Exercise 9: Summation and Sequences

Problem 1: Summation

1. Write the following expressions explicitly:

   (a) $\sum\limits_{n=1}^{5}\left(n^2-2n\right)$

   (b) $\sum\limits_{n=-3}^{3} 2^n$

   (c) $\sum\limits_{i=1}^{3}\sum\limits_{j=1}^{4} a_{ij}$

2. Use the summation form to write the product of two matrices $C = A \cdot B$, with dimensions $M \times N$ and $N \times K$, respectively.

3. Write in summation form the general real polynomial of order n: $P_n\left(x\right) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots, a_n$ are real numbers and $a_n \neq 0$.

4. The binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n!$ is defined as $n! = 1\times 2\times 3\cdots\times(n-1)\times n$. What is $\binom{4}{2}$?

5. The general expension formula for $\left(x+y\right)^n$ (where $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$) is:

$$\left(x+y\right)^n = \sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^k$$

   Show that for $n = 2$ the formula yields the known expression $\left(x+y\right)^2 = x^2 + 2xy + y^2$, and write the full formula for $\left(x+y\right)^4$.

Problem 2: Sequences

1. Write the first 10 elements of the following sequences:

$$a_n = 3n-2, \quad b_n = 1, \quad c_n = \frac{1}{n}, \quad d_n = \left(-1\right)^n, \quad e_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

2. Which of the above sequences are bounded from above and what are their upper boundaries? Which are bounded from below and what are their lower boundaries?

3. Which of the above sequences converge for $n \longrightarrow \infty$? For those that don't, find a sub-sequence that does.

4. Prove that the following sequences converge to the given limits:

   - $a_n = \frac{1}{n} \longrightarrow 0$.
   - $a_n = \frac{n+2}{n} \longrightarrow 1$.
   - $a_n = \frac{\sin(n)}{n} \longrightarrow 0$.

Problem 3: Series

Calculate the following expressions:

1. $\sum\limits_{n=0}^{\infty}\frac{5}{2^n}$.

2. $\sum\limits_{n=2}^{\infty}\frac{1}{n^2-n}$.

3. $\sum\limits_{n=0}^{\infty}\frac{n!}{2^n}$.