

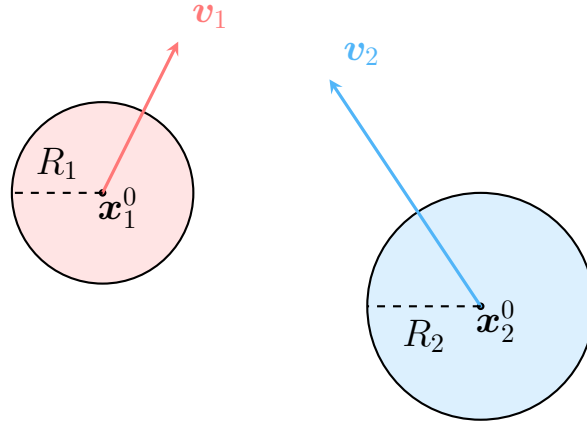
# Calculations

June 2, 2020

## 1 Collision of Two Spherical Particles Moving at Constant Velocities

### 1.1 Time to Collision

Two spherical particles with radii  $R_1$  and  $R_2$ , have initial positions  $\mathbf{x}_1^0$  and  $\mathbf{x}_2^0$  and move with constant velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively. At what time, if at all, will they collide?



The position of particle 1 as a function of time is

$$\mathbf{x}_1(t) = \mathbf{x}_1^0 + \mathbf{v}_1 t, \quad (1)$$

The position of particle 2 as a function of time is identical, i.e.

$$\mathbf{x}_2(t) = \mathbf{x}_2^0 + \mathbf{v}_2 t. \quad (2)$$

Therefore, the distance  $\mathbf{r}(t)$  between the particles is

$$\mathbf{r}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t) = \mathbf{x}_1^0 + \mathbf{v}_1 t - \mathbf{x}_2^0 - \mathbf{v}_2 t, \quad (3)$$

or in explicit vector form

$$\begin{aligned} \mathbf{r}(t) &= \begin{pmatrix} x_1^0 + v_1^x t - x_2^0 + v_2^x t \\ y_1^0 + v_1^y t - y_2^0 + v_2^y t \\ z_1^0 + v_1^z t - z_2^0 + v_2^z t \end{pmatrix} \\ &= \begin{pmatrix} (x_1^0 - x_2^0) - (v_1^x - v_2^x) t \\ (y_1^0 - y_2^0) - (v_1^y - v_2^y) t \\ (z_1^0 - z_2^0) - (v_1^z - v_2^z) t \end{pmatrix} \\ &= \begin{pmatrix} \Delta x_0 - \Delta v^x t \\ \Delta y_0 - \Delta v^y t \\ \Delta z_0 - \Delta v^z t \end{pmatrix}. \end{aligned} \quad (4)$$

The square of the norm of  $\mathbf{r}(t)$  is thus

$$\begin{aligned}
\|\mathbf{r}\|^2 &= (\Delta x_0 - \Delta v^x t)^2 + (\Delta x_0 - \Delta v^x t)^2 + (\Delta x_0 - \Delta v^x t)^2 \\
&= (\Delta x_0)^2 - 2\Delta x_0 \Delta v^x t + (\Delta v^x)^2 t^2 + \dots \\
&= (\Delta x_0)^2 + (\Delta y_0)^2 + (\Delta z_0)^2 - 2(\Delta x_0 \Delta v^x + \Delta y_0 \Delta v^y + \Delta z_0 \Delta v^z) t + \left( (\Delta v^x)^2 + (\Delta v^y)^2 + (\Delta v^z)^2 \right) t^2 \\
&= \|\Delta \mathbf{x}_0\|^2 - 2\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle t + \|\Delta \mathbf{v}\|^2 t^2.
\end{aligned} \tag{5}$$

The times  $t_{1,2}$  for which the particles collide can be calculated by solving equation 5 for the case

$$\|\mathbf{r}\|^2 = (R_1 + R_2)^2 = R^2, \tag{6}$$

i.e. when the distance between the particles is the sum of their radii.

Using the quadratic formula,

$$\begin{aligned}
t_{1,2} &= \frac{2\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle \pm \sqrt{4\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle^2 - 4\|\Delta \mathbf{v}\|^2 (\|\Delta \mathbf{x}_0\|^2 - R^2)}}{2\|\Delta \mathbf{v}\|^2} \\
&= \frac{\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle \pm \sqrt{\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle^2 - \|\Delta \mathbf{v}\|^2 (\|\Delta \mathbf{x}_0\|^2 - R^2)}}{\|\Delta \mathbf{v}\|^2}.
\end{aligned} \tag{7}$$

The condition for which a collision will happen is therefore

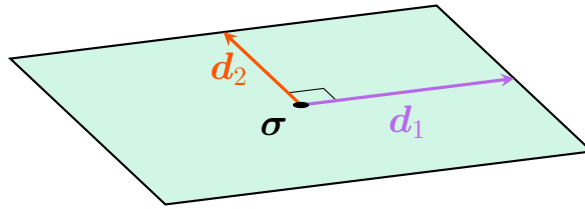
$$\langle \Delta \mathbf{x}_0, \Delta \mathbf{v} \rangle^2 \geq \|\Delta \mathbf{v}\|^2 (\|\Delta \mathbf{x}_0\|^2 - R^2). \tag{8}$$

## 1.2 Veclocity Change

# 2 Collision of a Spherical Particle and a Planar Wall

## 2.1 Definition of a Planar Wall

A plane can be defined by a point and a direction normal to the plane. However, we wish to use only a finite part of the plane. We therefore define a wall using a point  $\boldsymbol{\sigma}$  and two orthogonal vectors  $\mathbf{d}_1$  and  $\mathbf{d}_2$ :

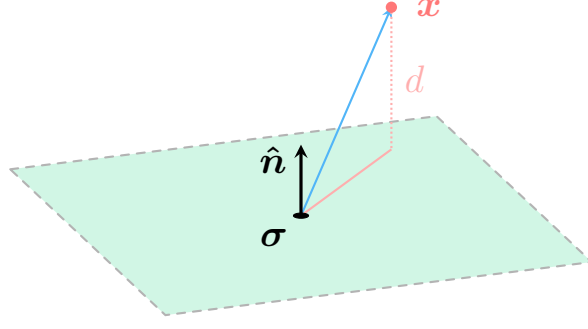


The normal to the wall surface can then be defined as

$$\hat{\mathbf{n}} = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{d}_2\|}. \tag{9}$$

## 2.2 Time to Collision

The distance  $d$  between a spherical particle and a wall can be calculated as the distance between its center (a point) and the plane of which the wall is a part of. This distance is the projection of the vector  $\mathbf{x} - \boldsymbol{\sigma}$  onto  $\hat{\mathbf{n}}$ :



Using the dot product, it can be written as

$$d = \langle \mathbf{x} - \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle, \quad (10)$$

since  $\|\hat{\mathbf{n}}\| = 1$ .

The position of the sphere as a function of time is

$$\mathbf{x}(t) = \mathbf{x}^0 + \mathbf{v}t, \quad (11)$$

and thus

$$\begin{aligned} d &= \langle \mathbf{x}^0 + \mathbf{v}t - \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle \\ &= \langle \mathbf{x}^0 - \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle + \langle \mathbf{v}, \hat{\mathbf{n}} \rangle t, \end{aligned} \quad (12)$$

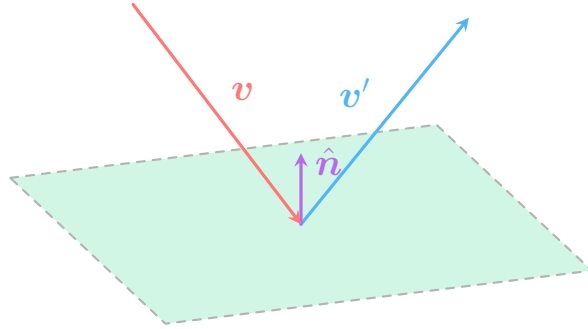
and the solution for  $d = R$  becomes

$$t = \frac{R - \langle \mathbf{x}^0 - \boldsymbol{\sigma}, \hat{\mathbf{n}} \rangle}{\langle \mathbf{v}, \hat{\mathbf{n}} \rangle}. \quad (13)$$

(note that both innerproducts implicitly contain a division by  $\|\hat{\mathbf{n}}\|$ , and so the units match)

PROJECTION OF XSIGMA ON D1, D2.

### 2.3 Velocity Change



The projection of  $\mathbf{v}$  onto  $\hat{\mathbf{n}}$  is

$$\text{proj}_{\hat{\mathbf{n}}} \mathbf{v} = \langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}, \quad (14)$$

and on a direction orthogonal to  $\hat{\mathbf{n}}$  is

$$\mathbf{v} - \langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}. \quad (15)$$

$\mathbf{v}$  can be reconstructed from these two components,

$$\mathbf{v} = \langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}} + \mathbf{v} - \langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}. \quad (16)$$

Similarly,  $\mathbf{v}'$  has the  $\hat{\mathbf{n}}$  opposite and equal in magnitude to that of  $\mathbf{v}$ , and the orthogonal component equal to that of  $\mathbf{v}$ , yielding

$$\begin{aligned} \mathbf{v}' &= -\langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}} + \mathbf{v} - \langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}} \\ &= \mathbf{v} - 2\langle \mathbf{v}, \hat{\mathbf{n}} \rangle \hat{\mathbf{n}}. \end{aligned} \quad (17)$$