Exercise 11: Derivatives of Real Functions (Solution)

1. Draw the following functions on a grid: $x^2,\ -x^2,\ x^2+3,\ x^2-5,\ x^2-2x,\ x^2-3x+5$.

Answer:
It is important to understand how parabolas of the form $f(x) = ax^2 + bx + c$ behave:

- The sign of a determines whether the parabola is upward-concaving or downward-concaving (See $-x^2$ in Figure 2), and its width, as the bigger a is, the narrower is the parabola.
- The coefficiecnt b affects the shape of the parabola in a more complicated way: the minimum point of a parabola of the form $x^2 + bx$ is at $x = -\frac{b}{2}$ with value $y = \frac{b^2}{4} - \frac{b^2}{2} = -\frac{b^2}{4}$. This means that changing b simultaneously moves the minimum linearly in the x-axis and on a parabolic path in the y-axis.
- Changing only c simply 'moves' the parabola up and down (See Figure 1).

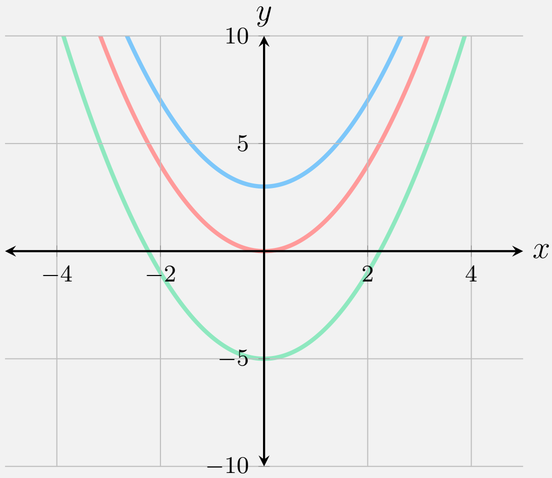


Figure 1: Graphing x^2, x^2+3 and x^2-5 for $x \in [-4, 4]$.

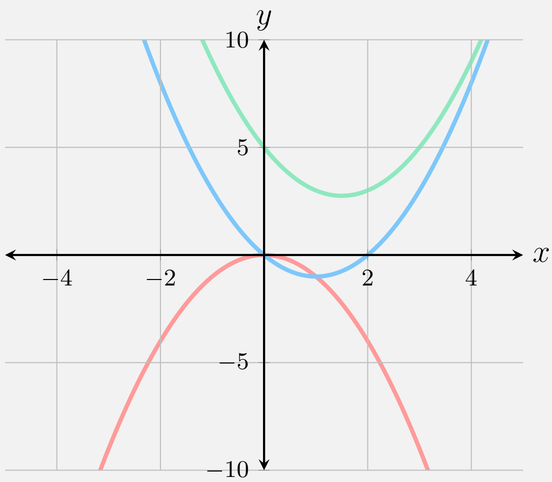


Figure 2: Graphing $-x^2$, x^2-2x and x^2-3x+5 for $x \in [-4, 4]$.

2. Calculate the following derivatives:

   i. $\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^4 - 3x^2 + 5\right)$

Answer:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^4-3x^2+5\right) &= \frac{\mathrm{d}}{\mathrm{d}x}5x^4-\frac{\mathrm{d}}{\mathrm{d}x}3x^2+\cancel{\frac{\mathrm{d}}{\mathrm{d}x}5}\\ &= 5\frac{\mathrm{d}}{\mathrm{d}x}x^4-3\frac{\mathrm{d}}{\mathrm{d}x}x^2\\ &= 5\times 4x^3-3\times 2x\\ &= 20x^3-6x.\end{aligned}$$

ii. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-6x+5}{x-7}\right)$

Answer:
For functions of the type $\frac{f}{g}$ we get $\frac{\mathrm{d}}{\mathrm{d}x}\frac{f}{g}=\frac{\frac{\mathrm{d}}{\mathrm{d}x}f\cdot g-f\cdot\frac{\mathrm{d}}{\mathrm{d}x}g}{g^2}$, and thus if we define $f(x)=x^3-6x+5,\ g(x)=x-7$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-6x+5}{x-7}\right) &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3-6x+5\right)(x-7)-\left(x^3-6x+5\right)\frac{\mathrm{d}}{\mathrm{d}x}(x-7)}{(x-7)^2}\\ &= \frac{\left(3x^2-6\right)(x-7)-\left(x^3-6x+5\right)(1)}{(x-7)^2}\\ &= \frac{3x^3-\cancel{6x}-21x^2+42-\left(x^3-\cancel{6x}+5\right)}{x^2-14x+49}\\ &= \frac{2x^3-21x^2+37}{x^2-14x+49}.\end{aligned}$$

iii. $\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)$, where $P_n(x)$ is a real polynomial of order n.

Answer:
Remembering that $P_n(x)=\sum\limits_{k=0}^{n}a_kx^k$ - we see that we can derive the terms of the polynomial separately:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{n}a_kx^k\\ &= \sum_{k=0}^{n}\frac{\mathrm{d}}{\mathrm{d}x}a_kx^k\\ &= \sum_{k=0}^{n}a_k\frac{\mathrm{d}}{\mathrm{d}x}x^k\\ &= \sum_{k=0}^{n}k\cdot a_k\cdot x^{k-1}.\end{aligned}$$

iv. $\frac{\mathrm{d}^n}{\mathrm{d}x^n}P_n(x)$

Answer:
Let's derive only the last term of the polynomial:

$$\frac{\mathrm{d}}{\mathrm{d}x}a_nx^n=na_nx^{n-1},$$

deriving the second time yields

$$\frac{\mathrm{d}}{\mathrm{d}x}na_nx^{n-1}=(n-1)\,na_nx^{n-2},$$

and then the third time...

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(n-1\right)na_nx^{n-2} = \left(n-2\right)\left(n-1\right)na_nx^{n-3}$$

...and so on. The n-th derivative will thus yield

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}a_nx^n = \left(\cancelto{=1}{n-n+1}\right)\times\left(\cancelto{=2}{n-n+2}\right)\cdots\times\left(n-2\right)\left(n-1\right)na_nx^{\cancel{n-n}}.$$

Since $n-n=0$ and $n-n+1=1$, the last term would be equal to

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}a_nx^n = \underbrace{1\times 2\times\cdots\times\left(n-2\right)\left(n-1\right)n}_{=n!}\,a_n\cancel{x^0},$$

as the expression highlighted with a curly beacket is simply $n!$, the last term is equal to $n!\cdot a_n$.

Since the rest of the terms have a power of x which is less than n they will all be lost during the derivation steps. Think for example about a_3x^3: after the first derivation it will become $3a_3x^2$, then $6a_3x$, then $6a_3$ and then simply 0.

Thus, the complete derivative is just the last term, i.e.

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}P_n\left(x\right) = n!\cdot a_n.$$

v. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2\sqrt{x}}$

Answer:
Since $\sqrt{x} = x^{\frac{1}{2}}$, we can simply use the power rule, and thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} &= \frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{2}}\\ &= \frac{1}{2}x^{\frac{1}{2}-1}\\ &= \frac{1}{2}x^{-\frac{1}{2}}\\ &= \frac{1}{2x^{\frac{1}{2}}}\\ &= \frac{1}{2\sqrt{x}}.\end{aligned}$$

Notice that a similar process can be applied for any expression of x (i.e. $f\left(x\right)$) yielding $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{f\left(x\right)} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}f(x)}{2\sqrt{f(x)}}$.
For $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2\sqrt{x}}$ we can use a similar process:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2\sqrt{x}} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}x^{-\frac{1}{2}}\\ &= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}\\ &= -\frac{1}{4x^{\frac{3}{2}}}\\ &= -\frac{1}{4\sqrt{x}^3}.\end{aligned}$$

vi. $\frac{\mathrm{d}}{\mathrm{d}x}e^{3x^3-2x}$, $\frac{\mathrm{d}}{\mathrm{d}x}e^{-2\sqrt{x}}$

Answer:
Generally, $\frac{\mathrm{d}}{\mathrm{d}x}e^{f(x)} = \left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]\cdot e^{f(x)}$ (and specifically: $\frac{\mathrm{d}}{\mathrm{d}x}e^{x} = e^{x}$).
Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{3x^3-2x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^3-2x\right)\cdot e^{3x^3-2x}\\ &= \left(9x^2-2\right)\cdot e^{3x^3-2x}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{-2\sqrt{x}} &= -\cancel{2}\frac{1}{\cancel{2}\sqrt{x}}e^{-2\sqrt{x}}\\ &= -\frac{1}{\sqrt{x}}e^{-2\sqrt{x}}.\end{aligned}$$

vii. $\frac{\mathrm{d}^7}{\mathrm{d}x^7}e^{-x}$

Answer:
Deriving e^{-x} yields $-e^{-x}$, which when derived yields back e^{-x}. Thus after 7 derivation we will get $-e^{-x}$.

viii. $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x-\sin(x)\right)$, $\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^2\right)$

Answer:
Since $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, we get simply

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3x-\sin(x)\right) = 3-\cos(x).$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^2\right) &= \cos\left(x^2\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^2\\ &= 2x\cos\left(x^2\right).\end{aligned}$$

ix. $\frac{\mathrm{d}^8}{\mathrm{d}x^8}\cos(x)$

Answer:
Since $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$, we can see that successive derivation of $\cos(x)$ will yield the follwing (the arrows here represent the derivative):

$$\cos(x)\to\sin(x)\to-\cos(x)\to-\sin(x)\to\cos(x)\to\cdots$$

We can see that after 4 derivatives, $\cos(x)$ becomes again $\cos(x)$, and so after 8 derivates the same will happen. Thus

$$\frac{\mathrm{d}^8}{\mathrm{d}x^8}\cos(x) = \cos(x).$$

3. Analyze the following functions (i.e. find points where the function intersects the axes, find all extrema and their type - including inflection points, and where the function is increasing or decreasing):
   (a) $f(x) = x^3-x^2-x+1$.
   (b) $f(x) = e^{-\frac{1}{2}x^2}$.

Answer:

(a) We start by finding the points where the function intercepts the axes. For the y-axis this will be when $x = 0$:

$$y = f(0) = \cancel{0^3} - \cancel{0^2} - \cancel{0} + 1 = 1.$$

For the x-axis this will be when $y = f(x) = 0$. Thus we need to solve the equation $x^3 - x^2 - x + 1 = 0$. The following is true:

$$x^3 - x^2 - x + 1 = \left(x^2 - 1\right)(x - 1),$$

and so $x^3 - x^2 - x + 1 = 0$ will be true when either $x^2 - 1 = 0$ or $x - 1 = 0$, which means $x = \pm 1$.

Next, we will find the extremum points of the function. Local minima and maxima have the property that in these points $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0$[1]. Thus, we will derive the function by x and find at which values of x our function has a local minimum/maximum:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0 &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[x^3 - x^2 - x + 1\right] = 0 \\
&\Rightarrow 3x^2 - 2x - 1 = 0 \\
&\Rightarrow x_{1,2} = \frac{2 \pm \sqrt{4 + 3 \cdot 4 \cdot 1}}{6} = \frac{2 \pm \sqrt{16}}{6} = \frac{2 \pm 4}{6} = \frac{1 \pm 2}{3} \\
&\Rightarrow x_{1,2} \approx -\frac{1}{3}, 1.
\end{aligned}$$

The corresponding y values of these two points are $y = \frac{32}{27}$ and $y = 0$.

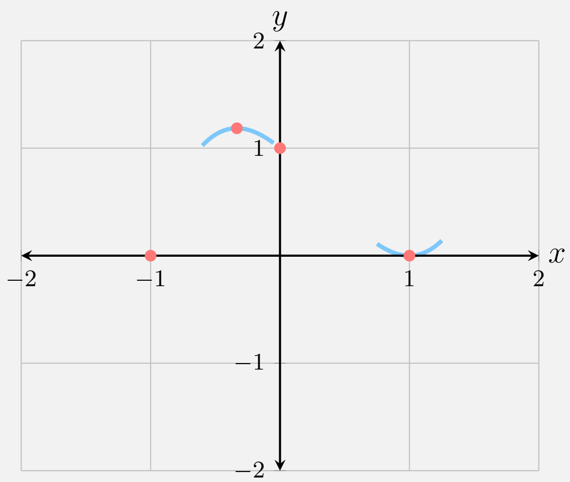
Let's summarize what we have so far:

- Point where the function crosses the x-axis: $(0, 1)$.
- Points where the function crosses the y-axis: $(-1, 0)$, $(1, 0)$.
- Minima and maxima: $\left(-\frac{1}{3}, \frac{32}{27}\right)$ and $(1, 0)$.

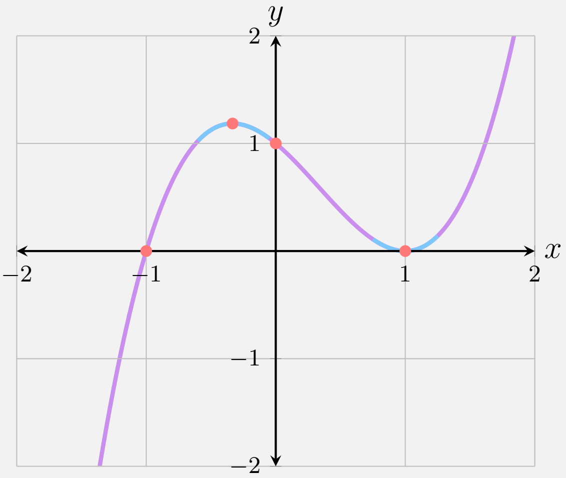
We should now categorize both points $\left(-\frac{1}{3}, \frac{32}{27}\right)$ and $(1, 0)$ into minimum and maximum points. We do this by either checking the function's behaviour in a small neighborhood around them, or by looking at the second derivative at that point: a positive value of the second derivative would mean a minimum, while a negative value would mean a maximum[2].

The second derivative of our function is $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = 6x - 2$. Substituting $x = -\frac{1}{3}$ yields $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f\left(-\frac{1}{3}\right) = -4$, and thus $\left(-\frac{1}{3}, \frac{32}{27}\right)$ is a maximum point. Substituting $x = 1$ into $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$ yields 4 and thus $(1, 0)$ is a minimum point.

Let's draw all the information we have so far:



The only thing remaining is to determine what are the limits of the function at $\pm\infty$. Since x^3 is the term with the highest power of x, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. Thus, the complete function looks as follows:



(b) Starting with points of intersection with the axes, we substitute $x = 0$ into the function and get $f(0) = e^0 = 1$. Solving $f(x) = 0$ should yield the points of intersection of the function with the x-axis:

$$\begin{aligned} f(x) = 0 \Rightarrow e^{-\frac{1}{2}x^2} &= 0 \\ \Rightarrow \log\left(e^{-\frac{1}{2}x^2}\right) &= \log(0). \end{aligned}$$

Since $\log(0)$ is undefined, we must look at the limits: $\lim_{x\to\pm\infty} e^{-\frac{1}{2}x^2} = 0$.

For the extremum points, we will derive the function.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= -\frac{1}{\cancel{2}} \cdot \cancel{2}x \cdot e^{-\frac{1}{2}x^2} \\ &= -xe^{-\frac{1}{2}x^2}. \end{aligned}$$

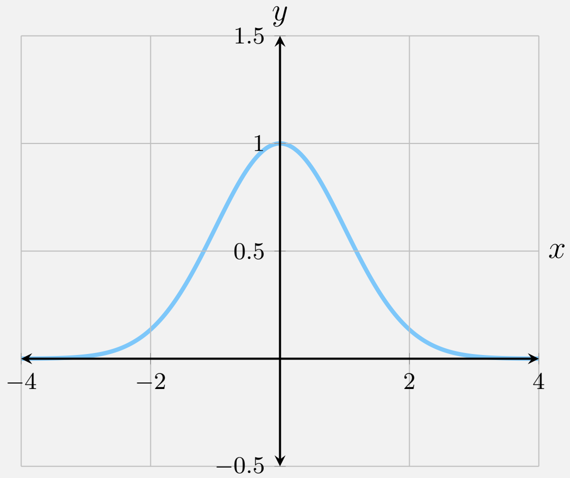
Solving $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0$ thus yields either $x = 0$ or $x = \pm\infty$.

Now let us check the type (minimum or maximum) of these points by deriving $f(x)$ again:

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) &= -\frac{\mathrm{d}}{\mathrm{d}x} xe^{-\frac{1}{2}x^2} \\ &= -\frac{\mathrm{d}}{\mathrm{d}x} x \cdot e^{-\frac{1}{2}x^2} - x\frac{\mathrm{d}}{\mathrm{d}x} e^{-\frac{1}{2}x^2} \\ &= -1 \cdot e^{-\frac{1}{2}x^2} + x \cdot xe^{-\frac{1}{2}x^2} \\ &= e^{-\frac{1}{2}x^2}\left(-1 + x^2\right) \\ &= \left(x^2 - 1\right) e^{-\frac{1}{2}x^2}. \end{aligned}$$

Subtituting $x = 0$ to $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$ yields $\left(0^2 - 1\right) e^{-\frac{1}{2}0^2} = -1 < 0$. Thus, the point $(0, 1)$ is a local maximum.

Using all this data, we can plot our function:



This is, of course, the normal distribution function (a.k.a. the 'bell curve', or the Gaussian distribution).

4. **Extra Question** (if time permits)

   i. Using matrix multiplication, show that if a line has slope m, a perpendicular line would have a slope $-\frac{1}{m}$.

   Answer:
   A line with slope m can be represented by a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $\frac{y}{x} = m$.
   From previous tutorials we know that the 90° (clockwise) rotation matrix is

   $$R_{90^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

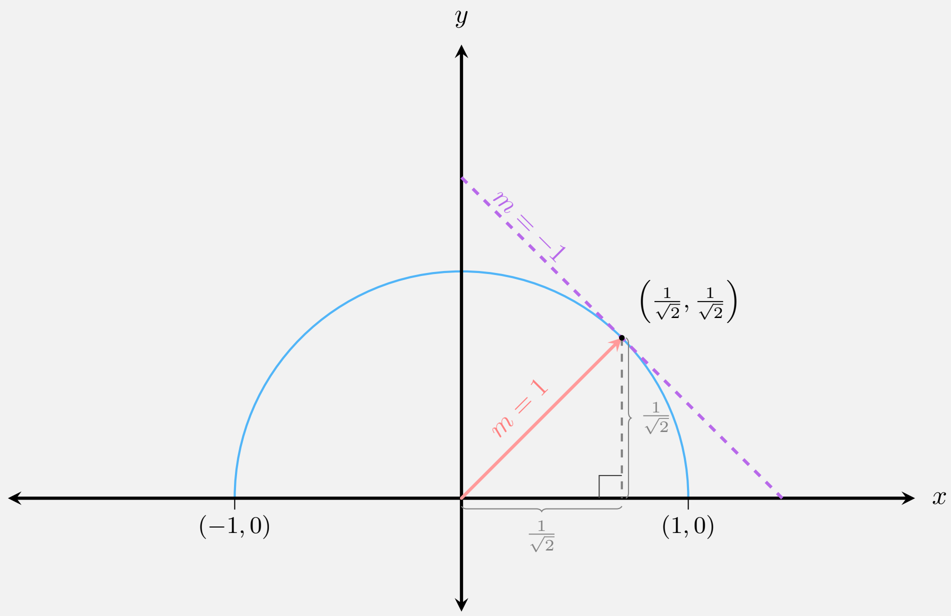
   Thus $R_{90^\circ} \cdot \vec{v} = \begin{pmatrix} -y \\ x \end{pmatrix}$, and the corresponding slope would be $m' = \frac{x}{-y} = -\frac{x}{y} = -\frac{1}{m}$.

   ii. Find the slope of the function $f(x) = \sqrt{1 - x^2}$ at $x = \frac{1}{\sqrt{2}}$ without using derivates.

   Answer:
   Rewriting $y = \sqrt{1 - x^2}$ and then squaring both sides yields $y^2 = 1 - x^2$, or $x^2 + y^2 = 1$. This is of course a circle of radius $r = 1$ centered at the origin. At $x = \frac{1}{2}$ $y = \frac{1}{2}$ as well, and this corresponds to an isosceles right triangle, with hypotenuse equal to 1 (since it is the circle's radius). Since the radius to the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is of slope 1, and the tangent line to the circle at any point must be at 90° to the radius, it follows that the tangent at that point has slope -1. This is exactly the derivative of the function at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

[1] If you understand why this is true, you have at least a good basic grasp of differential calculus! This is one of the most important concepts of the field.

[2] ...and if you understand why this is true, you have more than a good basic grasp of the topic! If you don't and are curious, don't hesitate to contact me at pelegs@gmail.com or watch this YouTube series: `https://youtu.be/WUvTyaaNkzM`.



iii. Find the derivative of $f(x)$ at $x = \frac{1}{\sqrt{2}}$ with derivation and compare the result to the one obtained in the previous section.

Answer:
$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = -\frac{x}{\sqrt{1-x^2}}$, and thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f\left(\frac{1}{\sqrt{2}}\right) &= -\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}} \\
&= -\frac{\frac{1}{\sqrt{2}}}{\sqrt{1-\frac{1}{2}}} \\
&= -\frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{1}{2}}} \\
&= -\frac{\cancel{\frac{1}{\sqrt{2}}}}{\cancel{\frac{1}{\sqrt{2}}}} \\
&= -1.
\end{aligned}$$

as expected.