

# Basic Maths for Non-mathematicians

Peleg Bar Sapi

$$\int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} \sum_{k=1}^N f(x_k) \Delta x$$

$$(AB)^\top = B^\top A^\top \quad \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

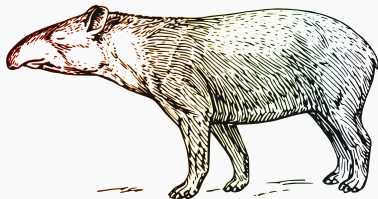
$$\vec{v} = \sum_{i=1}^n \alpha_i \hat{e}_i \quad A = Q \Lambda Q^{-1}$$

$$\frac{df}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\text{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad A\vec{v} = \lambda\vec{v}$$

$$\int_a^b f(x) dx = F(b) - F(a)$$

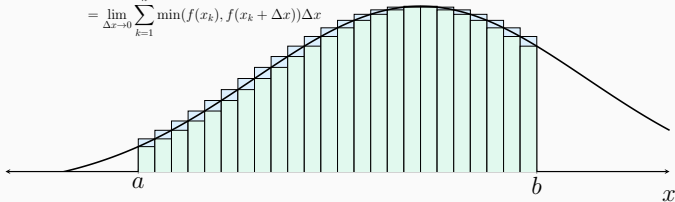
$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v}) \quad \langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$



## Chapter 8: Integrals

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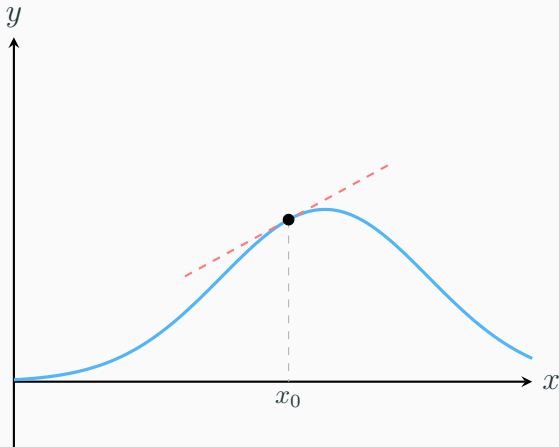
$$\begin{aligned}\int_a^b f(x) \, dx &= \lim_{\Delta x \rightarrow 0} \sum_{k=1}^n \min(f(x_k), f(x_k + \Delta x)) \Delta x \\ &= \lim_{\Delta x \rightarrow 0} \sum_{k=1}^n \min(f(x_k), f(x_k + \Delta x)) \Delta x\end{aligned}$$



## Reminder: The Derivative

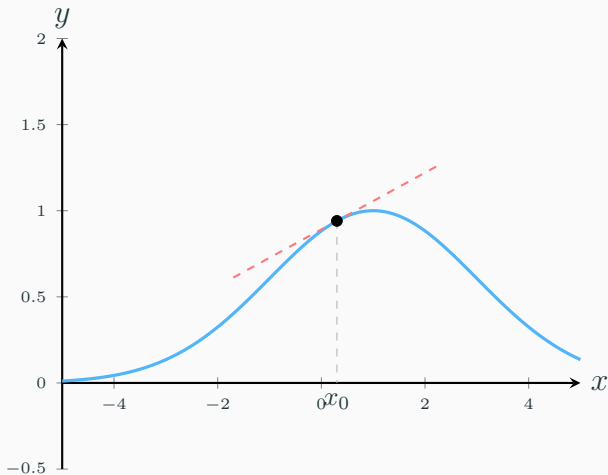
### Definition

The **derivative** of a function  $f$  at the point  $x_0$ , denoted  $f'(x_0)$ , is the **slope** of the tangent line to the function at  $x_0$ .



## Reminder: The Derivative

We can find the derivative by taking closer and closer points to  $x_0$ .



## Reminder: The Derivative

Thus, the derivative is the limit where  $\Delta x$  goes to 0:

### Definition

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\cancel{x_0} + \Delta x - \cancel{x_0}} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

### Note

The notation  $\frac{dy}{dx}$  **is not a fraction**. It only signifies that the derivative is the limit of  $\frac{\Delta y}{\Delta x}$  when  $\Delta x \rightarrow 0$ .

## Reminder: The Derivative

We can view the derivative as an **operator** acting on a function:



Recall that the following two statements are true:

$$\begin{cases} (\alpha f)' &= \alpha f', \\ (f + g)' &= f' + g'. \end{cases}$$

Similar to **linear transformations** acting on vectors, this means that the derivative operator is **linear**.

## Reminder: The Derivative

By viewing the derivative as an operator acting on functions, the notion of **higher order derivatives** becomes pretty clear:

### Definition

An  $n$ -th order derivative (where  $n \in \mathbb{N}$ ) of a function  $f$  is the result of applying the derivative operator  $n$  consecutive times on  $f$ .

We denote the  $n$ -th order derivative of  $f$  as either

$$f^{(n)},$$

or

$$\frac{d^n f}{dx^n}.$$

# The Antiderivative

Similar to linear transformations (with a non-zero determinant), one might try to find an inverse operator to derivative. We call such an operator the **Antiderivative**, or the **Non-definite integral**.

We can think of it a derivative of order  $-1$ , i.e.  $\frac{d^{-1}}{dx^{-1}}$ :

$$f'(x) \longrightarrow \left( \frac{d^{-1}}{dx^{-1}} \right) \longrightarrow f(x) + C$$

Why?



# The Antiderivative

Recall that constants are derived to 0, and thus don't affect the derivative:

$$(f(x) + C)' = f'(x) + \cancel{0'} = f'(x).$$

Therefore the antiderivative of a function is not a function itself, but a **family** of functions, differing from each other by a constant:

$$\frac{d^{-1}}{dx^{-1}} f = \{f(x) + C \mid C \in \mathbb{R}\}.$$

# The Antiderivative

A common way to denote the antiderivative of a function  $f$  by the variable  $x$  is

$$\int f(x) \, dx,$$

which is called the **indefinite integral**.

The reason for "multiplying" the function by  $dx$  will be made clear later.

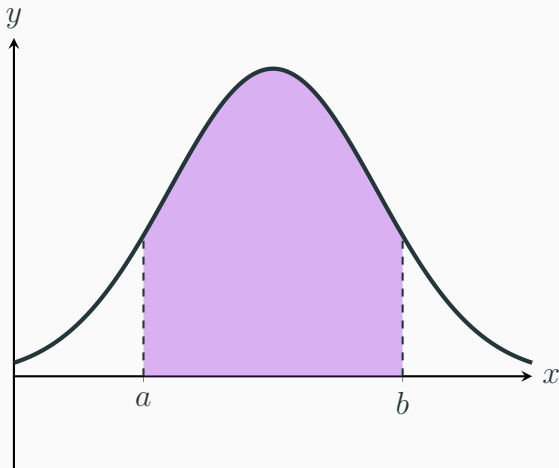
# Non-existing Antiderivatives

All **elementary** functions (e.g. polynomials, exponentials, trigonometric functions, etc. - and compositions of such functions) have well defined derivatives, except for specific points (i.e.  $x = 0$  for  $|x|$ , or  $x = \left(\frac{1}{2} + k\right)\pi, k \in \mathbb{Z}$  for  $\tan(x)$ ).

This is **NOT** true for antiderivatives: not every "well behaved" function has a known antiderivative.

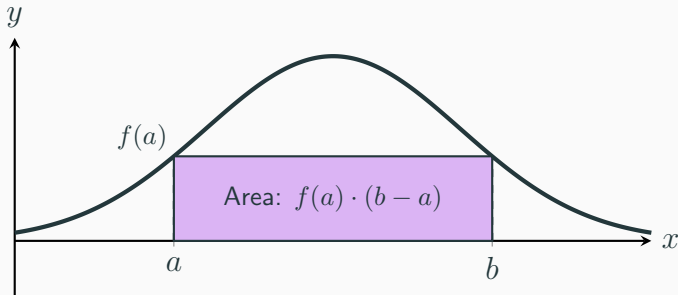
# The Definite Integral

What is the area under the curve of a function, between two points  $x_1 = a$  and  $x_2 = b$ ?



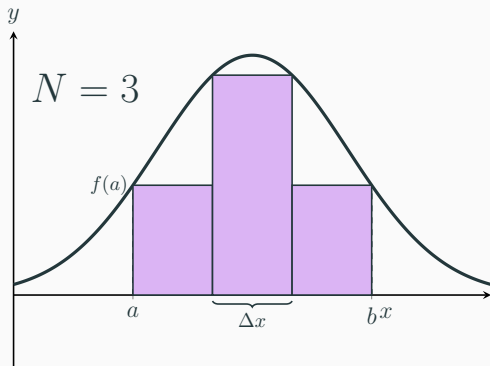
# The Definite Integral

We can start by approximating the area with a shape of a known area: a rectangle.



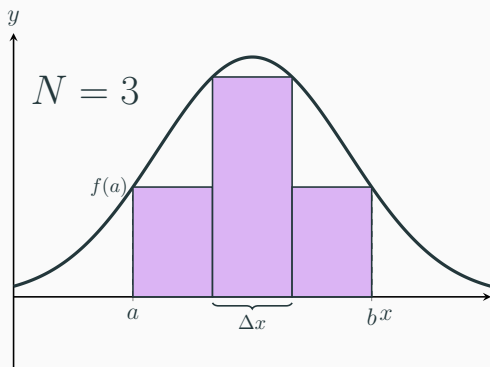
# The Definite Integral

Obviously, this is not a great approximation, so we can divide the interval  $[a, b]$  in three, and use the resulting rectangles (each with a base length  $\Delta x = \frac{(b-a)}{3}$ ) to approximate the area:



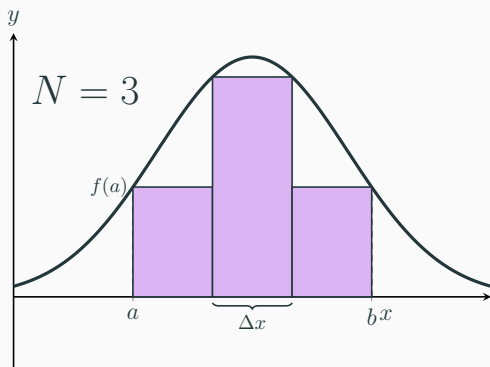
# The Definite Integral

Although each of the rectangles have a base  $\Delta x = \frac{(b-a)}{3}$ , their areas are dependent on their position: their height is either the value of  $f$  at their left side, or value of  $f$  at their right side.



# The Definite Integral

This will be determined by choosing the heights as either the minimum or maximum of the left and right values of  $f$ . For now we will arbitrarily choose using the minimum values.

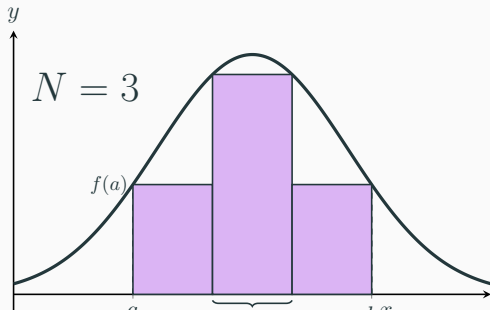




# The Definite Integral

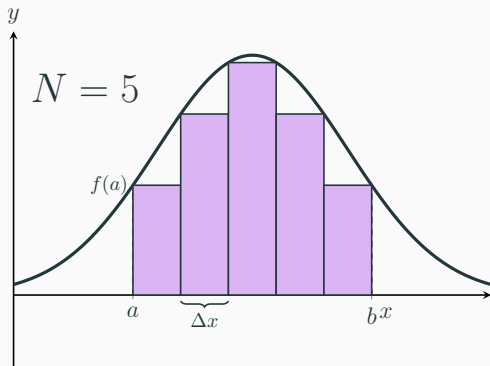
Writing  $x_1 = a$ ,  $x_2 = a + \Delta x$ ,  $x_3 = a + 2\Delta x$ ,  $x_4 = b$ , the total area approximated by the rectangles is:

$$\begin{aligned} S_{\text{approx}} &= \min(f(x_1), f(x_2))(b - a) + \min(f(x_2), f(x_3))(b - a) + \min(f(x_3), f(x_4))(b - a) \\ &= [\min(f(x_1), f(x_2)) + \min(f(x_2), f(x_3)) + \min(f(x_3), f(x_4))] \Delta x \\ &= \sum_{i=1}^3 \min(f(x_i), f(x_{i+1})) \Delta x. \end{aligned}$$



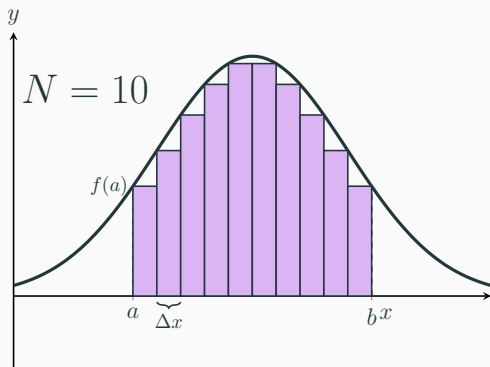
# The Definite Integral

Of course, we can refine the approximation by increasing the number of rectangles (which is equivalent to reducing  $\Delta x$ , since  $\Delta x = \frac{b-a}{N}$ ):



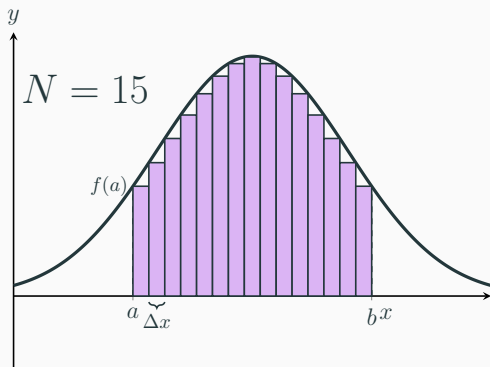
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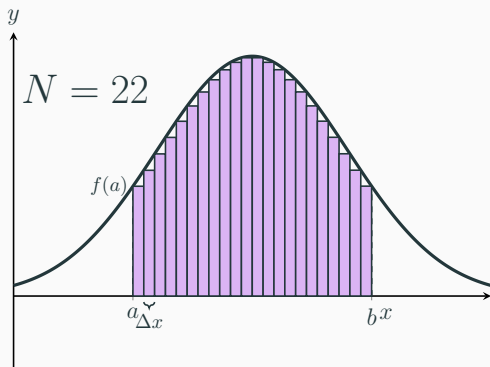
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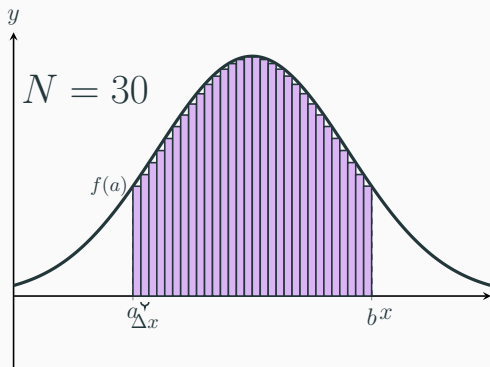
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# The Definite Integral

In the limit where  $N \rightarrow \infty$  (equivalently,  $\Delta x \rightarrow 0$ ), we get the exact area<sup>1</sup>, and write:

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^N \min(f(x_i), f(x_i + \Delta x)) \Delta x = \int_a^b f(x) dx.$$

The above sum is called the **lower Darboux sum** of  $f$  in the interval  $[a, b]$ .

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<sup>1</sup>if the function is well behaved...

# The Fundamental Theorem of Calculus

The connection between the two types of integrals mentioned so far is as follows: for a function  $f$ , its antiderivative  $F$  (i.e.  $\frac{d}{dx}F = f$ ) and a real interval  $[a, b]$ ,

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

This is a corollary of the **fundamental theorem of calculus**.



$$f(x) \longrightarrow \left( \frac{d}{dx} \right) \longrightarrow f'(x)$$

$$f'(x) \longrightarrow \left( \int dx \right) \longrightarrow f(x) + C$$

$$f(x) \longrightarrow \left( \int_a^b dx \right) \longrightarrow S_{a,b}$$