Exercise 9: Summation and Sequences (Solution)

Problem 1: Summation

1. Write the following expressions explicitly:

(a) $\sum_{n=1}^{5}\left(n^2-2n\right)$

Answer:

$$\begin{aligned}\sum_{n=1}^{5}\left(n^2-2n\right) &= \left(1^2-2\right)+\left(2^2-4\right)+\left(3^2-6\right)+\left(4^2-8\right)+\left(5^2-10\right)\\ &= -1+0+3+8+15\\ &= 25.\end{aligned}$$

(b) $\sum_{n=-3}^{3} 2^n$

Answer:

$$\begin{aligned}\sum_{n=-3}^{3} 2^n &= 2^{-3}+2^{-2}+2^{-1}+2^0+2^1+2^2+2^3\\ &= \frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^1}+\frac{1}{2^0}+2^1+2^2+2^3\\ &= \frac{1}{8}+\frac{1}{4}+\frac{1}{2}+1+2+4+8\\ &= 15.875.\end{aligned}$$

(c) $\sum_{i=1}^{3}\sum_{j=1}^{4} a_{ij}$

Answer:

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{4} a_{ij} &= \boxed{\sum_{j=1}^{4} a_{1j}}+\boxed{\sum_{j=1}^{4} a_{2j}}+\boxed{\sum_{j=1}^{4} a_{3j}}\\ &= \boxed{a_{11}+a_{12}+a_{13}+a_{14}}+\boxed{a_{21}+a_{22}+a_{23}+a_{24}}+\boxed{a_{31}+a_{32}+a_{33}+a_{34}}\end{aligned}$$

2. Use the summation form to write the product of two matrices $C = A \cdot B$, with dimensions $M \times N$ and $N \times K$, respectively.

Answer:
Let's define a_{ij} as the elements of matrix A, b_{ij} the elements of matrix B and c_{ij} the elements of the resulting matrix $C = AB$. The element cij is the result of the dot product between the i-th

row of matrix A and the j-th column of matrix B:

$$\begin{aligned} c_{ij} &= \vec{a}_{\text{row}=i} \cdot \vec{b}_{\text{column}=j} \\ &= \sum_{k=1}^{N} a_{ik} b_{kj} \end{aligned}$$

Let's look at a specific example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 10 & 9 \\ 8 & 7 \\ 6 & 5 \end{bmatrix}$. The element c_{12} of the resulting matrix C is a product of the first row of A: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, and the second column of B: $\begin{bmatrix} 9 \\ 7 \\ 5 \end{bmatrix}$.

We therefore set $i = 1, j = 2$ in the resulting general sum above, and get

$$\begin{aligned} c_{12} &= \sum_{k=1}^{3} a_{1k} b_{k2} \\ &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ &= 1 \times 9 + 2 \times 7 + 3 \times 5. \end{aligned}$$

3. Write in summation form the general real polynomial of order n: $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots, a_n$ are real numbers and $a_n \neq 0$.

Answer:

$$P_n(x) = \sum_{i=1}^{n} a_i x^i, \quad \{a_i\} \in \mathbb{R}, \quad a_n \neq 0.$$

4. The binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n!$ is defined as $n! = 1 \times 2 \times 3 \cdots \times (n-1) \times n$. What is $\binom{4}{2}$?

Answer:

$$\begin{aligned} \binom{4}{2} &= \frac{4!}{2!\,(4-2)!} \\ &= \frac{1 \times 2 \times 3 \times 4}{(1 \times 2) \times (2)!} \\ &= \frac{\cancel{1 \times 2} \times 3 \times 4}{\cancel{(1 \times 2)} \times (1 \times 2)} \\ &= \frac{12}{2} \\ &= 6. \end{aligned}$$

5. The general expension formula for $(x+y)^n$ (where $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$) is:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Show that for $n = 2$ the formula yields the known expression $(x+y)^2 = x^2 + 2xy + y^2$, and write the full formula for $(x+y)^4$.

Answer:
For $n = 2$ we get:

$$\begin{aligned}(x+y)^2 &= \sum_{k=0}^{2}\binom{2}{k}x^{2-k}y^k \\ &= \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 \\ &= \frac{2!}{0!2!}x^2 + \frac{2!}{1!1!}xy + \frac{2!}{2!0!}y^2 \\ &= \frac{1}{0!}x^2 + 2xy + \frac{1}{0!}y^2 \\ &= x^2 + 2xy + y^2.\end{aligned}$$

For $n = 4$ we get:

$$\begin{aligned}(x+y)^4 &= \sum_{k=0}^{4}\binom{4}{k}x^{4-k}y^k \\ &= \binom{4}{0}x^4y^0 + \binom{4}{1}x^{4-1}y^1 + \binom{4}{2}x^{4-2}y^2 + \binom{4}{3}x^{4-3}y^3 + \binom{4}{4}x^{4-4}y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

Problem 2: Sequences

1. Write the first 10 elements of the following sequences:

$$a_n = 3n - 2, \quad b_n = 1, \quad c_n = \frac{1}{n}, \quad d_n = (-1)^n, \quad e_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

Answer:

(a) $a = 1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ 22,\ 25,\ 28,\ \ldots$

(b) $b = 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \ldots$

(c) $c = 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \ldots$

(d) $d = -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \ldots$

(e) $e = \frac{1}{2},\ 2,\ \frac{1}{8},\ 4,\ \frac{1}{32},\ 6,\ \frac{1}{128},\ 8,\ \frac{1}{512},\ 10,\ \ldots$

2. Which of the above sequences are bounded from above and what are their upper boundaries? Which are bounded from below and what are their lower boundaries?

Answer:

- $\{a_n\}$ has a lower boundary equal to 1, and no upper boundary.
- $\{b_n\}$ has both a lower and an upper boundaries, and they are both equal to 1.
- $\{c_n\}$ has a lower boundary equal to 0 and an upper boundary equal to 1.
- $\{d_n\}$ has a lower boundary equal to -1 and an upper boundary equal to 1.
- $\{e_n\}$ has a lower boundary equal to 0 and no upper boundary.

3. Which of the above sequences converge for $n \longrightarrow \infty$? For those that don't, find a sub-sequence that does.

Answer:
The sequences b and c converge. The rest don't.
Subsequences convergence:

- $\{a_n\}$: does not have any converging subsequence.
- $\{d_n\}$: there are infinite converging subsequences, e.g. if one takes only the elements with even indices.
- $\{e_n\}$: an example of a converging subsequence would be only the elements with odd indices.

4. Prove that the following sequences converge to the given limits:

- $a_n = \frac{1}{n} \longrightarrow 0.$

Answer:
Let $\varepsilon > 0$ be a real number. For an $n > n_\varepsilon = \lceil \frac{1}{\varepsilon} \rceil$ (where $\lceil x \rceil$ of some $x \in \mathbb{R}$ is the smallest integer n that is bigger than x) we get

$$a_n = \frac{1}{n} \underset{\text{since } n > n_\varepsilon}{<} \frac{1}{n_\varepsilon} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} \underset{\text{since } \lceil \frac{1}{\varepsilon} \rceil \geq \frac{1}{\varepsilon}}{\leq} \frac{1}{\frac{1}{\varepsilon}} = \varepsilon,$$

meaning that for each real $\varepsilon > 0$, there is an $n_\varepsilon = \lceil \frac{1}{\varepsilon} \rceil$ for which for any $n > n_\varepsilon$ the sequence values are within ε of 0, and therefore this is the limit of the sequence.

- $a_n = \frac{n+2}{n} \longrightarrow 1.$

Answer:
Let $\varepsilon > 0$ be a real number. Then for $n \neq 0$,

$$\frac{n+2}{n} = \frac{\frac{\cancel{n}}{n} + \frac{2}{n}}{\frac{\cancel{n}}{n}} = \frac{1 + \frac{2}{n}}{1} = 1 + \frac{2}{n}.$$

Then, similarly to the previous sequence, for

$$n_\varepsilon = \lceil \frac{2}{\varepsilon} \rceil,$$

we get that any $n > n_\varepsilon$ will confirm to the following:

$$|a_n - 1| \underset{\text{since } n > n_\varepsilon,\, n > 1 \text{ and } a_{n+1} < a_n}{<} |a_{n_\varepsilon} - 1| = \left| \cancel{1} + \frac{2}{\lceil \frac{2}{\varepsilon} \rceil} - \cancel{1} \right| = \left| \frac{2}{\lceil \frac{2}{\varepsilon} \rceil} \right| \underset{\text{since } \lceil \frac{2}{\varepsilon} \rceil \geq \frac{2}{\varepsilon}}{\leq} \left| \frac{2}{\frac{2}{\varepsilon}} \right| = |\varepsilon| \underset{\text{since } \varepsilon > 0}{=} \varepsilon.$$

Thus, for any real number $\varepsilon > 0$, there exists an integer n_ε such that for each $n > n_\varepsilon$

$$|a_n - 1| < \varepsilon,$$

and $a_n \longrightarrow 1$.

Note: during this proof it is claimed that $a_{n+1} < a_n$. Let us show this by calculating the ratio $\frac{a_{n+1}}{a_n}$:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{n+1+2}{n+1}}{\frac{n+2}{n}} \\ &= \frac{\frac{n+3}{n+1}}{\frac{n+2}{n}} \\ &= \frac{n\,(n+3)}{(n+1)\,(n+2)} \\ &= \frac{n^2+3n}{n^2+3n+2}.\end{aligned}$$

For $n \geq 0$, as we have here, $\frac{n^2+3n}{n^2+3n+2} < 1$, i.e. $a_{n+1} < a_n$.

- $a_n = \frac{\sin(n)}{n} \longrightarrow 0$.

Answer:
For any $x \in \mathbb{R}$, and thus any $n \in \mathbb{N}$,

$$\sin(x) \in [-1, 1],$$

and therefore

$$\left|\frac{\sin(n)}{n}\right| \leq \left|\frac{1}{n}\right|.$$

Since we already proved that $\frac{1}{n} \longrightarrow 0$, this is true for $\frac{\sin(n)}{n}$ as well.

Problem 3: Series

Calculate the following expressions:

1. $\sum\limits_{n=0}^{\infty} \frac{5}{2^n}$.

Answer:
This is a geometric series with first time $a = 5$ and ratio $r = \frac{1}{2}$. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{5}{2^n} &= \frac{a}{1-r} \\ &= \frac{5}{1-\frac{1}{2}} \\ &= \frac{5}{\frac{1}{2}} \\ &= 5 \cdot 2 \\ &= 10.\end{aligned}$$

2. $\sum\limits_{n=2}^{\infty} \frac{1}{n^2-n}$.

Answer:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n^2 - n} &= \sum_{n=2}^{\infty} \frac{1}{n\,(n-1)} \\ &= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \lim_{n\to\infty} \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right).\end{aligned}$$

Note how in each term the last element is cancelling the first element of the next term, i.e.

$$\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n}\right),$$

and thus

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1.$$

3. $\sum\limits_{n=0}^{\infty} \frac{n!}{2^n}$.

Answer:
Note that

$$2^n = 2 \cdot 2 \cdot 2 \cdots 2,$$

while

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For any $n > 2$, $n! > 2^n$, and thus

$$\lim_{n\to\infty} \frac{n!}{2^n} = \infty,$$

meaning that

$$\sum_{n=0}^{\infty} \frac{n!}{2^n} = \infty.$$