Basic Maths for Non-mathematicians

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$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{k=1}^{N} f(x_k)\Delta x$$

$$(AB)^\top = B^\top A^\top$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$A = Q\Lambda Q^{-1}$$

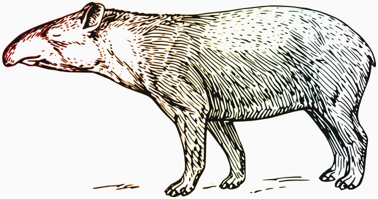
$$A\vec{v} = \lambda\vec{v}$$

$$\mathrm{Rot}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

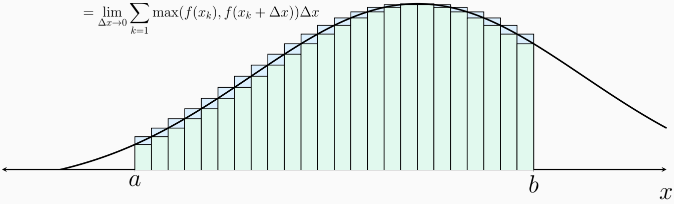
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$$



Chapter 8: Integrals

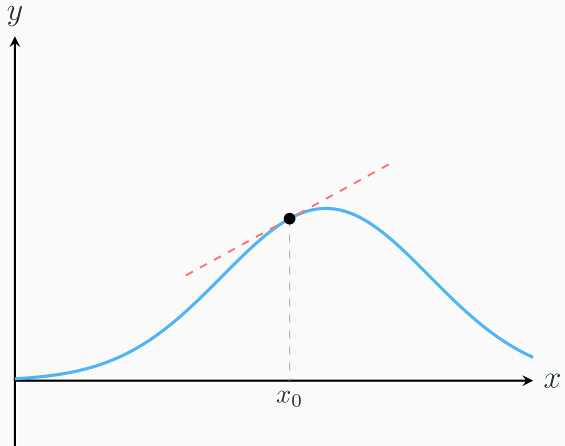
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{k=1}^{n} \min(f(x_k), f(x_k + \Delta x))\Delta x$$
$$= \lim_{\Delta x \to 0} \sum_{k=1}^{n} \max(f(x_k), f(x_k + \Delta x))\Delta x$$



Reminder: The Derivative

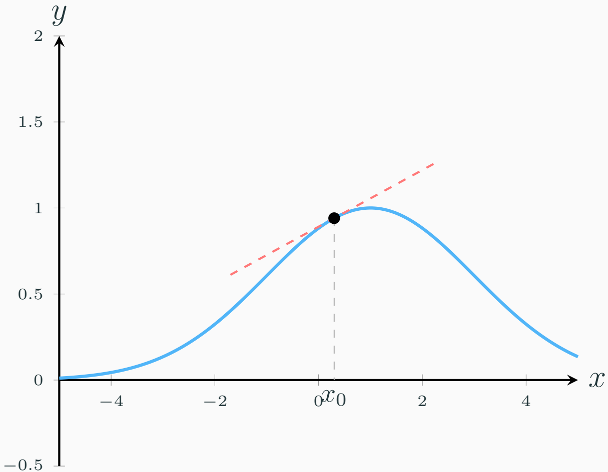
Definition

The **derivative** of a function f at the point x_0, denoted $f'(x_0)$, is the **slope** of the tangent line to the function at x_0.



Reminder: The Derivative

We can find the derivative by taking closer and closer points to x_0.



Reminder: The Derivative

Thus, the derivative is the limit where Δx goes to 0:

Definition

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\cancel{x_0} + \Delta x - \cancel{x_0}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Note

The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ **is not a fraction**. It only signifies that the derivative is the limit of $\frac{\Delta y}{\Delta x}$ when $\Delta x \to 0$.

Reminder: The Derivative

We can view the derivative as an **operator** acting on a function:

$$f(x) \longrightarrow \frac{\mathrm{d}}{\mathrm{d}x} \longrightarrow f'(x)$$

Recall that the following two statements are true:

$$\begin{cases} (\alpha f)' & = \alpha f', \\ (f+g)' & = f' + g'. \end{cases}$$

Similar to **linear transformations** acting on vectors, this means that the derivative operator is **linear**.

Reminder: The Derivative

By viewing the derivative as an operator acting on functions, the notion of **higher order derivatives** becomes pretty clear:

Definition

An n-th order derivative (where $n \in \mathbb{N}$) of a function f is the result of applying the derivative operator n consevutive times on f.

We denote the n-th order derivative of f as either

$$f^{(n)},$$

or

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}.$$

The Antiderivative (Indefinite Integral)

Similar to linear transformations (with a non-zero determinant), one might try to find an inverse operator to derivative. We call such an operator the **Antiderivative**, or the **Non-definite integral**.

We can think of it a derivative of order -1, i.e. $\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}$:

$$f'(x) \longrightarrow \boxed{\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}} \longrightarrow f(x) + C$$

Why?

The Antiderivative (Indefinite Integral)

Recall that constants are derived to 0, and thus don't affect the derivative:

$$\big(f(x) + C\big)' = f'(x) + \cancel{C'} = f'(x).$$

Therefore the antiderivative of a function is not a function itself, but a **family** of functions, differing from each other by a constant:

$$\frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} f = \left\{ f(x) + C \mid C \in \mathbb{R} \right\}.$$

The Antiderivative (Indefinite Integral)

A common way to denote the antiderivative of a function f by the variable x is

$$\int f(x)\,\mathrm{d}x,$$

which is called the **indefinite integral**.

The reason for "multiplying" the function by $\mathrm{d}x$ will be made clear later.

The Antiderivative (Indefinite Integral)

$f(x)$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	$\int f(x)\,\mathrm{d}x$ $(C=0)$
Linearity		
$Ag(x)$	$A\frac{\mathrm{d}g(x)}{\mathrm{d}x}$	$A\int g(x)\,\mathrm{d}x$
$f_1(x)+f_2(x)$	$\frac{\mathrm{d}f_1(x)}{\mathrm{d}x}+\frac{\mathrm{d}f_2(x)}{\mathrm{d}x}$	$\int f_1(x)\,\mathrm{d}x+\int f_2(x)\,\mathrm{d}x$
General Properties		
$f_1(x)f_2(x)$	$f_1'(x)f_2(x)+f_1(x)f_2'(x)$	Depends on f_1 and f_2
$\frac{f_1(x)}{f_2(x)}$	$\frac{f_1'(x)f_2(x)-f_1(x)f_2'(x)}{f_2^2}$	Depends on f_1 and f_2

The Antiderivative (Indefinite Integral)

$f(x)$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	$\displaystyle\int f(x)\,\mathrm{d}x\ (C=0)$
Polynomials		
B	0	Bx
x	1	$\frac{1}{2}x^2$
x^2	$2x$	$\frac{1}{3}x^3$
x^n	nx^{n-1}	$\boxed{\frac{1}{n+1}x^{n+1}}$
$\sum_{k=0}^{n} a_k x^k$	$\sum_{k=0}^{n} k a_k x^{k-1}$	$\sum_{k=0}^{n} \frac{1}{k+1} a_k x^{k+1}$

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The Antiderivative (Indefinite Integral)

$f(x)$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	$\int f(x)\,\mathrm{d}x\ (C=0)$
	Reciprocal Functions	
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\log(\lvert x\rvert)$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$	$-\frac{1}{x}$
	Exponents and Logarthims	
e^x	e^x	e^x
e^{-x}	$-e^{-x}$	$-e^{-x}$
$e^{g(x)}$	$\frac{\mathrm{d}g(x)}{\mathrm{d}x}e^{g(x)}$	Depends on $g(x)$
a^x	$a^x\log(a)$	$\frac{a^x}{\log(a)}$
$\log(x)$	$\frac{1}{x}$	$x\left(\log(x)-1\right)$

The Antiderivative (Indefinite Integral)

$f(x)$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$	$\int f(x)\,\mathrm{d}x \ (C=0)$
Trigonometric Functions		
$\sin(x)$	$\cos(x)$	$-\cos(x)$
$\cos(x)$	$-\sin(x)$	$\sin(x)$
$\tan(x)$	$\frac{1}{\cos^2(x)}$	$\frac{1}{1+x^2}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$x\arcsin(x)+\sqrt{1-x^2}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$x\arcsin(x)-\sqrt{1-x^2}$
$\arctan(x)$	$\frac{1}{x^2+1}$	$x\arctan(x)-\frac{\log(x^2+1)}{2}$

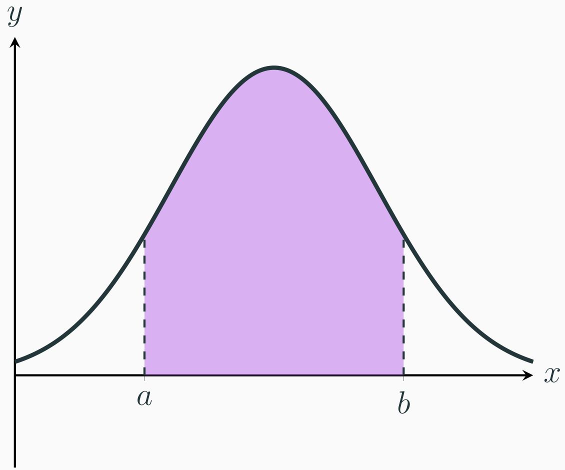
Non-existing Antiderivatives

All **elementary** functions (e.g. polynomials, exponentials, trigonometric functions, etc. - and compositions of such functions) have well defined derivatives, except for specific points (i.e. $x = 0$ for $|x|$, or $x = \left(\frac{1}{2} + k\right)\pi, k \in \mathbb{Z}$ for $\tan(x)$).

This in **NOT** true for antiderivatives: not every "well behaved" function has a known antiderivative.

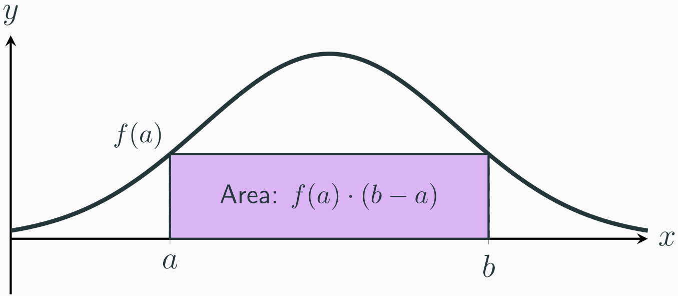
The Definite Integral

What is the area under the curve of a function, between two points $x_1 = a$ and $x_2 = b$?



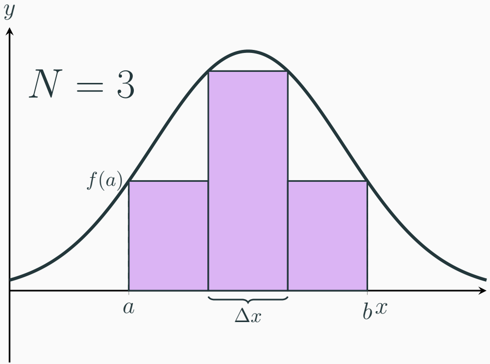
The Definite Integral

We can start by approximating the area with a shape of a known area: a rectangle.



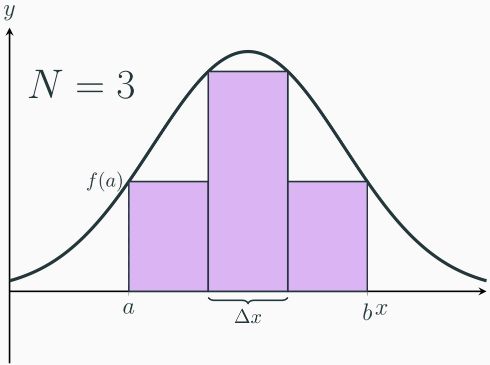
The Definite Integral

Obviously, this is not a great approximation, so we can divide the interval $[a,b]$ in three, and use the reulting rectangles (each with a base length $\Delta x = \frac{(b-a)}{3}$) to approximate the area:



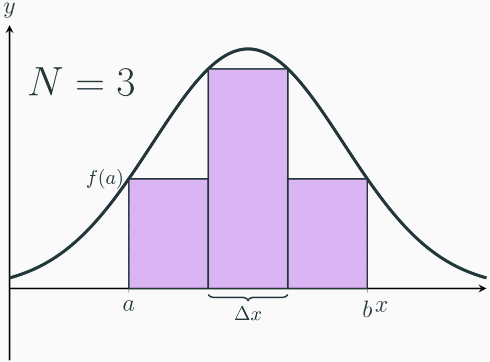
The Definite Integral

Although each of the rectangles have a base $\Delta x = \frac{(b-a)}{3}$, their areas are dependent on their position: their height is either the value of f at their left side, or value of f at their right side.



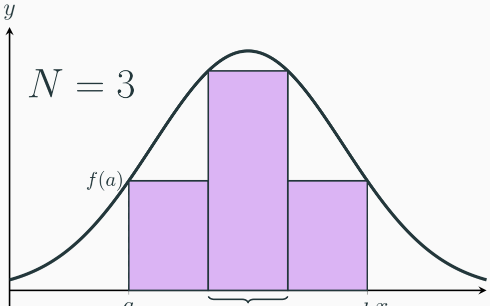
The Definite Integral

This will be determined by choosing the heights as either the minimum or maximum of the left and right values of f. For now we will arbitrarily choose using the minimum values.

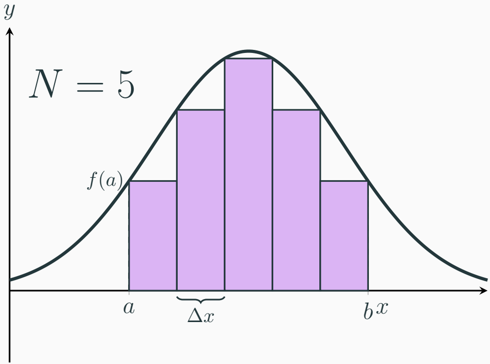


Writing $x_1 = a,\ x_2 = a + \Delta x,\ x_3 = a + 2\Delta x,\ x_4 = b$, the total area approximated by the rectangles is:

$$
\begin{aligned}
S_{\text{approx}} &= \min(f(x_1), f(x_2))(b-a) + \min(f(x_2), f(x_3))(b-a) + \min(f(\\
&= \left[\min(f(x_1), f(x_2)) + \min(f(x_2), f(x_3)) + \min(f(x_3), f(x_4))\right] (\\
&= \sum_{i=1}^{3} \min\left(f\left(x_i\right), f\left(x_{i+1}\right)\right) \Delta x.
\end{aligned}
$$



Of course, we can refine the approximation by increasing the number of rectangles (which is equivalent to reducing Δx, since $\Delta x = \frac{b-a}{N}$):



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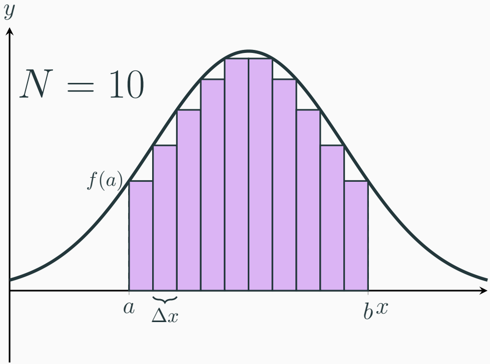

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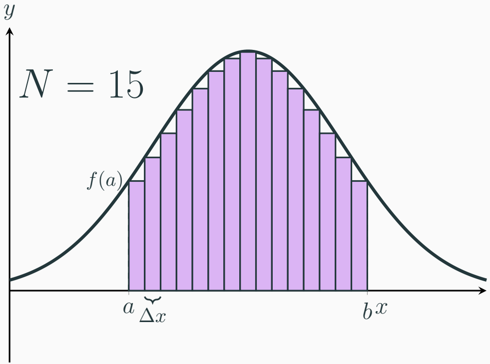

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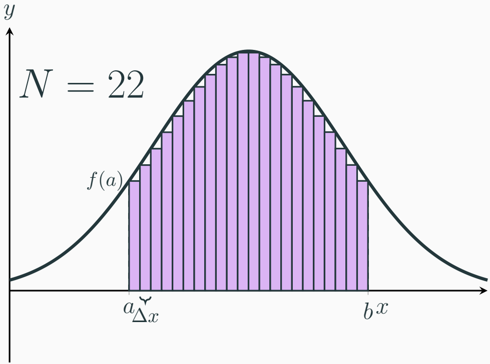

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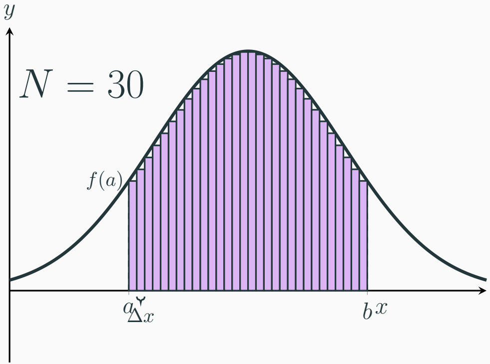

The Definite Integral

In the limit where $N \to \infty$ (equivalently, $\Delta x \to 0$), we get the exact area[1], and write:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{N} \min\big(f(x_i), f(x_i + \Delta x)\big)\, \Delta x = \int\limits_a^b f(x)\, \mathrm{d}x.$$

The above sum is called the **lower Darboux sum** of f in the interval $[a, b]$.

[1] if the function is well behaved...

The Funamental Theorem of Calculus

The connection between the two types of integrals mentioned so far is as follows: for a function f, its antiderivative F (i.e. $\frac{\mathrm{d}}{\mathrm{d}x}F = f$) and a real interval $[a, b]$,

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

This is a corollary of the **funamental theorem of calculus**.

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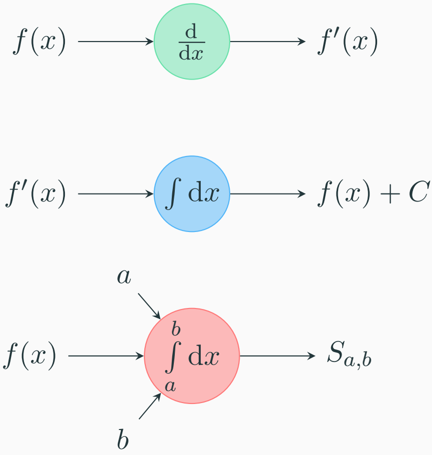
$f(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}$
$f'(x)$
$f'(x)$
$\int \mathrm{d}x$
$f(x) + C$
a
$f(x)$
$\int_a^b \mathrm{d}x$
$S_{a,b}$
b