RAFFLES INSTITUTION
2024 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 24

MATHEMATICS **9758/01**

Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper. **You may use the blank pages on page 22, 23 and 24 if necessary and you are reminded to indicate the question number(s) clearly.**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For Examiner's Use Only					
Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**
/ 4	**/ 5**	**/ 8**	**/ 8**	**/ 8**	**/ 7**
Q7	**Q8**	**Q9**	**Q10**	**Q11**	**TOTAL**
/ 10	**/ 12**	**/ 12**	**/ 14**	**/ 12**	**/ 100**

This document consists of **21** printed pages and **3** blank pages.

1 A function f is defined by $\mathrm{f}(x)=ax^3+bx^2+cx+d$. The graph of $y=\mathrm{f}(x)$ passes through the points $(-3,4)$ and $(1,8)$. Given that the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$, find the values of a, b, c and d. [4]

2 [The volume of a sphere with radius r is given by $\frac{4}{3}\pi r^3$ and the surface area of a sphere with radius r is given by $4\pi r^2$.]

(a) The volume of an expanding sphere is increasing at a constant rate of $5 \text{ cm}^3\text{s}^{-1}$. Show that, at any instant, the rate of increase of the surface area is $\frac{k}{r} \text{ cm}^2\text{s}^{-1}$, where r is the radius of the sphere and k is a constant to be determined. [3]

(b) Find the exact rate of change of surface area of the expanding sphere when the surface area is 20 cm^2. [2]

3 **(a)** Without using a calculator, solve exactly $\dfrac{x^2-x-1}{x+1} \leq 1$. [4]

(b) Hence solve exactly $\dfrac{x^2+|x|-1}{1-|x|} \leq 1$. [4]

4 **(a)** Find $\displaystyle\int \frac{\cos x}{\cos 3x + \cos x}\,\mathrm{d}x$. [3]

(b) Find $\int x \tan^{-1} x^2 \,\mathrm{d}x$. Hence find the exact value of $\int_{-1}^{1} \left| x \tan^{-1} x^2 \right| \mathrm{d}x$. [5]

5 **(a)** Using the formulae for $\sin(A \pm B)$, prove that

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\cos 2r\theta \sin\theta.$$ [1]

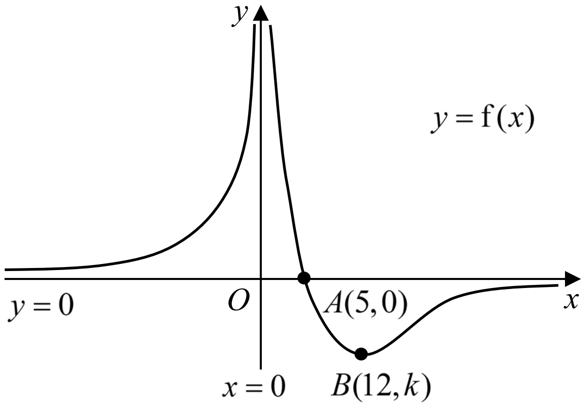
(b) Hence find a formula for $\sum_{r=1}^{n} \cos 2r\theta$, where $0 < \theta < \pi$, in terms of $\sin(2n+1)\theta$ and $\sin\theta$. [3]

(c) Using the formula found in part **(b)**, show that the sum of the series

$$\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + \ldots + \sin^2 20\theta, \text{ for } 0 < \theta < \pi$$

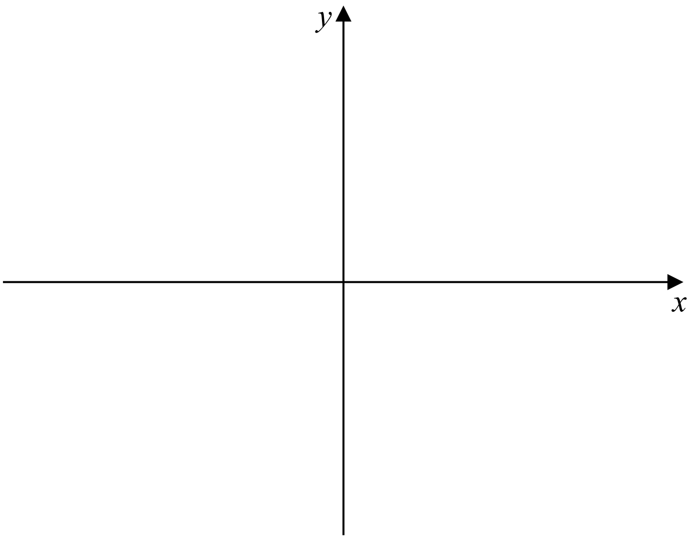
is $k - \dfrac{\sin(41\theta) - \sin(19\theta)}{4\sin\theta}$, where k is a constant to be determined. [4]

6 **(a)** The diagram below shows the graph of $y = \mathrm{f}(x)$.

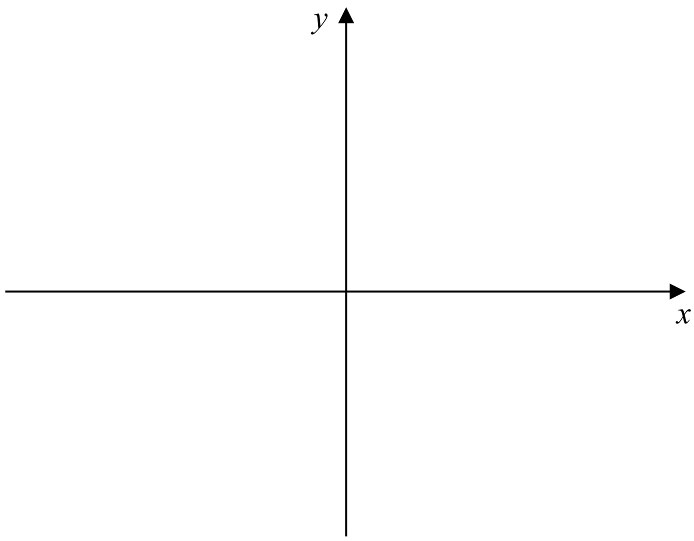


The graph cuts the x-axis at point $A(5,0)$. It has a turning point at $B(12,k)$, where $k < 0$ and asymptotes with equations $x = 0$ and $y = 0$. On separate diagrams, sketch the graph of

(i) $y = 2\mathrm{f}(x) + k$, stating the equations of any asymptotes and the coordinates of any turning point(s). [2]



(ii) $y = f'(x)$, stating the equations of any asymptotes and the coordinates of any point(s) where the curve crosses the axes. [2]



(b) The graph with equation $y = g(x)$, where $g(x) = x(x-1)^2$ undergoes a single transformation and the equation of the resultant graph is $y = h(x)$. Describe the transformation if

(i) $h(x) = -x(x+1)^2$, [1]

(ii) $h(x) = \frac{1}{8}x(x-2)^2$. [2]

7 An arithmetic series has first term a and common difference d, where $a > 0$ and $d \neq 0$. The first, sixth and ninth terms of the arithmetic series are consecutive terms of a geometric series.

(a) Show that $25d = -2a$. [2]

(b) The sum of the first n terms of the arithmetic series is denoted by S. Find the set of possible values of n for which S exceeds $6a$. [3]

(c) Find the common ratio of the geometric series, and deduce that the geometric series is convergent. [2]

(d) Hence find the smallest value of m such that the sum of the terms of the geometric series after, but not including, the mth term, is less than 1% of the sum to infinity. [3]

8 **Do not use a calculator in answering this question.**

(a) The complex number w is such that $w = a + \mathrm{i}b$, where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that

$$ww^* = 4 - 2\mathrm{i} + 2\mathrm{i}w^*,$$

find the two possible values of w. [4]

(b) The complex number z is given by $z = \dfrac{1 - \sqrt{3}\,\mathrm{i}}{-1 + \mathrm{i}}$.

(i) Find $\arg(z)$. [3]

(ii) Find z in cartesian form $x+\mathrm{i}y$. [2]

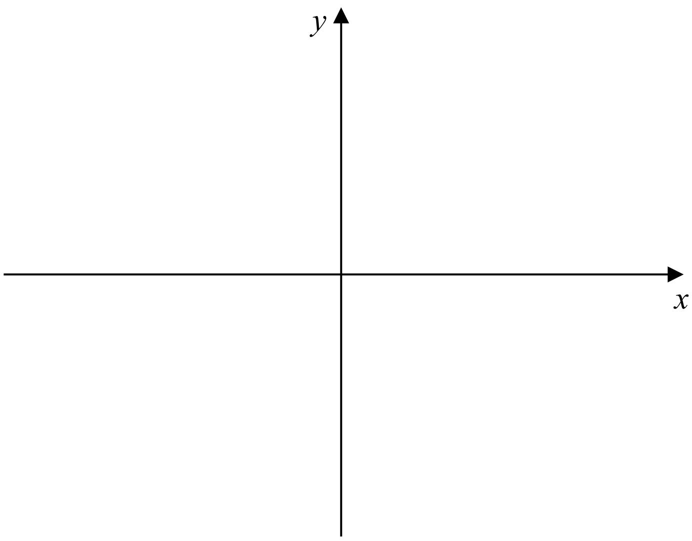
(iii) Hence find the value of $\tan\dfrac{\pi}{12}$ in the form $c+d\sqrt{3}$, where c and d are integers to be found. [3]

9 A curve C has equation $y = 2ax + \dfrac{b}{x}$ where a and b are non-zero real constants and $x \neq 0$.

(a) Using differentiation, determine whether C has any stationary points if $ab < 0$. [2]

It is now given that $b = \dfrac{1}{2}a$ where $a > 0$.

(b) Sketch C, stating the equations of any asymptotes, and the coordinates of any stationary points and points where C crosses the axes (if any). [3]



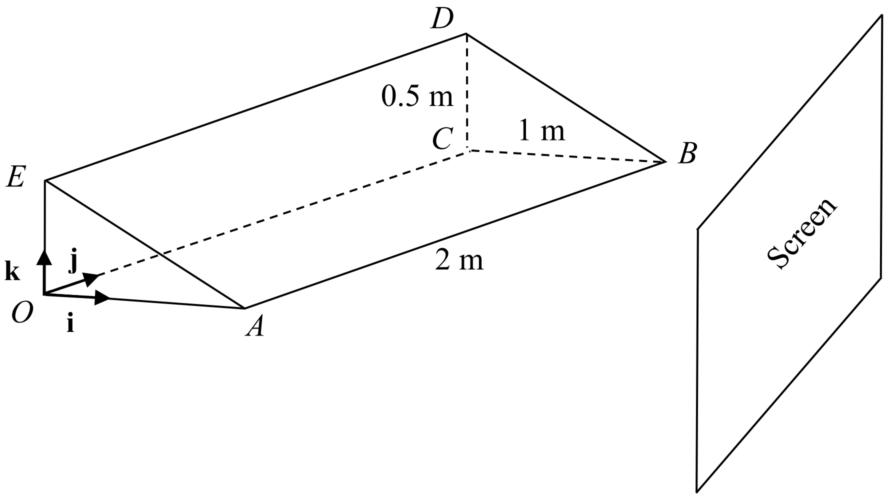
(c) State the range of values of k, in terms of a, for which the equation $2ax + \frac{b}{x} = kx$ has no real roots. [1]

(d) The region bounded by C, the axes, the lines $x = \frac{1}{2}$ and $y = 4a$ is rotated about the y-axis through 2π radians. Show that the volume generated is given by

$$\frac{1}{2}a\pi + \frac{\pi}{8a^2}\int_{2a}^{4a} y^2 - 2a^2 - y\sqrt{y^2 - 4a^2}\ \mathrm{d}y .$$

Hence, find, in terms of a and π, the exact volume generated. [6]

10 For an upcoming motor-car race, a spectators' gallery is to be set up near the racing track. As part of the preparations, a model of this gallery, shaped as a prism, is constructed.



The diagram above shows the model of the gallery with O as the origin and the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OC and OE respectively. Points (x, y, z) are defined relative to O, where units are in metres.

It is given that $OA = CB = 1$ m, $OC = AB = ED = 2$ m and $OE = CD = 0.5$ m.

(a) Find a cartesian equation of the plane $ABDE$. [2]

A shelter is to be constructed above the plane $ABDE$. On the model, this shelter is a rectangular plane that intersects plane $ABDE$ in the line ED.

(b) Given that the equation of the shelter is $-0.5x + z = h$, show that $h = 0.5$. [1]

(c) Find the acute angle between the plane $ABDE$ and the shelter. [2]

A spotlight is shone towards the gallery and the beam of light is in the form of a line l with cartesian equation $x - a = z,\ y = 1$ for some real number a. The beam lands on a point M on the rectangular surface $ABDE$.

(d) Find in terms of a, the position vector of M, and hence find the range of values of a. [5]

(e) A large rectangular flat screen is to be placed in front of the gallery at a distance away. The screen can be taken to be part of a vertical plane with equation $12x+5y=d$, where $d>50$. Using $a=\dfrac{1}{2}$, find the value of d so that the shortest distance between point M and the screen is 4 m. [4]

11 A wafer fabrication company uses the floating-zone method to purify polysilicon ingots, each having a uniform cross-sectional area and a length of 200 cm. The method involves placing a polysilicon ingot with impurity concentration C_0 atoms/cm^3 on top of a single seed crystal. The polysilicon ingot is then heated externally by an RF coil, which locally melts the ingot. The impurities prefer to stay in the molten state than in the solid state and thus as the coil and the melt zone move upwards, a single crystal, that is purer, solidifies on top of the seed crystal. A schematic illustration of the method is shown below in Fig. 1 and Fig. 2.

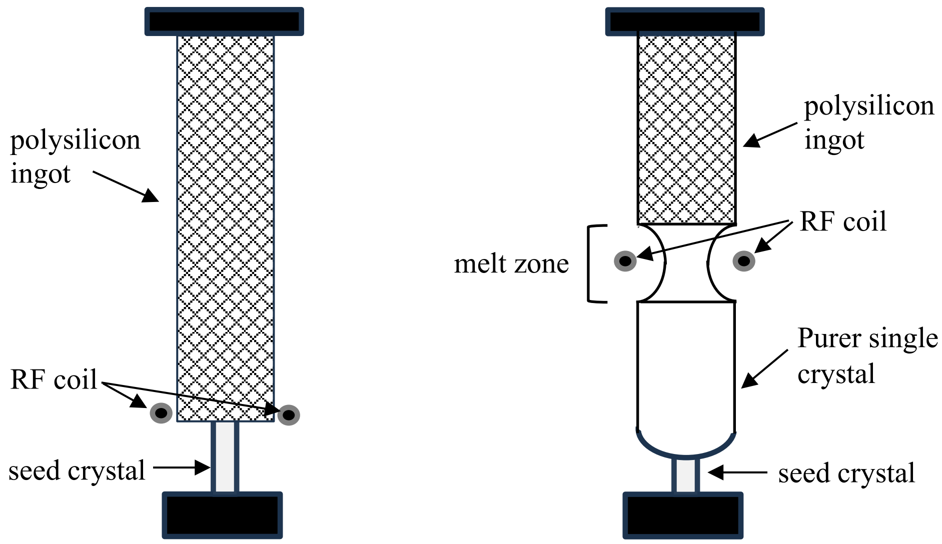


Fig. 1 Initial set-up   Fig. 2 During purification process

For a ‘floating’ melt zone of length L cm, the concentration of impurities in the melt zone, C atoms/cm^3, and the distance moved by the RF coil, x cm, are related by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}x} = \frac{1}{L}\left(C_0 - kC\right),$$

where k is a constant such that $0 < k < 1$.

The length of the “floating” melt zone, L cm, adopted by the company is 2 cm and $0 \le x \le 198$. It is also given that when $x = 0$, $C = C_0$.

(a) Solve the differential equation to find an expression for C in terms of C_0, k and x. [4]

(b) Sketch the graph of C against x. [2]

(c) Assume that $k = 0.3$ and that the RF coil moves upwards at a constant speed of 8 mm per hour. Find the time taken for the concentration of impurities in the melt zone to reach $2C_0$ and the rate of change of the concentration of impurities, in terms of C_0 at this instant. [5]

The company decides to change the length of the "floating" melt zone.

(d) Explain, with a reason, whether a shorter length is preferable over a longer one. [1]

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RAFFLES INSTITUTION
2024 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 24

MATHEMATICS — 9758/02
Paper 2 — **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

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<table>
<tr><th rowspan="2">Section A: Pure Math</th><th colspan="6">For Examiner's Use Only</th><th rowspan="2">TOTAL</th></tr>
<tr><th>Q1</th><th>Q2</th><th>Q3</th><th>Q4</th><th colspan="2">Q5</th></tr>
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<tr><th rowspan="2">Section B: Prob & Stats</th><th>Q6</th><th>Q7</th><th>Q8</th><th>Q9</th><th>Q10</th><th>Q11</th></tr>
<tr><td>/ 7</td><td>/ 8</td><td>/ 11</td><td>/ 10</td><td>/ 12</td><td>/ 12</td></tr>
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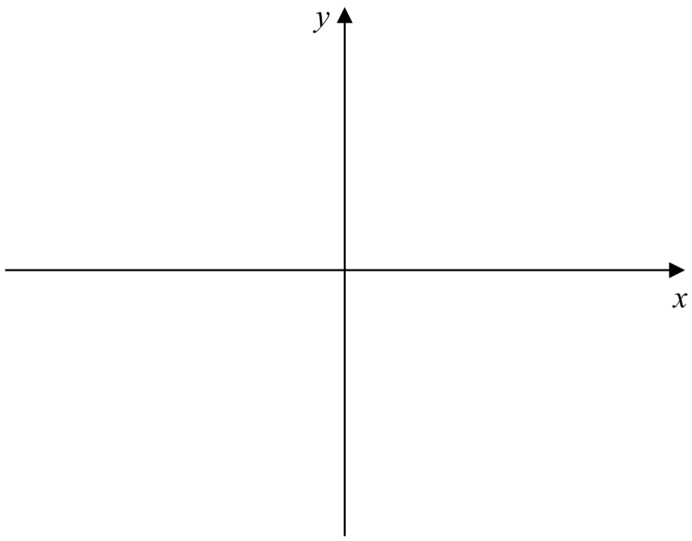
Section A: Pure Mathematics [40 marks]

1 The function f is defined by $\mathrm{f}: x \mapsto \dfrac{2x}{x-2}$, for $x \in \mathbb{R}$, $x \neq 2$.

(a) Sketch the graph of f and find its range. [3]



Another function g is defined by $\mathrm{g}: x \mapsto 3+|x+2|$, for $x \in \mathbb{R}$.

(b) Show that the composite function fg exists. Find $\mathrm{fg}(x)$ and state the domain and range of fg. [5]

2 The function f is defined by $\mathrm{f}(z) = z^4 + Az^3 + Bz^2 + Cz + 45$, where A, B and C are real numbers. Given that $2 + \mathrm{i}$ is a root of $\mathrm{f}(z) = 0$ and $(z - k)^2$ is a factor of $\mathrm{f}(z)$, where k is a positive real number, find the values of A, B, C and k. [5]

3 **(a)** The points A, B and C on the plane π have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Show that a vector perpendicular to π is parallel to

$$\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}.$$ [3]

(b) $\mathbf{p}$ and $\mathbf{q}$ are non-zero vectors such that $\mathbf{p}=(\mathbf{p}.\mathbf{q})\mathbf{q}$.

(i) Find the relationship between $\mathbf{p}$ and $\mathbf{q}$. [1]

(ii) Find $|\mathbf{q}|$. [1]

(c) $\mathbf{u}$ is the position vector of a fixed point U relative to the origin O. A variable point V has position vector $\mathbf{v}$ relative to O.

Given that $\mathbf{v}.(\mathbf{v}-\mathbf{u})=0$, describe geometrically the set of all possible positions of the point V. [2]

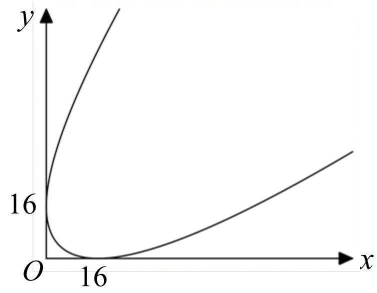
4 **(a)** Given that $y = \mathrm{e}^{\sqrt{1+2x}}$, show that

$$(1+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2 \quad \text{and} \quad (1+2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = y .$$ [3]

(b) Hence, or otherwise, obtain the series expansion for y in terms of x up to and including the term in x^3. [3]

(c) Verify that the same series expansion for y in part **(b)** is obtained if the standard series expansions for e^x and $(1+x)^n$ are used. [4]

5



The diagram shows the curve C with parametric equations

$$x = (1+t)^2, \qquad y = (3-t)^2.$$

The curve C meets the axes at $(16, 0)$ and $(0, 16)$.

(a) Show that the line $x = 16$ meets C at the point P where $t = -5$. [1]

The normal to C at P is denoted by l.

(b) Find the cartesian equation of l. [3]

(c) The line l meets C again at the point Q where $x = b$. Show that the area of the region bounded by l, the lines $x = 16$, $x = b$ and the x-axis is $\dfrac{266240}{81}$ units^2. [3]

(d) Show that the area (in units^2) of the region bounded by C and l can be given by

$$\frac{266240}{81} + \int_c^d \mathrm{f}(t)\, \mathrm{d}t\,,$$

where $\mathrm{f}(t)$, and the constants c and d are to be determined.
Hence find the value of this area. [3]

Section B: Probability and Statistics [60 marks]

6 Eleven cards each bears a single letter and together they can be made to spell the word COFFEEHOUSE. The 11 cards are arranged in a row.

(a) Find the number of different arrangements that can be made. [1]

(b) Find the number of different arrangements in which the 2 F's are next to each other and no E's are next to each other. [3]

Three cards are selected from the eleven cards and the order of selection is not relevant. Find the number of possible selections that can be made

(c) if the three cards all bear different letters, [1]

(d) if exactly two of the three cards bear the same letter. [2]

7 The probability of obtaining a head when a particular coin is tossed is p. A fair cubical die has the number '1' on one face, number '2' on two faces and number '3' on three faces.

The coin and die are thrown simultaneously. The random variable X is defined as follows.

If the coin shows a head, then X is thrice the score on the die.
If the coin shows a tail, then X is the score on the die.

(a) Show that $\mathrm{P}(X=3)=\frac{1}{2}-\frac{1}{3}p$, and find the probability distribution of X. [3]

(b) Given that $\mathrm{E}(X) = 5$, find the exact value of p. [2]

(c) Using the value of p found in part **(b)**, find the exact value of $\mathrm{Var}(X)$. [3]

8 **(a)** It is given that X is the number of times a student is late for school in a year and Y is the student's performance in the Mathematics Examination. The product moment correlation coefficient of a bivariate sample (x_i, y_i), $i = 1, 2, \ldots, n$, for n students is r.

State, giving a reason, whether each of the following statements is true or false.

(i) When the value of r is zero, it can be implied that the variables X and Y are not related. [1]

(ii) When the value of r is -1, it can be implied that late-coming causes poor performance in the subject. [1]

(b) The table below shows the daily sales of cups of iced coffee in a week by a shop and the maximum daily temperature.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Temperature ($°C$), t	30.5	31.3	31.9	34.8	25.9	28.5	29.2
Daily sales, y	95	101	115	79	81	86	88

(i) Sketch a scatter diagram of y against t, labelling the axes clearly. [2]

One of the values of y appears to be incorrect.

(ii) Indicate the corresponding point on your diagram by labelling it P. Omitting P, find the equation of the least squares regression line of $\frac{1}{y}$ on t, and the value of the product moment correlation coefficient between $\frac{1}{y}$ and t. Comment on this value. [5]

(iii) Use an appropriate regression line to give an estimate of the daily sales when the temperature is 20.4 $°C$.
State, with a reason, whether the estimate is reliable. [2]

9 **(a)** The masses of a randomly chosen bolt and a randomly chosen nut are denoted by M grams and W grams respectively. M and W are independent random variables with the distributions $\mathrm{N}(272, 8^2)$ and $\mathrm{N}(98, 5^2)$ respectively.

(i) Find the range of values of a for which $P(a < M < 275) > 0.2024$. [3]

(ii) Calculate the probability that twice the mass of a randomly chosen bolt differs from the total mass of 5 randomly chosen nuts by less than 80 grams. [3]

(b) Bolts are manufactured to fit into holes in steel plates. The bolts have diameters, in cm, that follow the distribution $\mathrm{N}(2.65, 0.03^2)$ and the diameters of the holes, in cm, follow the distribution $\mathrm{N}(2.72, 0.02^2)$. A manufacturer sells boxes of twenty pairs of these bolts and steel plates, where each pair consists of one randomly selected bolt and one randomly selected steel plate. A pair is acceptable only if the diameter of the hole in the steel plate is at least 0.02 cm larger, but no more than 0.15 cm larger, than the diameter of the bolt. Use a normal distribution to estimate the probability that the average number of acceptable pairs in 50 boxes is more than 18. [4]

10 **(a)** A company has a machine designed to fill bags with, on average, μ_0 kg of salt. The mass of salt in a randomly chosen bag has a normal distribution with population standard deviation denoted by σ kg. The production manager wishes to investigate if the machine is adjusted correctly. He takes a random sample of n bags and carries out a hypothesis test at the 1% level of significance.

(i) State null and alternative hypotheses for the manager's test, defining any parameters you use. [2]

(ii) Find, in terms of μ_0, σ and n, the critical region(s) for this test. [4]

(b) The company has a different machine which fills bags with, on average, 25 kg of low sodium salt. One of the company's production supervisor has reported that some of the workers suspect the machine is no longer set correctly, and the average mass of low sodium salt in the bags may in fact be more than 25 kg. The production supervisor decides to carry out a hypothesis test at the 0.5% level of significance with a random sample of 80 bags of low sodium salt. Summary data for the mass, y kg, of low sodium salt in these bags is as follows.

$$n = 80 \qquad \sum(y-25) = 27.2 \qquad \sum(y-25)^2 = 85.1$$

(i) Carry out the test and state the conclusion of the test in the context of the question. [5]

(ii) State, giving a reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

11 Two brothers Kai and Leo are duathlon (running and cycling) athletes who train regularly. For each day, the probability that Kai cycles is $\frac{3}{4}$, the probability that he runs is $\frac{3}{5}$ and the probability he does both is p.

(a) Write down, in terms of p, the probability that, on one day, Kai either runs or cycles but not both. [1]

(b) Find the range of possible values of p. [2]

On average, Leo cycles 5 out of 7 days in a week. The probability that Leo cycles when Kai cycles is 0.9.

(c) Find the probability that Leo cycles when Kai does not. [3]

(d) State, in context, two assumptions needed for the number of days Leo cycles over a period of 5 weeks to be well modelled by a binomial distribution. [2]

Assume now that the number of days Leo cycles in 5 weeks has a binomial distribution.

(e) Find the probability that, in the 5 weeks, Leo cycles on at least 20 days but fewer than 30 days. [2]

(f) Find the probability that, in the 5 weeks, there are exactly 5 days in which both brothers do not cycle and they both cycle on the other days. [2]

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1 A function f is defined by $\mathrm{f}(x)=ax^3+bx^2+cx+d$. The graph of $y=\mathrm{f}(x)$ passes through the points $(-3,4)$ and $(1,8)$. Given that the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$ find the values of a,b,c and d. [4]

[4]	$\mathrm{f}(x)=ax^3+bx^2+cx+d$ $\mathrm{f}(-3)=4\Rightarrow -27a+9b-3c+d=4$ ----- (1) $\mathrm{f}(1)=8\Rightarrow a+b+c+d=8$ ----- (2) Since the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$, the graph of $y=\mathrm{f}(x)$ has a turning point at $(2,4)$. $\therefore \mathrm{f}(2)=4$ and $\mathrm{f}'(2)=0$. $\mathrm{f}(2)=4\Rightarrow 8a+4b+2c+d=4$ ------ (3) $\mathrm{f}'(x)=3ax^2+2bx+c$ $\mathrm{f}'(2)=0\Rightarrow 12a+4b+c=0$ ------ (4) From the GC, $a=1,\ b=-1,\ c=-8$ and $d=16$.	Question was well done by most students. Some students had difficulty with understanding the statement: "$y=\dfrac{1}{\mathrm{f}(x)}$ has a turning point at $(2,\frac{1}{4})$" with some working out $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{f}'(x)}{[\mathrm{f}(x)]^2}=0$.

2 [The volume of a sphere with radius r is given by $\frac{4}{3}\pi r^3$ and the surface area of a sphere with radius r is given by $4\pi r^2$.]

(a) The volume of an expanding sphere is increasing at a constant rate of $5\ \text{cm}^3\text{s}^{-1}$. Show that, at any instant, the rate of increase of the surface area is $\frac{k}{r}\ \text{cm}^2\text{s}^{-1}$, where r is the radius of the sphere and k is a constant to be determined. [3]

(b) Find the exact rate of change of surface area of the expanding sphere when the surface area is $20\ \text{cm}^2$. [2]

(a) **[3]**	Volume, $V=\frac{4}{3}\pi r^3$ and Surface area, $A=4\pi r^2$ $\frac{\mathrm{d}V}{\mathrm{d}r}=4\pi r^2$ $\quad\frac{\mathrm{d}A}{\mathrm{d}r}=8\pi r$ $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $=\frac{\mathrm{d}A}{\mathrm{d}r}\times\left(\frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}\right)$ $=8\pi r\times\frac{1}{4\pi r^2}\times 5$ $=\frac{10}{r}$, where $k=10$. (shown)	Question was well done by most students. **Common error:** Some students related but then treated A or V as constants. So, by writing $V=\frac{Ar}{3}$ or $A=\frac{3V}{r}$, one should actually get $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{1}{3}\left(A+r\frac{\mathrm{d}A}{\mathrm{d}r}\right)$ or $\frac{\mathrm{d}A}{\mathrm{d}r}=3\left(-\frac{V}{r^2}+\frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\right)$ and not $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{A}{3}$ or $\frac{\mathrm{d}A}{\mathrm{d}r}=-\frac{3V}{r^2}$.
(b) **[2]**	When $A=20$, $4\pi r^2=20$ $r=\sqrt{\frac{5}{\pi}}$ (since $r>0$) $\therefore\frac{\mathrm{d}A}{\mathrm{d}t}=10\sqrt{\frac{\pi}{5}}=2\sqrt{5\pi}\ \text{cm}^2\text{s}^{-1}$.	

3 **(a)** Without using a calculator, solve exactly $\dfrac{x^2-x-1}{x+1} \le 1$. [4]

(b) Hence solve exactly $\dfrac{x^2+|x|-1}{1-|x|} \le 1$. [4]

(a) **[4]**

$$\frac{x^2-x-1}{x+1} \le 1$$

$$\frac{x^2-x-1}{x+1} - 1 \le 0$$

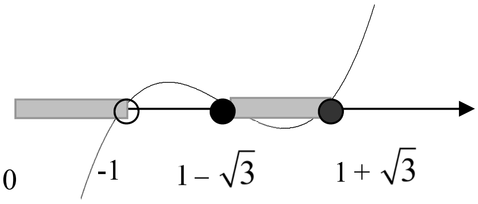
$$\frac{x^2-x-1-(x+1)}{x+1} \le 0$$

$$\frac{x^2-2x-2}{x+1} \le 0$$

Consider $x^2-2x-2=0$:

$$x = \frac{2 \pm \sqrt{(-2)^2-4(1)(-2)}}{2(1)} = 1 \pm \sqrt{3}$$

So $\dfrac{\left(x-1-\sqrt{3}\right)\left(x-1+\sqrt{3}\right)}{x+1} \le 0$



$\left(x-1-\sqrt{3}\right)\left(x-1+\sqrt{3}\right)(x+1) \le 0$

$x < -1$ or $1-\sqrt{3} \le x \le 1+\sqrt{3}$

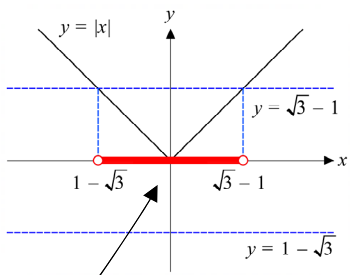
Most students are able to manipulate the inequality and solve for the roots. However, some students are unable to identify the regions correctly and not able to get the correct final answer.

(b) **[4]**

To solve $\dfrac{x^2+|x|-1}{1-|x|} \le 1$

$$\frac{(-|x|)^2-(-|x|)-1}{(-|x|)+1} \le 1$$

Replace x with $-|x|$ from part (a)



Hence, $-|x| < -1$ or $1-\sqrt{3} \le -|x| \le 1+\sqrt{3}$

i.e. $|x| > 1$ or $-1-\sqrt{3} \le |x| \le -1+\sqrt{3}$

$x < -1$ or $x > 1$ or $1-\sqrt{3} \le x \le -1+\sqrt{3}$

Majority of the students are able to identify the replacement correctly. However, many are not able to open up the modulus and get the correct answer.

Similar Question: Tut 2 Qn 8(c)

4 (a) Find $\int \frac{\cos x}{\cos 3x + \cos x}\, dx$. [3]

(b) Find $\int x \tan^{-1} x^2 \, dx$. Hence find the exact value of $\int_{-1}^{1} \left| x \tan^{-1} x^2 \right| dx$. [5]

<table>
<tr><td>(a)
[3]</td><td>

$$\int \frac{\cos x}{\cos 3x + \cos x}\, dx = \int \frac{\cos x}{2\cos 2x \cos x}\, dx$$
$$= \frac{1}{2}\int \sec 2x \, dx$$
$$= \frac{1}{4}\ln\left|\sec 2x + \tan 2x\right| + c,$$

where c is an arbitrary constant.

</td><td>Students are able to use the factor formula to simplify the integration to $\frac{1}{2}\int \sec 2x \, dx$.
However, quite a number of students are not able to integrate $\sec 2x$.</td></tr>
<tr><td>(b)
[5]</td><td>

$\int x \tan^{-1} x^2 \, dx$

$u = \tan^{-1} x^2 \qquad \frac{dv}{dx} = x$

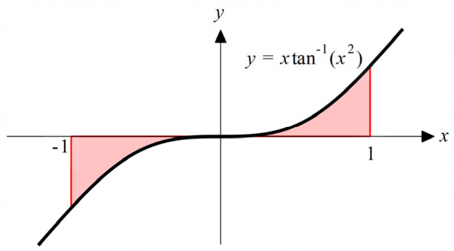
$\frac{du}{dx} = \frac{1}{1+(x^2)^2}(2x) \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\tan^{-1} x^2 - \int \frac{x^2}{2}\left(\frac{2x}{1+x^4}\right) dx$$
$$= \frac{x^2}{2}\tan^{-1} x^2 - \int \frac{x^3}{1+x^4}\, dx$$
$$= \frac{x^2}{2}\tan^{-1} x^2 - \frac{1}{4}\ln\left(1+x^4\right) + c,$$ where c is an arbitrary constant.

Method 1

$$\int_{-1}^{1} \left| x\tan^{-1} x^2 \right| dx = 2\int_0^1 x\tan^{-1} x^2 \, dx \quad (\because \text{ of symmetry})$$
$$= \left[x^2 \tan^{-1} x^2 - \frac{1}{2}\ln\left(1+x^4\right)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

[Note that $y = x\tan^{-1} x^2$ is an odd function since $(-x)\tan^{-1}(-x)^2 = -x\tan^{-1} x^2 = -\left(x\tan^{-1} x^2\right)$]



Method 2

$$\int_{-1}^{1} \left| x\tan^{-1} x^2 \right| dx$$
$$= -\int_{-1}^{0} x\tan^{-1} x^2 \, dx + \int_0^1 x\tan^{-1} x^2 \, dx$$
$$= -\left[\frac{1}{2}x^2 \tan^{-1} x^2 - \frac{1}{4}\ln\left(1+x^4\right)\right]_{-1}^{0} + \left[\frac{1}{2}x^2 \tan^{-1} x^2 - \frac{1}{4}\ln\left(1+x^4\right)\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

</td><td>Most students are able to use the integration by parts to solve this problem. There was quite a number of students who are unable to open up the modulus function.

Similar Question: Assignment 8B Qn 4, Mock Paper 1 Qn 5(b)</td></tr>
</table>

5 **(a)** Using the formulae for $\sin(A \pm B)$, prove that

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\cos 2r\theta \sin\theta. \qquad [1]$$

(b) Hence find a formula for $\sum_{r=1}^{n} \cos 2r\theta$, where $0<\theta<\pi$, in terms of $\sin(2n+1)\theta$ and $\sin\theta$. [3]

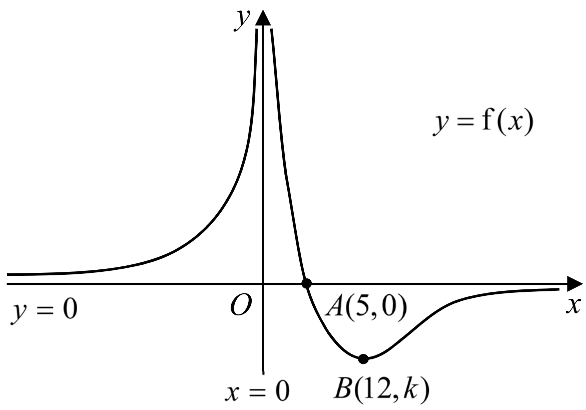
(c) Using the formula found in part **(b)**, show that the sum of the series

$$\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + \ldots + \sin^2 20\theta, \text{ for } 0<\theta<\pi$$

is $k - \dfrac{\sin(41\theta)-\sin(19\theta)}{4\sin\theta}$, where k is a constant to be determined. [4]

(a) **[1]**	$\sin(2r+1)\theta - \sin(2r-1)\theta$ $= \sin 2r\theta\cos\theta + \cos 2r\theta \sin\theta - (\sin 2r\theta\cos\theta - \cos 2r\theta\sin\theta)$ $= 2\cos 2r\theta\sin\theta$ (proven)	The question indicated the use of addition formulae, so it must be used instead of factor formulae Similar Question: Assignment 6B Qn 2
(b) **[3]**	$2\sum_{r=1}^{n}\cos 2r\theta\sin\theta = \sum_{r=1}^{n}\left[\sin(2r+1)\theta - \sin(2r-1)\theta\right]$ $= \cancel{\sin 3\theta} - \sin\theta$ $+ \cancel{\sin 5\theta} - \cancel{\sin 3\theta}$ $+ \cancel{\sin 7\theta} - \cancel{\sin 5\theta}$ $+ \ldots$ $+ \cancel{\sin(2n-1)\theta} - \cancel{\sin(2n-3)\theta}$ $+ \sin(2n+1)\theta - \cancel{\sin(2n-1)\theta}$ $= \sin(2n+1)\theta - \sin\theta$ $\therefore \sum_{r=1}^{n}\cos 2r\theta = \dfrac{\sin(2n+1)\theta - \sin\theta}{2\sin\theta}$	Most did well on this part, but do remember that cancellations must be clearly shown, and that the final two lines are in terms of n, not r, because r is the running index. Similar Question: Assignment 6B Qn 2
(c) **[4]**	$\sin^2 10\theta + \sin^2 11\theta + \sin^2 12\theta + \ldots + \sin^2 20\theta$ $= \sum_{r=10}^{20}\sin^2 r\theta$ $= \sum_{r=10}^{20}\dfrac{1-\cos(2r\theta)}{2}$ $= \dfrac{1}{2}(20-10+1) - \dfrac{1}{2}\left[\sum_{r=1}^{20}\cos(2r\theta) - \sum_{r=1}^{9}\cos(2r\theta)\right]$ $= \dfrac{11}{2} - \dfrac{1}{2}\left[\dfrac{\sin(41\theta)-\sin\theta}{2\sin\theta} - \dfrac{\sin(19\theta)-\sin\theta}{2\sin\theta}\right]$ $= \dfrac{11}{2} - \dfrac{\sin(41\theta)-\sin(19\theta)}{4\sin\theta}$, where $k = \dfrac{11}{2}$ (shown)	As this is a show question, the use of the answer in part (b) must be explicitly shown, especially the subtraction of the first 9 terms from the first 20 terms. Also, students should note that the number of terms from r = 10 to 20 is eleven.

6 **(a)** The diagram below shows the graph of $y = f(x)$.



The graph cuts the x-axis at point $A\,(5,0)$. It has a turning point at $B(12,k)$, where $k < 0$ and asymptotes with equations $x = 0$ and $y = 0$. On separate diagrams, sketch the graph of

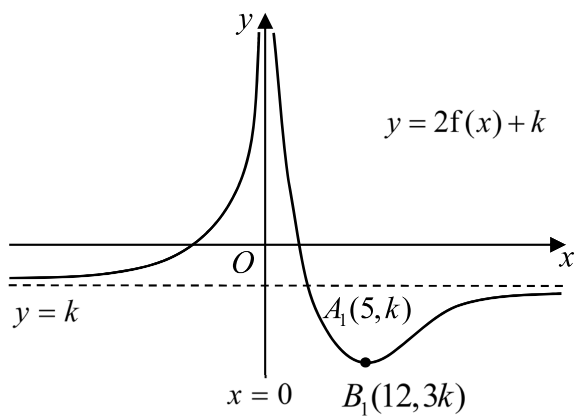
(i) $y = 2\mathrm{f}(x) + k$, stating the equations of any asymptotes and the coordinates of any turning point(s). [2]

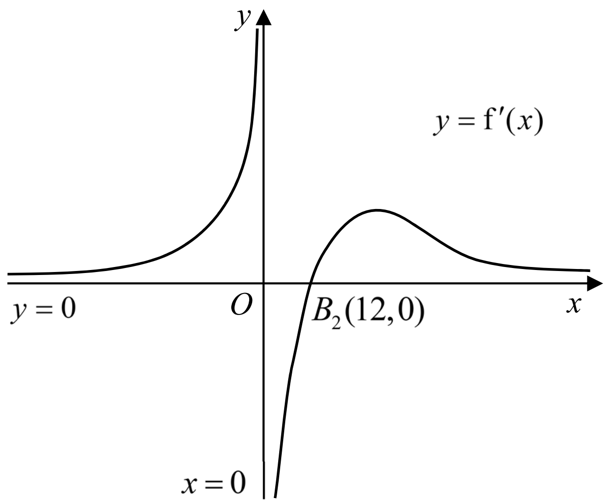
(ii) $y = \mathrm{f}'(x)$, stating the equations of any asymptotes and the coordinates of any point(s) where the graph crosses the axes. [2]

(b) The graph with equation $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = x(x-1)^2$ undergoes a single transformation and the equation of the resultant graph is $y = \mathrm{h}(x)$. Describe the transformation if

(i) $\mathrm{h}(x) = -x(x+1)^2$, [1]

(ii) $\mathrm{h}(x) = \dfrac{1}{8}x(x-2)^2$. [2]

(a)(i) **[2]**	Note that $k < 0$. 	One possible way to do this part is: $y = \mathrm{f}(x) \xrightarrow{\text{Replace } y \text{ by } y/2}$ $y = 2\mathrm{f}(x) \xrightarrow{\text{Replace } y \text{ by } y-k}$ $y = 2\mathrm{f}(x) + k$ which means that there is a stretch of factor 2 parallel to y-axis, and a translation $-k$ units in the negative y direction. Note that (5, k) was not asked in the question and so no marks were allocated to it.

(a)(ii) **[2]**		Most students did well on this part. Students are reminded to indicate the equations of the asymptotes as well as the *x*-intercept on the graph.
b(i) **[1]**	$g(x) = x(x-1)^2$ Consider $g(-x) = -x(-x-1)^2 = -x(x+1)^2 = h(x)$ Reflection of the graph of g in the *y*-axis to obtain the graph of h.	You are required to describe, not just indicate "replace *x* by -*x*".
b(ii) **[2]**	$g(x) = x(x-1)^2$ Consider $g\left(\frac{x}{2}\right) = \frac{x}{2}\left(\frac{x}{2}-1\right)^2 = \frac{x}{2}\left(\frac{x-2}{2}\right)^2 = \frac{1}{8}x\left(x-2\right)^2 = h(x)$ Scaling the graph of g parallel to the *x*-axis by factor of 2 to obtain the graph of h.	The question asked for a single transformation.

7 An arithmetic series has first term a and common difference d, where $a > 0$ and $d \neq 0$. The first, sixth and ninth terms of the arithmetic series are consecutive terms of a geometric series.

(a) Show that $25d = -2a$. [2]

(b) The sum of the first n terms of the arithmetic series is denoted by S. Find the set of possible values of n for which S exceeds $6a$. [3]

(c) Find the common ratio of the geometric series, and deduce that the geometric series is convergent. [2]

(d) Hence find the smallest value of m such that the sum of the terms of the geometric series after, but not including, the mth term, is less than 1% of the sum to infinity. [3]

(a) **[2]**	Given a, $a+5d$, $a+8d$ are consecutive terms of a GP. Then $\dfrac{a+5d}{a}=\dfrac{a+8d}{a+5d}$ $(a+5d)^2 = a(a+8d)$ $a^2+10ad+25d^2 = a^2+8ad$ $25d^2=-2ad$ Since $d \neq 0$, $25d=-2a$ (shown).	Common mistakes: • $\dfrac{a+5d}{a}=\dfrac{a+8d}{a+5d} \Rightarrow \dfrac{1}{a}=\dfrac{a+8d}{1}$ Similar question: Tut 6A Qn 2, Assignment 6A Qn 1
(b) **[3]**	$\dfrac{n}{2}\left[2a+(n-1)\left(-\dfrac{2}{25}a\right)\right] > 6a$ Since $a>0$, $n-\dfrac{n^2-n}{25}>6 \Rightarrow n^2-26n+150<0$ Using GC, $8.64 < n < 17.36$ Set of values of n $=\{n\in\mathbb{Z}: 9 \leq n \leq 17\}$ or $\{9,10,11,12,13,14,15,16,17\}$.	Common mistakes: • $\dfrac{n}{2}[2a+(n-1)d]=na+(n-1)d$ • $d\left(n^2-26n+150\right)>0$ $\Rightarrow n^2-26n+150>0$ Since $a>0$, from **(a)**, $d<0$. So, actually should be $n^2-26n+150<0$ • $n\in[9,17]$ There is a need to specify that the values of n are integers.
(c) **[2]**	Common ratio, $r=\dfrac{a+5d}{a}=\dfrac{a+(-\frac{2}{5}a)}{a}=\dfrac{3}{5}$ **Method 1** Since $\lvert r\rvert=\dfrac{3}{5}<1$, the geometric series is convergent. **Method 2** $S_n=\dfrac{a\left[1-\left(\frac{3}{5}\right)^n\right]}{1-\frac{3}{5}}=\dfrac{5}{2}a\left[1-\left(\dfrac{3}{5}\right)^n\right]$ As $n\to\infty$, $\left(\dfrac{3}{5}\right)^n\to 0$. Then, $S_n\to\dfrac{5}{2}a$. So, the geometric series is convergent.	Common mistakes: • $r<1$ for the series to be convergent $r<1$ would include values less than -1 which will not make the series convergent • $u_n=a\left(\frac{3}{5}\right)^{n-1}\to 0$ as $n\to\infty$ so, the series is convergent Note that $u_n\to 0$ does not imply series is convergent.

<table>
<tr>
<td>(d)
[3]</td>
<td>
Let b be the first term of the geometric series. It can be deduced that $b>0$ as a is a term ($a>0$) and $r>0$.

The sum of the terms of the series after, but not including, the mth term
$=u_{m+1}+u_{m+2}+u_{m+3}+\cdots$
$=S_\infty - S_m$
$=\dfrac{b}{1-r}-\dfrac{b\left(1-r^m\right)}{1-r}$
$=\dfrac{b}{1-r}\left[1-\left(1-r^m\right)\right]$
$=\dfrac{br^m}{1-r}$

$\dfrac{b\left(\frac{3}{5}\right)^m}{1-\frac{3}{5}}<0.01\left(\dfrac{b}{1-\frac{3}{5}}\right)$
Since $b>0$, $\left(\dfrac{3}{5}\right)^m<0.01$
$m\ln\dfrac{3}{5}<\ln 0.01$
$m>\dfrac{\ln 0.01}{\ln\frac{3}{5}}=9.02$
Smallest value of m = 10
</td>
<td>
Common mistakes:
• $\dfrac{b\left(\frac{3}{5}\right)^m}{1-\frac{3}{5}}<0.1\left(\dfrac{b}{1-\frac{3}{5}}\right)$
1% is equivalent to 0.01, and not 0.1
• $\dfrac{3}{5}^m<0.01$
Quite a few students wrote the above. This means that the power m is meant for 3 only. The proper way of writing should be $\left(\dfrac{3}{5}\right)^m<0.01$
• $m\ln\dfrac{3}{5}<\ln 0.01$
$\Rightarrow m<\dfrac{\ln 0.01}{\ln\frac{3}{5}}$
Note that $\ln\left(\dfrac{3}{5}\right)<0$
</td>
</tr>
</table>

8 **Do not use a calculator in answering this question.**

(a) The complex number w is such that where a and b are non-zero real numbers. The complex conjugate of w is denoted by w^*. Given that

$$ww^* = 4 - 2\text{i} + 2\text{i}w^*,$$

find the two possible values of w. [4]

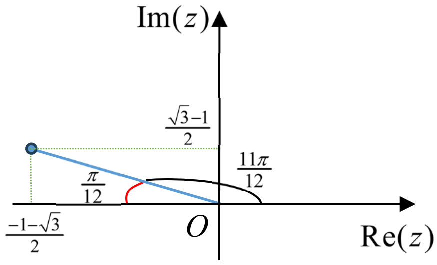
(b) The complex number z is given by $z = \dfrac{1-\sqrt{3}\,\text{i}}{-1+\text{i}}$.

(i) Find $\arg(z)$. [3]

(ii) Find z in cartesian form $x + \text{i}y$. [2]

(iii) Hence find the value of $\tan\dfrac{\pi}{12}$ in the form $c + d\sqrt{3}$, where c and d are exact integers to be determined. [3]

(a) **[4]**	$ww^* = 4 - 2\text{i} + 2\text{i}w^*$ Let $w = a + \text{i}b$, then $a^2 + b^2 = 4 - 2\text{i} + 2\text{i}(a - \text{i}b)$ $a^2 + b^2 = 4 + 2b + 2\text{i}(a-1)$ Comparing real and imaginary parts, $a^2 + b^2 = 4 + 2b$ and $0 = 2(a-1) \Rightarrow a = 1$ $1 + b^2 = 4 + 2b$ $b^2 - 2b - 3 = 0$ $(b-3)(b+1) = 0$ $b = 3$ or $b = -1$ $\therefore$ The 2 roots are $1 + 3\text{i}$ and $1 - \text{i}$.	Overall, this part was well-done by most students, demonstrating a clear understanding and application of the method for comparing real and imaginary parts. However, a few instances of carelessness in manipulation were noted.
(b)(i) **[3]**	$\arg\left(\dfrac{1-\sqrt{3}\text{i}}{-1+\text{i}}\right)$ $= \arg\left(1-\sqrt{3}\text{i}\right) - \arg\left(-1+\text{i}\right) + 2\pi$ $= -\dfrac{\pi}{3} - \dfrac{3\pi}{4} + 2\pi$ $= -\dfrac{13\pi}{12} + 2\pi$ $= \dfrac{11\pi}{12}$	Many students failed to connect this part to the application of argument properties. Among those who did, several were careless in identifying the correct quadrant for the complex number. Additionally, do note that no credit will be awarded for answers derived solely from a calculator.

(ii) **[2]**	$z=\dfrac{1-\sqrt{3}\mathrm{i}}{-1+\mathrm{i}}$ $=\dfrac{1-\sqrt{3}\mathrm{i}}{-1+\mathrm{i}}\times\dfrac{-1-\mathrm{i}}{-1-\mathrm{i}}$ $=\dfrac{-1-\mathrm{i}+\sqrt{3}\mathrm{i}-\sqrt{3}}{1+1}$ $=\dfrac{-1-\sqrt{3}}{2}+\dfrac{\sqrt{3}-1}{2}\mathrm{i}$	Quite many rationalize by multiplying (i + 1). While this worked, this is not the complex conjugate of $-1+\mathrm{i}$. Do revise the concept of complex conjugate.
(iii) **[3]**	 From the diagram, $\tan\dfrac{\pi}{12}=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ $=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}$ $=\dfrac{3-2\sqrt{3}+1}{3-1}$ $=2-\sqrt{3}$ where $c=2$ and $d=-1$	Full credit will not be given if the argument in **(b) (i)** was incorrectly determined. Many gave the final answer as $-2+\sqrt{3}$, which is clearly incorrect as $\tan\dfrac{\pi}{12}$ should give a positive value. Do try to double-check your answer whenever possible.

9 A curve C has equation $y = 2ax + \frac{b}{x}$ where a and b are non-zero real constants and $x \neq 0$.

(a) Using differentiation, determine whether C has any stationary points if $ab < 0$. [2]

It is now given that $b = \frac{1}{2}a$ where $a > 0$.

(b) Sketch C, stating the equations of any asymptotes, and the coordinates of any stationary points and points where C crosses the axes (if any). [3]

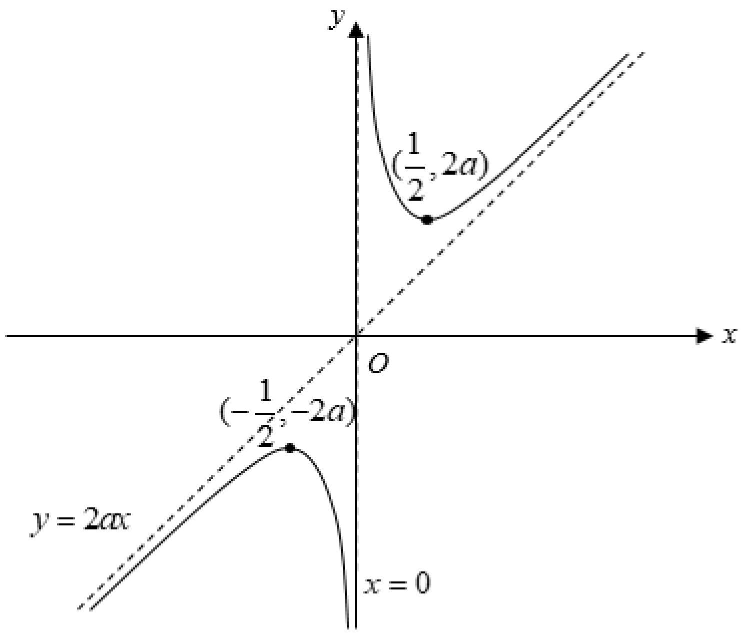
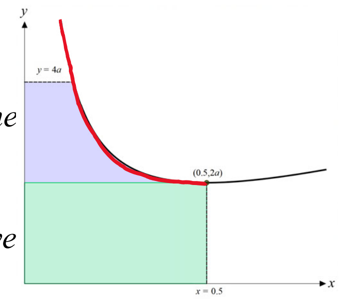
(c) State the range of values of k, in terms of a, for which the equation $2ax + \frac{b}{x} = kx$ has no real roots. [1]

(d) The region bounded by C, the axes, the lines $x = \frac{1}{2}$ and $y = 4a$ is rotated about the y-axis through 2π radians. Show that the volume generated is given by

$$\frac{1}{2}a\pi + \frac{\pi}{8a^2}\int_{2a}^{4a}\left(y^2 - 2a^2 - y\sqrt{y^2 - 4a^2}\right)\,\mathrm{d}y.$$

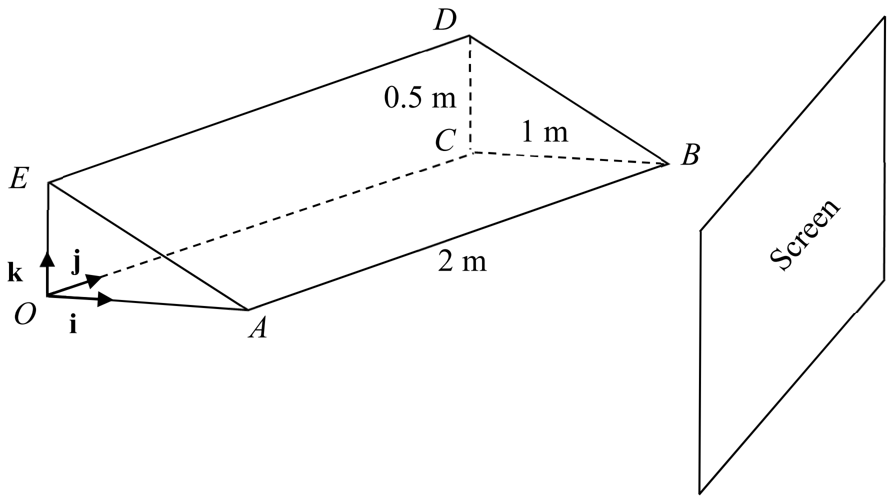
Hence, find, in terms of a and π, the exact volume generated. [6]

(a) **[2]**	$y = 2ax + \frac{b}{x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 2a - \frac{b}{x^2}$ If C has stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2a - \frac{b}{x^2} = 0$ $x^2 = \frac{b}{2a}$ However, if $ab < 0$, then $\frac{b}{a} < 0$. So $x^2 = \frac{b}{2a} < 0$ has no solutions, thus C has no stationary points if $ab < 0$.	Most students are able to differentiate and relate the given condition to deduce non-existence of stationary points.
(b) **[3]**	Given that $b = \frac{1}{2}a$, $y = 2ax + \frac{a}{2x} = a\left(2x + \frac{1}{2x}\right)$. Equations of asymptotes are $y = 2ax$ and $x = 0$. Since $b = \frac{1}{2}a$, thus $ab > 0$ and C has stationary points. Coordinates of stationary points = $\left(\frac{1}{2}, 2a\right)$ and $\left(-\frac{1}{2}, -2a\right)$.	While most students are able to deduce the shape of graph correctly, some students missed out the oblique asymptote which may caused by referring to GC without knowing the existence of 2 asymptotes. The drawing of stationary points and curve should be smooth

		and approaching the respective asymptotes. The graph should be fully labelled as stated in the question.
(c) **[1]**	Range of values of k is $k \le 2a$.	The answer can be deduced by referring to the gradient of oblique asymptote and graph in part **(b)**.
(d) **[6]**	$y = 2ax + \frac{a}{2x}$ $4ax^2 - 2yx + a = 0$ $x = \frac{2y \pm \sqrt{4y^2 - 4(4a)(a)}}{2(4a)} = \frac{2y \pm \sqrt{4y^2 - 16a^2}}{8a} = \frac{y \pm \sqrt{y^2 - 4a^2}}{4a}$ *Note for students :* *Coord. of turning point is* $(0.5, 2a)$. *We are interested in the portion of the curve that's highlighted in red, where* $x \le 0.5$. *Hence, equation of the required curve is* $x = \frac{y - \sqrt{y^2 - 4a^2}}{4a}$.  Volume generated	While students attempted to find x in term of y, some did not identity the correct x which disallow them to show the volume generated. Students were able to use the shown result to integrate correctly. Quite a number of students did not manage to simplify and obtain correct final answer.

	$=\pi\left(\frac{1}{2}\right)^2(2a)+\pi\int_{2a}^{4a}\left(\frac{y-\sqrt{y^2-4a^2}}{4a}\right)^2\ \mathrm{d}y$ $=\frac{1}{2}a\pi+\frac{\pi}{16a^2}\int_{2a}^{4a}\left(y^2+y^2-4a^2-2y\sqrt{y^2-4a^2}\right)\mathrm{d}y$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\int_{2a}^{4a}\left(y^2-2a^2-y\sqrt{y^2-4a^2}\right)\mathrm{d}y$ (shown) $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left[\frac{1}{3}y^3-2a^2y-\frac{1}{2}\frac{\left(y^2-4a^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{2a}^{4a}$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left[\frac{64}{3}a^3-8a^3-\frac{1}{3}\left(24\sqrt{3}a^3\right)-\frac{8}{3}a^3+4a^3+\frac{1}{3}(0)\right]$ $=\frac{1}{2}a\pi+\frac{\pi}{8a^2}\left(\frac{44}{3}a^3-8\sqrt{3}a^3\right)$ $=\left(\frac{7}{3}-\sqrt{3}\right)a\pi$	

10 For an upcoming motor-car race, a spectators' gallery is to be set up near the racing track. As part of the preparations, a model of this gallery, shaped as a prism, is constructed.



The diagram above shows the model of the gallery with O as the origin and the unit vectors **i**, **j** and **k** are parallel to OA, OC and OE respectively. Points (x, y, z) are defined relative to O, where units are in metres.

It is given that $OA = CB = 1$ m, $OC = AB = ED = 2$ m and $OE = CD = 0.5$ m.

(a) Find a cartesian equation of the plane $ABDE$. [2]

A shelter is to be constructed above the plane $ABDE$. On the model, this shelter is a rectangular plane that intersects plane $ABDE$ in the line ED.

(b) Given that the equation of the shelter is $-0.5x + z = h$, show that $h = 0.5$. [1]

(c) Find the acute angle between the plane $ABDE$ and the shelter. [2]

A spotlight is shone towards the gallery and the beam of light is in the form of a line l with cartesian equation $x - a = z$, $y = 1$ for some real number a. The beam lands on a point M on the rectangular surface $ABDE$.

(d) Find in terms of a, the position vector of M, and find the range of values of a. [5]

(e) A large rectangular flat screen is to be placed in front of the gallery at a distance away. The screen can be taken to be part of a vertical plane with equation $12x + 5y = d$, where $d > 50$. Using $a = \frac{1}{2}$, find the value of d so that the shortest distance between point M and the screen is 4 m. [4]

(a) **[2]**	$\overrightarrow{OA} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\overrightarrow{OE} = \begin{pmatrix}0\\0\\0.5\end{pmatrix}$ Normal to $ABDE$, $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AE}$ $= \begin{pmatrix}0\\2\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\0.5\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$	This part should have been done better. The normal vector to the plane can be obtained by taking the cross product of 2 vectors **parallel** to the plane, and not the position vectors of 2 points on the plane. Furthermore, the question asked for cartesian equation form of the plane as

	Equation of plane $ABDE$: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=1$ Cartesian equation: $x+2z=1$	the final answer which many students are not familiar with.
(b) **[1]**	Since the shelter plane contains line ED, point E $(0,0,0.5)$ lies on the shelter plane $-0.5x+z=h$ $\therefore -0.5(0)+0.5=h$ $h=0.5$ (shown)	This part was generally well done, but students should note that for a "show" question, the working should be clear and without doubt. Effort should be taken to explain which point is the position vector used to substitute into the equation of the plane.
(c) **[2]**	Acute angle between plane $ABDE$ and the shelter $=\cos^{-1}\dfrac{\left\lvert\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}-0.5\\0\\1\end{pmatrix}\right\rvert}{\sqrt{5}\sqrt{1.25}}$ $=53.1^\circ$ (1 d.p)	This part should have been done better. A few students chose to take 90° to subtract the answer. Perhaps a confusion with the formula between line and plane. Some others did not get the correct normal vector from part **(a)**, or copied the vector of the shelter wrongly from the question.
(d) **[5]**	line l : $\mathbf{r}=\begin{pmatrix}a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ At M, $\begin{pmatrix}a+\lambda\\1\\\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=1$ $a+3\lambda=1 \quad\Rightarrow\quad \lambda=\dfrac{1-a}{3}$ $\overrightarrow{OM}=\begin{pmatrix}a+\dfrac{1-a}{3}\\1\\\dfrac{1-a}{3}\end{pmatrix}=\begin{pmatrix}\dfrac{2a+1}{3}\\1\\\dfrac{1-a}{3}\end{pmatrix}$ From the diagram, $0\le\dfrac{2a+1}{3}\le 1$ and $0\le\dfrac{1-a}{3}\le\dfrac{1}{2}$ $-\dfrac{1}{2}\le a\le 1$ and $-\dfrac{1}{2}\le a\le 1$ $\therefore -\dfrac{1}{2}\le a\le 1$	The first part of this question was well done. Care should be taken to simplify the answer. Many students either left out the part on range of values or did not put the range fully for both the lower and upper bound.

(e) **[4]**	When $a=\frac{1}{2}$, $\overrightarrow{OM}=\begin{pmatrix}\frac{2}{3}\\1\\\frac{1}{6}\end{pmatrix}$. Let $S\left(\frac{d}{12},0,0\right)$ be a point on the screen. Then $\overrightarrow{SM}=\begin{pmatrix}\frac{2}{3}-\frac{d}{12}\\1\\\frac{1}{6}\end{pmatrix}$ Shortest distance from M to the screen $=\frac{\left\|\overrightarrow{SM}\cdot\begin{pmatrix}12\\5\\0\end{pmatrix}\right\|}{\sqrt{12^2+5^2}}=4$ $\Rightarrow\frac{\|8-d+5\|}{13}=4$ $\Rightarrow\|13-d\|=52$ $13-d=52$ or $13-d=-52$ $d=-39$ (reject as $d>50$) $d=65$	A few students who did well for this question simply imagined a vertical plane that M lies on, and used the distance between the 2 planes: $\frac{\left\|\overrightarrow{OM}\cdot\begin{pmatrix}12\\5\\0\end{pmatrix}-d\right\|}{\sqrt{12^2+5^2}}=4$ thus not needing to find a point on the screen. Many students who did not do the shortest method started well by letting F be the foot of perpendicular from M to the screen. Following which, many faltered by either: 1) Using $\overrightarrow{MF}$ as the position vector of the F. 2) Putting $\left\|\overrightarrow{OF}\right\|=4$. 3) Not putting a modulus around the parameter, thus not having another value to reject. 4) Using a wrong $\overrightarrow{OM}$ from earlier part.

11 A wafer fabrication company uses the floating-zone method to purify polysilicon ingots, each having a uniform cross-sectional area and a length of 200 cm. The method involves placing a polysilicon ingot with impurity concentration C_0 atoms/cm^3 on top of a single seed crystal. The polysilicon ingot is then heated externally by an RF coil, which locally melts the ingot. The impurities prefer to stay in the molten state than in the solid state and thus as the coil and the melt zone move upwards, a single crystal, that is purer, solidifies on top of the seed crystal. A schematic illustration of the method is shown below in Fig. 1 and Fig. 2.

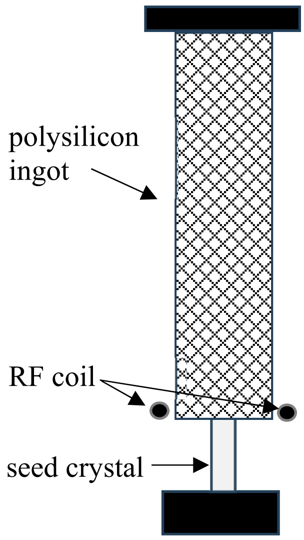


Fig. 1 Initial set-up

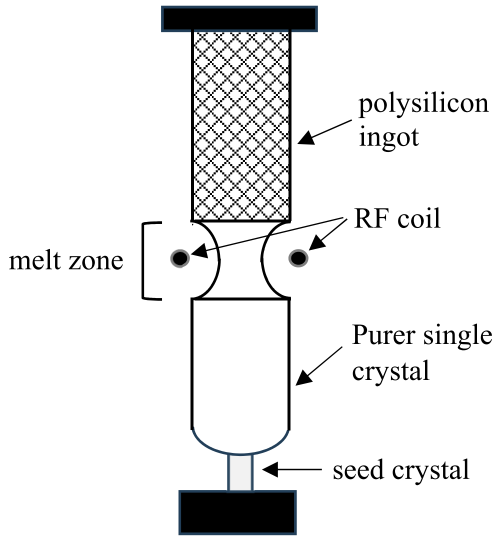


Fig. 2 During purification process

For a 'floating' melt zone of length L cm, the concentration of impurities in the melt zone, C atoms/cm^3, and the distance moved by the RF coil, x cm, are related by the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{L}\left(C_0-kC\right),$$

where k is a constant such that $0<k<1$.

The length of the "floating" melt zone, L cm, adopted by the company is 2 cm and $0\le x\le 198$. It is also given that when $x = 0$, $C=C_0$.

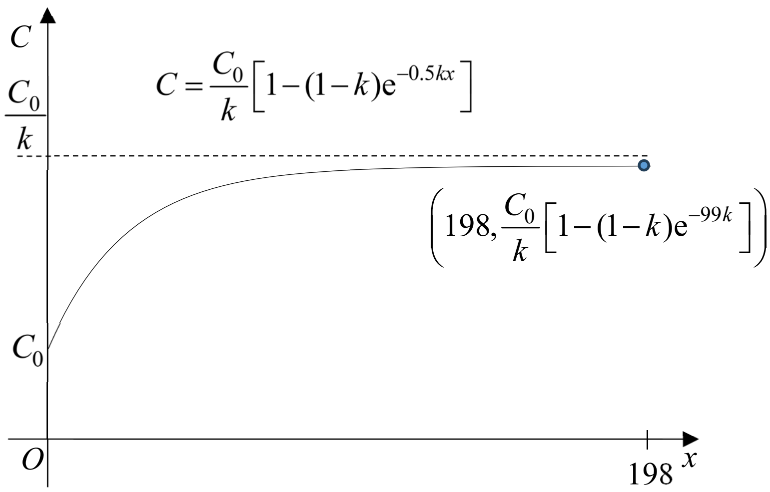
(a) Solve the differential equation to find an expression for C in terms of C_0, k and x. [4]

(b) Sketch the graph of C against x. [2]

(c) Assume that $k=0.3$ and that the RF coil moves upwards at a constant speed of 8 mm per hour. Find the time taken for the concentration of impurities in the melt zone to reach $2C_0$ and the rate of change of the concentration of impurities, in terms of C_0 at this instant. [5]

The company decides to change the length of the "floating" melt zone.

(d) Explain, with a reason, whether a shorter length is preferable over a longer one. [1]

(a) **[4]**	$$\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{2}\left(C_0-kC\right)$$ $$\int \frac{1}{C_0-kC}\ \mathrm{d}C=\int \frac{1}{2}\ \mathrm{d}x$$ Since $\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{L}\left(C_0-kC\right)>0,\ \left(C_0-kC\right)>0.$ $$-\frac{1}{k}\ln\left(C_0-kC\right)=0.5x+A$$ $$C_0-kC=B\mathrm{e}^{-0.5kx}$$ $$C=\frac{1}{k}\left(C_0-B\mathrm{e}^{-0.5kx}\right)$$ When $x=0,\ C=C_0$ and hence $B=C_0(1-k)$. Thus $C=\frac{C_0}{k}\left[1-(1-k)\mathrm{e}^{-0.5kx}\right]$.	Avoid using C for the arbitrary constant as C is already used to represent the concentration of impurities in this question. When solving for the arbitrary constant, the answer should be in terms of C_0.
(b) **[2]**		Note: It is not necessary to indicate the horizontal asymptote as we are sketching for $0\le x\le 198$.
(c) **[5]**	Assuming $k=0.3$, $C=\frac{C_0}{0.3}\left(1-0.7\mathrm{e}^{-0.15x}\right)$. When $C=2C_0$, $2C_0=\frac{C_0}{0.3}\left(1-0.7\mathrm{e}^{-0.15x}\right)$ $$\mathrm{e}^{-0.15x}=\frac{4}{7}$$ $$x=\frac{\ln\left(\frac{4}{7}\right)}{-0.15}=3.73077\ \text{cm}$$ Hence the time taken $=\frac{37.3077}{8}=4.66\ \text{h}$.	Give your answers with the units used. Some converted mm to cm incorrectly. You are expected to know the metric system.

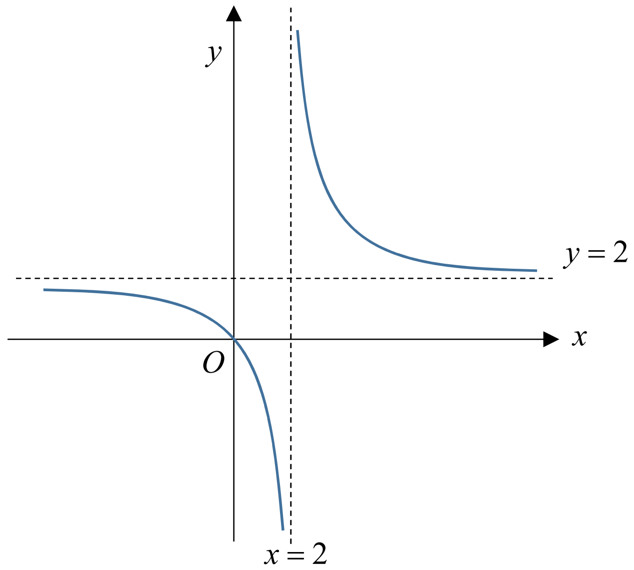
	$\frac{\mathrm{d}C}{\mathrm{d}t}=\frac{\mathrm{d}C}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$ $=0.8\times\frac{1}{2}\left(C_0-0.3(2C_0)\right)$ $=0.16C_0$ atoms/cm^3 per hour	Rate of change of the concentration of impurities refers to $\frac{\mathrm{d}C}{\mathrm{d}t}$.
(d) **[1]**	A shorter length for the floating melt zone is preferable since $\frac{\mathrm{d}C}{\mathrm{d}x}=\frac{1}{L}\left(C_0-kC\right)$ will be larger for this case. Thus for the same distance moved, the shorter melt zone has higher concentration of impurities *which means that the single crystal that solidifies has higher purity.*	It is insufficient to say that the concentration of impurities increases as that occurs regardless of length used. The key idea is the increased rate of change of concentration, with respect to x, when a shorter length is used.

1 The function f is defined by $\mathrm{f}: x \mapsto \dfrac{2x}{x-2}$, for $x \in \mathbb{R}$, $x \neq 2$.

(a) Sketch the graph of f and find its range. [3]

Another function g is defined by $\mathrm{g}: x \mapsto 3+|x+2|$, for $x \in \mathbb{R}$.

(b) Show that the composite function fg exists. Find $\mathrm{fg}(x)$ and state the domain and range of fg. [5]

<table>
<tr><td>(a)
[3]</td><td>

Range of f = $(-\infty,\infty)\setminus\{2\}$</td><td>Note that the curve passes through origin.

Other possible notations for range of f:
• $(-\infty,\infty)\setminus\{2\}$
• $\mathbb{R}\setminus\{2\}$</td></tr>
<tr><td>(b)
[5]</td><td>$\mathrm{R_g}=[3,\infty)$
$\mathrm{D_f}=(-\infty,2)\cup(2,\infty)$
Since $\mathrm{R_g}\subseteq \mathrm{D_f}$, function fg exists.
$\mathrm{fg}(x)=\mathrm{f}(3+|x+2|)=\dfrac{2(3+|x+2|)}{3+|x+2|-2}=\dfrac{6+2|x+2|}{1+|x+2|}$
$\mathrm{D_{fg}}=\mathbb{R}$
Note that $\mathrm{f}(3)=\dfrac{2(3)}{3-2}=6$.
$$\begin{array}{ccccc} \mathrm{D_{fg}}=\mathrm{D_g} & & \mathrm{R_g} & & \mathrm{R_{fg}} \\ \mathbb{R} & \xrightarrow{\quad g \quad} & [3,\infty) & \xrightarrow{\quad f \quad} & (2,6] \end{array}$$
Hence, $\mathrm{R_{fg}}=(2,6]$.</td><td>Give domain and range in set notations.

Some students are confused about the domain and range of a composite function.</td></tr>
</table>

2 The function f is defined by $\mathrm{f}(z)=z^4+Az^3+Bz^2+Cz+45$, where A, B and C are real numbers. Given that $2+\mathrm{i}$ is a root of $\mathrm{f}(z)=0$ and $(z-k)^2$ is a factor of $\mathrm{f}(z)$, where k is a positive real number, find the values of A, B, C and k. [5]

[5]	Since all coefficients of $\mathrm{f}(z)$ are real and $2+\mathrm{i}$ is a root of $\mathrm{f}(z)=0$, then $2-\mathrm{i}$ is also a root of the equation. The quadratic factor of the equation is $\left(z-(2+\mathrm{i})\right)\left(z-(2-\mathrm{i})\right)=z^2-4z+5$ Then, $z^4+Az^3+Bz^2+Cz+45=\left(z-k\right)^2(z^2-4z+5)$ $=\left(z-2kz+k^2\right)(z^2-4z+5)$ Comparing constants, $45=5k^2 \quad \Rightarrow \quad k=\pm 3$ Since $k>0$, therefore, $k=3$. So, $z^4+Az^3+Bz^2+Cz+45=\left(z-6z+9\right)(z^2-4z+5)$ $=z^4-10z^3+38z^2-66z+45$ $A=-10, B=38, C=-66$	This question is generally well done. Some students tried to substitute the root in, then compare the real and imaginary parts to obtain 2 equations. However, 2 equations are not enough to solve for 3 unknowns, without using the condition that $(z-k)^2$ is also a factor of $\mathrm{f}(z)$.

3 **(a)** The points A, B and C on the plane π have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Show that a vector perpendicular to π is parallel to

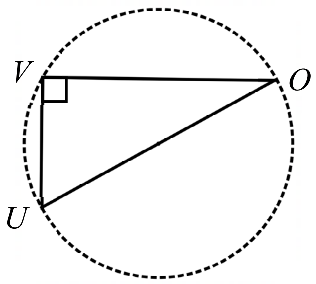
$$\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}.$$ [3]

(b) $\mathbf{p}$ and $\mathbf{q}$ are non-zero vectors and $\mathbf{p}=(\mathbf{p}\cdot\mathbf{q})\mathbf{q}$.

(i) Find the relationship between $\mathbf{p}$ and $\mathbf{q}$. [1]

(ii) Find $|\mathbf{q}|$. [1]

(c) $\mathbf{u}$ is the position vector of a fixed point U relative to a fixed origin O. A variable point V has position vector $\mathbf{v}$ relative to O.
Given that $\mathbf{v}\cdot(\mathbf{v}-\mathbf{u})=0$, describe geometrically the set of all possible positions of the point V. [2]

(a) **[3]**	$\overrightarrow{AB}\times\overrightarrow{BC}=(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})$ $=\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}$ $=\mathbf{b}\times\mathbf{c}-(-\mathbf{c}\times\mathbf{a})+\mathbf{a}\times\mathbf{b}$, since $\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $=\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$ Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel to π but not parallel to each other, so $\overrightarrow{AB}\times\overrightarrow{BC}$ is a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and hence perpendicular to π. Therefore a vector perpendicular to π is parallel to $\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}$ (Shown).	Some students did not show working clearly for a "show" question. Some students started from the given expression which is to "verify" not to "show" Some students find the cross product of two position vectors which are not parallel to the plane.
(b)(i) **[1]**	$\mathbf{p}=(\mathbf{p}\cdot\mathbf{q})\mathbf{q}\Rightarrow\mathbf{p}\,//\,\mathbf{q}$ since $\mathbf{p}\cdot\mathbf{q}$ is a scalar. Hence $\mathbf{p}$ is parallel to $\mathbf{q}$.	Note that parallel vectors may not be in the same direction, it can be opposite directions, since $\mathbf{p}\cdot\mathbf{q}$ could be negative.
(b)(ii) **[1]**	$\mathbf{p}=(\mathbf{p}\cdot\mathbf{q})\mathbf{q}\Rightarrow\|\mathbf{p}\|=\|(\mathbf{p}\cdot\mathbf{q})\mathbf{q}\|$ $\Rightarrow\|\mathbf{p}\|=\|\mathbf{p}\cdot\mathbf{q}\|\|\mathbf{q}\|$ $=\|\mathbf{p}\|\|\mathbf{q}\|\|\mathbf{q}\|$ $\Rightarrow\|\mathbf{q}\|^2=1$ $\Rightarrow\|\mathbf{q}\|=1$	$\|\mathbf{q}\|=\pm1$ is not accepted since the magnitude of a vector is positive.
(c) **[2]**	$\mathbf{v}\cdot(\mathbf{v}-\mathbf{u})=0\Rightarrow\mathbf{v}\perp(\mathbf{v}-\mathbf{u})$ $\Rightarrow\overrightarrow{OV}\perp\overrightarrow{UV}$  Since V is a variable point, so the set of all possible positions of the point V forms a sphere with OU as the diameter.	It would be a circle if the vectors are 2-dimensional, so it is also accepted.

4 **(a)** Given that $y = \mathrm{e}^{\sqrt{1+2x}}$, show that

$$(1+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2 \quad \text{and} \quad (1+2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = y. \qquad [3]$$

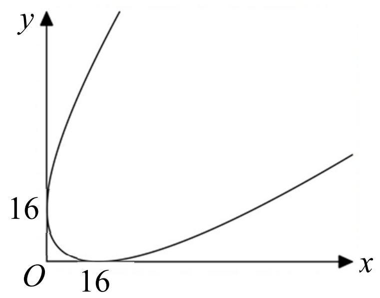
(b) By further differentiation, obtain the series expansion for y in terms of x up to and including the term in x^3. [3]

(c) Verify that the same series expansion for y in part **(b)** is obtained if the standard series expansions for e^x and $(1+x)^n$ are used. [4]

(a) **[3]**	$y = \mathrm{e}^{\sqrt{1+2x}}$ **Method 1** $y = \mathrm{e}^{\sqrt{1+2x}}$ $\ln y = (1+2x)^{\frac{1}{2}}$ Differentiating wrt x, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2) = (1+2x)^{-\frac{1}{2}}$ $(1+2x)^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x} = y$ $(1+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2$ (shown) **Method 2** $y = \mathrm{e}^{\sqrt{1+2x}}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{1+2x}}(2)\mathrm{e}^{\sqrt{1+2x}}$ $\sqrt{1+2x}\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\sqrt{1+2x}}$ $(1+2x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2$ (shown) Differentiating again wrt x, $(1+2x)(2)\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $(1+2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = y$ since $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ (shown)	Most students have done well for the first part of the question. Students must take note that $\left(\mathrm{e}^{\sqrt{1+2x}}\right)^2 \neq \mathrm{e}^{1+2x}$. Recall that $(a^m)^n = a^{mn}$ Unfortunately, many students did not realise that they could attempt this part using implicit differentiation. They attempted by finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in terms of x first, resulting in a need for tedious simplification.
(b) **[3]**	Differentiating again wrt x, $(1+2x)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}$ $(1+2x)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}$	Generally, the students are able to complete part **(b)** well.

	When $x=0,\ \ y=\mathrm{e},\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e},\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0,\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=\mathrm{e}.$ $y=\mathrm{e}+\mathrm{e}x+(0)\dfrac{x^2}{2!}+\mathrm{e}\dfrac{x^3}{3!}+\cdots=\mathrm{e}\left(1+x+\dfrac{x^3}{6}+\cdots\right)$	
(c) **[4]**	$\left(1+2x\right)^{\frac{1}{2}}=1+\frac{1}{2}(2x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{(2x)^2}{2}+\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(2x)^3}{3!}+\ ...$ $=1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ ...$ <u>**Method 1**</u> $\mathrm{e}^{\sqrt{1+2x}}=\mathrm{e}^{1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...}$ $=\mathrm{e}^1\mathrm{e}^{x-\frac{1}{2}x^2+\frac{1}{2}x^3+...}$ $=\mathrm{e}\left(1+\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3+...\right)\right.$ $\left.+\frac{1}{2!}\left(x-\frac{1}{2}x^2+...\right)^2+\frac{1}{3!}\left(x^3+...\right)^3+...\right)$ $=\mathrm{e}\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{2}\left(x^2-x^3\right)+\frac{1}{6}x^3+...\right)$ $=\mathrm{e}\left(1+x+\frac{1}{6}x^3+...\right)$ (same results as in part (a)) <u>**Method 2**</u> $\mathrm{e}^{\sqrt{1+2x}}=\mathrm{e}^{1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...}$ $=\mathrm{e}^1\mathrm{e}^x\mathrm{e}^{-\frac{1}{2}x^2}\ \mathrm{e}^{\frac{1}{2}x^3}\ \cdots$ $=\mathrm{e}\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\left(1+\left(-\frac{1}{2}x^2\right)+\cdots\right)$ $\left(1+\left(\frac{1}{2}x^3\right)+\cdots\right)\cdots$ $=\mathrm{e}\left(1+x+\frac{1}{2!}x^2-\frac{1}{2}x^2+\frac{1}{3!}x^3-\frac{1}{2}x^3+\frac{1}{2}x^3+...\right)$ $=\mathrm{e}\left(1+x+\frac{1}{6}x^3+...\right)$ (same results as in part (a))	Most of the students are able to expand $\left(1+2x\right)^{\frac{1}{2}}$ correctly Many students have tried to expand $\mathrm{e}^{\sqrt{1+2x}}=\mathrm{e}^{1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...}$ as $1+\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...\right)$ $+\frac{1}{2!}\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...\right)^2$ $+\frac{1}{3!}\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+...\right)^3$ $+...$ They did not realise that every subsequent term of this expansion will need to be taken in to account in order to obtain the correct constant term and the coefficients of x, x^2 and x^3 terms.

5



The diagram shows the curve C with parametric equations

$$x=(1+t)^2, \quad y=(3-t)^2.$$

The curve C meets the axes at $(16,0)$ and $(0,16)$.

(a) Show that the line $x=16$ meets C at the point P where $t=-5$. [1]

The normal to C at P is denoted by l.

(b) Find the cartesian equation of l. [3]

(c) The line l meets C again at the point Q where $x=b$. Show that the area of the region bounded by l, the lines $x=16$, $x=b$ and the x-axis is $\frac{266240}{81}$ units^2. [3]

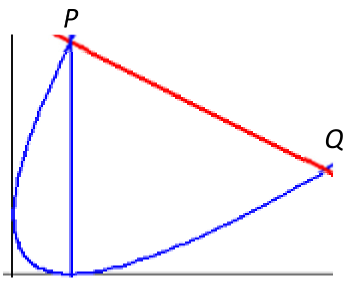
(d) Show that the area (in units^2) of the region bounded by C and l can be given by

$$\frac{266240}{81}+\int_c^d \mathrm{f}(t)\,\mathrm{d}t,$$

where $\mathrm{f}(t)$, and the constants c and d are to be determined.

Hence find the value of this area. [3]

(a) **[1]**	When the line $x=16$ meets C, $(1+t)^2=16$ $1+t=\pm 4$ $t=3$ or -5 Since $t=3$ gives the point $(16,0)$, so $t=-5$ at P.	To "show", need to <u>find</u> the answer assuming answer is not given. Substituting $t=-5$ into the equation to check the statement is true can only be used if the question says "verify".
(b) **[3]**	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{-2(3-t)}{2(1+t)}$ At $t=-5$, $x=16$, $y=64$, gradient of normal is $\dfrac{-1}{2}$ l: $\quad y-64=\dfrac{-1}{2}(x-16) \Rightarrow y=\dfrac{-1}{2}x+72$	Need to be careful when simplifying an expression involving negative signs.

(c) **[3]**	$(3-t)^2 = \frac{-1}{2}(1+t)^2 + 72$ $2\left(t^2-6t+9-72\right) = -t^2-2t-1$ $3t^2-10t-125=0$ $(t+5)(3t-25)=0$ since $t=-5$ at P At Q, $t=\frac{25}{3}$, $x=(1+t)^2=\frac{784}{9}$ **Method 1** Required area is $\int_{16}^{\frac{784}{9}} \frac{-1}{2}x+72 \; \mathrm{d}x = \frac{266240}{81}$ using GC **Method 2** At Q, $t=\frac{25}{3}$, $y=(3-t)^2=\frac{256}{9}$. Required area = Area of Trapezium $=\frac{1}{2}\left(\frac{784}{9}-16\right)\left(64+\frac{256}{9}\right)$ $=\frac{266240}{81}$	To find intersection, substitute parametric equations into cartesian equation. No need to find the corresponding cartesian equation of the parametric equations. Note that the question asks for area under the normal line, not area under curve.
(d) **[3]**	 The curve meets the y-axis at $t=-1$. The required area is: Area under l as found in **(c)** + Area under C from $t=-1$ to -5 − Area under C from $t=-1$ to $\frac{25}{3}$ $=\frac{266240}{81} + \int_{x=0,\,t=-1}^{x=16,\,t=-5} y \; \mathrm{d}x - \int_{x=0,\,t=-1}^{x=\frac{784}{9},\,t=\frac{25}{3}} y \; \mathrm{d}x$ $=\frac{266240}{81} + \int_{x=\frac{784}{9},\,t=\frac{25}{3}}^{x=16,\,t=-5} y \; \mathrm{d}x$ $=\frac{266240}{81} + \int_{\frac{25}{3}}^{-5} (3-t)^2 \cdot 2(1+t) \; \mathrm{d}t$ $=\frac{256000}{81}$ using GC	Question asks to "show", so working must be clear. Since we need to express in one integral and the trapezium is bounded by the x-axis, we should consider the different areas under the curve bounded by the x-axis so that they can be combined into one integral.

6 Eleven cards each bears a single letter and together they can be made to spell the word COFFEEHOUSE. The 11 cards are arranged in a row.

(a) Find the number of different arrangements that can be made. [1]

(b) Find the number of different arrangements in which the 2 F's are next to each other and no E's are next to each other. [3]

Three cards are selected from the eleven cards and the order of selection is not relevant. Find the number of possible selections that can be made

(c) if the three cards all bear different letters, [1]

(d) if exactly two of the three cards bear the same letter. [2]

(a) **[1]**	3 E, 2 F, 2 O, 1 C, 1 H, 1 U, 1 S Number of arrangements $=\dfrac{11!}{3!2!2!}=1663200$	Be careful when counting the number of repeated letters. Some students counted wrongly and the entire question was affected.
(b) **[3]**	**Method 1** _ FF _ O _ O _ C _ H _ U _ S _ Put the 3 E aside first. Group the 2 Fs as 1 unit and arrange it together with the other 6 letters. Then slot in the 3 E. Number of different arrangements where the 2 F are together $=\dfrac{7!}{2!}=2520$ Number of ways to slot in the 3 E in the 8 possible slots $={}^{8}C_{3}=56$ $\therefore$ Required number of arrangements $=2520\times 56=141120$ **Method 2** Number of arrangements where the 2 F are together $=\dfrac{10!}{2!3!}=302400$ Number of arrangements where the 2 F are together and 2 E are together $=\dfrac{7!}{2!}\times {}^{8}\mathrm{C}_{2}\times 2!=141120$ (note that the 2 Es that are together must be separated from the single E. So, arrange all other letters first then slot in the 2 Es and the single E) Number of arrangements where the 2 F are together and the 3 E are together $=\dfrac{8!}{2!}=20160$ Using complement method: Required number of arrangements $=302400-141120-20160=141120$	Drawing a simple diagram as shown helps in your counting process. For P&C questions, there are often numerous ways to solve the question. For this question, the first method is the fastest – employing the "grouping" and "slotting" method. All other methods are tedious and can be tricky. Some students did the alternative method with little success. Even if they succeeded, they have spent way too much time on a 3 marks question.

(c) **[1]**	There are 7 different letters to chose from. Number of selections such that the 3 cards all bear different letters $= {}^{7}C_{3} = 35$	Do read the question carefully. It was stated that **the order of selection is not relevant** for part **(c)** and **(d)**.
(d) **[2]**	Case 1 : E, E, _ (third card is not E) Case 2 : F, F, _ Case 3 : O, O, _ For each case, there are 6 choices for the third card. $\therefore$ Number of selections $= 3 \times 6 = 18$	Learn to explain your answer. In this case, list down the cases clearly. No method marks can be awarded if your answer is wrong and there was no explanation of how the string of numbers came about.

7 The probability of obtaining a head when a particular coin is tossed is p. A fair cubical die has the number '1' on one face, number '2' on two faces and number '3' on three faces.

The coin and die are thrown simultaneously. The random variable X is defined as follows.

If the coin shows a head, then X is thrice the score on the die.
If the coin shows a tail, then X is the score on the die.

(a) Show that $\mathrm{P}(X=3)=\frac{1}{2}-\frac{1}{3}p$, and find the probability distribution of X. [3]

(b) Given that $\mathrm{E}(X)=5$, find the exact value of p. [2]

(c) Using the value of p found in part **(b)**, find the exact value of $\mathrm{Var}(X)$. [3]

(a) [3]

$\mathrm{P}(X=3)$
$=\mathrm{P}(\text{tail and face is '3'})+\mathrm{P}(\text{head and face is '1'})$
$=(1-p)\frac{1}{2}+p\left(\frac{1}{6}\right)=\frac{1}{2}-\frac{1}{3}p$ (shown)

$\mathrm{P}(X=1)=\mathrm{P}(\text{tail and face is '1'})=\frac{1}{6}(1-p)$

$\mathrm{P}(X=2)=\mathrm{P}(\text{tail and face is '2'})=\frac{1}{3}(1-p)$

$\mathrm{P}(X=6)=\mathrm{P}(\text{head and face is '2'})=\frac{1}{3}p$

$\mathrm{P}(X=9)=\mathrm{P}(\text{head and face is '3'})=\frac{1}{2}p$

The probability distribution of X is:

x	1	2	3	6	9
$\mathrm{P}(X=x)$	$\frac{1}{6}(1-p)$	$\frac{1}{3}(1-p)$	$\frac{1}{2}-\frac{1}{3}p$	$\frac{1}{3}p$	$\frac{1}{2}p$

As with all word problems, do take time to read the question carefully to understand the scenario before attempting the question.

Getting the correct probability distribution of X is crucial – any wrong probability will affect subsequent parts.

(b) [2]

$$\mathrm{E}(X)=5$$

$$\frac{1}{6}(1-p)+2\left(\frac{1}{3}(1-p)\right)+3\left(\frac{1}{2}-\frac{1}{3}p\right)+6\left(\frac{1}{3}p\right)+9\left(\frac{1}{2}p\right)=5$$

$$\frac{7}{3}+\frac{14}{3}p=5$$

$$p=\frac{4}{7}$$

Recall : $\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$

Generally well done.

(c) [3]

x	1	2	3	6	9
$\mathrm{P}(X=x)$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{13}{42}$	$\frac{4}{21}$	$\frac{2}{7}$

$$\mathrm{E}\left(X^2\right)=\frac{1}{14}+2^2\left(\frac{1}{7}\right)+3^2\left(\frac{13}{42}\right)+6^2\left(\frac{4}{21}\right)+9^2\left(\frac{2}{7}\right)=\frac{234}{7}$$

$$\mathrm{Var}\left(X\right)=\mathrm{E}\left(X^2\right)-[\mathrm{E}(X)]^2=\frac{234}{7}-5^2=\frac{59}{7}$$

As the exact value of $\mathrm{Var}(X)$ was required, the use of the graphing calculator is not allowed. Hence working showing how $\mathrm{E}\left(X^2\right)$ was obtained is required.

8 **(a)** It is given that X is the number of times a student is late for school in a year and Y is the student's performance in the Mathematics Examination. The product moment correlation coefficient of a bivariate sample (x_i, y_i), $i = 1, 2, \ldots, n$, for n students is r.

State, giving a reason, whether each of the following statements is true or false.

(i) When the value of r is zero, it can be implied that the variables X and Y are not related. [1]

(ii) When the value of r is -1, it can be implied that late-coming causes poor performance in the subject. [1]

(b) The table below shows the daily sales of cups of iced coffee in a week by a shop and the maximum daily temperature.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Temperature ($°C$), t	30.5	31.3	31.9	34.8	25.9	28.5	29.2
Daily sales, y	95	101	115	79	81	86	88

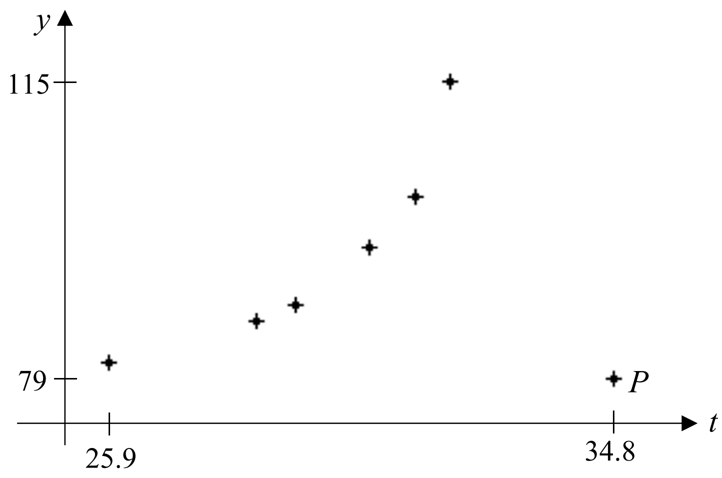
(i) Sketch a scatter diagram of y against t, labelling the axes clearly. [2]

One of the values of y appears to be incorrect.

(ii) Indicate the corresponding point on your diagram by labelling it P. Omitting P, find the equation of the least squares regression line of $\frac{1}{y}$ on t, and the value of the product moment correlation coefficient between $\frac{1}{y}$ and t. Comment on this value. [5]

(iii) Use an appropriate regression line to give an estimate of the daily sales when the temperature is 20.4 $°C$.
State, with a reason, whether the estimate is reliable. [2]

(a)(i) **[1]**	False. $r = 0$ indicates that X and Y are not linearly correlated, but they may have a non-linear relation.	Students are reminded to state whether the statement is *True* or *False*. It is important to remember that the value of r is a measurement of the **linear** correlation between two variables.
(a)(ii) **[1]**	False. The value of r measures strength of linear correlationship, it does not not indicate any causal relationship betweeen X and Y.	Similarly, the value of r does not imply any causal relationship or dependence between the two variables.

(b)(i) **[2]**		Students are reminded to label the max/min end points on both the axes. Students should also take note of the relative positions of the data points. (Note: the data points are not equally spaced along the horizontal axis)
(b)(ii) **[5]**	Label P (see diagram in **(b)(i)**). From GC, the least squares regression line $\frac{1}{y}$ on t is $\frac{1}{y}=-0.0005611311906t+0.0273247901$ $\frac{1}{y}=-0.000561t+0.0273$ (3 s.f.) $r=-0.934$ is very close to -1. Hence, there is a *strong*, *negative* linear correlation between $\frac{1}{y}$ and t.	Students are reminded to leave their final answers to 3 s.f. In the description of the *linear* correlation, the *strength* and *sign* must be mentioned.
(b) **(iii)** **[2]**	When t = 20.4, $\frac{1}{y}=-0.0005611311906(20.4)+0.0273247901$ $y=62.98$ Hence, the estimated daily sales is 63. Since t = 20.4 is outside the given temperature range $25.9\le t\le 31.9$, extrapolation is used to estimate. Hence, the estimation is not reliable.	Students are reminded to use at least 5 s.f in their intermediate workings, otherwise the accuracy of the answer might be affected. It is important to state that it is the **value of *t*** that is outside the given data range for the estimation to be reliable.

9 **(a)** The masses of a randomly chosen bolt and a randomly chosen nut are denoted by M grams and W grams respectively. M and W are independent random variables with the distributions N(272, 8^2) and N(98, 5^2) respectively.

(i) Find the range of values of a for which $P(a < M < 275) > 0.2024$. [3]

(ii) Calculate the probability that twice the mass of a randomly chosen bolt differs from the total mass of 5 randomly chosen nuts by less than 80 grams. [3]

(b) Bolts are manufactured to fit into holes in steel plates. The bolts have diameters, in cm, that follow the distribution N(2.65, 0.03^2) and the diameters of the holes, in cm, follow the distribution N(2.72, 0.02^2). A manufacturer sells boxes of twenty pairs of these bolts and steel plates, where each pair consists of one randomly selected bolt and one randomly selected steel plate. A pair is acceptable only if the diameter of the hole in the steel plate is at least 0.02 cm larger, but no more than 0.15 cm larger, than the diameter of the bolt. Use a normal distribution to estimate the probability that the average number of acceptable pairs in 50 boxes is more than 18. [4]

(a)(i) **[3]**	$M \sim \text{N}(272, 8^2)$ $\text{P}(a < M < 275) > 0.2024$ $\text{P}(M < 275) - \text{P}(M < a) > 0.2024$ $\text{P}(M < a) < \text{P}(M < 275) - 0.2024$ $= 0.44377$ From GC, $\text{P}(M < 270.8687) = 0.44377$ Thus, $0 < a < 271$ (3sf)	You are reminded of the need for clear presentation in your answers. As a is not an integer, students should not be using GC function : $\boxed{\text{2nd}}$$\boxed{\text{table}}$ in solving the question. Many forget that non-exact numerical answer ought to be corrected to 3 sf. Many wrongly wrote $\text{P}(a < M < 275) = \text{P}(M < 275) - \text{P}(a < M)$.
(a)(ii) **[3]**	Let $R = 2M - (W_1 + W_2 + W_3 + W_4 + W_5)$. $\text{E}(R) = 2\text{E}(M) - 5\text{E}(W) = 2(272) - 5(98) = 54$ $\text{Var}(R) = 4\text{Var}(M) + 5\text{E}(W) = 4(64) + 5(25) = 381$ Hence, $R \sim \text{N}(54, 381)$ Required probability $= \text{P}(\lvert R \rvert < 80)$ $= \text{P}(-80 < R < 80)$ $= 0.909$ (3 sf)	It is advisable to define a new r.v. in terms of M and W, but **(i)** do not use Z, **(ii)** do not include modulus expression at this stage. $\text{P}(\lvert R \rvert < 80)$ and $\text{P}(R < 80)$ may give similar numerical answers (as the difference in this case is very small), but for correctness, only the former version is acceptable. Some students may have expanded $\text{P}(\lvert R \rvert < 80)$ wrongly in calculation, but for reason mentioned in previous para., the final answer is not affected. This too will be penalized. E.g. $\text{P}(\lvert R \rvert < 80) \neq \text{P}(R < 80) + \text{P}(R < -80)$ $\text{P}(\lvert R \rvert < 80) \neq \text{P}(R < 80) + \text{P}(R > -80)$

(b) **[4]**	Let the diameter of a bolt be X cm, and the diameter of a hole be Y cm. $X \sim \mathrm{N}(2.65, 0.03^2)$; $Y \sim \mathrm{N}(2.72, 0.02^2)$ $Y - X \sim \mathrm{N}(0.07, 0.0013)$ $\mathrm{P}(0.02 < Y - X < 0.15) = 0.903990211$ Let V be the number of acceptable pairs in a box of 20. $V \sim \mathrm{B}(20,\ 0.903990211)$ $\mathrm{E}(V) = 18.0798184$ $\quad$ $\mathrm{Var}(X) = 1.73584$ **Method 1** Since sample size 50 is large, by Central Limit Theorem, $\overline{V} \sim \mathrm{N}\left(18.0798184,\ \frac{1.73584}{50}\right)$ approximately i.e. $\overline{V} \sim \mathrm{N}(18.0798184,\ 0.0347165)$ approximately Required probability = $\mathrm{P}\left(\overline{V} > 18\right) = 0.666$ (3sf) **Method 2** Let $K = V_1 + V_2 + ... + V_{50}$. Since sample size 50 is large, by Central Limit Theorem, $K \sim \mathrm{N}(50 \times 18.0798184,\ 50(1.73584))$ approximately i.e. $K \sim \mathrm{N}(903.99092,\ 86.792)$ approximately Required probability = $\mathrm{P}(K > 900) = 0.666$ (3sf)	Many students seem to be not too familiar with the application of CLT. Such questions are pretty common. Do learn it well, taking care of the presentation requirements. Gentle reminder: As CLT involves either $\sum V_i$ or $\frac{1}{n}(\sum V_i)$, it is therefore better to first define V properly, as demonstrated in the suggested solution. There were instances where the students tried defining $W \sim \mathrm{B}(1000,\ 0.90399)$, and then solve for $P(W > 900)$ directly. This is outright <u>ignoring</u> the instruction "use a normal distribution to estimate…. ". Some students try use a normal distribution to estimate $W \sim \mathrm{B}(1000,\ 0.90399)$. However, this is only possible under stringent conditions (which need to be explicitly discussed in the presentation), which is out of syllabus.

10 **(a)** A company has a machine designed to fill bags with, on average, μ_0 kg of salt. The mass of salt in a randomly chosen bag has a normal distribution with population standard deviation denoted by σ kg. The production manager wishes to investigate if the machine is adjusted correctly. He takes a sample of n bags and carries out a hypothesis test at the 1% level of significance.

(i) State null and alternative hypotheses for the manager's test, defining any parameters you use. [2]

(ii) Find in terms of μ_0, σ and n, the critical region(s) for this test. [4]

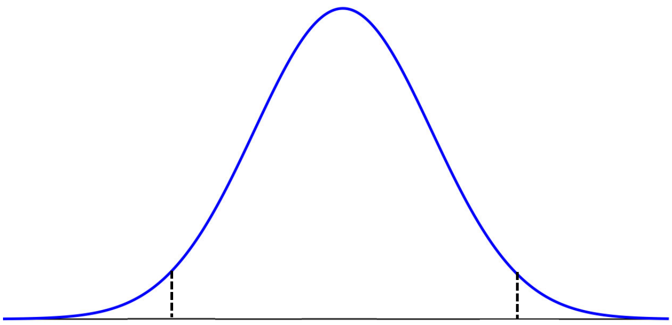
(b) The company has a different machine which fills bags with, on average, 25 kg of low sodium salt. One of the company's production supervisor has reported that some of the workers suspect the machine is no longer set correctly, and the average mass of low sodium salt in the bags may in fact be more than 25 kg. The production supervisor decides to carry out a hypothesis test at the 0.5% level of significance with a random sample of 80 bags of low sodium salt. Summary data for the mass, y kg, of low sodium salt in these bags is as follows.

$$n = 80 \qquad \sum(y-25) = 27.2 \qquad \sum(y-25)^2 = 85.1$$

(i) Carry out the test and state the conclusion of the test in the context of the question. [5]

(ii) State, giving a reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

(a)(i) **[2]**	Let μ kg be the population mean mass of salt in a bag. Null hypothesis, $H_0 : \mu = \mu_0$ Alternative hypothesis, $H_1 : \mu \neq \mu_0$	Most students can state the hypotheses correctly. However, a high number of students missed out the word "population" when defining μ, which students are strongly reminded not to forget.
(ii) **[4]**	Let X kg be the mass of salt in a bag. $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$ Perform a 2-tail test at 1% significance level. Under H_0, $\bar{X} \sim N\left(\mu_0, \dfrac{\sigma^2}{n}\right)$. $p\text{-value} = \begin{cases} 2P(\bar{X} \le \bar{x}) & \text{if } \bar{x} \le \mu_0 \\ 2P(\bar{X} \ge \bar{x}) & \text{if } \bar{x} \ge \mu_0 \end{cases}$ To reject H_0 at the 1% significance level, $p\text{-value} \le 0.01$	Quite a few students struggled with this part with the following common mistakes: 1. Notations issues like $P(\bar{X} \le \mu)$, $P(X \le \mu_0)$, $P(X \neq \mu)$, etc. 2. Missing out "2" for the p-value. 3. Use CLT when it is not required. 4. Gave values as the final answer instead of regions. 5. Standardize wrongly as $\dfrac{\bar{x}-\mu_0}{\sigma}$, $\dfrac{\bar{x}-\mu_0}{\sigma^2/n}$ 6. Use $\bar{X} \sim N\left(\mu_0, \dfrac{n}{n-1}\sigma^2\right)$ 7. Write σ as "6" and so 2.5758σ became 2.58. Students are reminded that in order to end up with regions, they need to use inequality in their working or indicate clearly on the diagram if they are using the critical values.

	$\Rightarrow 2\mathrm{P}\left(\bar{X} \le \bar{x}\right) \le 0.01$ or $2\mathrm{P}\left(\bar{X} \ge \bar{x}\right) \le 0.01$ $\Rightarrow \mathrm{P}\left(Z \le \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right) \le 0.005$ or $\mathrm{P}\left(Z \ge \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right) \le 0.005$ $\Rightarrow \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \le -2.57583$ or $\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \ge 2.57583$ $\Rightarrow \bar{x} \le \mu_0 - \dfrac{2.576\sigma}{\sqrt{n}}$ or $\bar{x} \ge \mu_0 + \dfrac{2.576\sigma}{\sqrt{n}}$ The critical regions for this test such that H_0 is rejected in favour of H_1 at the 1% level of significance is $\left(0, \mu_0 - \dfrac{2.58\sigma}{\sqrt{n}}\right] \cup \left[\mu_0 + \dfrac{2.58\sigma}{\sqrt{n}}, \infty\right)$ (3 s.f. for the coeff). Alternatively, let the critical regions be $(0,\ a] \cup [b\ , \infty)$, where *a* and *b* are as shown below.  Then we have $2\mathrm{P}\left(\bar{X} \le a\right) = 0.01$ or $2\mathrm{P}\left(\bar{X} \ge b\right) = 0.01$ $\Rightarrow \mathrm{P}\left(Z \le \dfrac{a-\mu_0}{\sigma/\sqrt{n}}\right) = 0.005$ or $\mathrm{P}\left(Z \ge \dfrac{b-\mu_0}{\sigma/\sqrt{n}}\right) = 0.005$ $\Rightarrow \dfrac{a-\mu_0}{\sigma/\sqrt{n}} = -2.57583$ or $\dfrac{b-\mu_0}{\sigma/\sqrt{n}} = 2.57583$ $\Rightarrow a = \mu_0 - \dfrac{2.576\sigma}{\sqrt{n}}$ or $b = \mu_0 + \dfrac{2.576\sigma}{\sqrt{n}}$ Hence, the critical regions for this test is $\left(0, \mu_0 - \dfrac{2.58\sigma}{\sqrt{n}}\right] \cup \left[\mu_0 + \dfrac{2.58\sigma}{\sqrt{n}}, \infty\right)$ (3 s.f. for the coeff).	

(b)(i) **[5]**	Let Y kg be the mass of low sodium salt in a bag. $\bar{y} = \dfrac{27.2}{80} + 25 = 25.34$ $s^2 = \dfrac{1}{79}\left(85.1 - \dfrac{27.2^2}{80}\right) = \dfrac{75.852}{79} \approx 0.960152$ $H_0: \mu = 25$ vs $H_1: \mu > 25$ Perform a 1-tail test at 0.5% significance level. Under H_0, $\bar{Y} \sim N\left(25, \dfrac{s^2}{n}\right)$ approx. by Central Limit Theorem since $n = 80$ is large. From the sample, $\bar{y} = 25.34$. Using a z-test, p-value $= P\left(\bar{Y} \geq 25.34\right) \approx 0.000956$ Since p-value = 0.000956 (3 sf) < 0.005, we reject H_0 and conclude there is sufficient evidence, at the 0.5% level of significance, that the mean mass of low sodium salt in the bags is more than 25 kg.	This part is pretty well done with many students demonstrated that they had put in effort to remember the 5 steps in carrying out a hypothesis test. However, a handful of students still committed some of these common mistakes: 1. $s^2 = \dfrac{80}{80-1}\left(\dfrac{85.1}{80}\right)$ 2. $s^2 = \dfrac{1}{80-1}\left(27.2^2 - \dfrac{85.1^2}{80}\right)$ 3. $s^2 = \dfrac{1}{80-1}\left(85.1 - \dfrac{25.34^2}{80}\right)$ 4. $0.000956 > 0.0005$ 5. $\bar{Y} \sim N(25, 0.960152)$ 6. $\bar{Y} \sim N\left(25.34, \dfrac{0.960152}{80}\right)$
(ii) **[1]**	No assumption about the population is needed. Since sample size 80 is large, by Central Limit Theorem, $\bar{Y}$ follows a normal distribution approximately.	Not well done as many students committed the following common mistakes: 1. Used assumptions meant for binomial distribution, like mass of salt in the bags are independent of each other, constant probability of success. 2. Tried to mention sample being random or being a good representative of the population. 3. Missing out the words "sample" or "sample mean". 4. Mentioned that population is large to apply CLT. 5. Assume the sample size = 80 is large when it is in fact large.

11 Two brothers Kai and Leo are duathlon (running and cycling) athletes who train regularly. For each day, the probability that Kai cycles is $\frac{3}{4}$, the probability that he runs is $\frac{3}{5}$ and the probability he does both is p.

(a) Write down, in terms of p, the probability that, on one day, Kai either runs or cycles but not both. [1]

(b) Find the range of possible values of p. [2]

On average, Leo cycles 5 out of 7 days in a week. The probability that Leo cycles when Kai cycles is 0.9.

(c) Find the probability that Leo cycles when Kai does not. [3]

(d) State, in context, two assumptions needed for the number of days Leo cycles over a period of 5 weeks to be well modelled by a binomial distribution. [2]

Assume now that the number of days Leo cycles in 5 weeks has a binomial distribution.

(e) Find the probability that, in the 5 weeks, Leo cycles on at least 20 days but fewer than 30 days. [2]

(f) Find the probability that, in the 5 weeks, there are 5 days in which both brothers do not cycle and they both cycle on the other days. [2]

(a) **[1]**	$\mathrm{P}(\text{Kai cycles only})+\mathrm{P}(\text{Kai runs only})=\left(\frac{3}{4}-p\right)+\left(\frac{3}{5}-p\right)$ $=\frac{27}{20}-2p$	This is generally well done.
(b) **[2]**	$\mathrm{P}(\text{Kai cycles or runs})=\frac{3}{4}+\frac{3}{5}-p$ When p is minimum, $\frac{3}{4}+\frac{3}{5}-p=1 \quad \Rightarrow \quad p=\frac{7}{20}$ Hence $\frac{7}{20}\le p\le \min\left\{\frac{3}{5},\frac{3}{4}\right\}$, i.e. $\frac{7}{20}\le p\le \frac{3}{5}$	Many students found the minimum value of p and wrote down 1 as the upper bound. This is clearly wrong because p must be smaller than both $\frac{3}{5}$ and $\frac{3}{4}$.
(c) **[3]**	Let K be the event that Kai cycles, and L the event that Leo cycles. It's given $\mathrm{P}(L\mid K)=\frac{9}{10}$, and here we want to find $\mathrm{P}(L\mid K')$: $\mathrm{P}(L\mid K')=\frac{\mathrm{P}(L\cap K')}{\mathrm{P}(K')}$ $=\frac{\mathrm{P}(L)-\mathrm{P}(L\cap K)}{\mathrm{P}(K')}$ $=\frac{\mathrm{P}(L)-\mathrm{P}(K)\cdot\mathrm{P}(L\mid K)}{\mathrm{P}(K')}=\frac{\frac{5}{7}-\frac{3}{4}\cdot\frac{9}{10}}{\frac{1}{4}}=\frac{11}{70}$	A lot of students stopped at $\mathrm{P}(L\cap K')$. Take note that the required probability is actually the conditional probability $\mathrm{P}(L\mid K')$ instead.

	Or from tree diagram, $\mathrm{P}(L)=\mathrm{P}(K)\cdot\mathrm{P}(L\mid K)+\mathrm{P}(K')\cdot\mathrm{P}(L\mid K')$ $\frac{5}{7}=\frac{3}{4}\cdot\frac{9}{10}+\frac{1}{4}\mathrm{P}(L\mid K')$ $\mathrm{P}(L\mid K')=\dfrac{\frac{5}{7}-\frac{3}{4}\cdot\frac{9}{10}}{\frac{1}{4}}=\frac{11}{70}$	
(d) **[2]**	Assume that: 1. The probability that Leo cycles on a day is constant at $\frac{5}{7}$ for all days over the 5 weeks. 2. The event that Leo cycles on a day is independent of the event that he cycles on another day.	1. A handful of students lost mark for not specifying the probability that Leo cycles refers to probability he cycles **on a day**. 2. Some made the mistakes in stating that the **probability** is independent rather than the **event** is independent.
(e) **[2]**	Let X be the number of days in 5 weeks that Leo cycles. $X\sim\mathrm{B}\left(35,\frac{5}{7}\right)$ $\mathrm{P}(20\le X<30)=\mathrm{P}(X\le 29)-\mathrm{P}(X\le 19)$ $=0.937\quad(3\text{sf})$	This is generally well done.
(f) **[2]**	$\mathrm{P}(K'\cap L')=\mathrm{P}(K')\cdot\mathrm{P}(L'\mid K')=\frac{1}{4}\left(1-\frac{11}{70}\right)=\frac{59}{280}$ $\mathrm{P}(K\cap L)=\mathrm{P}(K)\cdot\mathrm{P}(L\mid K)=\frac{3}{4}\cdot\frac{9}{10}=\frac{27}{40}$ Required probability is ${}^{35}C_5\left(\frac{59}{280}\right)^5\left(\frac{27}{40}\right)^{30}=0.00102\quad(3\text{sf})$	This part is poorly done as many students wrongly assume that the random variable is modelled by a binomial distribution.