Part I

Background Topics

1 Linear Algebra

1.1 Preface

THE GOAL OF THIS CHAPTER is not to teach you, the reader, linear algebra from scratch - nor to be a thorough source of information on the topic. Rather, my aim is to introduce important "advanced" concepts for those who took a basic linear algebra course as part of an undergraduate university program. These concpets should help you gain a basic knowledge of the topics needed for understanding the rest of the background material, as well as the topic of spinors itself.

My approach to teching topics in linear algebra - and in mathematics as a whole - is to first build an intuition and only then formalize and generalize the ideas as needed. In my personal experiences, when I was studying linear algebra I completely failed to understand it (and indeed, failed the course) until it "clicked" for me in regards to 2- and 3-dimensional real spaces, i.e. - visible geometry. After that I didn't even have to study for exams anymore, as everything became clear enough to grasp and develop on the spot even during an exam (except for later, more advances concepts). That is why, for example, I absolutely adore courses and study materials of the topic[1] which use animation, such as *3Blue1Brown* great video essay series Essence of linear algebra[2].

[1] And other mathematical topics as well.
[2] Temporary sidenote which should become a citation for the mentioned 3B1B video series

There are very few proofs in this chapter, and those that are shown are not completely rigorous. For more in-depth materials, see the last section (further read). With that out of the way - let's begin!

1.2 Dual Vectors and Dual Spaces

1.2.1 Measurements and rulers
(or: why should I care about dual vectors?)

USUALLY, DUAL VECTORS ARE TAUGHT by hitting the students with
the definition of a dual space and then analyzing its properties.
It's all very abstract and often leaves the students with a constant
question in mind: "why do we care about dual vectors?"

I would like to take a different approach here: instead of con-
fronting you with the definition and then discuss practical details, I
will start with explaining *why* we care about dual vectors in the first
place.

Let us begin with discussing rulers[3]. A ruler is essentially a ge-
ometric measuring tool which allows one to measure the *lengths*
of different objects by counting the number of graduation lines be-
tween the beginning and end of an object (section 1.2.1). Of course
in linear algebra, the geometric objects we measure using rulers are
vectors.
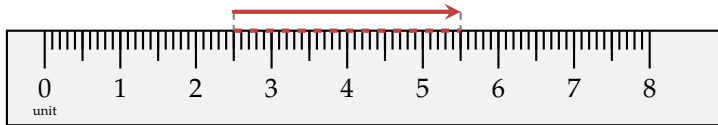
[3] The idea for this approach comes
from a beautiful answer by *Aloizio
Macedo* to a question in the mathemat-
ics stack exchange website.

Figure 1.1: Measuring a vector using
a ruler: the start of the vector sits at
2.5 units, while its head is at 5.5 units.
Therefore we say that the vector is
$5.5 - 2.5 = 3$ units in length.

We can represent a ruler in $\mathbb{R}^2$ as follows: the 0-graduation,
which I denote as L_0 here, is represented by a line going through
the origin with a direction orthogonal to the that of the ruler.
The rest of the graduation marks are then drawn as the lines that
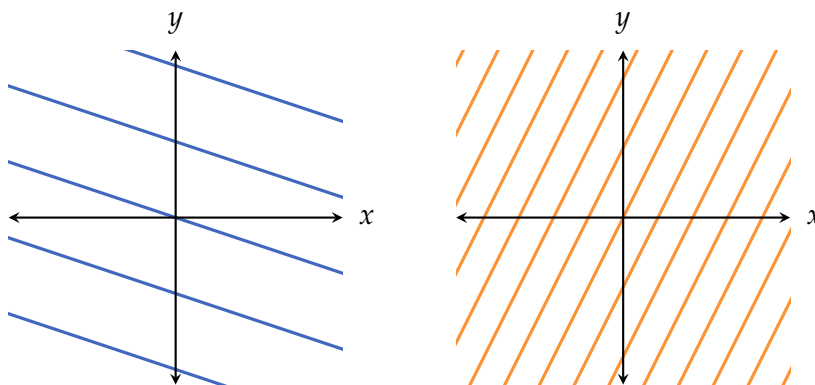are parallel to L_0 such that they are spaced equaly apart (see sec-
tion 1.2.1).

Figure 1.2: Two rulers in $\mathbb{R}^2$, each
represented as the infinite set of
lines which are parallel to the zero
graduation line L_0, and are equaly
spaced apart.

Using this representation, each ruler is uniquely described using

two properties: an orientation and the distance between its graduation lines. These two properties essentially correspond to a vector in $\mathbb{R}^2$. However, while a ruler can be uniquely described by a vector in $\mathbb{R}^2$, it is not the same as a geometric vector! It is a ruler. One way of keeping the difference between the two types of vectors is by denoting the graduations using their "frequency" per unit length instead of their distance. That way, the greater the frequency (i.e. the smaller the distance between the graduations) - the greater the length of the vector representing the ruler.

In $\mathbb{R}^3$ the graduations are represented in the same way as in $\mathbb{R}^n$, except that *planes* are used instead of line (section 1.2.1). The planes together are also called a "stack".
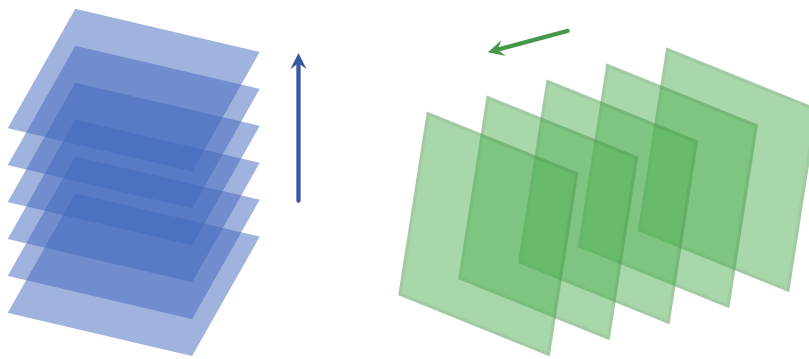

Figure 1.3: Representation of rulers in $\mathbb{R}^3$; as with the lines in the case of $\mathbb{R}^2$, I added a vector showing the direction and density of the graduation marks. Note how the planes (stacks) in blue are more densely spaced compared to the green planes, and therefore the vector representing that respective ruler is longer compared to the vector representing the ruler in green.

In general, a ruler in $\mathbb{R}^n$ is represented by a set of parallel and equally spaced *hyperplanes* of $(n-1)$-dimensions.

Using a ruler to measure a vector is done by projecting the vector onto the ruler (section 1.2.1): given a ruler, vectors are measured as greater the more they are pointing at a parallel direction to the ruler (and, of course, the longer they are in general).
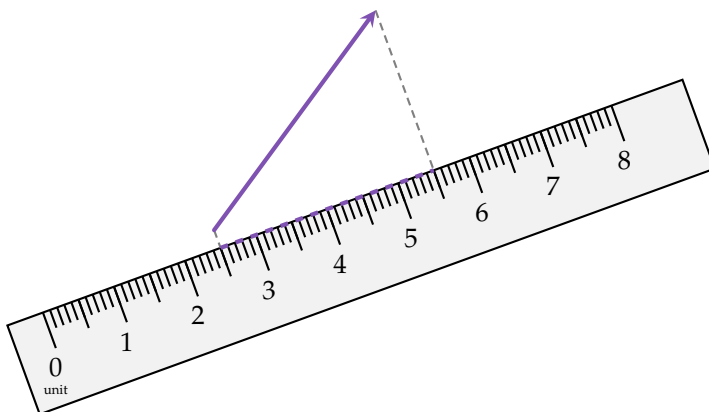

Figure 1.4: Projecting a vector onto a ruler in $\mathbb{R}^2$.

In linear algebra a projection is measured using the *inner product*, and that is exactly what we use to measure vectors using a given ruler: we simple calculate the inner product of the two quantities.

By the way, you probably already guessed that rulers are just another name for *dual vectors*. In this book I denote them using an asterisk: a^*, v^*, etc.

In any case, dual vectors are written as row vectors so they can be distinguished from vectors in actual calculations. That way, the inner product of a dual vector and a vector is consistent with the rules of matrix multiplication: a dual vector is interpreted as a matrix with dimensions $1 \times n$ (since it has one row and n columns), while a vector is interpreted as a matrix with dimensions $n \times 1$. Their product is therefore has dimension 1×1, neatly fitting the idea of the inner product generating scalars.

The most general form of an inner product between a dual vector and a vector in $\mathbb{R}^n$ looks as follows:

$$\langle \boldsymbol{\alpha}^*, \boldsymbol{v} \rangle = \begin{bmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.1}$$

$$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n. \tag{1.2}$$

This is in fact also the most generic for of a linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}$ acting on a vector in $\mathbb{R}^n$. Indeed, dual vectors are usually interpreted as *functionals*, taking vectors as arguments. In that interpretation, Equation (1.2) can be written as

$$\boldsymbol{\alpha}^* (\boldsymbol{v}) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, \tag{1.3}$$

and dual vectors are usually called *1-forms*.

1.2.2 Some formalism

To be written:

1. Dual vectors form a vector space V^*.

2. Formal definition of dual spaces.

3. Examples of dual vectors of functions?..

1.2.3 Basis sets and coordinate transformations

To be written:

1. Dual basis: converting from a basis set in V to its dual in V^*.

2. Covariance of dual vectors basis change vs. contra-variance of vectors.

1.3   Further Reading

2 Geometric Algebra

2.1 Preface

This is a temp text.

3 Abstract Algebra

3.1 Preface

This is a temp text.

4 Lie Groups and Algebras

4.1 Preface

This is a temp text.

Part II

Spinors