KU LEUVEN

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Morse Theory

A visual guide from handlebodies to the generalized Poincaré conjecture

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Preface

This thesis is on Morse theory, the study of nice real valued functions on manifolds, called Morse functions. While elementary, they provide great insight in the structure of manifolds, eventually leading to some of the most important theorems in differential topology. The goal of this thesis is to prove one of them, namely the generalized Poincaré conjecture in higher dimensions, stating that a homotopy sphere is a topological sphere.

This thesis could not have been written without the help of many people.

First and foremost, I would like to thank Dr. Charlotte Kirchhoff-Lukat for proposing this subject and supervising me throughout my journey. Charlotte, thank you for exposing me to the world of Morse theory and its many related topics. Thank you for your advice, patience, and the weekly chats which kept me motivated through the year.

I would like to thank Prof. Joeri Van der Veken and Dr. François Thilmany for reading and evaluating this thesis.

I would like to thank my family for their support, giving me the opportunity to make this possible.

Lastly, Marie, thank you for the whiteboard in front of which we have spent many hours together. It was essential for finishing this thesis. Thank you for listening to my never ending rambles on Morse theory, proofreading my thesis, being the test subject for my presentations and for the much needed distraction.

Gilles, June 2021

Summary

In 1925, Marston Morse wrote his seminal paper titled 'Relations between the critical points of a real function of n independent variables,'[1] founding the field that is nowadays known as Morse theory. In his paper, he studies smooth functions $f : M \to \mathbb{R}$ that satisfy a non-degeneracy condition. The core idea is to examine how the topology of $f^{-1}(t)$ changes when changing t, in order to infer something about the manifold M itself.

Over time, the ideas of Morse have been proven very useful. For example, handlebody decompositions and Heegaard splittings form an indispensable tool used in low-dimensional topology. Morse functions also give rise to a homology theory that has inspired many others, including Floer homology. Further developments of Morse theory led to Smale's proof of the h-cobordism theorem and the generalized Poincaré conjecture in higher dimensions.[2]

The aim of this thesis is twofold. Firstly, we introduce Morse functions and Morse homology and prove that the latter is isomorphic to singular homology. Secondly, we prove the generalized Poincaré conjecture in higher dimensions, following the ideas of Smale. By including more than a hundred figures and providing many details, we aim to make this subject accessible to all.

In **Chapter 1**, we talk about the basics of Morse theory. We give several equivalent definitions of a Morse function and give some examples. We also show that Morse functions give rise to so-called handle decompositions: there is a one-to-one correspondence between critical points of a Morse function and so-called handles, building blocks from which any manifold can be built. We give some very concrete examples of handle decompositions in dimensions one, two and three. We end the chapter by showing that any manifold admits (infinitely many) Morse functions. Even more, we show that they are generic and stable, meaning that any function can be approximated by a Morse function and if we perturb a Morse function, it stays Morse.

Chapter 2 concerns the concept of stable and unstable manifolds. They form a way of understanding interaction of handles. For example, two handles are 'independent' if the intersection of the associated stable and unstable manifolds is empty, allowing us to reorder them, as we prove in Theorem 2.15. This idea leads to the existence of self-indexing Morse functions, asserting that we can always build a manifold by first attaching 0-handles, then 1-handles, 2-handles, et cetera, in that order. We end the chapter by proving a first cancellation theorem, stating that under certain circumstances, we can cancel pairs of critical points.

[1] Marston Morse. "Relations between the critical points of a real function of n independent variables". In: *Transactions of the American Mathematical Society* 27.3 (1925), pp. 345–396

[2] Stephen Smale et al. "The generalized Poincaré conjecture in higher dimensions". In: *Bulletin of the American Mathematical Society* 66.5 (1960), pp. 373–375

In **Chapter 3**, we introduce Morse homology. We first define the Morse complex $C_k(f)$ and the Morse differential ∂_X, which is based on counting the number of trajectories along the (pseudo-)gradient vector field X between critical points of a Morse function f. Morse homology $HM(C_\bullet(f), \partial_X)$ is the homology associated to this complex. As an illustration, we compute some examples in two and three dimensions. The rest of the chapter covers three important theorems in Morse theory. We prove that the Morse complex is actually a complex ($\partial_X^2 = 0$), that Morse homology is independent of the Morse function and gradient, and lastly that Morse homology is actually isomorphic to singular homology. The proofs of these theorems are very geometrical in nature and their ideas have inspired many other theories.

Chapter 4 discusses some applications of Morse homology. While we now know that it is isomorphic to singular homology, and hence it enjoys all the properties of singular homology, it still can be illuminating to derive these facts directly from Morse homology. For example, we can prove Poincaré duality by simply changing the Morse function $f \rightsquigarrow -f$, i.e. turning the manifold upside down. This has the effect of k-handles becoming $n-k$-handles and flow lines reversing direction, from which the desired result follows rapidly. We also discuss the Morse inequalities, giving a lower bound for the number of critical points of a Morse function in terms of the singular homology of M:

$$\# \operatorname{Crit}_k f \geq \operatorname{rank} H_k(M; \mathbb{Z}).$$

We end the chapter by proving some related facts including a stronger version of the Morse inequalities.

In **Chapter 5**, we prove that under certain conditions, the Morse inequalities can be attained by some Morse function. In other words, there always exists a Morse function such that $\# \operatorname{Crit}_k f = \operatorname{rank} H_k(M; \mathbb{Z})$. We prove this by considering an arbitrary Morse function, and then cancelling pairs of critical points until the Morse inequalities have been reached. To this purpose, this chapter mostly contains stronger cancellation results. In Section 5.5, all these cancellation theorems come together and we prove the minimality of the Morse inequalities. This has as an immediate corollary two of the most important theorems in differential topology: the h-cobordism theorem and the generalized Poincaré conjecture in dimension $n \geq 5$, stating that a homotopy sphere is a topological sphere. To end this thesis, we also discuss some of the historical aspects of these theorems.

List of symbols

$M \pitchfork N$	Transverse intersection
$\langle \cdot, \cdot \rangle$	Riemannian metric on a manifold or, Inner product on space of critical points $\langle c, d \rangle = \delta_{cd}$
$N \cdot N'$	Intersection number of two manifolds
A	A $\mathbb{Z}$-module, i.e. an abelian group
B^n	Closed disk of dimension n
$C_k(f, \mathbb{Z})$	Free module over $\mathbb{Z}$ generated by index k critical points of f, i.e. the space of formal sums of index k critical points
$C_k(f, \mathbb{Z}_2)$	Vector space over $\mathbb{Z}_2$ generated by the index k critical points of the Morse function f
$\operatorname{codim} N$	Codimension of N
$\dim N$	Dimension of N
$\operatorname{Crit}_k f$	Critical points of f of index k
$\operatorname{Crit} f$	Critical points of f
$C^\infty(M, N)$	Smooth maps from M to N
$\partial_{X,k}$	Morse differential associated to pseudo-gradient X
$[\partial_k]$	Matrix of the Morse differential $\partial_k : C_k \to C_{k-1}$
D^n	Open disk of dimension n
$\operatorname{grad} f$	Gradient of f, i.e. $(df)^\sharp$
$HM(M; \mathbb{Z}_2)$	Morse homology of a manifold M with coefficients in $\mathbb{Z}_2$
$HM(M; \mathbb{Z})$	Morse homology of a manifold M with coefficients in $\mathbb{Z}$
$HM(C_\bullet(f), \partial_X)$	Morse homology of Morse function f and pseudo-gradient X
$H_k(M, N)$	Singular homology of M relative to N
$H_k(M; \mathbb{Z})$	Singular homology of M over $\mathbb{Z}$
$H_k(M; \mathbb{Z}_2)$	Singular homology M over $\mathbb{Z}_2$
$\operatorname{Ind} a$	Index of critical point a

$\mathcal{L}(p,q)$	Moduli space of unbroken trajectories between p and q, i.e. $\mathcal{M}(p,q)/\mathbb{R}$, where $\mathbb{R}$ acts by time translations
$\overline{\mathcal{L}}(p,q)$	Space of broken and unbroken trajectories between p and q, the compactification of $\mathcal{L}(p,q)$
M	A smooth manifold
$\mathcal{M}(p,q)$	Set of all points on trajectories following a pseudo-gradient from p to q, $W^u(p) \pitchfork W^s(q)$
$N_X(p,q)$	Signed number of trajectories of X connecting p to q
$n_X(p,q)$	Number of trajectories of X connecting p to q
$\pi_k(M)$	Homotopy group of a manifold
$r_0(A)$	Free rank of a $\mathbb{Z}$-module, i.e. $\dim_{\mathbb{Q}} A \otimes \mathbb{Q}$
$r_p(A)$	p-torsion rank of a $\mathbb{Z}$-module, i.e. cardinality of a maximal set of independent elements of order p^k for some k
$r_t(A)$	Total torsion rank of a $\mathbb{Z}$-module, i.e. $\sum r_t$
$r(A)$	Total rank of a $\mathbb{Z}$-module, i.e. $r_t(A) + r_0(A)$
$S^s(p)$	Stable sphere associated to a critical point p, alternatively called the belt sphere
$S^u(p)$	Unstable sphere associated to a critical point p, alternatively called the attachment sphere
S^n	Sphere of dimension n
$W^s(p)$	Stable manifold of a critical point p
$W^u(p)$	Unstable manifold of a critical point p
$\overline{W}^u(p)$	Compactification of the unstable manifold associated to a critical point p
X	Pseudo-gradient vector field

Contents

CHAPTER ZERO

Preliminaries

In this thesis, we assume the reader is familiar with basic concepts of differential geometry such as smooth manifolds, vector fields, flows, bundles, differential forms, et cetera. In this chapter, we discuss some concepts that might be unfamiliar.

We will first focus on two concepts from differential geometry, namely transversality and intersection numbers. The second section is on algebraic topology, discussing homology, homotopy and their relation. We will use these concepts from Chapter 3 and onwards.

Differential Geometry

Transversality

The first concept we want to introduce is transversality, introduced by Thom in 1954.[3] For a more in depth overview, we refer the reader to 'Differential Manifolds' by Kosinski.[4]

[3] René Thom. "Quelques propriétés globales des variétés différentiables". In: *Commentarii Mathematici Helvetici* 28.1 (1954), pp. 17–86

[4] Antoni A Kosinski. *Differential manifolds*. Courier Corporation, 1993, Chapter IV

> **Definition 0.1** (Transversality)**.** Let M be a manifold and N_1, N_2 be two submanifolds. Then N_1 intersects N_2 *transversely* if and only if for all points of intersection $p \in N_1 \cap N_2$, we have $T_pM = T_pN_1 + T_pN_2$. We denote this by $N_1 \pitchfork N_2$.[5]

[5] The notation is suggestive: the line $\cap$ intersects $|$ transversely $\pitchfork$.

Note in particular that this depends on the ambient manifold M, and that if N_1 and N_2 do not intersect, their intersection is vacuously transverse. We give some examples below.

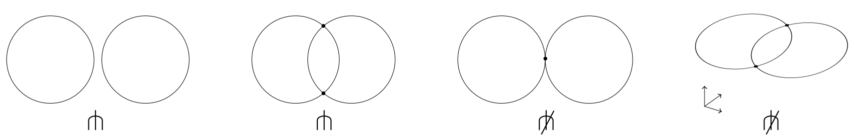


Figure 1: Examples and non-examples of transverse intersections. Multiple configurations of two circles are shown: thrice with ambient manifold $\mathbb{R}^2$, and once embedded in $\mathbb{R}^3$.

If the intersection of two manifolds N_1 and N_2 is transverse, it is more well-behaved than in the general case. For example the transverse intersection

$N_1 \pitchfork N_2$ is again a manifold, which is not true in general, as Figure 2 illustrates. Moreover, we have the following:

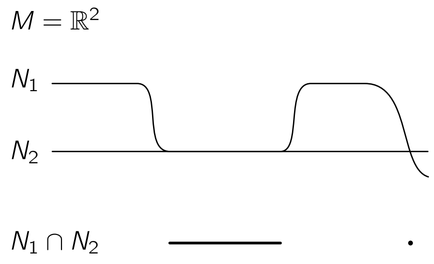


Figure 2: Let $M = \mathbb{R}^2$ and let N_1 and N_2 be submanifolds as in the figure. Then N_1 and N_2 do not intersect transversely and their intersection is not a manifold: it is the union of a point and an interval.

Proposition 0.2. Let M be a manifold and N_1, N_2 be two submanifolds. If N_1 intersects N_2 transversely, then $N_1 \pitchfork N_2$ is again a manifold and

$$\operatorname{codim}(N_1 \pitchfork N_2) = \operatorname{codim} N_1 + \operatorname{codim} N_2,$$

where the codimensions are to be taken with respect to M, by which we mean $\operatorname{codim} N_i = \dim M - \dim N_i$. Moreover, $T(N_1 \pitchfork N_2) = TN_1 \pitchfork TN_2$.

Proof. As this is a local statement, we prove it for $M = \mathbb{R}^m$. We may assume that $N_1 = f_1^{-1}(0)$ and $N_2 = f_2^{-1}(0)$ where f_i are submersions from $\mathbb{R}^m$ to $\mathbb{R}^{n_i}$, where n_i is the codimension of N_i. Then we can also consider $F = (f_1, f_2) : \mathbb{R}^m \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Notice that $N_1 \cap N_2 = F^{-1}(0)$. Then

$$\begin{aligned}
\dim N_1 \cap N_2 &= \dim \operatorname{Ker} d_0 F \\
&= \dim(\operatorname{Ker} d_0 f_1 \cap \operatorname{Ker} d_0 f_2) \\
&= \dim \operatorname{Ker} d_0 f_1 + \dim \operatorname{Ker} d_0 f_2 - \dim(\operatorname{Ker} d_0 f_1 + \operatorname{Ker} d_0 f_2) \\
&= (m - n_1) + (m - n_2) - m \\
&= m - (n_1 + n_2).
\end{aligned}$$

This also shows that $T_p(N_1 \pitchfork N_2) = T_p N_1 \pitchfork T_p N_2$ and that $N_1 \pitchfork N_2$ is a manifold, which is not always the case when the intersection is not transverse. □

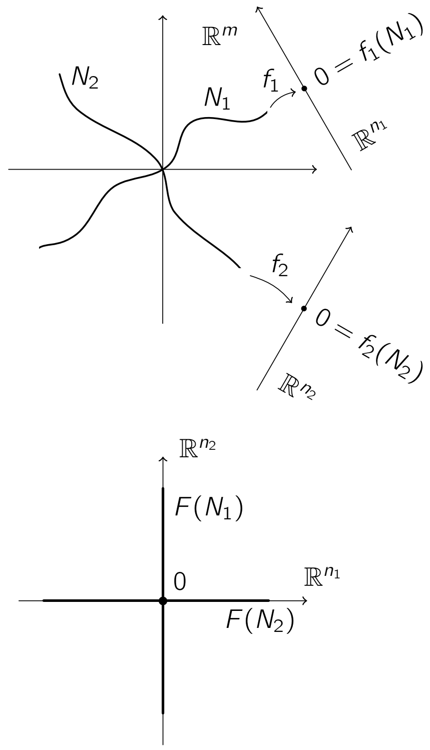


Figure 3: When $N_1 \pitchfork N_2$, the codimension of the intersection is the sum of their codimensions. By using the implicit function theorem, we straighten the situation.

An interesting property of transversality is that it is both *generic* and *stable*. By generic we mean that any two manifolds intersecting can be made to intersect transversely by perturbing one of the manifolds. Stability on the other hand means that when we perturb a transverse intersection, it stays transverse. At the core of the proof of stability and genericness of transversality lies the theorem of Sard:

Theorem 0.3 (Sard's theorem)**.** Let $f : M \to N$ be a C^∞ map. Then the set of critical values of f has measure zero in N. The set of regular values of f is open and dense.

For a proof, see 'Smooth manifolds and their applications in homotopy theory' by Pontrjagin.[6] Note that it is the set of critical *values* that has measure zero, not the set of critical points. Stability and genericness correspond to the set of regular values being open and dense respectively. We will use this theorem for showing that Morse functions exist, are abundant, and can be used to approximate any other function.

[6] Lev Semenovič Pontrjagin. "Smooth manifolds and their applications in homotopy theory". In: *Topological Library: Part 1: Cobordisms and Their Applications*. World Scientific, 2007, pp. 1–130

Intersection number

The second concept we want to introduce is the intersection number of two manifolds. We again follow 'Differential Manifolds' by Kosinski. The idea is to count the number of intersection points with signs:

Definition 0.4 (Intersection number)**.** Let N and N' be r- and s-dimensional submanifolds of an $(r+s)$-dimensional manifold M. Suppose N is oriented, N' is co-oriented and their intersection is transverse.

Let $p \in N \pitchfork N'$. Then T_pN is oriented both by the orientation of N and the co-orientation of N'. If these orientations match, we define the intersection number at p to be $+1$, otherwise we define it to be -1.

The intersection number $N' \cdot N$ is then defined as the sum of all the intersection numbers at the points in $N \pitchfork N'$.

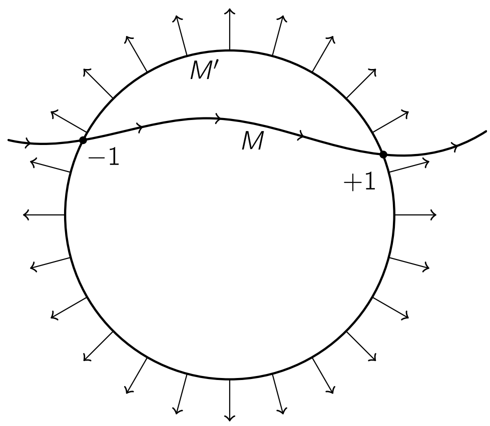


Figure 4: The intersection number is defined by comparing the orientation of N with the co-orientation of N' at the points of transverse intersection. In this case, the ambient manifold $M = \mathbb{R}^2$, $N = (0,1)$ and $N' = S^1$ and $N' \cdot N = -1 + 1 = 0$.

Remark 0.5. If we change the order, the intersection number might change sign: $N \cdot N' = \pm N' \cdot N$.

By counting the number of points in the intersection with signs, we have managed to associate a number to an intersection that is invariant under ambient isotopies, which is not true if we simply count the number of points in the intersection.

Proposition 0.6. The intersection number $N \cdot N'$ does not change under ambient isotopies of N or N'. In particular, we can define the intersection number of two manifolds that do not intersect transversely.

Example 0.7. Consider the intersection of $M = S^1$ and an interval embedded in $\mathbb{R}^2$. Consider the orientation of M and co-orientation of M as defined in Figure 4. Then $M' \cdot M = -1 + 1 = 0$. It is also clear that the intersection number does not change under isotopies of the manifolds, even though the number of intersection points does.

We will use the concept of intersection numbers in the last chapter where we prove the generalized higher dimensional Poincaré conjecture.

Algebraic topology

Roughly stated, algebraic topology is the study of functors from the category of topological spaces to the category of groups. In other words, we seek ways to associate groups (and often objects with even more structure such as R-modules) to topological spaces in such a way that homeomorphisms give rise to homomorphisms. In this section, we will discuss two ways to do so, resulting in homology and homotopy groups.

Homology theory

In the context of topology, homology was constructed as a way to detect holes in topological spaces. However, the methods and ideas generalize to other settings as well, and we can for example define homology of groups.

In its most general form, homology measures non-exactness of a chain complex. The following definitions can be found in any textbook on algebraic topology or homological algebra.

Definition 0.8 (Chain complex)**.** Let R be a commutative ring. A chain complex of R-modules is a sequence $C_\bullet$ of the form

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \cdots$$

where each term C_i is an R-module and $d_i : C_i \to C_{i-1}$ is an R-module homomorphism such that $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$.

We often suppress the indices of the maps d_i and the last condition then becomes $d^2 = 0$.

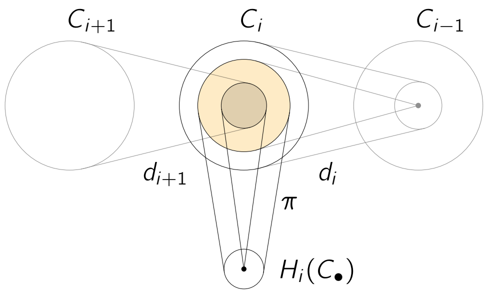


Figure 5: Homology measure exactness of a chain complex.

Definition 0.9 (Homology)**.** Let $C_\bullet$ be a chain complex of R-modules. The i-th homology of $C_\bullet$ is

$$H_i(C_\bullet) = \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i+1}}.$$

This is well defined because $\operatorname{Im}(d_{i+1}) \subset \operatorname{Ker}(d_i)$.

It is clear that this measures the exactness of the sequence. Indeed, if the sequence is exact, i.e. $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$, then $H_i(C_\bullet) = 0$.

Definition 0.10 (Chain map)**.** A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a collection of R-module homomorphisms which makes the following diagram commute:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{i+1} & \xrightarrow{d^C_{i+1}} & C_i & \xrightarrow{d^C_i} & C_{i-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \longrightarrow & D_{i+1} & \xrightarrow{d^D_{i+1}} & D_i & \xrightarrow{d^D_i} & D_{i-1} & \longrightarrow & \cdots
\end{array}$$

In other words, suppressing indices, $f \circ d^C = d^D \circ f$.

It is easy to check that chain maps induce a map on the level on homology which we denote by $H_i(f_\bullet) : H_i(C_\bullet) \to H_i(D_\bullet)$.

The following is a useful criterion for determining whether two different chain maps $f_\bullet, g_\bullet$ induce the same maps on the level on homology, i.e. $H_i(f_\bullet) = H_i(g_\bullet)$.

Definition 0.11 (Chain homotopic)**.** Let $f_\bullet, g_\bullet$ be chain maps between $C_\bullet$ and $D_\bullet$. A chain homotopy from $f_\bullet$ to $g_\bullet$ is a collection of R-module homomorphisms $h_i : C_i \to D_{i+1}$ such that $g_i - f_i = d^D_{i+1} \circ h_i + h_{i-1} \circ d^C_i$ for all i. If such a map exists, we say that $f_\bullet$ and $g_\bullet$ are chain homotopic.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "d^C_{i+1}"] \arrow[d, shift right, "f_{i+1}"'] \arrow[d, shift left, "g_{i+1}"] & C_i \arrow[r, "d^C_i"] \arrow[d, shift right, "f_i"'] \arrow[d, shift left, "g_i"] \arrow[dl, dashed, "h_i"'] & C_{i-1} \arrow[r] \arrow[d, shift right, "f_{i-1}"'] \arrow[d, shift left, "g_{i-1}"] \arrow[dl, dashed, "h_{i-1}"'] & \cdots \\
\cdots \arrow[r] & D_{i+1} \arrow[r, "d^D_{i+1}"] & D_i \arrow[r, "d^D_i"] & D_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Suppressing indices, this becomes $g - f = dh + hd$.

Proposition 0.12. Let $f_\bullet, g_\bullet$ be two chain homotopic chain maps from $C_\bullet$ to $D_\bullet$. Then $H_i(f_\bullet) = H_i(f_\bullet)$, i.e. the maps induced on the level of homology are identical.

Proof. Let h_i be a chain homotopy between $f_\bullet$ and $g_\bullet$. Let $x \in \mathrm{Ker}(d^C_i)$. Then, suppressing indices,

$$\begin{aligned}
g(x) + \mathrm{Im}(d^D) &= f(x) + (h \circ d^C)(x) + (d^D \circ h)(x) + \mathrm{Im}(d^D) \\
&= f(x) + h(0) + \mathrm{Im}(d^D) \\
&= f(x) + \mathrm{Im}(d^D).
\end{aligned}$$

□

Singular homology

In Algebraic topology, the most important form of homology is singular homology. In order to define the chain complex, we first need to introduce the concept of a simplex.

Definition 0.13 (Standard n-simplex)**.** We define the standard n-simplex to be

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0\}.$$

A singular n-simplex is a continuous map $\phi : \Delta^n \to M$, where M is a topological space.

Remark 0.14. The name 'singular' comes from the fact that ϕ does not need to be a homeomorphism. Hence, ϕ can 'squash' simplices, making them singular.

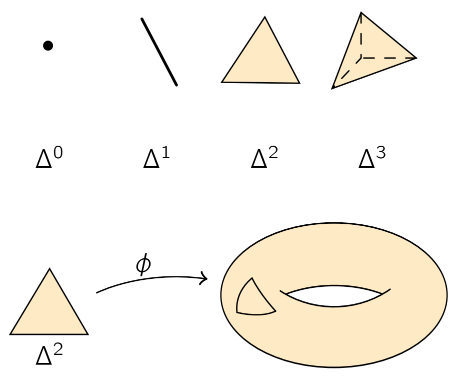


Figure 6: Top: Standard n-simplex for $n = 0, 1, 2, 3$. Bottom: a singular n-simplex in the torus.

Definition 0.15 (Boundary of a singular n-simplex)**.** Let $\phi : \Delta^n \to M$ be a singular n-simplex. We define the i-th boundary of ϕ to be

$$\partial_i\phi : \Delta^{n-1} \to X : (t_0, \ldots, t_{p-1}) \mapsto \phi(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1}).$$

We define the boundary of ϕ to be

$$\partial\phi = \sum(-1)^i \partial_i\phi.$$

Example 0.16. Consider the singular 2-symplex $\phi : \Delta^2 \to \mathbb{R}^2$ defined by $\phi(t_0, t_1, t_2) = t_0x_0 + t_1x_1 + t_2x_2$ where x_0, x_1, x_2 are points in $\mathbb{R}^2$ as indicated in the figure below. Then $\partial_0\phi$ is given by

$$(\partial_0\phi)(t_0, t_1) = \phi(0, t_0, t_1).$$

Viewing this singular 1-simplex as a path via the homeomorphism $f : [0, 1] \to \Delta^1 : t \mapsto (1 - t, t)$, we can see that $\partial_0\phi$ travels from x_1 to x_2. Similarly, $\partial_1\phi$ travels from x_0 to x_2 and $\partial_2\phi$ from x_0 to x_1. Combining these results with the correct signs, we get $\partial\phi = \partial_0\phi - \partial_1\phi + \partial_2\phi$, a linear combination of three paths travelling counter-clockwise around ϕ.

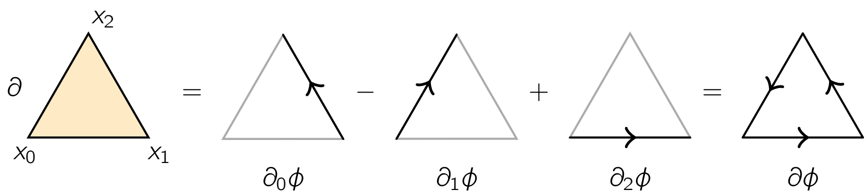


Figure 7: Illustrating the definition of boundary of a singular 2-simplex.

One can show that $\partial^2 = 0$, which allows us to define singular homology as follows:

Definition 0.17 (Singular homology)**.** Let M be a manifold. Let $C_i(M)$ be the free abelian group generated by all singular i-simplices. Elements in $C_i(M)$ are called chains. Then the following sequence is a chain complex

$$\cdots \xrightarrow{\partial} C_n(M) \xrightarrow{\partial} C_{n-1}(M) \xrightarrow{\partial} C_{n-2}(M) \xrightarrow{\partial} \cdots,$$

and its homology is called singular homology of M.

Remark 0.18. We take $C_{-1}(M) = C_{-2}(M) = \cdots = 0$. This implies that if M is an n-dimensional manifold, then only $H_0(M), \ldots, H_n(M)$ can be non-zero.

Let us give an interpretation of singular homology. The homology of the above chain complex is given by

$$H_k(M) = \frac{\{\text{singular } k\text{-symplices without boundary}\}}{\{\text{singular } k\text{-symplices that bound a } k+1\text{-simplex}\}}.$$

So a non-trivial element in $H_k(M)$ is a cycle (a chain without boundary) that does not bound a $k+1$ simplex. This should make clear how singular homology finds holes in manifolds.

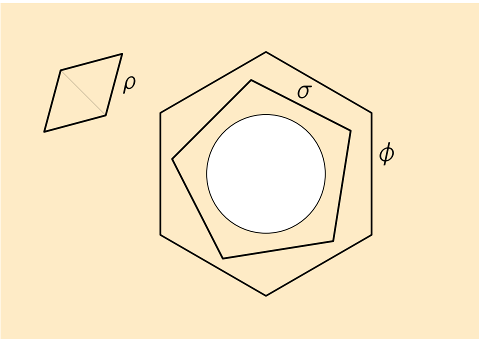


Figure 8: Some 1-chains in $M = \mathbb{R}^2 \setminus D^2$.

Example 0.19. Consider the situation in Figure 8 with $M = \mathbb{R}^2 \setminus D^2$.

- The chain ρ (which is the sum of four 1-simplices) has no boundary and bounds itself a 2-chain (the sum of two 2-simplices). Therefore, ρ is trivial in homology.
- The chain σ also has no boundary, but it does not bound a 2-simplex. The chain σ represents the hole in the plane and corresponds to a non-trivial element in homology.
- The chain ϕ is similar to σ. It also represent the hole in the plane and corresponds to a non-trivial element in homology. In fact, ϕ and σ represent the same element—we say that they are homologous. Indeed, their difference $\phi - \sigma$ bounds the region between the hexagon and the pentagon and we can see this as a 2-simplex by triangulating it.

Turning these ideas into a proof, one can show that

$$H_0(M) = \mathbb{Z} \qquad H_1(M) = \mathbb{Z} \qquad H_2(M) = 0 \qquad H_3(M) = 0 \qquad \cdots$$

Example 0.20. For a sphere, we have

$$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 0, k \\ 0 & \text{else.} \end{cases}$$

Computing singular homology

Homology is in many cases not so difficult to compute because it is essentially a local, as the following theorem shows:

Theorem 0.21 (Mayer-Vietoris)**.** Let M be a topological space covered by the interiors of two subspaces A, B, i.e. $M = \mathring{A} \cup \mathring{B}$. Then the following long exact sequence relates the homology of M with that of A, B and $A \cap B$:

$$\cdots \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_* j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - \ell_*} H_n(M) \xrightarrow{\partial_*}$$
$$\xrightarrow{\partial_*} H_{n-1}(A \cap B) \xrightarrow{(i_* j_*)} H_{n-1}(A) \oplus H_{n-1}(B) \xrightarrow{k_* - \ell_*} H_{n-1}(M) \xrightarrow{\partial_*} \cdots$$

where $i : A \cap B \hookrightarrow A$; $j : A \cap B \hookrightarrow A$; $k : A \hookrightarrow M$; $\ell : B \hookrightarrow M$ are inclusions and ∂_* takes a chain in M, splits it up in a chain lying entirely in A and one entirely in B and then takes the boundary of the chain lying in A.

Example 0.22. Let us illustrate the boundary map ∂_*. Consider the torus T^2 and the open cover by cilinders A and B are in Figure 9. Consider the one-chain ϕ that goes around the hole once. We can split it up in two chains: $\phi = \rho + \sigma$ where ρ lies entirely in A and σ lies entirely in B. Then $\partial_*[x] = [\partial\rho]$, which in this case is the difference of the points indicated in the figure. While we have made many choices, the end result is well-defined in homology.

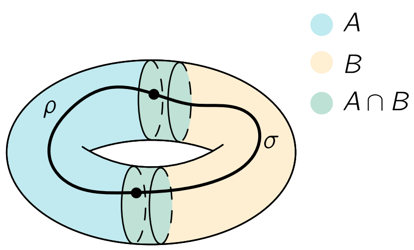


Figure 9: We can compute the homology of a torus by covering it with two open sets A and B.

Writing down the long exact sequence and using the fact that the homology of a cylinder is the same as that of S^1, we find

$$H_0(T^2) = \mathbb{Z} \qquad H_1(T^2) = \mathbb{Z}^2 \qquad H_2(T^2) = \mathbb{Z}.$$

Another tool for computing homology is the Künneth formula, allowing us to compute $H_\bullet(M \times N)$. For this, let us introduce the tensor product of complexes.

Definition 0.23 (Tensor product of complexes)**.** Let $C_\bullet$ and $D_\bullet$ be two complexes. Their tensor product is defined as

$$(C \otimes D)_k = \bigoplus_{i+j=k} C_i \otimes D_j,$$

with boundary operator $\partial_C \otimes 1 + 1 \otimes \partial_D$.

Under the condition that $C_\bullet$ and $D_\bullet$ are vector spaces, i.e. that we are working over a field, we have the following result:

Proposition 0.24. Let $C_\bullet$ and $D_\bullet$ be complexes over a field. The homology of the tensor product complex is the tensor product of the homologies:

$$H_\star(C_\bullet \otimes D_\bullet) = H_\star(C_\bullet) \otimes H_\star(D_\bullet).$$

The Künneth formula then states the following:

Theorem 0.25 (Künneth formula)**.** Let X and Y be topological spaces and F a field. Then

$$H_\bullet(X \times Y; F) = H_\bullet(X; F) \otimes H_\bullet(Y; F).$$

A proof of this can be found in any textbook on algebraic topology.[7]

[7] Allen Hatcher. *Algebraic topology*. 2005

Example 0.26. Applying this to $T^n = S^1 \times S^1 \times \cdots \times S^1$, we get that

$$H_k(T^n; \mathbb{Q}) = \mathbb{Q}^{\binom{n}{k}},$$

where we consider homology over $\mathbb{Q}$, which is defined similarly as over $\mathbb{Z}$, except we allow for formal rational sums of simplices.

Relative homology and singular cohomology

Let us lastly discuss relative homology and singular cohomology.

Definition 0.27 (Relative homology)**.** Let A be a submanifold of M. The inclusion of $A \hookrightarrow M$ induces an inclusion $C_\bullet(A) \hookrightarrow C_\bullet(M)$. Then the relative homology of M w.r.t. A is the homology of the chain complex $C_\bullet(M)/C_\bullet(A)$ and is denoted with $H_\bullet(M, A)$.

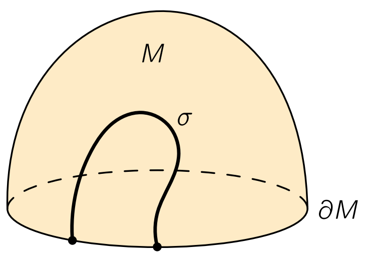


Figure 10: The chain σ is an example of a 1-cycle in the relative homology $H(M, \partial M)$.

If M and A are 'nice', you can think of the relative homology of M w.r.t A as the homology of M/A. We will often be considering the relative homology $H(M, \partial M)$, where M is a manifold with boundary as illustrated in Figure 10. As an illustration, we have drawn a chain that is not a cycle in M, but is a cycle relative to ∂M. Indeed, relatively seen, it has no boundary, because its boundary forms a chain in ∂M.

Singular cohomology is the dual of singular homology in the following sense:

Definition 0.28 (Singular cohomology)**.** Let M be a manifold and let R be a ring. The singular cohomology of M with coefficients in R is defined to be the homology of following complex

$$\cdots \leftarrow C^*_{i+1} \xleftarrow{d^*_i} C^*_i \xleftarrow{d^*_{i-1}} C^*_{i-1} \leftarrow \cdots$$

where $C^*_i = \mathrm{Hom}_R(C_i, R)$ and $d^*(f) = f \circ d$. We denote singular cohomology with $H^k(M, R)$.

Remark 0.29. If R is a field then $H^k(M; R)$ is dual to $H_k(M; R)$. In particular, if the homology is finite dimensional, we have $H^k(M; R) \cong H_k(M; R)$.

Remark 0.30. Poincaré duality states that if M is an n-dimensional oriented closed manifolds, then $H_k(M, R) \cong H^{n-k}(M, R)$.

Homotopy theory

Homotopy theory is similar to homology theory in the sense that it also is a way of capturing holes in a topological space. While the ideas are similar, homotopy turns out to be much harder to compute. For example, while the homology of S^n is easily computed, the homotopy groups of spheres are surprisingly complex and we still do not understand them fully.

Let us start by defining homotopy groups:

Definition 0.31 (Homotopy group)**.** Choose a basepoint x on S^n. Let M be a manifold and choose a basepoint $p \in M$. Then we define the n-th homotopy group $\pi_n(M, p)$ to be the set of homotopy classes of based maps $f : (S^n, x) \to (M, p)$, i.e. maps $f : S^n \to M$ that map x to p.

Example 0.32. Consider the situation in Figure 11. The manifold M is given by $\mathbb{R}^2 \setminus D^2$, and we have chosen a base point x_0. The maps ρ and σ are homotopic and capture the existence of the hole. The map ϕ corresponds to a trivial element in the homotopy group $\pi_1(\mathbb{R}^2 \setminus D^2)$. One can prove that $\pi_1(M) \cong \mathbb{Z}$, where the isomorphism is defined by mapping a map ϕ to the number of times it wraps around the hole.

Example 0.33. If M is connected, then $\pi_0(M) = 0$. If M is simply connected, then $\pi_1(M) = 0$.

If $n \geq 1$, $\pi_n(M)$ indeed forms a group:

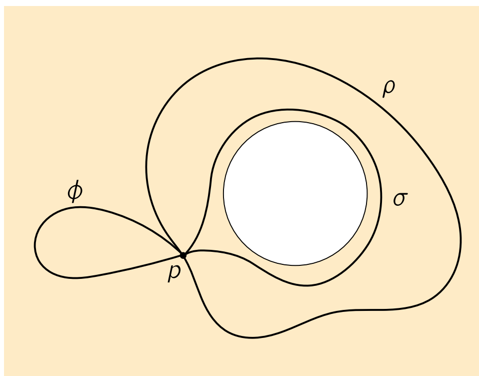


Figure 11: Examples of paths in $\mathbb{R}^2 \setminus D^2$. The paths ρ and σ are homotopic. The path ϕ is null-homotopic.

Proposition 0.34. The homotopy group $\pi_n(M)$ is—as its name suggests—a group with the following operation

$$[\rho] + [\sigma] = [h \circ \phi],$$

where $\phi : S^n \to S^n \vee S^n$ is a map that collapses the equator of S^n and $h : S^n \vee S^n \to M$ is defined to be ρ on the first copy of S^n and σ on the second copy.

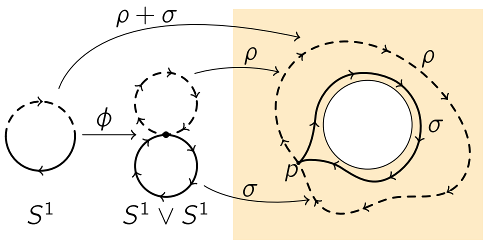


Figure 12: The group operation for homotopy groups.

Remark 0.35. In the case of $n = 1$, this is simply concatenation of paths.

It is not hard to prove the following:

Proposition 0.36. Let S^n be an n-dimensional sphere. Then $\pi_n(S^n) = \mathbb{Z}$ and all the lower order homotopy groups are 0.

While this seems contradictory to the claims made in the introduction of this section, we should note that higher order homotopy groups do not vanish: $k > n$ does not imply that $\pi_k(S^n) = 0$. Computing these higher order groups is where the difficulty lies.

Definition 0.37. A homotopy sphere is a manifold which has the homotopy groups of a sphere.

An interesting question that arises is: 'Is a homotopy sphere an actual sphere, i.e. homeomorphic to a sphere?' This is exactly what the Poincaré conjecture is about and in the last chapter of this thesis, we will prove that this is indeed the case if the dimension of the homotopy sphere is ≥ 5.

To do so, we will in fact only use that a homotopy sphere S is a homology sphere that is connected and simply connected. The last two properties are clear as $\pi_0(S) = \pi_1(S) = 0$. In order to see that a homotopy sphere is a homology sphere, we introduce the following notion of connectedness:

Definition 0.38 (k-connected)**.** A manifold M with basepoint p is k-connected if $\pi_i(M, p) = 0$ for $i \leq k$.

Clearly k-connectedness implies $k-1$-connectedness and a n-dimensional homotopy sphere is $n-1$-connected. With this definition, we can state the relation between homotopy and homology as follows:[8]

[8] Allen Hatcher. *Algebraic topology*. 2005

Theorem 0.39 (Hurewicz theorem)**.** Let M be a manifold. Let $\alpha \in H_n(S^n)$ be a canonical generator. Then we can define the Hurewicz homomorphism

$$\begin{aligned} h_n : \pi_n(M) &\longrightarrow H_n(M) \\ [f] &\longmapsto [f_*(\alpha)]. \end{aligned}$$

If M is $k-1$ connected with $k \geq 2$, then h_k is an isomorphism and h_{k+1} is an epimorphism.

Remark 0.40. In the case of $n = 1$, $H_1(M)$ is the abelianization of $\pi_1(M)$.

This allows us to prove the result we were after:

Proposition 0.41. A homotopy sphere is a homology sphere.

Proof. An n-dimensional homotopy sphere S is $n-1$-connected, hence the morphisms $h_1, h_2, \ldots, h_{n-1}$ are isomorphisms and h_n is an epimorphism. We conclude that

$$H_0(S) = \mathbb{Z} \qquad H_1(S) = 0 \qquad \cdots \qquad H_{n-1}(S) = 0 \qquad H_n(S) = \mathbb{Z},$$

so S has the homology of a sphere S^n. $\square$

Let us end our short discussion of homotopy theory by stating the theorem of Van Kampen, showing that fundamental groups are in fact fairly easy computable (contrary to higher order homotopy groups).

Theorem 0.42 (Van Kampen)**.** Let A_α be a cover of M by path-connected open sets, all of which containing the basepoint p. If each double and triple intersection ($A_\alpha \cap A_\beta$ and $A_\alpha \cap A_\beta \cap A_\gamma$) is path-connected, then

$$\pi_1(X) \cong *_\alpha A_\alpha / N,$$

where N is the normal subgroup generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the morphism induced by the inclusion.

We will use this result in Chapter 5 to show that certain spaces are simply connected.

CHAPTER ONE

Morse theory

When studying manifolds it often proves useful to examine simple structures that live on them. For example, differential forms give rise to the de Rham cohomology, which tells us something about the global structure of the manifold. In this thesis the objects of interest will be nice real valued functions $f : M \to \mathbb{R}$, which we will call Morse functions, named after the American mathematician Marston Morse.

We will start off this chapter by giving several definitions of Morse functions and showing that they are equivalent. We will show that Morse functions give rise to handlebody decompositions which decompose any manifold in standard building blocks called handles. We will end the chapter by showing that any manifold admits a Morse function, and in fact infinitely many.

1.1 Definition of a Morse function

A Morse function $f : M \to \mathbb{R}$ is a 'nice' real valued function on a manifold, which will in this context mean that f behaves nicely around its critical points. Let us first recall what a critical point is.

Definition 1.1 (Critical point)**.** Let M and N be a manifolds and f a map from M to N. The set of critical points of f is given by

$$\operatorname{Crit} f = \{p \in M \mid df_p \text{ is not surjective}\}.$$

Note that in particular, if $N = \mathbb{R}$ we have that the set of critical points is given by $\operatorname{Crit} f = \{p \in M \mid df_p = 0\}$. The maps we will be considering are then characterised as follows:

Definition 1.2 (Morse function[9])**.** Let M be a manifold. A function $f : M \to \mathbb{R}$ is a *Morse function* if for all critical points p, there exists a chart centred around p such that f is locally given by

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

Such a chart is said to be a *Morse chart* and we call k the index of p, which we will denote with $\operatorname{Ind} p$.

[9] Marston Morse. "Relations between the critical points of a real function of n independent variables". In: *Transactions of the American Mathematical Society* 27.3 (1925), pp. 345–396

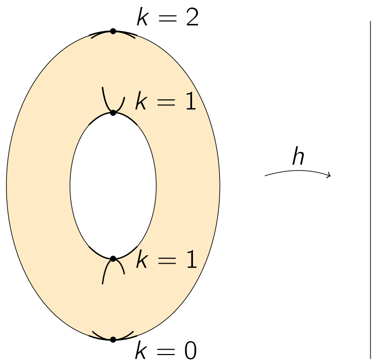


Figure 1.1: Example of a Morse function on the torus. At each critical point, the index k, the number of downward directions is indicated.

Intuitively, the index of a critical point p is 'the number of downward directions'. Let us give some examples of Morse functions.

Example 1.3. Let M be the torus T^2 embedded in $\mathbb{R}^3$ as illustrated in Figure 1.1. Then the height function $h : T^2 \to \mathbb{R}$ which is the projection on the z-axis is a Morse function with four critical points. We have a minimum, two saddle points and a maximum, whose indices are $0, 1, 1, 2$ respectively.

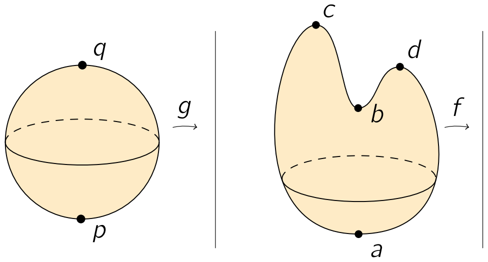


Figure 1.2: Two Morse functions on S^2 with a different number of critical points.

Example 1.4. In Figure 1.2, we have illustrated two embeddings of S^2 in $\mathbb{R}^3$, and considering the corresponding height functions, we get two Morse functions $S^2 \to \mathbb{R}$. The first one has only two critical points: a maximum and a minimum. The second one has two maxima, a saddle point and a minimum. Later on we will prove that any manifold admitting a Morse function with only two critical points is homeomorphic to the sphere.

Nonexample 1.5. Let $M = \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^2$. Then all points $(0, y)$ for $y \in \mathbb{R}$ are critical points of this function. In particular, $(0, 0)$ is a critical point. As it is impossible to find local coordinates (x_1, x_2) for which f can be written as $\pm x_1^2 \pm x_2^2$, we conclude that f is not Morse.

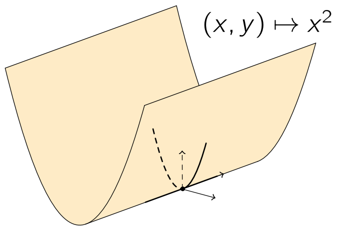


Figure 1.3: An example of an embedding where the height function is not Morse.

Nonexample 1.6. Let $M = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Then $x = 0$ is a critical point, but f is not Morse. Note however that if we add a small perturbation to f, say $g_t : x \mapsto x^3 + tx$, then for small non-zero t, g is Morse. For $t < 0$, g_t has two critical points: one of index 1 and one of index 0. If $t > 0$, g_t has no critical points.

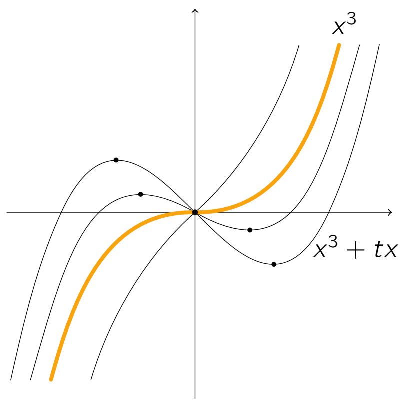


Figure 1.4: An example of a function that is not Morse: $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Small perturbations of f are Morse.

Note that this last case where f has no critical points cannot happen if M is compact. Indeed, any function attains it maximum and minimum on a compact manifold, so we have at least two critical points. On the other hand, the number of critical points is at most finite. This is because of the definition of a Morse function: it implies that critical points are isolated, which on a compact manifold implies that their number is finite. This also immediately rules out the situation we had in the other example, where the set of critical points was a straight line.

1.2 Coordinate-free definition

The attentive reader will have noticed that the notion of the index of a critical point could possibly be coordinate dependent and hence ill-defined. In order to show that it is not, we will give an equivalent coordinate-free definition. For this, let us first define the Hessian:

Definition 1.7 (Hessian)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a function. Let p be a critical point of f. Then we define the Hessian H_p to be the bilinear form

$$\begin{aligned} H_p : T_pM \times T_pM &\longrightarrow \mathbb{R} \\ (X, Y) &\longmapsto X(\tilde{Y} f)|_p, \end{aligned}$$

where $\tilde{Y}$ is a local extension of Y around p.

Because we are only considering the Hessian H_p at critical points, this is a well defined symmetric bilinear form.[10] In case of a Morse function given locally by $f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$, the Hessian at p is

$$H_p = 2(-dx_1^2 - \cdots - dx_k^2 + dx_{k+1}^2 + \cdots + dx_n^2),$$

where $dx_i^2 = dx_i \otimes dx_i$. Note in particular that H_p is non-degenerate and its signature is $(i_-, i_+) = (k, n-k)$, as we have k negative eigenvalues and $n-k$ positive eigenvalues. As the signature of a symmetric bilinear form is coordinate independent, this shows that the index of a critical point is as well.

[10] The difference between $H_p(X,Y)$ and $H_p(Y,X)$ is given by

$$\begin{aligned} H_p(X,Y) - H_p(Y,X) &= X(\tilde{Y}f)|_p - Y(\tilde{X}f)|_p \\ &= [\tilde{X},\tilde{Y}]f|_p \\ &= df_p[\tilde{X},\tilde{Y}]|_p = 0. \end{aligned}$$

The value of H_p also does not depend on the extension of the vector field. Indeed, suppose $\tilde{Y}$ and $\overline{Y}$ are two different extensions of Y. Then by symmetry of H_p, we have

$$X(\tilde{Y}f)|_p = Y(\tilde{X}f)|_p = X(\overline{Y}f)|_p.$$

This also shows linearity of the second component.

Interestingly, the converse is also true: if H_p is non-degenerate for all critical points p of f, then f is a Morse function. Many authors take this to be the definition of a Morse function, and then prove the so-called Morse lemma stating that there always exist local coordinates such that f is given by

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2,$$

which is our definition of a Morse function. With our choice, the Morse lemma takes on the following form:

Lemma 1.8 (Morse Lemma)**.** Let M be a manifold and $f : M \to \mathbb{R}$ a smooth function. If for all $p \in \operatorname{Crit} f$, the Hessian H_p is non-degenerate, then f is Morse.

Proof. We follow the proof of Milnor[11]. We may assume that $M = \mathbb{R}^n$, p is the origin and $f(p) = 0$. Then by a version of Taylor's theorem, we can write

[11] John Milnor. *Morse theory.(AM-51)*. Vol. 51. Princeton university press, 2016, p. 6

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n}(x_i - p_i)g_i(x) \\ &= \sum_{i=1}^{n} x_i g_i(x), \end{aligned}$$

where g_i are smooth functions. Now, as $g_i(0) = \partial_i f(0) = 0$, we can repeat this for each g_i, giving us the following:

$$f(x) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x).$$

Because this sum is symmetric in i and j, we may assume that h_{ij} is symmetric as well.[12] Note that

$$h_{ij}(0) = \frac{1}{2}\partial_{ij} f(0),$$

which is non-degenerate by assumption.

[12] If h_{ij} is not symmetric, we can replace it by $h_{(ij)} = \frac{1}{2}(h_{ij} + h_{ji})$. Then $h_{(ij)}$ is symmetric and we still have $\sum x_i x_j h_{ij} = \sum x_i x_j h_{(ij)}$.

Now we imitate the proof of diagonalization of a non-degenerate quadratic form. We do this by induction. Suppose we have coordinates $u_1, \cdots, u_n$ in a neighbourhood of 0 such that

$$f = \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u),$$

where H_{ij} is a symmetric matrix. After a linear change, we may assume by non-degeneracy that $H_{rr} \neq 0$. Then define new coordinates $v_1, \ldots, v_r$ as follows:

$$v_i = \begin{cases} u_i & \text{if } i \neq r \\ \sqrt{|H_{rr}|}(u_r + \sum_{i>r} u_i H_{ir}/H_{rr}) & \text{if } i = r. \end{cases}$$

Note that we may need to restrict the neighbourhood such that $\sqrt{|H_{rr}|} \neq 0$. Then we have that

$$f = \sum_{i \leq r} \pm v_i^2 + \sum_{i,j>r} v_i v_j H'_{ij}(v).$$

for some symmetric matrix H'_{ij}. By induction, we can find coordinates such that $f = \sum_{i=1}^n \pm v_i^2$, completing the proof. □

1.3 Handle decompositions

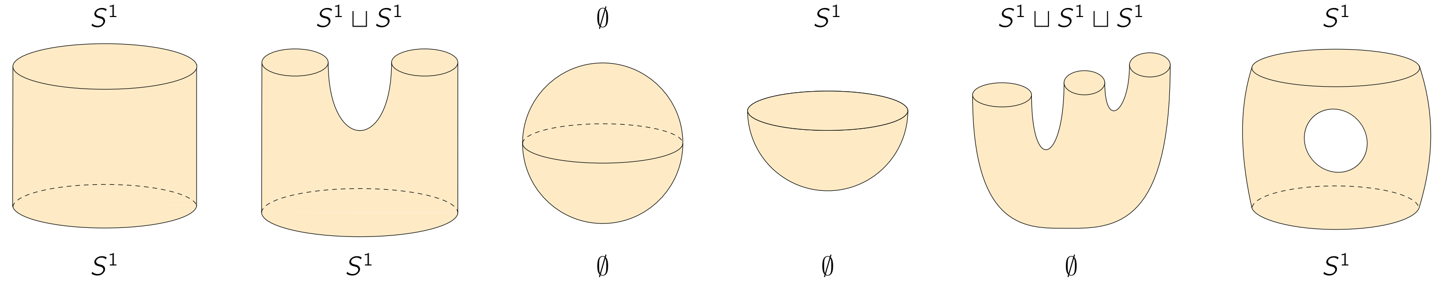
In this section we show how a Morse function $f : M \to \mathbb{R}$ gives rise to a step by step guide on how to build the manifold M out of basic building blocks. In order to show how this is done, we need to introduce the notion of a cobordism.

Definition 1.9. A cobordism between two compact manifolds M_0 and M_1 is a compact manifold M with boundary $\partial M = M_0 \sqcup M_1$.

In this text, we will often say that M is a cobordism *from* M_0 *to* M_1. The term 'cobordism' comes from the fact that $M_0 \sqcup M_1$ are the boundary of M, so we can think of M as the 'co-boundary' of M_0 and M_1. Cobordisms are an interesting topic in their own right. For example, they define an equivalence relation on all compact manifolds of the same dimension. Two manifolds M_0 and M_1 are then said to be equivalent (cobordant) if there exists a cobordism M connecting the two. This equivalence relation is much coarser than diffeomorphism and is generally easier to study. Cobordisms also form a category where the objects are manifolds and the morphisms are cobordisms.

We have illustrated some examples below. Note that we may also take M_0 or $M_1 = \emptyset$. In particular, any closed manifold is a cobordism from $\emptyset$ to $\emptyset$.

Figure 1.5: Some examples of cobordisms. Notice that if the height function does not have critical points, the topology of the bottom is the same as that of the top.



When we consider the height function $f : M \to [0, 1]$ on these examples, we observe that the topology of the bottom is the same as that of the top if the height function does not have critical points. In the case when the height function has a single critical point, the topology does change and it will turn out we can very concretely describe *how* this topology changes if f is Morse (as is the case here).

Cobordisms without critical points

Let us first consider the case where the height function has no critical points. For simplicity, we introduce the following refined definition of a Morse function on a cobordism:

Definition 1.10 (Morse function on a cobordism[13]**).** Let M be a cobordism from M_0 to M_1. A function $f : M \to [a, b]$ is Morse if

- $M_0 = f^{-1}(a)$, $M_1 = f^{-1}(b)$
- All critical points lie interior in M and are non-degenerate.

[13] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015, p. 8

Let us now prove our first observation:

Proposition 1.11. Suppose $f : M \to [a, b]$ is a Morse function on a cobordism M from M_0 to M_1. If f has no critical points, then M is diffeomorphic to $[a, b] \times M_0$.

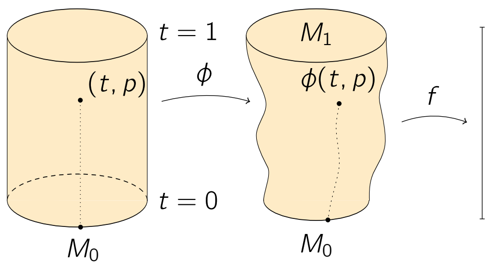


Figure 1.6: When a cobordism has no critical points, it is diffeomorphic to a product manifold.

Proof. Choose a Riemannian metric $\langle \cdot, \cdot \rangle$ on M. Because f has no critical points, the gradient vector field $W = \operatorname{grad} f := (df)^\sharp$ never vanishes.[14] Now we normalize the vector field as follows:

$$V = \frac{1}{\langle W, W \rangle} W.$$

Then $df(V) = \frac{1}{\langle W,W \rangle} df(W) = 1$. Consider

$$\begin{aligned} \phi : [0, 1] \times M_0 &\longrightarrow M \\ (t, p) &\longmapsto \theta_V^t(p), \end{aligned}$$

where θ_V^t is the flow along V. Then $f(\phi(t, p)) = t$ for all $p \in M_0$, as $df(V) = 1$. The map ϕ is injective because of the uniqueness of flows, and surjective because given a point p in M, we can always flow back along V for a time t to find a $p_0 \in M_0$ which then satisfies $\phi(t, p_0) = p$. Because this is also smooth, this proves that ϕ is a diffeomorphism. $\square$

[14] Here we use $\sharp$ and $\flat$ for the 'musical isomorphisms' induced by a metric $\langle \cdot, \cdot \rangle$, given by

$$\begin{aligned} \flat &: TM \longrightarrow T^*M : X \longmapsto X^\flat = \langle X, - \rangle \\ \sharp &: T^*M \longrightarrow TM : \alpha \longmapsto \alpha^\sharp, \end{aligned}$$

where $\alpha^\sharp$ is uniquely defined by $\langle \alpha^\sharp, X \rangle = \alpha(X)$. In other words $\sharp = (\flat)^{-1}$.

Remark 1.12. While almost trivial, this result is very useful and lies at the basis of the proof of the h-cobordism theorem, stating that a cobordism satisfying certain properties is a trivial cobordism.[15] We will prove this in Chapter 5 by showing that an h-cobordism admits a Morse function without any critical points, implying the result.

[15] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015, p. 107

In general, this result can be used to show that two manifolds are diffeomorphic. If we can construct a cobordism from M_0 to M_1 and we can prove the existence of a Morse function without critical points, then $M_0 \cong M_1$.

Cobordisms with a single critical point

Let us now investigate the situation when a Morse function f does have a single critical point, say p and assume $f(p) = 0$. Then we know f is locally of the following form:

$$f(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

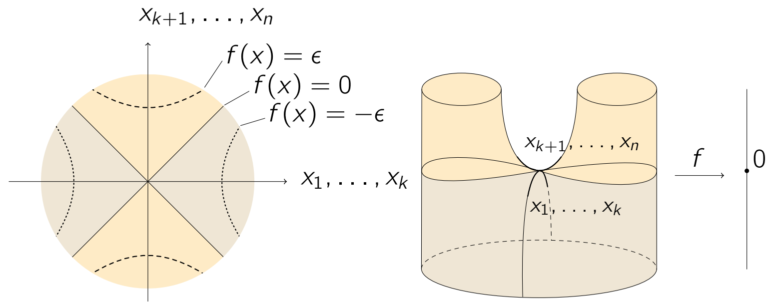


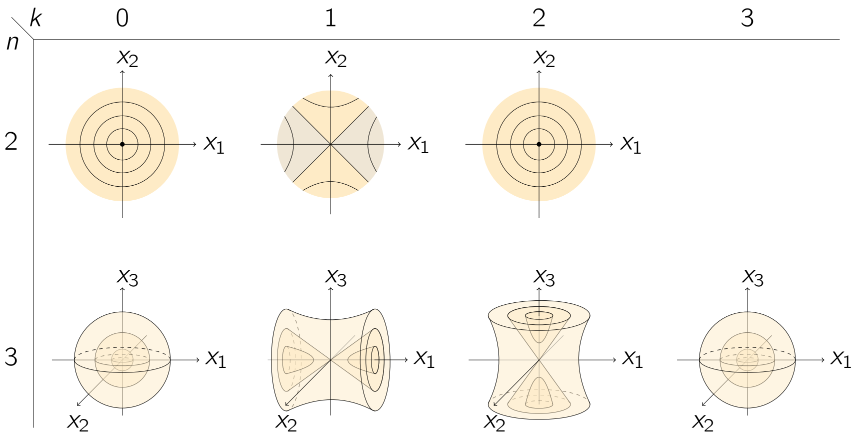
Figure 1.7: Around a critical point of a Morse function, we can find a chart as on the left. The dark region corresponds to $f(x) < 0$ and the light region to $f(x) > 0$. The level set $f^{-1}(0)$ is not a manifold.

Above, we have plotted the first k variables (the 'downward directions') on the horizontal axis, and the last $n - k$ variables on the vertical axis. Then the level set $f^{-1}(0)$ is given by

$$x_1^2 + \cdots + x_k^2 = x_{k+1}^2 + \cdots + x_n^2,$$

which consists of the two crossing lines in the figure, corresponding to the crossing part of the figure-eight on the right. Level sets of values slightly above and below 0 look locally like hyperbolas on our figure, but recall that the axes consist of multiple dimensions, so it would be more precise to call them hyperboloids. We give some more concrete examples for $n = 2, 3$ and $k = 0, 1, 2, 3$ in the following table:

Table 1.1: Some examples of Morse charts in dimension two and three. We have indicated the level sets of the Morse function.



To better understand the effect of the critical point, we want to isolate its behaviour. We will do this by cutting out a part of a Morse neighbourhood of p of a very specific form, as shown in yellow in Figures 1.8 and 1.9. The part we cut out is bounded by level sets $f(p) = \pm\epsilon$ and negative gradient flow lines of f.[16] The advantage of cutting in this specific way, is that we have split up the cobordism in three parts which we understand well:

(A) The part below $-\epsilon$ and the part above ϵ have a product structure, because we assumed the only critical value was 0. We can simply repeat the proof above.

(B) The part of M that lies between $-\epsilon$ and ϵ without (H) also has a product structure. To see this, extend the metric introduced on (H) for defining the gradient to the whole manifold. Then, because f has no critical values outside (H), we can flow along gradient lines giving us a product structure. Now it should be clear why we wanted to cut along gradient lines.

Riemannian metrics that are an extension of standard metrics around critical points are called 'adapted.' Using a partition of unity argument, one can show any manifold equipped with a Morse function admits such metrics. In most cases, we will often forget about the metric itself and simply consider vector fields which are gradient-like for f:

> **Definition 1.13** (Pseudo-gradient)**.** Let $f : M \to \mathbb{R}$ be a Morse function on a manifold M. A pseudo-gradient is a vector field X such that
>
> – $df(X) \leq 0$ and $df(X) = 0$ only at critical points
>
> – X coincides in Morse charts with the usual negative gradient for the standard metric on $\mathbb{R}^n$.

(H) The yellow part is what we call a *handle of index k*, and is homeomorphic to $B^k \times B^{n-k}$. This is the only non-trivial part of our cobordism. We call the bottom part of the border the *attachment region*, which is homeomorphic to $S^{k-1} \times B^{n-k}$.

> **Definition 1.14** (k-handle)**.** A k-handle is a thickened up k-cell: $B^k \times B^{n-k}$. The bottom of the handle, $S_k = \partial B^k \times B^{n-k} = S^{k-1} \times B^{n-k}$ is called the *attachment region*.

We conclude that a cobordism with exactly one critical point consists of product structures and a k-handle, where k is the index of the critical point.

In this context the concept of a handle is something that arises from an existing Morse function $f : M \to \mathbb{R}$ on a manifold. However we can also view them as standalone objects that we can attach to an existing manifold with boundary, in this case $N = f^{-1}(-\infty, -\epsilon]$. In general, the act of attaching a k-handle to a manifold N is gluing a handle $B^k \times B^{n-k}$ to the boundary of N along its attachment region S_k via a chosen embedding $S_k \to \partial N$. We should remark that the resulting manifold is not always smooth, although there exist techniques to 'smooth the corners'.. An overview is given in Figure 1.10. We

[16] Two remarks. First, while it does not really matter if we choose the negative or positive gradient, it is customary in Morse theory to use the latter. The gradient flow of a height function flows then from top to bottom which is arguably more intuitive when one thinks of it like water flowing down a mountain.

Secondly, a gradient is a choice, and here we choose the gradient w.r.t. the standard metric on $\mathbb{R}^n$. More explicitly, we know that locally around a critical point p, f is given by

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2.$$

Then the standard negative gradient of f is

$$-\operatorname{grad} f = 2x_1\partial_1 + \cdots + 2x_k\partial_k - 2x_{k-1}\partial_{k-1} - \cdots - 2x_n\partial_n.$$

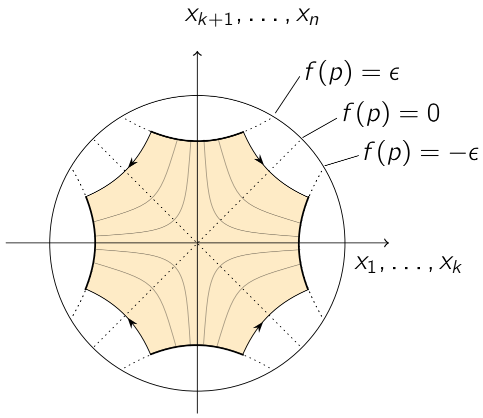


Figure 1.8: A Morse chart with negative gradient flow lines added. The coloured area represents a handle. Its boundary consists of parts of level sets of f and flow lines of grad f.

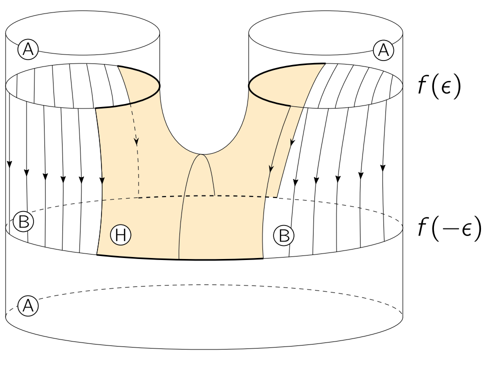


Figure 1.9: A cobordism with one critical point splits up in product structures and a handle.

see that the cobordism M with one critical point of index k is diffeomorphic to N with a 'smoothed $k-$handle' attached. This is homeomorphic to $N \cup H$, which is the same as $N \cup B^k \times B^{n-k}$. Finally, this is homotopy equivalent to N with a k-cell added (i.e. $N \cup B^k$). Proofs of these statements can be found in Morse Theory and Floer Homology by Audin and Damian[17].

[17] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014, p. 32

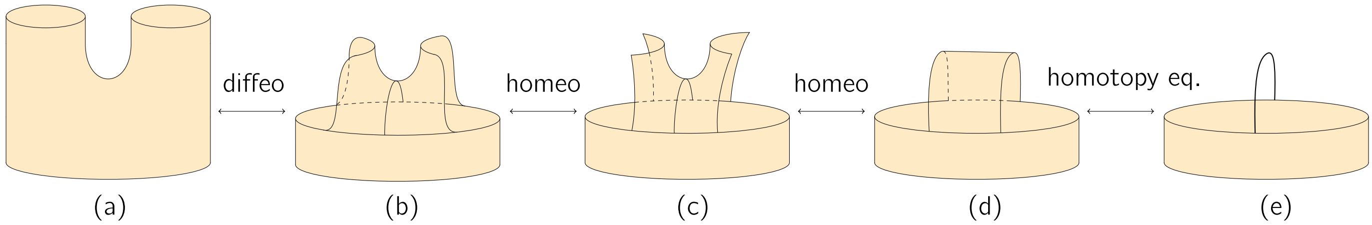


Figure 1.10: Equivalences of handle structures.

Handle decomposition of closed manifolds

Our goal now is to prove that any closed manifold M admits a handle decomposition. We will do this by splitting M into cobordisms that only have a single critical point. For this to work, we will need that the critical values are isolated, which is exactly what the following proposition states:

Proposition 1.15. Let $f : M \to \mathbb{R}$ be a Morse function on a closed manifold. Then there exists another function g arbitrarily close to f in C^1 sense that takes distinct values at all of its critical points. Moreover, if X is a pseudo-gradient field adapted to f, then we can choose g such that X is adapted to g as well.

Proof. We follow the proof of Audin and Damian[18]. Let $U = \bigcup_{p \in \mathrm{Crit}\, f} U_p$ be the union of Morse neighbourhoods of critical points. Let X be a pseudo-gradient field. Then by compactness, $df(X) < -\epsilon_0$ outside U for some $\epsilon_0 > 0$. Let $h : M \to \mathbb{R}$ be a function that is constant on each Morse chart such that $|dh(X)| < \frac{1}{2}\epsilon_0$ and

$$f(p) + h(p) \neq f(q) + h(q),$$

where $p \neq q$ are critical points. Then $f + h$ is still a Morse function, because it has exactly the same critical points as f.[19] Furthermore the critical points of $f + h$ are distinct and the vector field X is still adapted. □

[18] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014, p. 40

[19] If p is a critical point of $f + h$, then $df(X) + dh(X) = 0$ in p. Outside of the Morse charts, $df(X) + dh(X) < 0$, so p lies in a Morse chart. This implies that $df(X) + dh(X) = df(X) = 0$, so p is a critical point of f. Conversely, if p is a critical point of f, then because h is constant on Morse charts, it is also a critical point of $f + h$.

Now we are ready to prove one of the fundamental theorems in Morse theory.

Theorem 1.16. Any closed manifold has a handle decomposition.

Proof. We will later prove that that any closed manifold admits a Morse function $f : M \to \mathbb{R}$. By the previous proposition, we may assume that the critical values of f are distinct. Then we can split M in cobordisms each having a single critical point. More explicitly, let p_i be the critical points of

f and $a_i = f(p_i)$. As a_i are distinct, we can choose regular values b_i lying between a_i:

$$a_0 < b_0 < a_1 < \cdots < a_n < b_n.$$

Then we split $M = \bigcup M_i$ as follows:

$$M_0 = f^{-1}[a_0, b_0] \qquad M_1 = f^{-1}[b_0, b_1] \quad \cdots \quad M_{n-1} = f^{-1}[b_{n-1}, b_n].$$

Each of these sets is a cobordism with exactly one critical value. As we have seen earlier, we can split each of these cobordisms in product structures and handles, giving rise to a handle decomposition of the total manifold M. $\square$

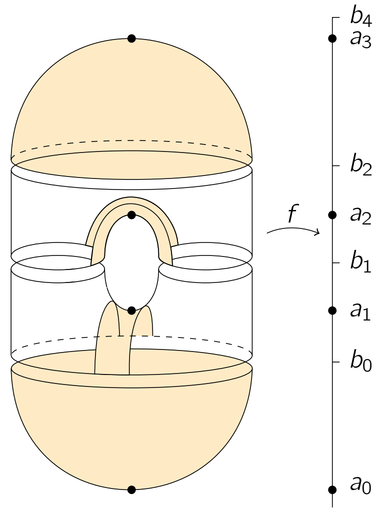


Figure 1.11: By splitting up a closed manifold in cobordisms with a single critical point, we find a handle decomposition.

Handles in low dimensions

Let us now discuss some examples of handle decompositions in low dimensions. We will consider manifolds of dimension one, two and three and our examples will give rise to some interesting questions to which we will find an answer in the next chapter. Let us start with one dimensional handles.

Handles in dimension one

Handles in dimension one are the simplest ones. We only have two types:

 0-handle $B^0 \times B^1 = [-1, 1]$, with attachment region $\emptyset$.

 1-handle $B^1 \times B^0 = [-1, 1]$ with attachment region $S^0 = \{-1, 1\}$.

Combining these two handles, we can build any compact one dimensional manifold, one example being S^1. Apart from the obvious way to do so, more involved handle decompositions also exist, as we show in the next example:

Example 1.17. Below in Figure 1.12, two handle decompositions of S^1 are illustrated. The first one is based on the height function f when embedding S^1 in $\mathbb{R}^2$. For the second decomposition, g is again defined as a height function, but this time we embedded S^1 in $\mathbb{R}^2$ in a non-standard way. This results in two different handle decompositions of the same manifold.

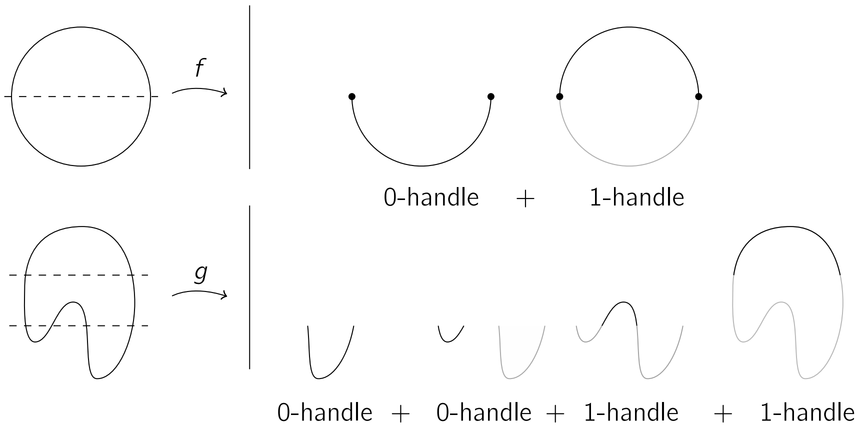


Figure 1.12: Two handle decompositions of S^1.

This example illustrates that handle decompositions are not unique. We can however construct an isotopy from the second to the first embedding that cancels two critical points. Interestingly, the height function is Morse for all t, except at the exact moment when the cancellation happens, indicated in yellow in below. A local model for this is $t \mapsto x^3 + tx$ as in Figure 1.4. This raises then the question: when is it possible in general to find an isotopy that cancels two critical points? What are possible obstructions?

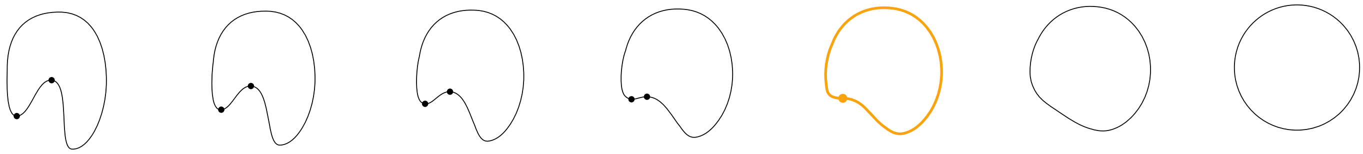


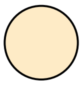
Figure 1.13: There exists an isotopy between the two handle decompositions described above.

Handles in dimension two

In dimension two, we have three types of handles:

 0-handle $B^0 \times B^2$ with attachment region $\emptyset$.

 1-handle $B^1 \times B^1$ with attachment region $B^1 \times S^0$.

 2-handle $B^2 \times B^0$ with attachment region $B^0 \times S^1$.

Example 1.18 (Handle decomposition of the torus)**.** Consider the height function on the torus embedded in $\mathbb{R}^3$. This Morse function gives rise to the handle decomposition in Figure 1.14: a 0-handle, two 1-handles and one 2-handle.

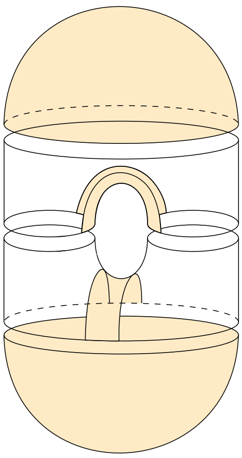


Figure 1.14: The standard embedding of the torus in $\mathbb{R}^3$ gives rise to a handle decomposition consisting of a 0-handle, two 1-handles and one 2-handle.

Example 1.19 (The 'other sphere')**.** In Figure 1.15, an embedding of S^2 in $\mathbb{R}^3$ is given, and we again consider the height function. This function is Morse, and gives rise to a handle decomposition with a 0-handle, a 1-handle and two 2-handles.

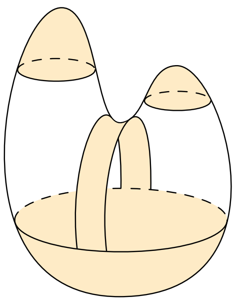
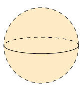
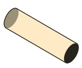


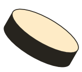
Figure 1.15: Embedding S^2 in $\mathbb{R}^3$ in an unusual way gives rise to a handle decomposition with one 0-handle, one 1-handle and two 2-handles.

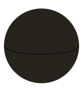
Handles in dimension three

The last examples we will give will be three dimensional and quite a bit more interesting than the previous ones. We have four types of handles:

 0-handle $B^0 \times B^3$ with attachment region $\emptyset$

 1-handle $B^1 \times B^2$ with attachment region $S^0 \times B^2$

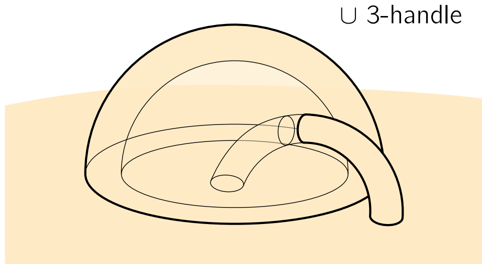
 2-handle $B^2 \times B^1$ with attachment region $S^1 \times B^1$

 3-handle $B^3 \times B^0$ with attachment region $S^2 \times B^0$

Drawing three-dimensional handle decompositions is sometimes difficult. Most of the times we suppress the 0-handle and also the 3-handle is not drawn. If a 3-handle is to be attached, to 'cap it off', we usually write "$\cup$ 3-handle". The first three examples will be (products of) spheres, i.e. S^3, $S^1 \times S^2$ and $S^1 \times S^1 \times S^1$.

Example 1.20 (Handle decomposition of S^3)**.** Consider $S^3 \subset \mathbb{R}^4$ and the height function f, the projection on the last coordinate. Then f has two critical points: $(0, 0, 0, \pm 1)$ of index 0 and 3. The handle decomposition is then a 0-handle glued to a 3-handle, which is in this case two balls B^3 glued along their boundary.

Example 1.21 (Handle decomposition of $S^1 \times S^2$)**.** An example of a handle decomposition of $S^1 \times S^2$ is illustrated in Figure 1.16. The construction starts with a 0-handle, which is just a three dimensional ball. Then we glue on a 2-handle. The resulting manifold is then a thickened up two sphere, $B^1 \times S^2$. Then we add a 1-handle connecting the inside to the outside and we cap off with a 3-handle, which has the effect of identifying the end points of B^1, i.e. this transforms $B^1 \times S^2$ into $S^1 \times S^2$.



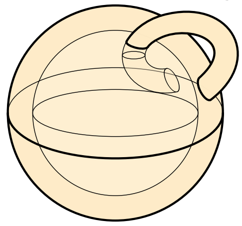


Figure 1.16: Top: $S^1 \times S^2$ can be decomposed using a 0-handle, 1-handle, 2-handle and 3-handle. Bottom: same handle decomposition, but drawn differently to show that a 0-handle and a 1-handle are diffeomorphic to $S^2 \times [0, 1]$.

Example 1.22 ($T^3 = S^1 \times S^1 \times S^1$)**.** To find a handle decomposition of T^3, we identify T^3 with $\mathbb{R}^3/\mathbb{Z}^3$ and consider the following map

$$\begin{aligned} f : T^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto 2\cos(2\pi x) + 3\cos(2\pi y) + 4\cos(2\pi z). \end{aligned}$$

Then we have

$$df = -4\pi \sin(2\pi x) dx - 6\pi \sin(2\pi y) dy - 8\pi \sin(2\pi z) dz,$$

so critical points are all $(x, y, z) \in \mathbb{R}^3/\mathbb{Z}^3$ for which $x, y, z \in \left\{0, \frac{1}{2}\right\}$. In total we have 8 distinct critical points and they are all non-degenerate, because the Hessian of f, given by

$$H_{(x,y,z)}f = -8\pi^2 \cos(2\pi x) dx^2 - 12\pi^2 \cos(2\pi y) dy^2 - 16\pi^2 \cos(2\pi z) dz^2$$

is non-degenerate at each of the critical points. To find the handle decomposition induced by f, we have to determine the index of the critical points. As discussed earlier, the index is the number of downward directions, which is also equivalent to the number of negative eigenvalues of $H_{(x,y,z)}f$. We find that the index of a critical point (x, y, z) is the number of zeros occurring in the triplet (x, y, z). Hence we get the following table:

x	y	z	$f(x, y, z)$	Index
0	0	0	9	3
$\frac{1}{2}$	0	0	5	2
0	$\frac{1}{2}$	0	3	2
0	0	$\frac{1}{2}$	1	2
$\frac{1}{2}$	$\frac{1}{2}$	0	-1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	-3	1
0	$\frac{1}{2}$	$\frac{1}{2}$	-5	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-9	0

We conclude that we can decompose T^3 in one 0-handle, three 1-handles, three 2-handles and one 3-handle. We have illustrated the construction below.

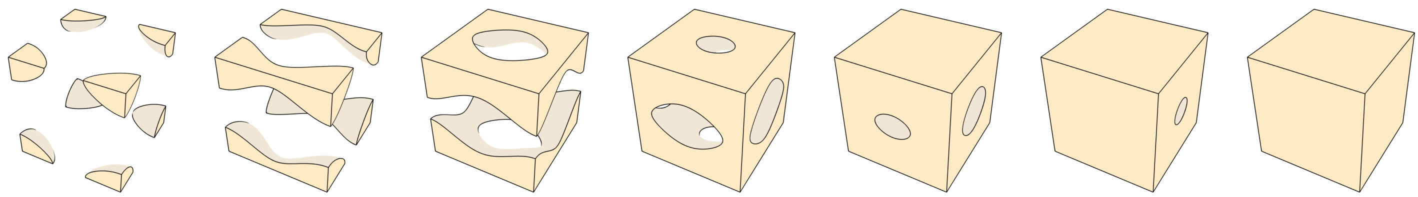


Figure 1.17: Step by step construction of T^3 adding one handle at the time. Here, $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, and $\left[-\frac{1}{2}, \frac{1}{2}\right]^3$ is drawn.
The construction starts with a 0-handle. In the first three steps, each time a 1-handle is added. In the consequent steps, three holes are filled with 2-handles, leaving us with a single void we fill up with a 3-handle.

While this choice of Morse function may seem unmotivated, it is actually based on an example we saw before, namely S^1 with its height function. Indeed, if we identify $S^1 = \mathbb{R}/\mathbb{Z}$, then $g_r(x) = r\cos(2\pi x)$ is exactly the height function of a sphere of radius r. Taking the product of S^1 three times and adding their corresponding Morse functions (with carefully chosen coefficients in order to have distinct critical values), we end up with the given example: $S^1 \times S^1 \times S^1$ and $f = g_2(x) + g_3(y) + g_4(z)$. This method of constructing a Morse function on the product is something that works in general and we will be considering this in more detail later on.

Example 1.23. As a last example, we have illustrated a 3-manifold with boundary in Figure 1.18. It consists of a 0-handle, a 1-handle and a 2-handle. Here, the 2-handle is placed on top of the 1-handle. It is also easy to see that the 1- and 2-handle can be cancelled by an isotopy.

This is an example where the order of the handles is important: we cannot attach the 2-handle before the 1-handle. This is not always the case: in the example discussing the handle decomposition of $S^1 \times S^2$, we see that the 1- and 2-handle are independent. This raises the question: 'Under what conditions can we reorder the attachment of handles?' We will answer this question in the next chapter.

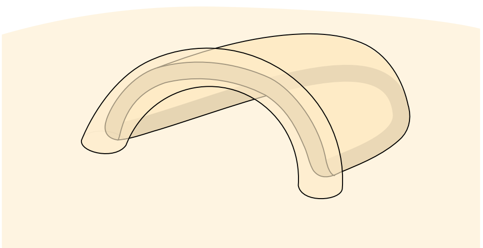


Figure 1.18: An example of a 3-manifold where a 1- and 2-handle can be cancelled.

1.4 Existence and abundance of Morse functions

We end this chapter with a more technical result. In this section we show that any manifold admits uncountably many Morse functions and that the set of Morse functions form a dense subset of all smooth functions $M \to \mathbb{R}$. In other words, almost any function is a Morse function and we can approximate any function by a Morse function.

Let us first recall the following result, due to Whitney.

Theorem 1.24 (Whitney Embedding Theorem)**.** Any smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$.

This allows us to assume that M is a submanifold of $\mathbb{R}^n$, making the constructions more straightforward. The following theorem says that there is an abundance of Morse functions on any manifold.

Proposition 1.25. Let $M \subset \mathbb{R}^n$ be a submanifold. Then for almost every point $p \in \mathbb{R}^n$, we have that

$$f_p : M \to \mathbb{R} : x \mapsto \|x - p\|^2$$

is a Morse function.

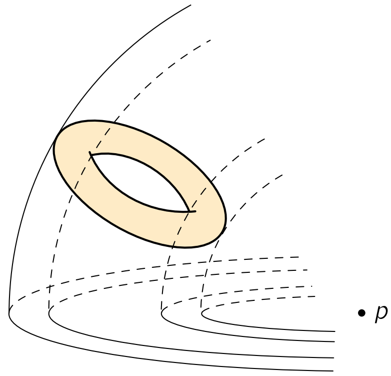


Figure 1.19: An embedding of the torus T^2 in $\mathbb{R}^3$. The level sets of f_p are spheres. We see that f_p has four critical points: a maximum, a minimum and two saddle points.

Example 1.26. Consider Figure 1.19, where we have embedded the torus $T^2 \hookrightarrow \mathbb{R}^3$ and have drawn some level sets of f_p. If should be intuitively clear that f_p is a Morse function.

Let us also give an example when f_p is not a Morse function:

Nonexample 1.27. Let $M = S^1 = \{\|x\|^2 = 1 \mid x \in \mathbb{R}^2\}$. Then the map $f_{(0,0)} : (x_1, x_2) \mapsto x_1^2 + x_2^2$ is not a Morse function. Indeed, $f_{(0,0)} \equiv 1$, so in particular the second derivative is degenerate (it vanishes everywhere).

More generally, f_p is not a Morse function when infinitesimally close normals intersect in p, because then the distance (squared) to that point f_p is constant up to second order. Such points p are called focal points. To make this precise, we will show that H_p is non-degenerate if p is a critical value of the map

$$E : NM \to \mathbb{R}^n : (x, v) \mapsto x + v,$$

where NM is the normal bundle of M. Then Sard's theorem will immediately imply that f_p is Morse for almost all p.

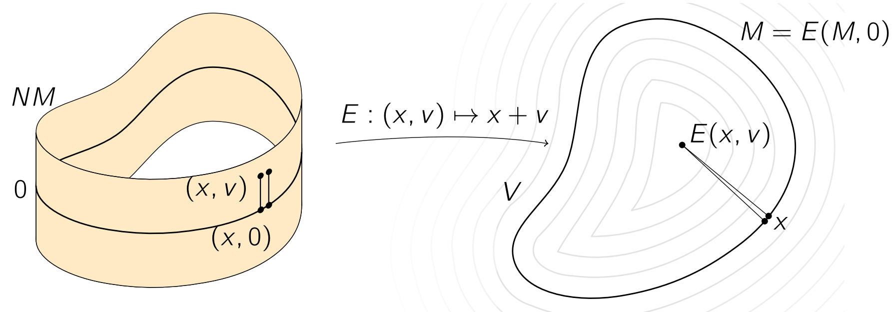


Figure 1.20: Visualization of the normal bundle of M, a submanifold of $\mathbb{R}^n$, and the map E used in the proof.

Proof. With this idea, the proof reduces to a straightforward although tedious calculation. First note that x is a critical point of f_p only if $x - p \perp T_x M$. Indeed:

$$df_p = \sum_{i=1}^n d(x_i - p_i)^2 = 2\sum_{i=1}^n (x_i - p_i)dx_i = 0 \qquad \text{if } x - p \perp T_x M.$$

Let $d = \dim M$ and $(u_1, \dots, u_d) \mapsto x(u_1, \dots, u_d)$ be a local parametrization of M. Then we have

$$\partial_i f_p = 2(x - p) \cdot \partial_i x$$

and for the Hessian we have

$$H_p = \partial_{ij} f_p = 2\left(\partial_j x \cdot \partial_i x + (x - p)\partial_{ij} x\right),$$

where we denoted $\partial_i = \frac{\partial}{\partial u_i}$. We will show that H_p is not of full rank if and only if p is a critical value of

$$E : NM \to \mathbb{R}^n : (x, v) \mapsto x + v,$$

where NM is the normal bundle to M w.r.t. the Euclidean metric on $\mathbb{R}^n$.

First we define a local parametrization of NM:

$$(u_1, \dots, u_d, t_1, \dots, t_{n-d}) \mapsto \Big(x(u_1, \dots, u_d), \sum_{i=1}^{n-d} t_i v_i(u_1, \dots, u_d)\Big),$$

where the v_i form a local orthonormal basis at each point, normal to TM. Then in these coordinates,

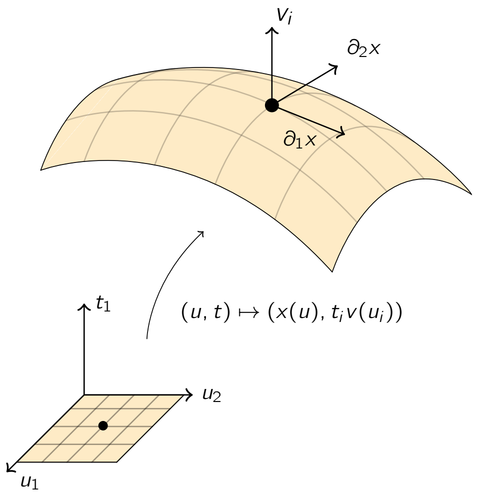


Figure 1.21: Parametrization of the normal bundle of M.

$$\partial_i E = \partial_i x + \sum_{k=1}^{n-d} t_k \partial_i v_k \qquad\qquad \partial_{t_j} E = v_j.$$

To see whether these vectors are independent, we compute the inner products with the n independent vectors $\partial_1 x, \dots, \partial_d x, v_1, \dots, v_{n-d}$. This gives the following matrix with the same rank as E_*:

$$\begin{pmatrix} (\partial_i x \cdot \partial_j x + \sum_k t_k \partial_i v_k \cdot \partial_j x) & \sum_{k=1}^{n-d} \partial_i v_k \cdot v_\ell \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Therefore E_* is of full rank iff the matrix with entries

$$A_{i,j} = \partial_i x \cdot \partial_j x + \sum_{k=1}^{n-d} t_k \partial_i v_k \cdot \partial_j x$$

is of full rank. Now in the second term, we can move ∂_i from v_k to x and get a minus sign in return:

$$\begin{aligned} \partial_i v_k \cdot \partial_j x &= \partial_i (v_k \cdot \partial_j x) - v_k \cdot \partial_i \partial_j x \\ &= -v_k \cdot \partial_i \partial_j x, \end{aligned}$$

as $v_k \perp \partial_j x$. This allows us to rewrite the matri

$$\begin{aligned} A_{ij} &= \partial_i x \cdot \partial_j x - \sum_{k=1}^{n-d} t_k v_k \cdot \partial_i \partial_j x \\ &= \partial_i x \cdot \partial_j x - v \cdot \partial_i \partial_j x \\ &= \partial_i x \cdot \partial_j x + (x - p) \cdot \partial_i \partial_j x \\ &= (H_p)_{ij}. \end{aligned}$$

Therefore, E_* is of full rank iff H_p is of full rank. This concludes the proof. □

Apart from the abundance of Morse functions, we also have that any smooth function can be approximated by a Morse function. By this we mean the following:

Proposition 1.28. Let $f : M \to \mathbb{R}$ be a function. Let k be an integer. Then f and all its derivatives of order $\leq k$ can be uniformly approximated by Morse functions on every compact subset.

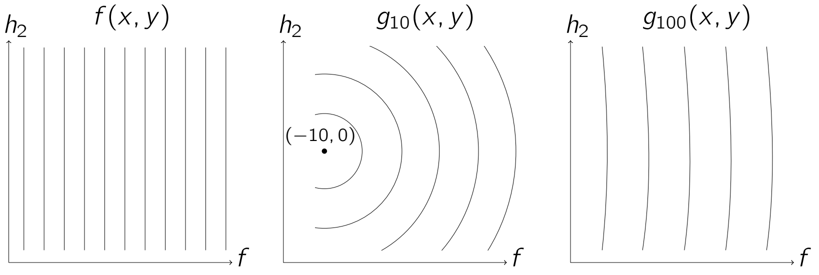


Figure 1.22: We can approximate any smooth function with a Morse function. On the left, we plotted the level sets of f itself. Because the coordinates are f and h_2, these level sets are vertical planes. The two right plots show level sets of g_c for $c = 10$ and $c = 100$, which are circles. We see that g_c approximates f if $c \to \infty$.

The idea of the proof goes as follows. We choose an embedding of M where f is the first coordinate on M, so we can think of f as a simple projection: $x \mapsto x_1$.[20] Then this function can be approximated in the following way:

$$x_1 \approx \frac{(x_1 + c)^2 - c^2}{2c} \qquad \text{as } c \to \infty,$$

and even when taking extra dimensions into account, the approximation still works:

$$x_1 \approx \frac{\|x - p\|^2 - c^2}{2c} \qquad \text{with } p = (-c, 0, \dots, 0) \text{ and } c \to \infty.$$

Now, note that the right hand side is almost always a Morse function. Hence, we can the proposition as follows.

Proof. Using the Whitney embedding theorem, embed M in $\mathbb{R}^n$ for n sufficiently large such that f is the first coordinate:

$$h(x) = (f(x), h_2(x), \dots, h_n(x)).$$

Let $c \in \mathbb{R}$. For almost every point $p = (-c + \epsilon_1, \epsilon_2, \dots, \epsilon_n)$, the function

$$g_c(x) = \frac{\|x - p\|^2 - c^2}{2c}$$

is Morse. Then

$$\begin{aligned} g_c(x) &= \frac{1}{2c} \sum_{i=1}^{n} (x_i - p_i)^2 - c^2 \\ &= \frac{1}{2c} \left((f(x) + c - \epsilon_1)^2 + (h_2(x) - \epsilon_2)^2 + \cdots + (h_n(x) - \epsilon_n)^2 \right) \\ &= f(x) + \frac{f(x)^2 + \sum h_i(x)^2}{2c} - \frac{\epsilon_1 f(x) + \sum \epsilon_i h_i(x)}{c} + \sum_{i=1}^{n} \epsilon_i^2 - \epsilon_1. \end{aligned}$$

This concludes the proof. Indeed, let K be a compact subset of M. The functions $\frac{d^j}{dx^j}(f(x)^2 + \sum h_i(x)^2)$ for $j = 1, \dots, k$ all attain their maximum on K, so by choosing c big enough, we can make them simultaneously arbitrarily small in a uniform way. Similarly for the third term. Lastly, we can also make ϵ_i arbitrarily small while still retaining that g is a Morse function. $\square$

[20] Start with an embedding $\phi : M \hookrightarrow \mathbb{R}^n$ for large enough n. Then consider $\psi : M \to \mathbb{R}^{n+1} : x \mapsto (f(x), \Phi(x))$. Then the height function of this embedding is exactly f.

CHAPTER TWO

Stable and unstable manifolds

In this chapter, we address two natural questions that came up when discussing examples of handle decompositions: 'When can we reorder handles in a handle decomposition?' and 'Under what conditions can we cancel two critical points?' To answer these questions, we introduce the concept of stable and unstable manifolds. While we previously viewed handles as a local phenomenon, these new concepts will allow us to understand how multiple handles can interact on a more global scale. Moreover, stable and unstable manifolds will give rise to trajectory spaces between critical points which will play a key role in the upcoming chapters.

2.1 Definition of stable and unstable manifolds

Stable and unstable manifolds associated to a critical point p consist of points that under the flow of the (negative) gradient reach p in the limit.

Definition 2.1. Let p be a critical point of a Morse function f. Denote by ψ^t the flow of a pseudo-gradient. Then the unstable manifold is defined as

$$W^u(p) = \{x \in M \mid \lim_{t\to-\infty} \psi^t(x) = p\},$$

and its stable manifold is defined as

$$W^s(p) = \{x \in M \mid \lim_{t\to\infty} \psi^t(x) = p\},$$

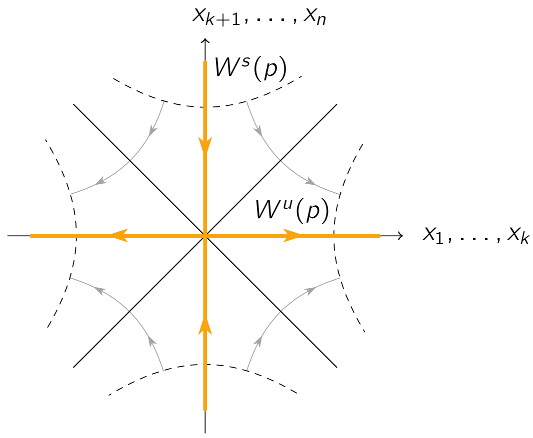


Figure 2.1: Locally in a Morse chart, stable and unstable manifolds are given by the vertical and horizontal axis, i.e. $x_1 = \cdots = x_k = 0$ and $x_{k+1} = \cdots = x_n = 0$.

Just like their names imply, these sets are indeed manifolds. Around a critical point p, the unstable manifold $W^u(p)$ is in a Morse chart U given by $x_{k+1} = \cdots = x_n = 0$, so it is diffeomorphic to an open disk D^k. The points in $W^u(p)$ that lie outside U can then be obtained by flowing the boundary of D^k along the pseudo-gradient, which is diffeomorphic to $S^{k-1} \times \mathbb{R}$. All in all when gluing this to D^k, we end up with something that is indeed a manifold diffeomorphic to D^k. A similar reasoning for the stable manifold shows that $W^s(p) \cong D^{n-k}$. In summary, we have obtained the following result:

Proposition 2.2. Stable and unstable manifolds of a critical points are submanifolds diffeomorphic to open disks. Moreover,

$$\dim W^u(p) = \operatorname{codim} W^s(p) = \operatorname{Ind} p.$$

Example 2.3. Let us consider T^2 embedded in $\mathbb{R}^3$ in the standard way and consider the height function. This function has 4 critical points and is clearly Morse. Let $X = -\operatorname{grad} f$ be the negative gradient of f w.r.t. the standard metric on $\mathbb{R}^3$. Then the stable and unstable manifolds of the critical points of f are illustrated in Figure 2.2. We have similarly done so for an embedding of S^2 in Figure 2.3.

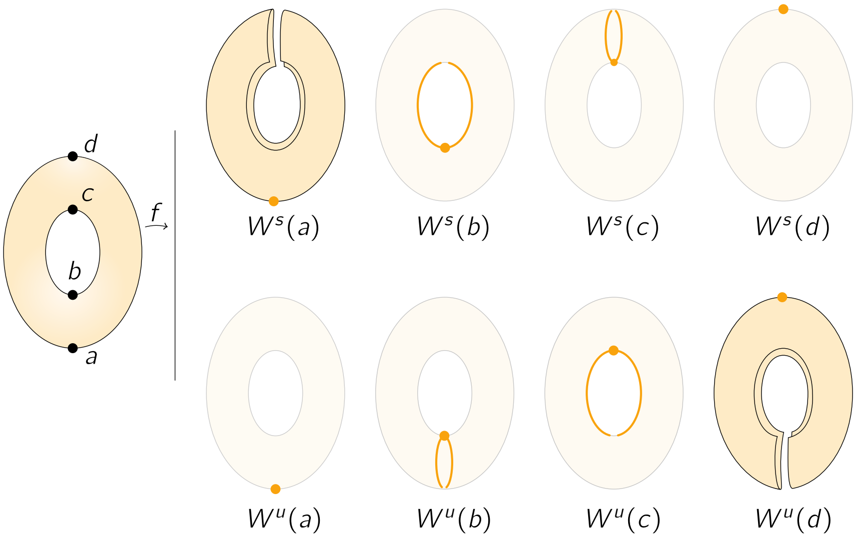


Figure 2.2: Stable and unstable manifolds for all critical points of the height function on the torus with the standard gradient on $\mathbb{R}^3$. All of them are diffeomorphic to either D^2, D^1 or D^0. Note that $W^s(b)$ and $W^u(c)$ do not intersect transversely.

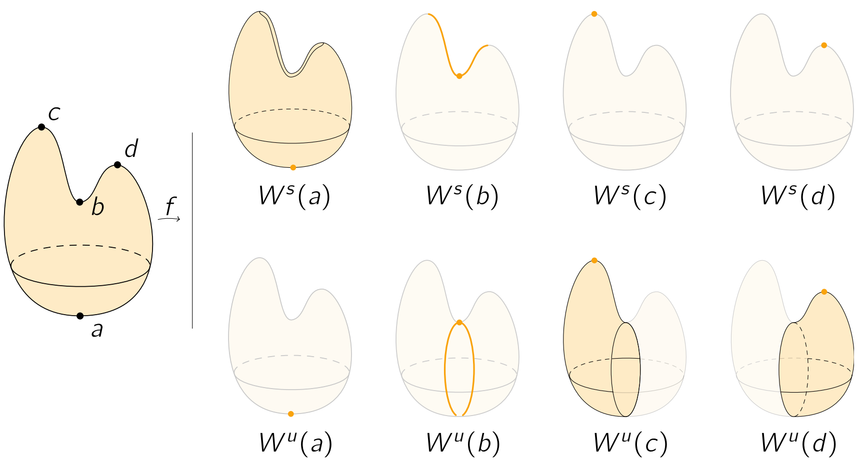


Figure 2.3: Stable and unstable manifolds of the 'other sphere'. Note that all of them intersect transversely.

2.2 Intersections of (un)stable manifolds

Stable and unstable manifolds give us information about interaction of handles. For example, consider the situation in Figure 2.4. None of the stable and unstable manifolds intersect, indicating the independence of the two handles. This allows us attach both handles at the same time or attach them in a different order. In terms of the Morse function, this means we can modify f such that $f(p) > f(q)$.

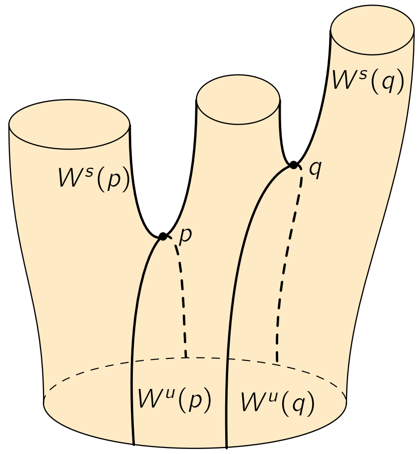


Figure 2.4: A cobordism from S^1 to $S^1 \sqcup S^1 \sqcup S^1$. Stable and unstable manifolds do not intersect, which implies we can reorder the critical points p and q.

In Figure 2.5, the situation is different. The intersection of $W^s(p)$ and $W^u(q)$ is not empty, indicating the dependence of the 2-handle associated to q on the 1-handle associated to p. We cannot first attach the 2-handle first and then attach the 1-handle.

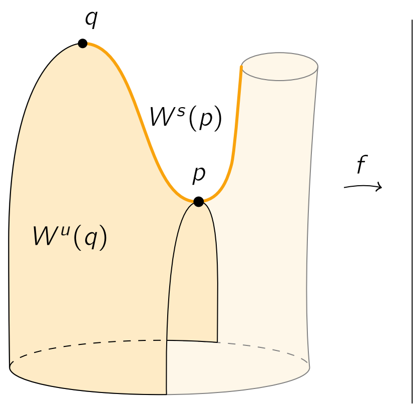


Figure 2.5: The intersection of stable and unstable manifolds is not empty, indicating the dependence of the 2-handle on the 1-handle.

This motivates us to in investigate necessary and sufficient conditions for the stable and unstable manifolds to intersect. Suppose p and q are two critical points of a Morse function $f : M \to \mathbb{R}$ and let us consider $W^s(p) \cap W^u(q)$. If $f(p) > f(q)$, then $W^s(p) \cap W^u(q) = \emptyset$, because $f(W^s(p)) > f(W^u(q))$. In the other case when $f(p) \leq f(q)$, the intersection $W^s(p) \pitchfork W^u(q)$ can be non-empty. Moreover, if we assume that the intersection is transverse, we can say something about its dimension. Indeed, Proposition 0.2 implies that

$$\operatorname{codim}(W^s(p) \pitchfork W^u(q)) = \operatorname{codim} W^s(p) + \operatorname{codim} W^u(q),$$

so we have

$$\dim(W^s(p) \pitchfork W^u(q)) = \operatorname{Ind} q - \operatorname{Ind} p.$$

In particular, we have that if $\operatorname{Ind} q < \operatorname{Ind} p$, $W^s(p) \pitchfork W^u(q) = \emptyset$, which in other words means that lower index handles do not depend on higher index handles, i.e. we can always attach lower index handles before higher index ones.

This transversality assumption has a name:

Definition 2.4 (Smale condition)**.** A pseudo-gradient field addapted to a Morse function f is said to satisfy the *Smale condition* if for all pairs of critical points $\{p, q\} \subset \operatorname{Crit} f$, we have that $W^s(p)$ intersects $W^u(q)$ transversely, i.e.

$$W^s(q) \pitchfork W^u(p) \text{ for all } p, q \in \operatorname{Crit} f.$$

It turns out that this condition is not at all restricting: we can always perturb the pseudo-gradient field such that it satisfies the Smale condition.

Theorem 2.5. Any Morse function $f : M \to \mathbb{R}$ admits a pseudo-gradient field that satisfies the Smale condition

This should be intuitively clear as transverse intersections are generic and stable. This means that a small perturbation of $W^s(q)$ and $W^u(p)$ is enough to make their intersection transverse, and we can accomplish this perturbation by perturbing X. A rigorous proof of this is rather technical and we refer the reader to Audin and Damian.[21]

[21] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014

Almost all of the previous examples we have given satisfy the Smale condition with the one exception being the torus.

Nonexample 2.6. The gradient vector field in Example 2.3 does not satisfy the Smale condition: the intersection of $W^s(b)$ and $W^u(c)$ is not transverse. Even more, if the condition were satisfied, there would be no trajectories connecting b and c, because both have index 1, so $\dim(W^s(b) \pitchfork W^u(c)) = 0$. However, as illustrated in the figure, we have two such paths.

Example 2.7. Instead of considering the embedding as in the previous example, we can embed the torus at a slight angle. Then the gradient of the height function (by using the standard metric on $\mathbb{R}^3$) does satisfy the Smale condition. The intersection of $W^s(b)$ and $W^u(c)$ is no longer tangent: indeed, they do not even intersect at all. We have illustrated this below.

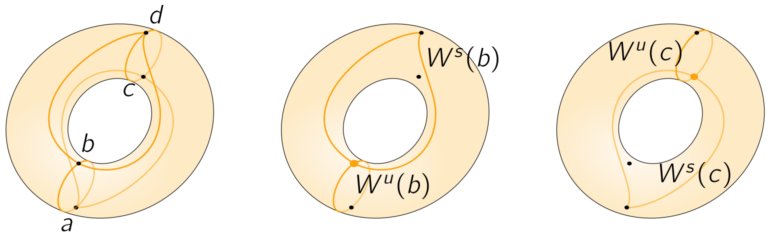


Figure 2.6: When embedding the torus in $\mathbb{R}^3$ tilted, the Smale condition is satisfied: all stable and unstable manifolds intersect transversely. Indeed, stable and unstable manifolds of c and b don not intersect at all.

Example 2.8. As illustrated in Figure 2.3, the 'other sphere' with gradient of the height function w.r.t the metric of $\mathbb{R}^3$ does satisfy Smale condition.

The Smale condition also has another interesting consequence, which we have not touched upon. If stable and unstable manifolds intersect transversely, we know that the intersection is again a submanifold. So $W^s(q) \pitchfork W^u(p)$ is a manifold for all critical points $p, q \in \operatorname{Crit} f$. This submanifold consists of all points on the trajectories connecting p to q.

Definition 2.9. Let $f : M \to \mathbb{R}$ be a Morse function and ψ^t the flow of a pseudo-gradient that satisfies the Smale condition. Then we define

$$\begin{aligned} \mathcal{M}(p, q) &= W^s(q) \pitchfork W^u(p) \\ &= \left\{x \in M \mid \lim_{t\to-\infty} \psi^t(x) = p, \ \lim_{t\to\infty} \psi^t(x) = q\right\}, \end{aligned}$$

which is a submanifold of dimension $\operatorname{Ind} p - \operatorname{Ind} q$.

Example 2.10. Consider the 'other sphere'. We have illustrated $\mathcal{M}(p, q)$ below for some of the critical points of the height function. Here we see that these type of submanifolds do not need to be connected. For example, $\mathcal{M}(b, a)$ is diffeomorphic to the disjoint union of two open intervals.

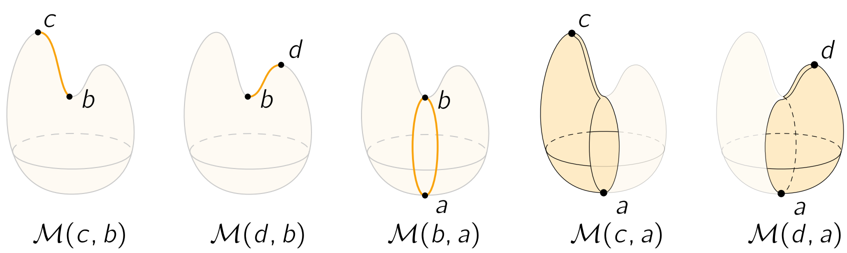


Figure 2.7: Illustration of $\mathcal{M}(p, q)$ for critical points of the 'other sphere'.

Instead of considering a manifold consisting of of points lying on all trajectories from p to q, $\mathcal{M}(p,q)$, it is often more interesting to construct a manifold each point corresponds to exactly one trajectory, that is a so-called moduli space of trajectories. We can do this by modding out $\mathcal{M}(p,q)$ by $\mathbb{R}$-action of translations in time. We denote the resulting space with $\mathcal{L}(p,q)$. More explicitly, we have the following:

Proposition 2.11. Let $f : M \to \mathbb{R}$ be a Morse function and ψ^t the flow of a pseudo-gradient field satisfying the Smale condition. Then the group $(\mathbb{R},+)$ of time translations acts on $\mathcal{M}(p,q)$ by $t \cdot x = \psi^t(x)$. If $p \neq q$ then the action is free and we can define $\mathcal{L}(p,q) = \mathcal{M}(p,q)/\mathbb{R}$. The dimension of $\mathcal{L}(p,q)$ is $\operatorname{Ind} p - \operatorname{Ind} q - 1$.

Proof. It is clear that $\mathbb{R}$ acts on $\mathcal{M}(p,q)$ by time translations. If $p \neq q$, $\mathcal{M}(p,q)$ does not contain any critical point, so flowing along a pseudo-gradient field, the value of f is strictly decreasing. This proves freeness. □

Remark 2.12. If the index of two points only differs by one, say $\operatorname{Ind} p = \operatorname{Ind} q + 1$, then the dimension of $\mathcal{L}(p,q)$ is 0, so it is a discrete set. This proves that the number of trajectories from p to q is always countable. We will later prove that it is in fact finite.

Remark 2.13. Another way to look at $\mathcal{L}(p,q)$ is to consider a regular value $a \in \mathbb{R}$ such that $f(p) < a < f(q)$. Then every flow line from p to q intersects $f^{-1}(a)$ exactly once (this is because the value of f is decreasing along the way), so we can identify $\mathcal{L}(p,q)$ with $\mathcal{M}(p,q) \cap f^{-1}(a)$.

Example 2.14. Below, we have illustrated the moduli space of trajectories for the 'other sphere'. When there is a single trajectory for example between c and b, $\mathcal{L}(c,b)$ consist of a single point. If the indices of critical points differ by two, as is the case for d and a, $\mathcal{L}(d,a)$ is a one-dimensional manifold describing a one-parameter family of flow lines between d and a.

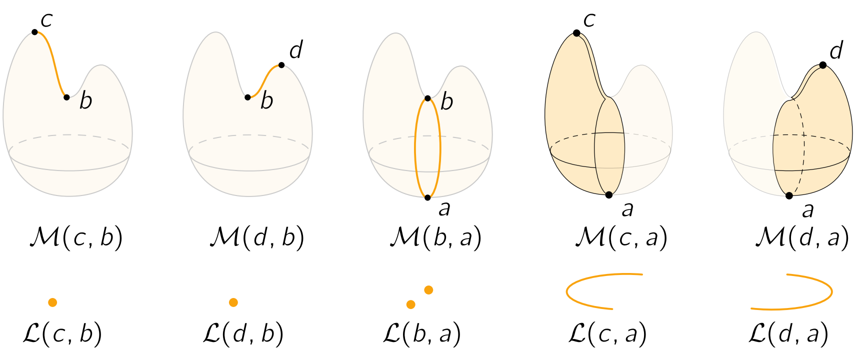


Figure 2.8: Illustration of $\mathcal{M}(p,q)$ and $\mathcal{L}(p,q)$ for critical points of the 'other sphere'.

2.3 Reordering critical points

Now that we have introduced the (un)stable manifolds, we are ready to prove a first reordering theorem. The statement and proof can be found in notes of 'Lectures on the h-Cobordism Theorem' by Milnor.[22]

[22] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015

Theorem 2.15. Let $f : M \to \mathbb{R}$ be a Morse function on a cobordism M from M_0 to M_1 with two critical points p and p'. Suppose that for some choice of pseudo-gradient field X, the stable and unstable manifolds do not intersect. Let $a, a' \in (0, 1)$ be arbitrary. Then there exists a new Morse function g such that

(a) X is a gradient-like vector field for g

(b) The critical points of g are still p, p' and $g(p) = a$, $g(p') = a'$.

(c) g agrees with f near $M_0 \sqcup M_1$ and equals f plus a constant in some neighbourhood of p and some neighbourhood of p'.

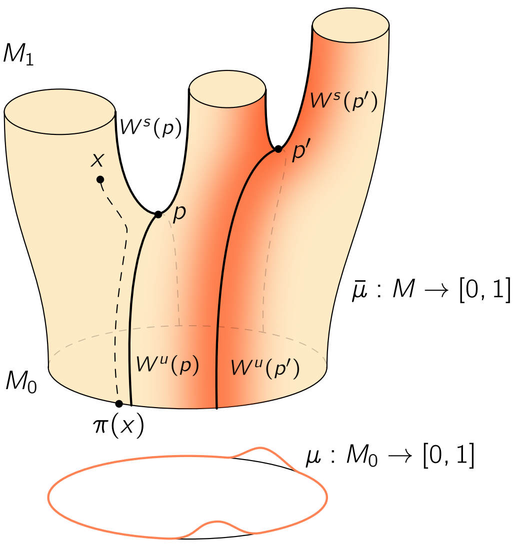


Figure 2.9: Construction of $\overline{\mu}$ and π in the proof on reordening critical points.

Proof. We want to mask out the area around the stable and unstable manifolds of one of the critical points. Let $\mu : M_0 \to [0, 1]$ be a smooth map that vanishes around $M_0 \cap W^u(p)$ and 1 around $M_0 \cap W^u(p')$, as illustrated in Figure 2.9. Then we can smoothly extend this to a function on the whole manifold M as follows. Define $\pi : M \to M_0$ by flowing along the pseudo-gradient field until we reach M_0. Then we can extend μ uniquely to a smooth function that is constant on each trajectory by defining

$$\begin{aligned} \overline{\mu} : M &\longrightarrow [0,1] \\ x &\longmapsto \begin{cases} 0 & \text{if } x \text{ in stable or unstable manifold of } p \\ 1 & \text{if } x \text{ in stable or unstable manifold of } p' \\ \mu(\pi(x)) & \text{else} \end{cases} \end{aligned}$$

We have illustrated this in the figure by indicating the value of $\overline{\mu}$ in red.

Define a new Morse function $g : M \to [0, 1]$ by $g(q) = G_{\overline{\mu}(q)}(f(q))$, where $G_s(x)$ is a smooth family of smooth functions $G_s : [0, 1] \to [0, 1]$ with $s \in [0, 1]$ that has the following properties, also indicated in Figure 2.10.

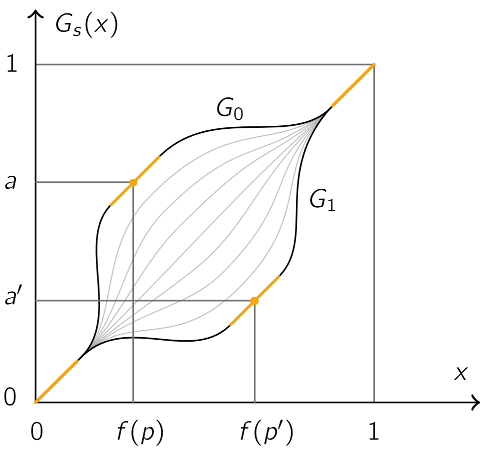


Figure 2.10: Necessary properties of G in the proof on reordering critical points are indicated in yellow.

(1) For all s, $G_s' > 0$ and $G_s(0) = 0, G_s(1) = 1$

(2) $G_0(f(p)) = a$
$G_1(f(p')) = a'$

(3) $G_s(x) = x$ for x near 0 or 1 and for all s

(4) $G_0'(x) = 1$ for x in a neighbourhood of $f(p)$
$G_1'(x) = 1$ for x in a neighbourhood of $f(p')$

Claim (b) and (c) in the statement of the theorem are clear: they follow immediately from (2), (3) and (4). For (a), consider

$$dG = \frac{\partial G}{\partial \overline{\mu}} d\overline{\mu} + \frac{\partial G}{\partial f} df.$$

Plugging in X, we have

$$\begin{aligned} dg(X) &= \frac{\partial G}{\partial \overline{\mu}} d\overline{\mu}(X) + \frac{\partial G}{\partial f} df(X) \\ &= \frac{\partial G}{\partial f} df(X) < 0, \text{ except at critical points of } f, \end{aligned}$$

where we used that $d\overline{\mu}(X) = 0$ by construction of $\overline{\mu}$, $\frac{\partial G}{\partial f} > 0$ by (1) and $df(X) < 0$ everywhere except at critical points of f by definition of pseudo-gradient field. Because of (4), this implies that g and f share the same behaviour around their critical points, proving that g is also Morse. $\square$

Remark 2.16. This theorem can be extended to a more general setting. Suppose we have a set of points $\mathbf{p} = \{p_1, \dots, p_k\}$ and $\mathbf{p}' = \{p'_1, \dots, p'_\ell\}$, with all p_i at the same level and all p'_i at a single level. Then the theorem remains valid, with exactly the same proof.

Applying the previous theorem repeatedly and using the fact that if $\operatorname{Ind} p \leq \operatorname{Ind} q$, the intersection of $W^u(p)$ and $W^s(q)$ is empty, we find the following result:

Theorem 2.17. Any closed manifold admits a Morse function such that for all critical points, $\operatorname{Ind} p < \operatorname{Ind} q \Rightarrow f(p) < f(q)$. In particular, it admits a Morse function which satisfies $\operatorname{Ind} p = f(p)$ for all critical points $p \in \operatorname{Crit} f$.

In other words, this asserts that lower index handles can always be attached before higher index ones. Morse functions as described in the second part of the theorem have a name:

Definition 2.18 (Self-indexing Morse function)**.** A Morse function $f : M \to \mathbb{R}$ is *self-indexing* if $\operatorname{Ind} p = f(p)$ for all critical points p of f.

Example 2.19. Consider $T^2 \subset \mathbb{R}^3$ embedded at an angle. We have illustrated the side view in Figure 2.11. Then the height function is a self-indexing Morse function.

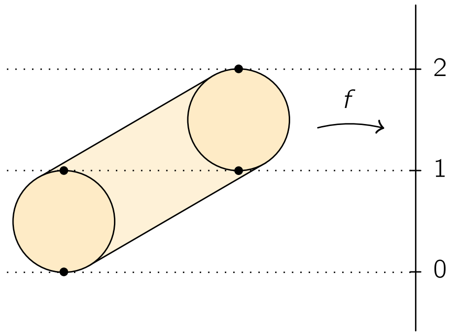


Figure 2.11: When tilting the torus to the right angle, the height function becomes self-indexing.

2.4 Heegaard splittings

In the three-dimensional setting, the existence of self-indexing Morse functions give rise so-called Heegaard splittings. Heegaard splittings form an essential tool for a low-dimensional topologist (studying manifolds of dimension ≤ 4) and give a way to understand 3-manifolds. As Heegaard splittings will not play an important role in the upcoming chapters, we will only discuss them briefly.

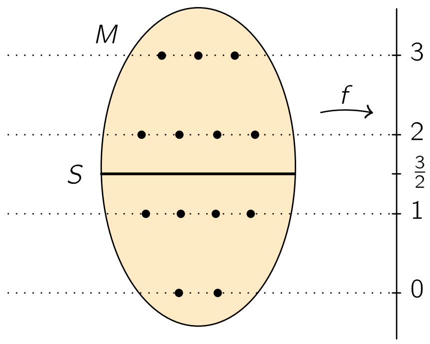


Figure 2.12: Schematic visualization of a self-indexing Morse function on a 3-manifold M. The manifold $S = f^{-1}(\frac{3}{2})$ is called the splitting surface of M.

A self-indexing Morse function gives rise to a handle decomposition schematically shown in Figure 2.12. When splitting the manifold along $f^{-1}(\frac{3}{2})$, we decompose M in two parts: a part that consists of 0- and 1-handles and a part consisting only of 2- and 3-handles. To build M we glue these two parts along their boundary. The part consisting of 2- and 3-handles can also be seen as being constructed of 0- and 1-handles, simply by building M from top to bottom by considering the Morse function $-f$ instead of f. This then interchanges k- and $n-k$-handles, which in the three-dimensional case results in $2 \leftrightarrow 1$ and $3 \leftrightarrow 0$. All things considered, we have a decomposition of M in two so-called handlebodies:

Definition 2.20 (Genus k handlebody[23]). A genus k handlebody is a compact connected orientable 3-manifold with boundary that possesses a handle decomposition consisting of 0-handles and 1-handles such that its boundary is a surface of genus k.

[23] Jennifer Schultens. *Introduction to 3-manifolds*. Vol. 151. American Mathematical Soc., 2014

A Heegaard splitting is then defined as follows.

Definition 2.21 (Heegaard splitting[23]). A *Heegaard splitting* of a closed 3-manifold M is a decomposition $M = V \cup_S W$ such that V and W are genus k handlebodies and $S = \partial V = \partial W$. Here S is called the *splitting surface of M*. Two Heegaard splittings are considered *equivalent* if their splitting surfaces are isotopic. The *genus* of a Heegaard splitting is the genus of S.

With these definitions, we can summarize our findings as follows:

Theorem 2.22. Every closed orientable 3-manifold admits a Heegaard splitting.

Proof. Let f be a self-indexing Morse function on M. Then $M = f^{-1}\left[0, \frac{3}{2}\right] \cup f^{-1}\left[\frac{3}{2}, 3\right]$ is a Heegaard splitting of M by duality of k- and $n-k$-handles. $\square$

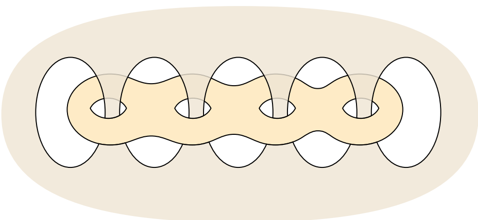


Figure 2.13: A genus four Heegaard splitting of S^3, seen as the one point compactification of $\mathbb{R}^3$. This way, we can obtain a Heegaard splitting of S^3 of any genus.

Let us give some examples.

Example 2.23. As seen earlier, the 'height' function on S^3 gives rise to a handle decomposition $S^3 = B^2 \cup B^2$. This is a Heegaard splitting of genus 0 with splitting surface S^2.

Example 2.24. The previous example is not the only way to decompose S^3. Indeed, we can arbitrarily add cancelable 1- and 2-handles (as in Example 1.23) to increase the genus. For example a genus four Heegaard splitting of S^3 is illustrated in Figure 2.13. Here, we visualized S^3 as the one point compactification of $\mathbb{R}^3$.

Example 2.25. If we split the handle decomposition of T^3 given in Example 1.22 between index 1 and 2 critical points, then we find a splitting surface that has genus three, as illustrated in Figure 1.17.

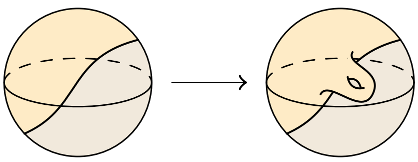


Figure 2.14: The act of stabilization is replacing a ball near the boundary as illustrated, increasing the genus of the splitting surface by one.

Remark 2.26. The act of increasing the genus, as done in Example 2.24 is called stabilization. It can be done in as in Figure 2.14. The reverse procedure, called destabilization cannot be done in general.

2.5 Cancellation of critical points

Let us now answer the second question that came up when discussing examples in the previous chapter: 'When can we cancel a pair of critical points p, q?' It turns out that the existence of a unique trajectory connecting p to q is a sufficient condition, as the following theorem states.

Theorem 2.27 (Cancellation theorem[24]). Let M be a cobordism from M_0 to M_1. Let $f : M \to [0, 1]$ be a Morse function with exactly two critical points p, q of index $k + 1$ and k and let X be a pseudo-gradient adapted to M.

If $\#\mathcal{L}(p, q) = 1$, that is if there is a single trajectory $\ell = \mathcal{M}(p, q)$ connecting p and q, then we can cancel p and q. More specifically, we can alter X around ℓ and f away from M_0, M_1 such that f has no critical points.

[24] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015

Below, we have illustrated the alteration of the pseudo-gradient X:

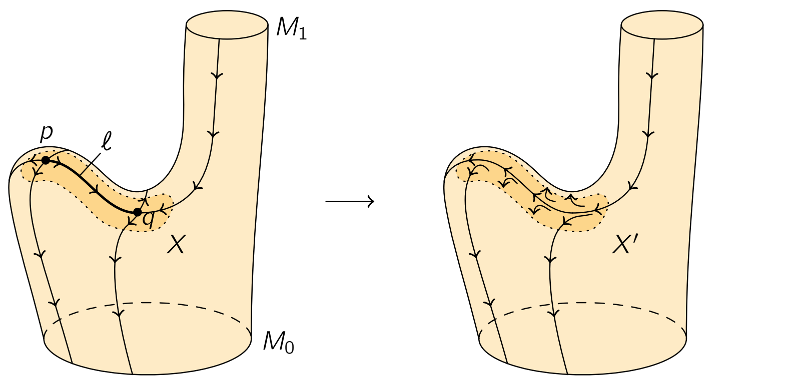


Figure 2.15: The pseudo-gradient before and after the alteration. Initially, X vanishes twice (once at p and once at q). After the alteration, X' does vanishes nowhere.

Before proving this theorem, let us give an example showing the importance of the conditions in the theorem.

Example 2.28. Consider the 'other sphere' as in Figure 2.16, where we have also drawn trajectories between critical points of consecutive index.

For the pair (d, b), let $N = f^{-1}([f(b)-\epsilon, f(d)+\epsilon])$ be a cobordism containing only b and d as critical points. There is a unique flow line connected d to b, and the theorem allows us to cancel these two critical points.

At first glance, the theorem does not allow us to cancel c and b: while there is a unique trajectory connecting c to b, we cannot cut out a part of M only containing c and b as critical points. This however turns out not be a problem, because we can lower c and raise d using methods discussed before such that $f(d) > f(c)$. Once that is done, we can consider $N = f^{-1}([f(b)-\epsilon, f(c)+\epsilon])$ and apply the theorem.

The last pair of critical points of adjacent index is (b, a). In this case, there

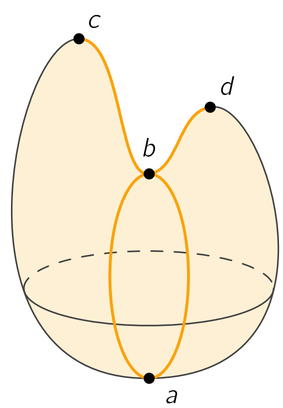


Figure 2.16: The 'other sphere' with trajectories between critical points whose index differ by exactly one. We can only cancel b and c or d and b. Cancelling b and a is impossible.

are two flow lines connecting b to a and we cannot cancel the critical points.

Proof. We will follow the proof given by Milnor which is based on the original proof by Morse.[25] For a proof focussing more on handles than on critical points, we refer the reader to 'Differential Geometry' by Kosinski.[26] There is also a nice proof by Laudenbach[27] reducing the general case to dimension one, where the problem is easy to solve, as we have seen in Example 1.6.

We prove the statement in a local model. Let U_ℓ be an open neighbourhood of ℓ. We may assume that there are coordinates such that:

- The critical points are given by $p = (0, \dots, 0)$ and $q = (1, 0, \dots, 0)$
- The pseudo-gradient is given by $X = (v(x_1), x_2, \dots, x_k, -x_{k+1}, \dots, -x_n)$ where $v(x_1)$ is a smooth function of x_1, such that v is positive on $(0, 1)$, vanishes at 0 and 1 and is negative elsewhere. Moreover, we assume that $v'(x_1) = 1$ near 0 and 1.

We have illustrated these properties in Figure 2.17 and formal proof of this fact can be found in the notes by Milnor.

[25] Stewart Scott Cairns. *Differential and Combinatorial Topology: A Symposium in Honor of Marston Morse (PMS-27)*. Princeton University Press, 2015; William Huebsch and Marston Morse. "The bowl theorem and a model nondegenerate function". In: *Proceedings of the National Academy of Sciences of the United States of America* 51.1 (1964), p. 49

[26] Antoni A Kosinski. *Differential manifolds*. Courier Corporation, 1993

[27] François Laudenbach. "A proof of Morse's theorem about the cancellation of critical points". In: *Comptes Rendus Mathematique* 351.11-12 (2013), pp. 483–488

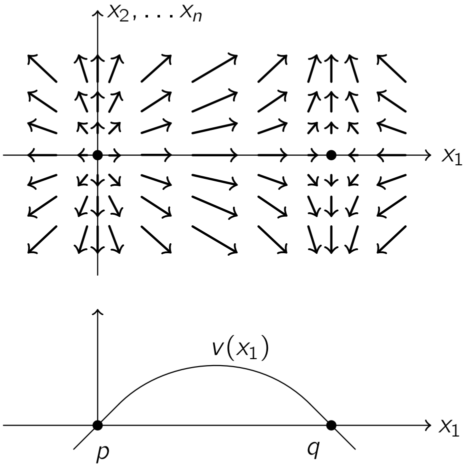


Figure 2.17: Model of the pseudo-gradient vector field X. In local coordinates it is given by $X = (v(x_1), x_2, \dots, x_k, -x_{k+1}, \dots, -x_n)$, where v is as illustrated above.

Assertion 1. Given an open neighbourhood U of ℓ, we can find a smaller neighbourhood U' such that no trajectory exiting U' enters U' again.

Note that this can be false when there are multiple trajectories connecting p to q, as is the case with $p = b, q = a$ in Figure 2.16.

Proof. Suppose this was not the case. Then there would exists a sequence of trajectories ℓ_k that pass through points r_k, s_k, t_k with $s_k \notin U$ and r_k and t_k approaching ℓ. We may assume that $s_k \to s$ because $M \setminus U$ is compact. The trajectory through s, call it ℓ_∞ comes from M_0 or reaches M_1 (or both), as it would otherwise be a trajectory connecting p and q not equal to ℓ. Suppose without loss of generality that it comes from M_1. Then the tail of the sequence of trajectories ℓ_k passing through points s_k near s also originate in M_1. This means that the minimal distance between points on ℓ_k and points on ℓ is bounded from below by a positive number for k big enough. Now, because $r_k \in \ell_k$, the points r_k cannot approach ℓ. This is a contradiction. ■

Let U and U' be neighbourhoods of ℓ such that $\ell \subset U' \subset U \subset \overline{U} \subset U_\ell$ such that U' satisfies the conditions of the previous assertion.

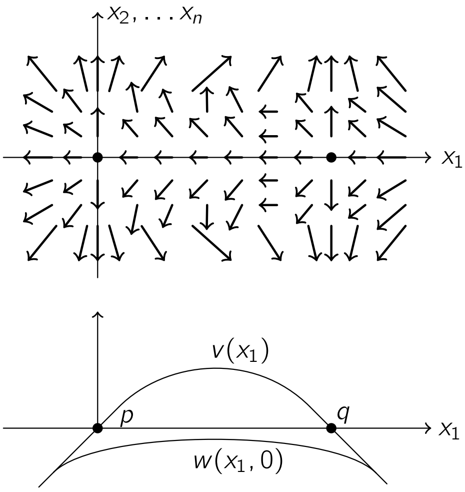


Figure 2.18: Alteration of the pseudo-gradient. In particular, notice that the alteration vanishes nowhere.

Assertion 2. We can alter X on a compact subset of U' in such a way that any point in U will exit U when flowing both backwards and forwards in time. In other words, any trajectory that contains a point in U has entered U and will exit U. Note that the altered vector field is not longer adapted to f.

Proof. Replace X by $(w(x_1, \rho), x_2, \dots, -x_n)$. Here, $\rho = \sqrt{x_2^2 + \cdots + x_n^2}$ is the norm of the coordinates normal to ℓ and $w(x_1, \rho)$ satisfies the following properties, also illustrated in Figure 2.18.

- The vector field $w(x_1, \rho(x))$ is equal to $v(x_1)$ outside a compact neighbourhood of ℓ in U'.
- For $\rho = 0$, the vector field $w(x_1, 0)$ is everywhere negative.

To prove the claim, let $x^0 = (x_1^0, \dots, x_n^0) \in U$ and let $x(t)$ the unique trajectory such that $x(0) = x^0$.

(a) If one of $x_{k+2}^0, \dots, x_n^0$ is non-zero, say x_m. Then $|x_m(t)|$ increases exponentially, so the trajectory leaves U.

(b) If all of $x_{k+2}^0, \dots, x_n^0$ are zero, then ρ decreases exponentially, so it gets closer to the x_1-axis. Since $w(x_1, \rho(x))$ is strictly negative on the x_1-axis, it is also negative on a compact set $K = \{x \in U \mid \rho(x) \leq \delta\}$ for some small δ. Therefore, $w(x_1, \rho(x))$ has a negative upper bound $-\alpha < 0$ on K. As ρ decreases exponentially, eventually $x \in K$ and $x_1'(t) < -\alpha$ from then onwards. As U is bounded, we conclude that x leaves U. ■

Assertion 3. Every trajectory of X' goes from M_0 to M_1.

Proof. Integral curves that do not enter U follow X and hence always go from M_1 to M_0. Any integral curve through a point in U eventually exits U by Assertion 2. Moreover, it cannot re-enter U by Assertion 1, and hence once exited, it follows X. ■

Assertion 4. The vector field X' determines a diffeomorphism $\phi : [0,1] \times M_0 \to M$ that maps $0 \times M_0$ to M_0 and $1 \times M_0$ to M_1.

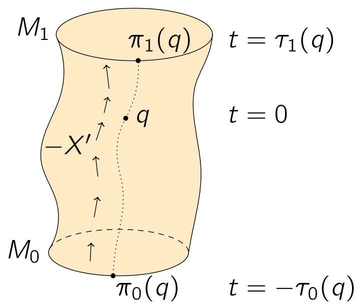


Figure 2.19: The nowhere vanishing vector field X' determines a diffeomorphism between $[0,1] \times M_0$ and the manifold M.

Proof. We will reverse X' and normalize it such that flowing along it for time 1, it it sends a point in M_0 to a point in M_1. Let $\tau_0(q)$ be the time needed to flow q back to M_0 along $-X'$ and similarly define $\tau_1(q)$. The maps τ_0 and τ_1 depend smoothly on q. Let π_0 be the projection $M \to M_0$ by flowing along $-X'$ and similarly define π_1. Then the vector field $Y(q) = -\tau_1(\pi_0(q))X'(q)$ has flow lines that go from M_0 to M_1 in unit time. This defines a diffeomorphism in the following way:

$$\begin{aligned} \phi : [0,1] \times M_0 &\longrightarrow M \\ (t, q) &\longmapsto \psi_Y^t(q), \end{aligned}$$

where ψ_Y^t is the flow of Y. The inverse of ϕ is $q \mapsto (\tau_0(q), \pi_0(q))$. ■

Assertion 5. The vector field X' is a pseudo-gradient vector field for some Morse function g on M that agrees with f near M_0 and M_1. Moreover, g has no critical points.

Proof. By using the previous assertion, it suffices to define a Morse function g on $[0,1] \times M_0$ such that $\frac{\partial g}{\partial t} > 0$ and g corresponds to $f_1 = f \circ \phi$ around $0 \times M_1$ and $1 \times M$. To make sure that $\frac{\partial g}{\partial t} > 0$, we will define g as the integral of a positive function, and to make sure it corresponds to f_1 around the boundary of $[0,1] \times M_0$, we will define g in the following way:

$$g(u,q) = \int_0^u \lambda(t)\frac{\partial f_1}{\partial t} + (1-\lambda(t))\, k(q)\, dt.$$

Here, $\lambda : [0,1] \to [0,1]$ is a bump function supported in the interior of $[0,\delta) \cup (1-\delta, 1]$, with δ small enough such that $\frac{\partial f_1}{\partial t} < 0$ in $[0,\delta) \cup (1-\delta,1]$ and $k : M_0 \to [0,1]$ is some yet unknown function that only depends on q. We have illustrated the situation below.

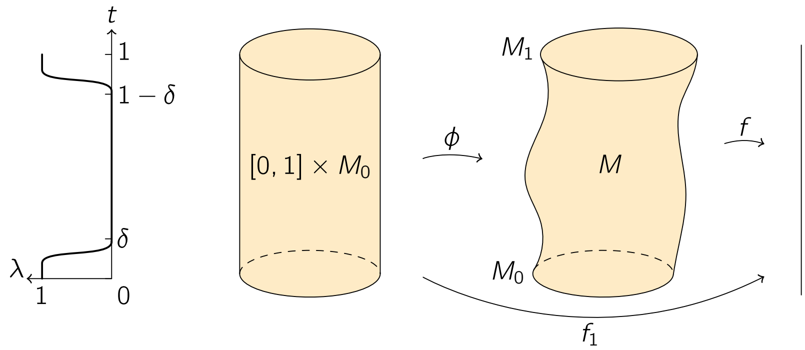


Figure 2.20: Construction of a Morse function that is adapted to X' and equals f in a neighbourhood of the boundary of $[0,1] \times M_0$. Here ϕ is the diffeomorphism obtained in Assertion 4.

The idea is to integrate the derivative of the original Morse function near the bottom and the top, and in the middle we will integrate an appropriately large function $k(q)$ such that $g(1,q) = 1$ for all q, i.e. the speed is adjusted so that at $t = 1$, we reach the top of $[0,1] \times M_0$. Expanding this condition and solving for $k(q)$, we get

$$\begin{aligned} g(1,q) = \int_0^1 \lambda(t)\frac{\partial f_1}{\partial t} + \int_0^1 (1-\lambda(t))k(q)dt = 1 \\ \int_0^1 (1-\lambda(t))dt\ k(q) = 1 - \int_0^1 \lambda(t)\frac{\partial f_1}{\partial t} \\ k(q) = \frac{1-\int_0^1 \lambda(t)\frac{\partial f_1}{\partial t}}{\int_0^1 (1-\lambda(t))dt}. \end{aligned}$$

Taking δ small enough ensures that $k(q) > 0$, hence $\frac{\partial g}{\partial t} > 0$, concluding the proof of the last assertion. ■

This finishes the proof of the first cancellation theorem. □

CHAPTER THREE

Morse homology

Morse homology is a way of associating global invariants to a manifold equipped with a Morse function f and an adapted pseudo-gradient X. In this long chapter, we will define Morse homology and prove three fundamental theorems. Let us give an outline.

As with any homology theory, Morse homology is based on a chain complex, called the Morse complex, which we introduce in Section 1. This complex consists of spaces $C_k(f)$ generated by the critical points of f, and the differential $\partial_X : C_k \to C_{k-1}$ counts trajectories between critical points. After introducing these concepts, we compute the Morse homology of some examples.

In Section 2, we prove that the Morse complex is actually a complex which comes down to proving that $\partial_X^2 = 0$. The proof, while rather long and technical, is based on illuminating geometrical ideas that have inspired many other homology theories, including Floer homology.

In Section 3, we show that Morse homology does not depend on the chosen Morse function and pseudo-gradient. This result is two-sided. On the one hand, it is disappointing that Morse homology does not capture any information about the dynamics of the Morse function and pseudo-gradient. On the other hand, it can be very useful to have a lot of freedom choosing input parameters without changing the resulting homology. For example, changing $f \rightsquigarrow -f$ does not change the homology and this will lead to a proof of Poincaré duality.

Section 4 concerns Morse homology over $\mathbb{Z}$. We will see that homology over $\mathbb{Z}$ captures more information than homology over $\mathbb{Z}_2$. For example $\mathbb{Z}$-homology can distinguish a Klein bottle from a torus while (the easier to define) $\mathbb{Z}_2$-homology does not.

In the last section of this chapter, we show that Morse homology is actually isomorphic to singular homology. We will do this by showing that it is isomorphic to de Rham homology which is based on currents. This last fundamental theorem concludes this chapter.

3.1 Morse complex

In order to define the Morse complex, we need a sequence of modules over a certain ring and maps between these modules. Most of the time, we will be working over $\mathbb{Z}/2\mathbb{Z}$, which we will denote by $\mathbb{Z}_2$, but sometimes the ring $\mathbb{Z}$ will be used instead. As will turn out, working over $\mathbb{Z}_2$ will allow us not to worry about orientation and it being a field also has some advantages.

The modules we will be considering depend on a Morse function f and consist of formal sums of critical points of a certain index:

$$C_k(f) = \Big\{ \sum_{p \in \mathrm{Crit}_k f} n_p p \mid n_p \in \mathbb{Z}_2 \Big\} = \bigoplus_{p \in \mathrm{Crit}_k f} \mathbb{Z}_2 p.$$

Note that this implies that $C_{-1} = C_{-2} = \cdots = 0$ and $C_{n+1} = C_{n+2} = \cdots = 0$, where n is the dimension of the manifold.

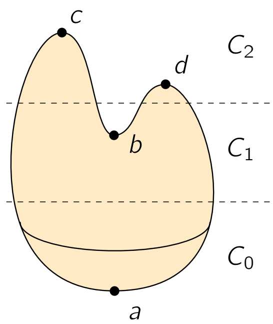


Figure 3.1: The critical points of the height function can be split up depending on their index and form the generators of the modules in the Morse complex.

Example 3.1. Consider the 'other sphere' in Figure 3.1. We have

$$C_0 = \{0, a\} \qquad C_1 = \{0, b\} \qquad C_2 = \{0, c, d, c+d\}.$$

The definition of the differential is based on a pseudo-gradient X: it counts trajectories connecting critical points of lower index. Because critical points of index k generate C_k, it suffices to define $\partial_{X,k}$ on these critical points and extend linearly:

$$\begin{aligned} \partial_{X,k} : C_k &\longrightarrow C_{k-1} \\ p &\longmapsto \sum_{q \in \mathrm{Crit}_{k-1} f} n_X(p,q) q, \end{aligned}$$

where $n_X(p,q)$ is the number of trajectories of X connecting p and q, modulo 2. If we require that X satisfies the Smale condition, we will later show that this is well defined, by which we mean that $n_X(p,q)$ is an integer (see also Remark 2.12). If it is clear from the context, we will often drop X, k or both from the notation.

Example 3.2. Consider again the 'other sphere' with its height function. Let X be the gradient induced from the standard gradient in $\mathbb{R}^3$. Then ∂_X is defined as follows, keeping in mind that we are working over $\mathbb{Z}_2$ and that $C_{-1} = 0$:

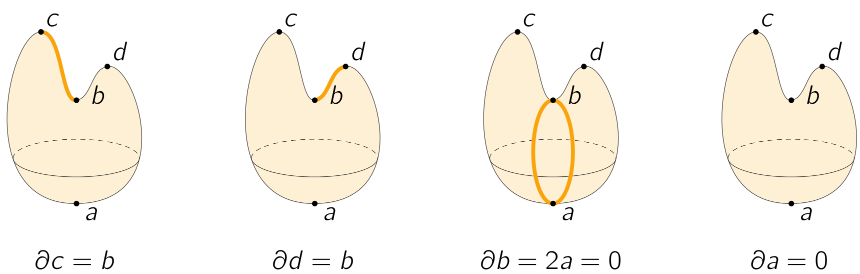


Figure 3.2: The Morse differential ∂_X counts the number of trajectories connecting critical points.

This allows us to define the Morse complex and Morse homology.

Definition 3.3 (Morse complex)**.** Let $f : M \to \mathbb{R}$ be a Morse function and X a pseudo-gradient with the Smale property. Then the Morse complex is given by

$$\cdots \xrightarrow{\partial} C_2(f) \xrightarrow{\partial} C_1(f) \xrightarrow{\partial} C_0(f) \xrightarrow{\partial} 0 \xrightarrow{\partial} 0 \xrightarrow{\partial} \cdots$$

The Morse homology $HM(C_\bullet(f), \partial_X)$ is the homology of this chain complex.

The attentive reader will have noticed that for this to be a chain complex, we need $\partial^2 = 0$, which is not obvious at all. Apart from this, we also would to like to prove that this homology does not depend on the choice of the Morse function f and the pseudo-gradient field X. We will address these topics in the two following sections, but let us first compute the Morse homology of some examples.

Example 3.4 (Homology of the (other) sphere)**.** We have already computed the differential in the previous example, so computing the homology is just a matter of applying the definition.

- $HM_0(C_\bullet(f), \partial_X) = \dfrac{\operatorname{Ker} \partial : C_0 \to C_{-1}}{\operatorname{Im} \partial : C_1 \to C_0} = \dfrac{\{0, a\}}{\{0\}} \cong \mathbb{Z}_2$
- $HM_1(C_\bullet(f), \partial_X) = \dfrac{\operatorname{Ker} \partial : C_1 \to C_0}{\operatorname{Im} \partial : C_2 \to C_1} = \dfrac{\{0, b\}}{\{0, b\}} \cong 0$
- $HM_2(C_\bullet(f), \partial_X) = \dfrac{\operatorname{Ker} \partial : C_2 \to C_1}{\operatorname{Im} \partial : C_3 \to C_2} = \dfrac{\{0, c+d\}}{\{0\}} \cong \mathbb{Z}_2$

In summary, we have

$$H_0 = \mathbb{Z}_2 \qquad H_1 = 0 \qquad H_2 = \mathbb{Z}_2.$$

Instead of embedding S^2 in this strange way, we can also repeat the same calculation with its standard embedding, illustrated in Figure 3.3. We have $C_2 = \{0, b\}$, $C_1 = \{0\}$ and $C_0 = \{0, a\}$, and $\partial(b) = 0$, $\partial(a) = 0$. This way, we obtain

- $HM_0(C_\bullet(g), \partial_Y) = \dfrac{\operatorname{Ker} \partial : C_0 \to C_{-1}}{\operatorname{Im} \partial : C_1 \to C_0} = \dfrac{\{0, a\}}{\{0\}} \cong \mathbb{Z}_2$
- $HM_1(C_\bullet(g), \partial_Y) = \dfrac{\operatorname{Ker} \partial : C_1 \to C_0}{\operatorname{Im} \partial : C_2 \to C_1} = \dfrac{\{0\}}{\{0\}} \cong 0$
- $HM_2(C_\bullet(g), \partial_Y) = \dfrac{\operatorname{Ker} \partial : C_2 \to C_1}{\operatorname{Im} \partial : C_3 \to C_2} = \dfrac{\{0, b\}}{\{0\}} \cong \mathbb{Z}_2$

In summary, we have

$$H_0 = \mathbb{Z}_2 \qquad H_1 = 0 \qquad H_2 = \mathbb{Z}_2,$$

exactly the same homology as with the other Morse function and other gradient. As mentioned earlier, we will prove that this is in general the case.

Example 3.5 (Homology of the three-torus)**.** Let us recall Example 1.22, which discusses the three-torus $T^3 = S^1 \times S^1 \times S^1 = \mathbb{R}^3/\mathbb{Z}^3$ and the following

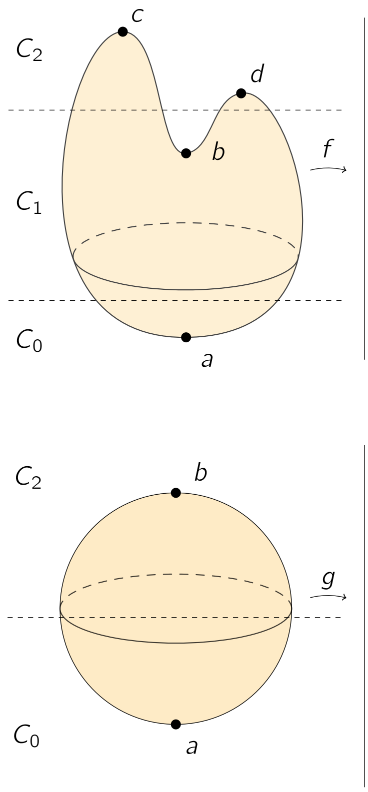


Figure 3.3: Two embeddings of the sphere in $\mathbb{R}^3$. The corresponding height functions are Morse functions and give rise to a different Morse complex. However, the resulting Morse homology is the same.

Morse function

$$\begin{aligned} f : T^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\longmapsto 2\cos(2\pi x) + 3\cos(2\pi y) + 4\cos(2\pi z), \end{aligned}$$

with critical values repeated here for convenience:

x	y	z	$f(x,y,z)$	Index
0	0	0	9	3
$\frac{1}{2}$	0	0	5	2
0	$\frac{1}{2}$	0	3	2
0	0	$\frac{1}{2}$	1	2
$\frac{1}{2}$	$\frac{1}{2}$	0	-1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	-3	1
0	$\frac{1}{2}$	$\frac{1}{2}$	-5	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-9	0

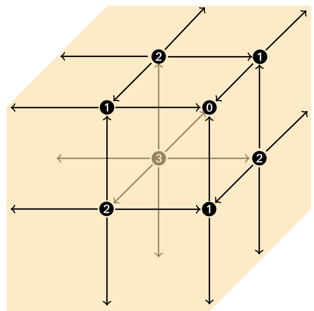


Figure 3.4: Trajectories connecting critical points whose index differ by exactly 1. Here $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ and we have drawn $\left(-\frac{1}{2}, \frac{1}{2}\right]^3$ as representative cube.

The differential equation for trajectories along $\operatorname{grad} f$ is

$$\begin{aligned} \dot{x} &= -4\pi \sin(2\pi x) \\ \dot{y} &= -6\pi \sin(2\pi y) \\ \dot{z} &= -8\pi \sin(2\pi z). \end{aligned}$$

Note that $\dot{x}$ only depends on x, $\dot{y}$ on y and $\dot{z}$ on z, so this is a decoupled system that is easy to solve. Requiring that for $t \to \pm\infty$ we end up in critical points, we find the trajectories in Figure 3.4. To have a better overview, we can also make a graph of critical points and trajectories between them, as is done in Figure 3.5. This graph provides all the information we need to compute the Morse homology of T^3. Note that all critical points are connected with two gradient lines, which means that each differential $\partial_3, \partial_2, \partial_1, \partial_0$ is the zero map, making it very easy to compute the homology. In the end, we get

$$H_0 = \mathbb{Z}_2 \qquad H_1 = \mathbb{Z}_2^3 \qquad H_2 = \mathbb{Z}_2^3 \qquad H_3 = \mathbb{Z}_2,$$

which corresponds to the usual homology.

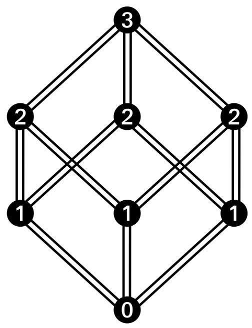


Figure 3.5: Graph of the critical points of f on T^3. Each edge represents a flow line between points whose indices differ by one.

3.2 The Morse complex is a complex: $\partial^2 = 0$

In this section, we will prove that the Morse complex is actually a complex, by which we mean that $\partial^2 = 0$. The core idea of the proof is very geometrical and depends on the fact that compact one-dimensional manifolds with boundary have an even number of boundary points.[28] We will first clarify this core idea, and then make this argument precise.

Let us first state clearly what we have to prove. Let $a \in \operatorname{Crit}_k f$ be a critical point of a Morse function $f : M \to \mathbb{R}$. We need to prove that $\partial^2(a) = 0$, so

$$\begin{aligned} \partial^2(a) &= \partial\Big(\sum_{c \in \operatorname{Crit}_{k-1} f} n_X(a,c)c\Big) \\ &= \sum_{b \in \operatorname{Crit}_{k-2} f} \sum_{c \in \operatorname{Crit}_{k-1} f} n_X(a,c) n_X(c,b) b. \end{aligned}$$

[28] Compact one-dimensional manifolds consist of disjoint unions of copies of S^1 and closed intervals, as proven in John Milnor and David W Weaver. *Topology from the differentiable viewpoint*. Princeton university press, 1997, p.55.

We see that $\partial^2(a)$ counts trajectories from a to points of index $k-2$ that are once broken in a critical point of index $k-1$. We are working over $\mathbb{Z}_2$, so if we can prove that these once broken trajectories always occur in pairs, we are done.

The core idea of the proof is to notice that there is a one parameter family $\mathcal{L}(a,b)$ of unbroken trajectories connecting the broken trajectories, as illustrated in Figure 3.6. More specifically the space of unbroken trajectories $\mathcal{L}(a,b)$ is a one-dimensional manifold, and by adding in the once broken trajectories between a and b, we can compactify this manifold, resulting in the space which we will denote with $\overline{\mathcal{L}}(a,b)$:

$$\overline{\mathcal{L}}(a,b) = \mathcal{L}(a,b) \cup \bigcup_{c \in \mathrm{Crit}_{k-1} f} \mathcal{L}(a,c) \times \mathcal{L}(c,b).$$

This turns out to be a compact 1-manifold, and its boundary consists exactly of the broken trajectories. Because a 1-manifold always has an even number of boundary points, this shows that once-broken trajectories come in pairs, proving that $\partial^2 = 0$.

Before we proceed with this idea, let us give an overview of the steps that need to be taken.

1. Define $\overline{\mathcal{L}}(a,b)$, the space of all (broken and unbroken) trajectories.
2. Define a topology on $\overline{\mathcal{L}}(a,b)$.
3. Prove that $\overline{\mathcal{L}}(a,b)$ is the compactification of $\mathcal{L}(a,b)$.
4. Prove that if $\mathrm{Ind}\, a - \mathrm{Ind}\, b = 2$, then $\overline{\mathcal{L}}(a,b)$ is a 1-dimensional manifold with boundary the once broken trajectories between a and b.

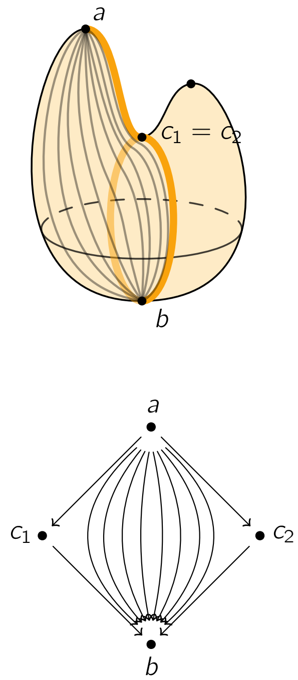


Figure 3.6: Core idea of the proof stating $\partial^2 = 0$. There are two unbroken trajectories from a to b, one passing through c_1 and one through c_2. There is a one parameter family of unbroken trajectories interpolating between the two broken ones. Together, they form a compact 1-dimensional manifold with boundary, which has an even number of boundary points.

3.2.1 The space of broken trajectories

Definition 3.6. The space of broken trajectories between a and b is

$$\overline{\mathcal{L}}(a,b) = \bigcup_{\substack{\{c_1,\dots,c_\ell\} \subset \mathrm{Crit} f \\ c_i \text{ distinct}}} \mathcal{L}(a,c_1) \times \mathcal{L}(c_1,c_2) \times \cdots \times \mathcal{L}(c_\ell,b).$$

Remark 3.7. Notice that $\mathcal{L}(p,q) = \emptyset$ if $\mathrm{Ind}\, p \leq \mathrm{Ind}\, q$, so the only broken trajectories that contribute to the union satisfy $\mathrm{Ind}\, a > \mathrm{Ind}\, c_1 > \cdots > \mathrm{Ind}\, c_\ell > \mathrm{Ind}\, b$. Moreover, note that this set also contains the 'zero-times' broken trajectories, i.e. $\mathcal{L}(a,b)$ itself.

The number ℓ describes how many times the trajectory is broken. In other words, each broken trajectory in $\mathcal{L}(a,c_1) \times \mathcal{L}(c_1,c_2) \times \cdots \times \mathcal{L}(c_\ell,b)$ has $\ell + 1$ segments, and we will denote such a trajectory by $(\lambda_1, \lambda_2, \cdots, \lambda_{\ell+1})$. As discussed above, in the case of $\mathrm{Ind}\, a = k$, $\mathrm{Ind}\, b = k-2$, this definition results in

$$\overline{\mathcal{L}}(a,b) = \mathcal{L}(a,b) \cup \bigcup_{c \in \mathrm{Crit}_{k-1} f} \mathcal{L}(a,c) \times \mathcal{L}(c,b),$$

which makes it clear it is the union of unbroken and once broken trajectories.

3.2.2 Topology of $\overline{\mathcal{L}}(a,b)$

To define a topology on $\overline{\mathcal{L}}(a,b)$, we will describe a basis around a (possibly broken) trajectory. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ be a trajectory in $\overline{\mathcal{L}}(a,b)$. Consider Morse charts around each critical point $a, c_1, c_2, \dots, c_{q-1}, b$. Take opens around entry and exit points of the trajectories, lying in level sets of f, indicated in the figure with thick black lines. We declare all trajectories passing through these opens to be in a open neighborhood of $(\lambda_1, \lambda_2, \dots, \lambda_q)$. Doing this for all possible 'entrance and exit opens' defines a basis of the topology.

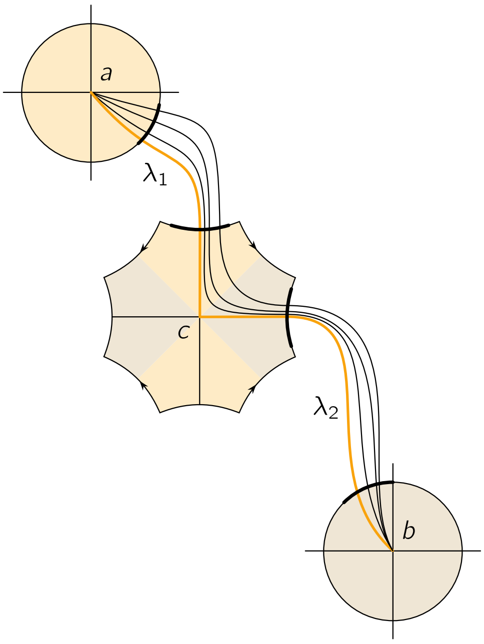


Figure 3.7: The topology on $\overline{\mathcal{L}}(a,b)$ is defined by looking at the entrance and exit points in the Morse charts. Here we have shown in black paths that lie in a neighborhood of the broken path $(\lambda_1, \lambda_2) \in \overline{\mathcal{L}}(a,b)$.

Remark 3.8. If λ is unbroken, we have $\mathcal{L}(a,b) \cong \mathcal{M}(a,b) \cap f^{-1}(\alpha)$ for some regular value α, and topology on $\mathcal{L}(a,b)$ corresponds to the subspace topology on $\mathcal{M}(a,b) \cap f^{-1}(\alpha) \subset M$.

In conclusion, the essence of this topology on $\overline{\mathcal{L}}(a,b)$ is: 'trajectories are nearby if entrance and exit points in Morse charts are nearby'.

3.2.3 $\overline{\mathcal{L}}(a,b)$ is the compactification of $\mathcal{L}(a,b)$

Now we are ready to prove that $\overline{\mathcal{L}}(a,b)$ is compact, and in fact a compactification of $\mathcal{L}(a,b)$. By this last statement, we mean that there are points in $\mathcal{L}(a,b)$ arbitrary close to ones in $\overline{\mathcal{L}}(a,b)$. In topological terms, any open around a broken trajectory in $\mathcal{L}(a,b)$ contains unbroken trajectories, i.e. elements of $\mathcal{L}(a,b)$.

Theorem 3.9. $\overline{\mathcal{L}}(a,b)$ is compact.

In the proof of this theorem, we will need the following lemma, stating that points y_n on trajectories that pass through a convergent sequence of points $x_n \to x$ also converge, at least if y_n all lie on the same level:

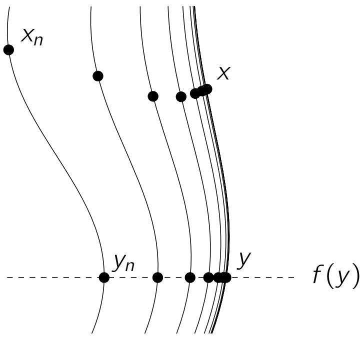


Figure 3.8: A convergent sequence $x_n \to x$ defines a sequence of trajectories. If y_n is a sequence of points that lie on these trajectories, then it also converges to a point y lying on the trajectory that passes through x.

Lemma 3.10. Let x be a regular point of f and $x_n \to x$. Let y_n and y be points lying on the same trajectory of X as x_n and x. Suppose all y_n lie on the same level as y, i.e. $f(y_n) = f(y)$. Then $y_n \to y$.

Proof. The idea of the proof is to flow y_n to x_n and y to x so that convergence of x_n implies convergence of y_n. Let ψ_t be the flow of $-\frac{1}{df(X)}X$ on a subset of M that contains x_n, y_n, x, y for large enough n and does not contain critical points. Then $f(\psi_t(z)) = f(z) - t$, so

$$\begin{aligned} y_n &= \psi_{-f(y_n)+f(x_n)}(x_n) \\ &= \psi_{-f(y)+f(x_n)}(x_n) \\ &\big\downarrow n\to\infty \\ &\psi_{-f(y)+f(x)}(x) = y. \end{aligned}$$

□

Let us now prove that $\overline{\mathcal{L}}(a, b)$ is compact. We will do this by proving that this space is sequentially compact. This is enough, because the topology on $\overline{\mathcal{L}}(a, b)$ is second countable (coming from the topology on M), which implies that compactness, countable compactness, sequentially compactness, etc. are equivalent properties.

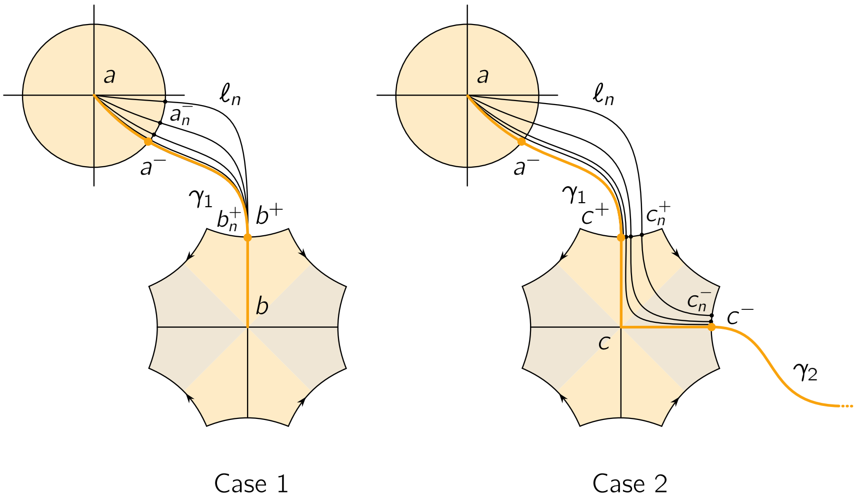


Figure 3.9: To show that $\overline{\mathcal{L}}(a, b)$ is compact, we consider a sequence of paths ℓ_n in $\mathcal{L}(a, b)$ and find an accumulation trajectory, i.e. a limit of a subsequence. In case 1, the accumulation trajectory lies in $\mathcal{L}(a, b)$. Case 2 shows the situation when the accumulation trajectory is broken in c (and possibly in multiple other critical points).

Proof. Let us first consider a sequence of unbroken trajectories, i.e. a sequence ℓ_n in $\mathcal{L}(a, b)$. Let $\Omega(a)$ be a Morse chart around a. Consider the points where ℓ_n exits the Morse chart a, call them a_n^-. Using compactness of the sphere, extract a subsequence of ℓ_n such that a_n^- converges and call the point of convergence a^-. We will denote the extracted subsequence again by ℓ_n and will do so continuously in the proof. Similarly, extract a subsequence such that b_n^+, defined as the entry point in the Morse chart of b converges to a point b^+. In conclusion, we have a subsequence whose exit point in $\Omega(a)$ converges to a^- and whose entrance in $\Omega(b)$ converges to b^+.

Let us now try to construct a trajectory that is the 'accumulation trajectory'[29] of ℓ_n. An obvious starting point would be to consider the unique trajectory γ_1 passing through a^- going from a to another critical point. Suppose for a moment that this other critical points is in fact b, so γ_1 connects a and b, as in case 1 in Figure 3.9. Lemma 3.10 implies that the entry points of ℓ_n in $\Omega(b)$ converge to the entry point of γ_1. This then proves that $\ell_n \to \gamma_1$[30].

In the other case , when γ_1 connects a and another critical point $c \neq b$, the accumulation trajectory of ℓ_n will be a trajectory that is at least broken in c. We have again that the entry points of ℓ_n in $\Omega(c)$ (call them c_n^+) converge to the entry point of γ_1, so the first segment of the accumulation trajectory will be γ_1. To find the second segment of the accumulation trajectory, we would again want a starting point (like we had a^- before) in the unstable manifold of c to flow from in order to find γ_2. For this, extract a subsequence of ℓ_n such that their exit points c_n^- in $\Omega(c)$ converge to a point c^-. We claim that c^- lies in the unstable manifold of c.[31] This means that when we flow back, we indeed get to c like we wanted, and when we flow forward, we get to another critical point, which may be b, or d, yet another critical point. This way we go on and find a subsequence of ℓ_n that converges to $(\gamma_1, \gamma_2, \dots, \gamma_k)$.

[29] By this, we simply mean an accumulation point of the sequence ℓ_n, but this terminology can be confusing because it is not a point in the geometrical sense of the word.

[30] By definition of our topology, a sequence of unbroken trajectories connecting a to b (ℓ_n) converges to an unbroken trajectory (γ_1) if entry and exit points converge.

[31] Suppose that it does not. Then we could flow back c^- to a point $c^\star$ with $f(c^\star) = f(c^+)$. Note that $c^\star$ is not in the stable manifold of c. (Otherwise, flowing forward again we would end up in c) Applying the lemma again, $c_n^+ \to c^\star$ meaning that $c^\star = c^+$. This cannot be right since $c^\star$ is not in the stable manifold of c, but c^+ is, by definition.

In order to complete the proof for a sequence ℓ_n in $\overline{\mathcal{L}}(a,b)$ (instead of $\mathcal{L}(a,b)$), note that for sufficiently large n and after extracting a subsequence, the critical points where ℓ_n is broken do not change. Then apply the proof above to the first segment, then to the second, ... □

Note that this proof also encapsulates the fact that $\overline{\mathcal{L}}(a,b)$ is actually the compactification of $\mathcal{L}(a,b)$, in the sense that there are elements of $\mathcal{L}(a,b)$ that are arbitrarily close to a fixed element of $\overline{\mathcal{L}}(a,b)$.

3.2.4 $\overline{\mathcal{L}}(a,b)$ is a 1-dimensional manifold with boundary

The last step we need to take is showing that that the topological space $\overline{\mathcal{L}}(a,b)$ actually has the structure of a manifold. More specifically if $\operatorname{Ind} a - \operatorname{Ind} = 2$, then it is a one-manifold with boundary.

Theorem 3.11. Let a, b be critical points of M such that $\operatorname{Ind} a - \operatorname{Ind} b = 2$. Then $\overline{\mathcal{L}}(a,b)$ is a compact 1-dimensional manifold with boundary.

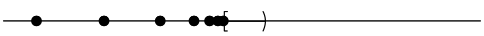


Figure 3.10: We can embed a half open interval $[a,b)$ in $\mathbb{R}$, but that does not mean that a is a boundary point of $\mathbb{R}$. Clearly, there exists $x_n \to a$ that is not eventually contained in $[a,b)$. Requiring that this last condition always holds ensures that a is actually a boundary point.

Remark 3.12. In general, the space $\mathcal{L}(a,b)$ when $\operatorname{Ind} a - \operatorname{Ind} b > 2$ is a compact manifold with corners. Its interior consists of unbroken trajectories, its codimension 1-stratum of the once-broken trajectories, its codimension 2-stratum of the twice-broken trajectories, etc.

We already know that $\mathcal{L}(a,b)$ is a 1-manifold, so the following proposition immediately implies the theorem.

Proposition 3.13. Let M be a compact manifold and $f : M \to \mathbb{R}$ a Morse function with adapted pseudo-gradient X satisfying the Smale condition. Let a, c, b be three critical points of index $k+1, k$ and $k-1$. Let $\lambda_1 \in \mathcal{L}(a,c)$ and $\lambda_2 \in \mathcal{L}(c,b)$. There exists a continuous embedding ψ from $[0,\delta)$ to a neighborhood of (λ_1,λ_2) in $\overline{\mathcal{L}}(a,b)$ such that

$$\begin{cases} \psi(0) = (\lambda_1,\lambda_2) \in \overline{\mathcal{L}}(a,b) \\ \psi(s) \in \mathcal{L}(a,b) \quad \text{for } s \neq 0. \end{cases}$$

Moreover if (ℓ_n) is a sequence in $\mathcal{L}(a,b)$ that tends to (λ_1,λ_2), then ℓ_n is eventually contained in the image of ψ.

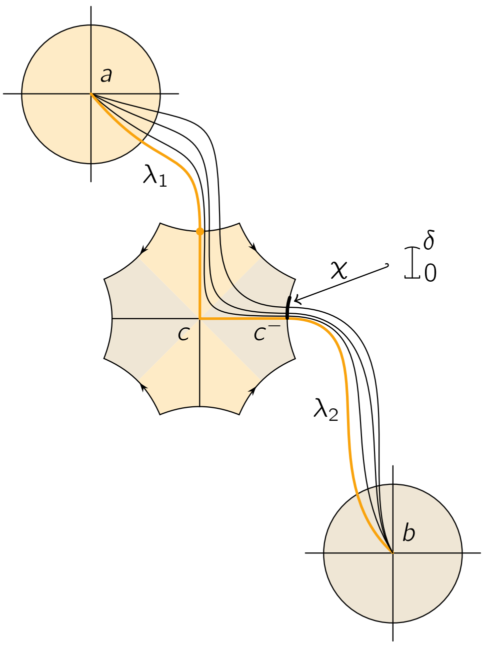


Figure 3.11: The map χ is an embedding of a half-open interval $[0,\delta)$ in the level set $\alpha-\epsilon$. Considering the trajectories passing through these points, we get an embdding $\psi : [0,\delta) \to \overline{\mathcal{L}}(a,b)$.

The last part is important in order to show that broken trajectories actually form the boundary, as illustrated in Figure 3.10.

Proof. Let a, b, c be critical points of index $(\operatorname{Ind} a, \operatorname{Ind} c, \operatorname{Ind} b) = (k+1, k, k-1)$. Consider a Morse chart around c that lies between level sets $\alpha \pm \epsilon$ where $\alpha = f(c)$. Let c^+ be the entry point of λ_1 in the Morse chart of c.

The idea is to embed an interval $[0,\delta)$ as in Figure 3.11 in the level set $\alpha - \epsilon$. Call this embedding χ. We will then define $\psi(t)$ as the trajectory passing through $\chi(t)$ (for $t > 0$) and $\psi(0) = (\lambda_1,\lambda_2)$. The difficulty lies in determining how to embed $[0, \delta)$ inside the level set $\alpha - \epsilon$ such that ψ is continuous at 0.

Parametrizing trajectories near $\boldsymbol{\lambda_1}$ Let D^k be a small k-dimensional disk around c^+ in $f^{-1}(\alpha + \epsilon)$, parametrizing trajectories starting in a near λ_1. Now consider $D^k \setminus c^+$ and flow this punctured disk along X until it reaches the level set $\alpha - \epsilon$. Points in the resulting manifold $Q \subset f^{-1}(\alpha - \epsilon)$ parametrize trajectories near λ_1 but not λ_1 itself. By adding the points lying on $S^- := W^u(c) \pitchfork f^{-1}(\alpha - \epsilon)$, we get the manifold $\overline{Q} = Q \cup S^-$ which is a manifold with boundary S^-. This submanifold $\overline{Q}$ of $f^{-1}(\alpha - \epsilon)$ parametrizes trajectories near λ_1 (but notice that λ_1 itself is represented multiple times).

Parametrizing trajectories ending in $\boldsymbol{b}$ Let $P = f^{-1}(\alpha - \epsilon) \pitchfork W^s(b)$. This is again a subspace of $f^{-1}(\alpha - \epsilon)$ that parametrizes trajectories going to b.

Parametrizing trajectories near $\boldsymbol{(\lambda_1, \lambda_2)}$ The intersection of P and $\overline{Q}$ is transverse (because of the Smale conditions) and parametrizes (broken and unbroken) trajectories near (λ_1, λ_2). Because of transversality, this intersection is in fact a manifold (with boundary c^-) and counting dimensions, we find that it has dimension 1. Now it is easy to define the required embedding $\chi : [0, \delta) \to f^{-1}(\alpha - \epsilon)$, which in turn gives rise to $\psi : [0, \delta) \to \overline{\mathcal{L}}(a, b)$. □

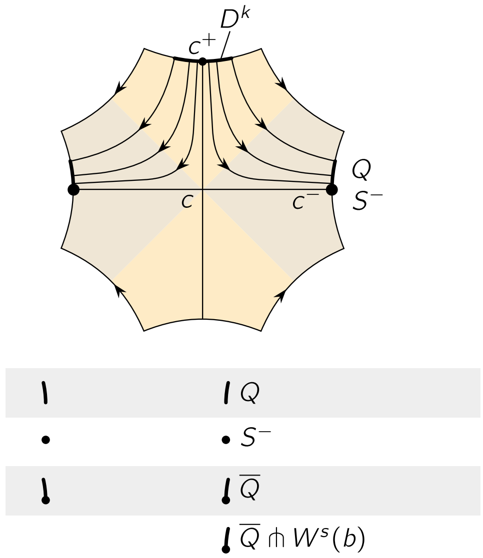


Figure 3.12: An overview of the different submanifolds considered in the proof.

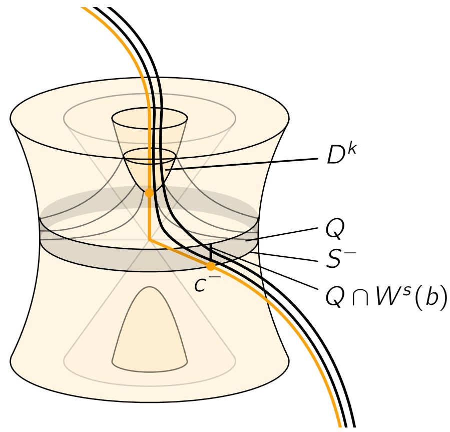


Figure 3.13: The situation in three dimensions.

3.2.5 Conclusion

Let us conclude this section by finally proving that $\partial^2 = 0$.

Theorem 3.14. Let X be a pseudo-gradient adapted to a Morse function $f : M \to \mathbb{R}$ that satisfies the Smale condition. Then the Morse differential ∂_X squares to zero.

Proof. Let a be a critical point of f. Then by definition of ∂,

$$\partial^2(a) = \sum_{b \in \mathrm{Crit}_{k-2} f} \; \sum_{c \in \mathrm{Crit}_{k-1} f} n_X(a, c) n_X(c, b)\, b.$$

By definition of n_X,

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \; \sum_{c \in \mathrm{Crit}_{k-1} f} \#(\mathcal{L}(a, c) \times \mathcal{L}(c, b))\, b$$

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \# \left(\bigcup_{c \in \mathrm{Crit}_{k-1} f} \mathcal{L}(a, c) \times \mathcal{L}(c, b) \right) b.$$

By definition of $\overline{\mathcal{L}}(a, b)$

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \#(\partial \overline{\mathcal{L}}(a, b))\, b.$$

Now, $\overline{\mathcal{L}}(a, b)$ is a compact 1-manifold and hence has an even number of boundary points. Working over $\mathbb{Z}_2$, this implies that $\partial^2(a) = 0$. □

3.3 Morse homology is independent of the Morse function and gradient

Theorem 3.15. Let M be a compact manifold and $f_0, f_1 : M \to \mathbb{R}$ two Morse functions. Let X_0, X_1 be pseudo-gradients adapted to f_0 and f_1 respectively with the Smale property. Then there exists a morphism of complexes

$$\Phi_\bullet : (C_\bullet(f_0), \partial_{X_0}) \to (C_\bullet(f_1), \partial_{X_1}),$$

that induces an isomorphism on the level of homology.

Proof. The proof of this theorem features an intricate interplay between homological algebra and differential geometry and is truly something to behold. It is perhaps the most beautiful proof in this thesis.

Construction of a morphism of complexes In order to find a relation between the two complexes, we geometrically connect f_0 and f_1 via a particular type of homotopy, namely a 'stable interpolation' by which we mean a smooth map

$$F : [0,1] \times M \to \mathbb{R} : (s,m) \mapsto F_s(m),$$

such that $F_s = f_0$ for $s \in \left[0, \frac{1}{3}\right]$ and $F_s = f_1$ for $s \in \left[\frac{2}{3}, 1\right]$. On of the reasons of looking specifically at stable interpolations is that we can concatenate them and again get a C^∞ map that is a stable interpolation.

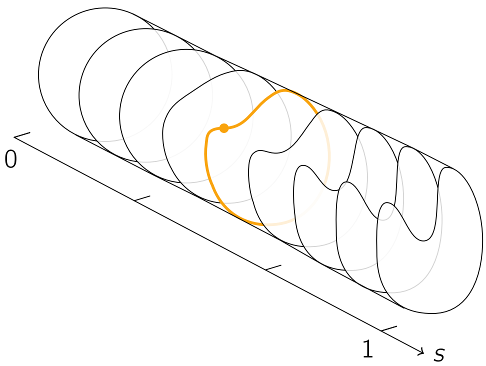


Figure 3.14: An interpolation between f_0 and f_1 can result in degenerate critical points, as shown in the figure in orange: a homotopy between Morse functions is not necessarily Morse for all times s.

We can visualize an interpolation between two Morse functions by embedding $[0,1] \times M$ in $\mathbb{R}^\ell$ in such a way that the height function in each slice corresponds to F_s. For example, doing this for the circle and the other circle, we get Figure 3.14.

Seen from a Morse perspective, the result is less than desirable: the function F is not a Morse function: critical points in the stationary parts of F are degenerate as $\frac{\partial F}{\partial s} = 0$. Furthermore, an interpolation of two Morse functions need not be Morse at each point in time which gives even more degenerate critical points. We have highlighted an example of this in the figure.

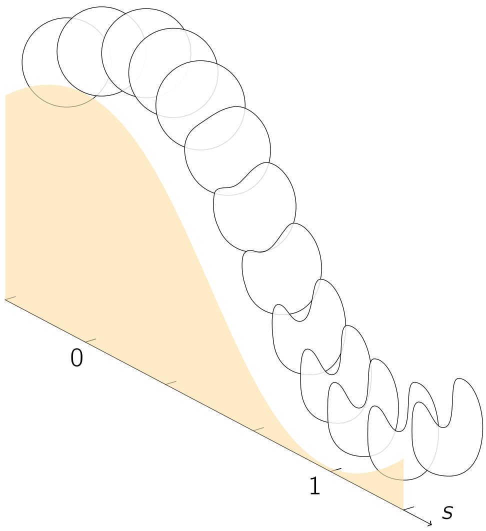


Figure 3.15: Adding a function (illustrated below) in the s-direction creates a slide overcoming the problem of degenerate critical points.

We can fix this problem by replacing the horizontal tube by a 'slide', as seen in Figure 3.15. We do this by extending F to $\left[-\frac{1}{3}, \frac{4}{3}\right]$ and adding a function g (illustrated in Figure 3.16) along the s-direction, i.e. $\tilde{F}_s(p) = F_s(p) + g(s)$.[32] Whatever kind of tube we start with, if we make the slide steep enough, we will always slide down and never have flat spots, except at the top and bottom of the slide. This means if we choose g appropriately[33], the only critical points lie in the slices $s = 0$ and $s = 1$ and correspond to critical points of f_0 and f_1 respectively. Because g is Morse, these critical points remain nondegenerate. We conclude that $\tilde{F}$ is in fact Morse with critical points $\mathrm{Crit}(\tilde{F}) = \{0\} \times \mathrm{Crit}(f_0) \cup \{1\} \times \mathrm{Crit}(f_1)$.

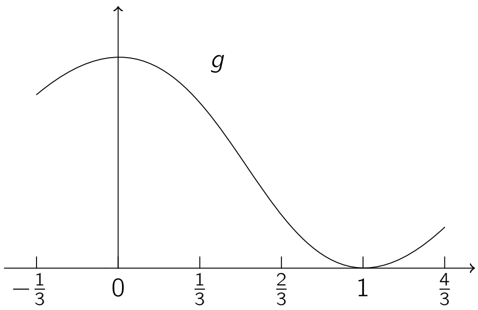


Figure 3.16: The function g used to transform the tube into a slide.

We can also determine the index of these critical points. Because we have created an extra downward direction at the top of the slide, the indices of these critical points have increased by 1. At the bottom, the indices stay the

[32] We extend the function to the interval $\left[-\frac{1}{3}, \frac{4}{3}\right]$ to make sure that the critical values of $\tilde{F}$ do not lie on the boundary.

[33] More explicitly, we want $\frac{\partial F}{\partial s}(p,s) + g'(s) < 0$ for all $p \in M, s \in (0,1)$.

same, giving us

$$C_{k+1}(\tilde{F}) = C_k(f_0) \oplus C_{k+1}(f_1).$$

Apart from the critical points, we are also interested in constructing a pseudo-gradient on $[0,1] \times M$, as this will give rise to a differential. On $[-\frac{1}{3}, \frac{1}{3}] \times M$, we set $X = X_0 - \operatorname{grad} g$, and on $\left[\frac{2}{3}, \frac{4}{3}\right] \times M$ we set $X = X_1 - \operatorname{grad} g$.[34] A partition of unity argument then fills in the gaps. Note that this pseudo-gradient is transverse to the boundary of $\left[-\frac{1}{3}, \frac{4}{3}\right] \times M$. We can slightly perturb X to make it satisfy the Smale condition and we can furthermore assume that the resulting vector field, $\tilde{X}$ is transversal to $\{s\} \times M$ for $s \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right\}$. We can also make this perturbation small enough such that $\partial_X = \partial_{\tilde{X}}$, that is to say, the number of X-trajectories between critical points is the same as the number of $\tilde{X}$-trajectories.

[34] Here, $\operatorname{grad} g$ is the Euclidian gradient.

Having a Morse function $\tilde{F}$ and a pseudo-gadient $\tilde{X}$ that is adapted to $\tilde{F}$, we can consider the associated Morse complex $(C_\bullet(\tilde{F}), \partial_{\tilde{X}})$. There are two types of trajectories connecting critical points of $\tilde{F}$: ones that stay in the same section ($s = 0$ or $s = 1$) and ones that connect critical points of f_0 to critical points of f_1, i.e. ones that 'slide down the slide'. This means we can decompose $\partial_{\tilde{X}}$ as follows:

$$\begin{aligned} \partial_{\tilde{X}} : C_k(f_0) \oplus C_{k+1}(f_1) &\longrightarrow C_{k-1}(f_0) \oplus C_k(f_1) \\ (p_0, p_1) &\longmapsto (\partial_{X_0}(p_0), \partial_{X_1}(p_1) + \Phi^F(p_0)), \end{aligned}$$

where Φ^F counts the trajectories connecting critical points of f_0 to ones of f_1. We can also write this as a matrix:

$$\partial_{\tilde{X}} = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1} \end{pmatrix}.$$

Readers familiar with homological algebra will recognize this construction as the mapping cone of the map $\Phi^F : C_\bullet(f_0) \to C_\bullet(f_1)$.

Let us now look at what the identity $\partial_{\tilde{X}}^2 = 0$ means in this context. Let $p \in C_k(f_0)$. Then

$$\begin{aligned} \partial_{\tilde{X}}^2(p, 0) &= \partial_{\tilde{X}}(\partial_0(p), \Phi^F(p)) \\ &= (\partial_0^2(p), \Phi^F \partial_0(p) + \partial_1 \Phi^F(p)) \\ &= (0, \Phi^F \partial_0(p) + \partial_1 \Phi^F(p)). \end{aligned}$$

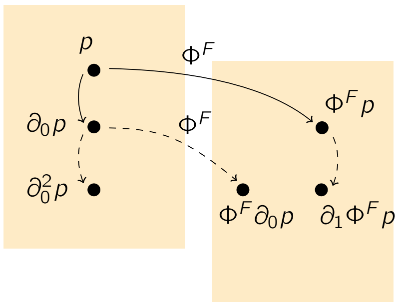


Figure 3.17: A visual depiction of the calculation $\partial^2(p)$.

Because we are working over $\mathbb{Z}_2$, this means that $\Phi^F \circ \partial_0 = \partial_1 \circ \Phi^F$, i.e. the following diagram commutes for all k:

$$\begin{tikzcd} C_k(f_0) \arrow[r, "\partial_0"] \arrow[d, "\Phi^F"] & C_{k-1}(f_0) \arrow[d, "\Phi^F"] \\ C_k(f_1) \arrow[r, "\partial_1"] & C_{k-1}(f_1) \end{tikzcd}$$

This proves that Φ^F is a morphism of complexes.

Φ^F induces an isomorphism on the level on homology We will now prove that this map induces an isomorphism on the level of homology. Let f_0, f_1, f_2 be Morse functions $M \to \mathbb{R}$. Suppose F interpolates between f_0 and f_1, G between f_1 and f_2 and H between f_0 and f_2, i.e. we are in the following situation:

$$\begin{tikzcd} f_0 \arrow[r, "F"] \arrow[rr, bend right, "H"'] & f_1 \arrow[r, "G"] & f_2. \end{tikzcd}$$

We claim that the maps induced by $\Phi^G \circ \Phi^F$ and Φ^H on the level of homology coincide, or equivalently, they are chain homotopic, meaning that there exists an operator S such that

$$\Phi^G \circ \Phi^F - \Phi^H = \partial S + S\partial.$$

This is sufficient to prove that Φ^F induces an isomorphism. Indeed, it is easy to check that if I is a constant interpolation between (f_0, X_0) and itself,

$$I : [0, 1] \times M \to \mathbb{R} : (s, p) \mapsto f_0(p),$$

then $\Phi^I = \mathrm{Id}$. So consider F a stationary interpolation between f_0 and f_1 and G, the reverse interpolation from f_1 to f_0 and $H = I$. Then the induced homological maps Φ^F and Φ^G are inverses of each other.

Let us prove that $\Phi^G \circ \Phi^F$ and Φ^H are chain homotopic. The idea of this part of the proof is very similar to the first part. Instead of creating one slide from f_0 to f_1 by adding an extra dimension, we create a two-dimensional slide with as sides four slides: $f_0 \xrightarrow{F} f_1$, $f_1 \xrightarrow{G} f_2$, $f_0 \xrightarrow{H} f_2$ and $f_2 \xrightarrow{I} f_2$.

More concretely, we create a map

$$K : \left[-\tfrac{1}{3}, \tfrac{4}{3}\right] \times \left[-\tfrac{1}{3}, \tfrac{4}{3}\right] \times M \to \mathbb{R} : (s, t, p) \mapsto K_{s,t}(p),$$

with the following properties, as illustrated in Figure 3.18:

- $K_{s,t} = H_t$ for $s \in \left[-\frac{1}{3}, \frac{1}{3}\right]$
- $K_{s,t} = G_t$ for $s \in \left[\frac{2}{3}, \frac{4}{3}\right]$
- $K_{s,t} = F_s$ for $t \in \left[-\frac{1}{3}, \frac{1}{3}\right]$
- $K_{s,t} = f_2$ for $t \in \left[\frac{2}{3}, \frac{4}{3}\right]$

Note that these properties are not contradictory because we are working with stationary interpolations.

Now, to make a slide, we modify K as follows:

$$\tilde{K}_{s,t}(p) = K_{s,t}(p) + g(s) + g(t),$$

with g defined similarly as before, making $\tilde{K}$ a Morse function with critical points in the yellow regions in the figure. The points correspond to critical points of f_0, f_1, f_2 and f_2 with indices raised by $2, 1, 1, 0$ respectively. Also similarly as before, we can construct a pseudo-gradient vector field X adapted to $\tilde{K}$, by adding $-\operatorname{grad} g(s)$, $-\operatorname{grad} g(t)$ at the appropriate regions and perturbing it in order to have the Smale property, again making sure that the

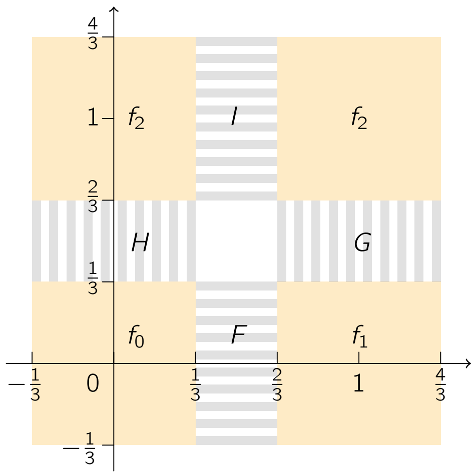


Figure 3.18: The map $K_{s,t}$ is a two-dimensional homotopy between f_0, f_1, f_2, f_2.

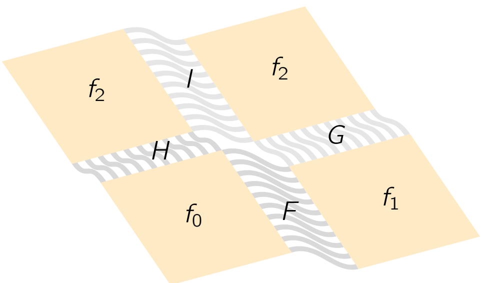


Figure 3.19: By adding the slide function g in s- and t-directions, we create a two-dimensional slide, eliminating the possibility of degenerate critical points.

perturbation is small enough such that $\partial_X = \partial_{\tilde{X}}$, where $\tilde{X}$ is the perturbed vector field.

While the resulting manifold $[-\frac{1}{3}, \frac{4}{3}]^2 \times M$ does not have a smooth boundary we can still define the Morse complex. In summary, we have

$$C_{k+1}(\tilde{K}) = C_{k-1}(f_0) \oplus C_k(f_1) \oplus C_k(f_2) \oplus C_{k+1}(f_2),$$

and the differential can be written as

$$\partial_{\tilde{X}} = \begin{pmatrix} \partial_0 & 0 & 0 & 0 \\ \Phi^F & \partial_1 & 0 & 0 \\ \Phi^H & 0 & \partial_2 & 0 \\ S & \Phi^G & \mathrm{Id} & \partial_2 \end{pmatrix}.$$

Now, computing $\partial^2_{\tilde{X}}(p, 0, 0, 0)$ is like water trickling down four spillway bowls, as illustrated in Figure 3.20. We get that

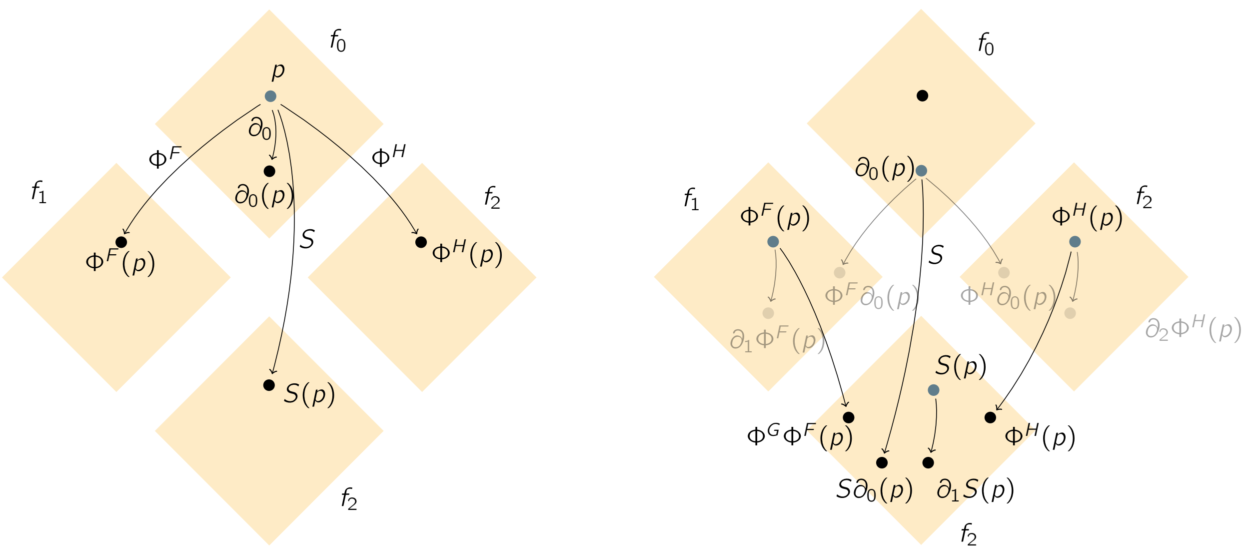
$$\Phi^G \circ \Phi^F + \Phi^H + S\partial_0 + \partial_1 S = 0,$$

or as we are working over $\mathbb{Z}_2$,

$$\Phi^G \circ \Phi^F - \Phi^H = S\partial_0 + \partial_1 S,$$

proving that $\Phi^G \circ \Phi^F$ and Φ^H induce the same map on the level of homology. This concludes the proof. $\square$

Figure 3.20: Calculating $\partial^2(p)$. On the left $\partial(p)$ is illustrated, and on the right $\partial^2(p)$. Because $\partial^2 = 0$, we find that $\Phi^G \circ \Phi^F + \Phi^H + S\partial_0 + \partial_1 S = 0$. Considering this over $\mathbb{Z}_2$ implies that $\Phi^G \circ \Phi^F$ and Φ^H are chain homotopic.



3.4 Morse homology over $\mathbb{Z}$

In this section, we will define Morse homology over $\mathbb{Z}$. As is to be expected, this homology theory will be less coarse than the one over $\mathbb{Z}_2$. For example, Morse homology over $\mathbb{Z}_2$ cannot distinguish a torus from a Klein bottle, while homology over $\mathbb{Z}$ can. The main difficulty in defining homology over $\mathbb{Z}$ is keeping track of signs and orientations.

Definition 3.16 (Integral Morse homology)**.** Let $f : M \to \mathbb{R}$ be a Morse function and X an adapted pseudo-gradient satisfying the Smale condition. Define

$$C_k(f, \mathbb{Z}) = \Big\{ \sum_{p \in \operatorname{Crit}_k f} n_p p \mid n_p \in \mathbb{Z} \Big\} = \bigoplus_{p \in \operatorname{Crit}_k f} \mathbb{Z} p,$$

with differentials

$$\begin{aligned} \partial_{X,k} : C_k &\longrightarrow C_{k-1} \\ p &\longmapsto \sum_{q \in \operatorname{Crit}_{k-1} f} N_X(p, q) q, \end{aligned}$$

where $N_X(p, q)$ is the *signed* number of trajectories between p and q. The homology of the complex $C_\bullet$ is called the integral Morse homology.

The count $N_X(p, q)$ is defined by orienting $\mathcal{L}(p, q)$. Because $\mathcal{L}(p, q)$ is discrete if the indices of p and q differ by one, this orientation comes down to assigning a sign to each trajectory in $\mathcal{L}(p, q)$. We can then define $N_X(p, q) = \sigma(\mathcal{L}(p, q))$ where σ adds up the signs of the points in $\mathcal{L}(p, q)$.

The orientation of $\mathcal{L}(p, q)$ is defined in the following way. First choose an (arbitrary) orientation for each of the stable manifolds. This is possible since stable manifolds are diffeomorphic to open disks. Orientations of stable manifolds induce co-orientations of unstable manifolds, because $W^u(p) \pitchfork W^s(q)$ and unstable manifolds are contractible. Now, the transverse intersection of an oriented and a co-oriented submanifold is also oriented, hence $\mathcal{M}(p, q) = W^u(p) \pitchfork W^s(q)$ is oriented. Finally fix an orientation on $\mathbb{R}$. Because $\mathcal{L}(p, q) = \mathcal{M}(p, q)/\mathbb{R}$, the orientation of $\mathcal{M}(p, q)$ induces an orientation on $\mathcal{L}(p, q)$.

Moreover, we have the following:[35]

[35] Michael Hutchings. *Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)*. 2002, p. 10

Proposition 3.17. The space of broken trajectories $\overline{\mathcal{L}}(a, b)$ is orientable. If $\operatorname{Ind} a = \operatorname{Ind} b + 2$, the space $\overline{\mathcal{L}}(a, b)$ is one-dimensional and as oriented manifolds we have

$$\partial \overline{\mathcal{L}}(a, b) = \bigcup_{c \in \operatorname{Crit}_{k-1} f} \mathcal{L}(a, c) \times \mathcal{L}(c, b).$$

This allows us to prove that $\partial^2 = 0$, using the fact that the signed number of boundary points of a compact oriented 1-manifold is always zero.

Theorem 3.18. Let X be a pseudo-gradient adapted to a Morse function $f : M \to \mathbb{R}$ that satisfies the Smale condition. Then the differential ∂_X of the integral Morse complex squares to zero.

Proof. Let a be a critical point of f. Then by definition of ∂,

$$\partial^2(a) = \sum_{b \in \mathrm{Crit}_{k-2} f} \sum_{c \in \mathrm{Crit}_{k-1} f} N_X(a, c) N_X(c, b)\, b.$$

By definition of N_X,

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \sum_{c \in \mathrm{Crit}_{k-1} f} \sigma(\mathcal{L}(a, c) \times \mathcal{L}(c, b))\, b$$

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \sigma\Big(\bigcup_{c \in \mathrm{Crit}_{k-1} f} \mathcal{L}(a, c) \times \mathcal{L}(c, b)\Big)\, b.$$

By definition of $\overline{\mathcal{L}}(a, b)$

$$= \sum_{b \in \mathrm{Crit}_{k-2} f} \sigma(\partial \overline{\mathcal{L}}(a, b))\, b.$$

Now, $\overline{\mathcal{L}}(a, b)$ is an oriented compact 1-manifold and hence $\sigma(\overline{\mathcal{L}}(a, b)) = 0$. This implies that $\partial^2(a) = 0$. ☐

Remark 3.19. While we are free to choose the orientations of $W^s(p)$, reversing it only changes the sign of $N_X(p, q)$ and $N_X(q, p)$ for any q, implying that the Morse homology is independent of the choice of orientation.

Let us end this section with some examples. We will illustrate the calculation for the signs of trajectories, and discuss Morse homology of the torus and the Klein bottle both over $\mathbb{Z}_2$ and $\mathbb{Z}$ to show that working over $\mathbb{Z}$ can have its advantages.

Example 3.20. Consider the 'other sphere' and more specifically $\mathcal{L}(p, q)$ with p and q as in Figure 3.21 below. We start off with an orientation of $W^s(q)$ and $W^s(p)$. To orient $\mathcal{M}(p, q)$, we use the orientation of $W^s(p)$, which co-orients $W^u(p)$, indicated in the figure with the thick black horizontal arrows. The orientation of $\mathcal{M}(p, q)$ is indicated with the vertical arrows tangent to $\mathcal{M}(p, q)$ and is defined by requiring that the horizontal and vertical vectors form a positive basis w.r.t. the orientation of $W^s(q)$. Finally, the orientation on $\mathcal{L}(p, q)$ is defined as follows: if the arrows are going up (i.e. in the negative time direction), the sign is negative, else it is positive.

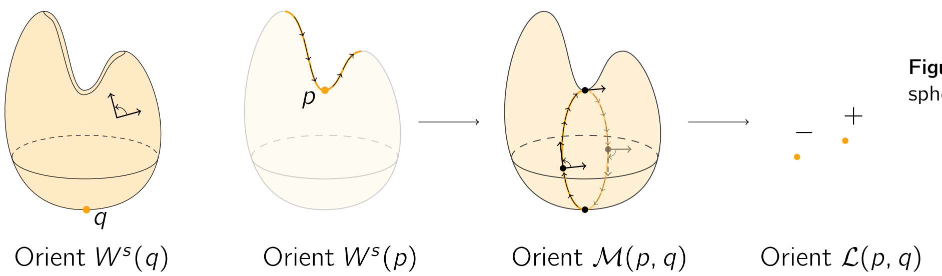


Figure 3.21: Orienting $\mathcal{L}(p, q)$ on the 'other sphere'.

Example 3.21 (Homology of T^2 over $\mathbb{Z}_2$)**.** We use the Morse function of Example 2.7, where the torus is slightly tilted, as illustrated in Figure 3.22, where we also added the graph of the critical points. Recall that each edge represents a trajectory connecting two critical points of consecutive index. The complex is given by $\mathbb{Z}_2 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2$, and because each critical point is connected twice to any other critical point of consecutive index, $\partial_k = 0$ for all k. This means that we have

$$HM_0(T^2;\mathbb{Z}_2) = \mathbb{Z}_2 \qquad HM_1(T^2;\mathbb{Z}_2) = \mathbb{Z}_2^2 \qquad HM_2(T^2;\mathbb{Z}_2) = \mathbb{Z}_2.$$

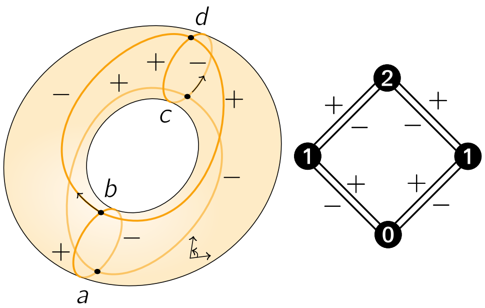


Figure 3.22: The height function on a tilted torus is a Morse function giving rise to the illustrated flow lines. On the right, an abstract depiction of the critical points and the signed flow lines connecting them.

Example 3.22 (Homology of the Klein bottle over $\mathbb{Z}_2$)**.** The Klein bottle K is a non-orientable surface. We cannot embed it in $\mathbb{R}^3$, but the strong version Withney's theorem shows that we are able to immerse it, which is what we have done in Figure 3.23. If we tilt the bottle, the height function h is Morse, and the gradient induced by the standard metric on $\mathbb{R}^3$ is adapted to h and satisfies the Smale condition. We have also included the graph of the critical points. Disregarding sign differences (which will only be important when discussing the complex over $\mathbb{Z}$), this graph is identical to the one we obtained for T^2. We conclude that the Morse homology of K and T^2 over $\mathbb{Z}_2$ are identical:

$$HM_0(K;\mathbb{Z}_2) = \mathbb{Z}_2 \qquad HM_1(K;\mathbb{Z}_2) = \mathbb{Z}_2^2 \qquad HM_2(K;\mathbb{Z}_2) = \mathbb{Z}_2.$$

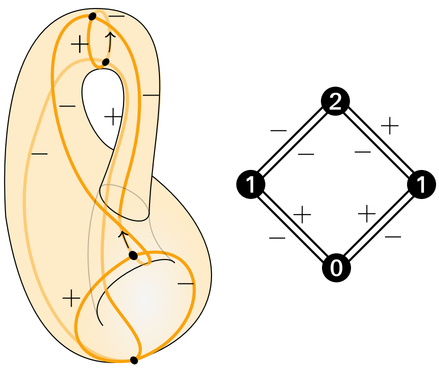


Figure 3.23: The height function of a tilted Klein bottle immersed in $\mathbb{R}^3$ is a Morse function. We have illustrated the flow lines connecting critical points. Not considering signs, we get the same complex as for the torus. If we do consider signs, we can distinguish one from another.

Example 3.23 (Homology of T^2 over $\mathbb{Z}$)**.** We use the Morse function of Example 2.7, where the torus is slightly tilted, as illustrated in Figure 3.22. We have indicated the chosen orientations as before and have assigned a positive orientation to the single point $W^s(d) = \{d\}$. For each trajectory, the sign has been added, both on the figure and on the graph.

The complex is given by $\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}^2 \xrightarrow{\partial} \mathbb{Z}$, and because the signs of the trajectories cancel, $N_X(p,q) = 0$ for all p and q, so each differential is zero. In the end, we have

$$HM_0(T^2;\mathbb{Z}) = \mathbb{Z} \qquad HM_1(T^2;\mathbb{Z}) = \mathbb{Z}^2 \qquad HM_2(T^2;\mathbb{Z}) = \mathbb{Z}.$$

Example 3.24. We have done the same for the Klein bottle, but notice that this time, the signs do not cancel. It is easy to check that $\partial_1 = 0$ and that ∂_2 is defined by $\partial_2(d) = -2b$, where a, b, c, d are the critical points of h with increasing height. This means that $\operatorname{Im}\partial_2 = 2\mathbb{Z}$. Summarizing, we have

$$HM_0(K;\mathbb{Z}) = \mathbb{Z} \qquad HM_1(K;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2 \qquad HM_2(K;\mathbb{Z}) = 0.$$

The conclusion of this series of examples is that integral homology is less coarse than homology over $\mathbb{Z}_2$.

3.5 Morse homology is singular homology

In the final section of this chapter, we prove that Morse homology is isomorphic to singular homology. There are many ways to go about this. Some authors[36] do it by proving that Morse homology is isomorphic to cellular homology, which is in itself isomorphic to singular homology. This approach consists of two steps: first proving that (suitable compactifications of) unstable manifolds form a cellular decomposition of M, and second: proving that the corresponding map is a chain isomorphism. While the first step is intuitive, a rigorous proof is actually very technical. The second step on the other hand, is quite straightforward.

[36] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014, p.110

To mitigate these technical difficulties, many other authors[37] follow a different approach, based on cellular filtrations, currents and other techniques. Here we opt for the proof by Hutchings based on currents, also called de Rham homology. For a thorough introduction on currents, see 'Differentiable manifolds' by de Rham.[38]

[37] Augustin Banyaga and David Hurtubise. *Lectures on Morse homology*. Vol. 29. Springer Science & Business Media, 2013, p. 195

Michael Hutchings. *Lecture notes on Morse homology (with an eye towards Floer theory and pseudo-holomorphic curves)*. 2002, p. 13

Alberto Abbondandolo and Pietro Majer. "Lectures on the Morse complex for infinite-dimensional manifolds". In: *Morse theoretic methods in nonlinear analysis and in symplectic topology*. Springer, 2006, pp. 1–74, p. 66

[38] Georges De Rham. *Differentiable manifolds: forms, currents, harmonic forms*. Vol. 266. Springer Science & Business Media, 2012

Definition 3.25 (Current)**.** A current is a continuous linear functional on the space of compactly supported k-forms on a manifold M.

Remark 3.26. One should think of currents as distributions on manifolds. For example, the Dirac delta distribution is a current acting on 0-forms on $\mathbb{R}$ as follows: $f \mapsto f(0)$, where f is a 0-form, i.e. a function.

Example 3.27. Any compact manifold (with boundary) M of dimension n defines a current $[M]$ on n-forms in the following way:

$$[M](\omega) := \int_M \omega.$$

For two disjoint submanifolds M, N we have

$$[M \sqcup N] = [M] + [N].$$

Notice that by Stokes' theorem,

$$[\partial M](\omega) = \int_{\partial M} \omega = \int_M d\omega = [M](d\omega).$$

This motivates the following definition:

Definition 3.28. We can define a differential on the space of currents by defining

$$(\partial T)(\omega) := T(d\omega).$$

When $T = [M]$, this can be written as $\partial[M] = [\partial M]$.

This differential clearly squares to zero, and hence defines a complex. It turns out that its homology is actually isomorphic to singular homology.[39] With this set up, we are ready to prove the following theorem:

[39] See Georges De Rham. *Differentiable manifolds: forms, currents, harmonic forms*. Vol. 266. Springer Science & Business Media, 2012, p. 89, Theorem 16, or Mariano Giaquinta, Guiseppe Modica, Jiri Soucek, et al. *Cartesian Currents in the Calculus of Variations II: Variational Integrals*. Vol. 2. Springer Science & Business Media, 1998, p. 582, Theorem 2

> **Theorem 3.29.** Let M be a closed manifold. Then
> $$HM_\bullet(M;\mathbb{Z}) = H_\bullet(M;\mathbb{Z}),$$
> where $H_\bullet(M;\mathbb{Z})$ denotes the singular homology of M.

Proof. As stated before, we follow Hutchings.[40]

[40] Michael Hutchings. *Lecture notes on Morse homology (with an eye towards Floer theory and pseudoholomorphic curves)*. 2002

Idea of the proof We define two chain maps:

$$\begin{aligned} D : C_\bullet &\longrightarrow C^c_\bullet(M) \\ \text{critical point} &\longmapsto [\text{compactification of } W^u(c)] \end{aligned}$$

and

$$\begin{aligned} A : C^c_\bullet(M) &\longrightarrow C_\bullet \\ \text{generic simplex} &\longmapsto \text{sum of critical points on which the} \\ &\qquad \text{simplex hangs by flowing via } X. \end{aligned}$$

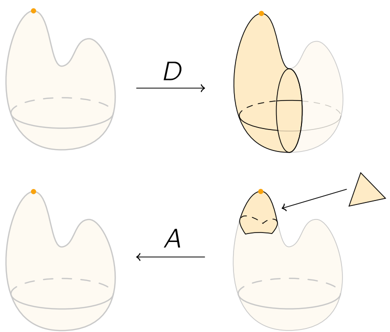


Figure 3.24: The map D is defined by mapping a critical point to the current of a compactification of $W^u(c)$. The map A maps a generic simplex to the critical points it 'hangs on'.

Here we define $C^c_i(M)$ as the subspace of all i-dimensional currents on M generated by *generic* i-simplices, by which we mean simplices that are smooth and whose faces are transverse to the stable manifolds of all critical points.

Then $A \circ D$ is the identity, and while $D \circ A$ is not, it is chain homotopic to the identity. The chain homotopy sends a singular chain to its entire forward orbit under the flow of X. This proves the theorem.

In summary, we will prove the theorem in 7 steps:

1. Compactification of $W^u(c)$
2. Definition of D
3. D is a chain map
4. Definition of A
5. A is a chain map
6. $A \circ D = \mathrm{Id}$
7. $D \circ A \cong \mathrm{Id}$

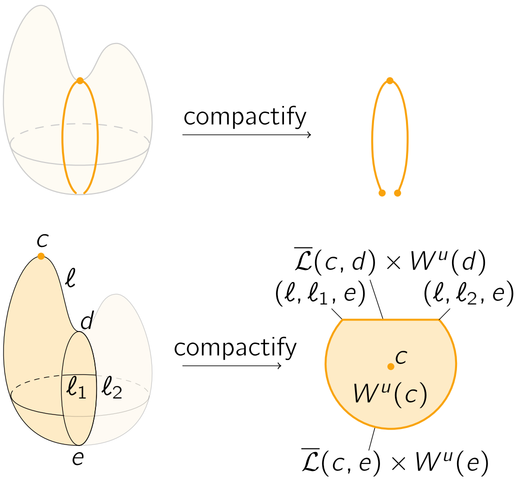


Figure 3.25: Two examples of compactifications of unstable manifolds in the 'other sphere'. On top we consider the index 1 critical point and on the bottom the index 2 critical point. Note that the compactifications are subtle and in particular are not diffeomorphic to S^1 and B^2 resp.

1. Compactification of $W^u(c)$ We can compactify $W^u(c)$ into a manifold with corners as follows:

$$\overline{W}^u(c) = W^u(c) \cup \bigcup_{d \neq c} \overline{\mathcal{L}}(c,d) \times W^u(d).$$

We have given some examples in Figure 3.25 which make it clear that this compactification possibly differs from the reader's expectations. For example, on top the compactification becomes a closed interval, and not a circle, even though the two end points map to the same point in M.

As an oriented manifold, its codimension-1 stratum[41] is given by

$$\partial \overline{W}^u(c) = \bigcup_{d \neq c} (-1)^{\operatorname{Ind} d + \operatorname{Ind} c + 1} \overline{\mathcal{L}}(c,d) \times W^u(d).$$

[41] The codimension k stratum of a manifold with corners M is the set of points p in M such that there exists a chart $f : U(p) \to \mathbb{R}^{n-k} \times [0,\infty)^k$ such that at least one of the last k coordinates of p is zero. The codimension 0 stratum is the interior of M, the codimension 1 stratum is its boundary, without the 'higher order' corners, etc.

2. Defining D Let $e : \overline{W}^u(c) \to M$ be the inclusion extending the inclusion of $W^u(c) \to M$.[42] Then we define the current $D(c) := e_*\left[\overline{W}^u(c)\right]$, i.e. integration over $\overline{W}^u(c)$:

$$D(c)(\omega) = \int_{\overline{W}^u(c)} e^*\omega.$$

This current is an element of $C_\bullet(M)$ because of the Smale condition.

[42] Note that by previous remarks, this extension is not necessarily injective

3. The map D is a chain map: $\partial D = D\partial^{\mathrm{Morse}}$ We have

$$\partial \overline{W}^u(c) = \bigcup_{d \neq c} (-1)^{\operatorname{Ind} d + \operatorname{Ind} c + 1} \overline{\mathcal{L}}(c,d) \times W^u(d),$$

which using the fact that $[M \sqcup N] = [M] + [N]$ implies that

$$\partial D(c) = \sum_{d \neq c} (-1)^{\operatorname{Ind} d + \operatorname{Ind} c + 1} e_* \left[\overline{\mathcal{L}}(c,d) \times W^u(d)\right].$$

We have three cases to consider based on the index of d:

$\operatorname{Ind} d > \operatorname{Ind} c - 1$: Then $\overline{\mathcal{L}}(c,d) = \emptyset$ by the Smale condition, so these terms vanish.

$\operatorname{Ind} d = \operatorname{Ind} c - 1$: We get a term of the form

$$e_*\left[\overline{\mathcal{L}}(c,d) \times \overline{W}^u(d)\right] = \#\mathcal{L}(c,d) \cdot e_*[W^u(d)],$$

because $\mathcal{L}(c,d) = \overline{\mathcal{L}}(c,d)$ is 0-dimensional.

$\operatorname{Ind} d < \operatorname{Ind} c - 1$: In this case, the term corresponds to a current of dimension less than $\operatorname{Ind} c - 2$, hence zero in $C^c_{\operatorname{Ind} c - 1}(M)$.

Summarizing, we have

$$\begin{aligned}\partial D(c) &= \sum_{d \in \mathrm{Crit}_{\operatorname{Ind} c-1} f} \#\mathcal{L}(c,d) \cdot e_*\left[\overline{W}^u(c)\right] \\ &= D(\partial^{\mathrm{Morse}}(c)).\end{aligned}$$

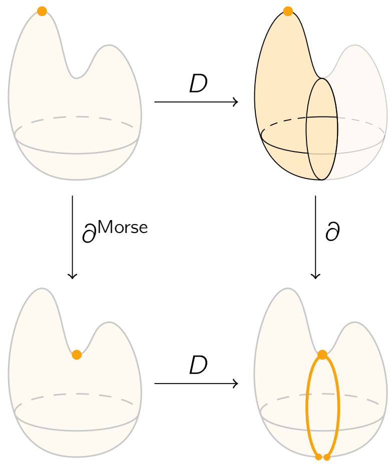


Figure 3.26: An example illustrating that D is a chain map.

4. Defining A The map A is defined by mapping a generic simplex σ to the sum of critical points on which the simplex hangs by flowing via X. Some examples of this vague definition are illustrated in Figure 3.27. More rigorously, we denote with $\mathcal{L}(\sigma, d)$ the moduli space of gradient flow lines starting in the i-simplex σ and ending in d. This space has a natural orientation and furthermore a compactification $\overline{\mathcal{L}}(\sigma, d)$, such that

$$\partial \overline{\mathcal{L}}(\sigma, d) = \overline{\mathcal{L}}(\partial\sigma, d) \cup \bigcup_{c \neq d} (-1)^{i + \operatorname{Ind} d} \mathcal{L}(\sigma, c) \times \mathcal{L}(c, d).$$

The dimension of this manifold is $i - \operatorname{Ind} d$, which means that if $\operatorname{Ind} d = i$, $\overline{\mathcal{L}}(\sigma, p) = \mathcal{L}(\sigma, p)$ consists of a finite number of points (with signs if we are considering orientation). The map A is then defined as

$$A(\sigma) = \sum_{p \in \mathrm{Crit}_i f} \#\mathcal{L}(\sigma, p) p,$$

which corresponds to the intuitive definition given earlier.

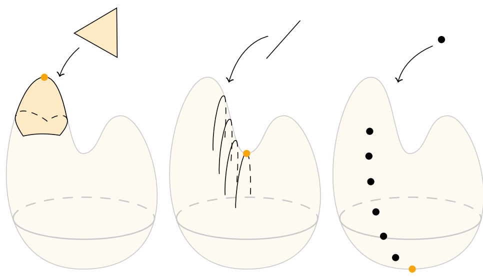


Figure 3.27: Examples illustrating the definition of A. From left to right, a 2, 1 and 0 simplex σ and the resulting critical point $A(\sigma)$ indicated in orange.

5. The map A is a chain map: $A\partial = \partial^{\text{Morse}} A$ We will show that $A\partial = \partial^{\text{Morse}} A$ component by component. To this purpose, we introduce the inner product on the space of critical points, defined on a basis as follows:

$$\langle c, d\rangle = \begin{cases} 1 & \text{if } c = d \\ 0 & \text{else.} \end{cases}$$

Let σ be a i-simplex and $d \in \mathrm{Crit}_{i-1} f$. Then $\dim \overline{\mathcal{L}}(\sigma, d) = 1$, so $\#\partial\overline{\mathcal{L}}(\sigma, d) = 0$, with signs taken into account. This implies that

$$\begin{aligned} \langle A(\partial\sigma), d\rangle - \left\langle \partial^{\text{Morse}} A(\sigma), d\right\rangle &= \#\overline{\mathcal{L}}(\partial\sigma, d) - \# \bigcup_{c\in \mathrm{Crit}_i f} \mathcal{L}(\sigma, c) \times \mathcal{L}(c, d) \\ &= \#\partial\overline{\mathcal{L}}(\sigma, d) = 0, \end{aligned}$$

showing that $A\partial = \partial^{\text{Morse}} A$.

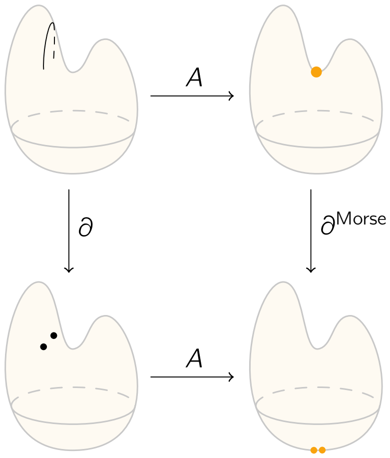


Figure 3.28: An example where σ is a 1-simplex illustrating that A is a chain map.

6. The map A is left inverse to D: $A \circ D = \mathrm{Id}$ This immediately follows from the definitions of A and D. Indeed, $\mathcal{L}(D(c), c) = \{c\}$ and $\mathcal{L}(D(c), d) = \emptyset$ for any other critical point d.

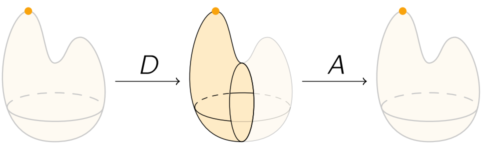


Figure 3.29: The map A forms a left inverse to D.

7. The composition $D \circ A$ is chain homotopic to Id While clearly $D \circ A \neq \mathrm{Id}$, these two maps are chain map homotopic, meaning that there exists a map $F : C_i(M) \to C_{i+1}(X)$ such that $\partial F + F\partial = D \circ A - \mathrm{Id}$. Intuitively, a chain homotopy is a chain of one dimension higher that connects $(D \circ A)(\sigma)$ and $\mathrm{Id}(\sigma) = \sigma$. The intuitive choice in this case would be the so-called forward orbit of σ:

$$G(\sigma) := [0, \infty) \times \sigma,$$

with the map $e : G(\sigma) \to M : (s, x) \mapsto \psi_s(\sigma(x))$. This set can be compactified to a smooth manifold with boundary

$$\partial\overline{G(\sigma)} = -\sigma \cup -G(\partial\sigma) \cup \bigcup_d \mathcal{L}(\sigma, d) \times W^u(d).$$

We can extend e to a smooth map and define

$$\begin{aligned} F : C_i^c(M) &\longrightarrow C_{i+1}^c(X) \\ \sigma &\longmapsto e_* \left[\overline{G(\sigma)}\right]. \end{aligned}$$

While intuitively, this is a chain map between σ and $(D \circ A)(\sigma)$, we should check that $\partial F + F\partial = D \circ A - \mathrm{Id}$. Calculating ∂F and $F\partial$, we have

$$\begin{aligned} (\partial F)(\sigma) &= -e_*[\sigma] - e_*[G(\partial\sigma)] + \sum_d e_* \left[\mathcal{L}(\sigma, d) \times W^u(d)\right] \\ (F\partial)(\sigma) &= e_* \left[\overline{G(\partial\sigma)}\right]. \end{aligned}$$

When we add this up, $[G(\partial\sigma)]$ and $\left[\overline{G(\partial\sigma)}\right]$ cancel as currents, leaving us with

$$(\partial F + F\partial)(\sigma) = -e_*[\sigma] + \sum_d e_* \left[\mathcal{L}(\sigma, d) \times W^u(d)\right].$$

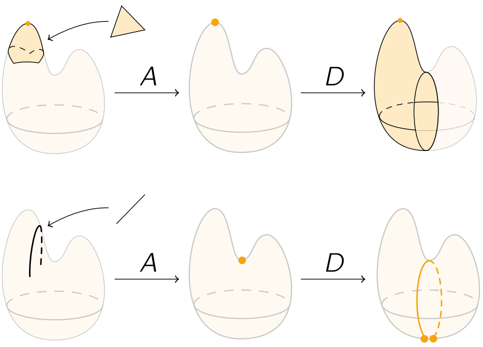


Figure 3.30: $D \circ A \neq \mathrm{Id}$.

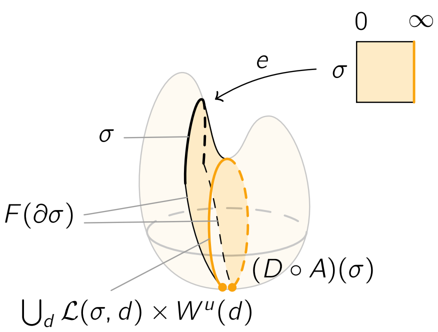


Figure 3.31: A compactification of the forward orbit of a chain σ forms a chain homotopy between σ and $(D \circ A)(\sigma)$.

Now, calculating $(D \circ A - \mathrm{Id})(\sigma)$, we get

$$\begin{aligned}(D \circ A - \mathrm{Id})(\sigma) &= D\left(\sum_d \#\mathcal{L}(\sigma, d) d\right) - e_*[\sigma] \\ &= \sum_d \#\mathcal{L}(\sigma, d) \cdot D(d) - e_*[\sigma] \\ &= \sum_d \#\mathcal{L}(\sigma, d) \cdot e_*[\overline{W}^u(d)] - e_*[\sigma].\end{aligned}$$

Keeping in mind that as currents $[\overline{W}^u(p)] = [W^u(p)]$, we find that $D \circ A - \mathrm{Id} = \partial F + F\partial$. $\square$

CHAPTER FOUR

Applications of Morse homology

This chapter discusses some applications of Morse homology. While we already know that it isomorphic to singular homology and hence it enjoys all the same properties, it can still be fruitful to derive these things directly from the definitions of Morse homology. We will do this for the Poincaré duality and the Künneth formula in the first two sections of this chapter. The third and last section discusses the Morse inequalities, giving a lower bound for the number of critical points of a Morse function $f : M \to \mathbb{R}$ based on the homology of M. This relation can also been seen in reverse: the number of critical points of a Morse function give bounds on the homology of M. These inequalities will play a major role in the next chapter.

4.1 Poincaré duality

The Poincaré duality was first stated by Henri Poincaré in his paper 'Analysis Situs'.[43] In this seminal paper, Poincaré introduces furthermore the concepts of fundamental group, simplicial complex, generalized Euler characteristic, founding the field of algebraic topology. Let us state and prove a modern version of Poincaré duality.

[43] Henri Poincaré. *Analysis situs.* Gauthier-Villars Paris, France, 1895

Theorem 4.1 (Poincaré duality)**.** Let M be a compact manifold, and $f : M \to \mathbb{R}$ a Morse function with adapted pseudo-gradient X. Then

$$H_k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2).$$

Moreover, if M is oriented, then

$$H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z}).$$

Proof. The idea of this proof is to turn the manifold upside down, i.e. $f \rightsquigarrow -f$. Then critical points of index k of f become critical points of index $n-k$ of $-f$. Moreover $-X$ is an adapted pseudo-gradient for $-f$, unstable manifolds of critical points of f become stable manifolds of corresponding critical points of $-f$ and vice versa. This immediately gives that $H_k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$.

Working over $\mathbb{Z}$, things are more subtle. In general $H_k(M;\mathbb{Z}) \ncong H_{n-k}(M;\mathbb{Z})$. The best we can do is saying that if M is orientable, then the ranks of $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ agree, as is proven in 'Morse theory and Floer homology' by Audin and Damian.[44].

[44] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Springer, 2014, p 84.

In order to get a more natural form of Poincaré duality, it is beneficial to consider Morse cohomology, as done in notes by Fushida-Hardy.[45] Recall from Chapter 0 that singular cohomology is defined as the homology of the dual of the singular chain complex, i.e. the homology of $C^k = \mathrm{Hom}(C_k, \mathbb{Z})$, with as differential the adjoint of ∂. The resulting homology is denoted with $H^k(M)$.

[45] Shintaro Fushida-Hardy. *Morse theory*

In Morse homology, it is more natural to define cohomology as follows:

$$C^k = \mathrm{Hom}(C_{n-k}(-f), \mathbb{Z}),$$

with differential $\partial^{k+1} : C^k \to C^{k+1}$ the adjoint of ∂_{k+1}. With these conventions, it is clear that there exists an isomorphism $t_k : C_k \to C^{n-k}$. Let us show that this is a chain map, from which the desired result follows. In other words we need to show that

$$t_{k-1} \circ \partial_k = \partial^{n-k+1} \circ t_k.$$

Let $b \in C_k(f)$ and $a \in C_{n-k+1}(f)$. Then the left hand side becomes

$$\begin{aligned}
(t_{k-1} \circ \partial_k)(b)(a) &= t_{k-1}\Big(\sum_{c \in \mathrm{Crit}_{k-1} f} N_X(b,c)c\Big)a \\
&= \sum_{c \in \mathrm{Crit}_{k-1} f} N_X(b,c)c^* a \\
&= N_X(b,a),
\end{aligned}$$

where c^* is the dual of c defined by $c^*(d) = \delta_{cd}$. The right hand side becomes

$$\begin{aligned}
(\partial^{n-k+1} \circ t_k)(b)(a) &= (\partial^{n-k+1} b^*)(a) \\
&= b^* \sum_{c \in \mathrm{Crit}_k f} N_{-X}(a,c)c \\
&= N_{-X}(a,b).
\end{aligned}$$

Hence, the last step of the proof is to show that $N_X(p,q) = N_{-X}(q,p)$. If we did not count signs, this would be immediately clear. However, when working over $\mathbb{Z}$ we need to check that the orientations work out. If M is oriented, this forms no problem. Indeed, in that case, the orientations of the stable manifolds not only induce a co-orientation of the unstable manifolds, but also an orientation. Now it is just a matter of carefully checking the definition of the orientation we put on $\mathcal{L}(p,q)$ to conclude that $N_X(p,q) = N_{-X}(q,p)$. This finishes the proof. □

If M is a cobordism we find the following theorem:

Theorem 4.2. Let M be an oriented cobordism from M_0 to M_1. Let $f : M \to [0,1]$ be a Morse function with adapted pseudo-gradient X. Then

$$H_k(M, M_0; \mathbb{Z}) \cong H^{n-k}(M, M_1; \mathbb{Z}).$$

A proof of this fact can be found in 'Differential Geometry' by Kosinski.[46]

[46] Antoni A Kosinski. *Differential manifolds*. Courier Corporation, 1993

4.2 The Künneth Formula

In homotopy theory, the homotopy groups of two spaces relate nicely to the homotopy group of their product: $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$. In homology the Künneth formula plays a similar role in homology theory.

> **Theorem 4.3** (Künneth formula)**.** Let M, N be two manifolds. Then
>
> $$H_k(M \times N; \mathbb{Z}_2) \cong \bigoplus_{i+j=k} H_i(M; \mathbb{Z}_2) \otimes H_j(N; \mathbb{Z}_2).$$
>
> In other words,
>
> $$H_\bullet(M \times N; \mathbb{Z}_2) \cong H_\bullet(M; \mathbb{Z}_2) \otimes H_\bullet(N; \mathbb{Z}_2),$$
>
> with the tensor product of chain complexes as defined in Chapter 0.

Remark 4.4. The Künneth formula as written above is not true when considering homology over $\mathbb{Z}$.

Remark 4.5. We can also express this in a different way using the Poincaré polynomial. For this, define $\beta_k(M) = \dim_{\mathbb{Z}_2} H_k(M; \mathbb{Z}_2)$ and let $P_M(t) = \sum_k \beta_k(M) t^k$. Then the Künneth formula states that $P_{M \times N}(t) = P_M(t) P_N(t)$.

For example, we have $P_{S^1}(t) = 1 + t$, so $P_{S^1 \times S^1 \times S^1} = (1+t)^3 = 1 + 3t + 3t^2 + 1t^3$, exactly the result we found in Example 3.5. More general, we have that $\beta_k(T^n)$ is the kth coefficient of $(1+t)^n$, i.e. $\binom{n}{k}$.

Proof. Let f, g be two Morse functions and X, Y two pseudo-gradient fields on M and N. Then $f + g$ is a Morse function on $M \times N$ and (X, Y) is an adapted pseudo-gradient field. If we assume that X and Y satisfy the Smale condition, then so does (X, Y). Critical points of $f + g$ are pairs of critical points of f and g and their indices are sums of the original indices. Furthermore, trajectories of (X, Y) correspond exactly to pairs of trajectories of X and Y. Now, in order to understand the differential $\partial_{(X,Y)}$ on $M \times N$, we are interested in gradient flow lines that connect critical points (a, b) and (c, d) whose index differ by exactly one. It's clear that the only way this can happen is when $a = c$ or $b = d$.[47]

[47] If $a \neq c$ and $b \neq d$, then $\operatorname{Ind} c \geq \operatorname{Ind} a + 1$ and $\operatorname{Ind} d \geq \operatorname{Ind} b + 1$, so $\operatorname{Ind}(c, d) \geq \operatorname{Ind}(a, b) + 2$.

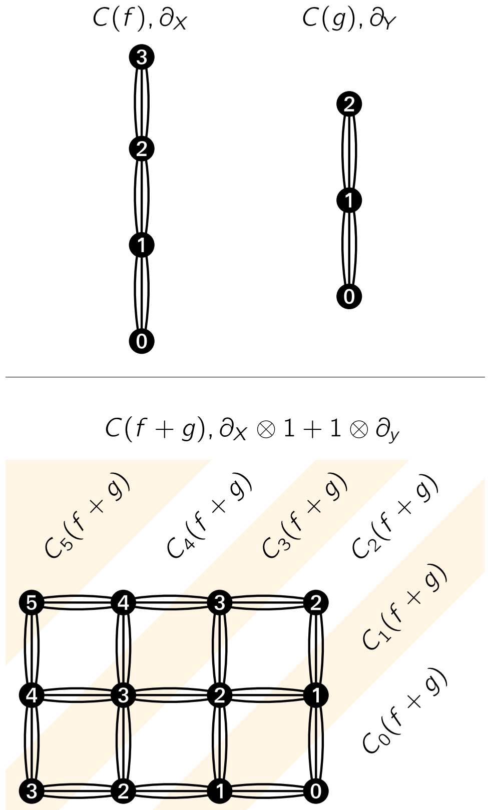


Figure 4.1: On top: the Morse complexes of (f, X) and (g, Y). On the bottom: the Morse complex of $(f + g, (X + Y))$.

When we think about this in terms of directed graphs of critical points, like we did in the example of T^3, we find that the graph of $M \times N$ is the Cartesian product of the graph of M and the graph of N. With these things in mind, it is easy to check that

$$\Phi : \bigoplus_{i+j=k} C_i(f) \otimes C_j(g) \longrightarrow C_k(f+g)$$
$$a \otimes b \longmapsto (a, b)$$

is an isomorphism of complexes with the following differentials:

$$(C_\bullet(f) \otimes C_\bullet(g), \partial_X \otimes 1 + 1 \otimes \partial_Y) \xrightarrow{\Phi} (C_\bullet(f+g), \partial_{(X,Y)}),$$

where $(C_\bullet(f) \otimes C_\bullet(g))_k := \bigoplus_{i+j=k} C_i(f) \otimes C_j(g)$. Now, taking the homology of both sides, we find

$$\bigoplus_{i+j=k} H_i(M; \mathbb{Z}_2) \otimes H_j(N; \mathbb{Z}_2) \cong H_k(M \times N; \mathbb{Z}_2),$$

where we have used that the homology of a tensor product complex is the tensor product of the homologies by Proposition 0.24. $\square$

Example 4.6. Let us illustrate the proof with a concrete example. Consider the height function f of S^1. The graph of critical points is illustrated below. There are two trajectories from the index 1 critical point the index 0 critical point, so $\partial = 0$. This means that $H_0(S^1; \mathbb{Z}_2) = H_1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$.

To find the homology of $T^2 = S^1 \times S^1$, we consider the Morse function $(\theta_1, \theta_2) \mapsto f(\theta_1) + f(\theta_2)$. We have illustrated the resulting graph of critical points below, which is just the Cartesian product of the graph associated to S^1. We have also done this for $T^3 = S^1 \times S^1 \times S^1$. Because all the trajectories occur in pairs, the differential ∂ vanishes. We conclude that $H_k(T^n; \mathbb{Z}_2) = \mathbb{Z}_2^{\binom{n}{k}}$.

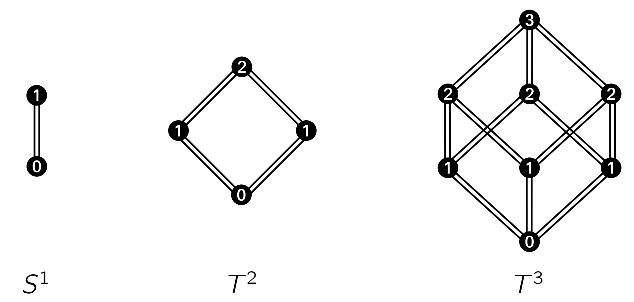


Figure 4.2: Graph of critical points of the Morse functions on S^1, T^2 and T^3.

4.3 Morse inequalities

The Morse inequalities give a lower bound for the number of critical points of a Morse function $f : M \to \mathbb{R}$ in terms of the homology of M. Put differently, the number of critical points of a Morse function give bounds on the dimension of the homology of M. In this section, we will discuss many different versions of the Morse inequalities, some stronger than others.

4.3.1 Morse inequalities over $\mathbb{Z}_2$

Let us start with the simplest version of the Morse inequalities.

Theorem 4.7 (Weak Morse inequalities)**.** Let $f : M \to \mathbb{R}$ be a Morse function. Then

$$\# \operatorname{Crit} f \geq \sum \dim H_k(M; \mathbb{Z}_2),$$

and more specifically,

$$\# \operatorname{Crit}_k f \geq \dim H_k(M; \mathbb{Z}_2).$$

Proof. With everything we have set up so far, this is actually a very straightforward result, following from the fact that

$$H_k(M; \mathbb{Z}_2) = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k-1}},$$

so $\dim H_k(M; \mathbb{Z}_2) = \dim \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k-1}} \leq \dim C_k = \# \operatorname{Crit}_k f$. □

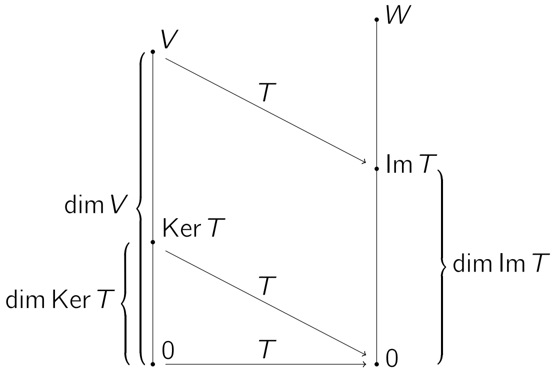


Figure 4.3: Visual depiction of the rank-nullity theorem stating that $\dim \operatorname{Im} T + \dim \operatorname{Ker} T = \dim V$.

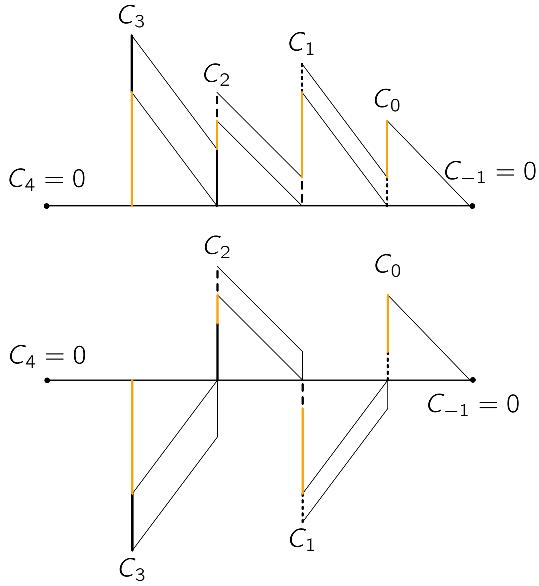


Figure 4.4: Top: same illustration as above, this time for the Morse complex with its differential. Each orange line corresponds to a homology group. Bottom: The alternating sum of the dimensions of C_k equals the alternating sum of the dimensions of H_k.

A result in similar vein is the following:

Theorem 4.8. Let $f : M \to \mathbb{R}$ be a Morse function. Then

$$\sum (-1)^k \# \operatorname{Crit}_k f = \sum (-1)^k \dim H_k(M; \mathbb{Z}_2) =: \chi(M).$$

Considering this equality modulo 2, we get

$$\# \operatorname{Crit} f \equiv \sum \dim H_k(M; \mathbb{Z}_2) \mod 2.$$

Proof. Use the fact that

$$H_k(M; \mathbb{Z}_2) = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k-1}} \quad \text{and} \quad \# \operatorname{Crit}_k f = \dim C_k,$$

together with the rank-nullity theorem for vector spaces. A simple computation completes the proof.

To visualize this proof, have a look at Figure 4.3, expressing the rank-nullity theorem, which in this context says that the two slanted lines are parallel, implying that $\dim V - \dim \operatorname{Ker} T = \dim \operatorname{Im} T$. Repeating this diagram for ∂_k, remembering that $\partial_k^2 = 0$, gives Figure 4.4 (top), where we have highlighted the dimension of the homology spaces in orange. The bottom part of Figure 4.4 shows that when we consider the alternating sum of the dimensions of C_k, the thick black lines cancel, leaving us with the alternating sum of the dimensions of the homology H_k. □

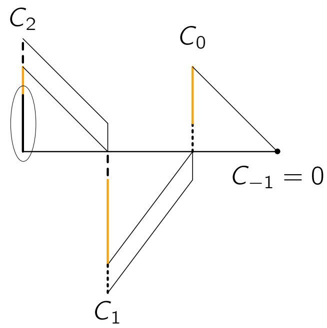


Figure 4.5: Truncating the above picture, we find that the alternating sum of the dimensions of C_k is greater than the alternating sum of the dimensions of H_k. All the thick black lines cancel, except the one circled, giving rise to the inequality.

We can also truncate this argument, considering only a partial alternating sum, illustrated in Figure 4.5. In this case, we do not have equality (the reason has been indicated in the figure), but we do have the following result:

Theorem 4.9 (Strong Morse inequalities)**.** For any Morse function $f : M \to \mathbb{R}$ and for any $m = 0, \ldots, n$, the following inequality holds:

$$\sum_{k=0}^{m} (-1)^{k+m} \# \operatorname{Crit}_k f \geq \sum_{k=0}^{m} (-1)^{k+m} \dim H_k(M; \mathbb{Z}_2).$$

It is easy to check that the strong Morse inequalities are in fact stronger: the weak version can be obtained by by subtracting the strong inequalities for m and $m+1$.

4.3.2 Morse inequalities over $\mathbb{Z}$ and $\mathbb{Z}_p$

In this section, we will generalize the Morse inequalities by working over other rings than $\mathbb{Z}_2$. More specifically we will consider the Morse inequalities over $\mathbb{Z}$ and $\mathbb{Z}_p$, where p is any prime.

Let us first consider the case of $\mathbb{Z}_p$ (or any other field for that matter). First of all, we should mention that defining Morse homology over $\mathbb{Z}_p$ gives no problems if we choose orientations of the stable manifolds, similarly as we have done for $\mathbb{Z}$. Moreover, because $\mathbb{Z}_p$ is a field, the proof given above can be simply repeated. In conclusion, we have the following:

Theorem 4.10. The (weak and strong) Morse inequalities hold over $\mathbb{Z}_p$ for any prime p.

When working over $\mathbb{Z}$, it not immediately clear how we can generalize the Morse inequalities: $\mathbb{Z}$ is not a field so the dimension of a $\mathbb{Z}$-module is not defined. Therefore, let us introduce some different notions of rank of a $\mathbb{Z}$-module, generalizing the concept of dimension.

[48] László Fuchs. *Infinite abelian groups*. Academic press, 1970

Definition 4.11 (Rank[48])**.** Let A be a $\mathbb{Z}$-module, and p be a prime. Then the following notions of rank are invariants of A:

$$\begin{aligned} r_0(A) &:= \text{cardinality of a maximal set of independent} \\ &\qquad \text{elements of infinite order} \\ &= \dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) && \text{(free rank)} \\ r_p(A) &:= \text{cardinality of a maximal set of independent} \\ &\qquad \text{elements of order } p^k \text{ for any } k \\ r_t(A) &:= \sum_p r_p(A) && \text{(torsion rank)} \\ r(A) &:= r_0(A) + \sum_p r_p(A). && \text{(total rank)} \end{aligned}$$

Example 4.12. Let $A = \mathbb{Z}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2$. Then

$$r_0(A) = 2 \qquad r_2(A) = 2 \qquad r_3(A) = 1.$$

The torsion rank is $r_t(A) = 3$ and the total rank is $r(A) = 5$.

We have the following relations between ranks of submodules:

Lemma 4.13. Let A and B be $\mathbb{Z}$-modules and suppose B is a submodule of A. Then

(i) $r(B) \leq r(A)$

(ii) $r(A) \leq r(B) + r(A/B)$

(iii) $r_0(A) = r_0(B) + r_0(A/B)$

Note in particular identity (iii), which is a rank-nullity theorem in the context of the r_0 rank. Also note that the modules $C_k(f, \mathbb{Z})$ over $\mathbb{Z}$ with as generators the critical points of f, are free, so $\#\operatorname{Crit}_k f = r_0(C_k(f, \mathbb{Z}))$. These two facts combined give the following result:

Theorem 4.14. The (weak and strong) Morse inequalities hold over $\mathbb{Z}$, in the following sense:

$$\#\operatorname{Crit}_k f \geq r_0(H_k(M; \mathbb{Z})),$$

and for any $m = 0, \dots, n$ we have

$$\sum_{k=0}^{m}(-1)^{k+m}\#\operatorname{Crit}_k f \geq \sum_{k=0}^{m}(-1)^{k+m} r_0(H_k(M; \mathbb{Z})).$$

Proof. Similar as before, replacing $\mathbb{Z}_2$ by $\mathbb{Z}$ and dim by r_0. □

4.3.3 Morse inequalities with torsion rank

Let us lastly give a stronger version of the Morse inequalities by using the torsion rank.

Theorem 4.15 (Pitcher inequalities[49]). Let $f : M \to \mathbb{R}$ be a Morse function. Then

$$\#\operatorname{Crit}_k f \geq r(H_k(M; \mathbb{Z})) + r_t(H_{k-1}(M; \mathbb{Z})).$$

[49] E. Pitcher. "Inequalities of critical point theory". In: *Bulletin of the American Mathematical Society* 64 (1958), pp. 1–30

Proof. We use $H_k = H_k(M; \mathbb{Z})$ for brevity.

$$r(H_k) + r_t(H_{k-1}) = r\left(\frac{\operatorname{Ker}\partial_k}{\operatorname{Im}\partial_{k+1}}\right) + r_t\left(\frac{\operatorname{Ker}\partial_{k-1}}{\operatorname{Im}\partial_k}\right).$$

Now, note that $\operatorname{Ker}\partial_k$ and $\operatorname{Im}\partial_k$ are free. Hence $r_t\left(\frac{\operatorname{Ker}\partial_{k-1}}{\operatorname{Im}\partial_k}\right) \leq r(\operatorname{Im}\partial_k)$.

Moreover, $r\left(\frac{\operatorname{Ker}\partial_k}{\operatorname{Im}\partial_{k+1}}\right) \leq r(\operatorname{Ker}\partial_k)$. This gives

$$\begin{aligned} r(H_k) + r_t(H_{k-1}) &\leq r(\operatorname{Im}\partial_k) + r(\operatorname{Ker}\partial_k) \\ &= r_0(\operatorname{Im}\partial_k) + r_0(\operatorname{Ker}\partial_k), \end{aligned}$$

again because $\operatorname{Im}\partial_k$ and $\operatorname{Ker}\partial_k$ are free. Now notice that $r_0(C_k) = r_0(\operatorname{Im}\partial_k) + r_0(\operatorname{Ker}\partial_k)$. This proves that

$$r(H_k) + r_t(H_{k-1}) \leq r_0(C_k),$$

from which the desired result follows. $\square$

CHAPTER FIVE

Generalized higher dimensional Poincaré conjecture

5.1 Introduction

In this chapter, we will prove three important theorems that are all closely related. The first one states that the Morse inequalities are in fact attainable under certain conditions. In other words, the bound on $\# \operatorname{Crit}_k f$ given by the (weak) Morse inequalities is in fact the best bound possible. This was first proven by Smale in 1960.

Theorem 5.1 (Smale's theorem[50])**.** If M is a simply connected closed manifold of dimension $m \geq 6$ and $H_\bullet(M)$ is free, then there exists a Morse function on M such that $\# \operatorname{Crit}_k f = r_0(H_k(M; \mathbb{Z}))$ for all $k = 0, \ldots, n$.

[50] Stephen Smale et al. "The generalized Poincaré conjecture in higher dimensions". In: *Bulletin of the American Mathematical Society* 66.5 (1960), pp. 373–375

Recall that we use $r_0(A)$ to denote the free rank of a $\mathbb{Z}$-module A. A closely related theorem is the h-cobordism theorem, stating

Theorem 5.2 (h-cobordism theorem[51])**.** If M is a cobordism from M_0 to M_1 that is simply connected, $\dim M \geq 6$ and $H_\bullet(M, M_0) = 0$, then M is a trivial cobordism, i.e. diffeomorphic to $M_0 \times I$.

[51] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015

The last—and perhaps most famous—theorem we will prove is the generalized Poincaré conjecture for dimensions $n \geq 5$. This conjecture (now a theorem) states that any homotopy sphere is a topological sphere.

Theorem 5.3 (Higher dimensional Poincaré conjecture)**.** If M is a homotopy sphere of dimension $n \geq 5$, then M is homeomorphic to S^n.

This was proven by Smale and in fact almost immediately follows from the minimality of the Morse inequalities. We will discuss the historical significance in Section 5.6. The proof of these theorems follows a handle cancellation approach, which roughly goes as follows:

(1) Let $f : M \to \mathbb{R}$ be an arbitrary Morse function.

(2) Under certain assumptions, we can cancel pairs of critical points.

(3) Repeatedly cancelling critical points allows us to conclude that the Morse inequalities are attained.

- On a cobordism with $H_\bullet(M, M_0) = 0$, this implies the existence of a Morse function without critical points, hence $M \cong [0, 1] \times M_0$
- On a homotopy sphere, this implies the existence of a Morse function with exactly two critical points: a minimum and a maximum, hence $M \cong S^n$ as we will show.

To this purpose, this chapter mostly contains cancellation results. The proof we give resembles the original proof of Smale but uses concepts and notation introduced earlier in this thesis. Most of this chapter is based on the somewhat more modern treatments by Milnor[52] and Kosinski.[53] The book of Kosinski is very thorough and proves things from the viewpoint of handles, isotopies of attachment regions, etc. Milnor on the other hand, does not even mention handle decompositions at all, and only works directly with the Morse function and its critical points.

[52] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015

[53] Antoni A Kosinski. *Differential manifolds*. Courier Corporation, 1993

5.2 Stronger cancellation result

Let us recall the first cancellation result we have proved in Chapter 2.

Theorem 2.27. Let $f : M \to [0, 1]$ be a Morse function on a cobordism with two critical points p, q of index k and $k - 1$. Let X be an adapted pseudo-gradient satisfying the Smale condition. If $\#\mathcal{L}(p, q) = n_X(p, q) = 1$, then p and q can be cancelled.

In the chapter on Morse homology, we have seen that by orienting $\mathcal{L}(p, q)$, we can define a signed count $N_X(p, q)$ of trajectories between p and q. A natural generalization of this theorem then would be to consider cases where $N_X(p, q) = \pm 1$, which is weaker than requiring $n_X(p, q) = 1$. Together with some additional assumptions, this turns out to be sufficient for cancelling p and q, giving the first strengthening of our cancellation result.

Theorem 5.4 (Second cancellation theorem)**.** Let $f : M \to [0, 1]$ be a Morse function on a cobordism M (from M_0 to M_1) with two critical points p, q. Suppose furthermore that M, M_0, M_1 are simply connected and that

$$2 \leq \operatorname{Ind} q = k \qquad \operatorname{Ind} p = k + 1 \leq n - 3.$$

If $N_X(p, q) = \pm 1$, then p and q are cancelable.

Remark 5.5. By changing f to $-f$, the theorem is also true if we replace the conditions with $k \geq 3$ and $k + 1 \leq n - 2$. This means that the theorem is true for

$$2 \leq \operatorname{Ind} q = k \quad \operatorname{Ind} p = k + 1 \leq n - 2.$$

Let us give some examples illustrating the idea of the theorem.

Example 5.6. Consider a 3-dimensional manifold M with a Morse function f and let α be a regular value such that $f^{-1}(-\infty, \alpha]$ consists of a 0, 1 and 2 handle as in the figures below.

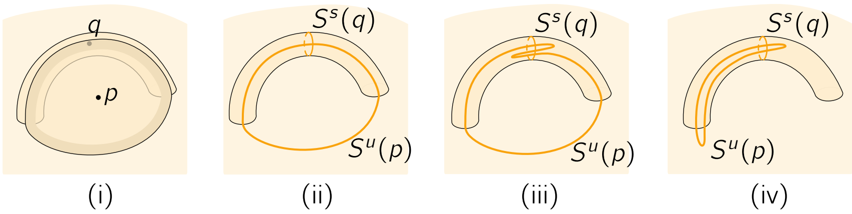


Figure 5.1: Examples illustrating the second cancellation theorem.

Let us consider the four situations from left to right.

(i) The first figure shows the 2- and 1- handle and their corresponding critical points p and q. The configuration is the same as in Figure 1.18, but from a different perspective. The 1-handle is a solid tube and the 2-handle is a dome (a thickened up disk) that rests on top of the 1-handle. It should be visually clear that these two critical points are cancelable as already discussed in Example 1.23.

(ii) The second figure shows the same situation, but we have hidden the 2-handle and indicated the so-called belt sphere $S^s(q)$ and the attaching sphere $S^u(p)$. Their intersection consists of a single point, corresponding to a single flow line from p to q. By the first cancellation theorem, we can confirm that p and q are indeed cancelable.

(iii) Next, we have drawn a situation where the number of intersection points is 3, but the intersection number (which corresponds to $N_X(p, q)$, as we will expand upon later) is 1. It is clear that we can isotope the attaching sphere such that it only intersects once with the belt sphere. This reduces the situation to (ii), so p and q are cancelable, as is also clear from the figure.

(iv) The last figure shows a situation where the number of intersections points is 2 and the intersection number is 0. In this situation, we cannot cancel the two critical points. In fact, we can isotope the attaching sphere off the 1-handle, showing that the two handles are completely independent.

As hinted at in the example, the proof of this stronger cancellation result will reduce the given situation where $N_X(p, q) = \pm 1$ to one that satisfies the condition of the first cancellation theorem, i.e. $n_X(p, q) = 1$. In order to do this we want to 'cancel' flow lines of opposite signs. We will do this by using a theorem of Whitney which allows to cancel intersection points of opposite intersection numbers.[54]

[54] For an introduction to the intersection number of two manifolds, we refer the reader to Chapter 0.

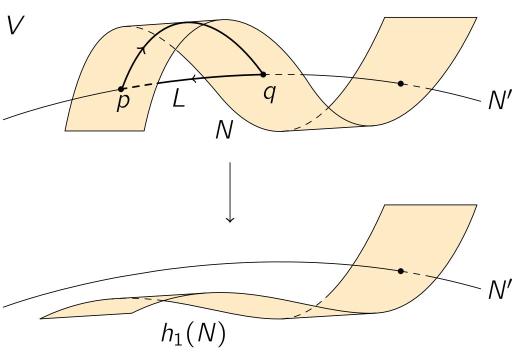


Figure 5.2: Under certain conditions, we can 'cancel' intersection points of opposite intersection number by deforming the manifold M by an isotopy.

Theorem 5.7 (Whitney)**.** Let N and N' be smooth, closed, tranversely intersecting submanifolds of dimensions r and s in a smooth $(r+s)$-dimensional manifold V. Suppose N is oriented and N' is co-oriented. Suppose $r+s \geq 5$, $s \geq 3$ and if $r < 3$ suppose that $\pi_1(V - N') \hookrightarrow \pi_1(N)$ is an isomorphism.

Let $p, q \in M \pitchfork N'$ points with opposite intersection number as in Figure 5.2 such that there exists a loop L contractible in V connecting p smoothly to q in N and then q smoothly to p in N' where both arcs miss other intersection points.

Then there exists an isotopy h_t of the identity $V \to V$ such that

(i) The isotopy is locally the identity around other intersection points;

(ii) At time $t = 1$, N' and N no longer intersect in p and q. In other words, $h_1(N) \cap N' = N \cap N' \setminus \{p, q\}$.

Proof. Postponed to page 75. □

In order to use this theorem, we will interpret $N_X(p,q)$ as the intersection number of two submanifolds. Let α be a regular value between $f(p)$ and $f(q)$ and let $V = f^{-1}(\alpha)$. Let $S^u(p) = W^u(p) \cap V$ and $S^s(q) = W^s(q) \cap V$ be spheres inside V. Then we have noted before that $\mathcal{L}(p,q) \cong S^u(p) \cap S^s(q)$ so that

$$n_X(p,q) = \#(S^u(p) \cap S^s(q)).$$

Moreover, we have that

$$N_X(p,q) = S^u(p) \cdot S^s(q),$$

where we denoted by $N \cdot N'$ the intersection number of two manifolds. Recall that for defining $N_X(p,q)$, we chose arbitrary orientations of the stable manifolds, inducing co-orientations of the unstable manifolds. These chosen orientations allow us to talk about the intersection number of $S^u(p)$ and $S^s(q)$. With this insight, we are ready to prove the second cancellation theorem.

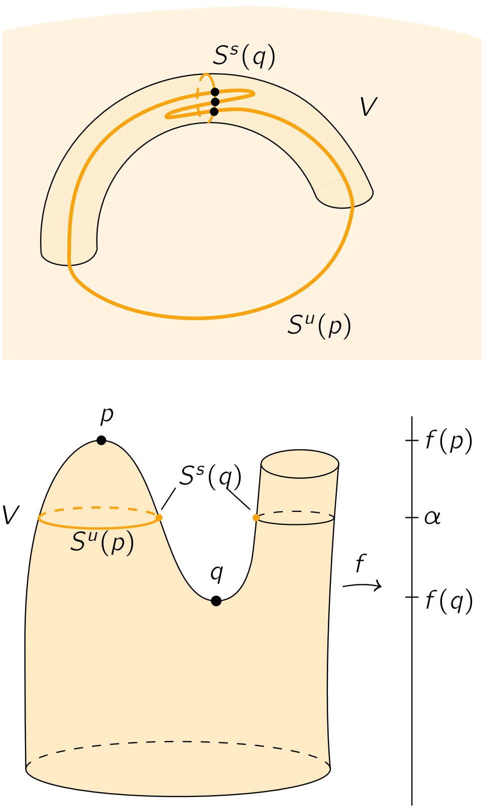


Figure 5.3: Setup of the second cancellation theorem in dimension two and three. Note the figure is somewhat misleading because of dimensionality reasons.

Proof of the second cancellation theorem. Let X be an adopted pseudo-gradient vector field adopted to f and satisfying the Smale condition. Let $N = S^u(q) = W^u(q) \cap V$ and $N' = S^s(p) = W^s(p) \cap V$.

We know that $S^u(q) \cdot S^s(p) = \pm 1$, which means that either there is only one flow line connecting p and q, in which case the proof reduces to the first cancellation theorem, or there are multiple flow lines with opposite signs. If we can show that the conditions of the theorem of Whitney are satisfied, we can cancel these intersection points pair by pair (by altering X) until we have reached the situation of the first cancellation theorem.

Let $V = f^{-1}(\alpha)$. If M is simply connected, then V is also simply connected.[55] If k, the index of the critical point q is greater than or equal to 3, then all the conditions of the theorem are satisfied and we are done. If $k = 2$, we need to show that $\pi_1(V - S^s(p)) \cong \pi_1(V) = 1$. Let $S = W^u(q) \cap f^{-1}(0)$.

[55] This follows from applying the Seifert–Van Kampen theorem: $\pi_1(V) \cong \pi_1(W^s(p) \cup V \cup W^u(q))$ (here we use that $k > 2, n - k \geq 3$). Then notice that $W^s(p) \cup V \cup W^u(q)$ is homotopic to M showing that $\pi_1(V) \cong \pi_1(M) = 1$.

Flowing $M_0 \setminus S$ via X gives $V - S^s(p)$, so we need to show that $M_0 \setminus S$ has trivial fundamental group. For this we use the Seifert–Van Kampen theorem. Let N be a product neighbourhood of S in M_0. Note that $k = 2$, so S is diffeomorphic to S^1 and $\dim M_0 = n - 1$, so the product neighbourhood is diffeomorphic to $S^1 \times \mathbb{R}^{n-2}$. As $k = 2$, $n - 2 \geq 4$, so $\mathbb{R}^{n-2} \setminus \{0\}$ has trivial fundamental group. Therefore $N \setminus S \cong S^1 \times (\mathbb{R}^{n-2} \setminus \{0\})$ has the same fundamental group as S^1, which is $\mathbb{Z}$. This allows us to use the Seifert–Van Kampen theorem in the following way: $(M_0 \setminus S) \cup N = M_0$, $\pi_1(M_0) = 1$, $\pi_1(N) = \mathbb{Z}$, $\pi_1(N \setminus S) = \mathbb{Z}$. This implies that $\pi_1(M_0 \setminus S) = 1$, completing the proof. ☐

Let us for completeness also give a proof of Whitney's theorem.

Proof of Whitney's theorem. We will construct two local models of the situation, where it will be easy to perform the isotopy of N. We will end the proof by extending this isotopy to the whole of V.

Plane model Let C, C' be the arcs in N and N' connecting p and q and extend them a little bit either way, as in Figure 5.4. For the plane model, let C_0 and C_1 be open curves in the plane intersecting transversely in two points, call them a and b. Let D be the disk with two corners enclosed by C_0 and C_0'. Then there exists an embedding ϕ_1 of these curves into $N \cup N'$ such that the following holds:

- $\phi_1(C_0) = C$, $\phi_1(C_0') = C'$,
- $\phi_1(a) = p$, $\phi_1(b) = q$.

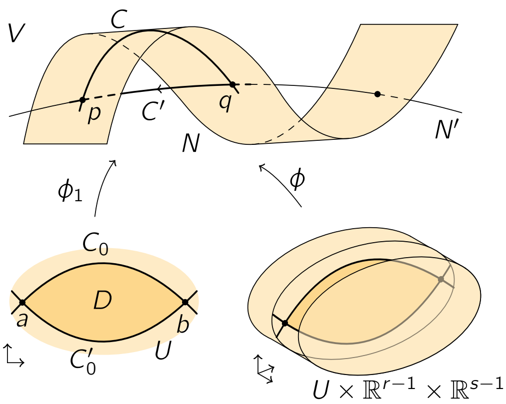


Figure 5.4: On the left: the plane model, on the right: the higher dimensional model.

Model Now we claim that we can extend this embedding by adding extra dimensions such that the following conditions are satisfied:

- The new embedding $\phi : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$ is an extension of $\phi_1|_{U \cap (C_0 \cup C_0')}$,
- $\phi^{-1}(N) = (U \cap C_0) \times \mathbb{R}^{r-1} \times 0$,
- $\phi^{-1}(N') = (U \cap C_0) \times 0 \times \mathbb{R}^{s-1}$.

This is actually quite subtle, and for a detailed proof of this claim, we refer the reader to the notes of Milnor.[56] With this model, the proof of the theorem follows quickly.

[56] John Milnor. *Lectures on the h-cobordism theorem*. Vol. 2258. Princeton university press, 2015, p. 75

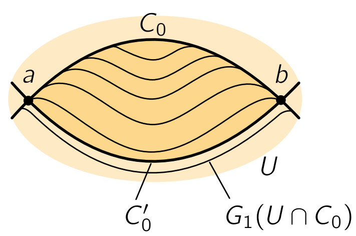


Figure 5.5: The isotopy G_t in the plane model moves C_0 below C_0', i.e. $G_1(U \cap C_0) \cap C_0' = \emptyset$.

Isotopy in the plane model Let $G_t : U \to U$ be an isotopy in the plane model that when applied to C_0, moves it under C_0' as in Figure 5.4. More specifically, we require the following:

- G_0 is the identity map,
- G_t is the identity near the boundary of U for all t,
- $G_1(U \cap C_0) \cap C_0' = \emptyset$.

Isotopy in the model To extend this isotopy to one on $U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1}$, define a bump function $\rho : \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \to [0,1]$ supported in $\{(x,y) \mid |x|^2 + |y|^2 \leq 1\}$ and set

$$\begin{aligned} H_t : U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} &\longrightarrow U \times \mathbb{R}^{r-1} \times \mathbb{R}^{s-1} \\ (u, x, y) &\longmapsto (G_{t\rho(x,y)}, x, y). \end{aligned}$$

Isotopy of V To finally find an isotopy of V, define $F_t : V \to V$ such that F_0 is the identity, F_t is the identity everywhere except away from $\operatorname{Im}\phi$, and on $\operatorname{Im}\phi$, define $F_t = \phi \circ H_t \circ \phi^{-1}$. This finishes the proof. $\square$

5.3 Sliding handles and diagonalizing $[\partial_k]$

If M is a cobordism with exactly two critical points, the second cancellation theorem allows us to cancel them if there is among other conditions, a single connecting flow line (counting with signs). When M has multiple critical points, we must also require that this pair does not interact with the other critical points. With this we mean the following. Let $[\partial_k]$ be the matrix associated to ∂_k with entries

$$[\partial_k]_{p,q} = N_X(p,q) \qquad (p,q) \in C_k(f,\mathbb{Z}) \times C_{k-1}(f,\mathbb{Z}).$$

Suppose $[\partial_k]$ has a row and a column of zeros, except for their intersection, which we require to be $N_X(p,q) = \pm 1$. Then it is easy to see that the conclusion of the second cancellation theorem still holds, i.e. we can cancel p and q, which comes down to removing the corresponding row and column in the matrix $[\partial_k]$. For example, in the case of $|C_k| = 4, |C_{k-1}| = 5$, the reduction could look as follows:

$$\begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & \pm 1 & 0 \\ * & * & 0 & * \\ * & * & 0 & * \end{pmatrix} \rightsquigarrow \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

This generalization of the second cancellation theorem leads us to our next goal: creating as many zeros in $[\partial_k]$ as possible. In other words, diagonalizing $[\partial_k]$ and showing that under certain conditions, $[\partial_k]$ is a diagonal matrix with only ± 1 on the diagonal. Then we can apply the second cancellation theorem multiple times, cancelling all the critical points (that lie in the middle dimensions, per assumption of the theorem).

Let us make very clear what we mean by diagonalizing. Starting off with a matrix $[\partial_k]$ based on f and X, we want to change it step by step such that the resulting $[\partial'_k]$ is diagonal (note that it does not need to be square). To accomplish these algebraic manipulations, we alter f and X.

We can diagonalize a matrix over $\mathbb{Z}$ by using three types of elementary row and column operations, as explained in Advanced modern algebra by Rotman.[57]

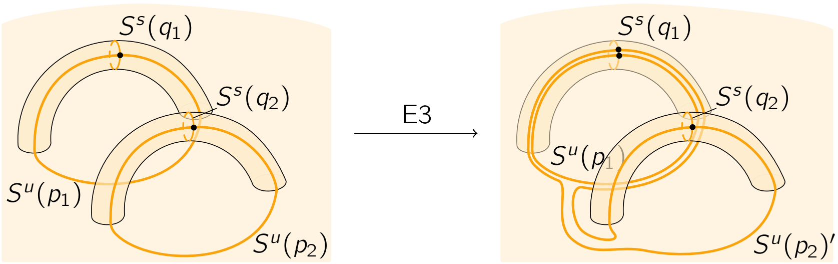
[57] Joseph J Rotman. *Advanced modern algebra*. Vol. 114. American Mathematical Soc., 2010, p. 688

E1 Interchange two rows or columns

E2 Multiplication of a row or column by -1

E3 Addition of a row (resp. column) to another row (resp. column)

It is clear that E1 and E2 can be done geometrically: we can just relabel the critical points for E1 and change the (arbitrary) orientation of the stable manifolds for E2. For the third operation, consider the following figure.



$$[\partial_2] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [\partial_2'] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Figure 5.6: By altering the attachment region of p_2, we can geometrically perform the addition of two columns in $[\partial]$.

By isotoping the attachment sphere of the second 2-handle, i.e. $S^u(p_2)$ over the first 2-handle to the sphere that is '$S^u(p_1)$ connected to $S^u(p_2)$ via a tube', we can add the first column of $[\partial_2]$ to the second one. Indeed, counting the intersections in the figure, we have that $\partial'(p_2) = q_1 + q_2 = \partial(p_1) + \partial(p_2)$.

To show that this works in general, we will use the following lemma:

Lemma 5.8. Let N be a connected closed manifold of dimension $n-1$ containing two embedded $(k-1)$-spheres S_1, S_2 $(1 < k < n)$. Assume that S_1 bounds a k-disk D disjoint from S_2. Then there is an isotopy in N of S_2 to a sphere S that is 'S_1 connected to S_2' with a tube. More explicitly, S consists of S_1 and S_2 with small discs removed and of a tube connecting these openings.

If V is a submanifold of dimension $n-k$ that does not disconnect N, then we can assume that the tube does not intersect V such that

$$V \cap S = (V \cap S_1) \cup (V \cap S_2).$$

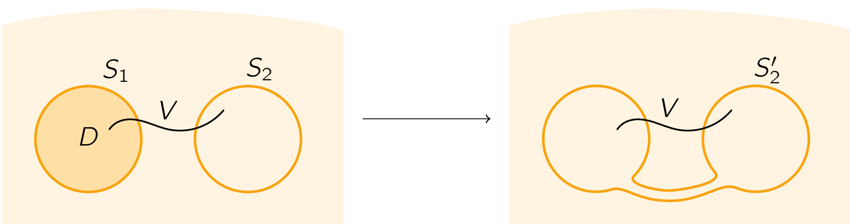


Figure 5.7: The lemma allows us to connect S_2 to S_1 while missing the submanifold V.

Proof. Let L be an arc in N that is disjoint from the interior of K connecting s_1 to s_2 points in S_1 and S_2. Let D_1, D_2 be disks around s_1 and s_2 in S_1 and S_2. First move s_2 along L to s_1 and extend this to an isotopy that moves a smaller disk $D \subset D_2$ to D_1, and keeps $S_2 - D_2$ fixed. Then move D_1 'in' D keeping its boundary fixed. Again extend to an isotopy of N. Composing these isotopies, we get the result. $\square$

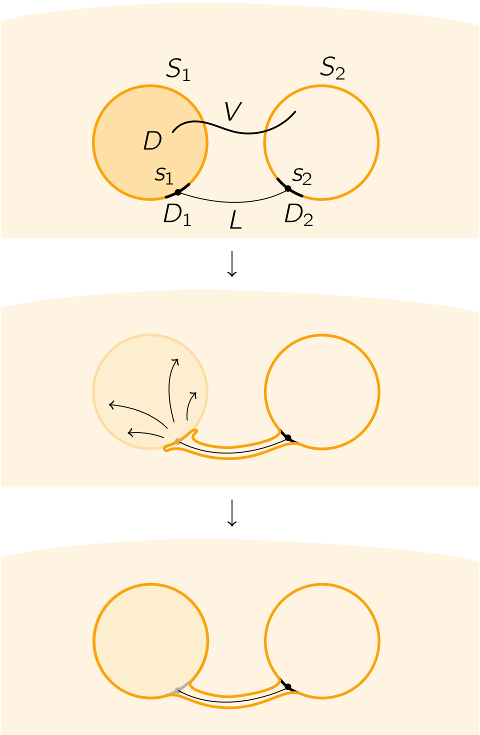


Figure 5.8: First we isotope S_2 by flowing along an extension of a vector field that is tangent to L. Then we use the fact that S_1 bounds a disk to move the end of the tube inside the disk until it reaches S_1.

Remark 5.9. It is also clear that $S \cdot V = S_1 \cdot V \pm S_2 \cdot V$, because the tube misses the submanifold V. If $k < n-1$ we can actually choose this sign freely by changing the first isotopy in the proof: we can either move D to D_2 with the same or the opposite orientation. If $k = n-1$, we do not have this freedom.

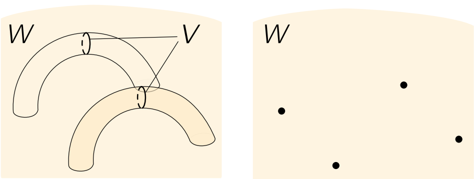
This lemma allows us to prove the following:

Theorem 5.10. Let $1 < k < n$. Then we can perform E1, E2, E3 geometrically on $[\partial_k]$, only affecting critical points of index k and $k+1$.

Proof. Operations E1 and E2 are clear. Assume the Morse function f is self-indexing. Let $M^t = f^{-1}(-\infty, t]$ and let V be the union of all belt spheres of critical points of index $k-1$,i.e.

$$V = \bigcup_{p \in \mathrm{Crit}_{k-1} f} S^s(p)$$

Suppose f is self-indexing and let $N = f^{-1}(k - \frac{1}{2})$. For dimensional reasons, V can only disconnect N if $k = 2$. However, $N \setminus V$ is diffeomorphic to $f^{-1}(k - \frac{3}{2})$ with a finite set of points removed, namely the attaching spheres of the one-handles, as illustrated in Figure 5.9. Therefore, as $f^{-1}(k - \frac{3}{2})$ is connected, so is $N \setminus V$. This means we can apply the previous lemma and we conclude that we can add/subtract one column to the other one. If $k < n-1$, we have control over the sign (determining addition or subtraction), and if $k = n-1$, we may need to change the orientation of the attaching sphere of k-handle first. Notice that this does only affect $[\partial_k]$ and $[\partial_{k+1}]$. $\square$



$$M^{3/2} \setminus V \quad \cong \quad M^{1/2} \setminus (S^u(q_1) \cup S^u(q_2))$$

Figure 5.9: In the case $k = 2$, the union of the belt spheres V does disconnect $M^{3/2}$. However, it is diffeomorphic to $M^{1/2}$ with a finite set of points removed, and since $M^{1/2}$ is connected, so is $M^{1/2} \setminus V$.

In order to also diagonalize $[\partial_1]$ and $[\partial_n]$, we will use the following:

Theorem 5.11. Let M be a connected cobordism from M_0 to M_1. If $M_0 = \emptyset$, there exists a handlebody decomposition with exactly one 0-handle. In the other case, there exists one without 0-handles. In both cases, $[\partial_1]$ is trivial.

Proof. Suppose $M_0 \neq \emptyset$, i.e. the cobordism has a bottom border and suppose $\# \mathrm{Crit}_0 f = 1$, i.e. there is a single 0-handle. Then because M is connected, there must exist an 1-handle connecting the 0-handle to another connected component of $f^{-1}(-\infty, \frac{1}{2}]$. The first cancellation theorem allows us to cancel the 0-and 1-handle. It is clear how to extend this to multiple 0-handles.

Next, suppose $M_0 = \emptyset$ and $\# \mathrm{Crit}_0 f = 2$. Then because M is connected, there must be a 1-handle connecting the two components in $f^{-1}(-\infty, \frac{1}{2}]$.

Again using the first cancellation theorem , we can cancel the 0- and 1-handle. Repeatedly applying this reasoning handles the case where $\# \operatorname{Crit}_0 f > 2$.

For the last part of the theorem, stating that $[\partial_1]$ is trivial, consider the following. If there are no 0-handles, $\partial_1 : C_1 \to C_0$ is clearly trivial. If there is a single 0-handle, all 1-handles have both ends attached to the same sphere, so the intersection number is 0, so $[\partial_1] = (0 \ \cdots \ 0)$. □

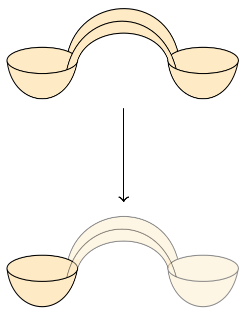


Figure 5.10: Assuming the manifold is connected, it cannot contain two zero handles without a one handle connecting them. We can then cancel the zero and one handle lowering the number of 0-handles by 1. Repeating this, we can find a handlebody decomposition with a minimal number of 0-handles, that is, zero 0-handles if $M_0 \neq \emptyset$, and one 0-handles if $M_0 = \emptyset$.

Remark 5.12. By turning the cobordism upside down, we can conclude the same for n-handles and $[\partial_n]$.

The two previous theorems allow us diagonalize all $[\partial_k]$ *simultaneously*.

Theorem 5.13. Let M be a cobordism from M_0 to M_1. Assume M, M_0, M_1 are connected and oriented. Then there exists a Morse function such that $[\partial_k]$ is diagonal for all k.

Proof by induction. Note that $[\partial_1]$ is diagonal by the previous theorem. Suppose $[\partial_i]$ is diagonal for $1 \leq i < k < n$. We have shown that we can do operations E1–E3 on columns geometrically. By turning the cobordism upside down, we can also do this for rows. These six operations are all that is needed for the diagonalizing $[\partial_k]$ by using the algorithm of Smith[58]. Diagonalizing $[\partial_k]$ does not change the already diagonalized matrices, because $[\partial_{k-1}][\partial_k] = 0$. Each column of $[\partial_k]$ with a non-zero element corresponds to a row of zeros in $[\partial_{k-1}]$. Lastly $[\partial_n]$ is already diagonal by the previous theorem. □

[58] Joseph J Rotman. *Advanced modern algebra*. Vol. 114. American Mathematical Soc., 2010

5.4 Trading 1-handles for 3-handles

The previous theorems allow us to diagonalize $[\partial_k]$ for all k. Moreover, the second cancellation theorem allows us to remove columns and rows of $[\partial_k]$ under certain conditions. These conditions do not cover the case $[\partial_2]$ and $[\partial_{n-1}]$. In this section, we will show that we can eliminate 1-handles by replacing them with 3-handles, solving these issues.

Theorem 5.14. Let f be a Morse function on M, a cobordism from M_0 to M_1. Assume that M, M_0, M_1 are connected and simply connected, and $\dim M \geq 5$. Then we can alter f such that 1-handles become 3-handles without changing the number of handles of index greater than three.

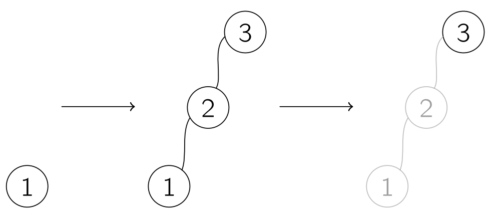


Figure 5.11: To change a 1-handle into a 3-handle, we first introduce a pair of cancelling auxiliary handles of index 2 and 3. Then we cancel the 1- and 2-handle, leaving us with a 3-handle.

Proof. The idea of the proof goes as follows and is illustrated in Figure 5.11. In order to change a 1-handle into a 3-handle, we introduce an auxiliary cancelling pair of 2- and 3-handles. Then we cancel the 1- and 2-handle, leaving us with a 3-handle.

Adding a cancelling pair of 2- and 3-handles Assume that f is self-indexing. By the first cancellation theorem, we can decompose an m-dimensional disk D^m as a 2- and 3-handle. (The attachment sphere of the 3-handle intersects the belt sphere of the 2-handle once transversely.) Therefore, we can remove a small disk from M and fill it up with a 2- and 3-handle. The attachment sphere L of the 2-handle bounds a 2-dimensional disk in $f^{-1}(\frac{3}{2})$ and we can make sure that it does not intersect belt spheres of 1-handles or attachment regions of other 2-handles.

Our next goal is to isotope this 2- and 3-handle such that the attachment region L of the 2-handle crosses a 1-handle exactly once. Then we will be able to cancel the 2-handle against the 1-handle.

Constructing the desired attachment region Let L' a path on top of a 1-handle that intersects the belt sphere once transversely. By Theorem 5.11, we can assume that there are a minimal number of 0-handles, implying that $f^{-1}(\frac{1}{2})$ is connected. Hence we can connect the endpoints of L' with a curve that lies in $f^{-1}(\frac{1}{2})$, moreover missing the other attachment spheres of 1- and 2-handles.[59] Because M is simply connected, $f^{-1}(\frac{3}{2})$ is as well, so this loop is null-homotopic.

[59] The attachment spheres of the 1-handles are a finite number of disjoint points, so it is easy to avoid them. For the second claim, we can assume that the loop is smooth and transversal to the attachment region of the 2-handles. Now note that transversal under these conditions means disjoint because of dimensional reasons.

[60] Hassler Whitney. "Differentiable manifolds". In: *Annals of Mathematics* (1936), pp. 645–680

Isotoping L to L' Both L and L' are null-homotopic in $f^{-1}(\frac{3}{2})$. Because $\dim f^{-1}(\frac{3}{2}) \geq 4$, they are in fact isotopic, by a theorem of Withney.[60] It states that if $f, g : M \to N$ be two *homotopic* embeddings of a compact manifold. If $\dim N \geq 2 \dim M + 2$, then f and g are *isotopic*. Both embeddings of L and L' are null homotopic, and $\dim f^{-1}(\frac{3}{2}) \geq 4 \geq 2 \dim S^1 + 2$, so the conditions are satisfied. Hence, we can isotope L to L', and assume that the 2- and 3-handle combination is actually attached along L', i.e. the attachment region of the 2-handle is L'.

Because L' crosses the 1-handle exactly one time, we can cancel the 1- and 2-handle as claimed before, ending the proof. $\square$

Remark 5.15. This idea can be extended to higher index critical points.

Remark 5.16. By reversing the cobordism ($f \rightsquigarrow -f$), we can do the same for $n-1$-handles.

5.5 Minimality of the Morse inequalities

Having discussed a multitude of cancellation theorems, we are ready to prove the minimality of the (weak) Morse inequalities over $\mathbb{Z}$, as first proven by Smale.[61]

[61] Stephen Smale et al. "The generalized Poincaré conjecture in higher dimensions". In: *Bulletin of the American Mathematical Society* 66.5 (1960), pp. 373–375

Theorem 5.17 (Smale)**.** Let M be a cobordism from M_0 to M_1. Assume M, M_0, M_1 are connected and simply connected and $\dim M \geq 6$. Assume moreover that the homology of M is free, i.e. $H_\bullet(M, M_0)$ and $H_\bullet(M, M_1)$ are free. Then there exists a Morse function such that

$$\# \operatorname{Crit}_k f = r_0(H_k(M, M_0; \mathbb{Z})).$$

In other words, under these conditions, the Morse inequalities are attainable.

Proof. Let f be an arbitrary Morse function on M. We show inductively that $[\partial_k]$ is trivial. This will then imply that $H_k = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}} = \frac{C_k}{0} = C_k$, hence $\# \operatorname{Crit}_k f = r_0 H_k$. Using Theorem 5.11, we alter f such that the number of 0-handles is minimized. Theorem 5.14 allows us to change 1-handles into 3-handles.

H_0 There are no -1-handles, so ∂_0 is trivial, hence $H_0 = C_0$.

H_1 There are no 1-handles, so ∂_1 is trivial, hence $H_1 = C_1$.

H_2 By definition, $H_2 = \frac{\operatorname{Ker} \partial_2}{\operatorname{Im} \partial_3}$. However, there are no 1-handles, so $\operatorname{Ker} \partial_2 = C_2$. By altering the Morse function and gradient, we can assume that $[\partial_3]$ is diagonal. This combined with the fact that the homology is free allows us to conclude that $\operatorname{Im} \partial_3$ is a matrix with as entries ± 1. (If it contained e.g. a 2, then the resulting quotient could be $\mathbb{Z}/2\mathbb{Z}$, which has torsion.) This allows us to cancel pairs of critical points, removing rows and columns until $[\partial_3]$ is trivial. We can do so without altering the triviality for $k = 1, 2$. We conclude that $H_2 = C_2$.

H_k Suppose $[\partial_k]$ is trivial. By definition $H_k = \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}} = \frac{C_k}{\operatorname{Im} \partial_{k+1}}$. Because H_k does not have torsion, this does mean that $[\partial_{k+1}]$ only contains ± 1's, so we can cancel critical points until $[\partial_{k+1}]$ is trivial. We can do this without changing the triviality of $[\partial_\ell]$ for $\ell < k$. We conclude that $H_k = C_k$.

While the proof now seems finished, we should be careful when k gets close to n, because then the second cancellation no longer applies. We can solve this by first doing the previous process for $k = 1, \ldots, n-2$. Then we turn the cobordism upside down ($f \rightsquigarrow -f$), and repeat the procedure, eventually making all matrices trivial resulting in $H_k(M, M_0) \cong C_k$. $\square$

Remark 5.18. If we do not assume that the homology is free, the weak Morse inequalities are not attainable. However, we can prove that the Morse inequalities including torsion rank in fact are.[62]

[62] Vladimir Vasilevich Sharko. *Functions on Manifolds: Algebraic and Topological Aspects: Algebraic and Topological Aspects*. 131. American Mathematical Soc., 1993

If we take $M_0 = M_1 = \emptyset$, we immediately have the following corollary,

Corollary 5.19. If M is a simply connected closed manifold of dimension $n \geq 6$ with free homology, then the Morse inequalities are attainable, i.e. there exists a Morse function such that

$$\# \operatorname{Crit}_k f = r_0(H_k(M; \mathbb{Z})).$$

If we assume that the homology vanishes completely, Theorem 5.17 becomes

Corollary 5.20 (h-cobordism theorem)**.** Let M be a cobordism from M_0 to M_1. If M, M_0, M_1 are connected and simply connected, $\dim M \geq 6$ and $H_\bullet(M, M_0) = 0$, then M is a trivial cobordism, i.e. M is diffeomorphic to $M_0 \times [0, 1]$.

Another direct corollary is the generalized Poincaré conjecture for dimension greater than 5.

Corollary 5.21 (Generalized Poincaré conjecture)**.** If M is a homotopy sphere of dimension $n > 5$, then M is homeomorphic to S^n.

Proof. A homotopy sphere is a homology sphere. Hence, by the previous theorem, there exists a Morse function with exactly one 0-handle and one n-handle, i.e. M consists of two disks glued along their boundary. This is homeomorphic to a sphere, as the following explicit homeomorphism shows:

$$\begin{aligned} h : S^n = D_1^n \cup_{\mathrm{Id}} D_2^n &\longrightarrow D_1^n \cup_\phi D_2^n \\ x &\longmapsto \begin{cases} x & \text{if } x \in D_1^n \\ \|x\|\phi\left(\frac{x}{\|x\|}\right) & \text{if } x \in D_2^n \setminus \{0\} \\ 0 & \text{if } x = 0 \in D_2^n. \end{cases} \end{aligned}$$

□

Remark 5.22. We cannot conclude that M is *diffeomorphic* to S^n. Indeed, there are so-called exotic spheres, which are topological spheres with a differential structure that is not equivalent to the standard differential structure on S^n. The first instances of an exotic spheres were constructed by Milnor in 1956.[63]

[63] John Milnor. "On manifolds homeomorphic to the 7-sphere". In: *Annals of Mathematics* (1956), pp. 399–405

The next result shows that—contrary to spheres—there is only a unique differential structure on disks:

Theorem 5.23. If M is contractible with a simply connected boundary and of dimension $n > 5$, then M is diffeomorphic to B^n

Proof. Consider M as a cobordism from $\emptyset$ to ∂M. We have that $H_\bullet(M) \cong H_\bullet(M, \emptyset)$. By Poincaré duality, this is also isomorphic to $H^\bullet(M, \partial M)$. Because $H^\bullet(M, \partial M)$ is finitely generated, it is isomorphic to $H_\bullet(M, \partial M)$.[64] In conclusion, $H_\bullet(M) \cong H_\bullet(M, \emptyset) \cong H_\bullet(M, \partial M)$. This means that $H_\bullet(M, \emptyset)$ and $H_\bullet(M, \partial M)$ are free. Applying Theorem 5.17 gives, together with the fact that M is contractible, a handlebody decomposition with exactly one 0-handle. Hence M is diffeomorphic to a disk. □

[64] Edwin H Spanier. *Algebraic topology*. Springer Science & Business Media, 1989, Chapter 5, Section 5, Corollary 4

This allows us to strengthen the generalized Poincaré theorem:

Theorem 5.24 (Generalized Poincaré conjecture for $n > 4$)**.** Let M be a homotopy sphere of dimension > 4. Then M is homeomorphic to a sphere S^n.

Proof. We claim that $M\#(-M)$ bounds a contractible manifold W of dimension greater than or equal to 6. Indeed, consider the following figure.

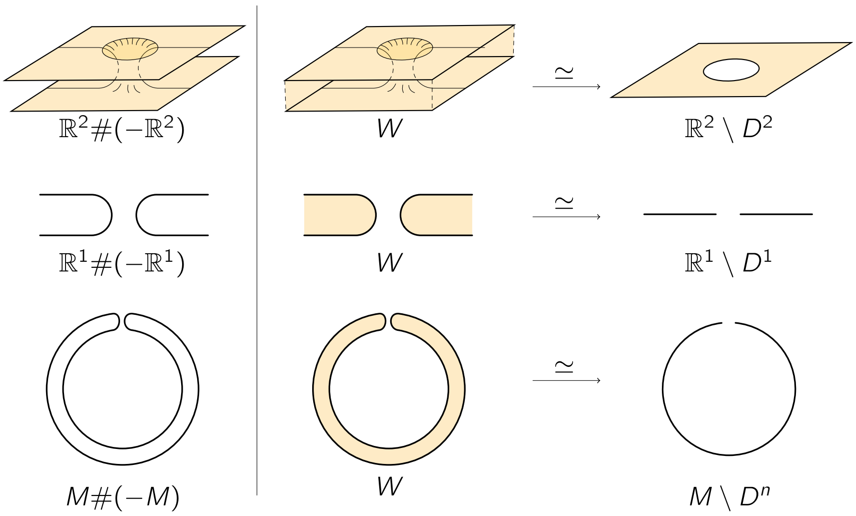


Figure 5.12: The connected sum of a manifold with itself bounds a manifold W which deformation retracts on $M \setminus D^n$.

The two top rows show a local model of a connected sum, considering the connected sum of two copies of $\mathbb{R}^n$ with reversed orientation on one of the copies. The resulting manifold bounds a manifold W and scaling in the vertical direction gives a deformation retract from W to $\mathbb{R}^n \setminus D^n$. On the bottom, we use this local model for $M\#(-M)$. This shows that $M\#(-M)$ bounds a manifold that is homotopy equivalent with $M \setminus D^n$. Because M is a homotopy sphere, we can easily use Mayer–Vietoris to compute that the homology of $M \setminus D^n$ is that of a contractible manifold.

By the previous theorem, this implies that W is diffeomorphic to D^{n+1}. Hence, $M\#(-M)$ is homeomorphic to S^n. Now, S^n is irreducible, meaning that if $M\#N$ is homeomorphic to S^n then both M and N are homeomorphic to S^n.[65] This proves that M is homeomorphic to S^n. $\square$

[65] Barry Mazur. "On embeddings of spheres". In: *Bulletin of the American Mathematical Society* 65.2 (1959), pp. 59–65

5.6 State of art of the Poincaré conjecture and *h*-cobordism theorem

Let us end this thesis by giving an overview of the history of the (generalized) Poincaré conjecture and *h*-cobordism theorem. We will consider the developments of these conjectures/theorems for three categories, namely

- $\mathrm{Man}_{\mathrm{top}}$, the category of topological manifolds,
- Man_{∞}, the category of smooth manifolds,
- $\mathrm{Man}_{\mathrm{PL}}$, the category of piecewise linear manifolds.[66]

For brevity, we will use P^n_C (resp. H^n_C) for the Poincaré conjecture (resp. *h*-cobordism theorem) of dimension n in category C.

[66] Piecewise linear manifolds are manifolds whose transition maps are piecewise linear. By this we mean a continuous map ϕ such that its domain can be split up in polytopes such that ϕ restricted to a single polytope is affine. A manifold having a PL structure is slightly stronger than admitting a triangulation. For an introduction on piecewise linear manifolds, see Colin P Rourke and Brian Joseph Sanderson. *Introduction to piecewise-linear topology*. Springer Science & Business Media, 2012

Dimensions 1, 2 In low dimensions, the categories Man_{∞}, $\mathrm{Man}_{\mathrm{top}}$, $\mathrm{Man}_{\mathrm{PL}}$ are equivalent, meaning for example that a topological 2-manifold has a unique differential structure.[67] The classification of one- and two-dimensional manifolds then immediately gives the Poincaré conjecture and *h*-cobordism theorem in all categories.

[67] Edwin E Moise. *Geometric topology in dimensions 2 and 3*. Vol. 47. Springer Science & Business Media, 2013

$\mathrm{P}^{1,2}_{\mathrm{top}}$	$\mathrm{P}^{1,2}_{\mathrm{PL}}$	$\mathrm{P}^{1,2}_{\infty}$	$\mathrm{H}^{1,2}_{\mathrm{top}}$	$\mathrm{H}^{1,2}_{\mathrm{PL}}$	$\mathrm{H}^{1,2}_{\infty}$
True	True	True	True	True	True

Dimension 3 The original statement of the Poincaré conjecture concerned manifolds of dimension three. More precisely, Henri Poincaré conjectured the following in 1904:

Theorem 5.25 (Poincaré conjecture)**.** Every simply connected, closed 3-manifold M is homeomorphic to the 3-sphere.

One can prove that these conditions imply that M is a three-dimensional homotopy sphere. This rephrasing allows for an easy generalization to higher dimensions and other categories: 'a homotopy sphere is homeomorphic/diffeomorphic/PL homeomorphic to a sphere'.

The Poincaré conjecture in three dimensions was one of the most important open problems in topology, until it was proved to be true by Perelman in 2006 in $\mathrm{Man}_{\mathrm{top}}$.[68] For his work, Perelman was offered a Fields Medal and the Millennium prize worth \$1 million, but he declined both.

[68] John Morgan, Gang Tian, and Ricci Flow. "The poincaré conjecture". In: *Clay Mathematics Monographs* 3 (2007)

His proof uses the concept of Ricci flow, introduced by Hamilton in 1982.[69] The idea is to put an arbitrary metric on a homotopy sphere. Then the Ricci flow equations tend to 'smoothen out' this metric, as illustrated in Figure 5.13. If the metric can get improved enough such that it has constant positive curvature, then the manifold is diffeomorphic to a sphere. However, problems can arise in the form of certain singularities. Perelman studied these singularities and found a way to deal with them with manifold surgery, proving the Poincaré conjecture in $\mathrm{Man}_{\mathrm{top}}$. The categories $\mathrm{Man}_{\mathrm{top}}$, Man_{∞} and $\mathrm{Man}_{\mathrm{PL}}$ are equivalent in dimension three, hence his proof implies P^3_{∞} and $\mathrm{P}^3_{\mathrm{PL}}$.[70] Moreover, it has been shown that $\mathrm{H}^3_C \Leftrightarrow \mathrm{P}^3_C$, giving the following summary:

[69] Richard S Hamilton et al. "Three-manifolds with positive Ricci curvature". In: *J. Differential geom* 17.2 (1982), pp. 255–306

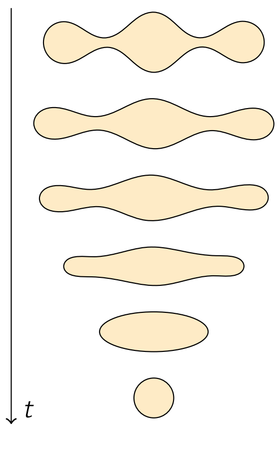


Figure 5.13: An illustration of the the Ricci flow equations.

[70] Edwin E Moise. *Geometric topology in dimensions 2 and 3*. Vol. 47. Springer Science & Business Media, 2013

$\mathrm{P}^{3}_{\mathrm{top}}$	$\mathrm{P}^{3}_{\mathrm{PL}}$	P^{3}_{∞}	$\mathrm{H}^{3}_{\mathrm{top}}$	$\mathrm{H}^{3}_{\mathrm{PL}}$	H^{3}_{∞}
True	True	True	True	True	True

Dimension 4 The h-cobordism theorem and Poincaré conjecture in dimension four are partly still an open problem. The Poincaré conjecture was proven by Freedman in 1982 in the category $\mathrm{Man}_{\mathrm{top}}$, for which he received a Fields Medal.[71] It was also proven that $\mathrm{P}^4_C \Leftrightarrow \mathrm{H}^4_C$ for all categories C, hence the four-dimensional h-cobordism theorem is also true. As of today, P^4_∞ and $\mathrm{P}^4_{\mathrm{PL}}$ (and hence H^4_∞ and $\mathrm{H}^4_{\mathrm{PL}}$) are still open and are tightly coupled to the unknown existence of exotic 4-spheres.

$\mathrm{P}^4_{\mathrm{top}}$	$\mathrm{P}^4_{\mathrm{PL}}$	P^4_∞	$\mathrm{H}^4_{\mathrm{top}}$	$\mathrm{H}^4_{\mathrm{PL}}$	H^4_∞
True	Open	Open	True	Open	Open

[71] Michael H Freedman and Frank Quinn. *Topology of 4-Manifolds (PMS-39), Volume 39.* Vol. 1085. Princeton University Press, 2014

Dimension 5 In dimension 5, the Poincaré conjecture holds in $\mathrm{Man}_{\mathrm{top}}$, as we have proven in this thesis. Moreover, it can be proven that a topological 5-sphere has a unique smooth structure, implying P^5_∞.[72] A theorem of Whitehead[73] states that a smooth manifold has a canonical PL structure, hence $\mathrm{P}^5_{\mathrm{PL}}$ follows as well. On the other hand, the h-cobordism theorem only holds in $\mathrm{Man}_{\mathrm{top}}$ (see Freedman[74]) and not in Man_∞ or $\mathrm{Man}_{\mathrm{PL}}$ (see Donaldson[75]).

$\mathrm{P}^5_{\mathrm{top}}$	$\mathrm{P}^5_{\mathrm{PL}}$	P^5_∞	$\mathrm{H}^5_{\mathrm{top}}$	$\mathrm{H}^5_{\mathrm{PL}}$	H^5_∞
True	True	True	True	False	False

[72] Guozhen Wang and Zhouli Xu. "The triviality of the 61-stem in the stable homotopy groups of spheres". In: *Annals of Mathematics* (2017), pp. 501–580

[73] John Henry C Whitehead. "On C1-complexes". In: *Annals of Mathematics* (1940), pp. 809–824

[74] Michael H Freedman and Frank Quinn. *Topology of 4-Manifolds (PMS-39), Volume 39.* Vol. 1085. Princeton University Press, 2014

[75] Simon K Donaldson and P Kronheimer. "The geometry of 4-manifolds". In: *Proceedings of the International Congress of Mathematicians (Berkeley 1986)(AM Gleason, ed.)* Vol. 1. Citeseer. 1986, pp. 43–54

Dimension 6 As we have shown in this thesis, the h-cobordism theorem in dimension six or greater is true in Man_∞ and this was first proven by Smale in 1960. It is also true in $\mathrm{Man}_{\mathrm{top}}$, proven in the book by Kirby and Siebenmann.[76] and also holds in $\mathrm{Man}_{\mathrm{PL}}$, as discussed in 'Introduction to piecewise-linear topology' by Rourke and Sanderson.[77]

The Poincaré conjecture in $\mathrm{Man}_{\mathrm{top}}$ is true in dimension 5 and higher as we have shown. It has been shown that the h-cobordism theorem $\mathrm{H}^{\geq 5}_{\mathrm{PL}}$ implies the Poincaré conjecture $\mathrm{P}^{\geq 5}_{\mathrm{PL}}$, hence $\mathrm{P}^{\geq 5}_{\mathrm{PL}}$ is true as well. The smooth Poincaré conjecture $\mathrm{P}^{\geq 5}_\infty$ is in general not true, because of the existence of exotic spheres. In particular, it is conjectured that spheres of a sufficiently high dimension always admit exotic structures, hence $\mathrm{P}^{\geq 5}_\infty$ would be usually false.[78]

$\mathrm{P}^{6+}_{\mathrm{top}}$	$\mathrm{P}^{6+}_{\mathrm{PL}}$	P^{6+}_∞	$\mathrm{H}^{6+}_{\mathrm{top}}$	$\mathrm{H}^{6+}_{\mathrm{PL}}$	H^{6+}_∞
True	True	Usually false	True	True	True

[76] Robion C Kirby and Laurence C Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations. (AM-88), Volume 88.* Princeton University Press, 2016

[77] Colin P Rourke and Brian Joseph Sanderson. *Introduction to piecewise-linear topology.* Springer Science & Business Media, 2012

[78] Guozhen Wang and Zhouli Xu. "The triviality of the 61-stem in the stable homotopy groups of spheres". In: *Annals of Mathematics* (2017), pp. 501–580

Summary Summarizing the current state of affairs, we find the following:

n	$\mathrm{P}_{\mathrm{top}}$	P_{PL}	P_∞	$\mathrm{H}_{\mathrm{top}}$	H_{PL}	H_∞
1,2,3	True	True	True	True	True	True
4	True	Open	Open	True	Open	Open
5	True	True	True	True	False	False
6+	True	True	Usually false	True	True	True

Conclusion

We have started this thesis with an introduction to Morse functions. While simple, they provide great insight in the structure of differential manifolds, as we have seen. Morse functions give rise to handlebody decompositions, the Morse complex and eventually Morse homology. We have shown that Morse homology does not depend on the Morse function nor on the pseudo-gradient vector field and that it is in fact isomorphic to singular homology. Morse homology gives rise to the Morse inequalities, providing a lower bound for the number of critical points of a Morse function.

In the last part of the thesis, we have witnessed the power of ideas of Morse. We have proven multiple cancellation theorems eventually leading to the proof of the minimality of the Morse inequalities, which was originally due to Smale. This in turn has the *h*-cobordism and the generalized higher dimensional Poincaré conjecture as an immediate corollary, forming the pinnacle of this thesis.

As of today, Morse theory remains an important subject in differential geometry. For example, handlebody decompositions and Heegaard splittings are used extensively for studying 3- and 4-manifolds. Moreover, the ideas of Morse homology have been extended to infinite dimensions by Andreas Floer, resulting in proofs of various versions of the Arnold conjecture.

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