The Eigenvectors of 2D and 3D Rotation Matrices

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1 Two-Dimensional Rotation Matrix

In $\mathbb{R}^2$ the matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{1}$$

represents a counter-clockwise rotation around the origin by an angle θ.

The eigenvalues of R_θ can be found by solving the equation

$$\begin{vmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} = 0, \tag{2}$$

i.e.

$$\left(\cos(\theta) - \lambda\right)^2 + \sin^2(\theta) = 0. \tag{3}$$

Expanding the left side of equation 3 yields

$$\cos^2(\theta) - 2\lambda\cos(\theta) + \lambda^2 + \sin^2(\theta) = \lambda^2 - 2\cos(\theta)\lambda + 1 = 0. \tag{4}$$

(since $\sin^2(\theta) + \cos^2(\theta) = 1$)

The solution of equation 4 for λ can be calculated by the well known quadratic formula:

$$\begin{aligned} \lambda_{1,2} &= \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} && (5) \\ &= \frac{2\cos(\theta) \pm \sqrt{4\left[\cos^2(\theta) - 1\right]}}{2} && (6) \\ &= \frac{\cancel{2}\cos(\theta) \pm \cancel{2}\sqrt{\cos^2(\theta) - 1}}{\cancel{2}} && (7) \\ &= \cos(\theta) \pm \sqrt{\cos^2(\theta) - 1}. && (8) \end{aligned}$$

For the solutions to equation 8 to have real value, the expression $\sqrt{\cos^2(\theta) - 1}$ must not be negative, i.e.

$$\cos^2(\theta) \geq 1. \tag{9}$$

Since the image of $\cos(\theta)$ is $[-1, 1]$, the image of $\cos^2(\theta)$ is $[0, 1]$. Thus, the only case for which a solution to equation 8 can be a real number is when $\cos^2(\theta) = 1$, or

$$\cos(\theta) = \pm 1. \tag{10}$$

This condition means that the only angles for which R_θ has real eigenvalues are

$$\theta = 0^\circ \text{ and } \theta = 180^\circ,$$

which are, respectively, the identity operation (rotation by 0°) and inverting all vectors (rotation by 180°). Indeed, these are the only angles for which a rotation around the origin has any eigenvectors.

2 Three-Dimensional Rotation Matrices

The three counter-clockwise rotation matrices around the axes by the angles θ, φ, ψ, respectively, are

$$R_\theta^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad R_\varphi^y = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}, \quad R_\psi^z = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

We expect R_θ^x to only have eigenvectors in the direction of the x-axis, with eigenvalue $\lambda_x = 1$. Similarly, the eigenvectors of R_φ^y should all lie on the y-axis with eigenvalues $\lambda_y = 1$, and the eigenvectors of R_ψ^z to be lying on the z-axis with eigenvalues $\lambda_z = 1$ (Figure 1).

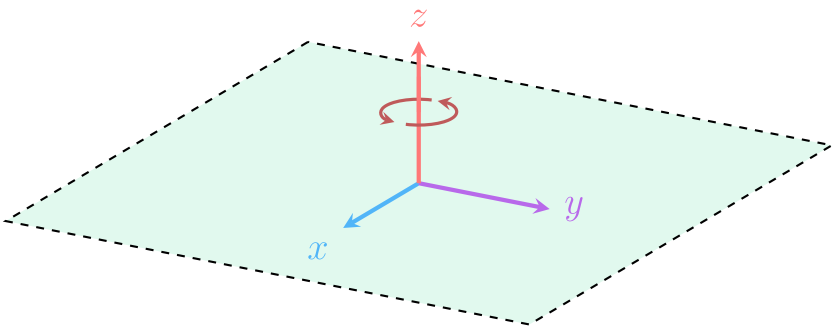

Figure 1: Rotation around the z-axis.

We will only calculate the eigenvectors and corresponding eigenvalues of R_θ^x. Calculation of the eigenvectors and eigenvalues of R_φ^y and R_ψ^z is left to the students.

We begin by solving the equation

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos(\theta)-\lambda & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta)-\lambda \end{vmatrix} = 0, \tag{12}$$

which is very similar to equation 3, except for the multiplication by the expression $1-\lambda$, i.e.

$$(1-\lambda)\left[\left(\cos(\theta)-\lambda\right)^2 + \sin^2(\theta)\right] = 0, \tag{13}$$

which has the same solutions as equation 4, and in addition $\lambda = 1$, as expected.

For the eigenvalue $\lambda = 1$, the corresponding eigenvector can be found by solving

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{14}$$

i.e.

$$\begin{pmatrix} 1 \\ \cos(\theta)y - \sin(\theta)z \\ \sin(\theta)y + \cos(\theta)z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{15}$$

This system has a solution for **any** angle θ **only** when

$$x = 1,\ y = 0,\ z = 0, \tag{16}$$

i.e. the vector

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{x}, \tag{17}$$

as expected.

For $y \neq 0$ and $z \neq 0$, the solution only exist when $\theta = 0°$, i.e. when the roation around the x-axis is by $\deg 0$. This case is the identity operataion, for which all vectors are eigenvectors with eigenvalue $\lambda = 1$.