Part I

Background Topics

1 Linear Algebra

1.1 Preface

The goal of this chapter is not to teach you, the reader, linear algebra from scratch - nor to be a thorough source of information on the topic. Rather, my aim is to introduce important "advanced" concepts for those who took a basic linear algebra course as part of an undergraduate university program. These concpets should help you gain a basic knowledge of the topics needed for understanding the rest of the background material, as well as the topic of spinors itself.

My approach to teching topics in linear algebra - and in mathematics as a whole - is to first build an intuition and only then formalize and generalize the ideas as needed. In my personal experiences, when I was studying linear algebra I completely failed to understand it (and indeed, failed the course) until it "clicked" for me in regards to 2- and 3-dimensional real spaces, i.e. - visible geometry. After that I didn't even have to study for exams anymore, as everything became clear enough to grasp and develop on the spot even during an exam (except for later, more advances concepts). That is why, for example, I absolutely adore courses and study materials of the topic[1] which use animation, such as *3Blue1Brown* great video essay series Essence of linear algebra[2].

[1] And other mathematical topics as well.

[2] Temporary sidenote which should become a citation for the mentioned 3B1B video series

There are very few proofs in this chapter, and those that are shown are not completely rigorous. For more in-depth materials, see the last section (further read). With that out of the way - let's begin!

1.2 Dual Vectors and Dual Spaces

An important aspect of vector spaces, which is sometimes waved away, is the question of *measurement*: how do we give vectors a sense of magnitude? By *magnitude* I mean a single real number we assign to each vector. Well, there's one obvious way: the norm. Normally[3] it is either given as is (due to the Pythagorean theorem), or in the case of more abstract vector spaces formalized as the square root of the inner product of a vector with itself, i.e.

$$\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}. \tag{1.1}$$

[3] pun kind-of intended.

Of course, other so-called "p-norms" are possible and often used:

$$\|\boldsymbol{v}\|_p = \left(|v_1|^p + |v_2|^p + \cdots + |v_n|^p \right)^{\frac{1}{p}}, \tag{1.2}$$

where $p = 2$ is the normal *Euclidean* norm we're used to.

However, with the exception of the case $p = 1$, these norms are non-linear. And if there's one insight that should be very clear to anyone who went through some university-level mathematics, it is that linear structures are so much easier to deal with than almost anything else[4].

[4] that's why linear approximations are so often used all throughout science

So instead of using the norm as a measurement, we would ideally like to use some linear function to measure our vectors. For example, let's try to come up with a linear way to measure vectors in $\mathbb{R}^3$: given a vector $\boldsymbol{v}$, we can derive a linear map $\phi : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$\phi\left(\boldsymbol{v}\right) = 3v_1 - v_2 + 5v_3. \tag{1.3}$$

Let's apply ϕ to some vectors and see the results:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to 3 \cdot 1 - 1 \cdot 0 + 5 \cdot 0 = 3.$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to 3 \cdot 0 - 1 \cdot 1 + 5 \cdot 0 = -1.$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to 3 \cdot 0 - 1 \cdot 0 + 5 \cdot 1 = 5.$$

$$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \to 3 \cdot 1 - 1 \cdot (-2) + 5 \cdot (-1) = 3 + 2 - 5 = 0.$$

We see that using this specific ϕ the standard basis vectors are "measured" to be of different values, and some vectors like $v = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ can measured to be 0.

A bit of thinking further shows that in fact, *any* linear map of the form $\phi : \mathbb{R}^3 \to \mathbb{R}$ we can apply to the vectors of $\mathbb{R}^3$ has to be in the following form:

$$\phi(v) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \tag{1.4}$$

where $\alpha_1, \alpha_2, \alpha_3$ are some real numbers. But this is exactly the definition of the inner product between $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ and v! So we see that *any* linear measurement on $\mathbb{R}^3$ will take the form ofan inner product with some given vector.

In other words, any vector in $\mathbb{R}^3$ can be used to linearly measure all vectors in $\mathbb{R}^3$ via the inner product, and in this context act as a *functional* on elements of the space. Moreover, any possible linear measurement of vectors in $\mathbb{R}^3$ would be of this form - and thus the space of linear maps of the form $\mathbb{R}^3 \to \mathbb{R}$ is itself a vector space with the same structure as $\mathbb{R}^3$[5] and in this context we call it the *dual space* of $\mathbb{R}^3$, and its elements *dual vectors*.

[5] In fact, *it is* $\mathbb{R}^3$ in any sensible meaning.

Now, of course choosing $\mathbb{R}^3$ was pretty arbitrary, so let us generalize somewhat the defintions:

Definition 1.1 Text

The dual space of $\mathbb{R}^n$ is the space of all linear maps of the form

$$\phi : \mathbb{R}^n \to \mathbb{R},$$

which can be represented as the inner product between the elements of $\mathbb{R}^n$.
We call the elements of the dual space *dual vectors*, and denote them with an asterisk: v^*.

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To distinguish between vectors and dual vectors in practice, we represent dual vectors as *row-vectors* instead of the usual *column-vector* for vectors. For example, a dual vector from $\mathbb{R}^3$ will have the form

$$\boldsymbol{a}^* = \begin{bmatrix} a_1, & a_2, & a_3 \end{bmatrix}. \tag{1.5}$$

This way, the action of dual vectors on vectors (as functionals) is clearly seen via writing the inner product: given the dual vector $\boldsymbol{a}^* \in \mathbb{R}^n$ and the vector $v \in \mathbb{R}^n$, their inner product can be written explicitely as

$$\langle \boldsymbol{a}^*, \boldsymbol{v} \rangle = \begin{bmatrix} a_1, & a_2, & \ldots, & a_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n. \tag{1.6}$$

This corresponds well to a product between two matrices, one of dimensions $1 \times n$ and the other of dimensions $n \times 1$ - which will become handy when we discuss *multi-linear forms*[6].

[6] yes, these things get more complicated...

Note 1.1 Product order

Using this formulation we must always multiply the dual vector and the vector such that the dual vector is *on the left*. Otherwise, we should expect to get a matrix of dimensions $n \times n$.

!

A way to visualize dual vectors is using *stacks*: since dual vectors have the same representation as "usual" vectors, they define a direction in a geometrical n-dimensional space. Since each direction in $\mathbb{R}^n$ has a single corresponding $n-1$-dimensional hyperplane, to differentiate vectors and dual vectors we visualize dual vectors using the hyperplanes.

This is a bit abstract, so let us look at our two favourite examples: $\mathbb{R}^2$ and $\mathbb{R}^3$. In $\mathbb{R}^2$ the $n-1$-dimensional hyperplanes are simply lines (since $2-1=1$), which "stack" on top of each other in the direction of the dual vector they represent. See Figure 1.1 for a graphical representation.

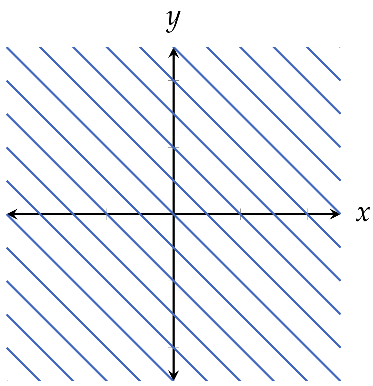

Figure 1.1: This is a test

1.3 The Braket Notation and Einstein's Summation

1.4 Quaternions and Octanions

1.5 Further Reading

2 *Geometric Algebra*

2.1 *Preface*

This is a temp text.

3 *Abstract Algebra*

3.1 *Preface*

This is a temp text.

4 Lie Groups and Algebras

4.1 Preface

This is a temp text.

Part II

Spinors