Calculations

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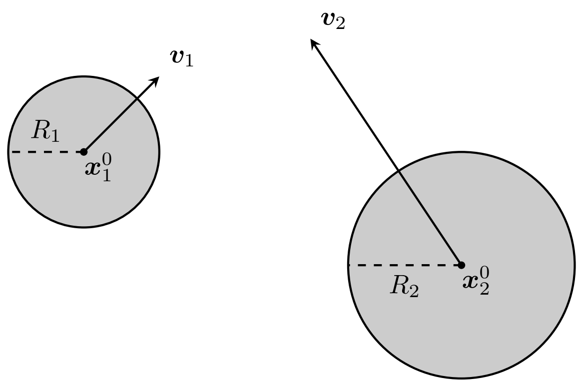
1 Collision of Two Spherical Particles Moving at Constant Velocities

Two spherical particles with radii R_1 and R_2, have initial positions $\boldsymbol{x}_1^0$ and $\boldsymbol{x}_2^0$ and move with constant velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. At what time, if at all, will they collide?



The position of particle 1 as a function of time is

$$\boldsymbol{x}_1(t) = \boldsymbol{x}_1^0 + \boldsymbol{v}_1 t, \tag{1}$$

The position of particle 2 as a function of time is identical, i.e.

$$\boldsymbol{x}_2(t) = \boldsymbol{x}_2^0 + \boldsymbol{v}_2 t. \tag{2}$$

Therefore, the distance $\boldsymbol{r}(t)$ between the particles is

$$\boldsymbol{r}(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t) = \boldsymbol{x}_1^0 + \boldsymbol{v}_1 t - \boldsymbol{x}_2^0 - \boldsymbol{v}_2 t, \tag{3}$$

or in explicit vector form

$$\begin{aligned}
\boldsymbol{r}(t) &= \begin{pmatrix} x_1^0 + v_1^x t - x_2^0 + v_2^x t \\ y_1^0 + v_1^y t - y_2^0 + v_2^y t \\ z_1^0 + v_1^z t - z_2^0 + v_2^z t \end{pmatrix} \\
&= \begin{pmatrix} \left(x_1^0 - x_2^0\right) - \left(v_1^x - v_2^x\right) t \\ \left(y_1^0 - y_2^0\right) - \left(v_1^y - v_2^y\right) t \\ \left(z_1^0 - z_2^0\right) - \left(v_1^z - v_2^z\right) t \end{pmatrix} \\
&= \begin{pmatrix} \Delta x_0 - \Delta v^x t \\ \Delta y_0 - \Delta v^y t \\ \Delta z_0 - \Delta v^z t \end{pmatrix}.
\end{aligned} \tag{4}$$

The square of the norm of $\boldsymbol{r}(t)$ is thus

$$
\begin{aligned}
\|r\|^2 &= \left(\Delta x_0 - \Delta v^x t\right)^2 + \left(\Delta x_0 - \Delta v^x t\right)^2 + \left(\Delta x_0 - \Delta v^x t\right)^2 \\
&= \left(\Delta x_0\right)^2 - 2\Delta x_0 \Delta v^x t + \left(\Delta v^x\right)^2 t^2 + \cdots \\
&= \left(\Delta x_0\right)^2 + \left(\Delta y_0\right)^2 + \left(\Delta z_0\right)^2 - 2\left(\Delta x_0 \Delta v^x + \Delta y_0 \Delta v^y + \Delta z_0 \Delta v^z\right) t + \left(\left(\Delta v^x\right)^2 + \left(\Delta v^y\right)^2 + \left(\Delta v^z\right)^2\right) t^2 \\
&= \|\Delta \boldsymbol{x}_0\|^2 - 2\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle t + \|\Delta \boldsymbol{v}\|^2 t^2. \qquad (5)
\end{aligned}
$$

The times $t_{1,2}$ for which the particles collide can be calculated by solving equation 5 for the case

$$\|r\|^2 = \left(R_1 + R_2\right)^2 = R^2, \qquad (6)$$

i.e. when the distance between the particles is the sum of their radii.

Using the quadratic formula,

$$
\begin{aligned}
t_{1,2} &= \frac{2\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle \pm \sqrt{4\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle^2 - 4\|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right)}}{2\|\Delta \boldsymbol{v}\|^2} \\
&= \frac{\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle \pm \sqrt{\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle^2 - \|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right)}}{\|\Delta \boldsymbol{v}\|^2}. \qquad (7)
\end{aligned}
$$

The condition for which a collision will happen is therefore

$$\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v}\rangle^2 \geq \|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right). \qquad (8)$$