Exercise 2: Vectors (Solution)

Problem 1: General Vectors Operations

The following column vectors are defined:

$$\vec{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$\vec{a} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

1. Calculate $\vec{u}+\vec{v}$, $\vec{u}-\vec{w}$, $\vec{u}\cdot\vec{v}$, $\vec{u}\cdot\vec{w}$.
   What does the result for $\vec{u}\cdot\vec{v}$ mean for these two vectors?

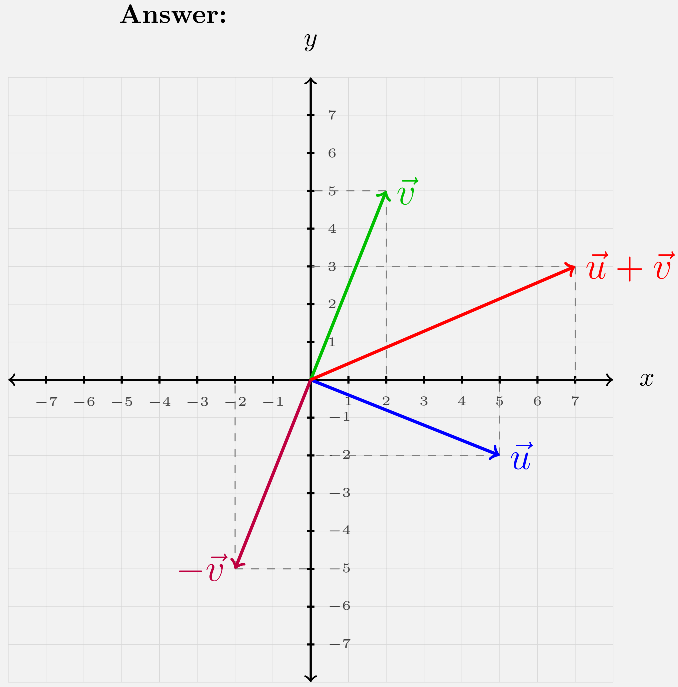
Answer:
(<u>Remember:</u> vectors are added/subtracted element-wise!)

$$\vec{u}+\vec{v} = \begin{pmatrix} 5+2 \\ -2+5 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$
$$\vec{u}-\vec{w} = \begin{pmatrix} 5+0 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$
$$\vec{u}\cdot\vec{v} = 5\times 2 + (-2)\times 5 = 10 - 10 = 0$$
$$\vec{u}\cdot\vec{w} = \cancel{5\times 0} + (-2)\times 1 = 0 - 2 = -2$$

Since $\vec{u}\cdot\vec{v} = 0$, these two vectors are orthogonal.
Generally, on a 2D plane ($\mathbb{R}^2$) any two vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} -y \\ x \end{pmatrix}$, where both $x \neq 0$ and $y \neq 0$, are orthogonal.

2. Draw $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$, $-\vec{v}$ on a cartesian coordinate system.

Answer:



3. Calculate $5\vec{a} - 3\vec{b}$.

Answer:
Multiplying a vector by a scalar is simply multiplying each of its elements by that scalar, hence:

$$\begin{aligned} 5\vec{a} &= 5\begin{pmatrix}1\\3\\7\end{pmatrix} \\ &= \begin{pmatrix}5\cdot 1\\5\cdot 3\\5\cdot 7\end{pmatrix} \\ &= \begin{pmatrix}5\\15\\35\end{pmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} 3\vec{b} &= \left(3\cdot(-2)\,, 3\cdot 0, 3\cdot 5\right) \\ &= \begin{pmatrix}-6\\0\\15\end{pmatrix}. \end{aligned}$$

and thus -

$$\begin{aligned} 5\vec{a} - 3\vec{b} &= \begin{pmatrix}5\\15\\35\end{pmatrix} - \begin{pmatrix}-6\\0\\10\end{pmatrix} \\ &= \begin{pmatrix}-11\\15\\20\end{pmatrix}. \end{aligned}$$

4. Calculate $\vec{a}+\vec{w}$, $\quad \vec{a}+\vec{b}$, $\quad \vec{b}\cdot\vec{w}$, $\quad \vec{a}\cdot\vec{c}$.

Answer:
$\vec{a}+\vec{w}$ is undefined since these vectors are of a different dimension (3 and 2, respectively).
The same is true for $\vec{b}\cdot\vec{w}$.

$$\begin{aligned} \vec{a}+\vec{b} &= \begin{pmatrix}1-2\\3+0\\7+5\end{pmatrix} \\ &= \begin{pmatrix}-1\\3\\12\end{pmatrix} \\ \vec{a}\cdot\vec{c} &= 1\cdot 1 + 3\cdot 1 + \cancel{7\cdot 0} \\ &= 1+3 \\ &= 4. \end{aligned}$$

5. What are the lengths of $\vec{u}$, $\vec{v}$, $\vec{a}$ and $\vec{c}$?

Answer:
The (Euclidean) length of a vector of N dimension is the square root of the sum of the squares of its elements, or as a general formula,

$$\|\vec{x}\| = \sqrt{\sum_{i=1}^{N} x_i^2}.$$

In the case of 2D and 3D vectors, this general formula simplifies to

$$\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2},$$

and

$$\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

respectively. Therefore:

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{5^2 + -2^2} = \sqrt{29} \approx 5.38516 \\
\|\vec{v}\| &= \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.38516 \\
\|\vec{a}\| &= \sqrt{1^2 + 3^2 + 7^2} = \sqrt{59} \approx 7.68115 \\
\|\vec{c}\| &= \sqrt{1^2 + 1^2 + \cancel{0^2}} = \sqrt{2} \approx 1.41421.
\end{aligned}$$

6. What is the angle between $\vec{v}$ and the x-axis?

Answer:
The angle between any vector in $\mathbb{R}^2$ and the x-axis is the inverse tan (i.e. arctan) of its y-component divided by its x-component. In this case,

$$\begin{aligned}
\theta_{\vec{v}} &= \arctan\left(\frac{v_y}{v_x}\right) \\
&= \arctan\left(\frac{5}{2}\right) \\
&\approx 1.19 \text{ [rad]} \\
&\approx 68.18^\circ.
\end{aligned}$$

7. What would be the cartesian coordinates of the vector $\vec{v}$ rotated by 42° counter clockwise?

Answer:
Rotating $\vec{v}$ by 42° will result in a vector with the same length (magnitude) as $\vec{u}$ and an angle of $68.18^\circ + 42^\circ = 110.18^\circ$ to the x-axis. Recalling that

$$\begin{aligned}
u_x &= \|\vec{u}\| \cos(\theta) \\
u_y &= \|\vec{u}\| \sin(\theta),
\end{aligned}$$

we can subtitute $\|\vec{u}\| \approx 5.39$ and $\theta = 110.18^\circ$ and get the components:

$$\begin{aligned}
u_x &= 5.39 \cos(110.18^\circ) = 5.39 \cdot \approx -0.35 \approx -1.89 \\
u_y &= 5.39 \sin(110.18^\circ) = 5.39 \cdot \approx 0.94 \approx 5.07,
\end{aligned}$$

which as a column vector is $\vec{u}' = \begin{pmatrix} -1.89 \\ 5.07 \end{pmatrix}$.

8. What is the angle between $\vec{a}$ and $\vec{b}$?

Answer:
For any two vectors $\vec{x}$, $\vec{y}$, the following always applies:

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos(\theta),$$

where θ is the angle between the two vectors.

Solving for θ we get

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}.$$

In our case

$$\vec{a} \cdot \vec{b} = 1 \times (-2) + \cancel{3 \times 0} + 7 \times 5 = 33,$$

and

$$\begin{aligned} \|\vec{x}\| \, \|\vec{y}\| &= \sqrt{1^2 + 3^2 + 7^2} \cdot \sqrt{2^2 + \cancel{0^2} + 5^2} \\ &= \sqrt{1 + 9 + 49} \cdot \sqrt{4 + 25} \\ &= \sqrt{59} \cdot \sqrt{29} \\ &= \sqrt{59 \times 29} \\ &= \sqrt{1711} \\ &\approx 41.36. \end{aligned}$$

Therefore,

$$\cos(\theta) \approx \frac{33}{41.36} \approx 0.798,$$

and thus

$$\theta \approx \arccos(0.789) \approx 37^\circ.$$

9. Calculate $\vec{c} = \vec{a} \times \vec{b}$. What is the general formula for all the vectors that are orthogonal to $\vec{c}$?

Answer:

$$\begin{aligned} \vec{c} = \vec{a} \times \vec{b} &= \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \cdot 5 - \cancel{7 \cdot 0} \\ 7 \cdot (-2) - 1 \cdot 5 \\ \cancel{1 \cdot 0} - 3 \cdot (-2) \end{pmatrix} \\ &= \begin{pmatrix} 15 \\ -14 - 5 \\ -(-6) \end{pmatrix} \\ &= \begin{pmatrix} 15 \\ -19 \\ 6 \end{pmatrix}. \end{aligned}$$

Every 3D-vector can be associated with a plane, to which it is orthogonal. All vectors on the plane will therefore also be orthogonal to that vector.
To define a plane in 3D-space, one needs two linearly independent vectors (meaning two non-zero vectors that are not on the same line).
$\vec{a}$ and $\vec{b}$ are indeed linearly independent, and thus all possible linear combinations of them would correspond to the general formula we are looking for, meaning that

$$\vec{d} = \alpha \cdot \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \beta \cdot \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix},$$

for any non-zero $\alpha, \beta \in \mathbb{R}$, is the general formula of the vectors we are looking for.
Verifying the result can be done by calculating the dot product between $\vec{c} = \begin{pmatrix} 15 \\ -19 \\ 6 \end{pmatrix}$ and $\alpha \cdot \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} +$

$\beta \cdot \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$:

$$\alpha \cdot \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \beta \cdot \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} \alpha - 2\beta \\ 3\alpha \\ 7\alpha + 5\beta \end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned} \begin{pmatrix} \alpha - 2\beta \\ 3\alpha \\ 7\alpha + 5\beta \end{pmatrix} \cdot \begin{pmatrix} 15 \\ -19 \\ 6 \end{pmatrix} &= 15\alpha - 30\beta + (-57)\,\alpha + 42\alpha + 30\beta \\ &= 15\alpha - 57\alpha + 42\alpha + 30\beta - 30\beta \\ &= 0, \end{aligned}$$

confirming that indeed $\alpha \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 15 \\ -19 \\ 6 \end{pmatrix}$ are orthogonal.

Problem 2: Linear Combinations of Vectors

Write the vector $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix}$ as a linear combination of the following vectors:

$$\vec{u}_1 = \begin{pmatrix} -2 \\ 5 \\ 0 \\ 5 \end{pmatrix}, \ \vec{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \ \vec{u}_3 = \begin{pmatrix} -4 \\ 4 \\ -8 \\ -2 \end{pmatrix}.$$

Answer:
Let's start with adding $\vec{u}_1$ and $\vec{u}_2$ together:

$$\begin{aligned} \vec{u}_1 + \vec{u}_2 &= \begin{pmatrix} -2 \\ 5 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2+1 \\ 5+0 \\ 0-1 \\ 5+1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 5 \\ -1 \\ 6 \end{pmatrix}. \end{aligned}$$

We can then see what is the result of subtracting $\vec{w}$ from $\vec{u}_1 + \vec{u}_2$:

$$\begin{aligned} \vec{w} - (\vec{u}_1 + \vec{u}_2) &= \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 1 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} 1-(-1) \\ 3-5 \\ 3-1 \\ 7-6 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

This is exactly equal to $-\frac{1}{2}\vec{u}_3$. Thus, adding $-\frac{1}{2}\vec{u}_3$ to $\vec{u}_1 + \vec{u}_2$ should give us $\vec{w}$. Let's check this:

$$\begin{aligned} \vec{u}_1 + \vec{u}_2 - \frac{1}{2}\vec{u}_3 &= \begin{pmatrix} -1 \\ 5 \\ -1 \\ 6 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -4 \\ 4 \\ -8 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -1+2 \\ 5-2 \\ -1+4 \\ 6+1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \\ 3 \\ 7 \end{pmatrix} \\ &= \vec{w}, \end{aligned}$$

as expected. Thus, with the coefficients $\alpha_1 = \alpha_2 = 1,\ \alpha_3 = -\frac{1}{2}$, we get

$$\vec{w} = \vec{u}_1 + \vec{u}_2 - \frac{1}{2}\vec{u}_3.$$

Problem 3: Linear Independence of Vectors

Which of the following sets of vectors are linearly independent?

1. $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 6 \\ 0 \\ 1 \end{pmatrix}$

Answer:
There is no $\alpha \in \mathbb{R}$ such that $\vec{b} = \alpha\vec{a}$, and thus these vectors are linearly independent.

2. $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 4 \\ -10 \end{pmatrix}$

Answer:
Since $\vec{b} = -2\vec{a}$, the two vectors are **not** linearly independent.

3. $\vec{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -9 \\ 1 \\ 6 \end{pmatrix}$

Answer:
Since $\vec{c} = \vec{a} - 2\vec{b}$, the three vectors are **not** linearly independent.

4. $\vec{a} = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} -1 \\ -7 \\ 7 \end{pmatrix}$

Answer:
For vectors of dimension N, any set of vectors with $M > N$ elements are **not** linearly independent. In this case $N = 3$ and $M = 4$, and thus these vectors are not linearly independent.