Part I

Background Topics

1 *Linear Algebra*

1.1 *Preface*

The goal of this chapter is not to teach you, the reader, linear algebra from scratch - nor to be a thorough source of information on the topic. Rather, my aim is to introduce important "advanced" concepts for those who took a basic linear algebra course as part of an undergraduate university program. These concpets should help you gain a basic knowledge of the topics needed for understanding the rest of the background material, as well as the topic of spinors itself.

My approach to teching topics in linear algebra - and in mathematics as a whole - is to first build an intuition and only then formalize and generalize the ideas as needed. In my personal experiences, when I was studying linear algebra I completely failed to understand it (and indeed, failed the course) until it "clicked" for me in regards to 2- and 3-dimensional real spaces, i.e. - visible geometry. After that I didn't even have to study for exams anymore, as everything became clear enough to grasp and develop on the spot even during an exam (except for later, more advances concepts). That is why, for example, I absolutely adore courses and study materials of the topic[1] which use animation, such as *3Blue1Brown* great video essay series Essence of linear algebra[2].

[1] And other mathematical topics as well.

[2] Temporary sidenote which should become a citation for the mentioned 3B1B video series

There are very few proofs in this chapter, and those that are shown are not completely rigorous. For more in-depth materials, see the last section (further read). With that out of the way - let's begin!

1.2 *Change of Coordinates*

In introductory linear algebra courses you should have learned about change of coordinate systems: a coordinate system is just another name for a basis set of whatever vector space is used (in this section it's $\mathbb{R}^n$). A change of coordinate system is the transformation of vectors from being represented in one basis set $B = \{e_1, e_2, \ldots, e_n\}$ to being represented in another basis set $\tilde{B} = \{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n\}$. Since such transformations are linear they are commonly represented in a matrix form.

In this section we will discuss *how* vectors and their components change under change of basis sets. There are many components involved in these kind of transformations, which causes them to be quite confusing. I will therefore color code the equations consistently as a visual guide. In addition, I will always introduce the $\mathbb{R}^2$ case first, before giving the generalized form for $\mathbb{R}^n$.

1.2.1 *Change of basis set in* $\mathbb{R}^2$

Suppose we use the standard basis set to represent $\mathbb{R}^2$:

$$B = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \tag{1.1}$$

and we want to change our coordinate system to use the following basis set:

$$\tilde{B} = \{\tilde{e}_1, \tilde{e}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \right\}. \tag{1.2}$$

(the two basis sets are shown in Figure 1.1)

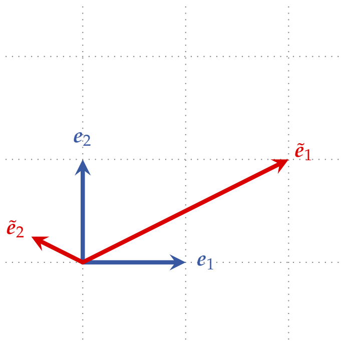

Figure 1.1: The standard basis set B and a new basis set $\tilde{B}$ shown together.

$$F = \begin{array}{c} \begin{array}{cc} \tilde{e}_1 & \tilde{e}_2 \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & \frac{1}{4} \end{bmatrix} \end{array}. \tag{1.3}$$

Now, if we want to transform $\tilde{e}_1$ and $\tilde{e}_2$ into $\tilde{e}_1$ and $\tilde{e}_2$ using F, we simply multiply them by F:

$$\begin{aligned} \tilde{e}_1 &= Fe_1, \\ \tilde{e}_2 &= Fe_2. \end{aligned} \tag{1.4}$$

To make **??** more concise, we can collect the two vectors into a matrix disguised as a row vecor:

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1, e_2 \end{bmatrix}. \tag{1.5}$$

We then get that **??** can be written in vector-matrix notation as

$$\tilde{e} = \begin{bmatrix} e_1, & e_2 \end{bmatrix} = \begin{bmatrix} e_1, & e_2 \end{bmatrix} \begin{bmatrix} 2 & -\frac{1}{2} \\ 1 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 2e_1 + e_2, & -\frac{1}{2}e_1 + \frac{1}{4}e_2 \end{bmatrix}. \tag{1.6}$$

The reverse transformation can be calculated by applying the inverse transformation $\boldsymbol{F^{-1}}$ on Equation 1.6:

$$\begin{bmatrix} \tilde{e}_1, & \tilde{e}_2 \end{bmatrix} \boldsymbol{F^{-1}} = \left(\begin{bmatrix} e_1, & e_2 \end{bmatrix} \boldsymbol{F} \right) \boldsymbol{F^{-1}} = \begin{bmatrix} e_1, & e_2 \end{bmatrix}. \tag{1.7}$$

(where in our case $\boldsymbol{F^{-1}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -1 & 2 \end{bmatrix}$)

To summarize: given a set B of basis vectors in $\mathbb{R}^2$, we can transform them into the basis vectors set $\tilde{B}$ by using the forward transformation $\boldsymbol{F} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$. To get back the original basis vectors from the transformed vectors we use the inverse transfomation $\boldsymbol{F^{-1}} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}^{-1}$. If the matrix *Forw* is invertible - i.e. if its two columns are not linearly depended (equivalently, if $F_{11}F_{22} \neq F_{12}F_{21}$), then $\boldsymbol{F^{-1}} = \frac{1}{F_{11}F_{22} - F_{12}F_{21}} \begin{bmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{bmatrix}$.

1.2.2 *The more general case, $\mathbb{R}^n$*

In $\mathbb{R}^n$ the transformations behave in a similar way: given the transformation rule that each new basis vector $\tilde{e}_i \in \tilde{B}$ is a linear combination of the old basis vector set B, i.e.

$$\begin{aligned} \tilde{e}_1 &= F_{11}e_1 + F_{12}e_2 + \cdots + F_{1n}e_n, \\ \tilde{e}_2 &= F_{21}e_1 + F_{22}e_2 + \cdots + F_{2n}e_n, \\ \vdots &= \vdots \\ \tilde{e}_n &= F_{n1}e_1 + F_{n2}e_2 + \cdots + F_{nn}e_n, \end{aligned}$$

we can write the transformation in matrix form as

$$\begin{bmatrix} \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n \end{bmatrix} = \begin{bmatrix} e_1, e_2, \dots, e_n \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1n} \\ F_{21} & F_{22} & \dots & F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1} & F_{n2} & \dots & F_{nn} \end{bmatrix}. \tag{1.8}$$

Per new basis vector $\tilde{e}_i$ Equation 1.8 has the form

$$\tilde{e}_j = \sum_{k=1}^{n} F_{kj} e_k. \tag{1.9}$$

Similarly, the inverse operation is given by

$$\begin{aligned}
e_1 &= F^{-1}_{11}\tilde{e}_1 + F^{-1}_{12}\tilde{e}_2 + \cdots + F^{-1}_{1n}\tilde{e}_n,\\
e_2 &= F^{-1}_{21}\tilde{e}_1 + F^{-1}_{22}\tilde{e}_2 + \cdots + F^{-1}_{2n}\tilde{e}_n,\\
\vdots &= \vdots\\
e_n &= F^{-1}_{n1}\tilde{e}_1 + F^{-1}_{n2}\tilde{e}_2 + \cdots + F^{-1}_{nn}\tilde{e}_n,
\end{aligned}$$

its matrix form is

$$\left[e_1, e_2, \ldots, e_n\right] = \left[\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n\right] \begin{bmatrix} F^{-1}_{11} & F^{-1}_{12} & \ldots & F^{-1}_{1n} \\ F^{-1}_{21} & F^{-1}_{22} & \ldots & F^{-1}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F^{-1}_{n1} & F^{-1}_{n2} & \ldots & F^{-1}_{nn} \end{bmatrix}, \tag{1.10}$$

and per basis vector the transformation is

$$e_i = \sum_{j=1}^{n} F^{-1}_{ji}\tilde{e}_j. \tag{1.11}$$

By subtituting Equation 1.9 into Equation 1.11, we get

$$\begin{aligned}
e_i &= \sum_{j=1}^{n} F^{-1}_{ji}\tilde{e}_j \qquad (1.12)\\
&= \sum_{j=1}^{n} F^{-1}_{ji}\left(\sum_{k=1}^{n} F_{kj}e_k\right)\\
&= \sum_{k=1}^{n} \underbrace{\left(\sum_{j=1}^{n} F_{kj}F^{-1}_{ji}\right)}_{\text{this is just a number!}} e_k.
\end{aligned}$$

?? just tells us something we already know: each basis vector e_i equals to a linear combination of the *same set* of basis vectors. This must mean that for $k = i$ the number in paranthesis is one, and for any other value of k it is zero - i.e. it equals δ_{ik}:

$$e_i = \sum_{k=1}^{n} \delta_{ik}e_k. \tag{1.13}$$

In turn, **??** means that the matrix $\boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I}_n$, the identity matrix in $\mathbb{R}^n$ - and thus the matrices $\boldsymbol{F}$ and $\boldsymbol{F}^{-1}$ are eachother's inverses.

1.3 Further Reading

2 *Geometric Algebra*

2.1 Preface

This is a temp text.

3 *Abstract Algebra*

3.1 Preface

This is a temp text.

4 *Lie Groups and Algebras*

4.1 *Preface*

This is a temp text.

Part II

Spinors