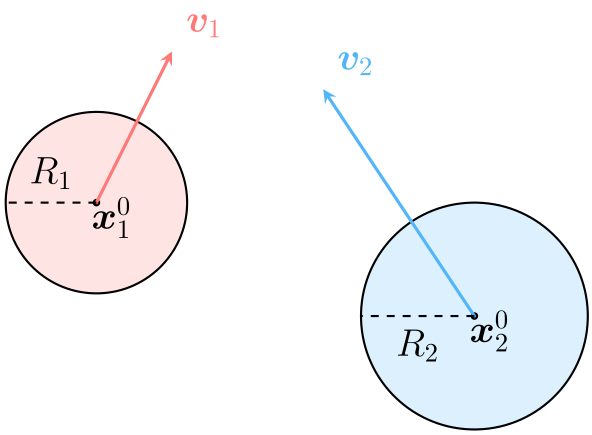
Calculations

June 1, 2020

1 Collision of Two Spherical Particles Moving at Constant Velocities

1.1 Time to Collision

Two spherical particles with radii R_1 and R_2, have initial positions $\boldsymbol{x}_1^0$ and $\boldsymbol{x}_2^0$ and move with constant velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, respectively. At what time, if at all, will they collide?



The position of particle 1 as a function of time is

$$\boldsymbol{x}_1(t) = \boldsymbol{x}_1^0 + \boldsymbol{v}_1 t, \tag{1}$$

The position of particle 2 as a function of time is identical, i.e.

$$\boldsymbol{x}_2(t) = \boldsymbol{x}_2^0 + \boldsymbol{v}_2 t. \tag{2}$$

Therefore, the distance $\boldsymbol{r}(t)$ between the particles is

$$\boldsymbol{r}(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t) = \boldsymbol{x}_1^0 + \boldsymbol{v}_1 t - \boldsymbol{x}_2^0 - \boldsymbol{v}_2 t, \tag{3}$$

or in explicit vector form

$$\begin{aligned}
\boldsymbol{r}(t) &= \begin{pmatrix} x_1^0 + v_1^x t - x_2^0 + v_2^x t \\ y_1^0 + v_1^y t - y_2^0 + v_2^y t \\ z_1^0 + v_1^z t - z_2^0 + v_2^z t \end{pmatrix} \\
&= \begin{pmatrix} \left(x_1^0 - x_2^0\right) - \left(v_1^x - v_2^x\right) t \\ \left(y_1^0 - y_2^0\right) - \left(v_1^y - v_2^y\right) t \\ \left(z_1^0 - z_2^0\right) - \left(v_1^z - v_2^z\right) t \end{pmatrix} \\
&= \begin{pmatrix} \Delta x_0 - \Delta v^x t \\ \Delta y_0 - \Delta v^y t \\ \Delta z_0 - \Delta v^z t \end{pmatrix}. \qquad (4)
\end{aligned}$$

The square of the norm of $\boldsymbol{r}(t)$ is thus

$$
\begin{aligned}
\|r\|^2 &= (\Delta x_0 - \Delta v^x t)^2 + (\Delta x_0 - \Delta v^x t)^2 + (\Delta x_0 - \Delta v^x t)^2 \\
&= (\Delta x_0)^2 - 2\Delta x_0 \Delta v^x t + (\Delta v^x)^2 t^2 + \cdots \\
&= (\Delta x_0)^2 + (\Delta y_0)^2 + (\Delta z_0)^2 - 2\left(\Delta x_0 \Delta v^x + \Delta y_0 \Delta v^y + \Delta z_0 \Delta v^z\right) t + \left((\Delta v^x)^2 + (\Delta v^y)^2 + (\Delta v^z)^2\right) t^2 \\
&= \|\Delta \boldsymbol{x}_0\|^2 - 2\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle t + \|\Delta \boldsymbol{v}\|^2 t^2. \qquad (5)
\end{aligned}
$$

The times $t_{1,2}$ for which the particles collide can be calculated by solving equation 5 for the case

$$
\|r\|^2 = (R_1 + R_2)^2 = R^2, \qquad (6)
$$

i.e. when the distance between the particles is the sum of their radii.

Using the quadratic formula,

$$
\begin{aligned}
t_{1,2} &= \frac{2\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle \pm \sqrt{4\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle^2 - 4\|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right)}}{2\|\Delta \boldsymbol{v}\|^2} \\
&= \frac{\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle \pm \sqrt{\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle^2 - \|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right)}}{\|\Delta \boldsymbol{v}\|^2}. \qquad (7)
\end{aligned}
$$

The condition for which a collision will happen is therefore

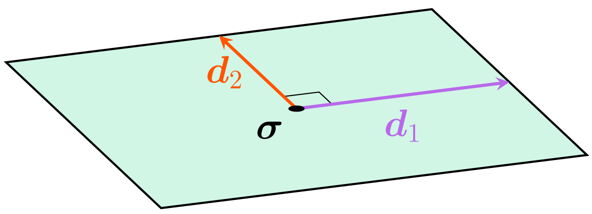
$$
\langle \Delta \boldsymbol{x}_0, \Delta \boldsymbol{v} \rangle^2 \geq \|\Delta \boldsymbol{v}\|^2 \left(\|\Delta \boldsymbol{x}_0\|^2 - R^2\right). \qquad (8)
$$

1.2 Veclocity Change

2 Collision of a Spherical Particle and a Planar Wall

2.1 Definition of a Planar Wall

A plane can be defined by a point and a direction normal to the plane. However, we wish to use only a finite part of the plane. We therefore define a wall using a point $\boldsymbol{\sigma}$ and two orthogonal vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$:

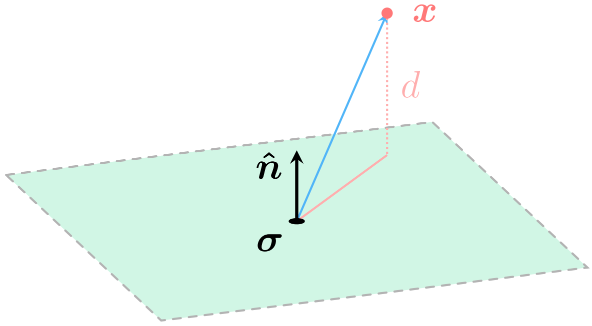


The normal to the wall surface can then be defined as

$$
\hat{\boldsymbol{n}} = \frac{\boldsymbol{d}_1 \times \boldsymbol{d}_2}{\|\boldsymbol{d}_1 \times \boldsymbol{d}_2\|}. \qquad (9)
$$

2.2 Time to Collision

The distance d between a spherical particle and a wall can be calculated as the distance between its center (a point) and the plane of which the wall is a part of. This distance is the projection of the vector $\boldsymbol{x} - \boldsymbol{\sigma}$ onto $\hat{\boldsymbol{n}}$:



Using the dot product, it can be written as

$$d = \langle \boldsymbol{x} - \boldsymbol{\sigma}, \hat{\boldsymbol{n}} \rangle, \tag{10}$$

since $\|\hat{\boldsymbol{n}}\| = 1$.

In order to have a concise notation for the following part, we define two new concepts:

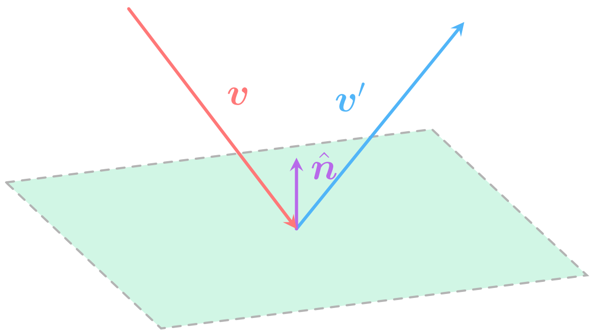
1. **Squared vector**:

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \Rightarrow \boldsymbol{v}^2 \equiv \begin{pmatrix} v_1^2 \\ v_2^2 \\ \vdots \\ v_n^2 \end{pmatrix}. \tag{11}$$

2. **Speical triple product**: the triple product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ is

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \equiv \sum_{i=1}^{n} u_i v_i w_i. \tag{12}$$

2.3 Velocity Change



The projection of $\boldsymbol{v}$ onto $\hat{\boldsymbol{n}}$ is

$$\mathrm{proj}_{\hat{\boldsymbol{n}}} \boldsymbol{v} = \langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}}, \tag{13}$$

and on a direction orthogonal to $\hat{\boldsymbol{n}}$ is

$$\boldsymbol{v} - \langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}}. \tag{14}$$

$\boldsymbol{v}$ can be reconstructed from these two components,

$$\boldsymbol{v} = \langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}} + \boldsymbol{v} - \langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}}. \tag{15}$$

Similarly, $\boldsymbol{v}'$ has the $\hat{\boldsymbol{n}}$ opposite and equal in magnitude to that of $\boldsymbol{v}$, and the orthogonal component equal to that of $\boldsymbol{v}$, yielding

$$\begin{aligned} \boldsymbol{v}' &= -\langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}} + \boldsymbol{v} - \langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}} \\ &= \boldsymbol{v} - 2\langle \boldsymbol{v}, \hat{\boldsymbol{n}} \rangle \hat{\boldsymbol{n}}. \end{aligned} \tag{16}$$