Part I

Background Topics

1 *Linear Algebra*

1.1 *Preface*

THE GOAL OF THIS CHAPTER is not to teach you, the reader, linear algebra from scratch - nor to be a thorough source of information on the topic. Rather, my aim is to overview the topic in such a way that new-comers, as well as those who studied linear algebra in an undergraduate university course, will gain the important insights of the topic needed for understanding the rest of the background material and spinors as well.

Instead of teaching the topic from the ground-up, like mathematicians tend to do[1], I prefer to stick to the geometric interpretation of the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ (and to a lesser extent $\mathbb{R}^n$ in general). These interpretations can be visualized relatively easily, and thus help in setting up the needed intuition in the student's mind, which becomes handy when the topic turns to more abstract constructs (such as for example vector spaces of matrices or functions).

[1] In my view, courses that "build" linear algebra step-by-step give the students good knowledge of the structures of vector spaces, but tend to miss the intuitive view of what these structures can *do*. This is exactly the difference between the "pure" mathematics of the mathematician and the mathematics as a tool of the scientist.

In my personal experiences, when I was studying the topic I completely failed to understand it (and indeed, failed the courses I took) until it "clicked" for me in regards to 2- and 3-dimensional real spaces, i.e. - visible geometry. Then I didn't even have to study for exams anymore, as everything became clear enough to grasp and develop on the spot even during an exam (except for later, more advances concepts). That is why, for example, I absolutely adore courses and study materials of the topic[2] which use animation, such as *3Blue1Brown* great video essay series Essence of linear algebra[3].

[2] And other mathematical topics as well.

[3] Temporary sidenote which should become a citation for the mentioned 3B1B video series

There are very few proofs in this chapter, and those that are shown are not completely rigorous. For more in-depth materials, see the last section (further read). With that out of the way - let's begin!

1.2 *Vectors and Vector Spaces*

Vectors are the most basic and important element in linear algebra. There are several different approaches to defining a vector: physicists like to talk about vectors as objects with magnitude and direction. Computer scientists and programmers tend to view vectors as lists of numbers (and sometimes other types of objects as well). To the mathematician, vectors are elements of *vector sets*, which are defined rigorously and precisely.

The existence of different definitions leads to much confusion[4]: for example, we usually think of matrices as *acting* on vectors, so how can matrices be vectors themselves? Also - are vectors just list of numbers, or are there some such lists that aren't vectors? Can we always reduce any vector to a list of numbers? And what about the case of functions as vectors? Etc., etc.

[4] I admit that this is a classic case of "citation needed", but it is something I come across often.

Therefore, for now I choose to limit the definition of vectors to the so-called "physicist's definition": a vector is an object which has both a magnitude (also *length* and *norm*), and a direction. These can be easily visualized in 2- and 3-dimensional real spaces as arrows. We will later see how this translates into lists of numbers (and later still how we can define more abstract and inclusive vectors). Since vectors don't have positions, we can freely move them around in space, and normally present them as originating from the same point in space (Figure 1.1).

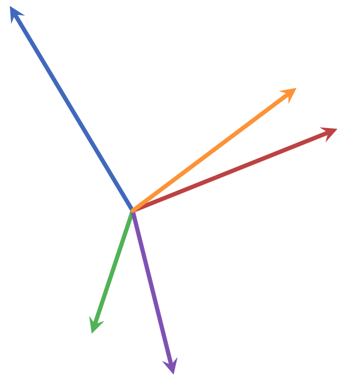

Figure 1.1: Some vectors placed in 2-dimensional space such that they all originate from the same point.

Let's now overview the basic operations that can be done with vectors. One operation is *scaling* by a real number: scaling a vector means that we're changing the norm of the vector without changing its direction (Figure 1.2). Note that by direction we mean the line going through the vector's origin and its head: when we scale a vector by a negative number $\alpha \in \mathbb{R}$ we flip the vector's orientation and scale its norm by the absolute value of α - the vector is then considered to stay in the same direction in space (Figure 1.2, again). Generally, the norm of a vector $\boldsymbol{u}$ is denoted using double vertical lines: $\|\boldsymbol{u}\|$.

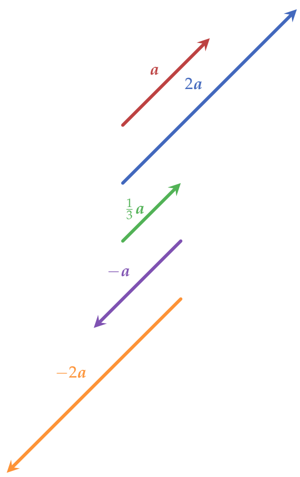

Figure 1.2: Some vectors placed in the origin of a 2-dimensional Cartesian coordinate system.

A vector which has norm of 1 is called a *unit vector*. We will denote unit vectors in the book using the regular bold notation with a "hat" on top of it: $\hat{\boldsymbol{a}}$. Any vector can be made into a unit vector (be *normalized*) by scaling the vector by the reciprocal of its norm (if it isn't zero): given a vector $\boldsymbol{u}$, we can scale it by $\|\boldsymbol{u}\|^{-1}$ and get the unit vector $\hat{\boldsymbol{u}}$, i.e.

$$\hat{\boldsymbol{u}} = \frac{1}{\|\boldsymbol{u}\|}\boldsymbol{u}. \tag{1.1}$$

Vectors can also be added together using the *parallelogram rule*: given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we first place $\boldsymbol{b}$ such that its origin lies on the head of $\boldsymbol{a}$. The vector $\boldsymbol{c}$ from the origin of $\boldsymbol{a}$ to the head of $\boldsymbol{b}$ is the result of the addition.

The addition rule of vectors is called the parallelogram rule because if we were to perdorm the addition of $\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{b}+\boldsymbol{a}$ we would get a parallelogram (Figure 1.3). This aslo indicates that vector addition is *commutative*:

$$\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}. \tag{1.2}$$

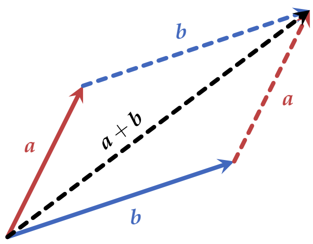

Figure 1.3: No matter in which order we perform the addition of two vectors, the result is the same. This is due to the parallelogram rule of vector addition.

1.3 *Linear Transformations*

1.4 *Matrices*

1.5 *Eigenvectors and Eigenvalues*

1.6 *Quaternions and Octanions*

1.7 *Some Formalism and General Vector Spaces*

Up until now in this chapter we only looked at vectors in $\mathbb{R}^n$ (and usually with $n \in \{2,3\}$). However, the ideas presented can be generalized to all kind of vector spaces. Yes - it is finally time to present the "mathematician's view of vectors"!

To do so, let us first overview some fundamental (trivial, even) properties of $\mathbb{R}^n$:

- **Closure of vector addition**: the sum of any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is also a vector in $\mathbb{R}^n$ - i.e.

$$\text{if } \boldsymbol{u}+\boldsymbol{v}=\boldsymbol{w},\ \text{then } \boldsymbol{w}\in\mathbb{R}^n. \tag{1.3}$$

- **Commutativity of vector addition**: resulting from the parallelogram rule, the addition of vectors is commutative - i.e.

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}. \tag{1.4}$$

- **Associativity of vector addition**: the order of adding multiple vectors does not matter: for any three vectors $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}$ in $\mathbb{R}^n$,

$$\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}. \tag{1.5}$$

- **Existence of zero**: the zero vector $\mathbf{0}$ is a neutral to addition - i.e.

$$\forall \boldsymbol{u}\in\mathbb{R}^n:\ \boldsymbol{u}+\mathbf{0}=\mathbf{0}+\boldsymbol{u}=\boldsymbol{u}. \tag{1.6}$$

- **Existence additive inverse**: for any vector $\mathbf{v} \in \mathbb{R}^n$ there's an inverse - i.e

$$\forall \mathbf{v} \in \mathbb{R}^n : \exists \left(-\mathbf{v}\right), \mathbf{v} + \left(-\mathbf{v}\right) = \mathbf{0}. \tag{1.7}$$

- **Closure of scalar multiplication**: the result of scaling by $\lambda \in \mathbb{R}$ of any vector $\mathbf{v} \in \mathbb{R}^n$ is also in $\mathbb{R}^n$ - i.e.

$$\forall \mathbf{v} \in \mathbb{R}^n \text{and } \forall \lambda \in \mathbb{R} : \lambda \mathbf{v} \in \mathbb{R}^n. \tag{1.8}$$

- **Associativity of scalar multiplication**: for any two scalars $\lambda, \mu \in \mathbb{R}$, the order of scaling a vector $\mathbf{v} \in \mathbb{R}^n$ doesn't matter - i.e.

$$\left(\lambda \mathbf{v}\right) \cdot \mu = \lambda \cdot \left(\mu \mathbf{v}\right). \tag{1.9}$$

- **Existnce of unity**: the number 1 is neutral with scaling - i.e.

$$\forall \mathbf{v} \in \mathbb{R}^n : 1\mathbf{v} = \mathbf{v}. \tag{1.10}$$

- **Distributive laws**: vector addition and scaling are distributive together with addition of scalars, i.e. for any $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$:

$$\lambda \left(\mathbf{v} + \mathbf{u}\right) = \lambda \mathbf{v} + \lambda \mathbf{u}, \text{ and} \tag{1.11}$$
$$\left(\lambda + \mu\right) \mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}. \tag{1.12}$$

There are many other mathematical objects that have the same properties, for example matrices of the same dimensions, with the usual matrix addition and scalar multiplication operations: in this case, the zero matrix is the neutral element to addition and the scalar 1 is the unity. Another example using functions can be seen in (EXAMPLE REF).

We use the properties we described to define a general idea of a vector space. First, we choose a *field* $\mathbb{F}$: a set whose elements will be our scalars. Usually the field is either $\mathbb{R}$ or $\mathbb{C}$, but there are many other options which we will ignore for now. Then, we choose a set V and define two operations: one operation between two elements of V (which we call *addition*) and one between elements of V and elements of the field $\mathbb{F}$ (which we call scalar multiplication).

Now, if we defined all of the above (field + set + two operations) **and** they behave according to the properties we described for $\mathbb{R}^n$ (closure, commutativity, associativity, etc.) **then** we say that the set V is a vector set *over* $\mathbb{F}$ with the two operations.

Example 1.1 Trivial vector space

What is the smallest vector space we can construct using a subset of $\mathbb{R}$ as a field? If we choose $V = \mathbb{F} = \emptyset$ and the usual addition and multiplication between, say, real numbers - this is a valid vector space (check for youself!). However,

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it's a very boring one. Let's try to build a small vector space with more than one element in V: ...

1.8 Dual Vectors and Dual Spaces

1.9 The Braket Notation and Einstein's Summation

1.10 Further Reading

2 *Geometric Algebra*

2.1 *Preface*

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3 *Abstract Algebra*

3.1 *Preface*

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4 Lie Groups and Algebras

4.1 Preface

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Part II

Spinors