HIGH ORDER FINITE DIFFERENCE SCHEMES FOR SOME OBSTACLE PROBLEMS

OLIVIER BOKANOWSKI AND KRISTIAN DEBRABANT

Abstract. New finite difference schemes, using Backward Differentiation Formula, are proposed for the approximation of one-dimensional non-linear diffusion equations with an obstacle term. Stability estimates are obtained for one of the schemes. Numerical experiments are given to illustrate the convergence of the proposed schemes and compared to more standard schemes. Application to the American option problem in mathematical finance is given. We also propose two related academic problems with explicit solution that can serve as reference test cases.

Keywords: diffusion equation, obstacle equation, viscosity solution, numerical methods, Crank Nicolson scheme, Backward Differentiation Formula, high order schemes.

1. Introduction

We consider a second order partial differential equation with an obstacle term, of the following form:

$$\min(u_t + \mathcal{A}u, u - \varphi(x)) = 0, \quad t \in (0,T), \quad x \in \Omega, \tag{1a}$$

$$u(0,x) = u_0(x), \quad x \in \Omega. \tag{1b}$$

with

$$\mathcal{A}u := -\frac{1}{2}\sigma^2(t,x)u_{xx} + b(t,x)u_x + r(t,x)u \tag{2}$$

where σ, b, r are assumed to be regular functions. The functions φ and u_0 are assumed to be Lipschitz continuous, with $u_0 \geq \varphi$. The set Ω can be either $\Omega = \mathbb{R}$, or $\Omega = [a,b] \subset \mathbb{R}$. In the second case we will consider (1) complemented by boundary conditions of Dirichlet type:

$$u(t,a) = u_\ell(t), \quad t \in (0,T) \tag{3}$$

$$u(t,b) = u_r(t), \quad t \in (0,T). \tag{4}$$

In the recent years there has been a lot of interest in the approximation of such obstacle problems. In particular, it is connected to optimal stochastic stopping time problems, such as the American put option problem, which is the particular case when $\Omega = [0,\infty[$, $\mathcal{A}u = -\frac{1}{2}\lambda^2 x^2 u_{xx} - rxu_x + ru$, with constant coefficients $\lambda > 0, r > 0$ and with initial data same as the obstacle function: $u_0(x) \equiv \varphi(x) = \max(K-x, 0)$. For this problem it is known [7] that the solution presents a singular point $x_s(t)$ moving with time, such that $u(t,x) = \varphi(x)$ for $x \leq x_s(t)$, and $u(t,x) > \varphi(x)$ for $x > x_s(t)$.

The convergence of a finite element scheme, in the american option setting, has been proved by Jaillet, Lamberton and Lapeyre in [11]. For a comprehensive study of finite difference schemes as well as finite element schemes in this context, we refer to Achdou and Pironneau [1]. A finite volume method is also precisely studied in Berton and Eymard [2].

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The solution u can be defined as the unique continuous viscosity solution [5]. Now we remark that in the case of $u_0 \equiv \varphi$, and for an operator of the form

$$\mathcal{A}u := -\frac{1}{2}\sigma^2(x)u_{xx} + b(x)u_x + r(x)u \tag{5}$$

with coefficients which do not depend on time, then u is also the unique viscosity solution of the following Hamilton-Jacobi (HJ) equation:

$$u_t + \min(0, \mathcal{A}u) = 0, \quad t \in (0,T), \quad x \in \Omega, \tag{6a}$$
$$u(0,x) = \varphi(x), \quad x \in \Omega. \tag{6b}$$

The equivalence between (1) and (6) was signaled to us by R. Eymard, and a sketch of the proof of independent interest will be given in appendix A.

In this paper we will first revisit some Crank-Nicolson schemes, and will propose some new second and third order order schemes for directly approximating the obstacle problem (1), for which furthermore some stability results can be proved.

More precisely, in Section 2, we consider two types of Crank-Nicolson schemes for the obstacle problem, that only differ in the way to treat the obstacle term. We show that a naive setting of the scheme may numerically switch back to a first-order behavior. We then propose a CN scheme with a particular treatment for the obstacle term in order to have a second order consistency error. Non trivial stability results for Crank-Nicolson scheme in the L^∞ exists for the heat equation [14]. But to the best of our knowledge, there is no stability result for the CN scheme in the case of the obstacle problem (unless a restrictive CFL condition holds). We will show that it can numerically lead to a first order behavior for high CFL numbers (ratio of time step τ over squared mesh step h^2, hereafter referred as the CFL number).

We then present new schemes for the obstacle problem, based on the use of Backward Difference Formula, for which second or third order convergence can be expected. In Section 3, we then consider Backward Differentiation Formula (BDF) for the approximation of the derivative u_t. We focus on second and third order BDF schemes. They are still non-linear and implicit but we explain how they can be solved by a simple Newton-type algorithm. An L^2 error analysis is furthermore carried out. This analysis utilizes techniques similar to the stability proof for a "Gear" scheme [1], but here we also deal with the non-linearity coming from the obstacle term.

In Section 4, we consider an other type of BDF types where the whole term $u_t + \mathcal{A}u$ is approximated by a BDF approach (second order and third order schemes are considered). We are then led to a new scheme that requires only an implicit treatment of linear terms (for the approximation of the diffusion part) and an explicit treatment of the max term. Numerical results are given, showing good behavior of the schemes. However, proving the stability is still an issue.

In Section 5 we focus on some numerical examples. Two basic models are introduced, with explicit solutions, yet very close to the american option model. These models will alllow us to study precisely and more easily the convergence behavior. In particular, for the new proposed BDF schemes, we observe second order behavior, in the L^∞ norm, *even near the singular region of the solution.* Third order behavior can also be observed but only for a particular type of model problem with smoother solutions than for the american option problem.

Our study concerns here only one-dimensional obstacle problems, but the proposed schemes based on BDF approximations can be extended to higher dimensions, which is the subject of ongoing works.

2. Crank-Nicolson finite difference type schemes

In this section we present Crank-Nicolson schemes for a diffusion plus obstacle equation, and show some numerical difficulties that may arise when considering such scheme for low CFL numbers.

Let us consider a domain $\Omega = [X_{min}, X_{max}]$ and a uniform mesh with $J \geq 1$ points inside:

$$x_j = X_{min} + jh, \quad j = 0, \dots, J+1,$$

where $h := \frac{X_{max} - X_{min}}{J+1}$. Let $N \geq 1$, $\tau = \frac{T}{N}$ and $t_n = n\tau$.

Denoting u_j^n an approximation of $u(t_n, x_j)$, we consider a centered finite difference approximation for the operator $\mathcal{A}u$:

$$(\mathcal{A}u)(t_n, x_j) \simeq \tag{7}$$
$$\frac{1}{2}\sigma^2(t_n, x_j)\left(\frac{-u_{j-1}^n + 2u_j^n - u_{j+1}^n}{h^2}\right) + b(t_n, x_j)\frac{u_{j+1}^n - u_{j-1}^n}{2h} + r(t_n, x_j)u_j^n$$

For simplicity we assume there is no time dependency in the coefficients of $\mathcal{A}$. The diffusion part will always dominate the advection part ($\frac{1}{2}\frac{\sigma^2}{h^2} \geq \frac{b}{2h}$) to avoid stability issues with the centered approximation. Note that for the American put option problem this requires $X_{min} \geq \frac{r}{\sigma^2}h$.

Let us denote by $Au^n + q(t_n)$ the approximation of $\mathcal{A}u(t_n, .)$ on a given set of grid points, with $u^n = (u_1^n, \dots, u_J^n)^T$ and with given Dirichlet boundary conditions, for $n = 0, \dots, N$:

$$u_0^n = u_\ell(t_n) \quad \text{and} \quad u_{J+1}^n = u_r(t_n). \tag{8}$$

(The vector $q(t_n)$ may depend of the time because of the time dependency in the boundary conditions (8).) We have second order consistency in space, that is,

$$(Au^n + q(t_n))_j = (\mathcal{A}u)(t_n, x_j) + O(h^2).$$

A first Crank Nicolson (CN) scheme is, for the obstacle equation (1): for $n = 0, \dots, N-1$:

$$\mathcal{S}_j^{1,n} := \min\left(\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{2}(Au^{n+1} + Au^n)_j + q(t_{n+1/2})_j,\ u_j^{n+1} - g_j\right) = 0,$$
$$1 \leq j \leq J \tag{9}$$

where we have denoted g_j the obstacle values at mesh points:

$$g_j := \varphi(x_j),$$

together with boundary conditions (8), and initialized with

$$u_j^0 := u_0(x_j), \quad 1 \leq j \leq J. \tag{10}$$

This scheme can be found in the literature [1].

Looking now at equation (6), an other possible CN scheme is

$$\mathcal{S}_j^{2,n} := \min\left(\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{2}(Au^{n+1} + Au^n)_j + q(t_{n+1/2})_j,\ \frac{u_j^{n+1} - u_j^n}{\tau}\right) = 0$$
$$1 \leq j \leq J \tag{11}$$

initialized with $u_j^0 := u_0(x_j)$. Because $\tau > 0$, this scheme is also equivalent to

$$\min\left(\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{2}(Au^{n+1} + Au^n)_j + q(t_{n+1/2})_j,\ u_j^{n+1} - u_j^n\right) = 0$$
$$1 \leq j \leq J \tag{12}$$

Remark 2.1. For both schemes, the unkown u^{n+1} can be obtained by using fix point methods (there exists a unique solution of the obstacle problem $\min(Bx-b, x-g) = 0$ as soon as, for instance, B is diagonaly dominant with $B_{ii} > 0$ (see [1]).

Remark 2.2. In the case when furthermore B is an M-matrix in the sense that $B_{ii} > \sum_{j\neq i} |B_{ij}|$, $\forall i$, and $B_{ij} \leq 0$, $\forall i \neq j$, then a Newton-like algorithm [3] can be implemented, which is particularly efficient for solving obstacle problems exactly (up to machine precision) in a few number of iterations. We refer also to [10] for convergence of semi-smooth Newton methods or related algorithms applied to solve discretized PDE obstacle problems.

The scheme (11) is naturally second order consistent in time and space. Precisely, taking v a regular function and denoting $v_j^n := v(t_n, x_j)$, we see that

$$\frac{v_j^{n+1} - v_j^n}{\tau} = v_t(t_{n+1/2}, x_j) + O(\tau^2 \|v_{3t}\|_\infty),$$

and

$$\begin{aligned} &\frac{v_j^{n+1} - v_j^n}{\tau} + \frac{1}{2}(Av^{n+1} + Av^n)_j + q(t_{n+1/2})_j \\ &\quad = (v_t + \mathcal{A}v)(t_{n+1/2}, x_j) + O(\tau^2\|v_{3t}\|_\infty) + O(h^2(\|v_{3x}\|_\infty + \|v_{4x}\|_\infty)), \end{aligned}$$

and hence

$$S_j^{2,n}(v) = \min(v_t + \mathcal{A}v, v_t)(t_{n+1/2}, x_j) + O(\tau^2 + h^2).$$

On the other hand, the scheme (9) is only first order consistent in time. Indeed, for the linear part $u_t + \mathcal{A}u$, the CN scheme is second order consistent only precisely at time $t_{n+1/2}$, while the obstacle term $v_j^{n+1} - g_j$ is evaluated at time t_{n+1} so is only first order consistent with the value $v(t_{n+1/2}, x_j) - g_j$.

The following assertions hold:

Lemma 2.3. *(i) If $u^{n+1} \geq u^n$ for the scheme (9) (with $u_j^n \geq g_j$), then u^{n+1} is also solution of the scheme (11) (hence both schemes give identical values).*

(ii) This is the case in particular when the vector $q(t) = q$ does not depend on time (that is the case of the American put option), the matrix $I - \frac{\tau}{2}A$ is positive componentwise (which is the case under an appropriate CFL condition of the form $\frac{\tau}{h^2} \leq c_0 := \frac{2}{\max_{j=1}^J \sigma^2(x_j)}$), and if $I + \frac{\tau}{2}A$ is an M-matrix (which is the case here).

The proof of this Lemma is given in appendix B.

This explains why the first order consistent CN scheme (9) can give second order results, in particular for low CFL number, since then the scheme is equivalent to a second order consistent scheme with respect to the second equivalent PDE (6).

Next, to verify the expected orders, we have tested the CN schemes numerically on the American put option model with parameters $\lambda = 0.2$, $r = 0.1$, $K = 100$ and $T = 1$.

The errors where computed on a discrete set of 13 reference values[1] in $[80, 110]$ spaced by 2.5 (this interval contains the singular point of the solution u.) Results are given in Table 1 with discretization parameters $J = N$, and in Table 2 with discretization parameters $N = J/10$ (this second case corresponds to higher CFL numbers). Note that the quite restrictive CFL condition in part (ii) of Lemma 2.3 is not fulfilled.

However, for lower N values (higher CFL numbers) the CN scheme is no more second order and goes back to first order behavior. This is illustrated in Table 2. In particular we observe that the pointwise inequality $u^{n+1} \geq u^n$ may not be true

[1] Reference values are computed using a BDF scheme of third order and with $N = J = 10240$, the method will be made precise in section 3

Mesh		Error L^1		Error L^2		Error L^∞		CPU time
J	N	error	order	error	order	error	order	
80	80	1.74E-02	-	2.16E-02	-	4.49E-02	-	0.14
160	160	3.15E-03	2.47	3.73E-03	2.54	5.25E-03	3.10	0.25
320	320	8.42E-04	1.91	9.87E-04	1.92	1.39E-03	1.92	0.52
640	640	2.23E-04	1.92	2.58E-04	1.93	3.56E-04	1.97	1.54
1280	1280	6.06E-05	1.88	6.96E-05	1.89	9.15E-05	1.96	6.65
2560	2560	1.66E-05	1.86	1.91E-05	1.86	2.69E-05	1.77	37.66

TABLE 1. Crank-Nicolson scheme (9) for (1) and $N = J$. The Crank-Nicolson scheme (11) for (6) gives same error table in this case.

(due to the fact that the amplification matrix $(I + \frac{\tau}{2}A)^{-1}(I - \frac{\tau}{2}A)$ may not have only positive coefficients anymore) .

Now, taking the obstacle function to be u^n instead of $g_j = (\varphi(s_j))$, hence solving the scheme (11), enforces that $u^{n+1} \geq u^n$. Results for the scheme (11) are similar that for scheme (9) for low CFL numbers (both schemes give identical values for the case $N = J$, here). However, for higher CFL numbers, results obtained with the scheme (11) differ and are given in Table 3. Although not second order, convergence is improved .

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	1.83E-02	-	2.47E-02	-	4.90E-02	-	0.05
160	16	4.67E-03	1.97	7.23E-03	1.77	2.18E-02	1.17	0.07
320	32	1.68E-03	1.48	2.97E-03	1.28	9.90E-03	1.14	0.13
640	64	6.89E-04	1.28	1.38E-03	1.11	4.75E-03	1.06	0.38
1280	128	2.99E-04	1.20	6.59E-04	1.06	2.31E-03	1.04	1.99
2560	256	1.37E-04	1.13	3.22E-04	1.03	1.14E-03	1.02	16.04

TABLE 2. Crank-Nicolson scheme (9) for (1), with higher CFL number

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.68E-02	-	3.11E-02	-	4.90E-02	-	0.25
160	16	6.91E-03	1.96	8.28E-03	1.91	1.14E-02	2.11	0.13
320	32	2.53E-03	1.45	2.97E-03	1.48	3.91E-03	1.54	0.18
640	64	9.35E-04	1.43	1.11E-03	1.43	1.50E-03	1.38	0.41
1280	128	3.74E-04	1.32	4.44E-04	1.32	6.07E-04	1.31	1.66
2560	256	1.56E-04	1.26	1.86E-04	1.26	2.56E-04	1.25	11.36

TABLE 3. Crank-Nicolson scheme (11) for (6), with higher CFL number

3. Implicit BDF obstacle schemes

In this section we consider BDF type approximations for the first derivative u_t, leading to implicit schemes. We aim to give in particular an implicit scheme which has the same complexity of the previous CN schemes but that gives better numerical results. Furthermore a stability analysis can be carried out for this particular scheme.

3.1. **BDF2 implicit obstacle scheme.** Our first scheme is therefore the following two-step implicit scheme, for $n \geq 1$:

$$\mathcal{H}_j^{n+1}(u) :\equiv \min\left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\tau} + (Au^{n+1} + q(t_{n+1}))_j,\ u_j^{n+1} - g_j\right) = 0$$

Such approximations for the linear term u_t, known as BDF approximations, are well known and used in various contexts [6, 9].

The use of a BDF scheme for a diffusion plus obstacle problems seems new, althouth the idea was already suggested by Seydel in [13, see page 187 and 217]. By construction the scheme has the following consistency error, when v is regular, for $v_j^n = v(t_n, x_j)$:

$$\begin{aligned} \mathcal{H}_j^{n+1}(v) &= \min(v_t + \mathcal{A}v, v - \varphi)(t_{n+1}, x_j) \\ &\quad + O(\tau^2 \|v_{3t}\|_\infty) + O(h^2(\|v_{3x}\|_\infty + \|v_{4x}\|_\infty)). \end{aligned} \tag{13}$$

Hence the scheme is second order consistent in time (and here also with space) with respect to the obstacle problem (1).

This consistency error justifies the introduction of BDF schemes, that precisely approximate $u_t + \mathcal{A}u$ at time t_{n+1} without the need of other particular requirements (like $v_t + \mathcal{A}v = 0$ at previous times, which would not hold in our case).

For convenience, the scheme will also be written as follows. Let $\min(X, Y) := (\min(x_j, y_j))_j$ denote the minimum of two vectors $X = (x_j), Y = (y_j)$ of $\mathbb{R}^J$. The scheme is equivalent to

$$\mathcal{S}^{n+1}(u) :\equiv \min\left((I_d + \frac{2}{3}\tau A)\, u^{n+1} - \frac{4}{3}u^n + \frac{1}{3}u^{n-1} + \frac{2}{3}\tau q(t_{n+1}),\ u^{n+1} - g\right) = 0.$$

(A multiplication by $\frac{2}{3}\tau > 0$ of the left part of the min term does not change the equation.)

Newton's method. Although non-linear, this scheme can be solved by Newton's method. More precisely, denoting $B := I_d + \frac{2}{3}\tau A$ (a real valued $J \times J$ matrix) and $b := \frac{4}{3}u^n - \frac{1}{3}u^{n-1} - \frac{2}{3}\tau q(t_{n+1})$, the problem is to solve for $x \in \mathbb{R}^J$

$$\min(Bx - b, x - g) = 0 \quad \text{in } \mathbb{R}^J. \tag{14}$$

Results for the BDF second order implicit obstacle scheme (or BDF-2), are given in Tables 4, with $N = J$, and 5, with $N = J/10$. They show robustness of the scheme with higher CFL numbers, and also an improvement of the convergence with respect to the CN schemes (the order is closer to two even for high CFL numbers).

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	80	1.72E-02	-	2.14E-02	-	4.48E-02	-	0.13
160	160	3.02E-03	2.51	3.57E-03	2.58	5.06E-03	3.15	0.26
320	320	7.93E-04	1.93	9.30E-04	1.94	1.32E-03	1.93	0.49
640	640	2.06E-04	1.94	2.40E-04	1.96	3.39E-04	1.97	1.54
1280	1280	5.46E-05	1.92	6.30E-05	1.93	8.75E-05	1.95	6.43
2560	2560	1.44E-05	1.93	1.67E-05	1.92	2.64E-05	1.73	36.70

TABLE 4. BDF2-implicit scheme for Eq.(1).

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	8.23E-03	-	1.25E-02	-	3.59E-02	-	0.06
160	16	9.61E-04	3.10	1.28E-03	3.29	2.20E-03	4.02	0.09
320	32	4.18E-04	1.20	5.41E-04	1.24	8.83E-04	1.32	0.13
640	64	1.54E-04	1.44	1.92E-04	1.49	2.99E-04	1.56	0.32
1280	128	4.77E-05	1.69	5.82E-05	1.73	8.71E-05	1.78	1.14
2560	256	1.14E-05	2.07	1.39E-05	2.06	2.01E-05	2.12	6.21

TABLE 5. BDF2-implicit scheme for Eq.(1), with high CFL numbers.

3.2. **BDF3 implicit obstacle scheme.** In the same way, we propose the following three-step (BDF3) implicit scheme, for $n \geq 2$:

$$\mathcal{H}_j^{n+1}(u) :\equiv \min\left(\frac{\frac{11}{6}u_j^{n+1} - 3u_j^n + \frac{3}{2}u_j^{n-1} - \frac{1}{3}u_j^{n-2}}{\tau} + (Au^{n+1} + q(t_{n+1}))_j,\ u_j^{n+1} - g_j \right) = 0$$

As we have done for the BDF2 scheme, we can multiply the left term by 6τ, define $B := 11\,Id + 6\tau A$, and obtain then an equivalent scheme in the following form, for $n \geq 2$:

$$\mathcal{S}_j^{n+1}(u) :\equiv \min\left((Bu^{n+1})_j - 18u_j^n + 9u_j^{n-1} - 2u_j^{n-2} + 6\tau q(t_{n+1})_j,\ u_j^{n+1} - g_j \right) = 0.$$

The scheme may be initialized by any second order approximation for the first two steps u^1 and u^2 (such as a Crank Nicolson scheme, or a RK2 explicit scheme). Then the unknown u^{n+1} can be solved by using again a semi-smooth Newton's method.

Numerical results with BDF3 implicit scheme are presented in Tables 6 and 7. However, they do not show improvement with respect to the BDF2 implicit scheme. (Furthermore the stability proof of the BDF2 implicit scheme given later on does not adapt to the case of BDF3).

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	80	1.74E-02	-	2.16E-02	-	4.49E-02	-	0.29
160	160	3.16E-03	2.46	3.73E-03	2.53	5.26E-03	3.09	0.28
320	320	8.52E-04	1.89	9.99E-04	1.90	1.40E-03	1.91	0.55
640	640	2.24E-04	1.93	2.60E-04	1.94	3.57E-04	1.97	1.60
1280	1280	6.07E-05	1.88	6.97E-05	1.90	9.15E-05	1.96	6.30
2560	2560	1.64E-05	1.89	1.88E-05	1.89	2.68E-05	1.77	37.19

TABLE 6. BDF3-implicit scheme for (1).

4. BDF-MAX SCHEMES

We now consider a BDF scheme for the direct treatment of the full operator $u_t + \mathcal{A}u$.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.25E-02	-	2.64E-02	-	4.35E-02	-	0.10
160	16	4.86E-03	2.21	5.76E-03	2.20	8.06E-03	2.43	0.12
320	32	1.49E-03	1.71	1.75E-03	1.72	2.32E-03	1.80	0.14
640	64	4.83E-04	1.62	5.65E-04	1.63	7.39E-04	1.65	0.32
1280	128	1.71E-04	1.50	2.00E-04	1.50	2.66E-04	1.47	1.17
2560	256	6.44E-05	1.41	7.59E-05	1.40	1.03E-04	1.37	6.18

TABLE 7. BDF3-implicit scheme for (1), with high CFL numbers.

Consider a first order scheme denoted S_τ to approximate $u_t+\mathcal{A}u=0$, of the form $u^{n+1}=S_\tau u^n$. Typically this can be an implicit Euler scheme, therefore solving

$$\frac{u^{n+1}-u^n}{\tau}+Au^{n+1}+q(t_{n+1})=0.$$

In that case $S_\tau u \equiv S^n_\tau u := (I_d+\tau A)^{-1}(u-\tau q(t_{n+1}))$. One could also consider a semi-Lagrangian scheme. The idea here is to use a scheme that is not too much CFL restrictive and easy to solve. Suppose there is furthermore a second (resp. third) order expansion for small τ of the form:

$$\frac{v_j^{n+1}-(\mathcal{S}_\tau v^n)_j}{\tau}=(v_t+\mathcal{A}v)(t_{n+1},x_j)+a(t_{n+1},x_j)\tau+O(\tau^2), \tag{15}$$

resp.

$$\frac{v_j^{n+1}-(\mathcal{S}_\tau v^n)_j}{\tau}=(v_t+\mathcal{A}v)(t_{n+1},x_j)+a(t_{n+1},x_j)\tau+b(t_{n+1},x_j)\tau^2+O(\tau^3). \tag{16}$$

(We use the notation $v_j^n=v(t_n,x_j)$ where v is a regular function). Then, using the idea of BDF approximations, we easily obtain the following orders, assuming (15):

$$\frac{3v_j^{n+1}-4(\mathcal{S}_\tau v^n)_j+(\mathcal{S}_{2\tau}v^{n-1})_j}{2\tau}=(v_t+\mathcal{A}v)(t_{n+1},x_j)+O(\tau^2), \tag{17}$$

or, assuming (16):

$$\frac{11v_j^{n+1}-18(\mathcal{S}_\tau v^n)_j+9(\mathcal{S}_{2\tau}v^{n-1})_j-2(\mathcal{S}_{3\tau}v^{n-2})_j}{6\tau}=(v_t+\mathcal{A}v)(t_{n+1},x_j)+O(\tau^3). \tag{18}$$

Moreover, it is not necessary to assume $v_t+\mathcal{A}v=0$ in order to obtain the above approximations.

Then a second order in time-consistent scheme is directly obtained by considering u^{n+1} solution of

$$\min\left(\frac{3u^{n+1}-4\mathcal{S}_\tau u^n+\mathcal{S}_{2\tau}u^{n-1}}{2\tau},\ u^{n+1}-g\right)=0. \tag{19}$$

Since the left part of the min can be multiplied by $\frac{2\tau}{3}$, the u^{n+1} term can be solved explicitly to obtain an explicit scheme.

In this way, we can consider

- an "explicit" two-step max scheme for $n\ge 1$:

$$u^{n+1}=\max\left(\frac{1}{3}\left(4S^n_\tau u^n-S^{n-1}_{2\tau}u^{n-1}\right),\ g\right). \tag{20}$$

  (with initialization $u^0=g$ and u^1 given by some first order scheme such as implicit Euler).

- an "explicit" three-step max scheme, for $n \geq 2$:

$$u^{n+1} = \max\left(\frac{6}{11}\big(3S_\tau^n u^n - \frac{3}{2}S_{2\tau}^{n-1}u^{n-1} + \frac{1}{3}S_{3\tau}^{n-2}u^{n-2}\big),\ g\right). \tag{21}$$

The "explicit" denomination is to emphasize the fact that the obstacle term is treated in an explicit way, to the contrary to the BDF schemes of the previous section.

Proposition 4.1. *For regular data, the scheme* (20) *has a second order consistency error $O(\tau^2)$ Similarly, the scheme* (21) *has third order consistency error $O(\tau^3)$.*

If the operator $\mathcal{A}$ is approximated by finite differences as in Section (2)*, then the spatial consistency error $O(h^2)$ is also present in both cases.*

Here we show numerical results using implicit Euler scheme for S_τ. The BDF2-max scheme is tested on Example 1 (American option), results are given in Table 8

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	80	1.78E-02	-	2.22E-02	-	4.90E-02	-	0.11
160	160	5.15E-03	1.79	6.47E-03	1.78	1.48E-02	1.73	0.22
320	320	1.52E-03	1.76	2.14E-03	1.59	6.10E-03	1.27	0.40
640	640	3.88E-04	1.97	5.33E-04	2.01	1.51E-03	2.01	0.97
1280	1280	1.18E-04	1.72	1.62E-04	1.72	4.61E-04	1.72	2.60
2560	2560	3.25E-05	1.86	4.10E-05	1.98	1.01E-04	2.19	8.64

TABLE 8. second order BDF-max scheme for (1).

For the similar BDF3 scheme, using implicit Euler scheme for $\mathcal{S}_\tau$, results are given in Tables 9 and 10.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
5120	20	2.72E-02	-	3.26E-02	-	6.16E-02	-	8.71
5120	40	8.48E-03	1.68	1.37E-02	1.25	4.29E-02	0.52	8.87
5120	80	2.39E-03	1.83	4.94E-03	1.47	1.70E-02	1.33	7.23
5120	160	7.76E-04	1.62	1.76E-03	1.48	6.22E-03	1.45	7.34
5120	320	2.54E-04	1.61	5.52E-04	1.68	1.93E-03	1.69	7.87
5120	640	8.09E-05	1.65	1.37E-04	2.02	4.42E-04	2.12	9.66
5120	1280	2.53E-05	1.68	2.94E-05	2.22	4.56E-05	3.28	13.87
5120	2560	1.14E-05	1.14	1.31E-05	1.17	1.63E-05	1.48	23.57

TABLE 9. third-order BDF-max scheme for (1) (test with fixed spatial mesh)

5. ERROR ESTIMATE FOR THE BDF2 IMPLICIT SCHEME

5.1. **Stability estimate.** Let us first start by considering an abstract obstacle problem of the form

$$\min(Bx - b, x - g) = 0 \quad \text{ for } \quad x \in \mathbb{R}^J,$$

where B is a square matrix of size I and b, g are given vectors of $\mathbb{R}^J$. We will use the following elementary result.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	80	1.68E-02	-	2.12E-02	-	4.90E-02	-	0.14
160	160	4.16E-03	2.02	4.93E-03	2.10	9.38E-03	2.39	0.24
320	320	1.20E-03	1.80	1.50E-03	1.71	3.61E-03	1.38	0.48
640	640	2.87E-04	2.06	3.51E-04	2.10	7.95E-04	2.18	1.19
1280	1280	7.34E-05	1.96	8.43E-05	2.06	1.08E-04	2.88	3.36
2560	2560	1.96E-05	1.91	2.32E-05	1.86	3.06E-05	1.82	11.61
5120	5120	5.86E-06	1.74	6.75E-06	1.78	8.68E-06	1.82	42.43

TABLE 10. third-order BDF-max scheme for Eq.(1).

Lemma 5.1. *For any matrix B, the following equivalence holds:*

$$\min(Bx-b,x-g)=0 \quad \Leftrightarrow \quad x\geq g \text{ and } \Big(\langle Bx-b,v-x\rangle \geq 0,\ \forall v\geq g\Big) \tag{22}$$

Proof. It is known [4] that if B is a positive definite symmetric matrix, the following equivalences hold:

$$\min(Bx-b,x-g)=0 \quad \Leftrightarrow \quad x \text{ solves } \min_{x\geq g} \frac{1}{2}\langle x,Bx\rangle - \langle b,x\rangle \tag{23}$$

$$\Leftrightarrow \quad x\geq g \text{ and } \Big(\langle Bx-b,v-x\rangle \geq 0,\ \forall v\geq g\Big) \tag{24}$$

When B is not symmetric, the equivalence between the min equation and (24) is still true. $\Rightarrow$: For $v\geq g$, $\langle Bx-b,v-x\rangle = \langle Bx-b,v-g\rangle + \underbrace{\langle Bx-b,g-x\rangle}_{=0}$ so is positive since $Bx-b\geq 0$ and $v-g\geq 0$.
$\Leftarrow$: By taking $v=x+\lambda e_j$ with $\lambda\to+\infty$ we get $(Bx-b)_j\geq 0$, hence $Bx-b\geq 0$. Then, $\langle Bx-b,x-g\rangle\geq 0$, and also $\langle Bx-b,x-g\rangle\leq 0$ by taking $v=g$ as a test function in the inequality. Hence $\langle Bx-b,x-g\rangle=0$. Together with $Bx-b\geq 0$, $x-g\geq 0$, this implies that $\min(Bx-b,x-g)=0$. □

The idea now is to use the inequality of Lemma 5.1 in order to obtain a stability estimate. For the linear case, it is possible for the Gear scheme - or BDF2 scheme - to obtain stability estimates in the L^2 norm (see for instance [8]). We are going to obtain similar estimates for the obstacle problem.

We write the exact scheme as

$$\min\big((I+\frac{2\tau}{3}A)u^{n+1}-\frac{4}{3}u^n+\frac{1}{3}u^{n-1}+\frac{2\tau}{3}q_{n+1},\ u^{n+1}-g\big)=0. \tag{25}$$

Assume that $v^n(x)=v(t_n,x)$ and $v^n=(v^n(x_j))$. Let $v(t,x)$ be a regular function, and $\bar\epsilon^n\in\mathbb{R}^J$ be such that

$$\frac{1}{2\tau}(3v^{n+1}-4v^n+v^{n-1})+Av^{n+1}+q_{n+1}=((v_t+\mathcal{A}v)(t_{n+1},x_j))_{1\leq i\leq I}+\bar\epsilon^n.$$

The term $\bar\epsilon^n$ corresponds to a "consistency error" for the linear part of the PDE, here written in discrete form on the grid mesh. Then we have

$$\min\big(\frac{1}{2\tau}(3v^{n+1}-4v^n+v^{n-1})+Av^{n+1}+q_{n+1}-\bar\epsilon^n,\ v^{n+1}-g\big)=0. \tag{26}$$

In the same way v^n thus satisfies a perturbed scheme:

$$\min\big((I+\frac{2\tau}{3}A)v^{n+1}-\frac{4}{3}v^n+\frac{1}{3}v^{n-1}+\frac{2\tau}{3}q_{n+1}-\frac{2\tau}{3}\bar\epsilon_n,\ v^{n+1}-g\big)=0. \tag{27}$$

Remark 5.2. Typically we expect that $\bar\epsilon_n$ is of order $O(\tau^2+h^2)$ where v is regular.

Our aim is now to show a stability estimate: control the error $\|u^n - v^n\|_2^2$ in terms of $\sum_{1\le k\le n-1} \tau\|\bar\epsilon^n\|^2$.

For a given vector $x = (x_j)_{1\le j\le J}$, let the semi-norm

$$N(x) := \left(\sum_{j=1}^{J+1} |x_j - x_{j-1}|^2\right)^{1/2} \tag{28}$$

(with the convention $x_0 := 0$ and $x_{J+1} := 0$).

We will suppose that the following coercitivity assumption: there exists constants $\eta > 0$, $\gamma \ge 0$ such that:

$$\langle e, Ae\rangle \ge \eta N(e/h)^2 - \gamma\|e\|_2^2. \tag{29}$$

Lemma 5.3. *If $a(x) \ge \eta_0 > 0$ with functions $a, b, c \in L^\infty$, and if a is Lipschitz continuous, then* (29) *holds true.*

Proof. Indeed, by direct calculation one has, in the case of $\mathcal{A}u = -u_{xx}$, $A = \frac{1}{h^2} tridiag(-1,\ 2,\ -1)$ and $\langle e, Ae\rangle = N(e/h)^2$. In the case of $\mathcal{A}u = -a(x)u_{xx} + b(x)u_x + c(x)u$,

$$A = \frac{1}{h^2} tridiag(-a_i,\ 2a_i,\ -a_i) + \frac{1}{2h} tridiag(-b_i,\ 0,\ b_i) + diag(c_i)$$

where $a_i = a(x_i)$, $b_i = b(x_i)$ and $c_i = c(x_i)$. Therefore

$$\langle e, Ae\rangle \;=\; \frac{1}{h^2}\sum_{i=1}^{J+1}(a_ie_i - a_{i-1}e_{i-1})(e_i - e_{i-1}) + \sum_{i=1}^{J} b_i(e_{i+1} - e_{i-1})e_i + \sum_{i=1}^{J} c_ie_i^2$$

Now we make use of $a_i = a_{i-1} + O(h)$ (since $a(.)$ is Lipschitz continuous), and $a_i \ge \eta_0$, to obtain:

$$\begin{aligned}\frac{1}{h^2}\sum_{i=1}^{J+1}(a_ie_i - a_{i-1}e_{i-1})(e_i - e_{i-1}) \;&\ge\; \frac{1}{h^2}\sum_{i=1}^{J+1}\eta_0(e_i - e_{i-1})^2 - \sum_{i=1}^{J} Ch|e_i|\,|e_i - e_{i-1}|\\ &\ge\; \eta_0 N(e/h)^2 - Ch\|e\|_2 N(e)\end{aligned} \tag{30}$$

We have also, by using $e_{i+1} - e_{i-1} = (e_{i+1} - e_i) + (e_i - e_{i-1})$:

$$\sum_{i=1}^{J} |b_i(e_{i+1} - e_{i-1})e_i| \le \|b\|_\infty 2N(e)\|e\|_2.$$

Hence a lower bound in form (for $0 < h \le 1$):

$$\langle e, Ae\rangle \ge \eta_0 N(e/h)^2 - CN(e)\|e\|_2 - C\|e\|_2^2$$

for some constant C. Using the inequality $N(e)\|e\|_2 \le \frac{\eta_0}{2Ch^2}N(e)^2 + \frac{Ch^2}{2\eta_0}\|e\|_2^2$ and $h^2 \le 1$ we finally obtain

$$\langle e, Ae\rangle \ge \frac{\eta_0}{2}N(e/h)^2 - (C + \frac{C^2}{2\eta_0})\|e\|_2^2$$

which gives the desired lower bound with $\eta = \eta_0/2$ and $\gamma = C + \frac{C^2}{2\eta_0}$. □

From now on we shall denote the error by

$$e^n := v^n - u^n.$$

Proposition 5.4. *Consider the scheme* (25), *and a perturbed scheme* (27)*. Let* $\tau > 0$ *be sufficiently small. Then there exists a constant* C_1 *independent of* n *and a constant* $\bar{\gamma} > 0$ *such that for all* $t_n \leq T$

$$
\begin{aligned}
& e^{-\bar{\gamma} t_n} \|e^n\|_2^2 + \tau \sum_{k=1}^{n} e^{-\bar{\gamma} t_k} N(e^k/h)^2 \\
& \qquad \leq \ C_1 \bigg(\|e^0\|_2^2 + \|e^1\|_2^2 + \tau \sum_{k=1,\ldots,n} e^{-\bar{\gamma} t_k} \|\bar{\epsilon}_n\|_2^2 \bigg). \qquad (31)
\end{aligned}
$$

where, by (28)*,* $N(e^k/h)^2 = \sum_{j=1}^{J+1} |\frac{e_i^k - e_{i-1}^k}{h}|^2$*.*

Proof of Proposition 5.4. Let

$$B := I + \frac{2\tau}{3} A,$$

and vectors b_u, b_v be such that

$$b_u := \frac{4}{3} u^n - \frac{1}{3} u^{n-1} - \frac{2\tau}{3} q_{n+1}$$

and

$$b_v := \frac{4}{3} v^n - \frac{1}{3} v^{n-1} - \frac{2\tau}{3} q_{n+1}.$$

Then, by Lemma 5.1, the first min is equivalent to $u^{n+1} \geq g$ and

$$\big\langle B u^{n+1} - b_u,\ w - u^{n+1} \big\rangle \geq 0, \quad \forall w \geq g. \qquad (32)$$

The second min equation (27) is equivalent to $v^{n+1} \geq g$ and

$$\big\langle B v^{n+1} - (b_v + \tau \bar{\epsilon}^n),\ w - v^{n+1} \big\rangle \geq 0, \quad \forall w \geq g. \qquad (33)$$

Taking $w = v^{n+1}$ in (32) gives

$$\big\langle B u^{n+1} - b_u,\ v^{n+1} - u^{n+1} \big\rangle \geq 0,$$

and $w := u^{n+1}$ in (33) gives

$$\big\langle B v^{n+1} - (b_v + \tau \bar{\epsilon}^n),\ u^{n+1} - v^{n+1} \big\rangle \geq 0.$$

Summing the last two relations gives

$$\big\langle B e^{n+1} - \frac{4}{3} e^n + \frac{1}{3} e^{n-1} - \frac{2\tau}{3} \bar{\epsilon}^n,\ e^{n+1} \big\rangle \leq 0 \qquad (34)$$

and therefore

$$\big\langle 3e^{n+1} - 4e^n + e^{n-1},\ e^{n+1} \big\rangle + 2\tau \langle e^{n+1}, A e^{n+1} \rangle \leq 2\tau \langle \bar{\epsilon}^n, e^{n+1} \rangle. \qquad (35)$$

Now let x_n and y_n be defined by

$$x_n := \|e^n\|^2, \qquad y_n := \|e^{n+1} - e^n\|^2, \qquad z_n := 2\tau \langle A e^n, e^n \rangle,$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^J$ and $\langle ., . \rangle$ the corresponding scalar product.

The following estimate holds:

$$3x_{n+1} - 4x_n + x_{n-1} + 2y_n + z_n \leq 2y_{n-1} + 2\tau \langle \bar{\epsilon}_n, e^{n+1} \rangle. \qquad (36)$$

To prove (36), we first use the properties $\langle a - b, a \rangle = \frac{1}{2}(\|a\|_2^2 + \|a - b\|_2^2 - \|b\|_2^2)$ as well as $\frac{1}{2}\|a + b\|_2^2 \leq \|a\|_2^2 + \|b\|_2^2$, to obtain

$$
\begin{aligned}
2\langle 3e^{n+1} - 4e^n + e^{n-1}, e^{n+1} \rangle \ &= \ 2(4\langle e^{n+1} - e^n, e^{n+1} \rangle - \langle e^{n+1} - e^{n-1}, e^{n+1} \rangle) \\
&= \ 4(x_{n+1} + y_n - x_n) - (x_{n+1} + \|e^{n+1} - e^{n-1}\|_2^2 - x_{n-1}) \\
&\geq \ 4(x_{n+1} + y_n - x_n) - (x_{n+1} + 2(y_n + y_{n-1}) - x_{n-1}) \\
&\geq \ 3x_{n+1} - 4x_n + x_{n-1} + 2y_n - 2y_{n-1}
\end{aligned}
$$

and we conclude by using (35).

Let

$$w_n := 4\tau\eta N(e^{n+1}/h)^2.$$

By using the bound $2\tau\langle\bar\epsilon_n, e^{n+1}\rangle \le 2\tau\|\bar\epsilon_n\|\|e^{n+1}\| \le \tau\|\bar\epsilon^n\|_2^2 + \tau x_{n+1}$ and the coercitivity (29), we obtain, for $n \ge 1$:

$$3x_{n+1} - 4x_n + x_{n-1} + 2y_n + w_n \le 2y_{n-1} + 2\tau\|\bar\epsilon^n\|_2^2 + (2\tau + 4\tau\gamma)x_{n+1}. \tag{37}$$

Let

$$\bar\gamma := 2 + 4\gamma \quad \text{and} \quad \beta := \tau\bar\gamma.$$

We have, for τ small enough (and there for β small):

$$(3-\beta)x_{n+1} - 4x_n + x_{n-1} + 2y_n + w_n \le 2y_{n-1} + 2\tau\|\bar\epsilon^n\|_2^2, \quad n \ge 1. \tag{38}$$

We multiply (38) by $e^{-n\beta}$ and sum up the inequalities from $n = 1$ to $n \ge 1$. Let $f(x) = x^2 - 4x + 3 - \beta$ and notice that for small $\beta > 0$, $f(e^{-\beta}) \sim \beta > 0$. We deduce that

$$\begin{aligned}
& e^{-n\beta}((3-\beta)x_{n+1} - (4-(3-\beta)e^{\beta})x_n) + \sum_{k=1}^{n} e^{-k\beta} w_k \\
& + \sum_{k=3}^{n-2} e^{-(k-1)\beta} f(e^{-\beta}) x_k + 2\sum_{k=1}^{n} e^{-k\beta} y_k \\
& \qquad \le\ C(x_0 + x_1) + 2\sum_{k=1}^{n} e^{-k\beta} y_{k-1} + 2\sum_{k=1}^{n} e^{-k\beta} 2\tau\|\bar\epsilon^k\|_2^2.
\end{aligned} \tag{39}$$

Using that $e^{-k\beta}y_{k-1} \le e^{-(k-1)\beta}y_{k-1}$ and $f(e^{-\beta}) > 0$ we deduce that

$$\begin{aligned}
& e^{-n\beta}((3-\beta)x_{n+1} - (4-(3-\beta)e^{\beta})x_n) + \sum_{k=1}^{n} e^{-k\beta} w_k \\
& \qquad \le\ C(x_0 + x_1) + 2e^{-\beta} y_0 + \sum_{k=1}^{n} e^{-k\beta} 2\tau\|\bar\epsilon^k\|_2^2 \\
& \qquad \le\ C(x_0 + x_1 + \tau\sum_{k=1}^{n} e^{-k\beta}\|\bar\epsilon^k\|_2^2) =: Q,
\end{aligned} \tag{40}$$

for some constant C (where we have used that $y_0 \le 2(x_0 + x_1)$).

Let us prove that $e^{-\bar\gamma t_n}x_n \le x_1 + CQ$, which will give the desired bound. By using $k\beta \le k\tau\bar\gamma \le \bar\gamma t_k$, we deduce from (40)

$$x_{k+1} \le \rho x_k + Ce^{\bar\gamma t_n}Q, \quad 1 \le k \le n.$$

where $\rho := \frac{4-(3-\beta)e^{\beta}}{3-\beta} \sim \frac{1}{3}$ as $\beta = \tau\bar\gamma \to 0$. By recursion we get for $1 \le k \le n$:

$$\begin{aligned}
x_k &\le \rho^{k-1}x_1 + Ce^{\bar\gamma t_n}Q(1 + \rho + \cdots + \rho^{k-2}) \\
&\le x_1 + Ce^{\bar\gamma t_n}Q\frac{1}{1-\rho}.
\end{aligned}$$

By using this bound for x_n into (40), we obtain the desired result (31) with a possibly different universal constant C_1. This concludes the proof of Proposition 5.4. $\square$

5.2. **BDF2 consistency and error estimate.** We now aim to estimate the consistency error $\bar{\epsilon}$ as it appears in (26).

Let us first consider the approximation in the x variable. In the region where $x \to v(t_{n+1}, x)$ is regular, assuming that $v_{4x}(t_{n+1}, .)$ is bounded on the interval $[x_{i-1}, x_{i+1}]$, it holds

$$(\mathcal{A}v)(t_{n+1}, x_i) = (Av^{n+1} + q_{n+1})_i + \bar{\epsilon}_i^{n,1}$$

where

$$\bar{\epsilon}_i^{n,1} = O(h^2)$$

is a consistency error in space. However because in the more general case where we expect to have no more that a bounded second derivative ($v_{xx} \in L^\infty_{loc}$), we will only have

$$\bar{\epsilon}_i^{n,1} = O(1)$$

as soon as the mesh interval $[x_{i-1}, x_{i+1}]$ encounter a singular point $x_s(t_{n+1})$, i.e.

$$x_s(t_{n+1}) \in [x_{i-1}, x_{i+1}].$$

We assume that there exists constants $C_{2x}, C_{4x} \geq 0$ and an integer $p \geq 0$, for all $t \geq 0$, $x \to v(t, x)$ is piecewise regular with at most p non regular points $X_{\min} < \bar{y}_1 < \cdots < \bar{y}_p < X_{\max}$ (we denote also $\bar{y}_0 = X_{\min}$ and $\bar{y}_{p+1} = X_{\max}$ for commodity), and such that for all $t \in [0, T]$: and

$$\max_{0 \leq k \leq p} \|v_{2x}(t, .)\|_{L^\infty((\bar{y}_k, \bar{y}_{k+1}))} \leq C_{2x},$$

and

$$\max_{0 \leq k \leq p} \|v_{4x}(t, .)\|_{L^\infty((\bar{y}_k, \bar{y}_{k+1}))} \leq C_{4x}.$$

(This still allows for $v_{2x}(t, .)$ to have "jumps" at singular points $x = y_k$.)

Then clearly

$$\sum_i |\bar{\epsilon}_i^{n,1}|^2 = \sum_{i,regular} |\bar{\epsilon}_i^{n,1}|^2 + \sum_{i,singular} |\bar{\epsilon}_i^{n,1}|^2 \tag{41}$$

$$\leq \sum_{i,regular} (C_{4x}h^2)^2 + C_{2x}p\,(O(1))^2 \tag{42}$$

$$\leq C \tag{43}$$

for some constant C that depends only of C_{2x}, C_{4x} and p. Hence the quantities $\|\bar{\epsilon}^{n,1}\|_2$ are uniformly bounded with respect to n.

Now we consider the approximation in time. Let $\bar{\epsilon}_i^{n,2}$ be such that

$$\frac{3v_i^{n+1} - 4v_i^n + v_{i-1}^n}{2\tau} = v_t(t_{n+1}, x_i) + \bar{\epsilon}^{n,2}.$$

In a region where $t \to v(t, x)$ is regular with bounded v_{3t} derivative, we have

$$|\bar{\epsilon}^{n,2}| \leq C\tau^2.$$

In general we will assume only that $t \to v(t, x)$ is piecewise regular, with $t \to v_t(t, x)$ is Lipschitz continuous, with at most one singular point t^* such that $x_s(t^*) = x$, and the a.e. derivative v_{3t} is bounded. Hence we assume

$$\|v_t(., x)\|_\infty \leq L, \quad \forall x \in (X_{min}, X_{max}),$$

and the following bounds in the regular regions (t_j^*, t_{j+1}^*),

$$\|v_{3t}(., x)\|_{L^\infty(t_j^*, t_{j+1}^*)} \leq C \tag{44}$$

(while $v(t, x) = g(x)$ in the other regions).

Then if there is one singularity $x_s(t)$ (assuming to be decreasing in time) that crosses x_i over the time interval $t \in [t_{n-1}, t_{n+1}]$, using the Lipschitz bound we obtain

$$|\bar{\epsilon}_i^{n,2}| \leq 2L.$$

In order to bound the number of singular estimations we need to control the cardinal of

$$J_s := \{i,\ x_i \in x_s([t_{n-1}, t_{n+1}])\}. \tag{45}$$

Then we have

$$\begin{aligned} \sum_i |\bar{\epsilon}_i^{n,2}|^2 &= \sum_{i \notin J_s} |\bar{\epsilon}_i^{n,2}|^2 + \sum_{i \in J_s} |\bar{\epsilon}_i^{n,2}|^2 && (46)\\ &\leq \sum_{i \notin J_s} C\tau + \sum_{i \in J_s} |\bar{\epsilon}_i^{n,2}|^2 && (47) \end{aligned}$$

Using the stability estimate (31), the fact that $e^0 = 0$ and that $\|e^1\|_2$ is clearly bounded, we deduce that

$$\|e^n\|_2^2 + \tau \sum_{k=1}^{n} N(e^k/h)^2 \leq \boxed{\text{... TO FINISH}}$$

Remark 5.5 (BDF3 scheme). We were not able to extend the previous stability estimate to the BDF3 implicit (obstacle) scheme.

6. Numerical results

6.1. **Two model test problems.** We first define two model test problems. In the case of (1) we do not known about exact solutions. Therefore we introduce a modified problem with a source term in order to construct simple model obstacle problems with explicit solutions (or solution that can be easily computed with machine precision) and also with the main features of the one-dimensional American option problem.

This is obtained by chosing an explicit function $v = v(t,x)$ and adding a corresponding source term $f = f(t,x)$ to the original PDE, thus considering

$$\min\left(v_t - \frac{\lambda^2}{2}x^2 v_{xx} - rxv_x + rv,\ v - g(x)\right) = f(t,x) \tag{48}$$

More precisely, let $g(x) := \max(K - x, 0)$ denote the payoff function. Let K, X_{max}, c_0 and T be given constants such that $0 < K < X_{max}$, $c_0 > 0$, $T > 0$ and such that $K - c_0 T > 0$. Let $x_s(t)$ be defined by

$$x_s(t) := K - c_0\, t \tag{49}$$

for $t \in [0,T]$ (in particular $x_s(t) \in]0,K[$ for $t \in]0,T]$).

We construct explicit function $v(t,x)$ defined for $x \in [0, X_{max}]$ and such that

(*i*) $v(t,x) = g(x) = K - x$ for $x \leq x_s(t)$,
(*ii*) $v(t,x) > g(x) = \max(K - x, 0)$ for $x \in]x_s(t), X_{max}]$,
(*iii*) v is at least C^1 on $[0, X_{max}]$,
(*iv*) $v(t, X_{max}) = 0$.

Note that requirement (*iii*) implies in particular $v_x(t, x_s(t)) = g'(x_s(t)) = -1$ for $t > 0$.

<u>Model 1</u> Let $v = v(t,x)$ be the function defined by:

$$v(t,x) := \begin{cases} g(x) & \text{for } x < x_s(t) \\ g(x_s(t)) - \dfrac{x - x_s(t)}{1 - (x - x_s(t))/C(t)} & \text{otherwise} \end{cases} \tag{50}$$

where $C(t) > 0$ is such that $v(t, X_{max}) = 0$:

$$C(t) := \left(\frac{1}{X_{max} - x_s(t)} - \frac{1}{g(x_s(t))} \right)^{-1}.$$

Then the requirements $(i) - (iv)$ are satisfied.

Model 2 Let $v = v(t, x)$ be the function defined by:

$$v(t, x) := \begin{cases} g(x) & \text{for } x < x_s(t) \\ g(x_s(t)) - C(t)\text{atan}\left(\dfrac{x - x_s(t)}{C(t)}\right) & \text{otherwise} \end{cases} \tag{51}$$

for a given $C(t) > 0$. Notice that $v(t, x)$ is a non-increasing function of the variable x. This function will satisfy requirements $(i) - (iv)$ if furthermore $C(t)$ is such that

$$\frac{g(x_s(t))}{C(t)} = \text{atan}\left(\frac{X_{max} - x_s(t)}{C(t)}\right). \tag{52}$$

Letting $a := X_{max} - x_s(t)$ and $b = g(x_s(t)) = K - x_s(t)$ it is clear that $0 < b < a$ and therefore there exists a unique $\theta > 0$ such that $b\theta = \text{atan}(a\theta)$. This value can be numerically obtained by using a fixed-point method. We then define $C(t) := 1/\theta$ to obtain a solution of (52). Therefore the function v is in explicit form but for the computation of the $C(t)$ function which can be computed to arbitrary precision.

Remark 6.1. For model 2, in order to compute $v_t(t, x)$ the derivative of $C(t)$, $\dot{C}(t)$, is needed. Denoting $a = a(t) = X_{max} - x_s(t)$ and $b = b(t) = g(x_s(t))$, and $\theta = \theta(t) = 1/C(t)$, by derivation of $b\theta = \text{atan}(a\theta)$ we obtain $\dot{C}/C = -\dot{\theta}/\theta = (q\dot{b} - \dot{a})/(qb - a)$ where $q = 1 + (a/C)^2$, with $\dot{a} = -\dot{x}_s(t)$ and $\dot{b} = g'(x_s(t))\dot{x}_s(t)$.

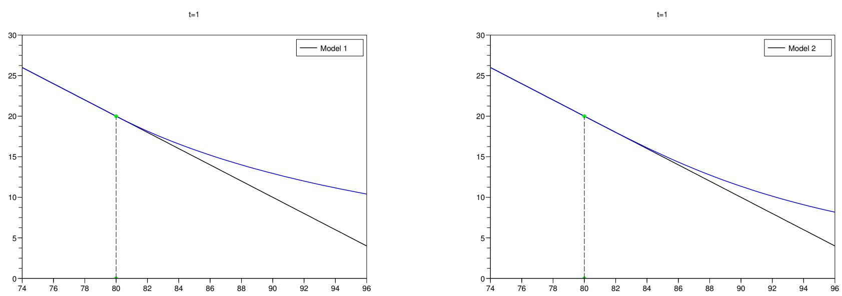

FIGURE 1. Zooming around the singular point $(x_s, g(x_s))$ for model 1 (left) and 2 (right).

Remark 6.2. The main difference between the two models is the regularity of the data near the singularity $x = x_s(t)$. More precisely, for the first model there is a jump in the second derivative: v is of class C^1 and v_{xx} is discontinuous. For the second model, v is of class C^2 and there is a jump in the third derivative v_{3x}.

6.2. **Using a 4th order approximation of the spatial operator $\mathcal{A}$.** We furthermore introduce a 4th order numerical matrix approximation of the $\mathcal{A}$ operator in order to better observe the time discretization error.

Let $D^2 u_j := \frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2}$. We use the fact that, by Taylor expansions,

$$-u_{xx} = \frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} + \frac{1}{12h^2}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}) + O(h^4)$$

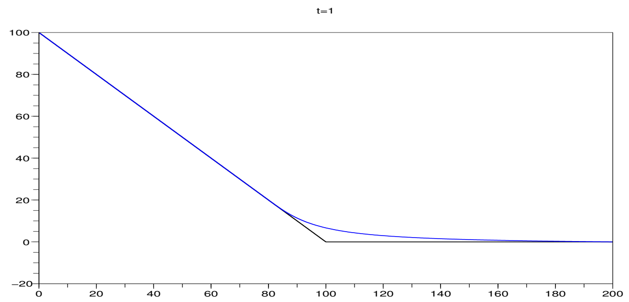

FIGURE 2. Model 2.

and also,

$$u_x = \frac{u_{j+1} - u_{j-1}}{2h} + \frac{1}{6h}(u_{j-2} - 3u_{j-1} + 3u_{j+1} - u_{j+2}) + O(h^4).$$

Therefore we have a 4-th order approximation of the spatial derivatives. At the boundaries, we use $u_{-1} = K - X_{Min} - h$ and $u_{J+2} = 0$ (the left boundary condition is consistent with fourth order because we expect that $v(t,x) = g(x) = K - x$ near the left boundary, also the right boundary condition is consistent with the fact that that the exact solution $v(t,x)$ (obtained for the case $X_{\max} = \infty$) decays faster than any polynomial as $x \to \infty$).

The BDF2 and BDF3 schemes are otherwise unchanged concerning the time discretization. Hence for $\tau \equiv c\ h$ we expect to mainly see the error of the time discretization (in particular to observe second or third order behavior, if possible).

6.3. **Numerical results for Model 1 and Model 2.** For testing on models 1 and 2 the parameters used are $\lambda = 0.3$, $r = 0.1$, $T = 1$, $X_{\min} = 50$ and $X_{\max} = 250$.

Numerical results for Model 1 with the BDF2 (resp. BDF3) implicit schemes are given in Tables 11 and 12. We observe that in Table 12 even with the third order scheme the numerical order is not greater than two. This can be explained by the fact that the solution has bounded second derivatives but no more.

Numerical results for Model 2 with the BDF2 (resp. BDF3) implicit schemes are given in Tables 13 and 14. Now the last Table clearly shows third order. For this model, by construction, the exact solution has bounded third derivatives.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.56E+00	-	5.93E-01	-	1.94E-01	-	0.10
160	16	6.48E-01	1.98	1.48E-01	2.00	5.11E-02	1.93	0.11
320	32	1.77E-01	1.87	4.00E-02	1.89	1.33E-02	1.95	0.24
640	64	4.72E-02	1.90	1.06E-02	1.92	3.41E-03	1.96	0.48
1280	128	1.25E-02	1.92	2.77E-03	1.94	8.60E-04	1.99	1.28
2560	256	3.24E-03	1.95	7.14E-04	1.96	2.16E-04	2.00	4.64
5120	512	8.06E-04	2.01	1.78E-04	2.00	5.39E-05	2.00	19.30

TABLE 11. (Model 1) BDF2 - second order - implicit scheme with 4th order spatial approximation for Eq.(1), using high CFL numbers.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.50E+00	-	5.99E-01	-	2.09E-01	-	0.09
160	16	6.42E-01	1.96	1.48E-01	2.01	5.34E-02	1.96	0.13
320	32	1.69E-01	1.93	3.85E-02	1.95	1.35E-02	1.99	0.24
640	64	4.41E-02	1.94	9.96E-03	1.95	3.36E-03	2.00	0.53
1280	128	1.15E-02	1.94	2.57E-03	1.96	8.43E-04	2.00	1.66
2560	256	2.96E-03	1.96	6.55E-04	1.97	2.11E-04	2.00	7.27
5120	512	7.33E-04	2.01	1.63E-04	2.01	5.28E-05	2.00	39.23

Table 12. (Model 1) BDF3 - third order - implicit scheme with 4th order spatial approximation for (1), using high CFL numbers.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.91E-01	-	5.31E-02	-	1.32E-02	-	0.07
160	16	5.80E-02	2.33	1.07E-02	2.31	2.72E-03	2.28	0.09
320	32	1.29E-02	2.16	2.42E-03	2.15	6.22E-04	2.13	0.23
640	64	3.07E-03	2.07	5.77E-04	2.07	1.48E-04	2.07	0.50
1280	128	7.46E-04	2.04	1.40E-04	2.04	3.62E-05	2.04	1.38
2560	256	1.84E-04	2.02	3.46E-05	2.02	8.92E-06	2.02	5.12
5120	512	4.55E-05	2.01	8.58E-06	2.01	2.21E-06	2.01	20.44

Table 13. (Model 2) BDF2 - second order - implicit scheme with 4th order spatial approximation for (1), using high CFL numbers.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	6.94E-02	-	1.31E-02	-	3.76E-03	-	0.10
160	16	1.96E-03	5.14	5.61E-04	4.54	2.75E-04	3.77	0.13
320	32	4.00E-04	2.29	7.59E-05	2.89	2.20E-05	3.65	0.25
640	64	7.85E-05	2.35	1.50E-05	2.34	4.07E-06	2.43	0.62
1280	128	1.14E-05	2.79	2.15E-06	2.80	5.70E-07	2.84	1.86
2560	256	1.51E-06	2.91	2.81E-07	2.93	7.39E-08	2.95	7.95
5120	512	1.99E-07	2.92	3.63E-08	2.95	9.50E-09	2.96	41.00

Table 14. (Model 2) BDF3 - third order - implicit scheme with 4th order spatial approximation for (1), using high CFL numbers.

6.4. **American option model.** For sake of completeness we detail here the parameters used for the American option model:

$$\lambda = 0.2, \quad r = 0.1, \quad T = 1, \quad \varphi(x) := \max(K - x, 0) \quad \text{with} \quad K := 100.$$

The one-dimensional American put option model is given by the following PDE

$$\min(u_t - \frac{1}{2}\lambda^2 x^2 u_{xx} - rxu_x + ru,\, u - \varphi(x)) = 0, \quad t \in (0, T), \quad x \in \Omega, \tag{53}$$

$$u(0, x) = \varphi(x), \quad x \in \Omega. \tag{54}$$

It is possible to see that the singular point $x_s(T)$ is greater than 80, so for practical reasons we consider a subdomain of the form $\Omega = [a, b] = [75, 275]$ of $(0, \infty)$, and boundary conditions of Dirichlet type:

$$u(t, X_{\min}) = K - X_{\min}, \quad 0 < t < T,$$

and

$$u(t, X_{\max}) = 0, \quad 0 < t < T$$

(the truncation error from the right, using $X_{max} = 275$ instead of $X_{max} = +\infty$, is estimated to be less than 10^{-8}).

Appendix A. An other HJ equation for obstacle problems

Proof of (6). We consider the problem after a change of variable $t \to T - t$:

$$\min(-u_t + \mathcal{A}u, u - \varphi(x)) = 0, \quad t \in (0, T), \quad x \in \Omega, \tag{55a}$$

$$u(T, x) = \varphi(x), \quad x \in \Omega \tag{55b}$$

where for simplification we consider here that $\Omega = \mathbb{R}$.

(*i*) Recall that u is given by the expectation formula

$$u(t, x) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}(e^{-\int_t^\tau r\, ds} \varphi(X_\tau^{t,x}) | \mathcal{F}_t).$$

(see for instance [12]) where we have considered a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t \geq 0}$, $\mathcal{T}_{[t,T]}$ is the set of stopping times taking values a.s in $[t, T]$, $X_\tau := X_\tau^{t,x}$ is the strong solution of the stochastic differential equation (SDE):

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s, \quad s \geq t,$$

with $X_t = x$, W_s denotes an $\mathcal{F}_t$-adapted Brownian motion on $\mathbb{R}$, and the "sup" is an essential supremum over $\mathcal{T}_{[t,T]}$. First one can see that

$$u(t, x) \leq \mathbb{E}(e^{-rh}\, u(t + h, X_{t+h}^{t,x}) | \mathcal{F}_t),$$

from which we deduce (in the viscosity sense), that

$$-u_t + \mathcal{A}u \geq 0.$$

(*ii*) Let us show that $u(t, x) \geq u(t + h, x)$, for any $h > 0$. This will imply (in the viscosity sense), that $-u_t \geq 0$. By definition,

$$u(t + h, x) = \sup_{\tau \in \mathcal{T}_{[t+h,T]}} \mathbb{E}(e^{-\int_{t+h}^\tau r\, ds} \varphi(X_\tau^{t+h,x}) | \mathcal{F}_{t+h}) \tag{56}$$

$$= \sup_{\tau \in \mathcal{T}_{[t,T-h]}} \mathbb{E}(e^{-\int_{t+h}^{\tau+h} r\, ds} \varphi(X_{\tau+h}^{t+h,x}) | \mathcal{F}_{t+h}) \tag{57}$$

$$= \sup_{\tau \in \mathcal{T}_{[t,T-h]}} \mathbb{E}(e^{-\int_t^\tau r\, ds} \varphi(X_\tau^{t,x}) | \mathcal{F}_t). \tag{58}$$

We have used the fact that the process $X^{t,x}$ satisfies a SDE with no time dependancy in the coefficients, and also, since $\tau \in \mathcal{T}_{[t,T-h]}$, $X_{\tau+h}$ a.s. stops before time T, the fact that $\mathbb{E}(X_{\tau+h}^{t+h,x} | \mathcal{F}_{t+h}) = \mathbb{E}(X_\tau^{t,x} | \mathcal{F}_t)$ - which corresponds to an averaging during a period of time $T - (t + h)$. Then, in particular,

$$u(t + h, x) \leq \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}(e^{-\int_t^\tau r\, ds} \varphi(X_\tau^{t,x}) | \mathcal{F}_t) = u(t, x). \tag{59}$$

At this point we therefore have shown than

$$\min(-u_t + \mathcal{A}u, -u_t) \geq 0.$$

(*iii*) Now let us assume that $-u_t(t, x) > 0$ (in the viscosity sense), and $t < T$. It implies that $u(t, x) > u(t + h, x)$ for all $h > 0$ small enough. Because $u(t, x) >$

$u(t+h,x) \geq \varphi(x)$, we have $u(t,x) > \varphi(x)$. The following dynamic programming principle holds:

$$u(t,x) = \mathbb{E}(e^{-\int_t^{\tau^*_{t,x}} r\,ds} \varphi(X^{t,x}_{\tau^*_{t,x}})|\mathcal{F}_t) = \mathbb{E}(e^{-\int_t^{\tau^*_{t,x}} r\,ds} u(\tau^*_{t,x}, X^{t,x}_{\tau^*_{t,x}})|\mathcal{F}_t)$$

where $\tau^*_{t,x}$ is the optimal stopping time for the obstacle problem, defined by

$$\tau^*_{t,x} = \inf\left\{\theta \geq t,\ u(\theta, X^{t,x}_\theta) = \varphi(X^{t,x}_\theta)\right\}.$$

It can be shown that $\tau^*_{t,x} > t$ a.s. (since $u(t,x) > \varphi(x)$, these functions beeing continuous). Then we obtain that $-u_t + \mathcal{A}u = 0$ at (t,x) in the viscosity sense. By writing the Ito formula between t and $\tau^*_{t,x}$, and from the dynamic programming principle, we deduce that

$$0 = \mathbb{E}\bigg(\int_t^{\tau^*_{t,x}} e^{-\int_t^\theta r\,ds}(u_t - \mathcal{A}u)_{(\theta, X^{t,x}_\theta)} d\theta \mid \mathcal{F}_t\bigg).$$

We already have proved that $u_t - \mathcal{A}u \leq 0$ a.s., so we deduce that $(u_t - \mathcal{A}u)(\theta, x) = 0$ a.e. for $\theta \in (t, \tau^*_{t,x})$. For some random parameter w we have $t^* := \tau^*_{t,x}(w) > t$, from which it is deduced that $(u_t - \mathcal{A}u)(t,x) = 0$. □

Remark A.1. *In the same way, it can be proved that the following PDE with source term and x-dependent coefficients in the operator $\mathcal{A}$:*

$$\min(-u_t + \mathcal{A}u, u - \varphi(x)) = f(x), \quad t \in (0,T), \quad x \in \Omega, \tag{60a}$$
$$u(T,x) = \varphi(x) + f(x), \quad x \in \Omega \tag{60b}$$

is equivalent to the following PDE

$$-u_t + \min(\mathcal{A}u,\ 0) = f(x), \quad t \in (0,T), \quad x \in \Omega, \tag{61a}$$
$$u(T,x) = \varphi(x) + f(x), \quad x \in \Omega \tag{61b}$$

Problem (60) *is associated with the control problem*

$$u(t,x) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}\bigg(e^{-\int_t^\tau r ds} \varphi(X^{t,x}_\tau) + \int_t^\tau e^{-\int_t^\theta r ds} f(X^{t,x}_\theta) d\theta | \mathcal{F}_t\bigg).$$

Appendix B. Proof of Lemma 2.3

Proof of (i): Let $c_n := \frac{1}{2}(Au^{n+1} + Au^n)_j + q(t_{n+1/2})_j$, so that the first scheme (9) reads $\min(\frac{u^{n+1}_j - u^n_j}{\tau} + c^n_j,\ u^{n+1}_j - g_j) = 0$. Assuming that u^{n+1} is solution of scheme (9), with $u^{n+1} \geq u^n$, if $u^{n+1}_j = g_j$, then it is clear that also $u^n_j = g_j = u^{n+1}_j$ and therefore since $\frac{u^{n+1}_j - u^n_j}{\tau} + c^n_j \geq 0$ it can be deduced that $S^{2,n}_j(u) = 0$. On the other hand, if $u^{n+1}_j > g_j$, then it implies $\frac{u^{n+1}_j - u^n_j}{\tau} + c^n_j = 0$, from which we conclude to $\mathcal{S}^{2,n}_j(u) = 0$. so u^{n+1} is also a solution of the scheme (11).

Proof of (ii): For the American option problem, the left boundary condition will be $u_\ell(t_n) = K - X_{min}$, and the right boundary condition $u_\ell(t_n) = 0$. So that the vector $q(t_{n+1/2}) = q$ does not depend on time. It is easy to see that $u^1 \geq u^0 = g$. Assuming $u^n \geq u^{n-1}$, for some $n \geq 1$, and denoting $F_n(x) := \min(Bx - b_n, x - g)$ where $B = I + \frac{\tau}{2}A$ and $b_n = (I - \frac{\tau}{2}A)u^n + \tau q$, the scheme is equivalent to $F_n(u^n) = 0$. Because B is an M-matrix, the function F_n is monotone in the sense that $F_n(x) \leq F_n(y) \Rightarrow x \leq y$ (componentwise). On the other hand, $u^{n-1} \leq u^n$ and assuming that $I + \frac{dt}{2}A \geq 0$, it holds $b_{n-1} \leq b_n$, and therefore $F_n(x) \leq F_{n-1}(x)$ for all x. In particular, $F_{n-1}(u_n) = 0 = F_n(u_{n+1}) \leq F_{n-1}(u_{n+1})$, and by the monotonicity of F_{n-1} we conclude to $u_n \leq u_{n+1}$.

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LABORATOIRE JACQUES-LOUIS LIONS UNIVERSITÉ PARIS-DIDEROT (PARIS 7), AND ENSTA PARISTECH (*BOKA@MATH.JUSSIEU.FR*).

UNIVERSITY OF SOUTHERN DENMARK, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, ODENSE (*DEBRABANT@IMADA.SDU.DK*).