Exercise 12: Exam Preparation

Problem 1: Set Theory

The following two sets are given:

$$A = \{x, y, z\}\,,\ B = \{\alpha, \beta, \gamma\}\,.$$

1. Write all the elements of the Cartesian Product $A \times B$.
2. Draw two figures: one representing an **injective** function $f : A \to B$, and one representing a **non-injective** function $g : A \to B$. Can g be surjective?

Problem 2: Linear Algebra I

The following three vectors in $\mathbb{R}^3$ are given:

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},\ \vec{b} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix},\ \vec{c} = \begin{pmatrix} -16 \\ 0 \\ -10 \end{pmatrix}$$

1. Calculate the vector $2\vec{a} + \vec{b}$.
2. Calculate the inner product $\vec{a} \cdot \vec{b}$.
3. What is the angle between $\vec{a}$ and $\vec{b}$?
4. Are the three vectors $\vec{a}, \vec{b}, \vec{c}$ linearly independent? Prove your answer.
5. Give a geometrical description of the shpae of the set of points in space described by $\{\vec{x} \in \mathbb{R}^3 \mid \vec{c} \cdot \vec{x} = 0\}$. (Note: $\vec{c} \cdot \vec{x}$ dontes the inner (scalar) product of $\vec{c}$ and $\vec{x}$)
6. Normalize the vector $\vec{a}$.

Problem 3: Linear Algebra II

The matrix M represents a transformation that affects $\hat{x}$ and $\hat{y}$ as follows:

$$M\hat{x} = \begin{pmatrix} -1 \\ 8 \end{pmatrix},\quad M\hat{y} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}.$$

1. Calculate the eigenvalues and eigenvectors of M.
2. Does the transformation represented by M flips the orientation of $\mathbb{R}^2$?.

Problem 4: Computer Science (Programming)

The following Java method `f` gets an integer array `x` and a single integer `a` as its arguments:

```
1  public int f(int x[], int a)
2  {
3      int i = 0;
4      boolean b = true;
5      while (b && (i < x.length))
6      {
7          if (x[i] == a)
8              b = false
9          else
10             i = i+1;
11     }
12     if (b)
13     {
14         println("Error!");
15         return -1;
16     }
```

```
17      else
18          return i;
19  }
```

What does the function return?

Problem 5: Computer Science (Representation of Numbers)

1. What is the binary representation of the decimal number 63?
2. What is the decimal representation of the hexadecimal number 2A5?
3. Give the 8-bit two's complement representation of the decimal number -84.
4. What is the exact value of the fraction which is represented in the ternary base (base-3) by $0.222222222\ldots$ (i.e. with infinitely many $2s$ after the period)?

Problem 6: Computer Science (Rule-Based Simulation)

1. Given the following L-system: A $\rightarrow$ (RF)FA,
   which string is produced after 3 steps of application of this rule to the start word 'A'?
   Draw the graphical structure in the place which is obtained from this string by the following turtle interpretation:
   - R: Rotate 45 degrees clockwise.
   - F: Move forward 1 unit.
   - A: Do nothing.
   - (: Add position and angle to stack.
   -): Pop position and angle from stack.
2. A modified L-system is given: A $\rightarrow$ [RFA]FA.
   Draw the corresponding graphical structure after 3 steps.
3. How does the number of single lines (obtained from an "F" symbol) grow (quantitatively) with the number of steps for L1? For L2?

Problem 7: Calculus (Sequences)

1. The following sequence is given:
$$a_n = \frac{4-n}{2n+3}.$$
   Is the sequence increasing, decreasing or neither for $n \geq 10$? Is it converging on a finite value, and if so - what is this value?
2. Suppose that the following sequence converges:
$$a_{n+1} = a_n^2 + \frac{1}{4}, \quad a_1 = \frac{1}{4}.$$
   What is its limit as $n \longrightarrow \infty$?

Problem 8: Calculus (Univariate Functions, Limits)

Evaluate the following limit:
$$\lim_{x \to -\infty} xe^x.$$

Problem 9: Calculus (Univariate Functions, Differentiation)

Given the following function: $f(x) = \frac{1}{3}x^3 - 4x^2 + 7x - 5$,

1. Find all x values where the function f has local extrema and classify them as minima or maxima.
2. Find where the function is increasing/decreasing, and all x values of inflection points.