Exercise 3: Linear Transformations and Matrices

Problem 1: Linear Transformations

Which of the following functions/transformations are linear? Prove your answer by showing that the two criteria for linearity are met (in the case of linear functions), or by giving an example which contraditcs a criterion or a property of linear functions (in the case of non linear functions).

1. $f(x) = |x|$.
2. $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$
3. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$.
4. $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2y \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.
5. $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x + y^2 - z$.

Problem 2: Matrices

1. The following matrices are defined:

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 0 & -4 & 1 \\ 2 & 2 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 7 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Calculate the following: $A + B,\ A - C,\ A \cdot B,\ B \cdot A, A \cdot C, C \cdot A$.

   (b) For each matrix write its transpose.

2. Which of the following products $A \cdot B = C$ are defined? Calculate the product if it is possible.

   (a) $\begin{pmatrix} 1 & 2 \\ -5 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 6 \\ -5 & 8 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 2 \\ 7 & 1 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 8 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 2 \\ 7 & 1 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 8 & 1 \end{pmatrix}^\top$

   (e) $\begin{pmatrix} 7 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$

   (f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$

Problem 3: Matrix-Vector Multiplication

1. The matrix $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is a 2D-rotation matrix: it rotates any 2D-vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ by the angle θ counter clockwise.
   (a) What do you expect the determinant of R_θ to be? Check your answer via direct calculation.
   (b) Show that for $\theta = 180°$ applying the matrix on a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ inverts the vector.
   (c) Show that for a vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ the resulting rotation by $\pm 90°$ using R is orthogonal to $\vec{v}$.
   (d) Show that applying R_θ to a 2D-vector $\vec{v}$ indeed results in a rotation of $\vec{v}$ by θ.
2. What operation does the following matrix perform? $A = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$
3. Construct a 2×2 matrix that scales a vector by 2 and then rotates it by 30°.