

Part I

Background Topics

1 Linear Algebra

1.1 Preface

THE GOAL OF THIS CHAPTER is not to teach you, the reader, linear algebra from scratch - nor to be a thorough source of information on the topic. Rather, my aim is to introduce important “advanced” concepts for those who took a basic linear algebra course as part of an undergraduate university program. These concepts should help you gain a basic knowledge of the topics needed for understanding the rest of the background material, as well as the topic of spinors itself.

My approach to teaching topics in linear algebra - and in mathematics as a whole - is to first build an intuition and only then formalize and generalize the ideas as needed. In my personal experiences, when I was studying linear algebra I completely failed to understand it (and indeed, failed the course) until it “clicked” for me in regards to 2- and 3-dimensional real spaces, i.e. - visible geometry. After that I didn’t even have to study for exams anymore, as everything became clear enough to grasp and develop on the spot even during an exam (except for later, more advanced concepts). That is why, for example, I absolutely adore courses and study materials of the topic¹ which use animation, such as *3Blue1Brown* great video essay series [Essence of linear algebra](#)².

¹ And other mathematical topics as well.

² Temporary sidenote which should become a citation for the mentioned 3B1B video series

There are very few proofs in this chapter, and those that are shown are not completely rigorous. For more in-depth materials, see the last section (further read). With that out of the way - let’s begin!

1.2 Change of Coordinates

IN INTRODUCTORY LINEAR ALGEBRA COURSES you should have learned about change of coordinate systems: a coordinate system is just another name for a basis set of whatever vector space is used (in this section it's \mathbb{R}^n). A change of coordinate system is the transformation of vectors from being represented in one basis set $B = \{e_1, e_2, \dots, e_n\}$ to being represented in another basis set $\tilde{B} = \{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n\}$. Since such transformations are linear they are commonly represented in a matrix form.

In this section we will discuss *how* vectors and their components transform under change of basis sets. These transformations can be quite confusing, so I color coded the equations consistently as a visual guide. In addition, the \mathbb{R}^2 case is introduced first, before giving the generalized form for \mathbb{R}^n .

Before we do this, there is one important idea to understand: vectors themselves do not change under change of basis - their magnitude and orientation stay the same no matter how *we* measure them. The component of a vector are a way to describe (or “measure”) the vector using a certain basis set, so it should not be surprising that they change under change of basis.

With that in mind, let us look at an example of a basis set change in \mathbb{R}^2 .

1.2.1 Change of basis set in \mathbb{R}^2

Suppose we use the standard basis set to represent \mathbb{R}^2 :

$$B = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad (1.1)$$

and we want to change our coordinate system to use the following basis set:

$$\tilde{B} = \{\tilde{e}_1, \tilde{e}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \right\}. \quad (1.2)$$

(the vectors composing the two basis sets are shown in [Figure 1.1](#))

The transformation between the basis sets can be represented in matrix form:

$$F = \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & \frac{1}{4} \end{bmatrix}. \quad (1.3)$$

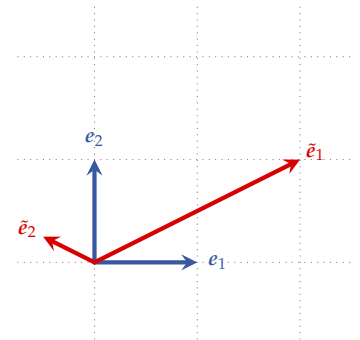


Figure 1.1: The standard basis set B and a new basis set \tilde{B} shown together.

Note 1.1 The components of F

The columns of F are exactly the two vectors of \tilde{B} because B is the standard basis. The transformation matrix is not that obvious between any two basis sets.



Now, if we want to transform e_1 and e_2 into \tilde{e}_1 and \tilde{e}_2 using F , we simply multiply each of the vectors by F :

$$\begin{aligned}\tilde{e}_1 &= Fe_1, \\ \tilde{e}_2 &= Fe_2.\end{aligned}\tag{1.4}$$

To make ?? more concise, we can collect the two vectors into a matrix disguised as a row vector:

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [e_1, e_2],\tag{1.5}$$

and find that ?? can be written in vector-matrix notation as

$$\tilde{e} = [\tilde{e}_1, \tilde{e}_2] = [e_1, e_2] \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & \frac{1}{4} \end{bmatrix} = [2e_1 + e_2, -\frac{1}{2}e_1 + \frac{1}{4}e_2].\tag{1.6}$$

The reverse transformation can be calculated by applying the inverse transformation F^{-1} on Equation 1.6:

$$\begin{aligned}[\tilde{e}_1, \tilde{e}_2] F^{-1} &= ([e_1, e_2] F) F^{-1} \\ &= [e_1, e_2] FF^{-1} \\ &= [e_1, e_2] I_2 \\ &= [e_1, e_2].\end{aligned}\tag{1.7}$$

(where in our case $F^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -1 & 2 \end{bmatrix}$)

To summarize: given a set B of basis vectors in \mathbb{R}^2 , we can transform it into the basis vectors set \tilde{B} by using the forward transformation

$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$. To get back the original basis vectors from the transformed vectors we use the inverse transformation

$F^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1}$ (if it exists - i.e. if F is invertible).

\tilde{e}_1, \tilde{e}_2
↓ ↓
The more general case: \mathbb{R}^n

In \mathbb{R}^n the transformations behave in a similar way: given the transformation rule that each new basis vector $\tilde{e}_i \in \tilde{B}$ is a linear combi-

nation of the old basis vector set B , i.e.

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= F_{11}\mathbf{e}_1 + F_{12}\mathbf{e}_2 + \cdots + F_{1n}\mathbf{e}_n, \\ \tilde{\mathbf{e}}_2 &= F_{21}\mathbf{e}_1 + F_{22}\mathbf{e}_2 + \cdots + F_{2n}\mathbf{e}_n, \\ &\vdots \\ \tilde{\mathbf{e}}_n &= F_{n1}\mathbf{e}_1 + F_{n2}\mathbf{e}_2 + \cdots + F_{nn}\mathbf{e}_n,\end{aligned}$$

we can write the transformation in matrix form as

$$\begin{bmatrix} \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1n} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} \end{bmatrix}. \quad (1.8)$$

Equation 1.8 can be written in index notation, which shows us how each vector is transformed:

$$\tilde{\mathbf{e}}_j = \sum_{k=1}^n F_{kj} \mathbf{e}_k. \quad (1.9)$$

Similarly, the inverse operation is given by

$$\begin{aligned}\mathbf{e}_1 &= F_{11}^{-1}\tilde{\mathbf{e}}_1 + F_{12}^{-1}\tilde{\mathbf{e}}_2 + \cdots + F_{1n}^{-1}\tilde{\mathbf{e}}_n, \\ \mathbf{e}_2 &= F_{21}^{-1}\tilde{\mathbf{e}}_1 + F_{22}^{-1}\tilde{\mathbf{e}}_2 + \cdots + F_{2n}^{-1}\tilde{\mathbf{e}}_n, \\ &\vdots \\ \mathbf{e}_n &= F_{n1}^{-1}\tilde{\mathbf{e}}_1 + F_{n2}^{-1}\tilde{\mathbf{e}}_2 + \cdots + F_{nn}^{-1}\tilde{\mathbf{e}}_n,\end{aligned}$$

with the matrix form being

$$\begin{bmatrix} \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \dots, \tilde{\mathbf{e}}_n \end{bmatrix} \begin{bmatrix} F_{11}^{-1} & F_{12}^{-1} & \cdots & F_{1n}^{-1} \\ F_{21}^{-1} & F_{22}^{-1} & \cdots & F_{2n}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1}^{-1} & F_{n2}^{-1} & \cdots & F_{nn}^{-1} \end{bmatrix}, \quad (1.10)$$

and the index notation being

$$\mathbf{e}_i = \sum_{j=1}^n F_{ji}^{-1} \tilde{\mathbf{e}}_j. \quad (1.11)$$

By substituting Equation 1.9 into Equation 1.11, we get

$$\begin{aligned}\mathbf{e}_i &= \sum_{j=1}^n F_{ji}^{-1} \tilde{\mathbf{e}}_j \\ &= \sum_{j=1}^n F_{ji}^{-1} \left(\sum_{k=1}^n F_{kj} \mathbf{e}_k \right) \\ &= \sum_{k=1}^n \left(\underbrace{\sum_{j=1}^n F_{kj} F_{ji}^{-1}}_{\text{this is just a number!}} \right) \mathbf{e}_k.\end{aligned} \quad (1.12)$$

Now, ?? simply tells us something we already know: each basis vector e_i is equal to a linear combination of the *same set* of basis vectors. This must mean that for $k = i$ the number in paranthesis is one, and for any other value of k it is zero - i.e. it equals δ_{ik} :

$$e_i = \sum_{k=1}^n \delta_{ik} e_k. \quad (1.13)$$

In turn, Equation 1.13 means that the matrix $FF^{-1} = I_n$, the identity matrix in \mathbb{R}^n - and thus the matrices F and F^{-1} are each other's inverses, exactly as we expect.

1.2.3 Components transformation

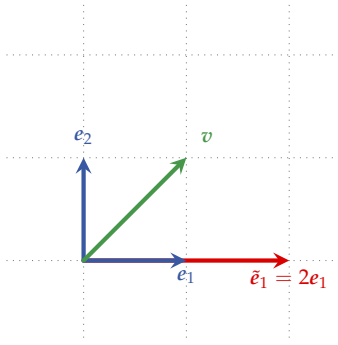


Figure 1.2: A vector v and two different sets of basis vectors: in blue the standard basis vectors and in red the basis set $\tilde{B} = \{2e_1, e_2\}$.

Now let us discuss how the *components* of a vector transform under change of basis: suppose we have the vector v as depicted in Figure 1.2. Using the standard basis set (depicted in blue) we can write $v = e_1 + e_2$, or simply $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. However, by using the basis set $\tilde{B} = \{2e_1, e_2\}$ (i.e. e_1 is scaled by 2), v is now $v = \frac{1}{2}\tilde{e}_1 + \tilde{e}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$.

We see that by scaling a basis vector by 2, its respective component is scaled by $\frac{1}{2}$. In general, when we scale the basis vector i by a non-zero scalar α , the respective i -th component is scaled by $\frac{1}{\alpha}$.

What about rotating the basis set? Again, we take the vector from Figure 1.2, but this time we rotate the standard basis set by some angle θ (Figure 1.3). The result is that the vector v is rotated by $-\theta$ with respect to the new basis set. If you don't see this, look at the angle between v and e_1 : it is exactly 45° (since $v = e_1 + e_2$). The angle between v and \tilde{e}_1 , on the other hand, is smaller than 45° - which means that v got relatively rotated towards it, which is the opposite direction of rotation from e_1 to \tilde{e}_1 .

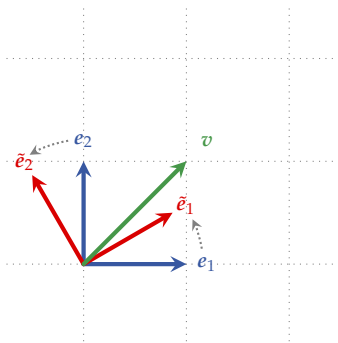


Figure 1.3: A vector v and two different sets of basis vectors: in blue the standard basis vectors and in red the basis set $\tilde{B} = \{2e_1, e_2\}$.

1.3 *Further Reading*

2 *Geometric Algebra*

2.1 *Preface*

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3 *Abstract Algebra*

3.1 *Preface*

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4 Lie Groups and Algebras

4.1 *Preface*

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Part II

Spinors

