Stochastic Modelling for Sterile Mosquitoes Technique

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Abstract

Vector-borne diseases such as malaria, yellow fever and dengue fever, etc, have always been long lasting public health problem for many regions in the world. One way to control the propagation of these diseases is to directly control the vector, which is the mosquito population. In this work, we are interested in using the *Sterile Insect Technique* (SIT) to control the mosquito population. We aim to develop and understand the stochastic model for the dynamic of mosquito population under the influence of SIT. We first introduce our stochastic models and then present some numerical simulations to illustrate the results. You can view the code of simulations on the `https://github.com/pelouse/SIT_Stochastic`.

Contents

1 Introduction

This project is centered on mathematical methods and models aimed at controlling mosquito populations to prevent the spread of vector-borne diseases such as malaria, yellow fever, dengue, Zika, and chikungunya. In various tropical regions, including French overseas territories with a longstanding history of problem, there have been significant outbreaks. Moreover, mainland France, including the Paris region, has recently experienced several cases. The urgent need for effective and sustainable surveillance and control strategies has become more pressing due to the rapid expansion of *Aedes albopictus*, a vector for many of these diseases, across Europe, including France, Italy, Germany, Spain, and others.

Mosquito abatement techniques, commonly employed, particularly during disease outbreaks, often rely on the widespread spraying of insecticides. However, the extensive use of these insecticides poses significant environmental pollution and biodiversity concerns. Additionally, mosquitoes swiftly develop resistance, diminishing the effectiveness of such control measures. Among the emerging approaches under exploration, sterile insect techniques (SIT) have gathered considerable attention within the scientific community. Mathematical models have proven invaluable in facilitating the safe and efficient implementation of these strategies, alongside other release methods for mosquitoes and pests. Notably, these techniques find applications beyond mosquito control, extending into agro-ecology to minimize pesticide usage in agriculture.

In SIT, the goal is to maintain mosquito (or pest) populations at sufficiently low levels to prevent the spread of diseases in human populations or fields. Since the zero population state is unstable, it is crucial to develop accurate models for small populations to effectively control them. This approach enables the reduction of both economic costs and environmental impacts associated with control measures. There have been multiple papers studying the dynamical system and the control strategies for the SIT in the deterministic framework such as [Alm+22], [ALV23], [BAC23]. Although the deterministic framework explains well the dynamics of large population systems, it fails to accurately capture the behaviour of the system in small population settings. In this case, the stochastic framework is considered and is expected to describe it better, since the evolution of the population is probabilistic by nature. To this end, this work aims to develop stochastic models of mosquito populations, in order to determine the appropriate way of implementing sterile insect releases.

This report is organized into multiple sections. In Section 1, we present the biological context of SIT: how it was first developed, its methodology and its benefits compared to other methods. In Section 2, we introduce our first stochastic model of the dynamics of mosquito populations based on the *linear birth and death process* and then we analyze this model under the presence of the sterile population. In Section 3, we study a more realistic model, in which many variables and parameters are integrated according to the mosquito's life cycle. Section 2.4 (first model) and 3.3 (second model) are dedicated to the numerical simulations of all our models, where multiple numerical techniques will be applied in order to analyze different release strategies and some properties of the system that are difficult to obtain theoretically.

Biological Context

History

The *sterile insect technique* (**SIT**) is a method of biological insect control that has been used as part of area-wide integrated pest management strategies to suppress, contain, locally eradicate or prevent the (re)invasion of insect pest populations and disease vectors worldwide for the past six–seven decades. This technique was first described theoretically by the Russian geneticist *A.S. Serebrovsky* in 1940. Around the 1950s, *Edward Knipling* and *Raymond Bushland* developed the SIT and applied it practically to eliminate screw-worms preying on warm-blooded animals that were devastating livestock herds across Mexico, Central America, and South America. Since then, integrated with other control methods, this technique has been successful in controlling a number of high-profile insect pests, including fruit flies (Mediterranean fruit fly, Mexican fruit fly, oriental fruit fly, melon fly); tsetse fly; moths (codling moth, pink bollworm, false codling moth, cactus moth, and the Australian painted apple moth).

Methodology

The SIT consists in mass-rearing and sterilization, using radiation, of a target pest, followed by the systematic area-wide release of the sterile males by air over defined areas, where they mate with wild females resulting in no offspring and a declining pest population. The released insects are preferably male, as this is more cost-effective and the females may in some situations cause damage by laying eggs in the crop, or, in the case of mosquitoes, taking blood from humans. The sterile males compete with fertile males to mate with the females. Females that mate with a sterile male produce no offspring, thus reducing the next generation's population.

Benefits

Unlike insecticide, larvicide and some other harmful control methods, the SIT is among the most environment-friendly methods ever developed. On the one hand, it leaves no residues and has no direct negative effect on nontarget species (breaking the pest's reproductive cycle, also called autocidal control, is by definition species-specific). Furthermore, the SIT does not introduce non-native species into an ecosystem. Indeed, sterile insects are not self-replicating and, therefore, cannot become established in the environment. On the other hand, sterilization is induced through the effects of X-ray or gamma ray photon irradiation on the reproductive cells of the insects and does not involve transgenic (genetic engineering) processes.

2 Preliminary Model

In this section, by following similar approaches and assumptions as in the original work of *Edward Knipling* [Kni55], we introduce a simple stochastic model for the SIT. We first start by studying a linear birth and death process of a pest population, then we introduce a sterile population into this native population and investigate its temporal evolution.

2.1 Linear Birth and Death Process

Let us consider a population of insects that evolves in a closed system in which the net flow of individuals across boundaries is negligible. We suppose that the growth of this population behaves as that of a linear birth and death process, and the sex ratio of the natural population is $1:1$ *i.e.* in the absence of migration, half of the population is female and the other half is male. Consequently, we will investigate only the dynamics of female population since there is a reflection in the declination between female, male and total population.

We denote by $F(t)$, the number of females alive at time t taking values in $\mathbb{N}$. Suppose that initially there are $m > 0$ females *i.e.* $F(0) = m$ and, λ and μ are the birth and death rate respectively. Then $(F(t))$ can be seen as a Markov jump process whose Q-matrix $Q = (q(\cdot,\cdot))$ is defined by

$$\begin{cases} q(x, x-1) = \mu x \\ q(x, x+1) = \lambda x \end{cases},$$

and $q(x, y) = 0$ when $|x - y| \geq 2$. With a little abuse of notation, let us define, for $t \geq 0$ and $x \in \mathbb{N}$,

$$P_x(t) = \mathbb{P}\left(F(t) = x | F(0) = m\right),$$

as the probability that there are x females at time t given that there are initially m females. By applying the Kolmogorov backward equation, one obtains

$$\frac{\mathrm{d}P_x(t)}{\mathrm{d}t} = \lambda(x-1)P_{x-1}(t) + \mu(x+1)P_{x+1}(t) - (\lambda+\mu)xP_x(t). \tag{1}$$

To solve (1), one can use the method of generating function. We define, for $z \in (0, 1]$,

$$G(z,t) = \sum_{k\geq 0} z^k P_k(t), \tag{2}$$

with the initial conditions $G(1, t) = 1$ and $G(z, 0) = z^m$. By deriving (2) with respect to t, one has

$$\begin{aligned}
\frac{\partial G}{\partial t}(z,t) &= \sum_{k\geq 0} z^k \frac{\mathrm{d}P_k}{\mathrm{d}t}(t) \\
&= \lambda z^2 \frac{\partial}{\partial z}\left(\sum_{k\geq 1} z^{k-1} P_{k-1} k(t)\right) + \mu \frac{\partial}{\partial z}\left(\sum_{k\geq 0} z^{k+1} P_{k+1} k(t)\right) - (\lambda+\mu) z \frac{\partial}{\partial z}\left(\sum_{k\geq 0} z^k P_k k(t)\right) \\
&= \left(\lambda z^2 - (\lambda+\mu)z + \mu\right) \frac{\partial G}{\partial z}(z,t) \\
&= (z-1)(\lambda z - \mu)\frac{\partial G}{\partial z}(z,t).
\end{aligned} \tag{3}$$

The Partial differential equation (3) can be solved using the method of characteristics of convenience, we present the solution of this equation in the Appendix. The explicit expression of the solution is

then given by

$$G(z,t) = \left(\frac{\mu(z-1) - (\lambda z - \mu)e^{-(\lambda-\mu)t}}{\lambda(z-1) - (\lambda z - \mu)e^{-(\lambda-\mu)t}} \right)^m .$$

By inverting and finding the coefficient of the power series $G(z,t)$, we finally obtain

$$P_x(t) = \sum_{j=0}^{m\wedge x} \binom{m}{j} \binom{m+x-j-1}{m-1} \alpha(t)^{m-j} \beta(t)^{x-j} \left(1 - \alpha(t) - \beta(t)\right)^j ,$$

where

$$\alpha(t) = \frac{\mu(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}, \quad \beta(t) = \frac{\lambda(e^{(\lambda-\mu)t} - 1)}{\lambda e^{(\lambda-\mu)t} - \mu}, \quad \text{and } m \wedge x = \min(m, x).$$

Now, we compute the mean and the variance of the process $(F(t))$ directly from the generating function

$$m(t) = \mathbb{E}[F(t)] = \left.\frac{\partial G}{\partial z}\right|_{z=1} = me^{(\lambda-\mu)t},$$

and

$$\sigma(t)^2 = \mathrm{Var}(F(t)) = \left.\frac{\partial^2 G}{\partial z^2}\right|_{z=1} + \left.\frac{\partial G}{\partial z}\right|_{z=1} - \left(\left.\frac{\partial G}{\partial z}\right|_{z=1}\right)^2 = \frac{m(\lambda+\mu)}{\lambda-\mu} e^{(\lambda-\mu)t}(e^{(\lambda-\mu)t} - 1).$$

2.2 Birth and Death Process in the Presence of Sterile Males Population

Now, we release a population of sterile males into the native population and we observe the temporal evolution of the system.

Suppose that at time $t = 0$, $M_0 > 0$ sterile males are released into this population and adequate mixing takes place. For simplicity, we suppose furthermore that the mating competitiveness between the native males and sterile males is the same, which means that there is no preference for mating between the males for a given female. In this situation, it is easily checked that the probability of a female mating with a fertile male is given by

$$\mathbb{P}(\text{A female mates with a fertile male}) = \frac{F(t)}{F(t) + M_s(t)},$$

where $M_s(t)$ is the number of sterile males alive at time t. Then, the modified birth rate of the process $(F(t))$ is given by

$$\lambda' = \lambda \frac{F(t)}{F(t) + M_s(t)} = \frac{\lambda}{1 + \frac{M_s(t)}{F(t)}} = \frac{\lambda}{1 + R(t)}.$$

The quantity $R(t) = M_s(t)/F(t)$ is the ratio of sterile to fertile males and also known as the *overflooding ratio*, see [Kni55; Kni59]. An accurate estimate of this ratio is another technical issue essential for any SIT programme. In control operations, the induction of sterility into the wild population becomes more efficient when the sterile males sufficiently outnumber the native population after the constant releases, ensuring progressive population decline over the generations [Kni55]. Several studies have provided estimates of the overflooding ratio needed to control specific species of pest and insect [Flo+14; Flo+17; SM16].

Without workable estimates, fixing $R(t)$ is one way of placing conservative bounds on the number of sterilised males to be released while ensuring that the *threshold limit* is met and that native population

decline takes place. For simplicity, in this preliminary model, we assume that $M_s(t)$ depends on the population size and it is controlled in such a way that the overflooding ratio $R(t)$ is constant. In this case, the new birth rate of $(F(t))$ is

$$\lambda' = \lambda_R = \frac{\lambda}{1+R}. \tag{4}$$

Since in the presence of sterile population the death rate of the native population remains unchanged, it is easily seen that the temporal dynamics of the population $F(t)$ behaves in the same manner as the linear birth and death process with a slightly modified birth. By replacing λ by λ_R in (1), one obtains

$$m(t) = \mathbb{E}[F(t)] = me^{(\lambda_R - \mu)t}$$

Now, we investigate the limiting behaviour of $m(t)$ as t tends to infinity

$$\lim_{t\to\infty} m(t) = \begin{cases} 0 & \text{if } \lambda_R < \mu \longrightarrow \text{extinction of population,} \\ m & \text{if } \lambda_R = \mu \longrightarrow \text{persistence of population,} \\ \infty & \text{if } \lambda_R > \mu \longrightarrow \text{explosion of population.} \end{cases}$$

We see that in order for the population to decline and consequently to be eradicated, we must attain the threshold limit

$$\lambda_R < \mu \quad \text{or equivalently,} \quad R > \frac{\lambda_R}{\mu} - 1.$$

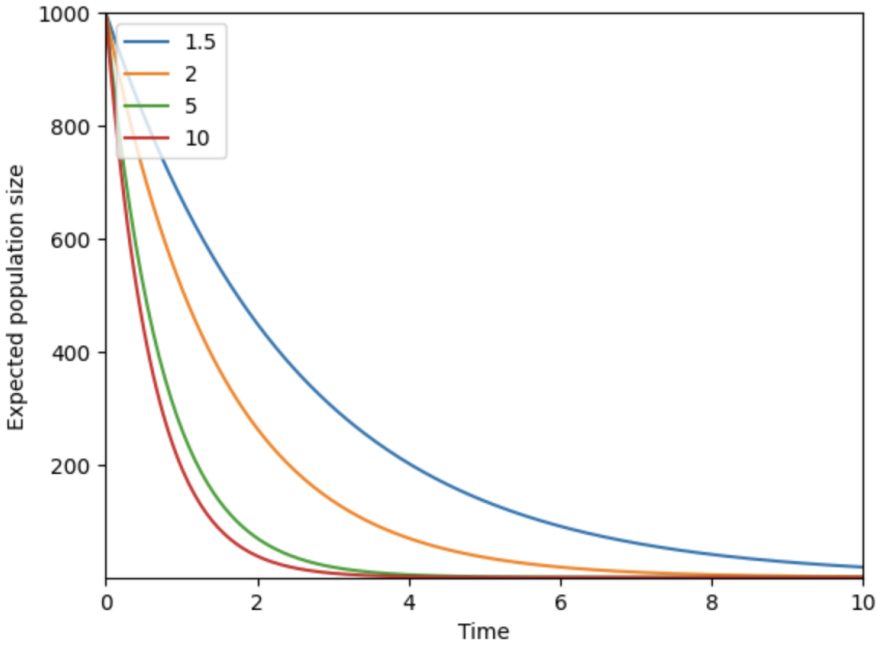

Figure 1: **Temporal Evolution of m(t)**: $m = 1000, \lambda = 4, \mu = 2$, and $R = 1.5, 2, 5, 10$.

Remarkably, for $1.5 < R < 5$, the curve drops drastically, but for $R > 5$ the variation is not as brutal as the former.

This phenomenon may suggest that the magnitude of sterile males needed to be released should be around 5 to 10 times more than the wild males in order to reach the extinction in a desirable amount of

time. Although this model is quite simple compared to the real situation, this result is corresponding to the practical work done in French Polynesia, where the quantity of sterile mosquitoes released is usually 10 times more than the wild male population.

2.3 Extinction Time

Let us define

$$T := \inf\Big\{t > 0 : F(t) = 0\Big\}$$

as the first time that the process $(F(t))$ hits 0 *i.e.* the extinction time of the native population (we assume that without migration, 0 is an absorbent state). It is trivial that if T is a stopping time with respect to the natural filtration of the process $(F(t))$, by definition, one has

$$\Big\{T \leq t \mid F(0) = m\Big\} = \Big\{F(t) = 0 \mid F(0) = m\Big\}.$$

By integrating both sides of the above equation, one obtains the probability distribution of the extinction time

$$\mathbb{P}\left(T \leq t \mid F(0) = m\right) = \mathbb{P}\left(F(t) = 0 \mid F(0) = m\right) = P_0(t) = \left[\frac{\mu(e^{(\lambda_R - \mu)t} - 1)}{\lambda_R e^{(\lambda_R - \mu)t} - \mu}\right]^m.$$

From this, it is possible to compute the probability that the population becomes extinct within a fixed time interval. Indeed, for a fixed $\tau > 0$, the probability that the population becomes extinct within the time interval $(0, \tau]$ is given by

$$\mathbb{P}(T \leq \tau) = \left[\frac{\mu(e^{(\lambda_R - \mu)\tau} - 1)}{\lambda_R e^{(\lambda_R - \mu)\tau} - \mu}\right]^m,$$

which depends only on R and τ.

2.4 Simulation

In the first model, we assume that the population of female and males are the same and we focus on one population. We focus on the female mosquitoes and we try to get this population to zero.

2.4.1 Linear Birth and Death Process

The first model has been implemented. It takes the initial number of the population, the birth function and the death function. These functions are linear and are in the form $birth(x) = \lambda x$ and $death(x) = \mu x$ with x the number of mosquitoes in the population.

The model can simulate the population until a time t_{max}. It is a jump process so the time between two events $T_n - T_{n-1}$ is exponentially distributed with the parameter $1/(birth(F(T_{n-1})) + death(F(T_{n-1})))$. During an event, we calculate the birth rate λx and the death rate μx and we choose one of them randomly. The probability to choose a birth is $\lambda x/(\lambda x + \mu x)$ and the probability to choose the death is $\mu x/(\lambda x + \mu x)$. Then, we add $+1$ to the population if it is a birth and -1 if it is a death.

The simulation consists in : calculate the time for the next event, calculate the event (birth or death) and do it again until the time is bigger than t_{max}. You can see in figure (2) different simulations with different values of λ and μ. The blue curve is the simulation with $\lambda = 1$ and $\mu = 1.4$, we can see the population grow down. The yellow curve corresponds to $\lambda = \mu = 1$, we can see the population stagnates. The green curve corresponds to $\lambda = 1.4$ and $\mu = 1$, we can see the population explodes.

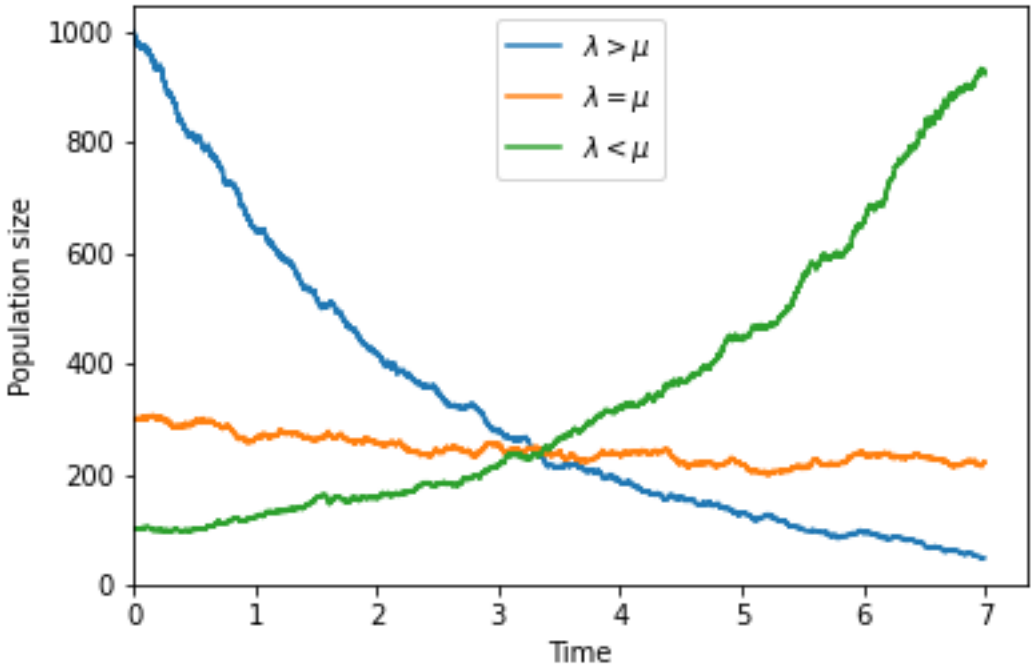

Figure 2: Linear Model with different values of λ and μ

2.4.2 Birth and Death Process in the Presence of Sterile Males Population

We now introduce the sterile mosquitoes by the constant R defined in the equation (4). We do not take functions for birth and death anymore. Now, we take three constants : λ, μ and R. We can reuse the previous code by defining the two functions by $birth(x) = x\lambda/(1+R)$ and $death(x) = x\mu$.

The curves of the models should look like the figure (1). We plot the process with the same parameters as the theoretical curves and obtain figure (3).

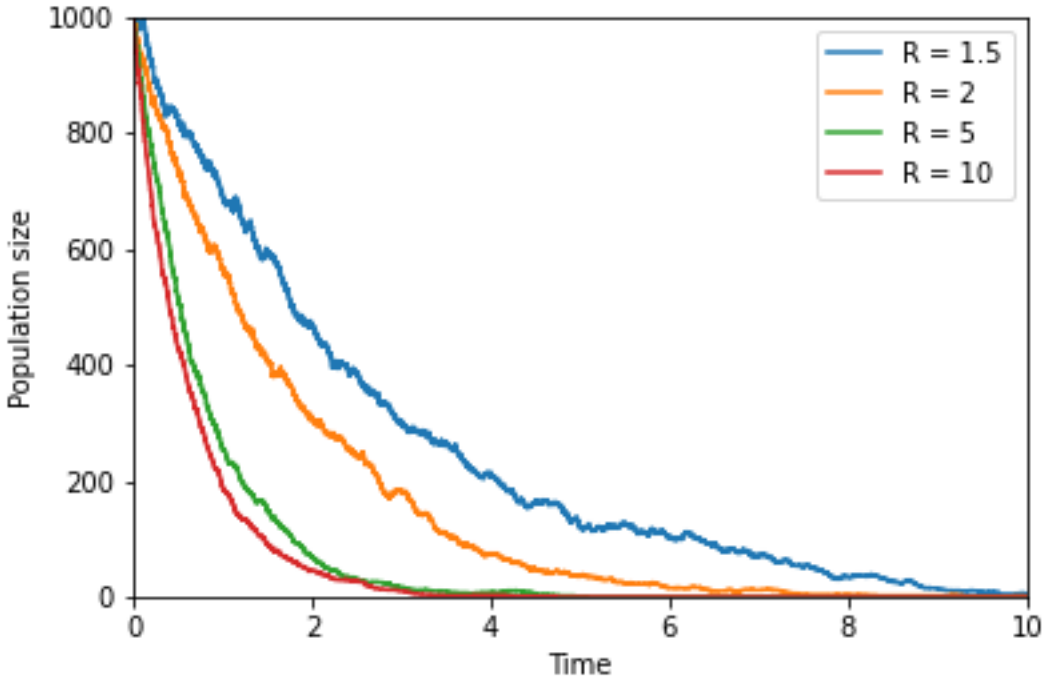

Figure 3: Simulations of the Linear Model with Sterile Mosquitoes for different values of R and the same parameters as the theoretical curves.

We can see four different extinctions. When $R = 10$, we have a faster extinction but we have a higher cost. Moreover, the extinction is almost the same with $R = 5$ so we do not need a very high value for R.

The question is to find for which value of R we can have an extinction.

2.4.3 Extinction Time

Let us define

$$T := \inf\Big\{t > 0 : F(t) = 0\Big\}$$

As we saw in section 2,

$$\mathbb{P}(T \leq \tau) = \left[\frac{\mu(e^{(\lambda_R - \mu)\tau} - 1)}{\lambda_R e^{(\lambda_R - \mu)\tau} - \mu}\right]^m, \tag{5}$$

To verify this equality, we can use a Monte-Carlo method. Indeed, let $F_i(t) \overset{d}{=} F(t) \forall i \in \mathbb{N}$ with all $F_i(t)$ independent. We also let $T_i := \inf\Big\{t > 0 : F_i(t) = 0\Big\}$ so the T_i are independent.

We can now estimate the probability

$$\begin{aligned}\mathbb{P}(T \leq \tau) &= \mathbb{E}\left[\mathbb{1}_{\{T\leq\tau\}}\right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{1}_{\{T_i\leq\tau\}}\end{aligned}$$

Let the estimator

$$\hat{p}_n = \sum_{i=1}^{n} \mathbb{1}_{\{T_i\leq\tau\}}$$

so that $\lim_{n\to\infty} \hat{p}_n = \mathbb{P}(T \leq \tau)$. We can now approximate the probability by simulating different trajectory of the process and count 1 if the population become extinct before τ and 0 otherwise. By calculating the average value, we obtain a estimation of the probability with a precision depending of n.

We fix $\tau = 10$, $\lambda = 4$, $\mu = 2$ and calculate $\hat{p}_n$ for different values of R. We also calculate the value of the probability using equation (5). By plotting in blue the estimator $\hat{p}_n$ depending on R and in yellow the theoretical result (5), we obtain figure (4).

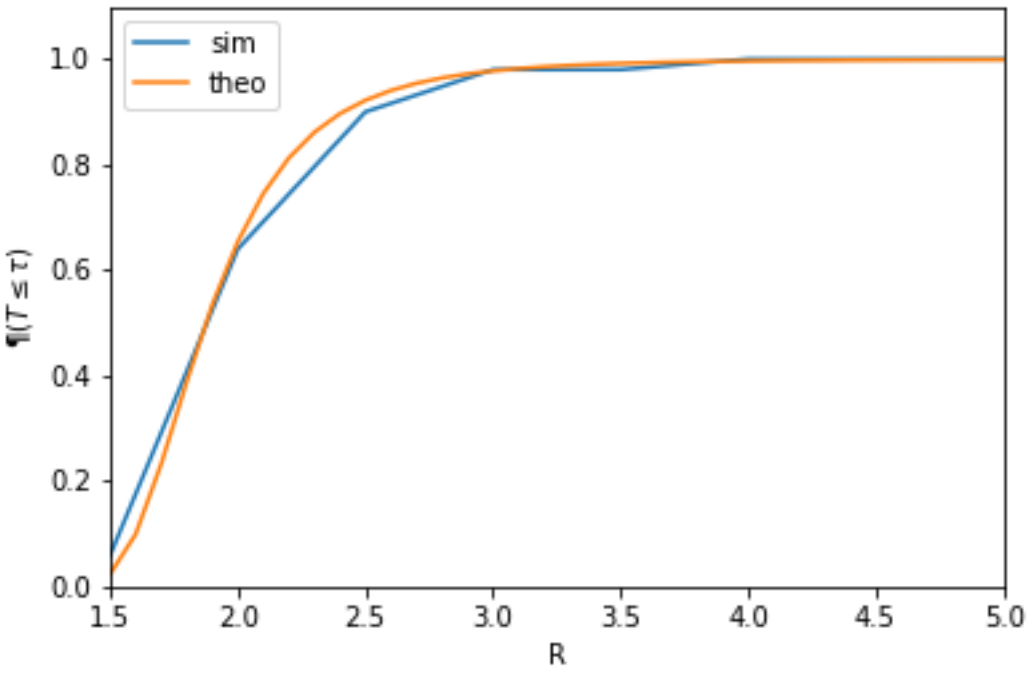

Figure 4: The curves of $\hat{p}_n$ and the theoretical result

We can see that the two curves are similar so the results are coherent.

3 Model for Sterile Mosquito Technique

In this model, we consider according to the mosquito life cycle, four different populations in the system: egg, wild adult male, fertilized adult female and sterilized adult male. The evolution of the system is described as follows:

- A fertilized female lays eggs in a limited environmental capacity.
- An egg can die or hatch and give birth to either a male or a female mosquito.
- Both wild male and fertilized female reproduce and die over time. Sterilized male does not reproduce and will die out after period of time.

3.1 Stochastic Model

Let us denote $E(t), M(t), F(t), M_s(t)$ as the number of eggs, wild adult, fertilized adult female and sterilized adult male mosquitoes, at time t, respectively. We define

$$X(t) \stackrel{\text{def.}}{=} (E(t), M(t), F(t), M_s(t)), \tag{6}$$

as the vector describing the system. We adopt similar choice of parameters as in the model of [Alm+22]:

- $\beta_E > 0$: the oviposition rate;
- $\delta_E, \delta_M, \delta_F, \delta_S > 0$: the death rates for eggs, adult males, fertilized females and sterilized males, respectively;
- ν_E: the hatching rates for eggs;
- ν and $(1-\nu) \in (0,1)$: the probability that a hatching egg gives rise to a female and a male, respectively;
- $K > 0$: the maximum number of eggs that females can lay in breeding sites (carrying capacity);
- γ_s: the preference factor of a female to reproduce with a fertile male.

Then, the transitions of the system are as follow:

- An egg is laid:
$$(E, M, F, M_s) \longrightarrow (E+1, M, F, M_s),$$
at rate $\beta_E \left(1 - \frac{E}{K}\right) F$.
- An egg hatches and gives rise to an adult male:
$$(E, M, F, M_s) \longrightarrow (E-1, M+1, F, M_s),$$
at rate $(1-\nu)\nu_E E$.
- An egg hatches and gives rise to a fertilized female:
$$(E, M, F, M_s) \longrightarrow (E-1, M, F+1, M_s),$$
at rate $\nu\nu_E \frac{M}{M+\gamma_s M_s} E$.
- An egg hatches and gives rises to a non fertilized female:
$$(E, M, F, M_s) \longrightarrow (E-1, M, F, M_s),$$
at rate $\nu\nu_E \frac{\gamma_s M_s}{M+\gamma_s M_s} E$.

- An egg dies:
$$(E, M, F, M_s) \longrightarrow (E-1, M, F, M_s),$$
at rate $\delta_E E$.
- An adult male dies:
$$(E, M, F, M_s) \longrightarrow (E, M-1, F, M_s),$$
at rate $\delta_M M$
- A fertilized female dies:
$$(E, M, F, M_s) \longrightarrow (E, M, F-1, M_s),$$
at rate $\delta_F F$.
- A sterilized male dies:
$$(E, M, F, M_s) \longrightarrow (E, M, F, M_s-1),$$
at rate $\delta_S M_s$.

The process $(X(t))$ can be seen as a Markov jump process whose $\mathcal{Q}$-matrix $Q = (q(\cdot,\cdot))$ is defined by

$$\begin{cases} q(x, x+e_1) = \beta_E \left(1 - \dfrac{x_1}{K}\right) x_3 \\ q(x, x-e_1+e_2) = (1-\nu)\nu_E x_1 \\ q(x, x-e_1+e_3) = \nu\nu_E \dfrac{x_2}{x_2+\gamma_s x_4} x_1 \\ q(x, x-e_1) = \left(\delta_E + \nu\nu_E \dfrac{\gamma_s x_4}{x_2+\gamma_s x_4}\right) x_1 \\ q(x, x-e_2) = \delta_M x_2 \\ q(x, x-e_3) = \delta_F x_3 \\ q(x, x-e_4) = \delta_S x_4 \end{cases}, \tag{7}$$

where, for $i = 1, 2, 3, 4$, $x = (x_i)$ and e_i is the ith unit vector of $\mathbb{N}^4$. If f is a function on $\mathbb{N}^4$, the $\mathcal{Q}$-matrix seen as a functional operator can be expressed as

$$\begin{aligned} Q(f)(x) = &\left(f(x+e_1) - f(x)\right) q(x, x+e_1) + \left(f(x-e_1+e_2) - f(x)\right) q(x, x-e_1+e_2) \\ &+ \left(f(x-e_1+e_3) - f(x)\right) q(x, x-e_1+e_3) + \sum_{i=1}^{4} \left(f(x-e_i) - f(x)\right) q(x, x-e_i). \end{aligned}$$

Let us denote $\mathcal{N}_+$ and $\mathcal{N}_-$ as two independent Poisson processes with intensity 1, it is easily checked that

$$\begin{aligned} \mathrm{d}f(X(t)) = &\lim_{s \to t} \left(f(X(t)) - f(X(s))\right) \\ = &\left(f(X(t-)+e_1) - f(X(t-))\right) \mathcal{N}_+ \left(\left(0, \beta_E \left(1 - \frac{E(t-)}{K}\right) F(t-)\right], \mathrm{d}t\right) \\ &+ \left(f(X(t-)-e_1) - f(X(t-))\right) \mathcal{N}_- \left((0, (\nu_E + \delta_E) E(t-)\right], \mathrm{d}t) \\ &+ \left(f(X(t-)+e_2) - f(X(t-))\right) \mathcal{N}_+ \left((0, (1-\nu)\nu_E E(t-)\right], \mathrm{d}t) \\ &+ \left(f(X(t-)-e_2) - f(X(t-))\right) \mathcal{N}_- \left((0, \delta_M M(t-)\right], \mathrm{d}t) \\ &+ \left(f(X(t-)+e_2) - f(X(t-))\right) \mathcal{N}_+ \left(\left(0, \nu\nu_E \frac{M(t-)}{M(t-)+\gamma_s M_s(t-)} E(t-)\right], \mathrm{d}t\right) \\ &+ \left(f(X(t-)+e_2) - f(X(t-))\right) \mathcal{N}_- \left((0, \delta_F F(t-)\right], \mathrm{d}t) \\ &+ \left(f(X(t-)-e_4) - f(X(t-))\right) \mathcal{N}_- \left((0, \delta_S M_s(t-)\right], \mathrm{d}t). \end{aligned} \tag{8}$$

Proposition 3.1. *The Markov process $X(t) = (E(t), M(t), F(t), M_s(t))$ with $\mathcal{Q}$-matrix Q starting from x_0 can be seen as the solution of the stochastic differential equation*

$$\begin{aligned} X(t) =& X(0) + \mathcal{M}(t) + e_1 \left(\beta_E \int_0^t \left(1 - \frac{E(s)}{K}\right) F(s)\mathrm{d}s - (\nu_E + \delta_E) \int_0^t E(s)\mathrm{d}s \right) \\ &+ e_2 \left((1-\nu)\nu_E \int_0^t E(s)\mathrm{d}s - \delta_M \int_0^t M(s)\mathrm{d}s \right) - e_4 \delta_S \int_0^t M_s(u)\mathrm{d}u \\ &+ e_3 \left(\nu\nu_E \int_0^t \frac{M(u)}{M(u) + \gamma_s M_s(u)} E(u)\mathrm{d}u - \delta_F \int_0^t F(s)\mathrm{d}s \right), \end{aligned}$$

or equivalently,

$$\begin{cases} E(t) = E(0) + \mathcal{M}_1(t) + \beta_E \displaystyle\int_0^t \left(1 - \frac{E(s)}{K}\right) F(s)\mathrm{d}s - (\nu_E + \delta_E) \int_0^t E(s)\mathrm{d}s \\ M(t) = M(0) + \mathcal{M}_2(t) + (1-\nu)\nu_E \displaystyle\int_0^t E(s)\mathrm{d}s - \delta_M \int_0^t M(s)\mathrm{d}s \\ F(t) = F(0) + \mathcal{M}_3(t) + \nu\nu_E \displaystyle\int_0^t \frac{M(u)}{M(u) + \gamma_s M_s(u)} E(u)\mathrm{d}u - \delta_F \int_0^t F(s)\mathrm{d}s \\ M_s(t) = M_s(0) + \mathcal{M}_4(t) - \delta_S \displaystyle\int_0^t M_s(u)\mathrm{d}u \end{cases},$$

where, for $i = 1, 2, 3, 4$, $\mathcal{M}(t) = (\mathcal{M}_i(t))$ is the martingale obtained from the compensation of the Poisson process.

Proof. By integrating (8), one obtains

$$f(X(t)) = f(X(0)) + R(t) + \int_0^t Q(f)(X(s))\mathrm{d}s,$$

where $R(t)$ is a local martingale obtained from the compensation of the Poisson processes and is given by (for convenience, we only define its first term here)

$$R(t) = \int_0^t \left(f(X(s-) - e_1) - f(X(s-))\right) \left[\mathcal{N}_- \left((0, (\nu_E + \delta_E)E(s-)], \mathrm{d}s\right) - (\nu_E + \delta_E)E(s-)\mathrm{d}s\right] + ...$$

Thus, the process

$$\left(f(X(t)) - f(X(0)) - \int_0^t Q(f)(X(s))\mathrm{d}s \right)$$

is a local martingale. This shows that the process $(X(t))$ is indeed a solution of the martingale problem associated with the $\mathcal{Q}$-matrix Q. By taking $f(x) = x$, for $x \in \mathbb{N}^4$, one gets that the desired result. □

We consider a renormalized version of the process where $K = \kappa N$.

$$\left(\bar{E}^{(N)}(t), \bar{M}^{(N)}(t), \bar{F}^{(N)}(t), \bar{M_s}^{(N)}(t)\right) = \left(\frac{E(Nt)}{N}, \frac{M(Nt)}{N}, \frac{F(Nt)}{N}, \frac{M_s(Nt)}{N}\right)$$

Assumption 3.2.

$$\frac{\bar{E}^{(N)}(0)}{N} \longrightarrow \bar{E}(0), \frac{\bar{M}^{(N)}(0)}{N} \longrightarrow \bar{M}(0), \frac{\bar{F}^{(N)}(0)}{N} \longrightarrow \bar{F}(0), \frac{\bar{M_s}^{(N)}(0)}{N} \longrightarrow \bar{M_s}(0).$$

Theorem 3.3. *Under assumption 3.2, as N tends to infinity, the processes $(\overline{E}(t), \overline{M}(t), \overline{F}(t), \overline{M}_s(t))_{t\geq 0}$ converge in distribution to the deterministic process $(x_E(t), x_M(t), x_F(t), x_{M_s}(t)_{t\geq 0}$ solution of the differential equation*

$$\begin{cases} \dot{x}_E(t) = \beta_E \left(1 - \dfrac{x_E(t)}{\kappa}\right) x_F(t) - (\nu_E + \delta_E) x_E(t) \\ \dot{x}_M(t) = (1-\nu)\nu_E x_E(t) - \delta_M x_M(t) \\ \dot{x}_F(t) = \nu\nu_E \dfrac{x_M(t)}{x_M(t) + \gamma_s M_s(t)} x_E(t) - \delta_F x_F(t) \\ \dot{x}_{M_s}(t) = -\delta_S x_{M_s}(t) \end{cases}. \tag{9}$$

Proof. **Part I.** Show that the martingale $\overline{\mathcal{M}}(t) = \frac{\mathcal{M}(t)}{N}$ converges in distribution to 0 as N tends to infinity.

Part II. Show that the processes $((\overline{E}(t), \overline{M}(t), \overline{F}(t), \overline{M}_s(t)))$ is tight.

Part III. The limiting point

□

3.2 Properties of the Dynamical System

The dynamical system (9) has been studied in detail in [Alm+22]. In this part, we will recall stability properties for equilibria of this system in order to qualitatively understand their behavior whenever initial data are chosen close to equilibria. First, we make the following assumption

$$\delta_S > \delta_M \quad \text{and} \quad \mathcal{R}_0 := \frac{\nu\beta_E\nu_E}{\delta_F(\nu_E + \delta_E)} > 1, \tag{10}$$

where $\mathcal{R}_0$ denotes the so-called *basic offspring number*(number of adult females produced by one adult female during her lifespan).

Proposition 3.4. *Under assumption (10)*

1. *If $x_{M_s}(0) = 0$, then, the system (9) has two equilibria:*
   - *the extinction equilibrium $(x_E^*, x_M^*, x_F^*, x_{M_s}^*) = (0,0,0,0)$, which is linearly unstable.*
   - *the persistence equilibrium $(x_E^*, x_M^*, x_F^*, x_{M_s}^*) = (\bar{x}_E, \bar{x}_M, \bar{x}_F, 0)$, where*

$$\bar{x}_E = \kappa\left(1 - \frac{1}{\mathcal{R}_0}\right), \quad \bar{x}_M = \frac{(1-\nu)\nu_E}{\delta_M} x_E, \quad \bar{x}_F = \frac{\nu\nu_E}{\delta_F}\bar{x}_E, \tag{11}$$

   which is locally asymptotically stable.

2. *If $x_{M_s}(0)$ is non-negative, then the corresponding solution (x_E, x_M, x_F, x_{M_s}) of the system (9) enjoys the following stability property: for all $t \geq 0$,*

$$\begin{cases} x_E(0) \in [0, \bar{x}_E) \\ x_M(0) \in [0, \bar{x}_M) \\ x_F(0) \in [0, \bar{x}_F) \\ x_{M_s}(0) \geq 0 \end{cases} \Rightarrow \begin{cases} x_E(t) \in [0, \bar{x}_E) \\ x_M(t) \in [0, \bar{x}_M) \\ x_F(t) \in [0, \bar{x}_F) \\ x_{M_s}(t) \geq 0 \end{cases}.$$

Finally, let U^* be defined by

$$U^* = \mathcal{R}_0 \frac{\kappa(1-\nu)\nu_E \delta_S}{4\gamma_s \delta_M} \left(1 - \frac{1}{\mathcal{R}_0}\right)^2 \tag{12}$$

and let $\bar{U}$ denotes any positive number such that $\bar{U} > U^*$. If $x_{M_s}(0)$ denotes the constant control function almost everywhere equal to $\bar{U}$ for all $t \geq 0$, then the corresponding solution (x_E, x_M, x_F, x_{M_s}) to system (9) converges to the extinction equilibrium as $t \to +\infty$.

Proof. The proof of this Proposition can be found in the Appendix of [Alm+22]. □

We verify from the first point of this proposition that $\mathcal{R}_0 > 1$ implies the population persistence while $\mathcal{R}_0 \leq 1$ expresses the population extinction.

3.3 Simulation

In this part, we will separate the population into four different types : The wild males, the sterile males, the fertilized females and the eggs.

3.3.1 Deterministic Model

First, the deterministic model described in [BAC23] has been implemented. The model looks like :

$$\begin{cases} \frac{dE}{dt} = \beta_E F\left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E, \\ \frac{dM}{dt} = (1-\nu)\nu_E E - \delta_M M, \\ \frac{dF}{dt} = \nu\nu_E E \frac{M}{M + \gamma_S M_s} - \delta_F F, \\ \frac{dM_s}{dt} = u - \delta_S M_s. \end{cases} \tag{13}$$

With u the deterministic function of the sterile mosquito release. This differential equation can be simulated and plotted using Euler's method. This method consists in creating a list of the different values along time. The first element of the list is X_0 and can be whatever we want. In our case, we start the operation when the mosquitoes are at their persistence equilibrium which is :

$$\bar{X} = \left(\bar{E}, \bar{M}, \bar{F}, 0\right) \tag{14}$$

with

$$\bar{E} = K\left(1 - \frac{\delta_F(\nu_E + \delta_E)}{\nu\beta_E\nu_E}\right), \qquad \bar{M} = \frac{(1-\nu)\nu_E}{\delta_M}\bar{E}, \qquad \bar{F} = \frac{\nu\nu_E}{\delta_F}\bar{E}.$$

We obtain the following Cauchy problem :

$$\begin{cases} X'(t) = f(X(t), t) \\ X(0) = X_0 = \bar{X} \end{cases} \tag{15}$$

with $f : \mathbb{R}^4 \times \mathbb{R}_+ \to \mathbb{R}^4$,

$$f(X,t) = \begin{pmatrix} \beta_E F\left(1 - \frac{E}{K}\right) - (\nu_E + \delta_E) E \\ (1-\nu)\nu_E E - \delta_M M \\ \nu\nu_E E \frac{M}{M + \gamma_S M_s} - \delta_F F \\ u - \delta_S M_s \end{pmatrix}$$

Then, we create a time discretization $0 = t_0 < t_1 < ... < t_N$ and we calculate recursively the elements of the list by :

$$X_{n+1} = X_n + (t_n - t_{n-1})\, f(X_n, t_{n-1})$$

By letting $u(t) = \delta_S M_s(t)$ we obtain the dynamic $\frac{dM_s}{dt} = 0$ which means we obtain a constant quantity of sterile mosquitoes. We show in figure (5) the curve of problem (15) with $\forall i \in \{0, 1, ..., 35'000\}, t_i = 0,01i$ such as $0 = t_0 < 0,01 = t_1 < ... < t_{35'000} = t_N = 350$. We also choose $X_0 = \bar{X} = \left(\bar{E}, \bar{M}, \bar{F}, 20'000\right)$ so $\forall t \in [0, 350], M_s(t) = 20'000$. We can observe an extinction of the natural population.

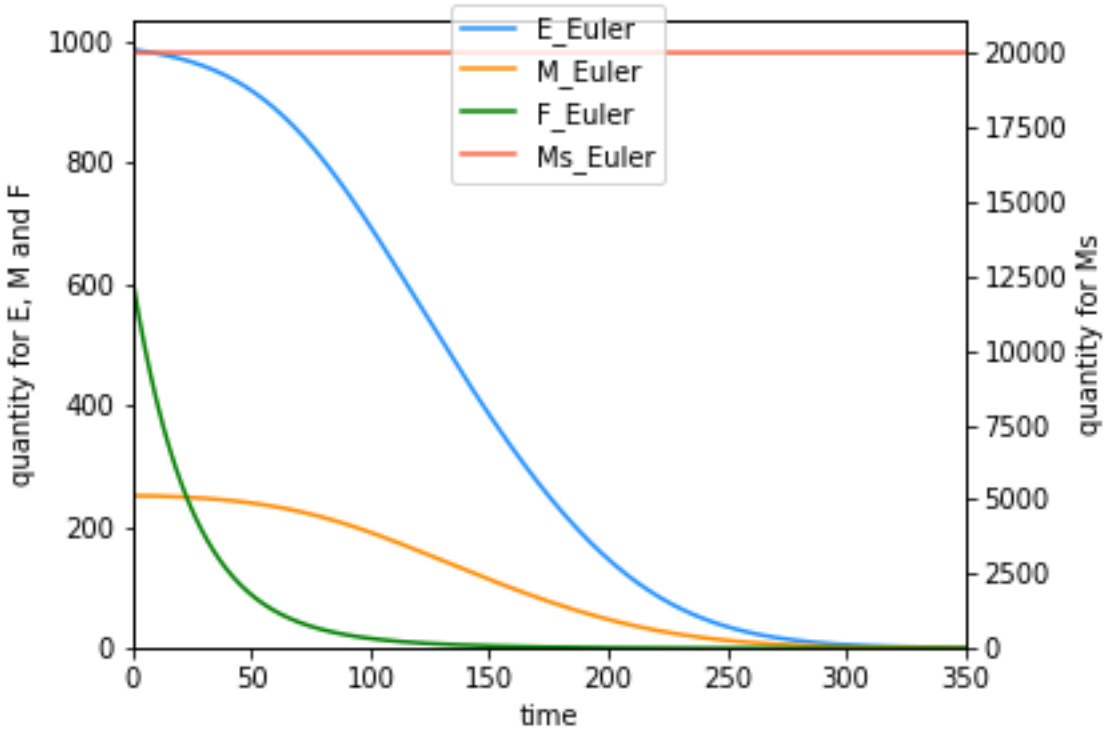

Figure 5: Euler's method on the deterministic model. $K = 1'000$, $M_s(t) = 20'000$, $t_N = 350$

This method is not realistic. Indeed, We can not just release a constant quantity of mosquitoes. The releases must be discrete, we choose at certain time the quantity to free and at all the other times, the quantity is zero.

The release of the sterile male mosquitoes should be a list of pair with the time and the quantity. For example,

$$sterile = [(0, 20'000), (35, 50'000), (80, 130'000)] \tag{16}$$

means we will free at time 0, $20'000$ mosquitoes, at time 35, $50'000$ mosquitoes and at time 80, $130'000$ mosquitoes. We so release sterile males three times.

To implement this method, we create a new algorithm. As the old one, we create a list for the values of X. We choose $X_0 = \bar{X}$ and we do recursively :

- $X_{n+1} = X_n + (t_n - t_{n-1})\, f(X_n, t_{n-1})$

- If $t_n >= sterile_{[0][0]}$ (first element of the list and first element of the pair, that means the first time of release) :
  - We add $sterile_{[0][1]}$ to M_s. (first element of the list and second element of the tuple, that means the quantity corresponding to the time)
  - We remove the first element of *sterile* to not add multiple times a quantity.
- We do it again until we reach t_N.

In figure (6) we can see in red the 3 releases at times 0, 35 and 80. We do not have extinction here because we did not release enough mosquitoes.

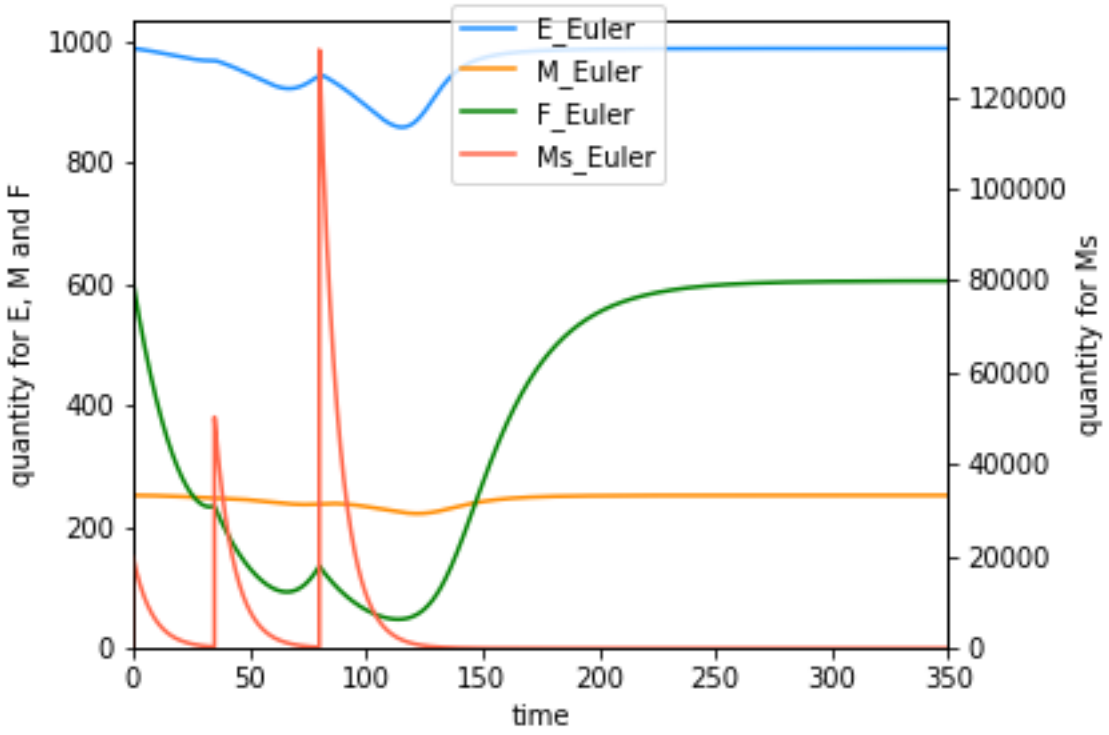

Figure 6: Euler's method on the deterministic model with 3 mosquitoes releases. $K = 1'000$, $t_N = 350$

Plotting the deterministic model will be the reference for the stochastic model because in high population, the two models should be equivalent.

3.3.2 Stochastic Model

The stochastic model is implemented like the linear model but with different transitions.

We first implemented the model with a constant quantity of sterile males. We first calculate the

quantities for time t

$$\begin{cases} p_1(t) = \beta_E \left(1 - \frac{E(t)}{K}\right) F(t) \\ p_2(t) = (1-\nu)\,\nu_E E(t) \\ p_3(t) = \nu\nu_E \frac{M(t)}{M(t)+\gamma_s M_s(t)} E(t) \\ p_4(t) = \left(\delta_E + \nu\nu_E \frac{\gamma_s M_s(t)}{M(t)+\gamma_s M_s(t)}\right) E(t) \\ p_5(t) = \delta_M M(t) \\ p_6(t) = \delta_F F(t) \\ p_7(t) = \delta_S M_s(t) \end{cases} \tag{17}$$

The p_i are the same as in the $\mathcal{Q}$-matrix defined at (7).

Let $p^{(k)}(t) = \sum_{i=1}^{k} p_i(t) \quad \forall t \in \mathbb{R}_+$.

The time between two events $T_{n+1} - T_n$ is exponentially distributed with the parameter $1/\big(p^{(7)}(T_n)\big)$. To simulate the process for all time T_n, we calculate the $p_i(T_n)$. We simulate an exponential random variable $\mathcal{E}$ with the parameter $1/\big(p^{(7)}(T_n)\big)$. We obtain $T_{n+1} = T_n + \mathcal{E}$. We randomly choose the transition between the p_i.

For this, we simulate an uniform random variable $\mathcal{U}$ between zero and one and we choose the corresponding transition :

$$\begin{cases} X_{n+1} = X_n + e_1 & \text{if } \mathcal{U} \le \frac{p^{(1)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 + e_2 & \text{if } \frac{p^{(1)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \le \frac{p^{(2)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 + e_3 & \text{if } \frac{p^{(2)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \le \frac{p^{(3)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 & \text{if } \frac{p^{(3)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \le \frac{p^{(4)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_2 & \text{if } \frac{p^{(4)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \le \frac{p^{(5)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_3 & \text{if } \frac{p^{(5)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \le \frac{p^{(6)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n & \text{otherwise} \end{cases} \tag{18}$$

where for $i \in \{1,2,3,4\}, e_i$ is the ith unit vector of $\mathbb{N}^4$.

The transitions are the same as the $\mathcal{Q}$-matrix defined at (7) except for the last one. This is because we do not want the quantity of sterile mosquitoes to decrease to get a constant quantity. This is like we release one mosquito every time one die.

The figure (7) shows the simulation of this method. We start at the equilibrium $\bar{X} = (\lfloor \bar{E} \rfloor, \lfloor \bar{M} \rfloor, \lfloor \bar{F} \rfloor, 20'000)$ and the quantity of sterile mosquitoes never drops. We can see that the curves are similar to the deterministic curves in figure (5).

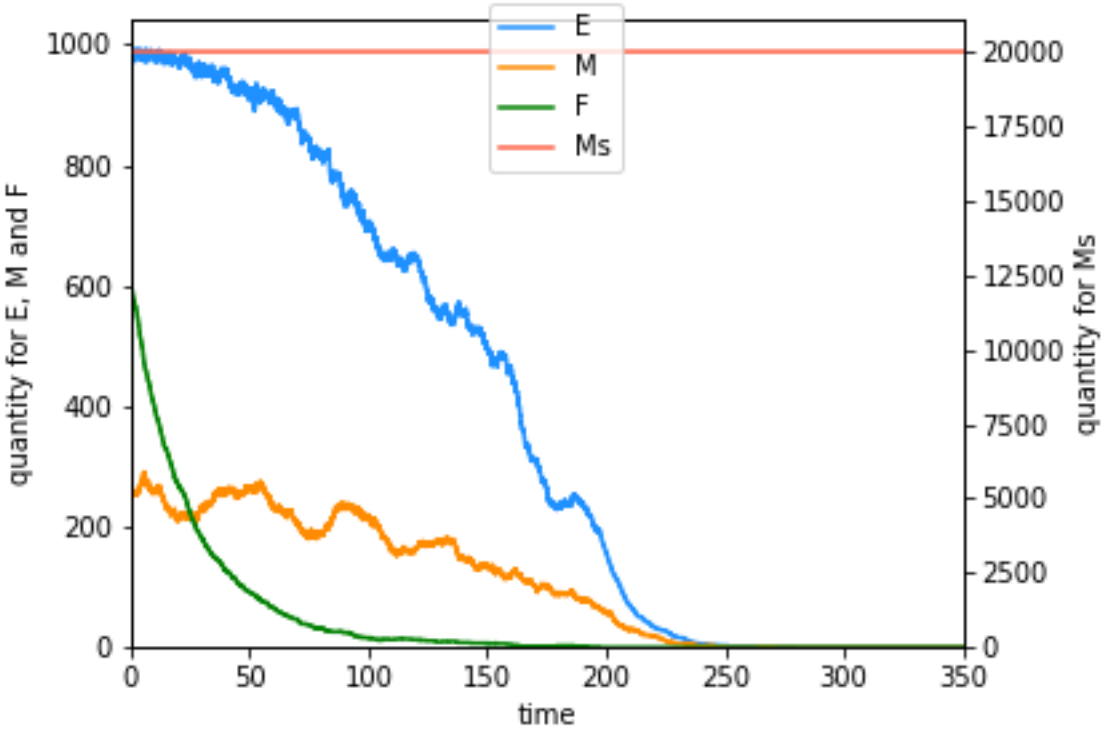

Figure 7: Simulation of the process with constant quantity of sterile = 20′000

We will now simulate with a variable quantity of sterile mosquitoes. To do so, we follow the same procedure but we change the transitions. Moreover, we reuse the same object *sterile* defined in (16) which is the list of times and quantities of releases.

- We calculate the $p_i(t)$ defined in (17).
- We calculate the next time with $\mathcal{E} \sim Exp\left(\frac{1}{p^{(7)}}\right)$.
- We choose the transition with :

$$\begin{cases} X_{n+1} = X_n + e_1 & \text{if } \mathcal{U} \leq \frac{p^{(1)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 + e_2 & \text{if } \frac{p^{(1)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \leq \frac{p^{(2)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 + e_3 & \text{if } \frac{p^{(2)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \leq \frac{p^{(3)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_1 & \text{if } \frac{p^{(3)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \leq \frac{p^{(4)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_2 & \text{if } \frac{p^{(4)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \leq \frac{p^{(5)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_3 & \text{if } \frac{p^{(5)}(T_n)}{p^{(7)}(T_n)} < \mathcal{U} \leq \frac{p^{(6)}(T_n)}{p^{(7)}(T_n)} \\ X_{n+1} = X_n - e_4 & \text{otherwise} \end{cases} \tag{19}$$

This time, the last transition modifies the value of the process. We will see that the sterile population drops during the simulation.

In figure (8) we can see the exact same three releases than in figure (6). We can see that the curves are similar and that there is also no extinction.

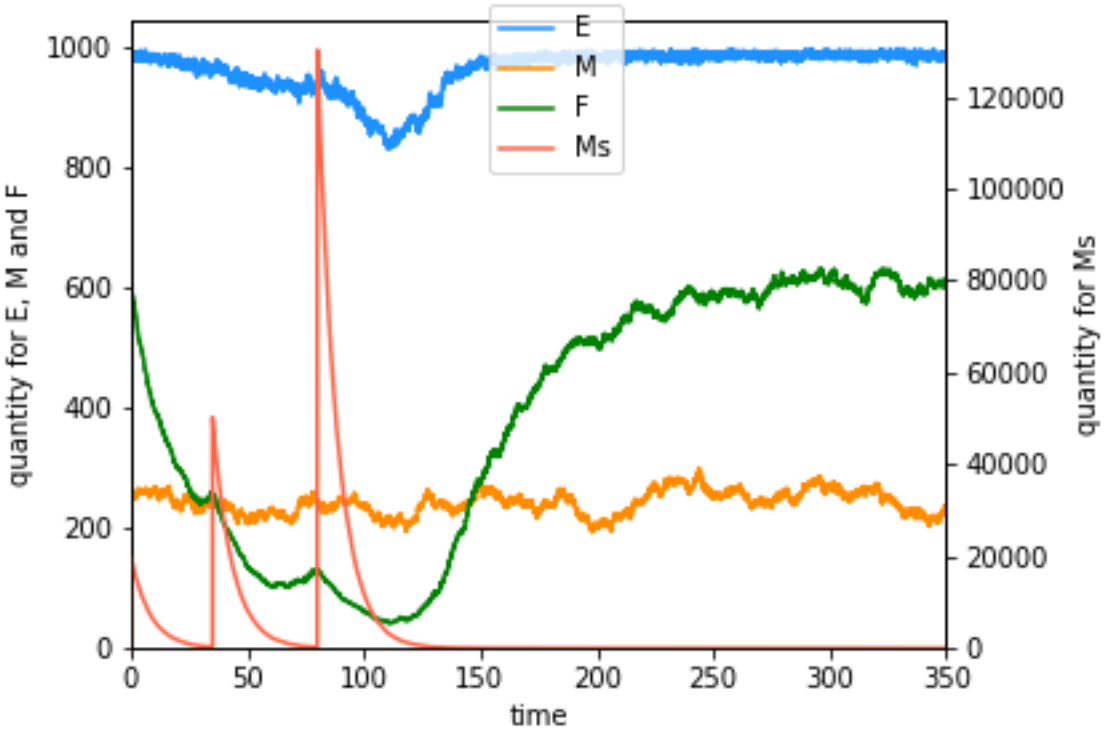

Figure 8: Simulation of the process with three releases

3.3.3 Proportionality between Males and Sterile Males

In this section, we want to release a quantity of sterile mosquitoes depending of the process' evolution. We want to release less mosquitoes if the population is small.

We can define a list of releases depending of the evolution of the process.

$$sterile = [(15, 10M(15)), (35, 5M(35)), (80, 15M(35))]$$

In this case, the program takes the list of the times and the release coefficients. For example, we can free 150 times more sterile mosquitoes than natural males and this every 10 time units until time 200. This will be written like

$$sterile = [(0, 150), (10, 150), (20, 150), ..., (200, 150)] \tag{20}$$

The quantity of releases conditioned to the quantity of males at the time is coded the same as precedent. The difference is that the program has to get the quantity of males and to add the coefficient times this quantity to M_s. It is only one more step. Figure (9) shows a trajectory of the process using release (20).

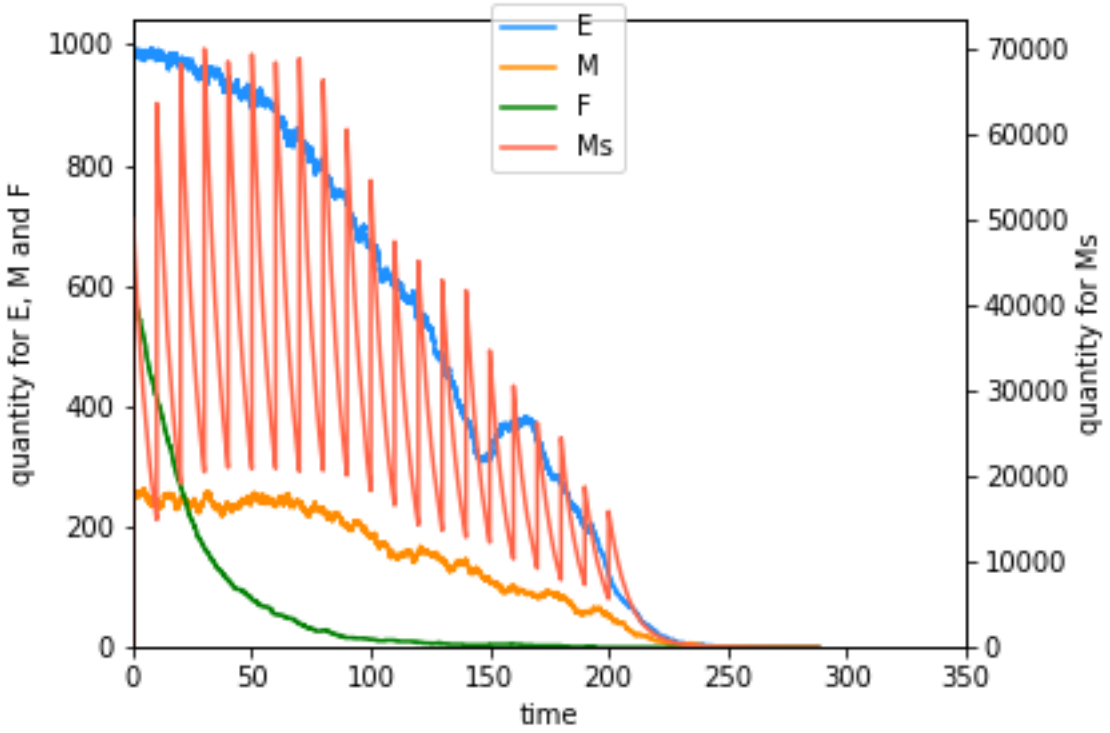

Figure 9: Simulation of the process with releases depending on the quantity of sterile mosquitoes

The advantages of this method is that we use less and less sterile mosquitoes but still obtain extinction.

3.3.4 Code Optimization

We know that in a big population, the stochastic process converges to the deterministic model as we saw in Theorem 3.3. The problem of the stochastic process is that in big population, there will are a lot of transitions due to the exponential variable. So the bigger the population is, the longer will take the program to simulate the process. The idea is to use the deterministic model for big population and when the population comes to low numbers, use the stochastic model. Moreover, the program is faster in small population.

We create a new model half deterministic and when the quantity of the eggs is smaller than a certain value, we change to the half stochastic. We choose the threshold to be 500 eggs.

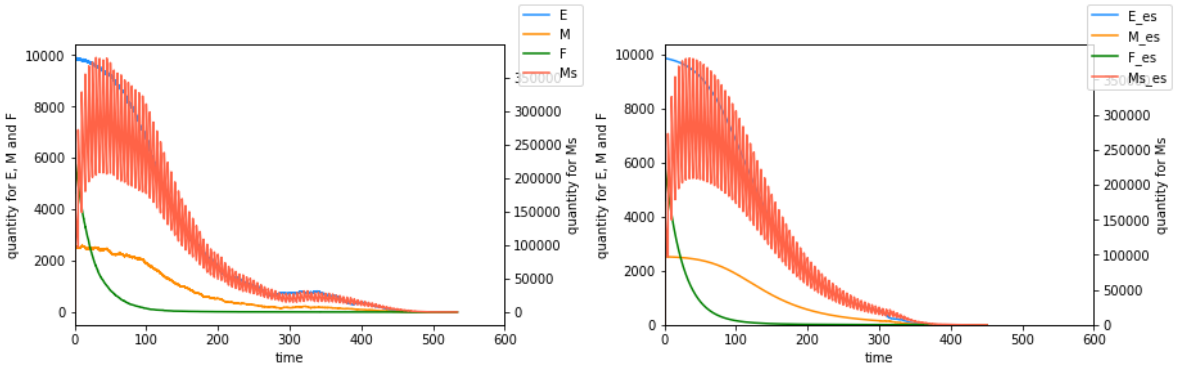

Figure 10: Extinction of the population with release dependent of the natural males = $S_l(5, 100, 100)$. On the left the stochastic model and on the right the half deterministic half stochastic model.

In figure (10) we can see on the left the stochastic model and on the right the half deterministic half stochastic model. We can see that the curves are similar at the beginning and start to vary at the end. Moreover, the stochastic model took one minute to be simulated and the half stochastic model took only two seconds.

With this method, it is possible to simulate multiple times a trajectory of the process to obtain a mean of some interesting values like the extinction time.

3.3.5 Minimum Female

In this section, we use the totally stochastic model.

Let define $S_l(t, x, N) = [(0, x), (t, x), (2t, x), ..., (Nt, x)]$ the releases of sterile mosquitoes depending to a constant interval of time, a constant quantity and the total number of releases. The question is how the population evolves depending on the number of intervals and quantities.

As we can see on the figure (5), the sterile mosquitoes affect firstly the quantity of fecundated females. Only after, the eggs and the males drop. This is why we focus on the fecundated females and how small the female population will be after each drop. We plot the curves with red crosses corresponding to the minimum of the female population on each interval of drop.

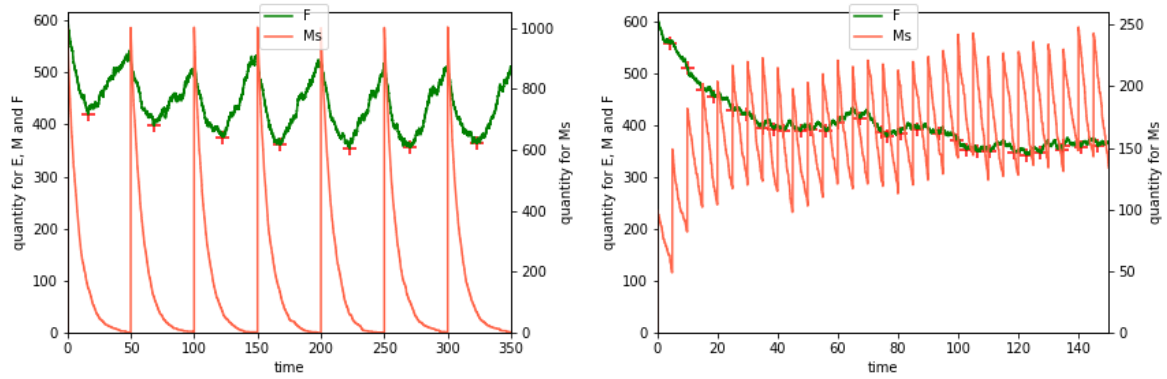

Figure 11: Drop of the female population after release = $S_l(50, 1'000, 7)$ and release = $S_l(5, 100, 30)$.

We can see in figure (12) two different behaviors. On the left, we release a lot of mosquitoes each time but with big time intervals. On the right we release less but more often.

Making the releases dependent on the population of natural males, we obtain two different curves with two behaviors similar to the previous ones.

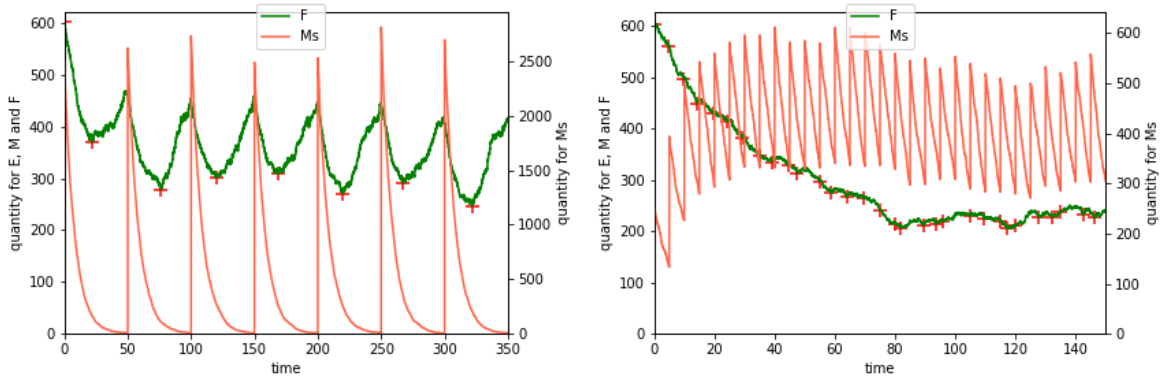

Figure 12: Drop of the female population after release $= S_l(50, 10, 7)$ and release $= S_l(5, 1, 30)$.

3.3.6 Extinction Time

With a certain $S_l(t, x, N)$ we can extinct the mosquitoes population. The extinction time can be calculated using Monte Carlo method.

Let $T(t, x, N) = \inf\{s \in \mathbb{R}_+ : E(s) = 0$ using $S_l(t, x, N)$ as releases$\}$ with $E(s)$ the quantity of eggs at time s.

We want to know the extinction probability defined by $\mathbb{P}\left(T(t, x, N) < \infty\right)$, the value of the extinction time when it is finite and the total quantity of sterile mosquitoes.

By supposing we release mosquitoes until $t_{max} \wedge T(t, x, N)$, we do not need the parameter N. Indeed, let $S_l(t, x) := \lim\limits_{N\to\infty} S_l(t, x, N)$, we now have $T(t, x) = t_{max} \wedge \lim\limits_{N\to\infty} T(t, x, N)$. It means that we will release mosquitoes until the extinction or we reach t_{max}.

In this section, we fix $K = 100$.

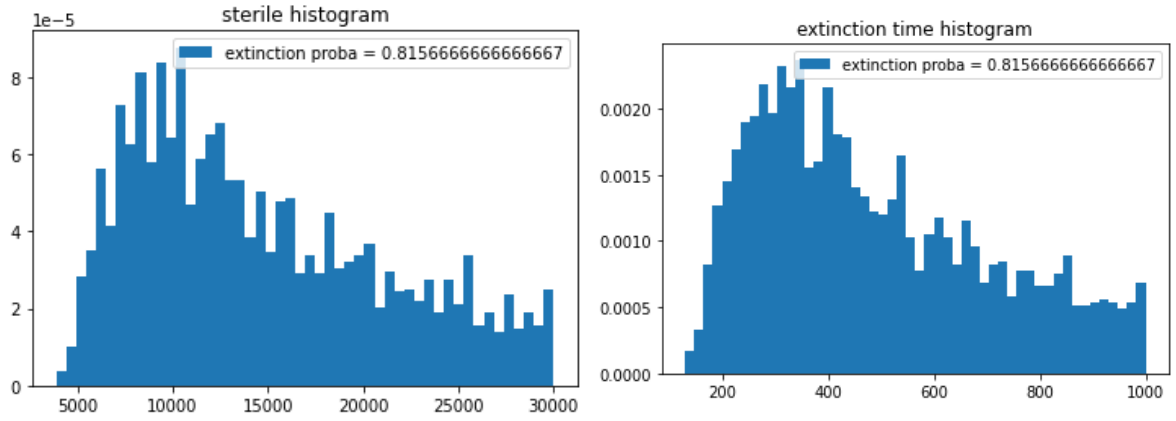

Figure 13: Histogram of extinction times and quantity of sterile mosquitoes using release $= S_l(5, 150)$. $t_{max} = 1000$.

On figure (13) we plot on the left the quantity of sterile mosquitoes used to extinct the population (if the population extincts). On the right, we plot the time $T(t, x) < t_{max}$. We see that the two histogram are similar because the quantity of sterile mosquitoes depends on the extinction time. We can also plot the histogram with quantity of releases conditioned to the quantity of males.

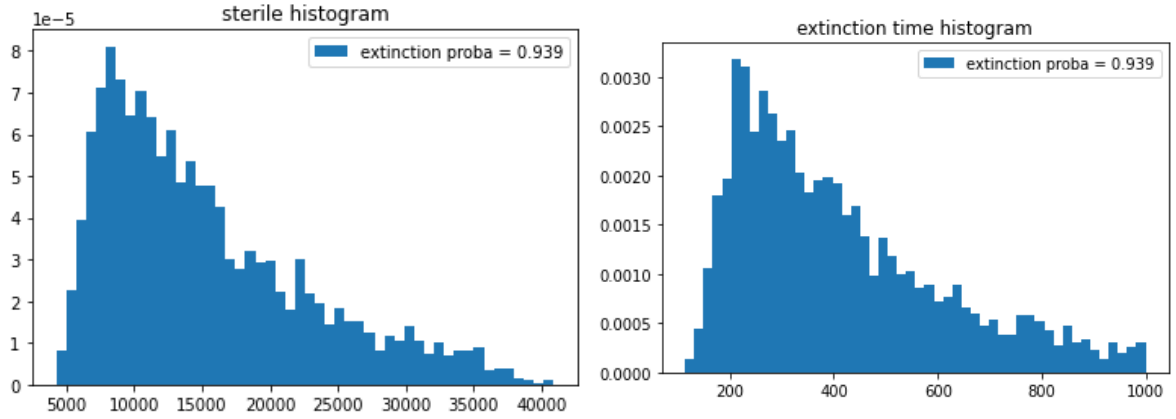

Figure 14: Histogram of extinction times and quantity of sterile mosquitoes using release $= S_l(5, 10)$ and the quantity of releases conditioned to the quantity of males. $t_{max} = 1000$.

On figures (13) and (14), we see that the extinction probability is not equal to one. On histograms, we remove the simulations where we do not have extinction. We also observe a higher variance when the quantity of sterile mosquitoes depends on the quantity of wild males.

3.3.7 Optimal parameters

Let $Q(t, x)$ the quantity of sterile mosquitoes to release to extinct the population. Note that if $T(t, x) = \infty$, $Q(t, x) = \infty$ too.

The problem here is to find the parameters t and x of $S_l(t,x)$ such as the quantity of sterile mosquitoes release is the lowest. We want to find for $p \in [0,1]$:

$$(t^*, x^*) := \underset{t,x}{\operatorname{argmin}}\ Q(t,x) + \infty \mathbb{1}_{\{\mathbb{P}(T(t,x)<\infty)<p\}} = \underset{t,x}{\operatorname{argmin}}\ J(t,x)$$

Two algorithms have been implemented to find the optimal parameters t_p^* and x_p^*.

The first one consists in testing the values of $Q(t,x)$ and $\mathbb{P}\left(T(t,x) < \infty\right)$ using Monte Carlo method on an interval of $\mathbb{R}_+ \times \mathbb{R}_+$. Then, we choose the parameters. This method takes a lot of time. For example, if we test for 10 values of t, 10 values of x and if we simulate 100 times for Monte Carlo. Assume that the simulation takes 5 seconds, this algorithm takes 14 hours.

The second algorithm consists in testing random values to find the minimum. Let $\Delta_i^{(t)}\ \mathcal{N}(0,\sigma^2)$ and $\Delta_i^{(x)}\ \mathcal{N}(0,\sigma^2)$ two independent sequences $\forall i \in \mathbb{N}$. We start to calculate $Q(t_0,x_0)$ for $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}_+$. Then, we calculate recursively $\forall n \in \mathbb{N}^*$

$$(t_n, x_n) = \begin{cases} \left(t_n + \Delta_n^{(t)}, x_n + \Delta_n^{(x)}\right) & \text{if } J\left(t_n + \Delta_n^{(t)}, x_n + \Delta_n^{(x)}\right) < J(t_n, x_n) \\ (t_{n-1}, x_{n-1}) & \text{otherwise} \end{cases}$$

We have of course $(t_n, x_n) \underset{n\to\infty}{\longrightarrow} (t^*, x^*)$. We can now find the optimal parameters such as we have a extinction probability more than p.

By choosing a minimum extinction probability, we can calculate the best parameters to minimise the quantity of sterile mosquitoes to release.

Appendix

Resolution of PDE (3)

$$\frac{\partial G}{\partial t}(z,t) - (z-1)(\lambda z - \mu)\frac{\partial G}{\partial z}(z,t) = 0. \tag{21}$$

Let t and z be functions of r *i.e.* $t = t(r)$ and $z = z(r)$. The Chain rule gives

$$\frac{\mathrm{d}G}{\mathrm{d}r} = \frac{\partial G}{\partial t}\cdot\frac{\mathrm{d}t}{\mathrm{d}r} + \frac{\partial G}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\partial G}{\partial t}(z,t) - (z-1)(\lambda z-\mu)\frac{\partial G}{\partial z}(z,t) = 0.$$

Then we see that

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1 \Rightarrow t(r) = r, \quad \text{for } t(0) = 0,$$

Similarly,

$$\frac{\mathrm{d}z}{\mathrm{d}r} = -(z-1)(\lambda z-\mu) \Rightarrow -\frac{\mathrm{d}z}{(z-1)(\lambda z-\mu)} = \mathrm{d}r, \tag{22}$$

By integrating both sides, one has

$$-\frac{1}{\lambda}\int_{z(0)}^{z(r)} \frac{\mathrm{d}u}{(u-1)(u-\frac{\mu}{\lambda})} = \int_0^r \mathrm{d}s = r = t.$$

Now, we compute the first term of the above equality

$$\begin{aligned} A = -\frac{1}{\lambda}\int_{z(0)}^{z(r)} \frac{\mathrm{d}u}{(u-1)(u-\frac{\mu}{\lambda})} &= -\frac{1}{\lambda-\mu}\int_{z(0)}^{z(t)} \left(\frac{1}{u-1} - \frac{1}{u-\frac{\mu}{\lambda}}\right)\mathrm{d}u \\ &= -\frac{1}{\lambda-\mu}\log\left(\frac{\lambda(z(t)-1)(\lambda z(0)-\mu)}{\lambda(\lambda z(t)-\mu)(z(0)-1)}\right). \end{aligned} \tag{23}$$

From (22) and (23), we solve for $z(0)$

$$z(0) = \frac{\mu(z(t)-1) - (\lambda z(t)-\mu)e^{-(\lambda-\mu)t}}{\lambda(z(t)-1) - (\lambda z(t)-\mu)e^{-(\lambda-\mu)t}}$$

By using the initial condition $G(z,0) = z^m$, the explicit expression of G is given by

$$G(z,t) = \left(\frac{\mu(z-1) - (\lambda z-\mu)e^{-(\lambda-\mu)t}}{\lambda(z-1) - (\lambda z-\mu)e^{-(\lambda-\mu)t}}\right)^m = \left(\frac{\alpha(t)-\beta(t)z}{1-\gamma(t)z}\right)^m = \left((\alpha(t)-\beta(t)z)\sum_{k\geq 0}\gamma(t)^k z^k\right)^m,$$

where

$$\alpha(t) = \frac{\mu(e^{(\lambda-\mu)t}-1)}{\lambda e^{(\lambda-\mu)t}-\mu}, \quad \beta(t) = \frac{\lambda(e^{(\lambda-\mu)t}-1)}{\lambda e^{(\lambda-\mu)t}-\mu}, \quad \text{and} \quad \gamma(t) = \frac{\lambda(1-e^{(\lambda-\mu)t})}{\lambda e^{(\lambda-\mu)t}-\mu}.$$

References

[Kni55] Edward F Knipling. "Possibilities of insect control or eradication through the use of sexually sterile males". In: *Journal of Economic Entomology* 48.4 (1955), pp. 459–462.

[Kni59] Edward F Knipling. "Sterile-Male Method of Population Control: Successful with some insects, the method may also be effective when applied to other noxious animals". In: *Science* 130.3380 (1959), pp. 902–904.

[Flo+14] Salvador Flores et al. "Estimation of populations and sterility induction in Anastrepha ludens (Diptera: Tephritidae) fruit flies". In: *Journal of Economic Entomology* 107.4 (2014), pp. 1502–1507.

[SM16] Todd Shelly and Donald McInnis. "Sterile insect technique and control of tephritid fruit flies: do species with complex courtship require higher overflooding ratios?" In: *Annals of the Entomological Society of America* 109.1 (2016), pp. 1–11.

[Flo+17] Salvador Flores et al. "Density estimation and optimal sterile-to-wild ratio to induce sterility in Anastrepha obliqua populations". In: *Entomologia Experimentalis et Applicata* 164.3 (2017), pp. 284–290.

[Alm+22] Luís Almeida et al. "Optimal control strategies for the sterile mosquitoes technique". In: *Journal of Differential Equations* 311 (2022), pp. 229–266.

[ALV23] Luis Almeida, Alexis Léculier, and Nicolas Vauchelet. "Analysis of the Rolling Carpet Strategy to Eradicate an Invasive Species". In: *SIAM Journal on Mathematical Analysis* 55.1 (2023), pp. 275–309.

[BAC23] Kala Agbo Bidi, Luis Almeida, and Jean-Michel Coron. "Global stabilization of sterile insect technique model by feedback laws". In: *arXiv preprint arXiv:2307.00846* (2023).