Exercise 1: Sets, Functions and Graphs (Solution)

Answer:
Important notations:

- Curly brackets (i.e. $\{\}$) represent a set.
- The empty set is written as $\varnothing$.
- $x \in A$ means x is an element of the set A, and $y \notin A$ means that y is not an element of A.
- $\wedge$ and $\vee$ mean *and* and *or*, respectively.
- Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$.
  (By some definitions $0 \in \mathbb{N}$, by others $0 \notin \mathbb{N}$)
- Integers: $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$.
- Rational numbers: $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$.
- Real numbers: $\mathbb{R} = \big\{x \mid x \in (-\infty, \infty)\big\}$.
  (The formal definition is too complicated for this course[a])
- Complex numbers: $\mathbb{C} = \big\{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\big\}$.

[a] Some formal definitions can be found here: `https://en.wikipedia.org/wiki/Construction_of_the_real_numbers`.

Problem 1: Sets

1. Write the following sets explicitly:

   (a) $A = \big\{x \in \mathbb{N} \mid 1 < x \leq 7\big\}$

   Answer:
   While 1 is not included in the set (since $1 < x$), the number 7 is, since $x \leq 7$.
   Hence: $A = \{2, 3, 4, 5, 6, 7\}$.

   (b) $A = \big\{x \in \mathbb{Z} \mid x < 5\big\}$

   Answer:
   $\mathbb{Z}$ is the set of all integers, including 0. Therefore: $A = \{\dots, -2, -1, 0, 1, 2, 3, 4\}$.

   (c) $A = \big\{x \in \mathbb{R} \mid x^2 = -1\big\}$

   Answer:
   The solution for $x^2 = -1$ is not a real number, and therefore $A = \varnothing$.

   (d) $A = \{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$

   Answer:
   All integers are also rational, and therefore $A = \mathbb{N}$.

   (e) $A = \big\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\big\}$

   Answer:
   Using the quadratic equation solution $\left(ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ yields $x_{1,2} = -1, 4$.
   This means $A = \{-1, 4\}$.

(f) $A = \left\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\right\}$

> **Answer:**
> $x = [2, 5)$

2. Determine the relation between the sets:

(a) $\mathbb{R},\ \mathbb{Q},\ \mathbb{Z},\ \mathbb{N}$

> **Answer:**
> $\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$

(b) $A = \{1,\ 2,\ 3\}\,,\ B = \{1,\ 2\}$

> **Answer:**
> $A \supset B$

(c) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$

> **Answer:**
> $A \subset B$

(d) $A = \mathbb{Z},\ B = \left\{\pm x \mid x \in \mathbb{N} \cup \{\,0\,\}\right\}$

> **Answer:**
> $A = B$

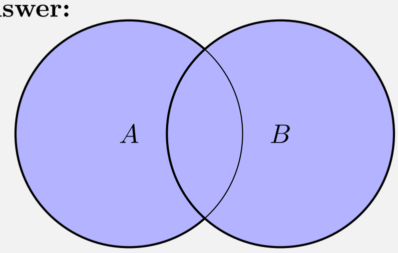
(e) $A = \left\{\pi, e, \sqrt(2)\right\},\ B = \mathbb{Q}$

> **Answer:**
> All the numbers in A are irrational and thus do not belong to B, meaning $A \cap B = \varnothing$.

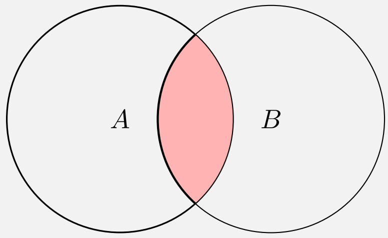
3. Using Venn diagrams, show the following relations:

$$\left.\begin{array}{ll}\text{(a)} & A \cup B = B \cup A \\ \text{(b)} & A \cap B = B \cap A\end{array}\right\} \text{Commutative laws}$$

Answer:



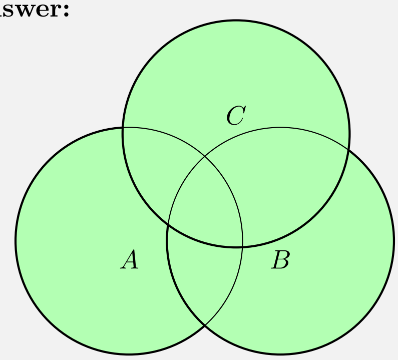
$A \cup B = B \cup A$



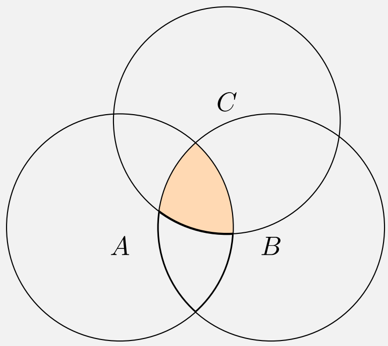
$A \cap B = B \cap A$

$$\left.\begin{array}{ll}\text{(c)} & (A \cup B) \cup C = A \cup (B \cup C) \\ \text{(d)} & (A \cap B) \cap C = A \cap (B \cap C)\end{array}\right\} \text{Associative laws}$$

Answer:



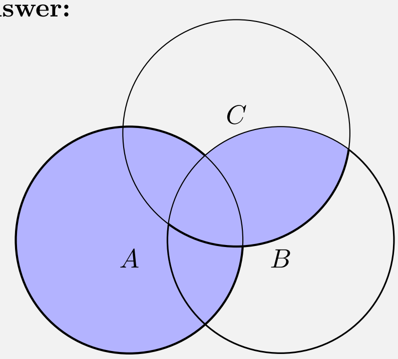
$(A \cup B) \cup C = A \cup (B \cup C)$



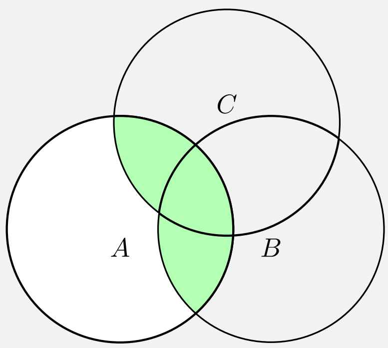
$(A \cap B) \cap C = A \cap (B \cap C)$

$$\left.\begin{array}{ll}\text{(e)} & A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ \text{(f)} & A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\end{array}\right\} \text{Distributive laws}$$

Answer:



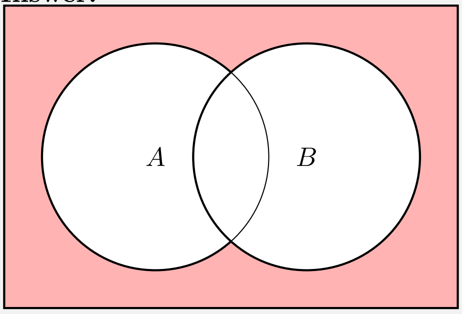
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$



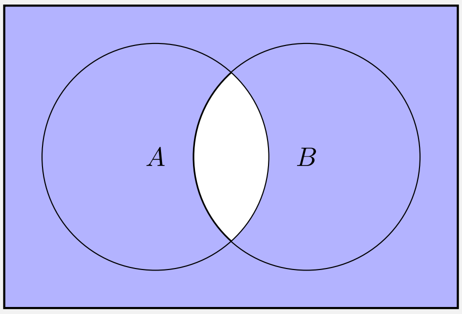
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\left.\begin{array}{ll}\text{(g)} & (A \cup B)^C = A^C \cap B^C \\ \text{(h)} & (A \cap B)^C = A^C \cup B^C\end{array}\right\} \text{De Morgan's laws}$$

Answer:



$(A \cup B)^C = A^C \cap B^C$



$(A \cap B)^C = A^C \cup B^C$

4. Cartesian products:

   (a) What is the Cartesian product of $A = \{x,\ y,\ z\}$ and $B = \{a,\ b,\ c\}$?

   Answer:
   $A \times B = \big\{(x,a)\,,\ (x,b)\,,\ (x,c)\,,\ (y,a)\,,\ (y,b)\,,\ (y,c)\,,\ (z,a)\,,\ (z,b)\,,\ (z,c)\big\}$

   (b) Is the following true: $(x,\ a) = (a,\ x)$?

   Answer:
   Generally not, only if $x = a$.

   (c) What is B^2? What is A^3?

   Answer:

   $$B^2 = \{(a,a)\,,\ (a,b)\,,\ (a,c)\,,\ (b,a)\,,\ (b,b)\,,\ (b,c)\,,\ (c,a)\,,\ (c,b)\,,\ (c,c)\}$$
   $$\begin{aligned} A^3 = \{&(x,x,x)\,,\ (x,x,y)\,,\ (x,x,z)\,,\ (x,y,x)\,,\ (x,y,y)\,,\ (x,y,z)\,,\ (x,z,x)\,,\ (x,z,y)\,,\ (x,z,z)\,,\\ &(y,x,x)\,,\ (y,x,y)\,,\ (y,x,z)\,,\ (y,y,x)\,,\ (y,y,y)\,,\ (y,y,z)\,,\ (y,z,x)\,,\ (y,z,y)\,,\ (y,z,z)\,,\\ &(z,x,x)\,,\ (z,x,y)\,,\ (z,x,z)\,,\ (z,y,x)\,,\ (z,y,y)\,,\ (z,y,z)\,,\ (z,z,x)\,,\ (z,z,y)\,,\ (z,z,z)\} \end{aligned}$$

5. Extra: Prove that $\sqrt{2}$ is irrational. (In formal writing: $\nexists p, q \in \mathbb{Z} \rightarrow \sqrt{2} = \frac{p}{q}$. Simply written: $\sqrt{2} \notin \mathbb{Q}$)

   Answer:

   Proof. Suppose $\sqrt{2}$ is rational. This means that there exist two integers p, q so that

   $$\sqrt{2} = \frac{p}{q} \tag{1}$$

   We may assume that p and q have no common factors, otherwise we cancel them out. Squaring both sides of equation (1) yields

   $$2 = \frac{p^2}{q^2} \tag{2}$$

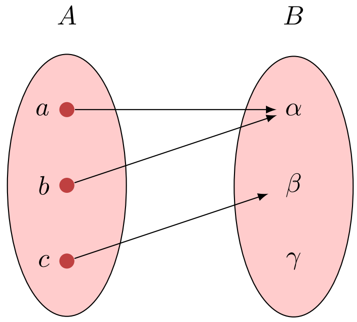
   which means

   $$p^2 = 2q^2 \tag{3}$$

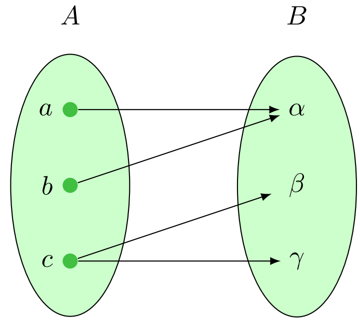
   i.e. p^2 is even. This is only possible if p is even, which by itself means that p^2 is divisible by 4.
   But if p^2 is divisible by 4 then $2q^2$ is divisible by 4 (due to equation (3)) and thus q^2 is divisible by 2, which means p and q have a common factor (i.e. 2), in contradiction to our assumption. Therefore $\sqrt{2} \notin \mathbb{Q}$. □

Problem 2: Functions

1. Which of the following figures represent functions? Which of these functions are injective, surjective and/or bijective?



(1)



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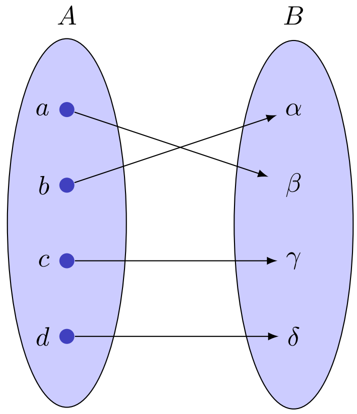

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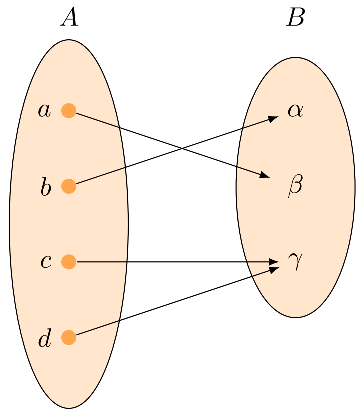


(4)

Answer:

- Functions: (1) (3) (4). Number (2) is not a function, since c has two corresponding values in B.
- Injective: (3). Both functions (1) and (4) have two values in A that corresponding to the same value in B.
- Surjective: (3) (4). Function (1) has a value in B, γ, that does not correspond to any value in A.
- Bijective: (3).

2. Which of the following functions are injective over $\mathbb{R}$?

   (a) $f(x) = x^2$

   (b) $f(x) = x^3$

   (c) $f(x) = \sin(x)$

   (d) $f(x) = \log(x)$
   (in the context of this course, $\log(x)$ is the natural logarithm, also known as $\ln(x)$)

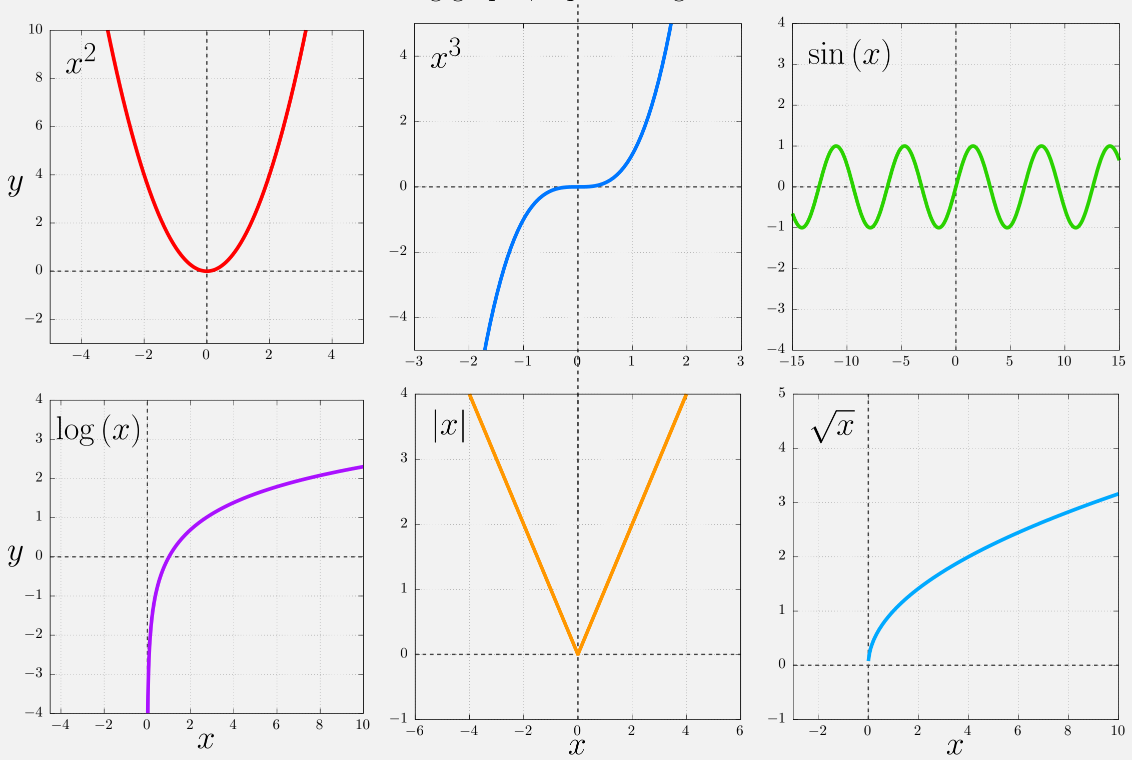
   (e) $f(x) = |x|$

   (f) $f(x) = \sqrt{x}$

Answer:
Functions (2) and (4) are injective. Function (1) is not injective, for example: $f(-2) = (-2)^2 = 4 = 2^2 = f(2)$.
Function (3) is also not injective, for example: $f(\pi) = \sin(\pi) = 0 = \sin(2\pi) = f(2\pi)$.

Function (5) is also not injective, for example: $\mathrm{abs}\,(-2) = \mathrm{abs}\,(2) = 2$.

See the following graphs, representing the functions 1-6:



3. Find the images of the functions from 2.

Answer:

(a) $\mathrm{Im}\left(x^2\right) = \mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\} = [0, \infty)$ (i.e. non-negative real numbers)

(b) $\mathrm{Im}\left(x^3\right) = \mathbb{R}$

(c) $\mathrm{Im}\left(\sin(x)\right) = [-1, 1]$

(d) $\mathrm{Im}\left(\log(x)\right) = \mathbb{R}$

(e) $\mathrm{Im}\left(|x|\right) = \mathbb{R}_0^+$

(f) $\mathrm{Im}\left(\sqrt{x}\right) = \mathbb{R}_0^+$

4. For each of the injective functions from 2, find its inverse.

Answer:

(a) $f(x) = x^3 \longrightarrow f^{-1}(y) = \sqrt[3]{y}$

(b) $f(x) = \log(x) \longrightarrow f^{-1}(y) = e^y$

(c) $f(x) = \sqrt{x} \longrightarrow f^{-1}(y) = y^2$

5. For each of the non injective functions from 2, find a domain over which it is injective.

Answer:

(a) $f(x) = x^2 : A \subseteq \left\{x \in \mathbb{R} \mid x \geq 0\right\}$ or $A \subseteq \left\{x \in \mathbb{R} \mid x \leq 0\right\}$ (i.e. $A = [0, 5], [-13, -4]$, etc.)

(b) $f(x) = \sin(x) : A = \left[\pi, \frac{3}{2}\pi\right]$

Find the inverse of the following functions (over $\mathbb{R}$):

1. $f(x) = x + 5$

Answer:
f takes any real number x and 'moves' it 5 numbers up. Therefore, $g(x) = x - 5$ will move it back, and so $g(x) = f^{-1}(x)$.

2. $f(x) = x^7$

Answer:
$f^{-1}(x) = x^{\frac{1}{7}} = \sqrt[7]{x}$

3. $f(x) = \frac{1}{3x-1}$

Answer:
Let's use a more general way of inverting a function: substituting $y = f(x)$ and solving for x. In this case:

$$y = \frac{1}{3x - 1}$$

and so

$$\frac{1}{y} = 3x - 1$$
$$\Downarrow$$
$$3x = \frac{1}{y} + 1$$
$$\Downarrow$$
$$x = \frac{1}{3}\left(\frac{1}{y} + 1\right)$$

Therefore, we expect that $f^{-1}(x) = \frac{1}{3}\left(\frac{1}{x} + 1\right)$.
Let's check some cases:

- $f(0) = \frac{1}{3 \cdot 0 - 1} = \frac{1}{0-1} = \frac{1}{-1} = -1 \longrightarrow f^{-1}(-1) = \frac{1}{3}\left(\frac{1}{-1} + 1\right) = \frac{1}{3}\left(\cancel{-1+1}\right) = 0$
- $f(1) = \frac{1}{(3 \cdot 1) - 1} = \frac{1}{3-1} = \frac{1}{2} = \longrightarrow f^{-1}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{1}{\frac{1}{2}} + 1\right) = \frac{1}{3}(2 + 1) = \frac{1}{3}(3) = 1$
- $f(-1) = \frac{1}{(3 \cdot -1) - 1} = \frac{1}{-3-1} = -\frac{1}{4} = \longrightarrow f^{-1}\left(-\frac{1}{4}\right) = \frac{1}{3}\left(\frac{1}{-\frac{1}{4}} + 1\right) = \frac{1}{3}(-4 + 1) = \frac{1}{3}(-3) = -1$

It seems like the answer is correct.

Let's check the general case by directly substituting $f^{-1}(x)$ into $f(x)$:

$$\begin{aligned} f\left(f^{-1}(x)\right) &= \frac{1}{3\left(f^{-1}(x)\right)-1} \\ &= \frac{1}{\cancel{3}\left(\frac{1}{\cancel{3}}\left(\frac{1}{x}+1\right)\right)-1} \\ &= \frac{1}{\frac{1}{x}+\cancel{1-1}} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

This verifies that we have indeed found the inverse of f.

4. $f(x) = x^2 + 1$

Answer:
Using the same method from the previous example we get:

$$\begin{gathered} y = x^2 + 1 \\ \Downarrow \\ x^2 = y - 1 \\ \Downarrow \\ x = \pm\sqrt{y-1} \end{gathered}$$

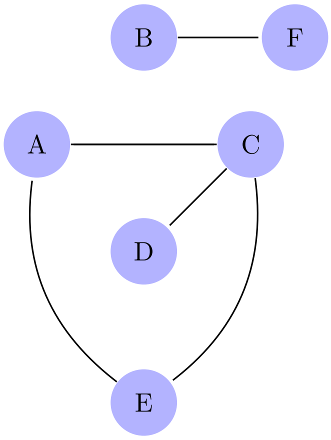
5. $f(x) = e^{-x}$

Answer:

$$\begin{gathered} y = e^{-x} \\ \Downarrow \\ \log(y) = \log\left(e^{-x}\right) = -x \\ \Downarrow \\ x = -\log(y) \\ \Downarrow \\ f^{-1}(x) = -\log(x) \end{gathered}$$

<u>Remark</u>: Generally, $\log_b(b) = 1$ for any base $b > 0$ and $b \neq 1$.
Two more special points are $x = 0$ and $x = 1$: $\log_b(0) = -\infty$ and $\log_b(1) = 0$, for any such base b.
(to be more mathematically precise: $\lim\limits_{x \to 0^-} \log_b(x) = -\infty$)

Problem 3: Graphs



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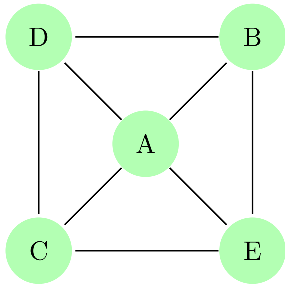

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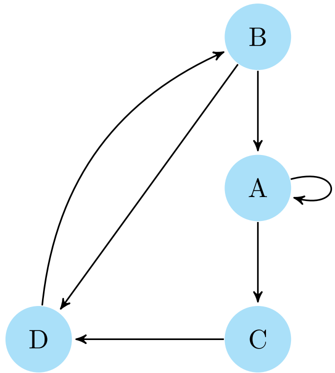

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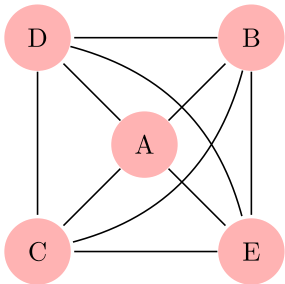

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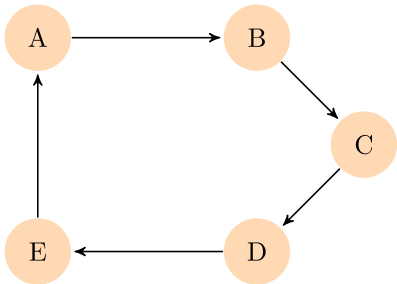

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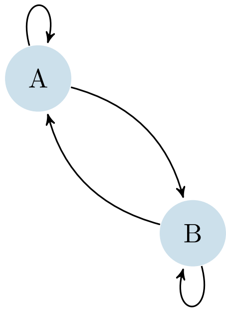

(6)

1. Which of the above graphs are:

   (1) Connected?

   > **Answer:**
   > 2, 3, 4, 5 and 6.

   (2) Complete?

   > **Answer:**
   > 4 and 6.

   (3) Directed?

   > **Answer:**
   > 3, 5 and 6.

2. What are $\operatorname{dist}(A, B)$ and $\operatorname{dist}(A, E)$ for these graphs?

   > **Answer:**
   >
   > (1) $\operatorname{dist}(A, B) = \text{not defined},\ \operatorname{dist}(A, E) = 1.$
   >
   > (2) $\operatorname{dist}(A, B) = 1,\ \operatorname{dist}(A, E) = 1.$
   >
   > (3) $\operatorname{dist}(A, B) = 3,\ \operatorname{dist}(A, E) = \text{not defined}.$
   >
   > (4) $\operatorname{dist}(A, B) = 1,\ \operatorname{dist}(A, E) = 1.$
   >
   > (5) $\operatorname{dist}(A, B) = 1,\ \operatorname{dist}(A, E) = 4.$

(6) $\operatorname{dist}(A, B) = 1$, $\operatorname{dist}(A, E) =$ not defined.