Basic Maths for Non-mathematicians

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$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(x_k)\Delta x$$

$$(AB)^\top = B^\top A^\top \qquad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

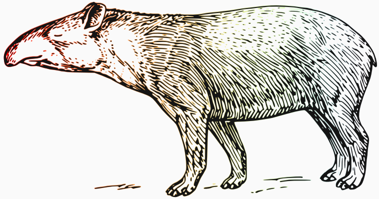
$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad A = Q\Lambda Q^{-1}$$

$$A\vec{v} = \lambda\vec{v}$$

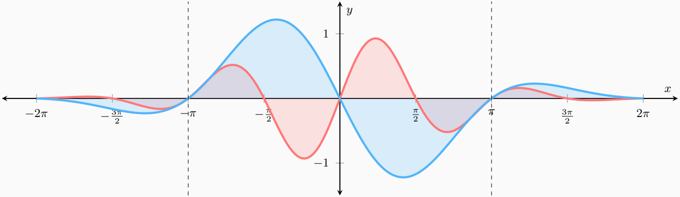
$$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v}) \qquad \langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$



Chapter 9: General Vector Spaces

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, \mathrm{d}x$$



Properties of $\mathbb{R}^n$

Let us review some properties of the space $\mathbb{R}^n$, some of them we already used implicetly without giving them too much thought.

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The addition of any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ yields a vector $\vec{w} = \vec{u} + \vec{v}$ that is also in $\mathbb{R}^n$.

Example

For $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ (both in $\mathbb{R}^3$),

$$\vec{w} = \vec{u} + \vec{v} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} \in \mathbb{R}^3.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is commutative: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$.

Example

For the same vectors as before:

$$\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + (-1) \\ 2 + 3 \\ -1 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$

$$\vec{v} + \vec{u} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 + 1 \\ 3 + 2 \\ 0 + (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is assosiative: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

Example

For $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:

$$\begin{aligned} \vec{a} + \left(\vec{b} + \vec{c}\right) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \end{aligned}$$

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Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The zero vector $\vec{0}$ is unique and has the property that $\vec{v} + \vec{0} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Any vector $\vec{v} \in \mathbb{R}^n$ has an opposite vector $(-\vec{v}) \in \mathbb{R}^n$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ 0 \\ -3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Any scale by $\alpha \in \mathbb{R}$ of a vector $\vec{v} \in \mathbb{R}^n$ is also in $\mathbb{R}^n$.

Example

$$-3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -6 \\ 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}.$$

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Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is associative: $\alpha\left(\beta\vec{v}\right) = \left(\alpha\beta\right)\vec{v}$.

Example

$$-3\left[2\begin{pmatrix}1\\-4\\5\end{pmatrix}\right] = -3\begin{pmatrix}2\\-8\\10\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}$$

$$(-3\cdot 2)\begin{pmatrix}1\\-4\\5\end{pmatrix} = -6\begin{pmatrix}1\\-4\\5\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is distributive in respect to scalar addition: $(\alpha + \beta)\,\vec{v} = \alpha\vec{v} + \beta\vec{v}$.

Example

$$(5-2)\begin{pmatrix}2\\-1\\0\end{pmatrix} = 3\begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}6\\-3\\0\end{pmatrix}.$$

$$5\begin{pmatrix}2\\-1\\0\end{pmatrix} - 2\begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}10\\-5\\0\end{pmatrix} - \begin{pmatrix}4\\-2\\0\end{pmatrix} = \begin{pmatrix}6\\-3\\0\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

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Example

$$5\left[\begin{pmatrix}1\\-1\\3\end{pmatrix}+\begin{pmatrix}0\\2\\-1\end{pmatrix}\right]=5\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}5\\5\\10\end{pmatrix}.$$

$$5\begin{pmatrix}1\\-1\\3\end{pmatrix}+5\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}5\\-5\\15\end{pmatrix}+\begin{pmatrix}0\\10\\-5\end{pmatrix}=\begin{pmatrix}5\\5\\10\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- The scalar $\alpha = 1$ is neutral in respect to scalar-vector products: $1\vec{v} = \vec{v}$.

Example

$$1 \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix}.$$

Abstract Vector Spaces

These properties are somewhat obvious on $\mathbb{R}^n$. However, many times it is worthwhile to use more abstract vector spaces, which can help us model diverse physical and theoretical systems, since once a construct behaves as a vector space, it is a relatively simple process to apply to it all the analysis tools learned so far.

We will not bother here with the formal definition of a vector space[1], but look at one example, which we will later expand on: the space of all real functions $f : \mathbb{R} \to \mathbb{R}$.

[1] For such definition, see here: http://www.math.niu.edu/ beachy/courses/240/06spring/vectorspace.html.

Real Functions as a Vector Space

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- For each real function $f(x)$ there exists an opposite function $(-f)(x) = -f(x)$, for which

$$f(x) + \big(-f(x)\big) = f(x) - f(x) = 0 = z(x).$$

Components (temp name)

Recall that a vector in $\mathbb{R}^n$ can be written using its component in any basis, e.g. the standard basis vectors $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$:

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots + v_n\hat{e}_n.$$

How can we "decompose" a function in a similar way?

The **Dirac delta function** comes in handy.

The Dirac Delta Function

Loosely speaking, we can define the Dirac delta function as

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0. \end{cases}$$