GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Mathematics and Computer Science (B.MES.108)
Summer Semester, 2020
Part 1: Linear Algebra for Non-Mathematicians

Peleg Bar Sapir

$(AB)^{\top} = B^{\top}A^{\top}$ $\quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

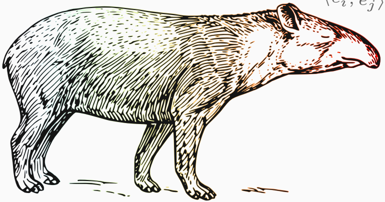
$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

$A = Q\Lambda Q^{-1}$

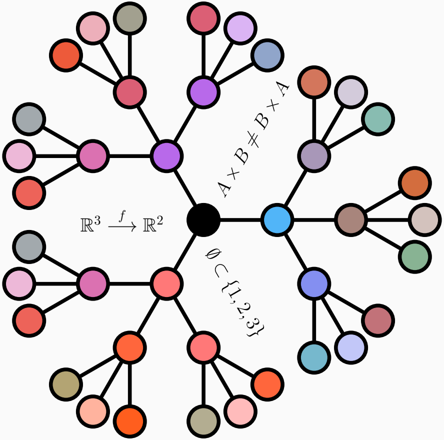
$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\quad A\vec{v} = \lambda\vec{v}$

$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$

$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$



Chapter 1: Introduction



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- $1 + 2 = 3$ (**true**)
- Protons have no electric charge (**false**)
- $13 > 37$ (**false**)

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- The **or** operator returns **true** if **at least** one of the statements it groups is true.

Operators: and, or

Example

$$\boxed{1+2=3} \text{ and } \boxed{3-5=-2} \Rightarrow \textbf{true}$$

Operators: and, or

Example

$\boxed{1+2=3}$ and $\boxed{3-5=-2}$ $\Rightarrow$ **true**

$\boxed{1+2=3}$ and $\boxed{2\times 4=7}$ $\Rightarrow$ **false**

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Operators: and, or

Example

$$1 + 2 = 3 \text{ or } 3 > 7 \Rightarrow \textbf{true}$$

$$0 + 3 = -1 \text{ or } 1 = 1 \Rightarrow \textbf{true}$$

$$2 \times 2 = 4 \text{ or } 2 + 0 = 2 \Rightarrow \textbf{true}$$

$$3 \times 7 = 10 \text{ or } \frac{1}{2} < \frac{1}{10} \Rightarrow \textbf{false}$$

Operators: Truth Table

We can summarize the behaviour of operators in a **truth table**:

A	B	AND	OR
true	**true**	**true**	**true**
true	**false**	**false**	**true**
false	**true**	**false**	**true**
false	**false**	**false**	**false**

Mathematical Notation

Other **notations** that will be used throught this course:

Symbol	In words
$\neg a$	**not** a
$a \wedge b$	a **and** b
$a \vee b$	a **or** b
$a \Rightarrow b$	a **implies** b
$a \Leftrightarrow b$	a **is equivalent to** b
$\forall x$	**For all** x (...)
$\exists x$	**There exists** x **such that** (...)
$a := b$	a **is defined to be** b

Sets

Definition

A **set** is a collection of **elements**. Elements of a set can be any concept - be it physical (a chair, a car, a tapir) or abstract (a number, an idea).

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Sets can have a **finite** or **infinite** number of elements.

Sets

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Example

$$\{1, 2, 3, 4\}\,, \quad \left\{-4, \frac{3}{7}, 0, \pi, i, 0.1\right\}, \quad \{\text{all even numbers}\}\,.$$

Sets

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Note

There is no repetition in sets, i.e. $\{1, 1, 3, 3, 3, 3, 5\}$ is not a proper set, contrary to e.g. $\{1, 3, 5\}$.

Sets

Sets can be denoted as conditions, using a vertical separator to denote **conditions**.

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$$\{x \text{ is odd} \mid 0 < x < 10\}.$$

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The following set contains all odd numbers between 0 and 10:

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It can also be written explicetly:

$$\{1, 3, 5, 7, 9\}.$$

Sets

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Example

For the two sets

$$A = \{1, 2, 5, 7\}, \quad B = \{\text{even numbers}\},$$

all the following propostions are true:

$$1 \in A, \quad 2 \in A, \quad 5 \in A, \quad 7 \in A,$$
$$2 \in B, \quad 1 \notin B, \quad 5 \notin B, \quad 7 \notin B.$$

Sets

The number of elements in a set (also called its **cardinality**) is denoted with two vertical bars.

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Example

$$S = \{-3, 0, -2, 7, 1\} \quad \Rightarrow \quad |S| = 5.$$

The Empty Set

An important special set is the **empty set**, which is set containing no elements. It is denoted by $\emptyset$, and has the unique proerty that

$$\left|\emptyset\right| = 0.$$

Subsets and Supersets

If a set A contains all the elements in a set B (and perhaps additional elements), then B is said to be a **subset** of A, and A a **superset** of B.

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Example

The sets

$$A = \{0, -3\}\,, \quad B = \{5, -3, 1\}\,, \quad C = \{-2, 2, 1\}\,,$$

are some of the subsets of

$$D = \{0, -3, 5, 1, 2, -2\}\,.$$

Equivalently, D is a super set of A, B and C.

Subsets and Supersets

Note

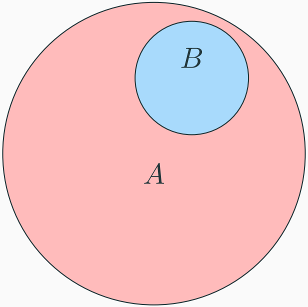
All sets are supersets and subsets of themselves. This is a direct consequence of the definition of supersets and subsets.

Subsets and Supersets

We denote that A is a superset of B as

$$B \subseteq A.$$

A **Venn Diagram** representation of this fact looks as following:



Subsets and Supersets

If for some two sets A, B both $A \subseteq B$ **and** $B \subseteq A$, then the sets are identical.

Formally, this fact is written as

$$A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B.$$

Intersections and Unions

Definition

The **intersection** of two sets A and B is the set of all elements that are **both** in A and in B.

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Example

Given the sets

$$A = \{1, 2, 5, 6, 7\}\,,\;\; B = \{-1, 0, 1, 5, 10, 13, 15\}\,,$$

the intersection of A and B is $\{1, 5\}$.

Intersections and Unions

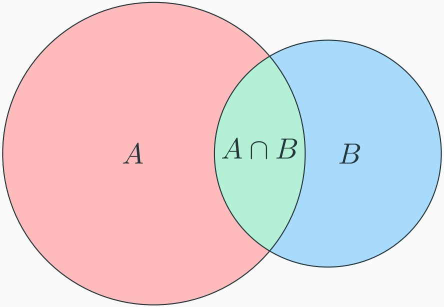
The symbol denoting intersection is $\cap$. An intersection can be formally defined as

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

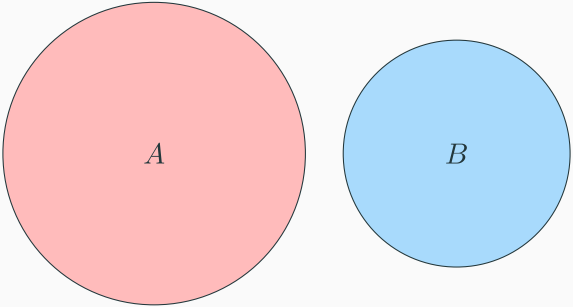
(read: "the intersection of A and B is the set containing all elements x, such that x is in A and x is in B")

Intersections and Unions

A Venn diagram visualization of $A \cap B$ (green area):



If the intersection of two sets is empty ($A \cap B = \emptyset$), then the sets are said to be **disjoint**:



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Example

The union of the sets

$$A = \{-5, 7, 1\}, \ B = \{10, -2, -5, 2\},$$

is

$$A \cup B = \{10, -2, -5, 2, 7, 1\}.$$

Intersections and Unions

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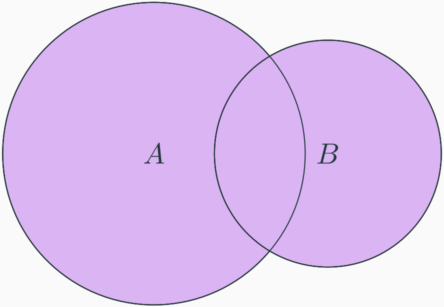
Intersections and Unions

A Venn diagram visualization of $A \cup B$ (purple area):



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The number of elements in a union of two sets A and B is

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Note

If A, B are disjoint, $|A \cup B| = |A| + |B|$ (because $|A \cap B| = 0$).

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Example

For the sets

$$A = \{1, 5, 9, 10\}\,,\ B = \{-3, 2, 5, 9, 13\}\,,$$

The differences are

$$A - B = \{1, 10\}\,,\ B - A = \{-3, 2, 13\}\,.$$

Difference of Sets

Formally:

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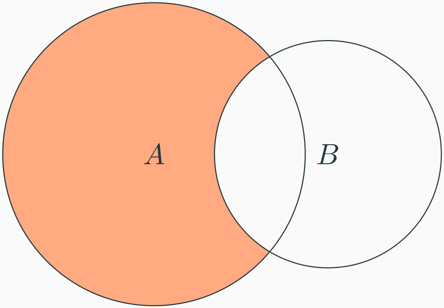
A Venn diagram visualization of $A - B$ (orange area):



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Example

For the sets

$$Z = \{1, 2, 3, 4, 5\}, \; A = \{1, 2, 3\},$$

The complement of A in relation to Z is

$$A^{\mathsf{c}} = \{4, 5\}$$

Complement

Formally:

$$A^{\mathsf{c}} = \left\{ x \in Z \mid x \notin A \right\}.$$

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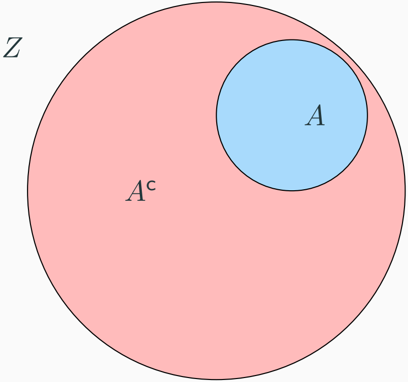

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Example

All the subsets of $A = \{1, 2, 3\}$ are:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Thus, the power set of A is

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Power Sets

Note

The empty set $\emptyset$ is a subset of all sets. Each set is also a subset of itself.

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- **The rational** numbers (symbol: $\mathbb{Q}$). As their name suggests, they are ratios between two integers (e.g. $\frac{1}{2}$, $\frac{-5}{3}$, $\frac{7}{13}$).
- **The real numbers** (symbol: $\mathbb{R}$). These are all the numbers on the number line (e.g. $2, \pi, \frac{\sqrt{3}}{17}, \sqrt{5}, -7.2, e^{\pi}$). A proper definition of the real numbers is beyond the scope of this course.

Additionaly, the **Complex Numbers** are the set of all numbers

$$z = a + bi,$$

where a and b are both real numbers, and i is the imaginary unit, i.e. $i = \sqrt{-1}$.

The complex number set has the notation $\mathbb{C}$.

Important Number Sets

Table summary:

Symbol	Name	Definition
$\mathbb{N}$	Natural numbers	$\{1, 2, 3, 4, \dots\}$
$\mathbb{Z}$	Integers	$\left\{0, \pm x \mid x \in \mathbb{N}\right\}$
$\mathbb{Q}$	Rational numbers	$\left\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\right\}$
$\mathbb{R}$	Real numbers	Not in this course
$\mathbb{C}$	Complex numbers	$\left\{a + ib \mid a, b \in \mathbb{R}, i = \sqrt{-1}\right\}$

Important Number Sets

Note

The relations between these sets are

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

(the symbol $\subset$ means "a proper subset")

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Note

Although each of these sets is infinite, the actual number of elements in $\mathbb{R}$ and $\mathbb{C}$ **is bigger** than the number of elements in $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$. There are different kinds of infinities.

Cartesian Products

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Example

Consider $A = \{1, 2, 3\}$, $B = \{x, y\}$. Then:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$$

Cartesian Products

Note

The cartesian product of two sets A, B is not commutative, i.e.

$$A \times B \neq B \times A,$$

unless $A = B$ or any one of the sets (or both) is the empty set.

Cartesian Products

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The definition of a cartesian product can be expanded to $n \in \mathbb{N}$ sets $A_1, A_2, \ldots, A_n$:

$$A_1 \times A_2 \times \cdots \times A_n = \left\{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\right\}$$

Cartesian Products

The definition can be made more compact by the use of the product symbol $\prod$:

$$\prod_{i=1}^{n} A_i = \left\{(a_1, a_2, \ldots, a_i) \mid a_i \in A_i, i = 1, 2, \ldots, n\right\}.$$

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Note

The symbol $\prod$ is a generalized product notation. It will be discussed in more details later in the course.

Cartesian Products

A cartesian product of the same set is written in an similar way to a power. For example

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2,$$
$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3.$$

These are, respectively, sets of pairs of real numbers, e.g. $(-3, 1)\,, (\pi, 2), (-\frac{\sqrt{7}}{13}, 0)$, and triples of real numbers, e.g. $(1, 2, -\pi)\,, \left(-6, \frac{1}{\sqrt{\pi}}, 0.2\right), \left(\frac{1}{51}, \sqrt{3}, -4\right)$.

Example

For the set $A = \{a, b\}$,

$$A^3 = \left\{(aaa), (aab), (aba), (abb), (baa), (bab), (bba), (bbb)\right\}.$$

Cartesian Products

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For the set $B = \{1, 2, 3\}$,

$$B^2 = \big\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\big\}.$$

Relations Between Sets

Definition

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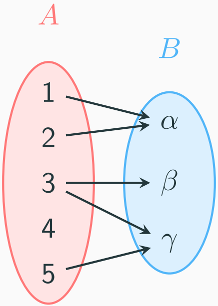
Example

An example relation between the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{\alpha, \beta, \gamma\}$ is

$$R = \left\{(1, \alpha), (2, \alpha), (3, \beta), (3, \gamma), (5, \gamma)\right\}.$$

Relations Between Sets

The previous relation can be visually represented as following:



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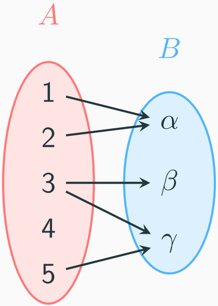

Note

Notice how not all elements are connected, and some elements in each set are connected to the same element in the other set.

Reversed Relations

The previous relation can be reversed, yielding a subset of $B \times A$:

$$R^{-1} = \left\{(\alpha, 1), (\alpha, 2), (\beta, 3), (\gamma, 3), (\gamma, 5)\right\}.$$

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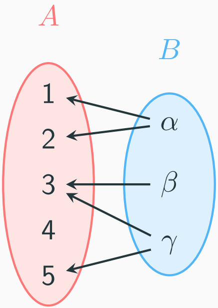
Graphically:



Functions

Definition

A **function** between the sets A, B is a relation in which for every element $a \in A$ there is exactly **one** connection to an element $b \in B$.

Functions

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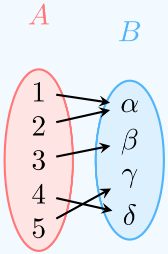
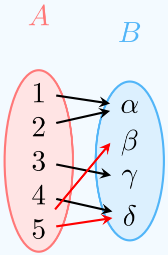
Example

A function from a set A to a set B:



Example

A relation which is **NOT** a function from A to B:



Functions

Two additional terms that are used interchangably with function are **transformation** and **map**.

Note

A function can have more than one element $a \in A$ connected to the same element $b \in B$. The only restriction is that no element $a \in A$ is connected to **more than one** element $b \in B$.

Functions

A common notation to a function f connecting between elements of the sets A and B is

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Image of f

Functions

When used in practice, a common notation to show that an element $x \in A$ is connected to another element $y \in B$ is

$$f(x) = y,$$

i.e. the function f applied to the element $x \in A$ returns the element $y \in B$.

Real Functions

In part 3 of the course we will deal with functions of the form

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

which we call **real functions**, i.e. functions that take a real number x and return a real number y.

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Example

The functions

$$f_1(x) = 2x^2 - 5, \quad f_2(x) = \sin\left(\frac{x}{3}\right), \quad f_3(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

are all real functions.

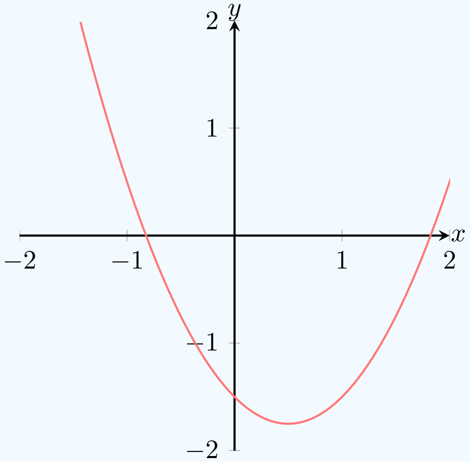
Plotting Real Functions

We can plot a real function f on a cartesian coordinate system by drawing a dot in each coordinate (x, y), where x is an element in the domain of f, and y is its image (i.e. $f(x) = y$).

Plotting Real Functions

Example

Plotting the function $f(x) = x^2 - x - 1.5$:



Injective, Surjective and Bijective Functions

A function is called **injective** if each of the elements in its **image** is connected to by a single element in its **domain**.

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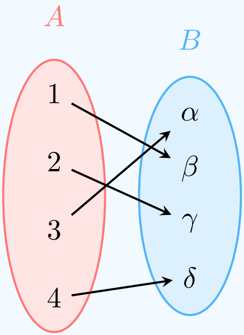
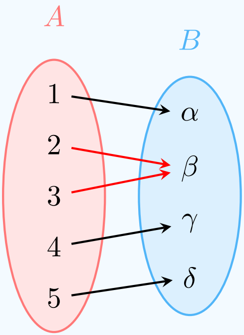
Example

Injective



Not injective



Injective, Surjective and Bijective Functions

A function is called **surjective** if all of the elements in its **image** are connected to by some element in its **domain**.

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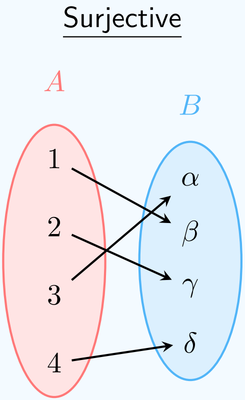
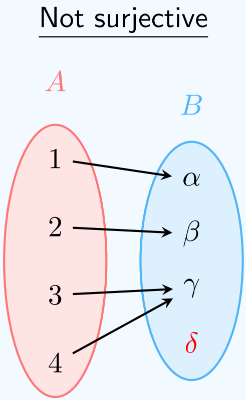




Injective, Surjective and Bijective Functions

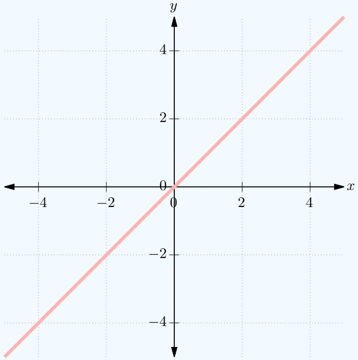
A function that is both **injective** and **surjective** is called **bijective** .

Injective, Surjective and Bijective Functions

Example

Let's look at a few examples of real injective, surjective and bijective functions:

- $f(x) = x$, injective + surjective (and thus bijective)



Injective, Surjective and Bijective Functions

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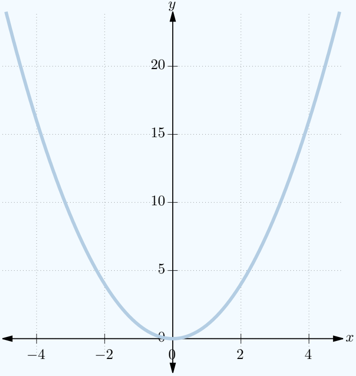
Let's look at a few examples of real injective, surjective and bijective functions:

- $f(x) = x^2$, not injective nor surjective.

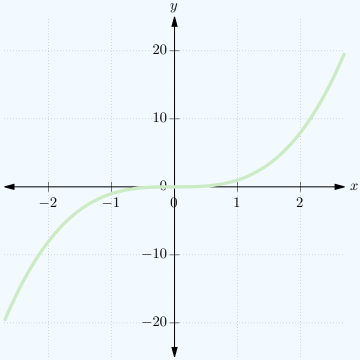


Injective, Surjective and Bijective Functions

Example

Let's look at a few examples of real injective, surjective and bijective functions:

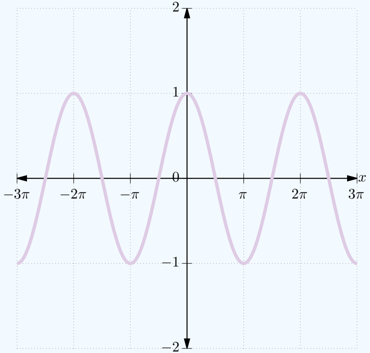
- $f(x) = x^3$: injective + surjective (and thus bijective)



Example

Let's look at a few examples of real injective, surjective and bijective functions:

- $f(x) = \cos(x)$, not injective nor subjective.



Multivariable Functions

Functions may have several arguments and return several arguments.

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The following functions take as input three real numbers, and return a single real number ($f : \mathbb{R}^3 \to \mathbb{R}$). The return value of some functions for a triplet of real numbers, $(-5, 7, 1)$, are:

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- $f(x, y, z) = \frac{x}{\sqrt{y}+z} \Rightarrow f(-5, 7, 1) = \frac{5}{\sqrt{7}+1}$

Example

The function $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$ is defined as

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- $f(0, 13) = \frac{0}{13} = 0$.

Graphs

Definition

A **graph** is a mathematical structure composed of **nodes** connected to other nodes by **edges**.

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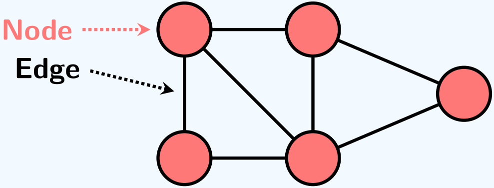
Example

A graph with 5 nodes and 7 edges:



Graphs

In the graphical representation of a graph, the actual position of nodes does not matter - what matters are the connections (edges) between them.

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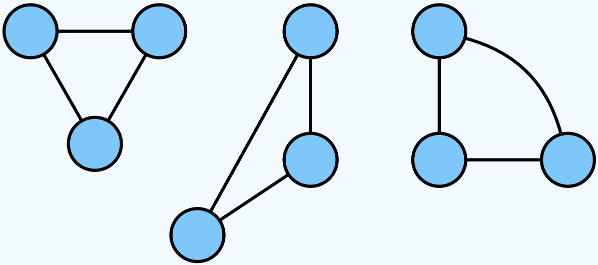
Example

The following three graphs are identical:



Definition

A graph in which edges have directions is called a **directed graph**.

Graphs

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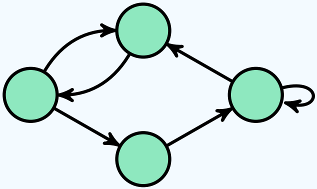
Example

A directed graph with 4 nodes and 6 edges:



Graphs

Definition

A **path** in a graph is a sequence of edges in which each edge shares a vertex with the previous edge (except the first edge).

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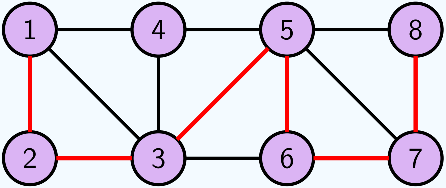
Example

A path in a graph (note that the nodes are labeled):



Definition

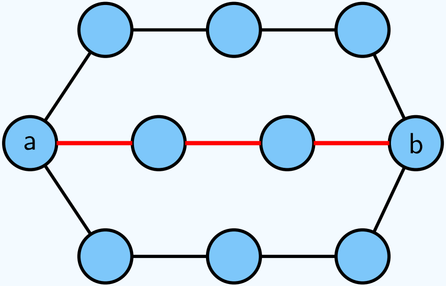
When the start and end vertices coincide the path is known as a **circle**. A directed circle is known as a **cycle**.

Definition

If one or more pathes exist between two vertices a, b in a graph, the number of edges in the shortest path is defined to be the **distance** between the two vertices, and is denoted as $\operatorname{dist}(a, b)$.

Example

In the following graph three paths between vertices a and b are shown. The number of edges in the shortest path, highlighted in red, is defined as the distance $\mathrm{dist}(a, b)$, and is equal to 3.



Definition

A **tree** is a graph with no circles.

Graphs

Definition

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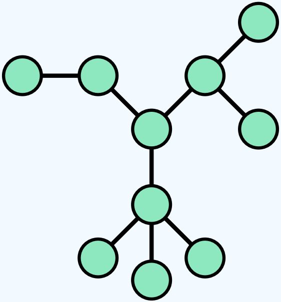
Example

A tree (notice that no circles are present):

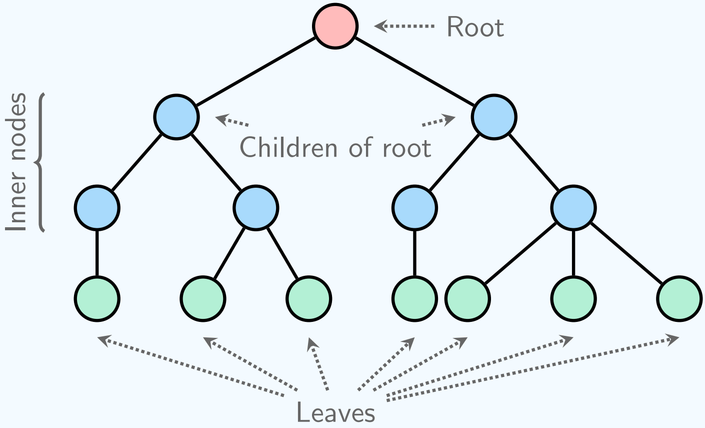


Graphs

Some trees have a distinctive **root** node, and are known as **rooted trees**. A node that is "branched" from a higher level node is called a **child node**. The last level nodes are called **leaves** (singular: leaf). The rest of the nodes are known as **inner nodes**.

Example

A rooted tree, with the root node highlighted in red and the leaves in green:

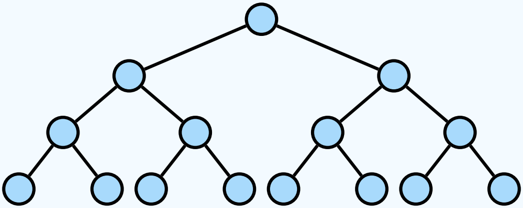


Definition

A tree with 2 children per node is called a **binary tree**. Similarly, trees can be trenary, quaternary, etc.

Example

A binary tree:



Graphs

Rooted trees are used to describe hierarchies, e.g. in biological systematics, organisations or nested directories of data.

Definition

The **complete graph** K_n is the graph with n vertices where every pair of different vertices is connected by an edge (Also called a **clique**).

Graphs

Definition

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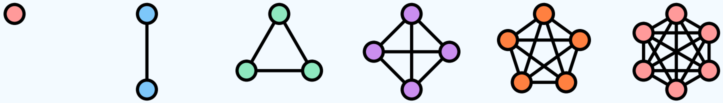
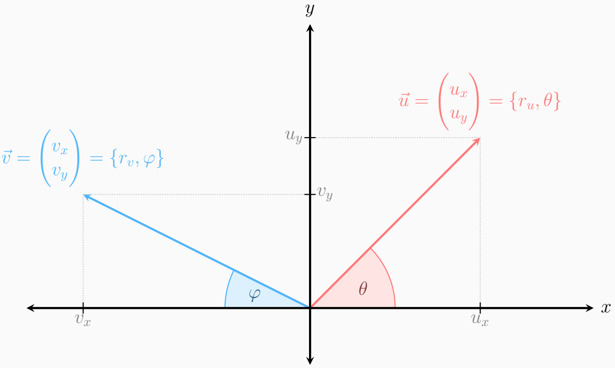
Example

The cliques $K_1, \ldots, K_6$:



K_1 K_2 K_3 K_4 K_5 K_6

Chapter 2: Vectors



Basics of Vectors

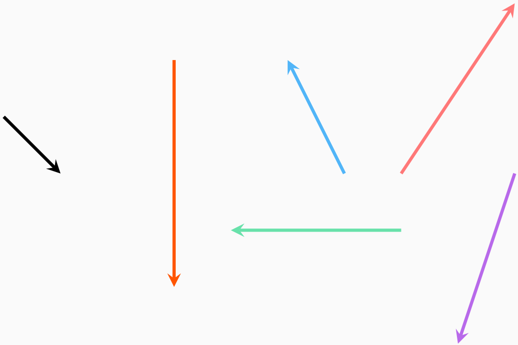
There are 3 distinct approaches to describe what a vector is:

- The physicist's approach (geometric)
- The computer scientist's approach (algebraic)
- The mathematician's approach (abstract)

Geometric Vectors

Definition

A vector is an object with a length and a direction.



Vector Notation

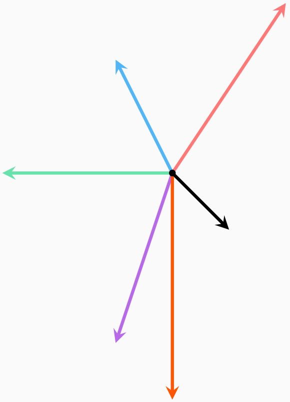
Vectors are denoted as latin letters with an arrow above them:

$$\vec{u}, \quad \vec{v}, \quad \vec{x}, \quad \vec{a}, \quad \ldots$$

In maths and physics the following notations are mostly used:

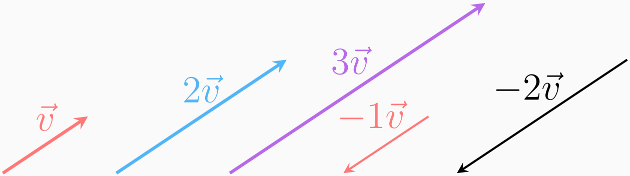
$$\boldsymbol{u}, \quad \boldsymbol{v}, \quad \boldsymbol{x}, \quad \boldsymbol{a}, \quad \ldots$$
$$\underline{\boldsymbol{u}}, \quad \underline{\boldsymbol{v}}, \quad \underline{\boldsymbol{x}}, \quad \underline{\boldsymbol{a}}, \quad \ldots$$

We consider all vectors starting at the same point, called the **origin**.



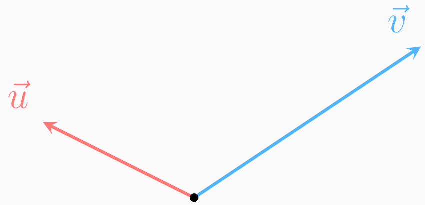
Scaling Vectors

We can multiply a vector by a real number, which we refer to as a **scalar**. This scales only the length of the vector while keeping its direction on the same line as before:



Vector Addition

Adding two vectors is done by placing the origin of one vector at the head of the other vector. The addition results in a vector starting at the first vector's origin and ending at the second vector's head:



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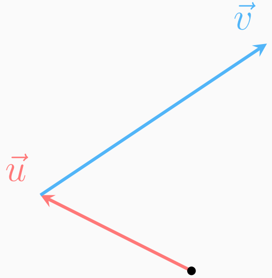

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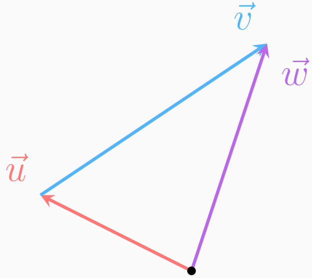

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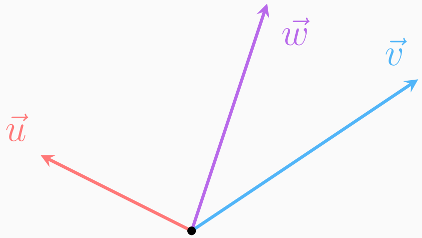
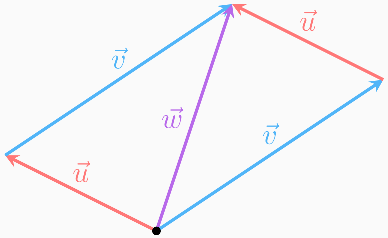


Vector Addition

Notice that adding vectors is a commutative operation, i.e.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$



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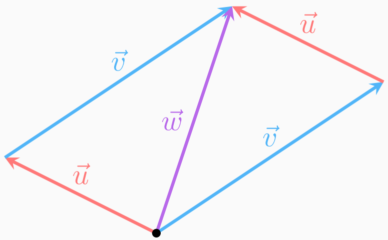

This is refered to as the **parallelogram law of vector addition**.

The Zero Vector

And important vector is the **zero vector**, which has a length of 0 and no direction. It is notated as $\vec{0}$, and is neutral to addition, i.e. for any vector $\vec{v}$:

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$

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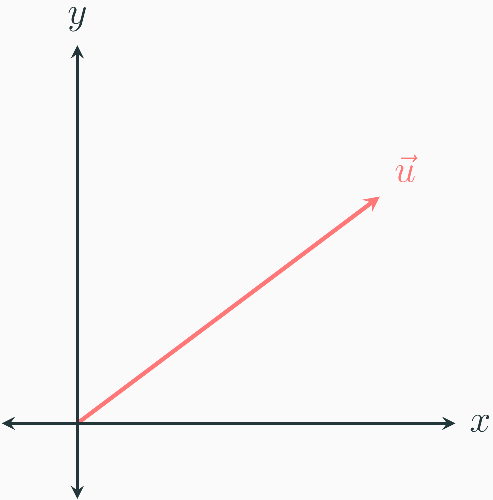
Similarily, any addition of a vector with its oppsite vector results in the zero vector:

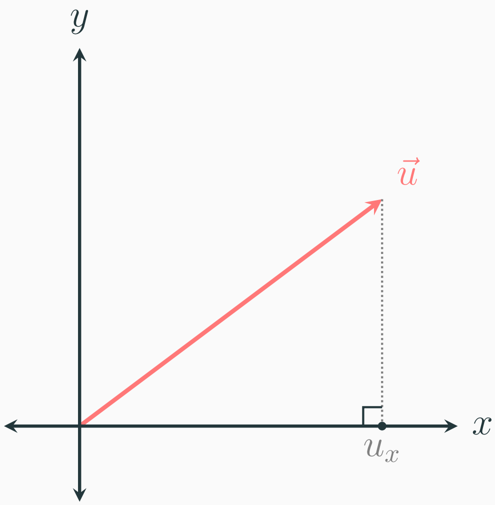
$$\vec{v} + (-\vec{v}) = -\vec{v} + \vec{v} = \vec{0}.$$

Algebraic Vectors

Placing a vector in a cartesian coordinate system:

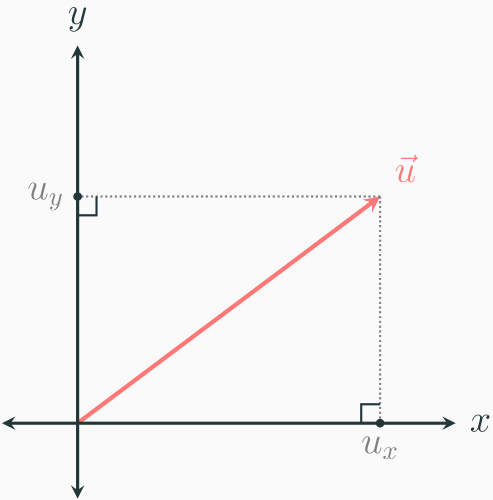


Then, drawing a perpendicular from $\vec{v}$ to the x-axis:



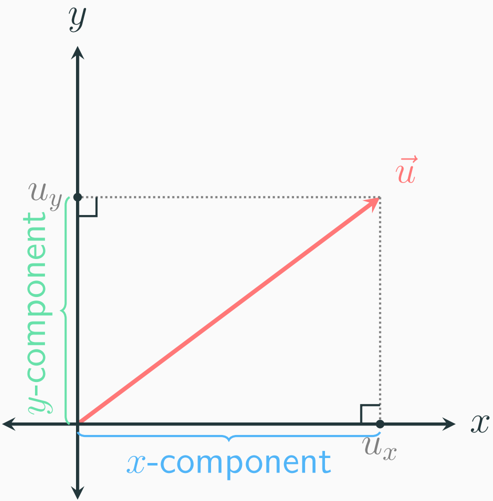
Algebraic Vectors

And similarily for the y-axis:



Algebraic Vectors

We call u_x and u_y the **components** of $\vec{u}$.



Column Vectors

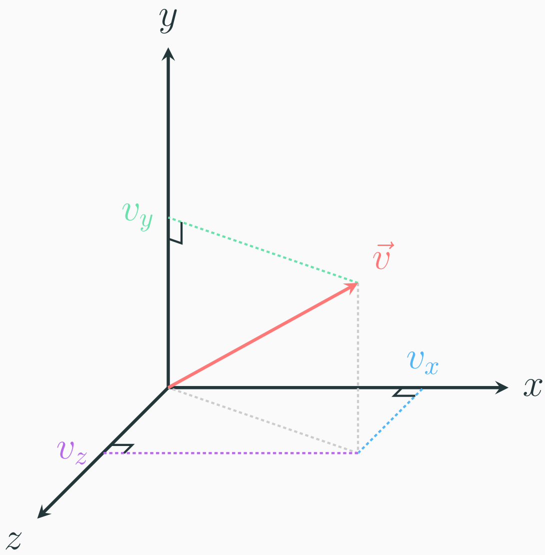
We then notate the vector $\vec{u}$ as a **column vector** with components u_x, u_y:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}.$$

Since $\vec{u}$ as two real components, it is a member of $\mathbb{R}^2$.

Higher-dimensional Vectors

This scheme can be extended to 3-dimensional vectors:



Higher-dimensional Vectors

A column vector in $\mathbb{R}^3$ looks as following:

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$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix},$$

and in $\mathbb{R}^4$:

$$\vec{a} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ v_w \end{pmatrix}.$$

Higher-dimensional Vectors

A general column vector in $\mathbb{R}^n$ looks as following:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$\Big\}$ n components

The Zero Vector

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In $\mathbb{R}^3$ it is

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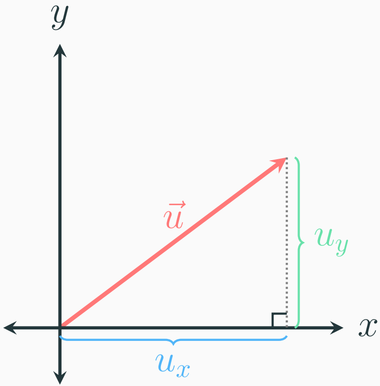
And generally, in $\mathbb{R}^n$, it is

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Bigg\} \; n \text{ components}$$

Length and Angle of a Vector

Using the Pythagorean theorem to calculate the length (norm) of a vector in $\mathbb{R}^2$:

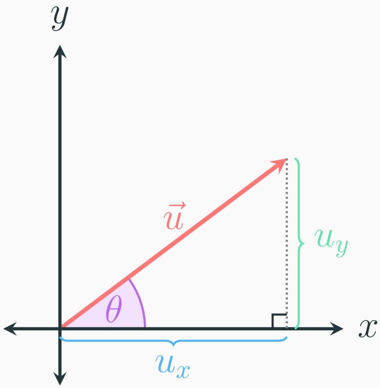
$$\|\vec{u}\| = \sqrt{u_x^2 + u_y^2}.$$



Length and Angle of a Vector

The angle θ is then:

$$\tan(\theta) = \frac{u_y}{u_x}.$$



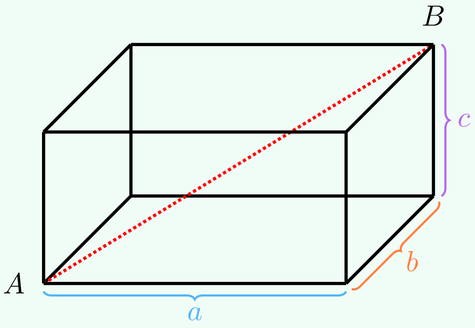
Length of a Vector

Similarly, the length of a column vector in $\mathbb{R}^3$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ is

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Challange

Show that the above given formula is true, i.e. show that for a box of sides a, b, c, the length of the line from A to B (see figure) is indeed $\sqrt{a^2 + b^2 + c^2}$.



Length of a Vector

For a general n-dimensional vector $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$,

$$\begin{aligned} \|\vec{w}\| &= \sqrt{w_1^2 + w_2^2 + \cdots + w_n^2} \\ &= \sqrt{\sum_{i=1}^{n} w_i^2}. \end{aligned}$$