Exercise 10: Limits of Real Functions (Solution)

Calculate the following limits:

1. $\lim\limits_{x\to\pm\infty} x^5 - 3$, $\lim\limits_{x\to\pm\infty} 4x^3 - 2x^7 + 103x^5$.

Answer:
For a big enough x, $x^5 \gg -3$ (the symbol $\gg$ means *much bigger than*), and thus $\lim\limits_{x\to\infty} x^5 - 3 = \infty$. The same argument works for a large negative x, although in this case it doesn't matter since we are subtracting 3. Thus $\lim\limits_{x\to-\infty} x^5 - 3 = -\infty$.

Generally, when we are faced with a limit at $\pm\infty$ of a polynomial expression, the largest power of x is the only thing that matters. Thus, for example, in $\lim\limits_{x\to\pm\infty} 4x^3 - 2x^7 + 103x^5$ the only component of the polynomial that matters is $-2x^7$. Since $\lim\limits_{x\to\pm\infty} -2x^7 = \mp\infty$ (i.e. when $x \to \infty$, $-2x^7 \to -\infty$, and when $x \to -\infty$, $-2x^7 \to \infty$), we get $\lim\limits_{x\to\pm\infty} 4x^3 - 2x^7 + 103x^5 = \mp\infty$.

2. $\lim\limits_{x\to\pm\infty} \frac{1}{x}$, $\lim\limits_{x\to 0} \frac{1}{x}$, $\lim\limits_{x\to\pm\infty} \frac{1}{x^2}$, $\lim\limits_{x\to 0} \frac{1}{x^2}$

Answer:
The bigger the value of x, the smaller the expression $\frac{1}{x}$ gets. The same is of course true for large negative values of x, the only difference being that the values approach 0 from the negative numbers. Thus, $\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$. The opposite occurs for small values of x: as x approaches 0 the expression $\frac{1}{x}$ becomes larger and larger.
However, the sign of x plays a role, as 1 over a positive number is positive, while 1 over a negative number is negative. Thus when approaching 0 from the positive numbers the limit tends towards ∞, while when approaching 0 from the negative numbers the limits will go to $-\infty$. In mathematical notation:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Therefore, the limit at 0 just does not exist. For more insight, look at a graph of $f(x) = \frac{1}{x}$ (Figure 1).

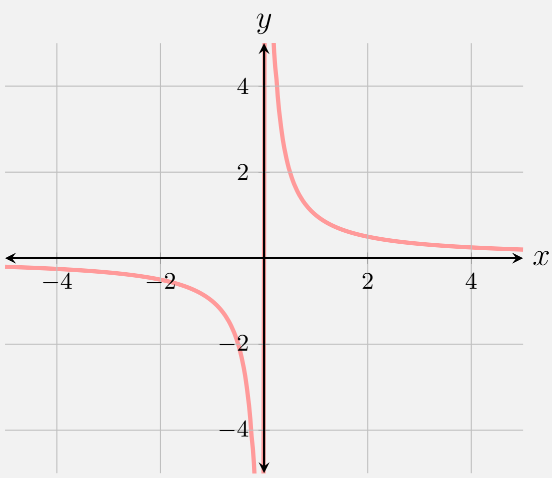

Figure 1: $f(x) = \frac{1}{x}$ for $x \in [-5, 5]$.

The same analysis is true for $f(x) = \frac{1}{x^2}$ except that since x^2 is always non-negative, so is $\frac{1}{x^2}$, and thus the graph looks as in Figure 2 (notice also how $\frac{1}{x^2}$ diverges/decays much faster than $\frac{1}{x}$).
Of course, since

$$\lim_{x\to 0^-} \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{x^2},$$

the limit at 0 is defined and simply equals 0.

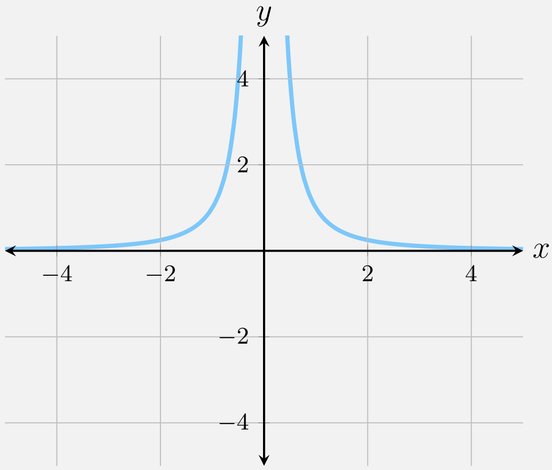

Figure 2: $f(x) = \frac{1}{x^2}$ for $x \in [-5, 5]$.

3. $\lim\limits_{x\to\infty} \frac{x^4-3x^2+10x}{-2x^2-5}$, $\lim\limits_{x\to-1} \frac{2x^2+x-1}{x+1}$

Answer:
The highest power of x in both the numerator and the denominator is x^4, so it controls the behavior of the function as $x \to \infty$. Since the coefficient of x^4 is positive (it is simply 1), we get

$$\lim_{x\to\infty} \frac{x^4 - 3x^2 + 10x}{-2x^2 - 5} = \infty.$$

The other expression, $\frac{2x^2+x-1}{x+1}$, is somewhat tricky. If we pay close attention we can see that $2x^2 + x - 1 = (2x - 1)(x + 1)$, and thus

$$\begin{aligned} \frac{2x^2 + x - 1}{x + 1} &= \frac{(2x - 1)\cancel{(x + 1)}}{\cancel{x + 1}} \\ &= 2x - 1. \end{aligned}$$

so we expect the two expressions to behave in the same way (i.e. be a simple line). However, $\frac{2x^2+x-1}{x+1}$ is still undefined at $x = -1$, and thus the function $f(x) = \frac{2x^2+x-1}{x+1}$ has a 'hole' at $x = -1$. Except for this point, however, it is well-behaved and looks exactly like $2x - 1$, and thus

$$\lim_{x\to-1} \frac{2x^2 + x - 1}{x + 1} = \lim_{x\to-1} 2x - 1 = -3,$$

even though $\frac{2x^2+x-1}{x+1}$ is undefined at $x = -1$ (limits do not care about values at specific points! They only care about the behaviour leading to a point).

4. $\lim\limits_{x\to\pm\infty} \frac{P_n(x)}{P_m(x)}$, where $P_k(x)$ is a real polynomial of order k, n is even, m is odd and $n > m$.

Note: A real polynomial $P_k(x)$ is defined as $P_k(x) = \sum\limits_{i=0}^{k} a_i x^i$, with $a_i \in \mathbb{R}$ and $a_k \neq 0$.

Answer:
Let's generalize what we saw in the previous paragraph: since for any polynomial the limit at $\pm\infty$ is depended only on the term with the highest power of x, we can write

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{P_n(x)}{P_m(x)} &= \lim_{x\to\pm\infty} \frac{a_n x^n}{b_n x^m} \\ &= \lim_{x\to\pm\infty} \frac{a_n}{b_n} x^{n-m}. \end{aligned}$$

where a_n and b_n are the coefficients of the terms x^n and x^m for $P_n(x)$ and $P_m(x)$, respectively.
We can see that there are three possibilities:

- $n > m$: the limit would be $\pm\infty$, depending on the sign of $\frac{a_n}{b_n}$.
- $n = m$: in this case $x^{n-m} = x^0 = 1$, and thus the limit would be $\frac{a_n}{b_n}$.
- $n < m$: the term x^m will 'win', and drag the limit to 0 (as in $\frac{1}{x}$, for example).

5. $\lim\limits_{x\to\pm\infty} \sin(x)$, $\lim\limits_{x\to\pm\infty} \tan(x)$

Answer:
Since $\sin(x)$ is periodic, there can be no limit when $x \to \pm\infty$: the function oscillates forever. The same is true for $\tan(x)$, except that the 'oscillation' in that case is between $-\infty$ and $+\infty$.

6. $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$, $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$

Answer:
There are several equivalent ways to approach this, but we will look at one involving infinite sums. Thanks to Taylor expansions[1]we know that the following is true:

$$\begin{aligned}\sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots\end{aligned}$$

We can see that all the terms like x^3, x^5, x^7, ... approach *zero* really fast, leaving x dominant. Therefore, for a small x $\sin(x) \approx x$, since both $\sin(x)$ and x look similar. This means that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = \lim\limits_{x\to 0} \frac{x}{x} = 1$.

The second limit, $\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ is not that simple. In short, since the smaller the x the bigger $\frac{1}{x}$ is, we get that more of the function is being 'condensed' near $x = 0$. Therefore, it oscillates faster and faster, and thus the limit at 0 is undefined. See Figure 3 for a graphical representation of this function.

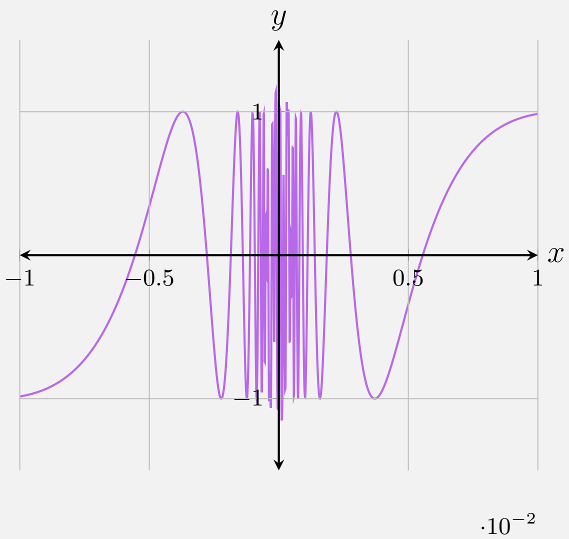

Figure 3: The function $\sin\left(\frac{1}{x}\right)$ graphed for $x \in [-0.01, 0.01]$.

7. $\lim\limits_{x\to\pm\infty} e^x$, $\lim\limits_{x\to\pm\infty} e^{-x}$, $\lim\limits_{x\to 0^+} \log(x)$, $\lim\limits_{x\to\pm\infty} \log(x)$

Answer:
Obviously, the bigger x is, the bigger will e^x be. Therefore

$$\lim_{x\to\infty} e^x = \infty.$$

On the other hand, when $x \to -\infty$, the expression e^x gets smaller and smaller, since for any negative

[1]Specifically, the McLaren series for $\sin(x)$. For further reading one should go to, of course, Wikipedia.

number $x = -a$, $e^x = \frac{1}{e^a}$, i.e. 1 over a very large (positive) number. Mathematically speaking

$$\lim_{x\to-\infty} e^x = 0.$$

For $f(x) = e^{-x}$ the exact opposite is true, since the expression $-x$ is an exact mirror of x about the y-axis. Therefore,

$$\lim_{x\to\infty} e^{-x} = 0$$

and

$$\lim_{x\to-\infty} e^{-x} = \infty.$$

See Figure 4 for a graph of both these functions.

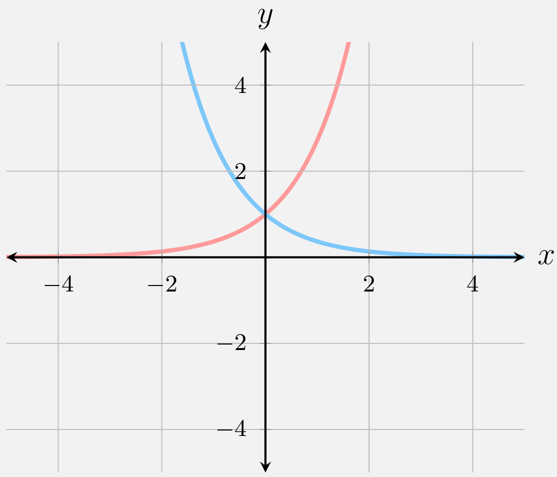

Figure 4: Graphing both e^x and e^{-x} for $x \in [-3, 3]$.

Since $\log(x)$ is the inverse function of e^x, we can infer the behavior of $\log(x)$ from that of e^x. First, we recall that inverse functions behave as if their axes were swapped, which is equivalent to a 90° rotation followed by flipping the y axis. In our case we get $\lim_{x\to0^+} \log(x) = -\infty$ and $\lim_{x\to\infty} \log(x) = \infty$. See Figure 5 for a graphical representation of $\log(x)$.

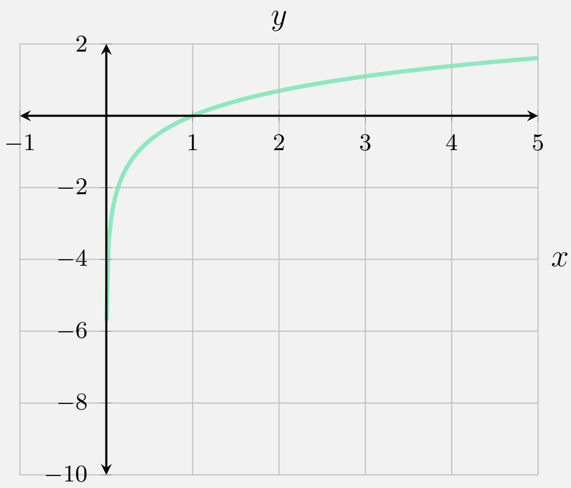

Figure 5: Graph of $\log(x)$ for $x \in (0, 5]$.