Problem 3

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CS [illegible]

lets look at only the terms of $\log p(\Psi, Z, W, X)$ that depend on W and denote that by:

$$\log q(W) = \sum_{n=1}^{N} -\frac{1}{2}(x_n - Wz_n)^T \operatorname{diag}(\Psi)(x_n - Wz_n)$$
$$-\frac{1}{2\alpha}\sum_{d=1}^{D} w_d^T w_d$$

thus:

$$\log q(w_d) = \sum_{n=1}^{N} -\frac{1}{2}\underbrace{(x_{nd} - w_d^T z_n)}_{\in\mathbb{R}}^T \Psi_d (x_{nd} - w_d^T z_n)$$
$$-\frac{1}{2\alpha} w_d^T w_d$$

$$= \sum_{n=1}^{N}\left(-\frac{1}{2}\Psi_d (x_{nd} - w_d^T z_n)^2\right) - \frac{1}{2\alpha} w_d^T w_d$$

$$= -\frac{1}{2}\Psi_d \sum_{n=1}^{N}\left(\underbrace{x_{nd}^2}_{\text{doesn't depend on } w_d} - 2x_{nd} w_d^T z_n + (w_d^T z_n)^2\right) - \frac{1}{2\alpha} w_d^T w_d$$

$$= -\frac{1}{2}\sum_{n=1}^{N}\left(\Psi_d\left(w_d^T\left(-2x_{nd} z_n\right) + w_d^T z_n z_n^T w_d\right)\right) - \frac{1}{2\alpha} w_d^T w_d$$

$$= -\frac{1}{2}\sum_{n=1}^{N}\left(\Psi_d w_d^T (z_n z^T) w_d\right) - \frac{1}{2\alpha} w_d^T w_d + w_d^T\left(\sum_{n=1}^{N}\Psi_d x_{nd} z_n\right)$$

$$= -\frac{1}{2} w_d^T \left(\sum_{n=1}^{N} \tau_d (z_n z_n^T) + \frac{1}{\alpha} I \right) w_d + w_d^T \left(\sum_{n=1}^{N} \tau_d x_{nd} z_n \right)$$

now replace τ and z with their expectations since they were given

$$\log q(w_d) = -\frac{1}{2} w_d^T \underbrace{\left(\sum_{n=1}^{N} \langle \tau_d \rangle \langle z_n z_n^T \rangle + \frac{1}{\alpha} I \right)}_{A} w_d$$

$$+ w_d^T \underbrace{\left(\sum_{n=1}^{N} \langle \tau_d \rangle x_{nd} \langle z_n \rangle \right)}_{b}$$

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$$= -\frac{1}{2} w_d^T A w_d + w_d^T b$$

again complete the square (assume $A = A^T$)

$$\Rightarrow \log q(w_d) = -\frac{1}{2} (w_d - A^{-1} b)^T A (w_d - A^{-1} b) + \underbrace{\frac{1}{2} b^T A^{-1} b}_{\text{does not depend on } w_d}$$

$$= -\frac{1}{2} (w_d - A^{-1} b)^T A (w_d - A^{-1} b)$$

$\Rightarrow$ gaussian

$$\boxed{\begin{array}{ll} \text{Thus: } & \log q(w_d) = \mathcal{N}_K(\mu_d, K_d) \\ \text{where} & K_d = A^{-1} = \left(\sum_{n=1}^{N} (\langle \tau_d \rangle \langle z_n z_n^T \rangle) + \frac{1}{\alpha} I \right)^{-1} \\ \text{and} & \mu_d = A^{-1} b = K_d \left(\sum_{n=1}^{N} \langle \tau_d \rangle x_{nd} \langle z_n \rangle \right) \end{array}}$$

$\square$