Basic Maths for Non-mathematicians

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$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(x_k)\Delta x$$

$$(AB)^\top = B^\top A^\top$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$A = Q\Lambda Q^{-1}$$

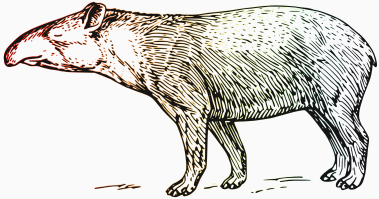
$$A\vec{v} = \lambda\vec{v}$$

$$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

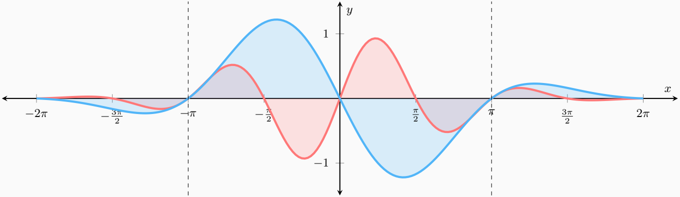
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$$



Chapter 9: General Vector Spaces

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(x) g(x) \, \mathrm{d}x$$



Properties of $\mathbb{R}^n$

Let us review some properties of the space $\mathbb{R}^n$, some of them we already used implicetly without giving them too much thought.

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The addition of any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ yields a vector $\vec{w} = \vec{u} + \vec{v}$ that is also in $\mathbb{R}^n$.

Example

For $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ (both in $\mathbb{R}^3$),

$$\vec{w} = \vec{u} + \vec{v} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} \in \mathbb{R}^3.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is commutative: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$.

Example

For the same vectors as before:

$$\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + (-1) \\ 2 + 3 \\ -1 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$

$$\vec{v} + \vec{u} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 + 1 \\ 3 + 2 \\ 0 + (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Vector addition is assosiative: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

Example

For $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$:

$$\begin{aligned} \vec{a} + \left(\vec{b} + \vec{c}\right) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \end{aligned}$$

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$$\begin{aligned} \left(\vec{a} + \vec{b}\right) + \vec{c} &= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- The zero vector $\vec{0}$ is unique and has the property that $\vec{v} + \vec{0} = \vec{v}$ for any vector $\vec{v} \in \mathbb{R}^n$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to vector-vector addition:

- Any vector $\vec{v} \in \mathbb{R}^n$ has an opposite vector $(-\vec{v}) \in \mathbb{R}^n$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

Example

$$\begin{pmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ 0 \\ -3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Any scale by $\alpha \in \mathbb{R}$ of a vector $\vec{v} \in \mathbb{R}^n$ is also in $\mathbb{R}^n$.

Example

$$-3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -6 \\ 0 \\ 3 \\ -9 \\ 6 \end{pmatrix}.$$

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Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is associative: $\alpha\left(\beta\vec{v}\right) = \left(\alpha\beta\right)\vec{v}$.

Example

$$-3\left[2\begin{pmatrix}1\\-4\\5\end{pmatrix}\right] = -3\begin{pmatrix}2\\-8\\10\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}$$

$$(-3\cdot 2)\begin{pmatrix}1\\-4\\5\end{pmatrix} = -6\begin{pmatrix}1\\-4\\5\end{pmatrix} = \begin{pmatrix}-6\\24\\-30\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- Scalar-vector multiplication is distributive in respect to scalar addition: $(\alpha + \beta)\,\vec{v} = \alpha\vec{v} + \beta\vec{v}$.

Example

$$(5-2)\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}.$$

$$5\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

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Example

$$5\left[\begin{pmatrix}1\\-1\\3\end{pmatrix}+\begin{pmatrix}0\\2\\-1\end{pmatrix}\right]=5\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}5\\5\\10\end{pmatrix}.$$

$$5\begin{pmatrix}1\\-1\\3\end{pmatrix}+5\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}5\\-5\\15\end{pmatrix}+\begin{pmatrix}0\\10\\-5\end{pmatrix}=\begin{pmatrix}5\\5\\10\end{pmatrix}.$$

Properties of $\mathbb{R}^n$

Relating to scalar-vector product:

- The scalar $\alpha = 1$ is neutral in respect to scalar-vector products: $1\vec{v} = \vec{v}$.

Example

$$1 \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 6 \\ -5 \\ 7 \\ -4 \end{pmatrix}.$$

Abstract Vector Spaces

These properties are somewhat obvious on $\mathbb{R}^n$. However, many times it is worthwhile to use more abstract vector spaces, which can help us model diverse physical and theoretical systems, since once a construct behaves as a vector space, it is a relatively simple process to apply to it all the analysis tools learned so far.

We will not bother here with the formal definition of a vector space[1], but look at one example, which we will later expand on: the space of all real functions $f : \mathbb{R} \to \mathbb{R}$.

[1] For such definition, see here:
http://www.math.niu.edu/ beachy/courses/240/06spring/vectorspace.html.

Real Functions as a Vector Space

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- For each real function $f(x)$ there exists an opposite function $(-f)(x) = -f(x)$, for which

$$f(x) + \big(-f(x)\big) = f(x) - f(x) = 0 = z(x).$$

Components (temp name)

Recall that a vector in $\mathbb{R}^n$ can be written using its component in any basis, e.g. the standard basis vectors $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$:

$$\begin{aligned}\vec{v} &= v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots + v_n\hat{e}_n \\ &= \sum_{i=1}^{n} v_i\hat{e}_i.\end{aligned}$$

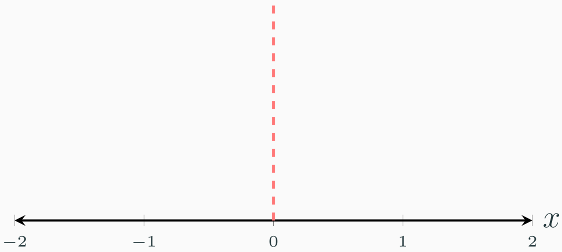
How can we "decompose" a function in a similar way?

For this purpose, the **Dirac delta function** comes in handy.

The Dirac Delta Function

Loosely speaking, we can define the Dirac delta function as

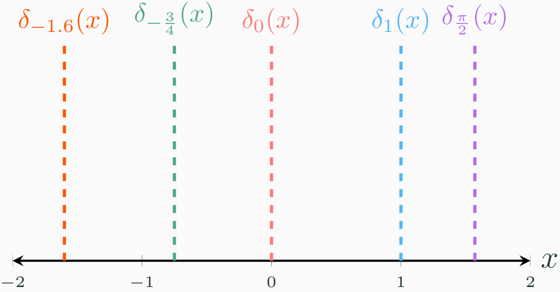
$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0. \end{cases}$$



The Dirac Delta Function

We can then define infinitaly many dirac functions, for each point $\tilde{x} \in \mathbb{R}$:

$$\delta_{\tilde{x}}(x) = \delta(x - \tilde{x}).$$



Spanning a Function Space

Using the Dirac delta function, we can now decompose a function to its components in a similar way we did with vectors:

$$\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i$$

$$\Downarrow$$

$$f(x) = \text{to do.}$$

Dot Product of Two Functions

Recall the definition of a dot product of two vectors $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\langle \vec{u}, \vec{v} \rangle &= u_1 v_1 + u_2 v_2 + \cdots + v_n u_n \\ &= \sum_{i=1}^{n} u_i v_i.\end{aligned}$$

The dot product of two functions $f(x), g(x)$ on the interval $[a, b]$ can be similarily defined:

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x) \, \mathrm{d}x.$$

Dot Product of Two Functions

Note

Over $\mathbb{C}^n$, the dot product of two vectors $\vec{u}, \vec{v}$ is defined as

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n \\ &= \sum_{i=1}^{n} \bar{u}_i v_i, \end{aligned}$$

where $\bar{z}$ is the **complex conjugate** of z. For a real number x, $\bar{x} = x$ - i.e. real numbers are their own complex conjugates.

Dot Product of Two Functions

Note

The definition of a dot product of two complex functions $f(z), g(z)$ is similar:

$$\langle f(z), g(z) \rangle = \int\limits_{\Omega} \bar{f}(z) g(z) \, \mathrm{d}z,$$

where Ω is the space over which the integration is done.

Sometimes, especially in physics, the complex conjugate of a function $f(z)$ is denoted as $f^*(z)$.

Dot Product of Two Functions

If the dot product of two functions $f(x), g(x)$ is zero, we say that the functions are **orthogonal** (just as with vectors).

Example

The functions $f(x) = x, f(x) = x^2$ are orthogonal over the entire real line, since

$$\langle x, x^2 \rangle = \int_{-\infty}^{\infty} x \cdot x^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} x^3 \, \mathrm{d}x = 0.$$

(recall that integrals over the real line of anti-symmetric functions, such as x^3, always equal 0)

Norm of Functions

The **norm** of a function f over an interval $[a, b]$ can be defined as

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f^2(x)\,\mathrm{d}x}.$$

Example

The norm of $f(x) = -x^2$ on $[-2, 2]$ is

$$\|f\| = \sqrt{\int_{-2}^{2} \left(-x^2\right)^2 \mathrm{d}x} = \sqrt{2\int_0^2 x^4\,\mathrm{d}x}$$

$$= \sqrt{\frac{2}{5}x^5\Big|_0^2} = \sqrt{\frac{2}{5}\left(32 - 0\right)} = \frac{8}{\sqrt{5}} \approx 3.578.$$

Norm of Functions

A function that has a norm of 1 is said to be **normalized**.

Example

The **Gaussian distribution**

$$\mathcal{G}(x) = e^{-\frac{x^2}{2}}$$

has a **squared** norm

$$\|\mathcal{G}\|^2 = \int_{-\infty}^{\infty} \left(e^{-\frac{x^2}{2}}\right)^2 \mathrm{d}x = \int_{-\infty}^{\infty} e^{-\cancel{2}\frac{x^2}{\cancel{2}}} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$