Basic Maths for Non-mathematicians

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$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(x_k)\Delta x$$

$$(AB)^\top = B^\top A^\top$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$A = Q\Lambda Q^{-1}$$

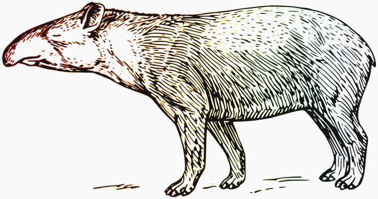
$$A\vec{v} = \lambda\vec{v}$$

$$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

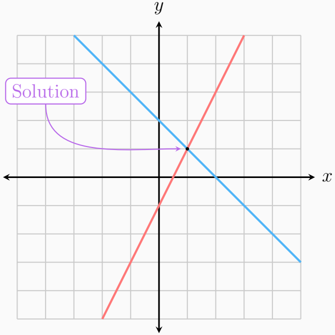
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$



Chapter 5: Systems of Linear Equations

$$\begin{cases} \boxed{2x - y = 1} \\ \boxed{x + y = 2} \end{cases} \Rightarrow \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



Linear Equations

Definition

A **linear equation** is an equation of the form

$$a_1 \; x_1 + a_2 \; x_2 + \cdots + a_n \; x_n = \; b$$

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Variables

Linear Equations

Definition

A **linear equation** is an equation of the form

$$\boxed{a_1} x_1 + \boxed{a_2} x_2 + \cdots + \boxed{a_n} x_n = b$$

Coefficients

Linear Equations

Definition

A **linear equation** is an equation of the form

$$a_1\ x_1 + a_2\ x_2 + \cdots + a_n\ x_n = \boxed{b}$$

ree coefficient

Linear Equations

Example

The following are three linear equations of the variables x, y and z:

$$2x - 7y + z = 26$$
$$-3x + y = -9$$
$$9y - 4z = -31$$

Note

In the second equation above the coefficient of z is zero, while in the last equation the coefficient of x is zero.

Definition

A **system of linear equation** is a set of linear equations of the same variables.

Example

The previous three equations can be combined together to form a system of three linear equations in three variables (x, y and z).

Systems of Linear Equations

We can write systems of linear equations as product of a matrix (the coefficients) and a vector (the variables) equatling a vector (the free coefficients).

Example

The previous system can be written in matrix form as

$$\begin{pmatrix} 2 & -7 & 1 \\ -3 & 1 & 0 \\ 0 & 9 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 26 \\ -9 \\ -31 \end{pmatrix}.$$

Systems of Linear Equations

A general system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

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In matrix form it is simply

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Solution Set

Definition

A **solution** is an ordered set of values which correspond to the variables of the system, such that all of its equations are satisfied.

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Example

The only solution for the previous system is

$$x = 2,\ y = -3,\ z = 1,$$

which in vector form can be written as $\vec{u} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Solution Set

Generally, a linear system might have any of the following:

- An **infinite** amount of distinct solutions.
- Only **a single** solution.
- **No solutions**.

The number of solutions depends on the properties of the system, which we will briefly explore in this chapter.

Geometric Interpretation of the Solution Set

A linear equation in **two** variables represents a **line** in $\mathbb{R}^2$, a linear equation in **three** variables represents a **plane** in $\mathbb{R}^3$, and so forth.

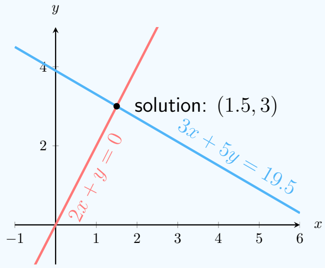
Thus, a solution of several linear equations represents a set of points where the respective shapes intersect.

Example

The equations

$$-2x + y = 0$$
$$3x + 5y = 19.5$$

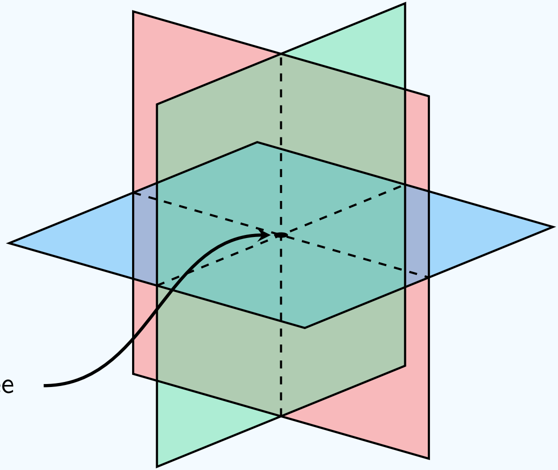
represent the following two lines:



Geometric Interpretation of the Solution Set

Example

Here will come explaination



Solution
of all three
equations

Gaussian Elimination Method

We will now introduct the **Gaussian elimination method** for solving linear systems.

In matrix form, a system of linear equations looks as

$$\boxed{A}\boxed{\vec{x}} = \boxed{\vec{b}}.$$

$$\boxed{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}\boxed{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} = \boxed{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}.$$

Gaussian Elimination Method

We can "stick" A and $\vec{b}$ together to form an augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

We then apply to the matrix a sequence of **row operations**, untill the matrix is in a form which we will introduce in a moment.

Gaussian Elimination Method

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

The three row operations are:

- Exchange any two rows i and j.
- Multiply any row i by a scalar $0 \neq \gamma \in \mathbb{R}$.
- Subtract any γ-scaled row j from a different row $i \neq j$.

Gaussian Elimination Method

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

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Gaussian Elimination Method

$$\left(\begin{array}{cccc|c}
a_{31} & a_{32} & \dots & a_{3n} & b_3 \\
a_{21} & a_{22} & \dots & a_{2n} & b_2 \\
a_{11} & a_{12} & \dots & a_{1n} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & a_{m2} & \dots & a_{mn} & b_m
\end{array}\right)$$

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Gaussian Elimination Method

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \gamma a_{31} & \gamma a_{32} & \dots & \gamma a_{3n} & \gamma b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

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Gaussian Elimination Method

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

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Gaussian Elimination Method

$$\left(\begin{array}{cccc|c} a_{11} - \gamma a_{31} & a_{12} - \gamma a_{21} & \dots & a_{1n} - \gamma a_{3n} & b_1 - \gamma b_3 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & b_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

The three row operations are:

- Exchange any two rows i and j.
- Multiply any row i by a scalar $0 \neq \gamma \in \mathbb{R}$.
- Subtract any γ-scaled row j from a different row $i \neq j$.

Gaussian Elimination Method

The process proceeds until the matrix is in a **row echelon form**, which has the following properties:

- all nonzero rows are above any row of zeroes, and
- the first nonzero number from the left, called the **leading coefficient**, of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Example

Matrices in row echelon form:

$$\begin{pmatrix} 3 & 4 & -1 & 7 \\ 0 & -2 & 9 & -1 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 5 \\ 0 & 7 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 6 & 2 & 5 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Gaussian Elimination Method

Further steps can be taken to bring a matrix to a **reduced row echelon form**, which is a row echelon form in which

- the matrix is in a row echelon form,
- the leading coefficients are all 1 (called a **leading 1**), and
- each column containing a leading 1 has only zeros in its other components.

Example

Matrices in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

The following system is given:

$$\begin{pmatrix} 0 & 1 & 7 \\ -2 & 0 & 2 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 14 \end{pmatrix}.$$

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

Rearranging into an augemented matrix:

$$\left(\begin{array}{ccc|c} 0 & 1 & 7 & -4 \\ -2 & 0 & 2 & -6 \\ 0 & 1 & 5 & -14 \end{array}\right)$$

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_1 \longleftrightarrow R_2$$

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & -6 \\ 0 & 1 & 7 & -4 \\ 0 & 1 & 5 & -14 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_1 \longrightarrow -\tfrac{1}{2}R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 7 & -4 \\ 0 & 1 & 5 & -14 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_3 \longrightarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & -2 & -10 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_3 \longrightarrow -\tfrac{1}{2}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_1 \longrightarrow R_1 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 7 & -4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

$$R_2 \longrightarrow R_2 - 7R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -39 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Gaussian Elimination Method

When a matrix is in its reduced row echelon form, the system can be solved more easily, starting from the bottom-most non-zero row.

Example

Thus the solution to this system is

$$
\begin{aligned}
x &= 8, \\
y &= -39, \\
z &= 5.
\end{aligned}
$$

Gaussian Elimination

The row operations introduced here do not change the rank and determinant of a matrix.

Thus, if the row echelon form of a matrix has one or more zero rows, then its determinant is zero, and in turn - the determinant of the original matrix is also zero.

$$\mathrm{rank}(A) < n \Leftrightarrow |A| = 0.$$

Gaussian Elimination

The differnce between the number of rows in a matrix A and its rank, $n - \mathrm{rank}(A)$, is the number of free variables in the solution.

Example

The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

has three rows and one zero rows. Thus, $\mathrm{rank}(A) = 2$, and the solution for the system it represents has $d = 3 - 2 = 1$ free variables. In this case $x = -3$, $y = 2$ and z can be chosen freely.