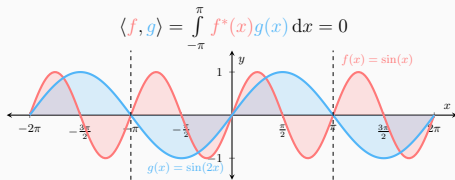# Part 1: Linear Algebra for Non-Mathematicians

Peleg Bar Sapir

$$(AB)^\top = B^\top A^\top$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$A = Q \Lambda Q^{-1}$$

$$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$A\vec{v} = \lambda \vec{v}$$

$$T\left(\alpha \vec{u} + \beta \vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

Chapter 8: General Vector Spaces

$$\langle f, g \rangle = \int\limits_{-\pi}^{\pi} f^*(x)g(x)\,\mathrm{d}x = 0$$

$f(x) = \sin(x)$

$g(x) = \sin(2x)$

Let us review some properties of the space $\mathbb{R}^n$, some of them we already used implicetly without giving them too much thought.

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The addition of any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ yields a vector $\vec{w} = \vec{u} + \vec{v}$ that is also in $\mathbb{R}^n$.

> **Example**
>
> For $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ (both in $\mathbb{R}^3$),
>
> $$\vec{w} = \vec{u} + \vec{v} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} \in \mathbb{R}^3.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is commutative: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$.

> **Example**
>
> For the same vectors as before:
>
> $$\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + (-1) \\ 2 + 3 \\ -1 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$
>
> $$\vec{v} + \vec{u} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 + 1 \\ 3 + 2 \\ 0 + (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is assosiative: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

> **Example**
>
> For $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ \vec{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:
>
> $$\vec{a} + \left(\vec{b} + \vec{c} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right]$$
>
> $$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is assosiative: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

> **Example**
>
> For $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:
>
> $$\left(\vec{a} + \vec{b} \right) + \vec{c} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
>
> $$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The zero vector $\vec{0}$ is unique and has the property that
  $\vec{v} + \vec{0} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Any vector $\vec{v} \in \mathbb{R}^n$ has an opposite vector $(-\vec{v}) \in \mathbb{R}^n$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ 0 \\ -3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Any scale by $\alpha \in \mathbb{R}$ of a vector $\vec{v} \in \mathbb{R}^n$ is also in $\mathbb{R}^n$.

Example

$$-3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -6 \\ 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Any scale by $\alpha \in \mathbb{R}$ of a vector $\vec{v} \in \mathbb{R}^n$ is also in $\mathbb{R}^n$.

Example

$$-3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -6 \\ 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is associative: $\alpha\left(\beta\vec{v}\right) = \left(\alpha\beta\right)\vec{v}$.

> **Example**
>
> $$-3\left[2\begin{pmatrix}1\\-4\\5\end{pmatrix}\right] = -3\begin{pmatrix}2\\-8\\10\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}$$
>
> $$(-3\cdot 2)\begin{pmatrix}1\\-4\\5\end{pmatrix} = -6\begin{pmatrix}1\\-4\\5\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is distributive in respect to scalar addition: $(\alpha + \beta)\,\vec{v} = \alpha\vec{v} + \beta\vec{v}$.

Example

$$(5 - 2) \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}.$$

$$5 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is distributive in respect to vector addition: $\alpha\left(\vec{v} + \vec{u}\right) = \alpha\vec{v} + \alpha\vec{u}$.

Example

$$5\left[\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right] = 5\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix}.$$

$$5\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 15 \end{pmatrix} + \begin{pmatrix} 0 \\ 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- The scalar $\alpha = 1$ is neutral in respect to scalar-vector products: $1\vec{v} = \vec{v}$.

Example

$$1\begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix}.$$

Abstract Vector Spaces

These properties are somewhat obvious on $\mathbb{R}^n$. However, many times it is worthwhile to use more abstract vector spaces, which can help us model diverse physical and theoretical systems, since once a construct behaves as a vector space, it is a relatively simple process to apply to it all the analysis tools learned so far.

We will not bother here with the formal definition of a vector space[1], but look at one example, which we will later expand on: the space of all real functions $f : \mathbb{R} \to \mathbb{R}$.

---

[1]For such definition, see here:

http://www.math.niu.edu/ beachy/courses/240/06spring/vectorspace.html.

Real Functions as a Vector Space

Real Functions as a Vector Space

- For any two real functions $f(x), g(x)$ the addition $(f + g)(x) = f(x) + g(x)$ is also a real function.

Real Functions as a Vector Space

- For any two real functions $f(x), g(x)$ the addition $(f + g)(x) = f(x) + g(x)$ is also a real function.
- The addition of two real functions is commutative: $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.

Real Functions as a Vector Space

- For any two real functions $f(x), g(x)$ the addition
  $(f + g)(x) = f(x) + g(x)$ is also a real function.
- The addition of two real functions is commutative:
  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.
- This addition is also associative: xxx.

Real Functions as a Vector Space

- For any two real functions $f(x), g(x)$ the addition
  $(f + g)(x) = f(x) + g(x)$ is also a real function.
- The addition of two real functions is commutative:
  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.
- This addition is also associative: xxx.
- The function $z(x) = 0$ is the zero function, since for any other
  real function $f(x)$,

$$(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x).$$

Real Functions as a Vector Space

- For any two real functions $f(x), g(x)$ the addition $(f + g)(x) = f(x) + g(x)$ is also a real function.
- The addition of two real functions is commutative: $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.
- This addition is also associative: xxx.
- The function $z(x) = 0$ is the zero function, since for any other real function $f(x)$,

$$(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x).$$

- For each real function $f(x)$ there exists an opposite function $(-f)(x) = -f(x)$, for which

$$f(x) + \big(-f(x)\big) = f(x) - f(x) = 0 = z(x).$$